PERTURBATION CLASSES OF SEMI-FREDHOLM OPERATORS
IN BANACH LATTICES

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Abstract. We prove some results giving positive answers to the perturbation classes problem for semi-Fredholm operators acting on Banach lattices satisfying certain conditions, and we show that these results can be applied to some Lorentz and Orlicz function spaces.

1. Introduction

In [19, Theorem 5.2] Kato proved that the upper semi-Fredholm operators $\Phi^+$ are stable under additive perturbation by strictly singular operators $SS$: given Banach spaces $X$ and $Y$ for which $\Phi^+(X,Y)$ is nonempty, the set of strictly singular operators $SS(X,Y)$ is contained in the perturbation class of $\Phi^+(X,Y)$, defined as follows:

$$P\Phi^+(X,Y) := \{ K \in L(X,Y): T + K \in \Phi^+ \text{ for every } T \in \Phi^+(X,Y) \}.$$ 

Vladimirkii [28, Corollary 1] proved that the lower semi-Fredholm operators $\Phi^-$ are stable under additive perturbation by strictly cosingular operators $SC$; i.e., $SC(X,Y) \subset P\Phi^-(X,Y)$ when $\Phi^-(X,Y)$ is nonempty. The question whether the equalities $SS(X,Y) = P\Phi^+(X,Y)$ and $SC(X,Y) = P\Phi^-(X,Y)$ are satisfied when the perturbation classes are defined was raised by Gohberg, Markus and Feldman [11, page 74] for $SS$ and $P\Phi^+$, and both questions were stated in [26, 26.6.12]; see also [27, Section 3]. These questions are referred to as the perturbation classes problem for semi-Fredholm operators.

Some partial positive answers to the perturbation classes problem were obtained in [20, 29, 1, 2], but it was proved in [12] that the answer is negative in general: There exists a separable, reflexive Banach space $Z$ for which $P\Phi^+(Z) \neq SS(Z)$ and $P\Phi^-(Z^*) \neq SC(Z^*)$. Further negative answers can be found in [10] and [13].

Although the answer to the perturbation classes problem for $\Phi^+$ and $\Phi^-$ is negative in general, it is still interesting to find spaces $X$ and $Y$ for which $P\Phi^+(X,Y) = SS(X,Y)$ or $P\Phi^-(X,Y) = SC(X,Y)$, because in these cases we have intrinsic characterizations of the operators $K$ in the perturbation classes, i.e., characterizations involving the action of $K$, instead of the properties of the sums of $K$ with all the operators in $\Phi^+(X,Y)$ or $\Phi^-(X,Y)$. Moreover, the spaces that appear in the counterexamples in [12, 10, 13] are very special: they involve finite products in which at least one of the factors is an indecomposable space. The existence of Banach spaces of this kind was only recently proved by Gowers and Maurey [17].

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Let \( P_\parallel \ell \) isomorphic to \( T \)exposition of the perturbation theory for semi-Fredholm operators. An operator \( Q \) surjective, where \( F \) given a closed infinite-codimensional subspace dimensional closed subspace \( E \) we will write \( A \) linear) operators between Banach spaces \( X \), \( Y \) and \( x \). These two results can be applied to some Lorentz and Orlicz function spaces.

Proof. If \( M \) holds when \( K = 0 \), so we only have to deal with the case where \( S(M) \cap N \) is finite-dimensional; by passing to a finite-codimensional subspace of \( N \), we can further assume that \( S(M) \cap N = 0 \).

First, since \( S(M) + N \) is not closed, there exist \( x_1 \in S_M \) and \( y_1 \in N \) such that \( \|S(x_1) - y_1\| < 1/2 \), and then there also exists \( x^*_1 \in S_{X^*} \) such that \( \langle x^*_1, x_1 \rangle = 1 \). Define \( P_0 \) to be the identity mapping on \( M \).

Positive results showing that \( SS(X, Y) = P\Phi_+(X, Y) \) or \( SC(X, Y) = P\Phi_-(X, Y) \) holds when \( X \) or \( Y \) satisfy some conditions have been recently obtained in [16], [15] and [10]. We refer to the introduction of [10] for a description of these results.

In this paper we apply some Banach lattice techniques to obtain further positive answers to the perturbation classes problem for semi-Fredholm operators. In Section 2 we prove a result for \( P\Phi_+ \) for spaces satisfying some technical conditions, and we derive a result for \( P\Phi_- \) from it (Theorems 7 and 8). In Section 3 we show that these two results can be applied to some Lorentz and Orlicz function spaces.

We will use standard notation. If \( X \) is a Banach space, \( S_X \) stands for its unit sphere, and \([x_1, \ldots, x_n]\) is the linear span of \( x_1, \ldots, x_n \in X \). The class of (bounded, linear) operators between Banach spaces \( X \) and \( Y \) will be denoted by \( L(X, Y) \), and we will write \( \mathcal{A}(X, Y) \) for \( \mathcal{A} \cap L(X, Y) \) for any class of operators \( \mathcal{A} \). Given \( T \in L(X, Y) \), the conjugate operator of \( T \) is \( T^* \in L(Y^*, X^*) \). An operator \( T \in L(X, Y) \) is upper semi-Fredholm if its kernel \( N(T) \) is finite-dimensional and its range \( R(T) \) is closed; and \( T \) is lower semi-Fredholm if \( R(T) \) is finite-codimensional and closed in \( Y \). The classes of all upper semi-Fredholm and lower semi-Fredholm operators will be denoted by \( \Phi_+ \) and \( \Phi_- \), respectively. It follows from the basic duality relations for operators that \( T \in \Phi_+ \) if and only if \( T^* \in \Phi_- \) and \( T \in \Phi_- \) if and only if \( T^* \in \Phi_+ \).

An operator \( T \in L(X, Y) \) is strictly singular if the restriction of \( T \) to an infinite-dimensional closed subspace \( E \) is never an isomorphism; \( T \) is strictly cosingular if given a closed infinite-codimensional subspace \( F \) of \( Y \) the composition \( Q_F T \) is never surjective, where \( Q_F \) is the quotient operator onto \( Y/F \). We refer to [14] for an exposition of the perturbation theory for semi-Fredholm operators. An operator \( T \in L(X, Y) \) is \( \ell_p \)-singular if it is not an isomorphism when restricted to any subspace isomorphic to \( \ell_p \).

If \( X \) is a Banach lattice, then \( T \in L(X, Y) \) is disjointly strictly singular if it is not an isomorphism when restricted to any subspace spanned by a disjoint sequence, and \( T \) is AM-compact if the image of every order interval is a relatively compact set.

2. Main results

The following perturbation result is essentially known; we include it here for the convenience of the reader.

**Lemma 1.** Let \( X, Y \) be Banach spaces, let \( M \) be a closed subspace of \( X \) and \( N \) be a closed subspace of \( Y \) and let \( S \in L(X, Y) \) be an operator such that \( S|_M \) is an isomorphism and \( S(M) + N \) is not closed. Then there is a compact operator \( K \in L(X, Y) \) such that \( (S + K)(M) \cap N \) is infinite-dimensional.

**Proof.** If \( S(M) \cap N \) is already infinite-dimensional, the proof is finished by taking \( K = 0 \), so we only have to deal with the case where \( S(M) \cap N \) is finite-dimensional; by passing to a finite-codimensional subspace of \( N \), we can further assume that \( S(M) \cap N = 0 \).

First, since \( S(M) + N \) is not closed, there exist \( x_1 \in S_M \) and \( y_1 \in N \) such that \( \|S(x_1) - y_1\| < 1/2 \), and then there also exists \( x^*_1 \in S_{X^*} \) such that \( \langle x^*_1, x_1 \rangle = 1 \). Define \( P_0 \) to be the identity mapping on \( M \).
Assume now that we have \( x_1, \ldots, x_n \in S_M \) and \( x^*_1, \ldots, x^*_n \in X^* \) such that \( \langle x^*_i, x_j \rangle = \delta_{ij} \) for all \( i, j \in \{1, \ldots, n\} \) and \( \|S(x_k) - y_k\| \|x^*_k\| < 1/2^k \) for all \( k \in \{1, \ldots, n\} \). Define \( M_n := M \cap \bigcap_{i=1}^n N(x^*_i) \), so that \( M = M_n \oplus [x_1, \ldots, x_n] \), and let \( F_n: M \to M_n \) be the projection from \( M \) onto \( M_n \) with kernel \([x_1, \ldots, x_n]\). Then \( S(M_n) + N \) is still not closed in \( Y \), so there exists \( x_{n+1} \in S_{M_n} \) and \( y_{n+1} \in N \) such that \( \|S(x_{n+1}) - y_{n+1}\| < 1/(2^{n+1} \|P_n\|) \); note that \( x_{n+1} \in M_n \) implies \( \langle x^*_i, x_{n+1} \rangle = 0 \) for all \( i \in \{1, \ldots, n\} \). Now, by the Hahn-Banach theorem, there exists \( x^*_n \in S_{X^*} \) such that \( \langle x^*_n, x_j \rangle = 0 \) for all \( j \in \{1, \ldots, n\} \) and \( \langle x^*_n, x_{n+1} \rangle = \text{dist}(x_{n+1}, [x_1, \ldots, x_n]) \geq \|P_n\|^{-1} \); scaling as needed, we can assume \( \langle x^*_n, x_{n+1} \rangle = 1 \) and \( \|x^*_n\| \leq \|P_n\| \), so \( \|S(x_{n+1}) - y_{n+1}\| \|x^*_n\| < 1/2^{n+1} \).

By induction, we now have a biorthogonal sequence \((x^*_n, x_n)_{n \in \mathbb{N}} \) in \( X^* \times M \) and a sequence \((y_n)_{n \in \mathbb{N}} \subseteq N \) such that \( \|x_n\| = 1 \) and \( \|S(x_n) - y_n\| \|x^*_n\| < 1/2^n \) for all \( n \in \mathbb{N} \). Then the expression \( K(x) = \sum_{n=1}^\infty \langle x^*_n, x \rangle (y_n - S(x_n)) \) defines a compact operator \( K: X \to Y \) such that \((S + K)(M) \cap N \) contains the infinite-dimensional subspace \([(y_n)_{n \in \mathbb{N}}] = [(S + K)(x_n)]_{n \in \mathbb{N}} \).

**Lemma 2.** Let \( X \) be a Banach space, let \( Y \) be a Banach space containing an isomorphic copy \( Z \) of \( X \), let \( M \) be an infinite-dimensional complemented subspace of \( X \) and let \( S: X \to Y \) be an operator such that \( S|_M \) is an isomorphism. Then:

(i) if \( S(M) \cap Z \) is finite-dimensional and \( S(M) + Z \) is closed, then \( S \notin P\Phi_+(X, Y) \);

(ii) if \( S \in P\Phi_+(X, Y) \), then there exists a compact operator \( K \in \mathcal{L}(X, Y) \) such that \((S + K)(M) \cap Z \) is infinite-dimensional.

**Proof.** (i) Let \( U: X \to Z \) be an isomorphism, let \( H \) be a complement of \( M \) in \( X \), and define the operator \( T: X = M \oplus H \to Y \) as \( T(m + h) = -S(m) + U(h) \). Then \( N(T) \subseteq S^{-1}(S(M) \cap Z) + U^{-1}(S(M) \cap Z) \) is finite-dimensional and \( R(T) \) is closed because \( R(T) = S(M) + U(H) \subseteq S(M) + Z \) and \( S(M) \cap Z \) is finite-dimensional, so \( T \in \Phi_+(X, Y) \). However, \( M \subseteq N(T + S) \), so \( T + S \notin \Phi_+(X, Y) \) and \( S \notin P\Phi_+(X, Y) \).

(ii) If \( S \in P\Phi_+(X, Y) \), then either \( S(M) \cap Z \) is infinite-dimensional or \( S(M) + Z \) is not closed, so taking \( K \) to be either 0 or the operator provided by Lemma 1 finishes the proof.

The following result is essentially contained in [8, Proposition 2.5], although not formally stated in this form. We refer to [21, Definition 1.e.12] for the concept of cotype of a Banach space.

**Proposition 3.** Let \( X \) be a Banach lattice with finite cotype, let \( Y \) be a Banach space and let \( T: X \to Y \) be an operator such that \((Tf_n)_{n \in \mathbb{N}} \) is relatively compact for any order-bounded sequence \((f_n)_{n \in \mathbb{N}} \) equivalent to the unit vector basis of \( \ell_1 \). Then \( T \) is AM-compact.

**Proof.** Let \( x \in X_+ \) and denote by \( E_x \) the closed ideal of \( X \) generated by \( x \). Since \( X \) is \( q \)-concave for some \( 2 < q < \infty \) [22, Corollary 1.f.9], we have \( L_q(\mu) \hookrightarrow E_x \hookrightarrow L_1(\mu) \) for a certain probability measure \( \mu \) [18, p. 14].

Let \((f_n)_{n \in \mathbb{N}} \) be a sequence in \([-x, x] \), which means that \((f_n)_{n \in \mathbb{N}} \) is uniformly bounded in \( E_x \). Since the order intervals in \( X \) are weakly compact, we can assume that \((f_n)_{n \in \mathbb{N}} \) is weakly null without loss of generality. If \((f_n)_{n \in \mathbb{N}} \) is not seminormalized, it has a convergent subsequence, and so has \((Tf_n)_{n \in \mathbb{N}} \).
Proposition 3 can be used, of course, under the stronger assumption that \((f_n)_{n \in \mathbb{N}}\) is equivalent to the unit vector basis of \(\ell_2\). Since a normalised disjoint sequence in \(L_p(\mu)\) spans a subspace isomorphic to \(\ell_p\), the span of \((f_n)_{n \in \mathbb{N}}\) has to be strongly embedded in \(L_p(\mu)\) [8, Proposition 1.1]. This means that the \(L_p(\mu)\) and \(L_1(\mu)\) topologies coincide on the span of \((f_n)_{n \in \mathbb{N}}\), and so \((f_n)_{n \in \mathbb{N}}\) is equivalent to the unit vector basis of \(\ell_2\) in \(E_x\), too. By hypothesis, \((Tf_n)_{n \in \mathbb{N}}\) is relatively compact in this case as well, finishing the proof.

Recall that a Banach lattice \(Y\) satisfies a lower 2-estimate if there exists a constant \(C\) such that for every choice of pairwise disjoint elements \((y_j)_{j=1}^n\) in \(Y\), we have
\[
\left\| \sum_{j=1}^n y_j \right\| \geq C^{-1} \left( \sum_{j=1}^n \|y_j\|^2 \right)^{1/2}.
\]

**Remark 4.** Proposition 3 can be used, of course, under the stronger assumption that \((Tf_n)_{n \in \mathbb{N}}\) is relatively compact for any sequence \((f_n)_{n \in \mathbb{N}}\) equivalent to the unit vector basis of \(\ell_2\), not just those that are order-bounded. This is equivalent to requiring that \(TU\) be compact for every isomorphic embedding \(U: \ell_2 \rightarrow X\), as is the case when \(T: X \rightarrow Y\) is \(\ell_2\)-singular and \(\text{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)\). This last condition \(\text{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)\) holds, for instance, whenever \(Y\) is a Banach lattice with a lower 2-estimate [8, Proposition 2.1]; note that all Banach lattices with cotype 2 satisfy a lower 2-estimate, although the converse is not true [22, Example 1.f.19].

This leads us to the following result.

**Proposition 5.** Let \(X\) be a Banach lattice with finite cotype such that every copy of \(\ell_2\) in \(X\) contains a complemented copy, let \(Y\) be a Banach space containing an isomorphic copy of \(X\) such that \(\text{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)\) and let \(S \in P\Phi_+(X, Y)\). Then \(S\) is AM-compact.

**Proof.** We will first prove that \(S\) is \(\ell_2\)-singular. Assume, to the contrary, that there exists a subspace \(M \subseteq X\) isomorphic to \(\ell_2\) such that \(S|_M\) is an isomorphism. By hypothesis, \(M\) can be assumed to be complemented. Let \(Z\) be an isomorphic copy of \(X\) in \(Y\); then, by Lemma 2, there exists a compact operator \(K \in \mathcal{L}(X, Y)\) such that \(\hat{S}(M) \cap Z\) is infinite-dimensional, where \(\hat{S} = S + K \in P\Phi_+(X, Y)\). By passing to a subspace of \(M\), we can further assume that \(\hat{S}|_M\) is an isomorphism and that \(\hat{S}(M) \subseteq Z\). Now, again by hypothesis, \(\hat{S}(M)\) must contain a subspace complemented in \(Z\), so it is possible to find subspaces \(N \subseteq M\) and \(H \subseteq Z\) such that \(Z = \hat{S}(N) \oplus H\) and \(H\) is isomorphic to \(X\). But then Lemma 2 provides \(\hat{S} \notin P\Phi_+(X, Y)\), a contradiction, so \(S\) must indeed be \(\ell_2\)-singular.

To finish the proof, take any isomorphism \(U \in \mathcal{L}(\ell_2, X)\); then \(SU \in \text{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)\), so Proposition 3 can be applied and \(S\) is AM-compact.

Throughout the following result we will make liberal use of this well-known fact (see, for instance, [3, Proposition 2.2.1]): If \(M\) is isomorphic to \(\ell_p\), for some \(1 \leq p < \infty\), then every infinite-dimensional closed subspace of \(M\) contains a further subspace \(N\) such that \(N\) is isomorphic to \(\ell_p\) and complemented in \(M\). As a consequence, if \(M\) is a complemented subspace of \(X\), then so is \(N\).
Proposition 6. Let $X$ be a Banach lattice and let $p \in (1, \infty)$ be such that

(i) every subspace of $X$ spanned by a disjoint sequence contains a further subspace that is complemented in $X$ and isomorphic to $\ell_p$;

(ii) for every subspace $M$ of $X$ isomorphic to $\ell_p$, there exist subspaces $N \subseteq M$ and $H \subseteq X$ such that $N$ is infinite-dimensional, $H$ is isomorphic to $X$, $N \cap H = 0$ and $N + H$ is closed.

Let $Y$ be a Banach space containing an isomorphic copy of $X$ and let $S \in P\Phi_+(X,Y)$. Then $S$ is disjointly strictly singular.

Proof. Assume, to the contrary, that there exists a subspace $M \subseteq X$, spanned by a sequence of disjoint elements, such that $S|_M$ is an isomorphism. By (i), we can assume $M$ to be complemented and isomorphic to $\ell_p$.

Take $Z$ to be an isomorphic copy of $X$ in $Y$; then, by Lemma 2, there exists a compact operator $K \in \mathcal{L}(X,Y)$ such that $\tilde{S}(M) \cap Z$ is infinite-dimensional, where $\tilde{S} = S + K \in P\Phi_+(X,Y)$. By passing to a subspace of $M$, we can further assume that $\tilde{S}|_M$ is an isomorphism and that $\tilde{S}(M) \subseteq Z$, while $M$ is still complemented in $X$.

Now, by hypothesis (ii), there exist subspaces $N \subseteq \tilde{S}(M)$ and $H \subseteq Z$ such that $N$ is infinite-dimensional, $H$ is isomorphic to $X$, $N \cap H = 0$ and $N + H$ is closed. Then $\tilde{S}|_M^{-1}(N) \subseteq M$, which is isomorphic to $\ell_p$ and complemented in $X$, so there is $G \subseteq S|_M^{-1}(N)$ again isomorphic to $\ell_p$ and complemented in $X$. But this means that $	ilde{S}(G) \cap H = 0$ and $\tilde{S}(G) + H$ is closed, contradicting $\tilde{S} \notin P\Phi_+(X,Y)$ by Lemma 2.

Combination of Propositions 5 and 6 brings the following.

Theorem 7. Let $X$ be a Banach lattice with finite cotype such that

(i) every copy of $\ell_2$ in $X$ contains a complemented copy;

(ii) there exists $p \in (1, \infty)$ such that every subspace of $X$ spanned by a disjoint sequence contains a further subspace that is complemented in $X$ and isomorphic to $\ell_p$;

(iii) for every subspace $M$ of $X$ isomorphic to $\ell_p$, there exist subspaces $N \subseteq M$ and $H \subseteq X$ such that $N$ is infinite-dimensional, $H$ is isomorphic to $X$, $N \cap H = 0$ and $N + H$ is closed.

Let $Y$ be a Banach space containing an isomorphic copy of $X$ and such that $SS(\ell_2,Y) = \mathcal{K}(\ell_2,Y)$. Then $P\Phi_+(X,Y) = SS(X,Y)$.

Proof. Let $S \in P\Phi_+(X,Y)$; then, by Propositions 5 and 6, $S$ is both AM-compact and disjointly strictly singular, so $S \in SS(X,Y)$ [8, Theorem 2.4].

Theorem 7 admits an immediate dual version, when $Y$ is reflexive, using [13, Theorem 2.3].

Theorem 8. Let $Y$ be a reflexive Banach lattice with finite type such that $Y^*$ satisfies conditions (i), (ii) and (iii) in Theorem 7, and let $X$ be a Banach space admitting a quotient isomorphic to $Y$ and such that $SC(X,\ell_2) = \mathcal{K}(X,\ell_2)$. Then $P\Phi_-(X,Y) = SC(X,Y)$.

Proof. Since $SC(X,\ell_2) = \mathcal{K}(X,\ell_2)$, by [13, Lemma 2.1] we have $SS(\ell_2,Y^*) = \mathcal{K}(\ell_2,Y^*)$, so Theorem 7 provides $P\Phi_+(Y^*,X^*) = SS(Y^*,X^*)$, from which it follows $P\Phi_-(X,Y) = SC(X,Y)$ [13, Theorem 2.3].
This proof relies on the fact that $\mathcal{SC}(X, \ell_2) = \mathcal{K}(X, \ell_2)$ implies $SS(\ell_2, X^*) = \mathcal{K}(\ell_2, X^*)$; it is not difficult to see that, in fact, the converse also holds.

3. Applications

While the conditions described in the hypotheses of Theorem 7 are somewhat technical, they are readily satisfied by large classes of Banach spaces, such as some Lorentz and Orlicz function spaces.

We begin with a straightforward lemma that will be useful later. If $X$ is a Köthe function space on $[0, 1]$, that is, a Banach lattice which is lattice-isomorphic to a (not necessarily closed) ideal of $L_1$, and $0 \leq a < b \leq 1$, we will write $X(a, b)$ for the subspace of $X$ consisting of all functions $f \in X$ such that $f = f_{[a,b]}$. We will say that a closed subspace $M$ of a Köthe function space on $[0, 1]$ is strongly embedded when the natural inclusion of $M$ into $L_1$ is an isomorphic embedding.

**Lemma 9.** Let $X$ be a Köthe function space on $[0, 1]$, and let $M$ be a reflexive, strongly embedded subspace of $X$. Then there exist $0 < a < b \leq 1$ such that $M \cap X(a, b) = 0$ and $M + X(a, b)$ is closed.

**Proof.** Assume otherwise; then, for every $n \in \mathbb{N}$, we can find $f_n \in S_M$ and $g_n \in X(\frac{1}{n+1}, \frac{1}{n})$ such that $\|f_n - g_n\| < 2^{-n}$. Since $M$ is strongly embedded in $L_1$, there is $C > 0$ such that $\|f_n\|_1 > C$ for all $n \in \mathbb{N}$; combined with $\|f_n - g_n\|_1 \leq \|f_n - g_n\| < 2^{-n}$, this means that (a subsequence of) $(g_n)_{n \in \mathbb{N}}$ is disjointly supported and seminormalised in $L_1$, hence equivalent to the unit vector basis of $\ell_1$, and so must be $(f_n)_{n \in \mathbb{N}}$. But this is impossible since $M$ is reflexive.

Our first application of Theorem 7 will be to the class of $L_{p,q}$ spaces, for suitable values of $p$ and $q$. Recall that, given $1 < p < \infty$ and $1 \leq q < \infty$, the space $L_{p,q}(I)$ is the space of (equivalence classes of) measurable functions $f$ on $I$ such that $\|f\|_{p,q} < \infty$, where

$$\|f\|_{p,q} = \left( \int_I \frac{q}{t} f^q(t)^\frac{q}{p} t^{\frac{p}{q} - 1} dt \right)^{1/q};$$

here, $f^*$ is the decreasing rearrangement of $|f|$. We will only be concerned with $I = [0, 1]$ or $I = [0, \infty)$, and will generally follow [4] when dealing with these spaces. Note that $L_{p,q}(0, 1)$ and $L_{p,q}(0, \infty)$ are not isomorphic unless $p = q$ [4, Corollary 2.2], although $L_{p,q}(0, 1)$ is a subspace of $L_{p,q}(0, \infty)$. Given $s \in (0, \infty)$, we will denote the restriction mapping from $L_{p,q}(0, \infty)$ to $L_{p,q}(0, s)$ by $P_s(f) = f|_{[0,s]}$.

**Theorem 10.** [4, Theorem 2.5] Let $1 < p < \infty$ and $1 \leq q < \infty$.

(i) If $M$ is a subspace of $L_{p,q}(0, 1)$, then $M$ contains a complemented copy of $\ell_q$ or $M$ is strongly embedded in $L_{p,q}(0, 1)$.

(ii) If $M$ is a subspace of $L_{p,q}(0, \infty)$, then $M$ contains a complemented copy of $\ell_q$ or there exists $s \in (0, \infty)$ such that $P_s$ is an isomorphism on $M$ and $P_s(M)$ is strongly embedded in $L_{p,q}(0, s)$.

**Proposition 11.** Let $1 < p < \infty$ and $1 \leq q < \infty$, and let $I$ be either $[0, 1]$ or $[0, \infty)$. Then

(i) every copy of $\ell_2$ in $L_{p,q}(I)$ contains a complemented copy;
(ii) every subspace of \( L_{p,q}(I) \) spanned by a disjoint sequence contains a complemented copy of \( \ell_q \).

(iii) for every copy \( M \) of \( \ell_q \) in \( L_{p,q}(I) \), there exist subspaces \( \mathcal{N} \subseteq M \) and \( H \subseteq L_{p,q}(I) \) such that \( \mathcal{N} \) is isomorphic to \( \ell_q \), \( H \) is isomorphic to \( L_{p,q}(I) \), \( \mathcal{N} \cap H = 0 \) and \( \mathcal{N} + H \) is closed.

**Proof.** (i) Let \( M \) be a copy of \( \ell_2 \) in \( L_{p,q}(I) \). If \( M \) contains a complemented copy of \( \ell_q \), it must be \( q = 2 \) and we are done.

Otherwise, if \( I = [0, 1] \), \( M \) is strongly embedded in \( L_{p,q}(0, 1) \). In this case, take any \( 1 < r < p \); then \( M \) is a copy of \( \ell_2 \) in \( L_r(0, 1) \) and there exist \( \mathcal{N} \subseteq M \) and \( H \subseteq L_r(0, 1) \) such that \( L_r(0, 1) = \mathcal{N} \oplus H \) [25, Theorem 3.1], so \( L_{p,q}(0, 1) = \mathcal{N} \oplus (H \cap L_{p,q}(0, 1)) \).

If \( I = [0, \infty) \), there exists \( s \in I \) such that \( P_s \) is an isomorphism on \( M \) and \( P_s(M) \) is strongly embedded in \( L_{p,q}(0, s) \). By the previous paragraph, there exist \( \mathcal{N} \subseteq P_s(M) \) and \( H \subseteq L_{p,q}(0, s) \) such that \( L_{p,q}(0, s) = \mathcal{N} \oplus H \), so \( L_{p,q}(0, \infty) = P_s \left |_{\mathcal{N}} \right. + \mathcal{N} \oplus P_s^{-1}(H) \).

(ii) See [4, Corollary 2.4].

(iii) Let \( M \) be a copy of \( \ell_q \) in \( L_{p,q}(I) \); if \( M \) contains a complemented copy of \( \ell_q \), it is easy to find a further subspace of \( M \) whose complement is isomorphic to \( L_{p,q}(I) \), so we need only check the case where \( M \) does not contain a complemented copy of \( \ell_q \).

If \( I = [0, 1] \), then \( M \) is strongly embedded in \( L_{p,q}(0, 1) \), and therefore in \( L_r \) for \( 1 < r < p \), hence \( M \) is reflexive. By Lemma 9, there exists \( H := L_{p,q}(a, b) \subseteq L_{p,q}(0, 1) \) such that \( M \cap H = 0 \) and \( M + H \) is closed, where \( H \) is isomorphic to \( L_{p,q}(0, 1) \) (so no \( \mathcal{N} \) is needed).

If \( I = [0, \infty) \), there exists \( s \in I \) such that \( P_s \) is an isomorphism on \( M \) and \( P_s(M) \) is strongly embedded in \( L_{p,q}(0, s) \). By the previous paragraph, there exists a subspace \( H \subseteq L_{p,q}(0, s) \) isomorphic to \( L_{p,q}(0, s) \) such that \( P_s(M) \cap H = 0 \) and \( P_s(M) + H \) is closed. Then \( P_s^{-1}(H) \) is isomorphic to \( L_{p,q}(0, \infty) \), \( M \cap P_s^{-1}(H) = 0 \) and \( M + P_s^{-1}(H) \) is closed.

It is now just one step away to prove the coincidence of the perturbation class of upper semi-Fredholm operators with the class of strictly singular operators when \( X = L_{p,q}(I) \) and \( Y \) meets the criteria of the last result. Note, however, that the requirement that \( SS(\ell_2, Y) = K(\ell_2, Y) \), where \( Y \) must contain a copy of \( L_{p,q}(I) \), excludes all values of \( p \) and \( q \) for which \( L_{p,q}(I) \) itself contains a copy of \( \ell_r \) for \( r > 2 \), so only \( p, q \leq 2 \) make sense.

**Proposition 12.** Let \( 1 < p \leq 2 \) and \( 1 \leq q \leq 2 \), let \( I \) be either \([0, 1]\) or \([0, \infty)\) and let \( Y \) be a Banach space containing an isomorphic copy of \( L_{p,q}(I) \) such that \( SS(\ell_2, Y) = K(\ell_2, Y) \). Then \( P\Phi_+(L_{p,q}(I), \mathcal{N}) = SS(L_{p,q}(I), \mathcal{N}) \).

**Proof.** \( L_{p,q}(I) \) is a Banach lattice with finite cotype [6, Theorems 3.5 and 3.6]; apply Theorem 7 and Proposition 11.

This loss in the range of \( p \) and \( q \) is partially compensated by the fact that, for \( 2 < p < \infty \), the space \( L_{p,q}(I) \) is strongly subprojective [15, Proposition 2.4], so a similar conclusion would follow from [15, Theorem 2.6].

This result can be applied, for instance, to the following spaces.

**Corollary 13.** Let \( 1 \leq q < p < 2 \). Then \( P\Phi_+(L_{p,q}(0, \infty), Y) = SS(L_{p,q}(0, \infty), Y) \) when \( Y \) is one of
(i) $L_q$.
(ii) $L_q(\ell_s)$, for some $p < s \leq 2$.
(iii) $L_{r,s}(0,1)$, for some $1 < r < q$ and $1 \leq s \leq 2$.

**Proof.** All of the aforementioned spaces $Y$ are Banach lattices with cotype 2, so $SS(\ell_2, Y) = K(\ell_2, Y)$ [8, Proposition 2.1]. In order to use Proposition 12, it remains to check that they contain an isomorphic copy of $L_{p,q}(0,\infty)$. For $Y = L_q$ and $Y = L_q(\ell_s)$, this can be seen in [5, Theorem 2, Corollary 2].

For $Y = L_{r,s}(0,1)$, we have that $L_q$ contains a copy of $L_{p,q}(0,\infty)$ [5, Corollary 2]. Take now $r < t < q$; then, in turn, $L_t$ contains a copy of $L_q$ [22, Corollary 2.5]. This copy must be strongly embedded in $L_t$, as it cannot contain $\ell_t$, so it must also be a closed subspace of $L_{r,s}$.

Theorem 7 can also be applied to Lorentz function spaces of type $\Lambda(W, p)$. To fix our notation, we will say that any unbounded, non-increasing function $W$ on $(0,1]$ with $\int_0^1 W(t) dt = 1$ and $W(1) > 0$ is a Lorentz weight function. Given $1 \leq p < \infty$ and a Lorentz weight function $W$, $\Lambda(W, p)$ will be the Lorentz function space of all measurable functions on $[0,1]$ such that $\|f\| = (\int_0^1 f^*(t)^p W(t) dt)^{1/p} \leq \infty$, where $f^*$ is the decreasing rearrangement of $|f|$. These spaces were studied in [7].

We need the following auxiliary result.

**Lemma 14.** Let $1 \leq p < \infty$, let $W$ be a Lorentz weight function and let $0 \leq a < b \leq 1$. Then $\Lambda(W, p)(a,b)$ is isomorphic to $\Lambda(W, p)$.

**Proof.** We will see that the expression $Tf(t) = f(a + t(b-a))$ defines a bijective isomorphism $T: \Lambda(W, p)(a,b) \to \Lambda(W, p)$. It is easy to check that $(Tf)^*(t) = f^*(t(b-a)) \geq f^*(t)$, so $\|Tf\|^p = \int_0^1 ((Tf)^*(t))^p W(t) dt \geq \|f\|^p$. On the other hand, since $W$ is non-increasing,

$$\|Tf\|^p = \int_0^1 ((Tf)^*(t))^p W(t) dt = \int_0^1 f^*(t(b-a))^p W(t) dt$$

$$= \int_0^{b-a} f^*(t)^p W\left(\frac{t}{b-a}\right) \frac{1}{b-a} dt \leq \frac{1}{b-a} \int_0^{b-a} f^*(t)^p W(t) dt = \frac{1}{b-a} \|f\|^p$$

So $T$ is well-defined, and clearly bijective.

In the following proposition, the conditions under which $\Lambda(W, p)$ has finite cotype can be seen in [24].

**Proposition 15.** Let $1 < p < 2$ and let $W$ be a Lorentz weight function such that $\Lambda(W, p)$ has finite cotype. Let $Y$ be a Banach space containing an isomorphic copy of $\Lambda(W, p)$ such that $SS(\ell_2, Y) = K(\ell_2, Y)$. Then $P\Phi_+(\Lambda(W, p), Y) = SS(\Lambda(W, p), Y)$.

**Proof.** We only need to show that $\Lambda(W, p)$ meets the requirements of Theorem 7 for $X$.

(i) Let $M$ be a copy of $\ell_2$ in $\Lambda(W, p)$. Since $M$ cannot contain a complemented copy of $\ell_p$ for $p < 2$, it must embed isomorphically into $L_p$ [7, Remark 5.6], so there exist $N \subseteq M$ and $H \subseteq L_p$ such that $L_p = N \oplus H$ [25, Theorem 3.1], and $\Lambda(W, p) = N \oplus (H \cap \Lambda(W, p))$, where $N$ is a complemented copy of $\ell_2$.

(ii) See [7, Theorem 5.1].
Let \( i \) for all \( \text{malised sequence in } L \).

By Lemma 9, there exists \( M \). Otherwise, if \( M \) embeds isomorphically into \( L_p \), it must be strongly embedded in \( L_p \) and, by Lemma 9, there exists \( H := \Lambda(W,p)(a,b) \subseteq \Lambda(W,p) \) such that \( M \cap H = 0 \) and \( M + H \) is closed, where \( H \) is isomorphic to \( \Lambda(W,p) \) by Lemma 14.

Orlicz function spaces are also good candidates for Theorem 7. Recall that, given an Orlicz function \( \varphi \), the space \( L_\varphi \) consists of all measurable functions \( f \) on \([0,1]\) such that \( \int_0^1 \varphi(|f(t)|/\rho) \, dt < \infty \) for some \( \rho > 0 \), where the norm is given by \( \|f\| = \inf\{ \rho > 0 : \int_0^1 \varphi(|f(t)|/\rho) \, dt < \infty \} \). The complementary function of \( \varphi \) will be denoted by \( \varphi^* \), so that \( L_\varphi^* = L_{\varphi^*} \) [21, Chapter 4], and \( E_\varphi^\infty \) will be the set of functions \( G(t) \) of the form \( \lim_ {n \to \infty} \varphi(t y_n)/\varphi(y_n) \), \( 0 < t < 1 \), for some sequence \( y_n \to \infty \) [23, Section 4]. We will write \( E_\varphi^\infty \equiv \{ F \} \) when every function in \( E_\varphi^\infty \) is equivalent to a certain function \( F \) at 0.

**Proposition 16.** Let \( \varphi \) be an Orlicz function such that \( E_\varphi^\infty \equiv \{ t^p \} \) for some \( 1 < p < 2 \), and let \( Y \) be a Banach space containing an isomorphic copy of \( L_\varphi \) such that \( SS(\ell_2,Y) = K(\ell_2,Y) \). Then \( P\Phi_+(L_\varphi,Y) = SS(L_\varphi,Y) \).

**Proof.** Again, we will prove that \( L_\varphi \) meets the requirements of Theorem 7 for \( X \). First of all, note that \( L_\varphi \) satisfies a lower 2-estimate [22, Proposition 2.b.5], so it has finite cotype.

(i) The Boyd indices for \( L_\varphi \) are \( p_{L_\varphi} = q_{L_\varphi} = p \) [22, Proposition 2.b.5] [23, Section 4]; if we take any \( 1 < q < p \), then \( L_\varphi \) is contained in \( L_q \) [22, Proposition 2.b.3], so any copy of \( \ell_q \) in \( L_\varphi \) must contain a complemented copy by the argument of the proof of Proposition 11 (i).

(ii) Let \( (f_n)_{n \in \mathbb{N}} \) be a normalised disjoint sequence in \( L_\varphi \), and take \( (g_n)_{n \in \mathbb{N}} \) a normalised sequence in \( L^* \) such that \( \supp f_n = \supp g_n \) for all \( n \in \mathbb{N} \) and \( \langle g_i, f_j \rangle = \delta_{ij} \) for all \( i, j \in \mathbb{N} \); note that \( E_\varphi^\infty \equiv \{ t^p \} \) implies \( E_\varphi^\infty \equiv \{ t^q \} \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). Since \( (f_n)_{n \in \mathbb{N}} \) is disjunctly supported, and \( E_\varphi^\infty \equiv \{ t^p \} \), by passing to a subsequence we can assume that \( (f_n)_{n \in \mathbb{N}} \) is equivalent to the unit vector basis of \( \ell_p \) [23, Proposition 3], and similarly that \( (g_n)_{n \in \mathbb{N}} \) is equivalent to the unit vector basis of \( \ell_q \). Then \( Q(f) = \sum_{n \in \mathbb{N}} \langle g_n, f \rangle f_n \) defines a projection from \( L_\varphi \) onto the span of \( (f_n)_{n \in \mathbb{N}} \).

(iii) Let \( M \) be a copy of \( \ell_p \) in \( L_\varphi \); then either \( M \) contains an almost disjoint sequence or \( M \) is strongly embedded in \( L_1 \) [7, Theorem 4.1]. If \( M \) contains an almost disjoint sequence, an argument similar to the previous paragraph shows that it contains a complemented copy of \( \ell_p \). Otherwise, if \( M \) is strongly embedded in \( L_1 \), by Lemma 9, there exists \( H := L_\varphi(a,b) \subseteq L_\varphi \) such that \( M \cap H = 0 \) and \( M + H \) is closed, where \( H \) is isomorphic to \( L_\varphi \).

Many regular Orlicz function satisfy the condition \( E_\varphi^\infty \equiv \{ t^p \} \). For example, any \( \varphi \) such that \( \lim_{t \to \infty} (t \varphi'(t))/\varphi(t) = p \), such as \( \varphi(t) = t^p \log^\alpha (1 + t) \) for \(-\infty < \alpha < \infty \) [9, Section 4].

**Dual results for \( P\Phi_- \).** Using Theorem 8, we can derive some results for \( P\Phi_- \) from Propositions 12, 15 and 16. These results are summarised below.
Proposition 17. Let $Y$ be one of the spaces

(i) $L_{p,q}(0, 1)$ or $L_{p,q}(0, \infty)$ with $2 \leq p, q < \infty$;

(ii) $\Lambda(W, p)$ with $2 < p < \infty$ and finite type;

(iii) $L_{\varphi}$ with $E_\varphi^\infty \equiv \{t^p\}$ for some $2 < p < \infty$,

and let $X$ be a Banach space satisfying $\mathcal{S}C(X, \ell_2) = \mathcal{K}(X, \ell_2)$ and admitting a quotient isomorphic to $Y$. Then $P\Phi_-(X, Y) = \mathcal{S}C(X, Y)$.

The proof is similar to that of Theorem 8. Note that $Y$ is reflexive in all cases.

References


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