Computing Hypercircles by Moving Hyperplanes

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Abstract

Let $K$ be a field of characteristic zero, $\alpha$ algebraic of degree $n$ over $K$. Given a proper parametrization $\psi$ of a rational curve $C$, we present a new algorithm to compute the hypercircle associated to the parametrization $\psi$. As a consequence, we can decide if $C$ is defined over $K$ and, if not, to compute the minimum field of definition of $C$ containing $K$. The algorithm exploits the conjugate curves of $C$ but avoids computation in the normal closure of $K(\alpha)$ over $K$.

1 Introduction

Let $K(\alpha)$ be a computable characteristic zero field with factorization such that $K$ is finitely generated over $\mathbb{Q}$ as a field and $\alpha$ is of degree $n$ over $K$.

Let $\psi(t) = (\psi_1(t), ..., \psi_m(t))$ be a proper parametrization of a rational spatial curve $C$, where $\psi_i \in K(\alpha)(t)$, $1 \leq i \leq m$. The reparametrization problem ask for methods to decide in $C$ is defined or parametrizable over $K$ and, if possible, compute a parametrization of $C$ with coefficients in $K$ easily from a parametrization of $U$ with coefficients in $K$.

In [1], the authors proposed a construction to solve this problem introducing a family of curves called hypercircles and avoiding any implicitization technique. Starting from the parametrization $\psi$, they construct an analog to Weil descente variety to compute a curve $U$ called the witness variety or the parametric variety of Weil. This curve exists if and only if $C$ is defined over $K$ and we can obtain a parametrization of $C$ with coefficients in $K$ easily from a parametrization of $U$ with coefficients in $K$. Efficient algorithms to compute a parametrization of $U$ with coefficients in $K$ are studied in [7], provided we are able to find a point in $U$ with coefficients over $K$.

The definition of $U$ is done under a parametric version of Weil’s descente method. In the proper parametrization $\psi = (\psi_1, ..., \psi_m), \psi_i \in K(\alpha)(t)$ with coefficients in $K(\alpha)$, we substitute $t = \sum_{i=0}^{n-1} \alpha^i t_i$, where $t_0, ..., t_{n-1}$ (where $n$ is the degree of $\alpha$ over $K$).

We can rewrite:

$$\psi_j \left( \sum_{i=0}^{n-1} \alpha^i t_i \right) = \sum_{i=0}^{n-1} \alpha^i \lambda_{ij}(t_0, ..., t_{n-1}), \lambda_{ij} = \frac{F_{ij}}{D} \in K(t_0, ..., t_{n-1})$$

In this context we have the following definition:
Definition 1. The parametric variety of Weil $Z$ of the parametrization $\psi$ is the Zariski closure of

$$\{F_{ij} = 0 \mid 1 \leq i \leq n - 1, \ 1 \leq j \leq N\} \setminus \{D = 0\} \subseteq \mathbb{F}^n.$$ 

Much is known about $Z$, it is always a set of dimension 0 or 1. It is of dimension one exactly in the case that $C$ is defined over $K$ (See [1], [2]). In this case, $Z$ contains exactly one component of dimension 1 that is the searched curve $U$. The computation of the curve $U$ from its definition is unfeasible except for toy examples. The curve $U$ is defined as the unique one dimensional component of the difference of two varieties $A - B$. This already is a hard enough problem to look for alternatives, but this method also uses huge polynomials. If $\psi_i(t) = n_i(t)/d(t)$ and $d = d(\alpha, t) \in K[\alpha, t]$. Let $M(x)$ be the minimal polynomial of $\alpha$ over $K$. In the generic case, the denominator $D$ is $D = \text{Res}_{z}(d(z, \sum_{i=0}^{n-1} z^t_i), M(z))$ which is typically a dense polynomial of degree $dn$ in $n$ variables. Hence, the number of terms of the polynomial $D$ alone is not polynomially bounded in $n$.

The aim of the article is to present an algorithm to compute the variety $U$ that is polynomial in $d$ and $n$ and, if $C$ is not defined over $K$, to compute the smallest field $L$, $K \subseteq L \subseteq K(\alpha)$ that defines $C$. The article is structured as follows. First we introduce in Section 2 the geometric construction that will allow us to derive an efficient algorithm. Then, we show in Section 3 how to compute efficiently some steps of the algorithm. Last, in Section 4 we study the complexity of the algorithm and some running times comparing with other approaches.

2 Synthetic construction of Hypercircles

The problem of parametrizing $C$ over $K$ can be translated to the problem of parametrizing $U$. In the case that $C$ can be parametrized over $K$, then $U$ is a very special curve called hypercircle.

Definition 2. Let $\frac{at+b}{ct+d} \in K(\alpha)(t)$ represent an isomorphism of $F(t)$, $a, b, c, d \in K(\alpha)$, $ad - bc \neq 0$. Write

$$\frac{at+b}{ct+d} = \lambda_0(t) + \alpha \lambda_1(t) + \cdots + \alpha^{n-1} \lambda_{n-1}(t)$$

where $\lambda_i(t) \in K(t)$. The hypercircle associated to $\frac{at+b}{ct+d}$ for the extension $K \subseteq K(\alpha)$ is the parametric curve in $F^n$ given by the parametrization $(\lambda_0, \ldots, \lambda_{n-1})$.

If $C$ cannot be parametrized over $K$ and $K$ is small enough (that means that it is finitely generated over $Q$ as a field, that we can always assume without loss of generality), then there always exists an element $\beta$ algebraic of degree 2 over $K$ such that $[K(\beta, \alpha) : K(\alpha)] = n$ and $C$ can be parametrized over $K(\beta)$, see [12] for
the details. In this situation $\mathcal{U}$ is a hypercircle for the extension $K(\beta) \subseteq K(\alpha, \beta)$. That is, there is an associated unit $\frac{a_1 + b_1 \tau}{a_1 + b_1 \tau}$, but with $a, b, c, d \in K(\beta)$.

Thus, the curve $\mathcal{U}$ is always a hypercircle for certain algebraic extension. So all the geometric properties of hypercircles studied in [6] hold for $\mathcal{U}$ except, maybe, the existence of a point in $\mathcal{U} \cap K^n$. We will exploit the geometric properties of hypercircles to derive our algorithm. We start with the fact that $\mathcal{U}$ is always a rational normal curve in $\mathbb{P}^n$ defined over $K$ (See [6]) and the synthetic construction of rational normal curves as presented in [5].

Let us recall the construction of conics by a pair of pencil of lines. Let $\mathcal{L}(t)$ and $\mathcal{F}(t)$ be two different pencils of lines in the plane with two different base points $l_0 \neq f_0$ and let $C$ be a conic passing through $l_0$ and $f_0$. Then, $C$ induces an isomorphism $u : \mathcal{L}(t) \to \mathcal{F}(t)$ given by extending the map $u(\mathcal{L}(t_0)) = \mathcal{F}(s_0)$ if

$$\mathcal{L}(t_0) \cap C - \{l_0\} = \mathcal{F}(s_0) \cap C - \{f_0\}.$$ 

Conversely, an isomorphism $u$ between $\mathcal{L}(t)$ and $\mathcal{F}(t)$ defines a line or a conic passing through the base points. There is a proper parametrization of this curve given by $t \mapsto \mathcal{L}(t) \cap \mathcal{F}(u(t))$.

**Example 3.** Let $C = x^2 + y^2 - 1$ be the unit circle. And take the pencils of lines that passes through the points at infinity of the circle $[1 : i : 0], [1 : -i : 0]$. 

$$\mathcal{L}(t) = \{ x + iy = t \}, \quad \mathcal{F}(t) = \{ x - iy = t \}. \quad \text{In this case} \quad C \cap \mathcal{L}(t) = (\frac{1 + 1}{2}, \frac{-r^2 + i}{2t}) \quad \text{and} \quad C \cap \mathcal{F}(t) = (\frac{1 + 1}{2}, \frac{r^2 - i}{2t}).$$ 

In this case, the isomorphism between the pencils is given by $u(t) = 1/t$. Now, let us take the isomorphism $u(t) = (t + i)/t$. Then, the conic defined by $u$ from the two pencils of lines is $x^2 + y^2 - x - iy - i$. Which is a conic passing through the base points, although not defined over $\mathbb{Q}$.

More generally, the same geometric construction applies to rational normal curves of degree $n > 2$ in $\mathbb{P}^n$ as explained in [5]. We only show the special case of this construction that is relevant for hypercircles. If $\mathcal{U}$ is a hypercircle, it is known that $\mathcal{U}$ can be parametrized by the pencil of hyperplanes $\mathcal{L}_0 = \{ \sum_{i=0}^{n-1} \alpha^i x_i = t \}$. This pencil of hyperplanes yield to a proper parametrization $\phi = (\phi_0(t), \ldots, \phi_{n-1}(t))$ of the hypercircle with coefficients in $K(\alpha)$ that is called the standard parametrization of the hypercircle and has been studied with detail in [7]. Since the hypercircle is always a curve defined over $K$, it is invariant under conjugation and it can also be parametrized by the conjugate pencil of hyperplanes.

Let us fix some notation. Let $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, the conjugates of $\alpha$ over $K$ in $\mathbb{P}$. Let $\sigma_i$, $0 \leq i \leq n - 1$ be $K$-automorphisms of $F$ such that $\sigma_i(\alpha) = \alpha_i$ and $\sigma_0 = 1d$. If we have a rational function $f(t) \in K(\alpha)(x_1, \ldots, x_r)$, we denote by $f^{\sigma_i} = \sigma_i(f) \in K(\alpha_j)(x_1, \ldots, x_r)$ that results applying $\sigma_i$ to the coefficients of $f$. If $C$ is the original curve, then we denote by $C^\sigma$ the conjugate curve $C^{\sigma} = \{ \sigma(x) | x \in C \}$, where $\sigma(x)$ is applied component-wise. $C^\sigma$ is clearly a rational curve with proper parametrization $\psi^\sigma$.

It is known [2] that $C$ is defined over $K$ if and only if $C = C^{\sigma_i}, 1 \leq i \leq n - 1$ if and only if $\psi^{\sigma_i}$ parametrizes $C$, $1 \leq i \leq n - 1$. 

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The conjugate pencil of hyperplanes \( \mathcal{L}_j(t) = \{ \sum_{i=0}^{n-1} \alpha_i^j x_i = t \} \), 1 \( \leq j \leq n-1 \) also parametrizes \( \mathcal{U} \), yielding the conjugate parametrization \( \phi^{\sigma_i}(t) = \sigma_i(\phi(t)) \).

The hypercircle then induces an isomorphism \( u_j(t) \) between \( \mathcal{L}_0(t) \) and \( \mathcal{L}_j(t) \) given by \((\mathcal{L}_0(t_0) \cap \mathcal{U}) - H = (\mathcal{L}_j(u_j(t_0)) \cap \mathcal{U}) - H \) for all but finitely many parameters \( t_0 \), where \( H \) is the hyperplane at infinity of \( \mathbb{P}(\mathbb{F})^n \). So \( \phi(t) = \phi^{\sigma_i}(u_j(t)) \), from which \( u_j(t) = (\phi^{\sigma_i})^{-1} \circ \phi \). But, by construction, \((\phi^{\sigma_i})^{-1}(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} \alpha_i^j x_i \) and \( u_j(t) = \sum_{i=0}^{n-1} \alpha_i^j \phi_i(t) \). Conversely, a set of isomorphisms \( u_j : \mathcal{L}_0(t) \to \mathcal{L}_j(t), 0 \leq j \leq n-1 \), \( u_0(t) = t \), defines a rational normal curve given by \( t \to \bigcap_{i=1}^{n-1} \mathcal{L}_j(u_j(t)) \). So, we can recover the standard parametrization of the hypercircle if we know the isomorphisms \( u_j, 0 \leq j \leq n-1 \), where \( u_0(t) = t \). The standard parametrization \( \phi \) is the unique solution of the Vandermonde linear system of equations:

\[
\begin{pmatrix}
1 & \alpha & \ldots & \alpha^{d-1} \\
1 & \alpha_2 & \ldots & \alpha_2^{d-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \alpha_n & \ldots & \alpha_n^{d-1}
\end{pmatrix}
\begin{pmatrix}
\phi_0 \\
\phi_1 \\
\vdots \\
\phi_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
u_0(t) = t \\
u_1(t) \\
\vdots \\
u_{n-1}(t)
\end{pmatrix}
\]  

(1)

with coefficients on the normal closure of \( \alpha \) over \( \mathbb{K} \).

As in the planar case, if the automorphisms are generic enough, the curve \( \mathcal{U} \) will be of degree \( n \). In this case we say that \( \mathcal{U} \) is a primitive hypercircle. There may be cases in which the curve \( \mathcal{U} \) is of degree less than \( n \). If this is the case, the degree of \( \mathcal{U} \) must be a divisor of \( n \) and is related with the field of definition of the place of \( \mathcal{C} \) corresponding to \( \psi(t) = \infty \), as showed in [12].

The good news is that we can compute easily the automorphisms \( u_j(t) \) from the parametrization \( \psi(t) \) alone.

**Theorem 4.** Let \( \psi(t) \in \mathbb{K}(\alpha)(t)^m \) be a proper parametrization of \( \mathcal{C} \) and assume that \( \mathcal{C} \) is defined over \( \mathbb{K} \). Let \( \phi(t) \) be the standard parametrization of the associated hypercircle \( \mathcal{U} \). Let \( \sigma_i \) be a \( \mathbb{K} \)-automorphism of \( \mathbb{F} \). Let \( \phi^{\sigma_i} = \sigma_i(\phi) \), \( \psi^{\sigma_i} = \sigma_i(\psi) \) be the conjugate parametrizations and \( u_{\sigma_i}(t) = (\phi^{\sigma_i})^{-1} \circ \phi \) be the conjugation isomorphism induced by \( \mathcal{U} \) in the pencil of hyperplanes \( \mathcal{L}_0 \) and \( \mathcal{L}_{\sigma_i} \). Then \( u_{\sigma_i} = (\psi^{\sigma_i})^{-1} \circ \psi \).

**Proof.** We identify \( \mathcal{C} \) with the diagonal curve \( \Delta \) in the variety \( \mathcal{C} \times \mathcal{C}^{\sigma_1} \times \ldots \times \mathcal{C}^{\sigma_{n-1}}, \Delta = \{(x, \ldots, x)|x \in \mathcal{C}\} \) the hypercircle \( \mathcal{U} \) is a curve such that the map

\[
(\mathcal{U}) \to \mathcal{C} \times \mathcal{C}^{\sigma_1} \times \ldots \times \mathcal{C}^{\sigma_{n-1}}
\]

\[
(x_0, \ldots, x_{n-1}) \mapsto \left( \psi\left( \sum_{j=0}^{n-1} x_j \alpha^j \right), \psi^{\sigma_1}\left( \sum_{j=0}^{n-1} x_j \alpha_1^j \right), \ldots, \psi^{\sigma_{n-1}}\left( \sum_{j=0}^{n-1} x_j \alpha_{n-1}^j \right) \right)
\]

Is a birational map between \( \mathcal{U} \) and \( \Delta = \mathcal{C} \). See [2] for the details. This means that \( \psi\left( \sum_{j=0}^{n-1} x_j \alpha^j \right) = \psi^{\sigma_i}\left( \sum_{j=0}^{n-1} x_j \alpha_i^j \right) \) for the points of the hypercircle. If we plug the standard parametrization of the hypercircle in this equality, we get that

\[
\psi(t) = \psi\left( \sum_{j=0}^{n-1} \phi_j \alpha^j \right) = \psi^{\sigma_i}\left( \sum_{j=0}^{n-1} \phi_j \alpha_i^j \right) = \psi^{\sigma_i}(u_{\sigma_i}(t))
\]
From which $u_{\sigma_i} = (\psi^{\sigma_i})^{-1} \circ \psi$.

Hence the isomorphism $u_j$ induced by the hypercircle in the pencil of hyperplanes $\mathfrak{L}(t)$ and $\mathfrak{L}_j(t)$ is the change of variables needed to transform the conjugate parametrization $\psi^{\sigma_j}(t)$ into $\psi(t)$. We can compute $u_j$ using gcd.

**Theorem 5.** Let $\psi_i(t) = n_i(t)/d_i(t)$ and $\psi_i^{\sigma_j}(t) = n_i^{\sigma_j}(t)/d_i^{\sigma_j}(t)$ be the numerators and denominators of $\psi$ and $\psi^j$. Then, if $C$ is defined over $K$, the numerator of $s - u_j(t)$ is a polynomial of degree 1 in $t$ and in $s$ that is the common factor of the set of polynomials

$$B_{\sigma_j} = \{n_i(t) \cdot d_i^{\sigma_j}(s) - n_i^{\sigma_j}(s) \cdot d_i(t), 1 \leq i \leq m\}.$$  

On the other hand, if $C$ is not defined over $K$, there is an index $1 \leq j \leq n - 1$ such that $\gcd(B_{\sigma_j}) = 1$.

**Proof.** This result follows directly from the geometric interpretation. First, assume that $C$ is defined over $K$. It is clear that the numerator of $s - u_j(t)$ is a common factor of the set $B_{\sigma_j}$. Let $f(t, s)$ be the gcd of $B_{\sigma_j}$ and let $p = \psi(t_0) \in C$ where $t_0$ is a generic evaluation of $t$. The roots of $f(t_0, s)$ are solutions of the system of equations $\psi_i^{\sigma_j}(s) = p_i$. But, since $\psi^{\sigma_j}$ is birational, for all but finitely many $t_0$ there is only one solution, $(\psi^{\sigma_j})^{-1}(p)$. Hence, the degree of $f$ with respect to $s$ is one. By symmetry, the degree of $f$ with respect to $t$ is also one. It follows that $f$ must be the numerator of $s - u(t)$.

Now, assume that $C$ is not defined over $K$. Then, there is an index $j$ such that $C \neq C^{\sigma_j}$. In this situation, for all but finitely many evaluations $t = t_0$, the system of equations $\psi^{\sigma_j}(s) = \psi(t_0)$ has no solution. It follows that $\gcd(B_{\sigma_j}) = 1$. ☐

So, we can compute $K$-definability and the standard parametrization of the hypercircle $\mathcal{U}$ by the following method:

- For each conjugate $\alpha_j$, Compute $a(t) + sb(t)$, the gcd of $n_i(t) \cdot d_i^{\sigma_j}(s) - n_i^{\sigma_j}(s) \cdot d_i(t)$, $1 \leq i \leq m$. If one of the gcd is one, then the curve is not defined over $K$ and we are done.

- Set $u_j = -a(t)/b(t)$.

- Solve the linear system of equations whose coefficients are rational functions in $t$ with coefficients in the normal closure of $K(\alpha)$.

However, computing these bivariate gcd are expensive and, moreover, in the worst case, we will have to solve a linear set of equations with coefficients in an extension of $K$ of degree $n!$. Next section address the problem of how to perform this algorithm efficiently.

## 3 Efficient Computation of the Hypercircle

We have shown how to compute $u_{\sigma_i}(t)$ by computing the gcd of the polynomials in $B_{\sigma_i}$. We already now that, if $C$ is $K$-definable, the gcd has degree 1 in $t$ and
s, so the best suited algorithms for computing the gcd seem to be interpolation algorithms. Since we are only interested in $u_\sigma$ and this linear fraction is an automorphism of $\mathbb{P}^1(F)$, we only need to know the image of three points $t_0, t_1, t_2$ under $u_\sigma$. From Theorem 5 for almost all $t_i$, $u_\sigma(t_i) = (s_i)$ if and only if $\psi(t_i) = \psi^\sigma(s_i)$. Hence, each $s_i$ is the common root of the polynomials:

$$
\psi(t_i) \cdot d_j^\sigma(s_i) - n_j^\sigma(s_i), 1 \leq i \leq m
$$

that can be computed by means of gcd of univariate polynomials in $K(\alpha, \sigma(\alpha))$.

If $C$ is defined over $K$ then only finitely many parameters $t_k$ will fail to provide a valid $s_k$. Essentially the parameters $t_k$ can fail if $\psi(t_k)$ is a singular point of the curve or if it cannot be attained by a finite parameter $s$ by the parametrization $\psi^\sigma$.

On the other hand, if $C$ is not defined over $K$, then there is an automorphism $\sigma$ such that $C \neq C^\sigma$. For this permutation, there are only finitely many parameters $t_k$ such that $\psi(t_k) \in C \cap C^\sigma$. Hence, if we want to follow this approach and do not depend on probabilistic algorithms that may fail or give wrong answers, we need bounds to detect that the curve is defined over $K$ or not.

**Theorem 6.** Let $C \subseteq \mathbb{F}^m$ be a rational curve of degree $d$ given by a parametrization $\psi \in (K(\alpha))^m$. Let $\alpha_i$ be any conjugate of $\alpha$ over $K$. Take $t_1, \ldots, t_k \in \mathbb{F}$ parameters then:

- If $C$ is definable over $K$, then we can compute $u_i$ from three correct solutions of the system of equations $\psi^\sigma(s) = t_k$.

- If $C$ is defined over $K$, then at most $d^2 - 2d + n + 1$ parameters can fail to give a correct answer.

- If $C$ is not defined over $K$, then at most $d^2$ parameters $t_k$ will give a fake answer $s_k$.

**Proof.** We have to compute the inverse of the point $\psi(t_j)$ under the parametrization $\psi^\sigma(s)$. For each $t_k$, this computation is done using univariate gcd. If we want to restrict to affine points, we have to eliminate $d$ potential parameters of the denominator of $\psi$. Then, for an affine point $\psi(t_j)$, there can only be one point that is not attained by a finite parameter of $\psi^\sigma$. Since we have $n - 1$ possible conjugates, then there may be $n - 1$ points that are not attained by a finite parameter in one of the conjugate parametrizations. So, if we get two different parameters $t_j$ such that $\psi(t_j)$ is well defined but that $\psi^\sigma(s) = t_j$ have no solution (the corresponding gcd is 1), then the curve is not defined over $K$.

Now, it may happen that the gcd is of degree $> 1$. This can only happen if the point is singular in $C$. Since $C$ is of genus 0 and degree $d$, it can have at most $(d-1)(d-2)/2$ singularities. The number of different parameters whose image is a singularity is maximal if every singularity is ordinary. We have to maximize

$$
\sum_{p \in \text{sing}(C)} \text{mult}_p(C)
$$

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subject to

\[ \sum_{p \in \text{sing}(C)} \text{mult}_p(C)(\text{mult}_p(C) - 1) = (d-1)(d-2) \]

See [9] Theorem 2.60 for details. But clearly, for any singular point \( \text{mult}_p(C) \leq \text{mult}_p(C)(\text{mult}_p(C) - 1) \) so \( \sum_{p \in \text{sing}(C)} \text{mult}_p(C) \leq (d-1)(d-2) \) and the equality is attained if every singularity is an ordinary double point.

Thus, the maximal number of parameters that cannot be used to compute \( u_i \) is bounded by \( d \) parameters corresponding to the points at infinity plus \( n-1 \) points that might not be attained by a finite parameter in a conjugate parametrization \( \psi^\sigma \) plus \( (d-1)(d-2) \) parameters whose image are singular points. This gives the bound \( d^2 - 2d + n + 1 \).

Suppose now that \( C \) is not defined over \( K \). Let \( \sigma_i \) be such that \( C \neq C^{\sigma_i} \). A parameter \( t_0 \) gives a fake answer for computing \( u_i \) if \( \psi(t_0) \) is smooth in \( C^{\sigma_i} \) and is attained by a unique parameter \( s_0 \) by \( \psi^{\sigma_i} \). But, by Bezout, \( C \cap C^{\sigma} \) contains at most \( d^2 \) different points. So, there can be at most \( d^2 \) such bad parameters.

**Remark 7.** In order to check that a parameter \( t_k \) is a good parameter or not we can do the following:

- If \( t_k \) is a root of the denominator of \( \psi \), then \( t_k \) is a bad parameter.
- If \( \gcd(\psi(t_k) \cdot d_i^n(s) - n_i^n(s), 1 \leq i \leq m) = 1 \), then it is a bad parameter. It is a point that is not attained by the parametrization \( \psi^\sigma \). If \( C \) is defined over \( K \) there can be at most one bad parameter that happens to be in this case that corresponds to \( \psi^\sigma(t = \infty) \).
- If \( \deg(\gcd(\psi(t_k) \cdot d_i^n(s) - n_i^n(s), 1 \leq i \leq m)) > 1 \) then \( t_k \) is a bad parameter, since \( \psi(t_k) \) is a singular point.
- If \( \deg(\gcd(\psi(t_k) \cdot d_i^n(s) - n_i^n(s), 1 \leq i \leq m)) = 1 \) but \( \psi(t_k) = \psi^\sigma(\infty) \) then \( t_k \) is a bad parameter, \( \psi(t_k) \) is singular.
- If \( \deg(\gcd(\psi(t_k) \cdot d_i^n(s) - n_i^n(s), 1 \leq i \leq m)) = 1 \) and \( \psi(t_k) \neq \psi^\sigma(\infty) \) then \( t_k \) is a good parameter, we compute \( s_k \) solving the linear equation in \( s \) given by the gcd.

Hence, if \( C \) is defined over \( K \), we can compute each \( u_j \) by interpolation. We will need at most \( d^2 - 2d + n + 1 + 3 = d^2 - 2d + n + 4 \) parameters. In practice however, we will almost always need only 3 parameters. Note also that if we choose the parameters in \( K \), then all computations needed to compute \( u_i \) are done in \( K(\alpha, \alpha_i) \), that is an extension of degree bounded by \( n(n-1) \).

If \( C \) is not defined over \( K \) it can happen two things while trying to compute \( u_j \). With high probability, we may find two different parameters such that \( \gcd(\psi(t_k) \cdot d_i^n(s) - n_i^n(s), 1 \leq i \leq m) = 1 \) and this is a certificate that the curve is not defined over \( K \). On the other hand, we may succeed computing \( u_j \). This may happen if \( C = C^{\sigma_j} \) for this specific \( \sigma_j \) or if we have chosen three
parameters $t_k$ such that $\psi(t_k) \in C \cap C^{\sigma_j}$. So, if we have computed all the linear fractions $u_j(t)$ but we want a certificate that $C$ is defined over $\mathbb{K}$, we only need to check that $\psi(t) = \psi^{\sigma_j}(u_j(t))$, $1 \leq i \leq n$. In the case that computing this composition may be expensive, we can try to check the equality evaluating in several parameters $t$. $\psi(t)$ and $\psi^{\sigma_j}(u_j(t))$ are rational functions of degree $d$, so if they agree on $2d + 1$ parameters where both parametrizations are defined, then $\psi(t) = \psi^{\sigma_j}(u_j(t))$ and $C = C^{\sigma_j}$. But there are $d$ parameters where $C$ is not defined and other $d$ where $C^{\sigma_j}$ is not defined. So, if we want a certificate that $C = C^{\sigma_j}$ by evaluation, we will need to try at most $4d + 1$ parameters in the worst case. So $(n - 1)(4d + 1)$ evaluations to check all conjugates.

**Example 8.** Let us show that the bounds given can be easily proven to be sharp if we allow the parameters to be in $F$ and $d \geq n$. Let $\mathbb{K}(\alpha)$ be normal over $\mathbb{K}$ of degree $n$ and $\sigma_1, \ldots, \sigma_{n-1}$ be $\mathbb{K}$-automorphisms that send $\alpha$ onto its conjugates. The common denominator of the parametrization of the curve will be $g = (t + 1)\ldots(t + d)$ so that the parameters $-1, \ldots, -d$ will fail in the algorithm. Let us write a component as $f(t) = (a_t^d + a_{d-1}t^{d-1} + \ldots + a_1t + a_0)/g(t)$, where the $a_i$ are indeterminates. Impose the conditions $f(i) = \sigma_i(\alpha)$. This is a linear system of equations in the $a_i$ representing an interpolation problem. We have $n - 1$ conditions and $d$ unknowns in the system and $n - 1 < d$. Hence, there are infinitely many solutions to the system and we can take two generic solutions $f_1(t), f_2(t)$. The curve $\psi(t) = (f_1(t), f_2(t))$ will fail to give a correct answer for $t = -1, \ldots, -d$ due to the denominator and for $t = i, i = 1, \ldots, n - 1$ because $\psi(t = \infty) = (\alpha, \alpha)$, so $\psi^\sigma(t = \infty) = (\sigma(\alpha), \sigma(\alpha)) = \psi(i)$. Finally, if we have chosen $f_1, f_2$ generic, the only singularities of $\psi$ will be simple nodes in the affine plane. Thus, there will be $(d - 1)(d - 2)$ parameters that will yield to a singularity.

For a specific example, take $\mathbb{K} = \mathbb{Q}$, $\alpha$ a primitive $5$-th root of unity, so that $n = 4$. Let the degree be $d = 4$. If we perform the construction above, we get the relations in the coefficients of $f$:

\[
\begin{align*}
a_0 &= -6a_3 + (1440\alpha^3 + 1080\alpha^2 + 1044\alpha + 1920) \\
a_1 &= 11a_3 + (-1740\alpha^3 - 1440\alpha^2 - 1380\alpha - 2700) \\
a_2 &= -6a_3 + (420\alpha^3 + 360\alpha^2 + 335\alpha + 780)
\end{align*}
\]

If we compute $f_0$ and $f_1$ substituting $a_3$ by 0 and 1 respectively, we get the parametrization $\phi(f_0, f_1)$ of a rational curve of degree 4 with three nodes, such that the nodes are attained by the roots of $t^6 + (-420\alpha^3 + 60\alpha^2 - 90\alpha - 102)t^5 + (-59220\alpha^3-171720\alpha^2+85110\alpha-214952)t^4+(688980\alpha^3+1237740\alpha^2-450750\alpha+1759626)t^3+(-2309580\alpha^3-3135240\alpha^2+714450\alpha-4869077)t^2+(2877600\alpha^3+3308280\alpha^2-391560\alpha+5387628)t+(-1197360\alpha^3-1231920\alpha^2+42840\alpha-2063124)$.

Now, we show how to avoid in some cases some computations of $u_j$ using conjugation.
**Proposition 9.** Assume that \( C \) is defined over \( K \). Let \( \alpha_i \neq \alpha_j \) be two conjugates of \( \alpha \) over \( K \). Suppose that \( \alpha_i, \alpha_j \) are also conjugated over \( K(\alpha) \) and that \( \tau \) is a \( K(\alpha) \)-automorphism of \( F \) such that \( \tau(\alpha_i) = \alpha_j \). Then \( \tau(u_i) = u_j \).

**Proof.** All operations to compute \( u_j \) are evaluating rational functions with coefficients in \( K(\alpha, \alpha_i) \) at parameters in \( K \) (or even \( \mathbb{Z} \)), compute gcd of univariate polynomials with coefficients also in \( K(\alpha, \alpha_i) \) and solving a linear system of equations. These operations commute with conjugation by \( \sigma \). Thus, if \( \tau \) is a \( K(\alpha) \)-automorphism such that \( \tau(\alpha_i) = \alpha_j \), we can conjugate by \( \tau \) at every step of the method to compute \( u_i \). Hence, \( \tau(u_i) = u_j \). \( \square \)

If the Galois group of \( \mathbb{K}(\alpha) \) over \( K \) is the permutation group \( S_n \), we will only need to compute one automorphism \( u_i \) making computations in a number field of degree \( n(n-1) \). On the other extreme, if \( K \subseteq \mathbb{K}(\alpha) \) is normal, we will have to compute \( n-1 \) different automorphisms \( u_i \), but the computations will be in the smaller field \( K(\alpha) \).

Now, we show how to avoid computing in the normal closure of \( K(\alpha) \) over \( K \) to solve the linear system of equations \([\mathbf{I}]\). This system is given by a Vandermonde matrix, so we are dealing with an interpolation problem. If the standard parametrization searched is \((\phi_0, \ldots, \phi_{n-1})\). Then, the polynomial

\[
F(x) = \phi_0 + \phi_1 x + \ldots + \phi_{n-1} x^{n-1} \in K(\alpha)(t)[x]
\]

is the unique polynomial of degree at most \( n-1 \) such that \( F(\alpha_i) = u_i(t), \) \( 0 \leq i \leq n-1 \). \( F \) can be computed by Lagrange interpolation

\[
F(x) = \sum_{i=0}^{n-1} \frac{(x - \alpha_0) \ldots (x - \alpha_{i-1})(x - \alpha_{i+1}) \ldots (x - \alpha_{n+1})}{(\alpha_i - \alpha_0) \ldots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \ldots (\alpha_i - \alpha_{n+1})} u_i(t).
\]

Let us take a look at each term:

\[
\frac{(x - \alpha_0) \ldots (x - \alpha_{i-1})(x - \alpha_{i+1}) \ldots (x - \alpha_{n-1})}{(\alpha_i - \alpha_0) \ldots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \ldots (\alpha_i - \alpha_{n-1})}
\]

The numerator is \( M(x)/(x - \alpha_i) = m(\alpha_i, x) \), where \( M(x) \) is the minimal polynomial of \( \alpha \) over \( K \) and the denominator is \( m(\alpha_i, \alpha_i) = M'(\alpha_i) \). For each conjugacy class \( \{\alpha_{i_1}, \ldots, \alpha_{i_j}\} \) of roots of \( M(x) \) over \( K(\alpha) \), we have that

\[
\sum_{k=1}^{j} \frac{m(\alpha_{i_k}, x)}{m(\alpha_{i_k}, \alpha_{i_k})} u_{i_k}(t) = trace \frac{m(\alpha_{i_k}, x)}{m(\alpha_{i_k}, \alpha_{i_k})} u_{i_k}(t).
\]

Where the trace is computed for the extension \( \mathbb{K}(\alpha, t, x) \subseteq \mathbb{K}(\alpha, t,x)(\alpha_i) \). Hence, we need to compute only one term of the Laurent interpolation for each conjugacy class of roots of \( M(x) \) over \( K(\alpha) \). These conjugacy classes are determined by the factorization of \( M(x) \) in \( K(\alpha)[x] \).
Remark 10. To compute fast the trace of \( v = \frac{m(\alpha, x)}{m'(\alpha, x)} u_i \in \mathbb{K}(\alpha, t, \alpha_i)[x] \), first, we can compute the Newton sums \( \sum_{k=1}^l \alpha_{l_k} \), \( 1 \leq l \leq n - 1 \) from the minimal polynomial of \( \alpha_{l_k} \) over \( \mathbb{K}(\alpha) \). If the coefficients of \( v \) are polynomials in \( t \), we compute easily the trace of \( v \) computing the trace of each coefficient of \( v \). If the coefficients of \( v \) are not polynomials in \( t \), we can write \( v \) as \( n/(t + b) \), \( b \in \mathbb{K}(\alpha, \alpha_i), n \in \mathbb{K}(\alpha, \alpha_i)[t] \). This is due to the fact that the variable \( t \) only appears on the term \( u_i \) and it is a linear fraction. Now, let \( g(t) \) be the minimal polynomial of \( -b \) over \( \mathbb{K}(\alpha) \) and \( g_1(t) = g(t)/(t + b) \in \mathbb{K}(\alpha, \alpha_i)[t] \). Then \( v = n/(t + b) = (n \cdot g_1(t))/g(t) \) and trace \( v \) = trace \( n \cdot g_1(t)/g(t) \) can be easily computed.

Thus, we can compute the polynomial \( F \) (i.e. the standard parametrization) computing gcd and traces and norms in some fields of the form \( \mathbb{K}(\alpha, \alpha_i) \). To sum up, our algorithm to compute the standard parametrization of \( \mathcal{U} \) is the following.

Algorithm 11. Input: A curve \( C \) given by a proper parametrization \( \psi(\alpha, t) \) with coefficients in \( \mathbb{K}(\alpha) \).

Output: Either \( C \) is not defined over \( \mathbb{K} \) or \( \phi \), the standard parametrization of the hypercircle associated to \( \psi(t) \).

1. Set \( M(x) \) the minimal polynomial of \( \alpha \) over \( \mathbb{K} \).
2. Set \( m(\alpha, x) = M(x)/(x - \alpha) \in \mathbb{K}(\alpha)[x] \).
3. Compute \( m(\alpha, x) = f_1(x) \cdots f_r(x) \) the factorization of \( m(\alpha, x) \) over \( \mathbb{K}(\alpha) \).
4. Set \( F = \frac{m(\alpha, x)}{m(\alpha, \alpha)} t \in \mathbb{K}(\alpha, t)[x] \).
5. For \( 1 \leq i \leq r \) do
   (a) Set \( \alpha_i \) a root of \( f_i(x) \).
   (b) Set \( \psi^{\alpha_i}(t) = \psi(\alpha_i, t) \) the parametrization of the curve \( C^{\alpha_i} \).
   (c) Compute three good parameters \( t_1, t_2, t_3 \) in the sense of remark 7.
   (d) If two parameters \( t_1, t_2 \) are found such that \( \psi(t_i) \) and \( \psi(t_j) \) are well defined but not attained by \( \psi^{\alpha_i} \) then Return \( C \) is not defined over \( \mathbb{K} \).
   (e) Compute \( s_k \) such that \( \psi(t_k) = \psi^{\alpha_i}(s_k), 1 \leq k \leq 3 \).
   (f) Compute \( u_i(t) = \frac{at + b}{ct + d} \) the linear fraction such that \( u(t_k) = s_k \).
   (g) If \( \psi \neq \psi^{\alpha_i}(u_i) \) the Return \( C \) is not defined over \( \mathbb{K} \).
   (h) Compute \( v = m(\alpha_i, x)/m(\alpha_i, \alpha_i) \cdot u_i(t) \in \mathbb{K}(\alpha, t, \alpha_i)[x] \).
   (i) Compute \( w = \text{trace}(v) \) for the extension \( \mathbb{K}(\alpha, t, x) \subseteq \mathbb{K}(\alpha, t, \alpha_i) \).
   (j) Set \( F = F + w \in \mathbb{K}(\alpha, t)[x] \).
6. Write \( F = \phi_0(t) + \phi_1(t)x + \ldots + \phi_{n-1}(t)x^{n-1} \).
7. Return \( \phi = (\phi_0, \ldots, \phi_{n-1}) \).
Example 12. Now we present a full small example of the algorithm. Let $K = \mathbb{Q}$, $\alpha$ a root of $M(x) = x^4 - 2$, consider the proper parametrization $\psi$ of a plane curve:

\[
x = ((11\alpha^3 + 15\alpha^2 + 9\alpha + 11)t^3 + (7\alpha^3 + 14\alpha^2 + 14\alpha + 7)t^2 + (\alpha^3 + 2\alpha^2 + 4\alpha + 1)t)/D
\]
\[
y = ((15\alpha^3 + 9\alpha^2 + 11\alpha + 22)t^3 + (25\alpha^3 + 29\alpha^2 + 16\alpha + 25)t^2 + (9\alpha^3 + 18\alpha^2 + 15\alpha + 9)t + \alpha^3 + 2\alpha^2 + 4\alpha + 1)/D.
\]

with $D = (7t^3 + (12\alpha^3 + 3\alpha^2 + 6\alpha + 12)t^2 + (6\alpha^3 + 12\alpha^2 + 3\alpha + 6)t + \alpha^3 + 2\alpha^2 + 4\alpha + 1)$.  

Now, $M(x) = (x-\alpha)(x+\alpha)(x^2+\alpha^2)$ is the factorization of $M(x)$ in $K(\alpha)[x]$. $m(\alpha, x) = (x + \alpha)(x^2 + \alpha^2)$ and $m(\alpha, \alpha) = 4\alpha^3 = M'(\alpha)$. start with $F = \frac{m(\alpha, x)}{m(\alpha, \alpha)} = 1/(8(\alpha x^3 t + \alpha^2 x^2 t + \alpha^3 x t + 2))$.

From the factors of $m(x)$ we have two conjugacy classes of roots of $m$ over $K(\alpha)$. The first one is $\{-\alpha\}$. Let $\sigma$ be a $\mathbb{Q}$-automorphism such that $\sigma(\alpha) = -\alpha$. Hence, we consider the conjugate parametrization $\psi^\sigma$:

\[
x = ((-11\alpha^3 + 15\alpha^2 - 9\alpha + 11)t^3 + (-7\alpha^3 + 14\alpha^2 - 14\alpha + 7)t^2 + (-\alpha^3 + 2\alpha^2 - 4\alpha + 1)t)/D_1,
\]
\[
y = ((-15\alpha^3 + 9\alpha^2 - 11\alpha + 22)t^3 + (-25\alpha^3 + 29\alpha^2 - 16\alpha + 25)t^2 + (-9\alpha^3 + 18\alpha^2 - 15\alpha + 9)t - \alpha^3 + 2\alpha^2 - 4\alpha + 1)/D_1,
\]

with $D_1 = (7t^3 + (-12\alpha^3 + 3\alpha^2 - 6\alpha + 12)t^2 + (-6\alpha^3 + 12\alpha^2 - 3\alpha + 6)t - \alpha^3 + 2\alpha^2 - 4\alpha + 1)$.

We have to compute the automorphism $u_\sigma$ such that $\psi(t) = \psi^\sigma(u_\sigma(t))$. we evaluate $(\psi^\sigma)^{-1}(\psi(t_k))$ and obtain:

\[
\psi(0) = \psi^\sigma(0)
\]
\[
\psi(1) = \psi^\sigma(8/31\alpha^3 - 4/31\alpha^2 + 2/31\alpha - 1/31)
\]
\[
\psi(2) = \psi^\sigma(128/511\alpha^3 - 32/511\alpha^2 + 8/511\alpha - 2/511)
\]

Hence, $u_\sigma = \frac{a_1 + b}{a_1 + \alpha}$ is such that $u_\sigma(0) = 0$, $u_\sigma(1) = 8/31\alpha^3 - 4/31\alpha^2 + 2/31\alpha - 1/31$, $u_\sigma(2) = 128/511\alpha^3 - 32/511\alpha^2 + 8/511\alpha - 2/511$. We can compute $u(t) = \frac{a_1 + b}{a_1 + \alpha}$ by solving a linear homogeneous system of equations and get the solution

\[
u(t) = \frac{\alpha^3 t}{4t + \alpha^3}
\]

In this case $\psi^\sigma(u_\sigma) = \psi$, so $C = C^\sigma$. We can update $F$ by adding:

\[
m(-\alpha, x)/m(-\alpha, -\alpha)u_\sigma(t) = \frac{-x^3 t + \alpha x^2 t - \alpha^2 x t + \alpha^3 t}{16t + 4\alpha^3}
\]

So now:

\[
F = \frac{\alpha x^2 t^2 + \alpha^2 x^2 t^2 + \alpha x^2 t + \alpha^3 x t^2 + 2t^2 + \alpha^3 t}{8t + 2\alpha^3}
\]

For this root, all operations are done in $K(\alpha)$ since $\sigma(\alpha) = -\alpha \in K(\alpha)$.

Now, we have to deal with the roots of $x^2 + \alpha^2$. Let $\beta$ be a root of $x^2 + \alpha^2$ and $\tau$ a $\mathbb{Q}$-automorphism such that $\tau(\alpha) = \beta$. Consider the conjugate parametrization $\psi^\tau$: $x = (((-11\alpha^2 + 9)\beta - 15\alpha^2 + 11)t^3 + ((-7\alpha^2 + 14)\beta - 14\alpha^2 + 7)t^2 + ((-\alpha^2 + 4)\beta - 2\alpha^2 + 1)\beta)/D_2$, $y = (((-15\alpha^2 + 11)\beta - 9\alpha^2 + 22)t^3 + ((-25\alpha^2 + 16)\beta - 29\alpha^2 + 25)t^2 + ((-9\alpha^2 + 15)\beta - 18\alpha^2 + 9)t + (-\alpha^2 + 4)\beta - 2\alpha^2 + 1)/D_2$, where $D_2 = \ldots$
From this data, we can compute:

\[ \psi(0) = \psi^7(0) \]

\[ \psi(1) = \psi^7((2/9a^2 - 2/9a + 1/9) \beta + 2/9a^3 - 1/9a + 1/9) \]

\[ \psi(2) = \psi^7((32/129a^2 - 16/129a + 4/129) \beta + 32/129a^3 - 4/129a + 2/129) \]

From this data, we can compute:

\[ u_t(t) = \frac{t}{(\alpha - \beta)t + 1} \]

If \( \gamma \) is the other root of \( x^2 + a^2 \) (i.e. \( \gamma = -\beta \)) and \( \delta \) is a \( \mathbb{Q} \)-automorphism such that \( \delta(\alpha) = \gamma \), then \( u_{\gamma}(t) = t/(\alpha - \gamma)t + 1 \). We have to compute the trace of

\[ v = \frac{m(\beta, x)}{m(\beta, \beta)} u_{\tau}(t) = \frac{\beta x^3 t - \alpha^2 x^2 t - \alpha^2 \beta xt + 2t}{(-8\beta + 8\alpha)t + 8} \]

over \( \mathbb{Q}(\alpha, t) \). This is done using the technique described in Remark 10.

\[ \text{trace}(v) = \frac{-\alpha^2 x^3 t^2 - \alpha^3 x^2 t^2 - \alpha^2 x^2 t + 2x^2 t + 2t}{8\alpha^2 t^2 + 8\alpha t + 4} \]

To compute this part, we have made computation in \( \mathbb{Q}(\alpha, \beta) \). We add \( \text{trace}(v) \) to \( \mathcal{F} \) and get

\[ \mathcal{F} = \phi_0 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3 \]

where

\[ \phi_0 = \frac{2t^4 + 3a^3 t^3 + 3a^3 t^2 + at}{8t^3 + 6a^3 t^2 + 4a^2 t + a}, \quad \phi_1 = \frac{a^3 t^4 + 2a^2 t^3 + at^2}{8t^3 + 6a^3 t^2 + 4a^2 t + a} \]

\[ \phi_2 = \frac{a^2 t^4 + at^3}{8t^3 + 6a^3 t^2 + 4a^2 t + a}, \quad \phi_3 = \frac{at^4}{8t^3 + 6a^3 t^2 + 4a^2 t + a} \]

And \( \phi = (\phi_0, \phi_1, \phi_2, \phi_3) \) is the standard parametrization of the hypercircle associated to \( \psi \).

So far, Algorithm 11 only computes the hypercircle \( \mathcal{U} \). The algorithm is able to detect if \( \mathcal{C} \) is not defined over \( \mathbb{K} \), but apart from that it does not provide much more useful information. In the rest of the section, we show that, if \( \mathcal{C} \) is not defined over \( \mathbb{K} \), how can we compute the minimum field \( \mathcal{L} \) such that \( \mathbb{K} \subseteq \mathcal{L} \subseteq \mathbb{K}(\alpha) \) and \( \mathcal{C} \) is defined over \( \mathcal{L} \). Note that \( \mathbb{K}(\alpha) \) always is a field of definition of \( \mathcal{C} \), so the existence of \( \mathcal{L} \) is always guaranteed.

**Theorem 13.** Let \( \mathcal{C} \) be a curve not \( \mathbb{K} \)-definable but \( \mathbb{K}(\alpha) \)-parametrizable. Let \( \mathcal{L} \) be the minimum field of definition of \( \mathcal{C} \) containing \( \mathbb{K} \). \( \mathbb{K} \subseteq \mathcal{L} \subseteq \mathbb{K}(\alpha) \). Then \( \mathcal{L} \) is the subfield of the normal closure \( \mathbb{K}(\alpha) \) over \( \mathbb{K} \) that is fixed by the \( \mathbb{K} \)-automorphisms \( \sigma \) of \( \mathbb{K}(\alpha) \) such that \( \sigma \mathcal{C} = \mathcal{C}^\sigma \).
Proof. First, we recall that the intersection of fields of definition of \( \mathcal{C} \) is a field of definition of \( \mathcal{C} \). Hence, since \( \mathbb{K}(\alpha) \) is a field of definition, there always exists a minimum field of definition \( \mathbb{L} \) of \( \mathbb{C} \) containing \( \mathbb{K} \).

From [2] it follows that if \( \mathbb{L}_1 \subseteq \mathbb{L}_2 \) is any algebraic finite normal extension and \( \mathbb{L}_2 \) is a field of definition of \( \mathcal{C} \), then \( \mathbb{L}_1 \) is a field of definition of \( \mathcal{C} \) if and only if \( \mathcal{C} = \mathcal{C}^\sigma \) for all \( \sigma \in \text{Aut}(\mathbb{L}_2/\mathbb{L}_1) \).

Let \( \mathbb{G} = \{ \sigma \in \text{Aut}(\mathbb{K}(\alpha)/\mathbb{K}) \mid \mathcal{C}^\sigma = \mathcal{C} \} \). Clearly, \( \mathbb{G} \) is a subgroup of \( \text{Aut}(\mathbb{K}(\alpha)/\mathbb{K}) \). This follows from the fact that \( (\mathcal{C}^\sigma)^\tau = \mathcal{C}^{\sigma \circ \tau} \). Let \( \mathbb{L} \) be the subfield of \( \mathbb{K}(\alpha) \) that is fixed by \( \mathbb{G} \). \( \mathbb{L} \subseteq \mathbb{K}(\alpha) \) is a normal extension and, if \( \sigma \) is a \( \mathbb{L} \)-automorphism of \( \mathbb{K}(\alpha) \) then \( \sigma \in \mathbb{G} \) so \( \mathcal{C} = \mathcal{C}^\sigma \). In this conditions, \( \mathbb{L} \) is a field of definition of \( \mathcal{C} \). Moreover, it is the smallest field of definition of \( \mathcal{C} \) containing \( \mathbb{K} \). If \( \mathbb{K} \subseteq \mathbb{L}_1 \subseteq \mathbb{L} \) is a subfield of \( \mathbb{L} \), then \( \mathbb{G}_1 \), the set of \( \mathbb{L}_1 \)-automorphisms of \( \mathbb{K}(\alpha) \), is \( \mathbb{G}_1 \supseteq \mathbb{G} \). Hence, there is an automorphism \( \tau \in \mathbb{G} \). But then \( \mathcal{C} \neq \mathcal{C}^\tau \) and \( \mathbb{L}_1 \) cannot be a field of definition of \( \mathcal{C} \). Now, since \( \mathbb{K}(\alpha) \) is also a field of definition of \( \mathcal{C} \), then \( \mathbb{L} \subseteq \mathbb{K}(\alpha) \).

If \( \sigma_0 = \text{Id}, \sigma_1, \ldots, \sigma_{n-1} \) are the automorphisms defined in Section [2] then for any \( \sigma \in \text{Aut}(\mathbb{K}(\alpha)/\mathbb{K}) \), it happens that \( \mathcal{C}^\sigma = \mathcal{C}^{\sigma_i} \) for some \( i, 0 \leq i \leq n - 1 \). Hence

\[
\mathbb{L} = \bigcap_{\sigma_i \in \mathbb{C}^{\mathcal{C}^\sigma}} \{ x \in \mathbb{K}(\alpha) \mid \sigma_i(x) = x \}
\]

If \( \mathcal{C} \) is not defined over \( \mathbb{K} \), we compute in step 5 of Algorithm [11] the set of automorphism \( \sigma_i \) such that \( \mathcal{C} = \mathcal{C}^{\sigma_i} \). For any such \( i \), let \( m \) be the degree of \( \alpha_i \) over \( \mathbb{K}(\alpha) \). If \( x \in \mathbb{K}(\alpha) \), we can write \( \sigma_i(x) = \sum_{j=0}^{m-1} l_i \alpha_j^j \), where \( l_i \in \mathbb{K}(\alpha) \). \( x \) is \( \sigma_i \) invariant if and only if \( x = l_0, l_i = 0, 1 \leq i \leq m - 1 \). This provide a set of \( \mathbb{K} \)-linear equations in the coordinates of \( x \) in \( \mathbb{K}(\alpha) \equiv \mathbb{K}^n \). Note also that if \( \alpha_i \) and \( \alpha_j \) are conjugate over \( \mathbb{K}(\alpha) \), the equations imposed by \( \sigma_i \) and \( \sigma_j \) are the same. Hence, we only need to compute them once for each set of conjugate roots of \( M(x) \) over \( \mathbb{K}(\alpha) \). Solving the system of linear equations provide a base of \( \mathbb{L} \) as a \( \mathbb{K} \)-subspace of \( \mathbb{K}(\alpha) \). From this equation, we may reapply Algorithm [11] but to the extension \( \mathbb{L} \subseteq \mathbb{K}(\alpha) \). In this case we already have computed the automorphisms \( u_i \) so we can reuse this computation.

4 Complexity and Running Time

We now compute the complexity of Algorithm [11] in terms of number of operations over the ground field \( \mathbb{K} \). The analysis is by no means sharp, we only intend to prove that there is a polynomial bound and that the main obstacle is the degree of \( \alpha \) over \( \mathbb{K} \).

Theorem 14. Let \( \mathbb{K} \) be a computable field with factorization of characteristic zero, \( \alpha \) algebraic of degree \( n \) over \( \mathbb{K} \) of minimal polynomial \( M(x) \). Let \( \psi(t) = (\psi_0, \ldots, \psi_{m-1}) \) be a proper parametrization of a spatial curve \( \mathcal{C} \) with coefficients in \( \mathbb{K}(\alpha) \). Then the number of operations over \( \mathbb{K} \) of Algorithm [11] is bounded by \( \mathbb{K} + \mathcal{O}(\text{md}^5n^3) \) where \( \mathbb{K} \) is the time needed to factor \( M(x) \) in \( \mathbb{K}(\alpha)[x] \).
Proof. We only use naive algorithms. The factorization of $M[x]$ can be performed standard methods \[\] from a factorization algorithm in $K[x]$. Addition in $K(\alpha)$ costs $n$ operations and multiplication costs $O(n^2)$ operations and inversion $O(n^2)$. If $\beta$ is a conjugate of $\alpha$, the worst case complexity of addition in $K(\alpha, \beta)$ is $O(n^2)$ while multiplication is $O(n^4)$ and inversion $O(n^6)$. If $f$ and $g$ are two polynomials of degree at most $d$, their gcd costs $O(d^3n^2 + n^3d^2)$ operations in $K(\alpha)$ or $O(d^3n^4 + n^5d^2)$ if their coefficients live in $K(\alpha, \beta)$. Steps $1-3$ of the algorithm cost $K + O(n^5)$. Step 4 is evaluating a polynomial in $K(\alpha)$, invert the result and multiply the polynomial this result. By Horner’s method it is $O(n^5)$. Step 5.b can be done in $O(dmn)$ operations. For a parameter $t_k$ doing steps $5.c - d$ is evaluating $m$ rational functions in $K(\alpha)$ and then compute $m - 1$ gcd in $K(\alpha, \beta)$ of degree $d$, this costs $O(md^3n^4 + mn^6d^2)$. From Theorem 15 we have to try at most $O(d^2 + n)$ times, so the total cost is bounded by $O(md^5n^7)$.

Computing step 5.f is just solving a system of 3 linear equations in 4 unknowns in $K(\alpha, \beta)$. This can be done in $O(n^4)$ operations. Now, comparing $\psi$ and $\psi''(u)$ in 5.e can be done evaluating both functions in $O(d)$ parameters. Each evaluation costs $O(n^6)$, so in total, this step can be done in $O(md^5n^7)$. Step 5.h we already have $m(\alpha_i, x)/m(\alpha_i, \alpha_i)$ precomputed by conjugation, so we only need to multiply the polynomials, which is dominated by computing $O(n)$ products ($u$ is always of degree $\leq 1$ and we do not need to do anything with the denominator). This costs $O(n^5)$. Now, instead of computing the minimal polynomial of the pole of $u$, we can compute its characteristic polynomial over $K(\alpha)$. Since the characteristic polynomial of an $n \times n$ matrix can be done in $O(n^4)$ operations and the matrix will have entries in $K(\alpha)$, we can compute this characteristic polynomial in $O(n^6)$ operations. Step 5.j can be done in $O(n^5)$ operations. Hence step 5 is bounded by $O(md^5n^7)$. Since we have to perform step 5 at most $n$ times. We get a bound of $O(md^5n^8)$ operations over $K$.

If $C$ is not $K$-definable. In step 5 we compute the automorphisms $\sigma_i$ such that $C = C^\sigma$. From this automorphisms, we can compute the field of definition $L$ in $O(n^4)$ operations and repeat the whole algorithm. It is clear that the running time for the extension $L \subseteq K(\alpha)$ is bounded by the case $K \subseteq K(\alpha)$. So the global bound does not change.

This result agrees with experimentation, the most important parameter is the degree of $\alpha$ over $K$ and the ambient dimension of $C$ tend to be not relevant in the algorithm compared to the other parameters.

Computing the hypercircle using Definition 1 is too slow, because we have to work with an ideal in $n$ variables over $K$ and make the quotient by the ideal defined by the denominator. In 8 the authors proposed a method to compute the parametrization of the hypercircle. It is based in the following result.

**Theorem 15.** Let $\psi$ be a proper parametrization of $C$ with coefficients in $K(\alpha)$. Let $G(x_1, \ldots, x_m) : C \to F$ be the inverse of the parametrization. $G \in K(\alpha)(x_1, \ldots, x_m)$. Write $G = \sum_{i=0}^{n-1} G_i \alpha^i$, $G_i \in K(x_1, \ldots, x_m)$, $0 \leq i \leq n - 1$. Consider $\phi = (G_0(\psi), \ldots, G_{n-1}(\psi))$. Then $C$ is defined over $K$ if and only if $\phi$ is well defined and parametrizes a curve in $F^n$. In this case $\phi$ is the standard
parametrization of the associated hypercircle to $\psi$.

Proof. See [8].

Algorithm [11] and the algorithm in Theorem [15] have been implemented in the Sage CAS [10], the code for the method presented in this paper can be obtained from [11]. We are interested in the average case, so we will assume that our curve is planar (since we can always make a generic projection). However, the method presented here also performs well for spatial curves. For the method based on the inverse of the parametrization of [8] we compute $(G_0, \ldots, G_{n-1})$ but we do not simplify the composition $G_i(\psi)$. This is done to avoid artifacts in the running time that appeared if we simplify the composition. The inverse of $\psi$ is computed using the resultant method explained in [3].

We show the results for random curves of degree 2, 5, 10, 25 and 50. First over an extension of $\mathbb{Q}$ of degree 2, over a cyclotomic extension of degree 6 and a random extension of degree 5. In all these cases $C$ is defined over $K$.

<table>
<thead>
<tr>
<th>Case: $\alpha^2 + 1 = 0$</th>
<th>Moving hyperplanes</th>
<th>Inverse-based method</th>
</tr>
</thead>
<tbody>
<tr>
<td>method \ degree of $C$</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Moving hyperplanes:</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>Inverse-based method:</td>
<td>0.03</td>
<td>0.15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case: $\alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$</th>
<th>Moving hyperplanes</th>
<th>Inverse-based method</th>
</tr>
</thead>
<tbody>
<tr>
<td>method \ degree of $C$</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Moving hyperplanes:</td>
<td>1.12</td>
<td>2.00</td>
</tr>
<tr>
<td>Inverse-based method:</td>
<td>0.13</td>
<td>28.61</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case: $\alpha$ of degree 5, random minimal polynomial</th>
<th>Moving hyperplanes</th>
<th>Inverse-based method</th>
</tr>
</thead>
<tbody>
<tr>
<td>method \ degree of $C$</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Moving hyperplanes:</td>
<td>1.04</td>
<td>2.01</td>
</tr>
<tr>
<td>Inverse-based method:</td>
<td>0.12</td>
<td>10.36</td>
</tr>
</tbody>
</table>

Now, we show a table with a random extension of degree 5 but $C$ is not defined over $K$. In this case is more evident that the moving hyperplanes method is better. With high probability it will detect that the curve is not defined over $K$ while trying to compute the automorphisms $u(t)$, on the other hand, the inverse-based method always has to compute the inverse of the parametrization $\psi$. In all cases, our algorithm computed the minimum field of definition of the corresponding curve.

<table>
<thead>
<tr>
<th>Case: $\alpha$ of degree 5, $C$ not defined over $K$</th>
<th>Moving hyperplanes</th>
<th>Inverse-based method</th>
</tr>
</thead>
<tbody>
<tr>
<td>method \ degree of $C$</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Moving hyperplanes:</td>
<td>0.36</td>
<td>0.59</td>
</tr>
<tr>
<td>Inverse-based method:</td>
<td>0.08</td>
<td>12.68</td>
</tr>
</tbody>
</table>

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References


