Super-Gaussian apodization in ground based telescopes for high contrast coronagraph imaging

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Abstract: We introduce the use of Super-Gaussian apodizing functions in the telescope pupil plane and/or the coronagraph Lyot plane to improve the imaging contrast in ground-based coronagraphs. We describe the properties of the Super-Gaussian function, we estimate its second-order moment in the pupil and Fourier planes and we check it as an apodizing function. We then use Super-Gaussian function to apodize the telescope pupil, the coronagraph Lyot plane or both of them. The result is that a proper apodizing masks combination can reduce the exoplanet detection distance up to a 45% with respect to the classic Lyot coronagraph, for moderately aberrated wavefronts. Compared to the prolate spheroidal function the Super-Gaussian apodizing function allows the planet light up to 3 times brighter. An extra help to increase the extinction rate is to perform a frame selection (Lucky Imaging technique). We show that a selection of the 10% best frames will reduce up to a 20% the detection angular distance when using the classic Lyot coronagraph but that the reduction is only around the 5% when using an apodized coronagraph.

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References and links
1. Introduction

Apodization has been a relevant topic for the last decades. In 1978 Harris [1] published an extensive review of apodizing functions for improving signal detectability. The apodizing functions there proposed are still of great interest and, in fact, some of them have been rediscovered lately. The field has moved to numerically optimized apodization in search for optimal performances, however well behaved functions can provide a good approximation to the optimal solutions and their performances can be evaluated analytically. They are thus tremendously useful when seeking theoretical insight regarding coronagraph designs. Apodization functions have application in different fields like the data storage [2], beam reshaping [3] or confocal microscopy [4].

We are interested in pupil apodization for high contrast imaging of astronomical objects in ground-based detection. From its first steps [5,6] until nowadays there has been an increasing interest in this field due to the prospects of direct imaging of object or structures around starts (e.g. faint companions, exoplanets or circumstellar disks) which/that are difficult to observe due to the angular proximity of the star and the very large luminosity ratios involved [7,8]. Special attention is placed on exoplanets imaging to study their atmospheres in sufficient detail to identify possible signs of biological activity. In particular, the pupil mapping concept has recently received considerable attention because of its performances. Consequently, there have been numerous studies over the past few years dealing with pupil mapping systems and its implementation [9–11].

Most of the apodizing functions have been developed for ideal systems, that is, for non-aberrated incoming wavefronts. However, much work is being done for the exoplanet ground based observation to be a realistic option [12]. In particular the technique based on the local suppression of the star halo [13] seems to be a clear solution to deal with atmospherically distorted wavefronts. An extended review of the state of art of ground-based coronagraphy can be found in reference [14].

In this context, in a previous work [15] we checked the use of adaptive coronagraphic masks. The adaptive mask blocked the light in those points of the coronagraphic plane where the intensity of star speckled PSF were over a previously determined threshold. The temporal average of the set of individual adaptive masks obtained in a time interval had a Super-Gaussian profile. We checked that the use of the average Super-Gaussian mask was more efficient than the use of individual adaptive masks. This result encouraged us to check the behavior of the Super-Gaussian transmission profile as an apodizing function for both the entrance pupil of the telescope and the Lyot stop. We selected Super-Gaussian functions not only because of their efficiency, but also because they are easy to be studied analytically and provide a good balance between the absorbed energy and contrast. We restricted our analysis to ground based telescopes in which the condition $D/r_0 < 8$ is fulfilled, where $D$ is the pupil...
diameter and $r_0$ the Fried parameter. This condition can be satisfied on visible or infrared bands depending on the telescope size and the degree of compensation achieved by the Adaptive Optics system. We consider that this type of apodization combined to a Lucky-Imaging detection strategy may provide new exciting results.

Although Super-Gaussian functions have been extensively employed for beam reshaping it is not easy to find a proper description of their properties and how they behave under the Fourier transform. In Section 2 we obtain Super-Gaussian functions as the 2D convolution of hard-edge circular and Gaussian functions and we estimate its normalized second order moment. In Section 3 we introduce atmosphere aberrations and we show that the best performing Super-Gaussian evolves with the $D/r_0$ value. In Section 4 we estimate the effects on the coronagraphic images of using a Super-Gaussian apodizing profile on the telescope entrance pupil and/or on the Lyot stop. Results for the prolate spheroidal apodizing function are also included for comparison. Section 5 analyzes the results obtained combining image selection (Lucky Imaging) and apodized coronagraphy. Finally, Section 6 summarizes the main results of the paper.

2. Super-Gaussian pupil functions

2.1 Theoretical description of Super-Gaussian and its Fourier transform

A two dimensional normalized Super-Gaussian function of order $n$ and half-width $\sigma$ is defined as:

$$SG(n,r) = \exp\left(-\frac{|r|}{\sigma^n}\right), \quad (1)$$

where $r$ is the usual radial coordinate. It can be shown that, for a particular width value, the Super-Gaussian function ranges from two limiting cases: It is a Gaussian function for $n = 2$ and a hard-edge circle for $n \to \infty$ (a hard-edge circle is a function having unit value inside the circle and zero value outside).

Although the two limiting cases are well known, it is necessary to pay some attention to intermediate orders. In fact, $SG(n,r)$ can be understood as the convolution of a hard-edge circle with a Gaussian function as suggested [16]. We have numerically checked that Super-Gaussian radial functions can be seen as the 2D convolution of a hard-edge circle and a Gaussian radial function. The width and order of the resulting Super-Gaussian function will only depend on the ratio of the Gaussian width and hard-edge circle diameter. This is shown in Fig. 1 where we represent the order (solid line and left scale) and width (dashed line and right scale) of the resulting Super-Gaussian functions in terms of the width ratio of the convolving Gaussian and hard-edge circle functions. It is clear that if the Gaussian function is much narrower than the hard-edge circle the resulting function has a very large order ($n \to \infty$) and the width tends to the hard-edge circle diameter ($G$/HE width tends to unity). On the contrary, when the Gaussian is wider than the hard-edge circle we basically obtain the Gaussian function.

Figure 2(a) shows a set of Super-Gaussian functions for different orders ($n = 2, 4, 10$) along with the hard-edge circle pupil of unit diameter (HEP) (the width of Super-Gaussian functions has been fixed so that all have the value $10^{-3}$ at the pupil border). We can see how it evolves from a Gaussian to a hard-edge circle as the order increases. When assuming the Super-Gaussian function as a convolution operation, its Fourier transform will be the product of the individual Fourier transforms:

$$SG(n,r) = HE(r) \otimes G(r) \rightarrow sg(w) = A(w) \cdot g(w),$$

where $A(w)$ is the Fourier transform of the hard-edge circle function (its square module $|A(w)|^2$ is the well-known Airy pattern) and $g(w)$ the Gaussian function resulting from Fourier transforming $G(r)$. 

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Figure 2b shows the set of square module Fourier transform corresponding to the set of functions appearing in Fig. 2(a). We can see that for $n = 2$ we get that $sg(\omega)$ is a pure Gaussian as expected. In the other limiting case ($n \to \infty$) we obtain that the Fourier transform is an Airy pattern. In the intermediate cases we find Airy-like patterns whose ripples have been dramatically reduced. This reduction is a consequence of the Gaussian $g(\omega)$ multiplying the Airy pattern as it can be seen in Eq. (2).

2.2. Moments analysis

An important aspect to take into account to choose an apodizing pupil function is the image spot size in the optical system image plane. The image intensity (within Fourier Optics approximation) is the square module Fourier transform of the optical system entrance pupil field and a standard measure of the image spot size is the second-order moment of the image intensity.
The k-order moment (in fact the central moment, since we do not consider spot displacements) of an integrable and square-integrable two dimensional function \( F(r, \theta) \) can be evaluated from (\( r \) and \( \theta \) are polar coordinates):

\[
m_k (F) = \int_0^{2\pi} \int_0^\infty r^k \left| F(r, \theta) \right|^2 r dr d\theta .
\]

So that the normalized second-order moment is given by:

\[
nm_2 (F) = \frac{m_2 (F)}{m_0 (F)} = \frac{\int_0^{2\pi} \int_0^\infty r^2 \left| F(r, \theta) \right|^2 r dr d\theta}{\int_0^{2\pi} \int_0^\infty \left| F(r, \theta) \right|^2 r dr d\theta} .
\]

If \( f(w, \vartheta) \) is the Fourier transform of \( F(r, \theta) \), and \( w \) and \( \vartheta \) are polar coordinates in the Fourier domain, the normalized second-order moment is expressed by:

\[
nm_2 (f) = \frac{m_2 (f)}{m_0 (f)} = \frac{\int_0^{2\pi} \int_0^\infty w^2 \left| f(w, \vartheta) \right|^2 w dw d\vartheta}{\int_0^{2\pi} \int_0^\infty \left| f(w, \vartheta) \right|^2 w dw d\vartheta} .
\]

Assuming an optical system where \( F \) and \( f \) are the pupil and image fields respectively, the numerator in Eq. (5) gives a measure of the image spot size whilst the denominator is the total signal intensity in the image plane that, thanks to the Parseval’s theorem, is the same as the total energy in the apodized pupil. Hence, an efficient apodizing function will provide the smallest possible \( nm_2 (f) \) value, that is, the narrowest function with the maximum energy. At this point it is interesting to recall the uncertainty principle namely:

\[
nm_2 (F) \cdot nm_2 (f) \geq 1 / (2\pi)^2 .
\]

The right part may differ from \( 1 / (2\pi)^2 \) depending on the normalization used. The equality in Eq. (6) is attained when \( F \) is a Gaussian function [17]. Hence, it would be interesting, to analyze how do behave the set of Super-Gaussian functions, evolving from the standard pupil (hard-edge circle) to the more balanced, from the point of view of the uncertainty principle, Gaussian function. However, only the second term of the product shown in Eq. (6) is useful for our purpose.

We will evaluate the image second-order moment for the set of Super-Gaussian apodizing pupil functions. If we call \( sg(w) \) to the Fourier transform of \( SG(n, r) \) shown in Eq. (1), thanks to the derivative property of the Fourier transform we will have:

\[
m_2 (sg) = \int_0^{2\pi} \int_0^\infty w^2 \left| sg(w) \right|^2 w dw d\vartheta = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^\infty \left( \frac{dSG(n, r)}{dr} \right)^2 r dr d\theta = \frac{n}{8\pi} .
\]

The zero-order moment is given by:

\[
m_0 (sg) = m_0 (SG) = \frac{2\pi \sigma^2}{2^{2/n}} \frac{\Gamma(2/n)}{n} .
\]

Hence, the expression for \( nm_2 (sg) \) will be:

\[
nm_2 (sg) = \frac{2^{2/n}}{16\pi^2 \sigma^2} \frac{n^2}{\Gamma(2/n)} .
\]

Equation (9) is a measure of the image spot size of an optical system with a Super-Gaussian pupil. Figure 3 shows the evolution of \( nm_2 (sg) \) (squares and left scale) as a function of the
Super-Gaussian order. The width of each Super-Gaussian (also shown by circles and right scale in Fig. 3) was calculated so that all functions have the value $10^{-3}$ at the unit-diameter circle (this value is the one providing the best contrast as it will be seen below). It can be seen that $nm^2_{\text{sg}}$ has a minimum value for $n$ about 5. This means that the fifth-order Super-Gaussian is the best balanced apodizing function from the point of view of spot size and energy lost in the optical system image plane.

The expression for the moments $m^2_{\text{SG}}$ can be also calculated:

$$m^2_{\text{SG}}(n,r) = \int_0^{2\pi} \int_0^\infty r^2 |SG(n,r)|^2 r \, dr \, d\theta = \frac{2\pi \sigma^4}{2^4 n} \frac{\Gamma\left(\frac{4}{n}\right)}{n} .$$

(10)

Now, since $m^0_{\text{SG}}$ is the same as $m^0_{\text{sg}}$ as stated in Eq. (8), we can obtain an expression for the uncertainty principle for Super-Gaussian functions:

$$nm^2_{\text{SG}} \cdot nm^2_{\text{sg}} = \frac{1}{16\pi^2} \frac{n^2 \Gamma\left(\frac{4}{n}\right)}{\Gamma\left(\frac{2}{n}\right)^2} .$$

(11)

This moment product does not depend on the width $\sigma$, like in the case of Gaussian functions, it depends only on the Super-Gaussian order $n$ and tends to $1/(2\pi)^2$ when $n = 2$. Similar results have already been obtained in the context of propagation of Super-Gaussian field distributions [18].

2.3. Prolate spheroidal comparison

Since Landau [19] analyzed the uncertainty principle of prolate spheroidal functions it is known that the use of these apodizing functions maximizes the integrated intensity within the central region of the focal plane with a given width. However, the comparison between the Super-Gaussian and prolate spheroidal functions is not easy since no exact prolate spheroidal solution exists. Although a cosine square function (also known as the Hanning window) may roughly approximate the prolate spheroidal shape [20], we used the Kaiser-Bessel function to get an approximation suitable in the context of the following analysis [21].

To measure the capability of a function to concentrate energy in a limited region of the image plane Landau estimates the energy contained in a circular area centered on the system axis whilst Soummer [20] measures the energy in the remaining image plane surface. Both measures could be considered more appropriated for our purpose than the normalized second order but, as we will confirm by computer simulation, predictions extracted from the analysis of the normalized second-order are also really significant.
Figure 4(a) shows a comparison between the transmission shapes of apodized pupils. The width of Super-Gaussian and prolate spheroidal functions has been fixed so that both have the same value at the pupil border and zero outside in order to reach the same contrast level. It is clear that the prolate spheroidal function blocks more energy than the Super-Gaussian and this more than the hard-edge circle. In Fig. 4(b), the square modules Fourier transforms of the previous pupils are shown. It can be seen that, although the best inner-working-angle (IWA) corresponds to the prolate spheroidal pupil, the Super-Gaussian pupil produces a throughput larger than the prolate spheroidal function and the same diffraction rings for angular distances larger than $9 \lambda/D$. Since, we are going to detect atmosphere distorted wavefronts, the coronagraphic mask needs to have an angular radius of several diffraction rings to provide a good contrast. Hence, under these conditions, it seems to be more advantageous to use Super-Gaussian apodizing functions when detecting at large angular distances.

![Figure 4](image_url)

Fig. 4. (a) Hard-edge circular pupil (HEP), Super-Gaussian apodized pupil (SGP), and prolate spheroidal apodized pupil (PSP). (b) Fourier transformed HEP (dots), SGP (solid), and PSP (dashed); Horizontal axis is in $\lambda/D$ units.

3. Apodizing atmospheric aberrated wavefronts

In a ground-based telescope the Point Spread Function (PSF) is composed of a coherent peak and a surrounding speckled halo dynamically changing with atmospheric seeing variation [22]. The halo introduced by atmospheric turbulence affects the image quality to a great extent and this effect becomes more important when the ratio between the telescope entrance pupil diameter ($D$) and the atmosphere Fried parameter ($r_0$) increases since the wavefront phase variance scales as $(D/r_0)^{5/3}$. Given that we are only able to act over the coherent part of the light, we expect that pupil apodization will be effective only when the coherent peak intensity contributes to the Strehl ratio more than the halo peak intensity. This condition is fulfilled for $D/r_0 < 7.8$ [23]. Hence, we expect that the effect of apodization will decrease when $D/r_0$ increases and it will become almost negligible for $D/r_0$ values larger than 7.8.

To take into account the distortion introduced by the atmosphere we consider that the telescope entrance pupil is multiplied by the phase screen $\Phi(x, y) = \exp(i\phi(x, y))$, where $\phi(x, y)$ is the wavefront aberration function, $x$ and $y$ are the spatial Cartesian coordinates.

Within the framework of Fourier Optics the telescope image is obtained by Fourier transform the entrance pupil field, $\phi(u, v) = FT(\Phi(x, y))$. Now the pupil function is the product between the apodizing function $SG(n, r)$ and the phase screen $\Phi(x, y)$:

$$SG(n, r) \cdot \Phi(x, y) = [HE(r) \otimes G(r)] \cdot \Phi(x, y) \xrightarrow{FT} \ [A(w) \cdot g(w)] \otimes \phi(u, v) = [A(w) \otimes \phi(u, v)] \cdot g(w),$$

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(for simplicity, we have mixed Cartesian and polar coordinates on spatial and Fourier domains, \( r = (x' + y')^{1/2} \) and \( w = (u' + v')^{1/2} \). The resulting image, obtained Fourier transforming the new pupil function, is the product between the result of convolving the function \( A(w) \) with \( \varphi(u,v) \) (which describes the degraded PSF composed by coherent peak and speckled halo) and the Gaussian function \( g(w) \). It is necessary to state that the function \( A(w) \) appearing in Eq. (12) is wider than the PSF of the clear pupil of size \( D \) since it corresponds to the Fourier transform of the function \( HE(r) \) which is always smaller than \( D \). Although the last term of Eq. (12) is an approximation only valid for slightly aberrated wavefronts, it can be helpful to understand how the Super-Gaussian apodizing function works. We see that the speckled PSF, described by the convolution product \( A(w) \otimes \varphi(u,v), \) is multiplied by a Gaussian function \( (g(w)) \) what produces an efficient noisy tails reduction.

Although it could be possible a theoretical estimate of the effect of applying a Super-Gaussian apodizing function on the telescope image quality, this estimate would be based on atmosphere models [23] that do not describe the PSF halo shape as exactly as necessary for attaining a proper moments estimate. Therefore, we preferred to use a numerical procedure previously checked [15] to simulate atmosphere distorted wavefronts and so to obtain a set of speckled PSF. Moments evaluated over this set are more accurate than those obtained from the theoretical model what allows a more actual image quality analysis. The field at the telescope entrance pupil can be expressed as \( E(x,y) = \exp(i\varphi(x,y)) \), where a constant field amplitude is assumed. This field is Fourier transformed to simulate the telescope outcome. Computer simulations were carried out using the FFT routine implemented in Matlab. To achieve a good spatial sampling and to avoid aliasing effects, 1024x1024 data samples were used. The telescope pupil was simulated with a 128x128 data sampling. We simulated series of \( \varphi(x,y) \) for a number of atmospheric conditions following the standard procedure established by [24]. In all cases piston, tip and tilt were corrected.

As an example, we show in Fig. 5 the effect of Super-Gaussian apodization for atmosphere conditions \( D/r_0 = 1, 5, \) and 9. It is clear that images obtained with the Super-Gaussian apodized pupil (b, d, f) show a sharper appearance than the corresponding unapodized ones (a, c, e), although the apodized image for \( D/r_0 = 9 \) remains highly aberrated. This sharpness will produce a reduction on the value of the parameter \( nm_{sg}(sg) \) as we will see later in Fig. 6.

A further detail that cannot be appreciated in Fig. 5 is that the Strehl of apodized pupil images is always higher than that of unapodized ones in a factor ranging from 4 for \( D/r_0 = 1 \) to 2 for \( D/r_0 = 9 \).

As it was stated, \( nm_{sg}(sg) \) is a measure of the size of the spot to be blocked by the coronagraphic mask. Consequently the smaller \( nm_{sg}(sg) \) value the better contrast. We used Fig. 6 to estimate the order of the Super-Gaussian apodizing function providing the smaller \( nm_{sg}(sg) \) value and accordingly the better contrast. It can be seen that the optimum Super-Gaussian order evolves from \( n = 5 \) to \( n = 2 \) as the ratio \( D/r_0 \) ranges from \( D/r_0 = 1 \) to 9. Hence, results in Fig. 6 can be used to choose the optimum value of the Super-Gaussian function order as a function of the atmosphere seeing conditions.

4. Apodized coronagraphy

After the theoretical analysis of the Super-Gaussian apodizing functions, we studied the detectability improvement produced when they are used in a coronagraph. The coronagraph behavior is modeled within the Fourier Optics theory. As we stated the field at the telescope entrance pupil is Fourier transformed to simulate the telescope outcome. The telescope image field is then multiplied by a coronagraphic mask (we will only consider circular hard-edge mask). The product is again transformed and then multiplied by the Lyot stop. Finally, a third Fourier transform is calculated to form the final image. The simulation conditions are the same as in Section 3. Since the coronagraph IWA is defined as the halfwidth at half-maximum of the intensity transmission profile and we are using hard-edge masks, the IWA value will be the radius mask in all cases.
The first step was to determine the optimum radius of the mask for the different atmospheric seeing conditions. This analysis has been carried out taking pupil and Lyot stop as hard-edge apertures. For this purpose, a set of one hundred images were obtained from the corresponding wavefront realizations for a particular D/r0 value and a particular mask radius. Then the calculated average image was used to determine the angular distance ρ at which a star companion with intensity 106 times weaker than the parent star can be detected. We consider that detection is possible only when the intensity ratio between the companion and the remaining star halo is larger than 5 (SNR>5). The process was repeated for different mask radii and we choose those radii providing the smallest ρ. The so obtained mask radii were 10, 14, 18, and 22 (in λ/D units) for D/r0 = 1, 3, 5, and 7 respectively [15].

In addition, a useful tool to compare performances is to evaluate the star contrast profile defined as [24]:

Fig. 5. Aberrated PSF for different seeing conditions with and without Super-Gaussian apodization: D/r0 = 1, (a) unapodized and (b) apodized; D/r0 = 5, (c) unapodized and (d) apodized; D/r0 = 9, (e) unapodized and (f) apodized.

Fig. 6. Normalized second-order moment as a function of the Super-Gaussian order, for different atmosphere conditions: D/r0 = 1,3,5,7, and 9.
where \( I(r) \) is the intensity at the radial coordinate in the final star image, \( I_s(0) \) is the peak intensity obtained without the coronagraphic mask in the optical train, and \( |M(r)|^2 \) is the mask intensity transmission that in our case takes value zero inside the radius mask and unity outside it.

Figure 7 shows the contrast profiles for different coronagraph and telescope apodizations in various atmosphere conditions: i) a hard-edge entrance pupil and a hard-edge Lyot stop fixed at 75% of the telescope pupil diameter (HEP-HEL, dashed line); ii) an entrance pupil apodized by a Super-Gaussian function and a hard-edge Lyot stop fixed at 75% of the telescope pupil diameter (SGP-HEL, dotted line); and iii) an entrance pupil apodized by a Super-Gaussian function and a Super-Gaussian profile Lyot stop (SGP-SGL, solid line). It is also included for comparison the PSF of the no mask case (dash-dotted). Figures 7(a-d) correspond to \( D/r_0 \) values of 1, 3, 5, and 7, respectively. We see that the introduction of apodizing Super-Gaussian (SGP-HEL and SGP-SGL cases) produces an important improvement with respect to hard-edge pupil case (HEP-HEL) for all \( D/r_0 \) values. Besides, the use of a Super-Gaussian Lyot function (SGP-SGL case) improves the values of hard-edge Lyot stop (SGP-HEL). This improvement is particularly important for angular positions close to the coronagraphic mask border.

Although to apodize pupil and Lyot stop at the same time (SGP-SGL case) produces companion light absorption, the energy loss using Super-Gaussian profiles is about one order of magnitude smaller than using prolate spheroidal apodizing profiles.

Table 1 shows positions and intensities of detectable companions for different entrance pupil and Lyot stop combinations. The positions (left numbers) correspond to the closest angular distance (in \( \lambda/D \) units) at which a companion \( 10^6 \) times fainter than the parent star can be detected (SNR>5) and the peak companion intensities (right numbers) are normalized to those obtained when entrance pupil and Lyot stop are both hard-edge. Data correspond to entrance pupil telescope: HEP, SGP, and apodized with a prolate spheroidal function (PSP); combined with HEL or SGL. Atmosphere conditions correspond to \( D/r_0 \) = 1, 3, 5, and 7.

It can be seen that apodization achieves a reduction of the detection distance at the expense of a companion peak intensity decrease. Apodized entrance pupil combined with hard-edge Lyot (SGP-HEL) provide smaller angular detection distances and higher companion intensities than clear pupil with apodized Lyot (HEP-SGL), for small \( D/r_0 \) values. However, when \( D/r_0 \) increases, both combinations provide equivalent distances whilst the combinations HEP-SGL offer the highest peaks intensities. For apodized pupils the introduction of Super-Gaussian Lyot stop does not improve angular detection distance but reduces the companion signal. Hence, for low \( D/r_0 \) values it is better to apodize the pupil whilst for large values to apodize the Lyot stop.

In spite of the IWA of the prolate spheroidal apodization derived from Fig. 4(b) was smaller than that of the Super-Gaussian apodization one, we see in Table 1 that, due to the atmosphere effect, both Super-Gaussian and prolate spheroidal pupils provide basically the same IWA. However, the planet peak is always highest for apodizing Super-Gaussian functions in a factor about 3. Hence, to have a clear signal that is less affected by detection noises it seems that Super-Gaussian apodizing functions should be used.
Fig. 7. Contrast curves for $D/r_0 = 1, 3, 5,$ and $7$ as a function of the angular distance from the optical axis in $\lambda/D$ units.

<table>
<thead>
<tr>
<th>$1.0E-6$</th>
<th>HEP</th>
<th>SGP</th>
<th>PSP</th>
<th>$D/r_0$</th>
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<tr>
<td>HEL</td>
<td>15.3 / 1.0</td>
<td>12.7 / 0.7</td>
<td>12.5 / 0.2</td>
<td>1</td>
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<td></td>
<td>19.2 / 1.0</td>
<td>17.1 / 0.6</td>
<td>17.0 / 0.2</td>
<td>3</td>
</tr>
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<td></td>
<td>22.9 / 1.0</td>
<td>20.7 / 0.6</td>
<td>20.4 / 0.2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>28.0 / 1.0</td>
<td>24.1 / 0.3</td>
<td>23.7 / 0.1</td>
<td>7</td>
</tr>
<tr>
<td>SGL</td>
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<td>12.6 / 0.3</td>
<td>12.4 / 0.1</td>
<td>1</td>
</tr>
<tr>
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<td>17.0 / 0.3</td>
<td>17.0 / 0.2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>20.7 / 0.4</td>
<td>20.4 / 0.3</td>
<td>20.4 / 0.1</td>
<td>5</td>
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<tr>
<td></td>
<td>24.1 / 0.4</td>
<td>23.7 / 0.2</td>
<td>23.7 / 0.1</td>
<td>7</td>
</tr>
</tbody>
</table>

Closest angular distance at which a companion $10^6$ times fainter than the parent star can be detected (left numbers) and companion peak intensity normalized by that obtained for both entrance pupil and Lyot stop hard-edges (right numbers). Data correspond to clear (HEP), Super-Gaussian apodized (SGP), and prolate spheroidal apodized (PSP) telescope pupils, and for hard-edge and Super-Gaussian apodized Lyot stops (HEL and SGL).
Table 2. Angular distance reduction

<table>
<thead>
<tr>
<th>%</th>
<th>HEP</th>
<th>SGP</th>
<th>PSP</th>
<th>D/r0</th>
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<td>16.4 / 28.0</td>
<td>18.0 / 33.3</td>
<td>1</td>
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<td></td>
<td>0.0 / 0.0</td>
<td>11.1 / 24.8</td>
<td>11.7 / 26.6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.0 / 0.0</td>
<td>9.3 / 28.6</td>
<td>11 / 31.9</td>
<td>5</td>
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<td></td>
<td>0.0 / 0.0</td>
<td>13.9 / 41.9</td>
<td>15.2 / 43.7</td>
<td>7</td>
</tr>
<tr>
<td>SGL</td>
<td>12.3 / 31.7</td>
<td>17.2 / 35.5</td>
<td>18.8 / 36.0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>9.7 / 26.6</td>
<td>11.7 / 27.5</td>
<td>11.7 / 28</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>9.3 / 31.5</td>
<td>10.9 / 31.9</td>
<td>10.9 / 32.2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>13.9 / 42.9</td>
<td>15.2 / 44.0</td>
<td>15.2 / 43.7</td>
<td>7</td>
</tr>
</tbody>
</table>

Reduction in percent of the closest angular distance at which a companion 10⁶ times (left numbers) and 10⁷ times (right numbers) fainter than the parent star can be detected. Values corresponding to entrance pupil and Lyot stop both hard-edge have been taken as reference. Data correspond to clear (HEP), Super-Gaussian apodized (SGP), and prolate spheroidal apodized (PSP) telescope pupils, and for hard-edge and Super-Gaussian apodized Lyot stops (HEL and SGL).

Table 3. Angular distance reduction with image selection

<table>
<thead>
<tr>
<th>%</th>
<th>HEP</th>
<th>SGP</th>
<th>PSP</th>
<th>D/r0</th>
</tr>
</thead>
<tbody>
<tr>
<td>HEL</td>
<td>0.0 / 0.0</td>
<td>0.0 / 0.8</td>
<td>1.0 / 6.4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>5.2 / 9.6</td>
<td>2.9 / 4.9</td>
<td>3.6 / 4.4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>5.5 / 11.6</td>
<td>6.0 / 5.0</td>
<td>4.3 / 4.3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>10.3 / 22.8</td>
<td>6.7 / 5.9</td>
<td>6.9 / 7.0</td>
<td>7</td>
</tr>
<tr>
<td>SGL</td>
<td>0.0 / 0.0</td>
<td>0.9 / 3.3</td>
<td>2.1 / 4.2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1.5 / 3.1</td>
<td>2.9 / 2.5</td>
<td>3.6 / 2.5</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>2.4 / 4.2</td>
<td>4.3 / 3.7</td>
<td>4.9 / 3.8</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4.6 / 4.6</td>
<td>5.3 / 5.1</td>
<td>7.4 / 7.0</td>
<td>7</td>
</tr>
</tbody>
</table>

The same as in Table 2 but when only the better 10% of the simulated frames are used.

Depending on the companion intensity these results may differ. In Table 2, values of the reduction in percent of the closest angular distance are shown for companions 10⁶ (left numbers) and 10⁷ (right numbers) times fainter than the parent star. Data are normalized to those values obtained when the telescope pupil and the Lyot stop are hard-edge. The main conclusion is that the distance reduction is more important when the companion intensity decreases. In this particular case the relative distance reduction in percent can reach a value of 44%. This tendency has also been confirmed for companions fainter than 10⁻⁷ times the parent star. It also can be seen that the general behavior of the detection angular distance reduction as a function of D/r₀ basically does not depend on the type of apodizing function used or even on the plane (pupil or Lyot) where they are applied.

5. Frame selection. Lucky imaging

It was already commented that a frame selection may improve previous results. Table 3 shows the improvement in percent of the closest angular distance at which a companion can be detected as a result of using the 10% of the detected frames with a higher Strehl ratio. Data for companions 10⁶ times fainter than the parent star appear on the left and those for a ratio of 10⁷ appear on the right. A general conclusion is that frame selection may reduce the angular detection distance up to a 20 percent when telescope pupil and Lyot stop are hard-edge.
However, when the pupil or the Lyot stop has been apodized the improvement falls to values around a 6 percent. Hence, the angular position where a companion can be detected may decrease an additional distance of $1 \lambda/D$ when using apodizing mask and frame selection.

6. Conclusions

A thorough analysis of Super-Gaussian functions obtained as the 2D convolution product of a circular hard-edge function and a Gaussian has been performed. As a way to estimate its ability to concentrate light we have carried out a theoretical analysis based on their second order moments.

Super-Gaussian functions have been used to apodize the telescope pupil or the Lyot stop in a high-contrast coronagraph. The angular distance at which a faint companion can be detected is significantly reduced in both cases although the companion transmitted intensity is higher when apodizing the telescope pupil. A comparison with prolate spheroidal apodizing functions has also been carried out. Improvement in the angular distance detection obtained using prolate spheroidal functions is similar to that obtained from Super-Gaussian functions although prolate spheroidal functions produce a larger absorption of the companion light intensity.

The reduction of the detection distance can be significant in particular for the faintest companions. Besides, a proper frame selection (Lucky Imaging technique) will allow an additional reduction on the angular detection distance. A combination of all these effects may suppose a relevant reduction of the detection angular distance. All the analysis has been carried out for moderately aberrated wavefronts, that is, for those cases in which the coherent part of the PSF is as large as the incoherent one.

Acknowledgments

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