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Classical relativistic statistical mechanics: The case of a hot dilute plasma

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Starting from predictive relativistic mechanics we develop a classical relativistic statistical mechanics. For a system of N particles, the basic distribution function depends, in addition to the 6 N coordinates and velocities, on N times, instead of a single one as in the usual statistical mechanics. This generalized distribution function obeys N (instead of 1) continuity equations, which give rise to N Liouville equations in the case of a dilute plasma (i.e., to lowest, nonzero order in the charges). Hence, the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for the reduced generalized distribution functions is derived. A relativistic Vlasov equation is obtained in this way. Thermal equilibrium is then considered for a dilute plasma. The calculation is explicitly worked out for a weakly relativistic plasma, up to order $1/c^2$, and known results are recovered.

I. INTRODUCTION

The lack of a classical relativistic theory of interacting particles for many years has had as a consequence that a satisfactory classical relativistic statistical mechanics does not exist at present. This situation has been emphasized by Havas.[1] Nevertheless, in the last few years a consistent framework for classical relativistic systems of interacting particles has been developed. The original work is in Refs. 2–5, and in Ref. 4 this new framework has been called "predictive relativistic mechanics" (PRM).

PRM is a "Newtonian type" theory of the classical N-particle interaction in Minkowski space M_4. By Newtonian type it is meant that the dynamics of the system is governed by a system of second-order ordinary differential equations. It can be formulated in a three-dimensional formalism as well as in a manifestly covariant formalism.

The fundamental facts of the latter approach[4,5] are as follows: consider a system of N classical pointlike interacting particles in M_4. Let x_a^α be the four-position of particle a $(a, b, \ldots =1, \ldots, N)$ and let s_a be its proper time. Then PRM states that the dynamics of the system is governed by a system of ordinary differential equations of the form

$$u_a^\alpha=\frac{dx_a^\alpha}{ds_a},\quad \frac{du_a^\alpha}{ds_a}=\xi_a^\alpha(x_b^\beta, u_b^\gamma)\quad (\alpha,\beta,\ldots=0,1,2,3)\,, \tag{1}$$

that is, u_a^α is the four-velocity and ξ_a^α is the four-acceleration of particle a, which depends on the four-positions and four-velocities of all particles.

Now we search for solutions to the system (1) of the form

$$x_a^\alpha=x_a^\alpha(s_a, \text{initial conditions})\,, \tag{2}$$

that is, we can parametrize the trajectory of particle a with one parameter: its own proper time. In order that Eq. (2) will be compatible with (1) we must have the integrability conditions

$$\frac{d\xi_a^\alpha}{ds_b}=0\,,\quad \forall b\neq a\,, \tag{3}$$

where d/ds_b means the differential operator

$$\frac{d}{ds_b}=u_b^\rho\frac{\partial}{\partial x_b^\rho}+\xi_b^\rho\frac{\partial}{\partial u_b^\rho}\,. \tag{4}$$

Throughout this paper the Einstein summation convention for *Greek* labels is assumed.

Choosing the signature -2 we have for the four-vector u_a^α

$$u_a^{\,2}=\eta_{\alpha\beta}u_a^\alpha u_a^\beta=1\,, \tag{5}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric tensor. [In the definition (4) the derivatives $\partial/\partial u_b^\rho$ must be understood as if the identity (5) did not exist, that is, as if the four components of u_b^ρ were independent.] Then

$$(u_a\xi_a)=\eta_{\alpha\beta}u_a^\alpha\xi_a^\beta=0\,. \tag{6}$$

Equations (3) and (6) are the fundamental restrictions that the theory imposes on the possible four-accelerations. They are true independent of whether or not an external field exists.

As an alternative to the manifestly covariant formalism of PRM, we can obtain from the ξ_a^α the three-accelerations $\vec{\mu}_a$ of the system through the relations

$$\mu_a^i=\gamma_a^{-2}(\xi_a^i-\xi_a^0 v_a^i)\,,\quad i=1,2,3 \tag{7}$$

where $\vec{v}_a$ is the three-velocity of the particle a and $\gamma_a=(1-v_a^{\,2})^{-1/2}$. Let c, the speed of light, be equal to 1. In this way we obtain μ_a^i as a function of the four-positions $x_b^\alpha=(t_b, \vec{x}_b)$ and four-velocities u_b^α. Then we can restrict ourselves to simultaneous configurations, $t_b=t$, $\forall b$, in order to obtain the physical three-accelerations "seen" by a given inertial observer: $\mu_a^i\big|_{t_b=t}$, $\forall t_b$. The canonical formulation of this three-dimensional formalism of PRM was given first by Hill.[6] In Ref. 7 the three-accelerations, $\mu_a^i|_{t_b=t}$, are supposed to exist as power series in the coupling constants and then a canonical formalism is proved to exist within this perturbative framework satisfying "good" asymptotic conditions (good meaning that for infinite particle separation in the past—alternatively in the future—the canonical coordinates recover their standard free-particle expressions). The role of good asymptotic conditions was originally brought out by Kerner and Hill.[8] According to the "noninteraction theorems"[9,10] the canonical coordinates $\vec{q}_a$ cannot be the three-positions $\vec{x}_a$.

PRM is consistent with the classical theory of fields at least in the perturbative framework.[11,12] For example, in the case of electromagnetism, given the retarded Liénard-Wiechert potentials (alternatively advanced or time symmetric) we have a unique four-acceleration ξ_a^α which reduces to that obtained from these potentials for isotropic configurations: $\eta_{\alpha\beta}(x_a^\alpha - x_b^\alpha)(x_a^\beta - x_b^\beta) = 0$, $\forall b, a$ with $b \neq a$.

Now consider a relativistic macroscopic system of N classical interacting particles, perhaps with an external field in the framework of the PRM. Since, under the conditions stated before, we have a canonical formalism corresponding to the three-accelerations $\mu_a^i|_{t_b=t}$, that is, systems of canonical coordinates $(\vec{q}_a, \vec{p}_a)$ and the corresponding Hamiltonian H, we can construct the statistical mechanics of the macroscopic system along the same lines as that in the nonrelativistic case. In the important case of equilibrium we can write for the partition function Z

$$Z = \text{const} \times \int e^{-\beta H} \prod_{a=1}^{N} d^3\vec{q}_a d^3\vec{p}_a , \tag{8}$$

where as usual $\beta = 1/kT$ with T the temperature and k the Boltzmann constant. Formula (8) supposes that the intrinsic angular momentum of the macroscopic system is zero and that we work in a frame relative to which the system is macroscopically at rest, i.e., a frame where the total momentum of the system is zero. As long as the system is isolated this is the correct definition of the rest frame, since in PRM the description of the interaction between particles can be made without the introduction of the field as something independent of the degrees of freedom of the particles. Now, as can be seen in Ref. 11, where the case without an external field is treated, $\vec{q}_a$, $\vec{p}_a$, and H are complicated functions of $\vec{x}_b, \vec{v}_b$ even at first order in the coupling constants. This is true for both short- and long-range scalar and vector interactions and also for the interesting special case of the electromagnetic interactions. Then the calculation of Z becomes involved. Nevertheless, in the case of a dilute completely ionized plasma we guess that a remarkable simplification is possible, as we will see in Sec. III.

In Sec. II we develop a general approach to classical relativistic statistical mechanics to treat both the equilibrium and nonequilibrium cases, which allows for practical calculations. We begin defining the generalized distribution function of N particles and then we obtain N continuity equations for this distribution function.

In Sec. III the case of a dilute relativistic plasma is considered. We work in the space of the three-positions $\vec{x}_a$ and three-velocities $\vec{u}_a = \gamma_a \vec{v}_a$ and obtain N Liouville equations from the N continuity equations although the $(\vec{x}_a, \vec{u}_a)$ are not canonical coordinates. Then we give the standard distribution function relative to the coordinates $(\vec{x}_a, \vec{u}_a)$ for a dilute plasma in equilibrium in a frame where the system is at rest, and finally we give the relativistic extension of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy.

In Sec. IV the relativistic BBGKY hierarchy is cut off in a standard way and some general results are established for the resulting solutions. Finally, in Sec. V we calculate the two-body distribution function of a slightly relativistic plasma in order to get a simple test for the theory.

II. THE GENERALIZED DISTRIBUTION FUNCTION AND THE CONTINUITY EQUATIONS

We now consider a relativistic macroscopic system of N classical particles which are interacting among themselves, with or without an external field, in the framework of the PRM. Let us define its "generalized distribution function" $F(t_1, \vec{x}_1, \vec{u}_1, t_2, \vec{x}_2, \vec{u}_2, \ldots) \equiv F(t_a, \vec{x}_a, \vec{u}_a)$ ($\vec{u}_a$ is the three-vector consisting of the space components of u_a^α) as the probability density of finding particle 1 at $\vec{x}_1$ with velocity $\vec{u}_1$ at time t_1, particle 2 at $\vec{x}_2$ with velocity $\vec{u}_2$ at time t_2, and so on. That is,

$$\delta P(t_a, \vec{x}_a, \vec{u}_a) = F(t_a, \vec{x}_a, \vec{u}_a) \prod_{b=1}^{N} d^3\vec{x}_b d^3\vec{u}_b \tag{9}$$

is the elementary probability of finding every particle a in the corresponding elementary volume $d^3\vec{x}_a d^3\vec{u}_a$ at time t_a.

Because of the Newtonian character of the equations of motion (1) and because of the equations (3) which allow for the existence of dynamical trajectories such as (2), we have a deterministic dynamical problem with a finite number of initial-value data: $(t_a, \vec{x}_a, \vec{u}_a)$, $a = 1, \ldots, N$. Then $\delta P(t_a, \vec{x}_a, \vec{u}_a)$ does not change by changing the arguments $(t_a, \vec{x}_a, \vec{u}_a)$ to new arguments representing the same dynamical trajectories, or correlatively, δP depends only on which ensemble of dynamical trajectories is considered. That is,

$$\frac{d\delta P}{ds_a}=u_a^\rho\frac{\partial\delta P}{\partial x_a^\rho}+\xi_a^\rho\frac{\partial\delta P}{\partial u_a^\rho}=0\,,\quad \forall a\,. \tag{10}$$

These N conservation equations for δP are actually compatible since Eqs. (3), satisfied by the four-accelerations, guarantee that the integrability conditions of (10) are satisfied.

From the N conservation equations (10) we obtain the N continuity equations

$$\frac{\partial F}{\partial t_a}+\frac{\partial}{\partial \vec{x}_a}\cdot\left(F\frac{d\vec{x}_a}{dt_a}\right)+\frac{\partial}{\partial\vec{u}_a}\cdot\left(F\frac{d\vec{u}_a}{dt_a}\right)=0\,, \tag{11}$$

that is,

$$\frac{\partial F}{\partial t_a}+\frac{\partial}{\partial \vec{x}_a}\cdot(F\vec{V}_a)+\frac{\partial}{\partial\vec{u}_a}\cdot(F\gamma_a^{-1}\vec{\xi}_a)=0\,, \tag{12}$$

where $\vec{\xi}_a$ is the three-vector consisting of the space components of ξ_a^α. [Note that at variance with Eq. (10), where the four components of u_a^α are considered as independent, in Eq. (11) the identity (5) is taken into account. That is, the three components of $\vec{u}_a$ are considered as independent while u_a^0 is given by $u_a^0=(1+\vec{u}_a^{\,2})^{1/2}$.]

Thus, instead of having a unique continuity equation for the standard distribution function $f(t,\vec{x}_a,\vec{u}_a)$ depending on $6N+1$ variables, we have N continuity equations for the generalized distribution function $F(t_a,\vec{x}_a,\vec{u}_a)$ depending on $7N$ variables.

When we put $t_a=t$, $\forall a$ in the generalized distribution function $F(t_a,\vec{x}_a,\vec{u}_a)$, we obtain the standard distribution function

$$f(t,\vec{x}_a,\vec{u}_a)=F(t_a=t,\vec{x}_a,\vec{u}_a)\,. \tag{13}$$

By adding the N continuity equations (12) and setting $t_b=t$, $\forall b$, we obtain the standard continuity equation for the standard distribution function

$$\frac{\partial f}{\partial t}+\sum_a\frac{\partial}{\partial\vec{x}_a}\cdot(f\vec{v}_a)+\sum_a\frac{\partial}{\partial\vec{u}_a}\cdot[f\gamma_a^{-1}(\vec{\xi}_a)_{t_b=t}]=0\,. \tag{14}$$

If the macroscopic system is in equilibrium then F must be invariant by time translations and f does not depend on time. If besides being in equilibrium the system is homogeneous, then F must be also invariant by space translations. That is, in this case, F will depend on the $\vec{x}_a$ only through the relative positions $\vec{x}_b-\vec{x}_c$. Of course if there is a container we will have some sort of inhomogeneity near the walls.

From the generalized distribution function, $F(t_a,\vec{x}_a,\vec{u}_a)$, we obtain the reduced generalized distribution function of order $s<N$,

$$F^{(s)}(t_A,\vec{x}_A,\vec{u}_A)=\int F\prod_{R=s+1}^{N}d^3\vec{x}_R\,d^3\vec{u}_R\,. \tag{15}$$

Here and in the following capital Latin letters from the first part of the alphabet, $A,B,\ldots$, take the values $1,\ldots,s$ and capital Latin letters from the latter part of the alphabet, $R,S,\ldots$, take the values $s+1,\ldots,N$. The integral is extended to all positions $\vec{x}_R$ and to all velocities $\vec{u}_R$.

Let us set $a=R$ in Eq. (12) and integrate over $d^3\vec{x}_R d^3\vec{u}_R$, the integral being extended to all values of $\vec{x}_R$, $\vec{u}_R$. After applying Gauss's theorem we obtain in an obvious notation

$$\frac{\partial F^{(N-1)}}{\partial t_R}=-\int d^3\vec{u}_R\int F\vec{v}_R\cdot d\vec{\Sigma}_x-\int d^3\vec{x}_R\int F\gamma_R^{-1}\vec{\xi}_R\cdot d\vec{\Sigma}_u\,, \tag{16}$$

that is, for any reasonable asymptotic conditions for the macroscopic system

$$\frac{\partial F^{(N-1)}}{\partial t_R}=0 \tag{17}$$

and then

$$\frac{\partial F^{(s)}}{\partial t_R}=0 \tag{18}$$

in agreement with the notation in the left-hand side of (15), where the labels t_R were omitted.

III. THE CASE OF A DILUTE PLASMA

Let us consider a system of N interacting charged particles without any external field. (Later we will relax somewhat this condition when we consider a system in equilibrium, in which case a container may be necessary, but we may suppose that the container produces a surface effect which will be negligible for particles not too near the walls.) Then, to first order in the products of the charges, the physical solution to Eq. (3) gives[13] for the four-acceleration ξ_a^α of the charge a

$$\xi_a^\alpha=\sum_b\xi_{ab}^\alpha\,,\quad a\neq b \tag{19}$$

where

$$\xi_{ab}^\alpha=\frac{e_ae_b}{m_aR_{ab}^{\;3}}[(u_au_b)x_{ab}^\alpha-(x_{ab}u_a)u_b^\alpha]\,. \tag{20}$$

Here e_a, e_b are the charges of particles a and b, m_a is the mass of particle a, and R_{ab} is the function

$$R_{ab}=[(x_{ab}u_b)^2-x_{ab}^{\;2}]^{1/2}\,. \tag{21}$$

For x_{ab}^α we have $x_{ab}^\alpha=x_a^\alpha-x_b^\alpha$, and (u_au_b), $(x_{ab}u_a)$, $(x_{ab}u_b)$, and $x_{ab}^{\;2}$ mean four-scalar products.

When no bound states are present, formal expansions in the charges, which underly the method to obtain the Eqs. (19) and (20), must be interpreted as expansions in the dimensionless parameter e_ae_b/mc^2h, where m is m_a or m_b and h stands for the typical impact parameter of the collisions.

Then, in order to assure the fast convergence of the expansions, we must have relatively high impact parameters, that is, we must have dilute enough systems.

For a dilute completely ionized plasma without external field, it is easy to verify that

$$\frac{\partial}{\partial \vec{u}_a}\cdot(\gamma_a^{-1}\vec{\xi}_a)=0 \tag{22}$$

with ξ_a^α given by Eqs. (19) and (20). [Note that Eq. (22) is true even in the presence of an external field.] Hence, in this plasma, the N continuity equations (12) can be written as the N Liouville equations

$$\frac{\partial F}{\partial t_a}+\vec{v}_a\cdot\frac{\partial F}{\partial \vec{x}_a}+\gamma_a^{-1}\vec{\xi}_a\cdot\frac{\partial F}{\partial \vec{u}_a}=0\,. \tag{23}$$

Multiplication of these equations by γ_a allow us to write more compactly

$$\frac{dF}{ds_a}=u_a^\alpha\frac{\partial F}{\partial x_a^\alpha}+\xi_a^\alpha\frac{\partial F}{\partial u_a^\alpha}=0 \tag{24}$$

according to Eq. (4).

A generalized distribution function such as the one defined here, $F(t_a, \vec{x}_a, \vec{u}_a)$, has been considered previously by Hakim[14] and van Kampen.[15] Both authors give N Liouville equations for this distribution function but they limit themselves to the case when the particles are free or only a prescribed external field is present. When the interaction among the particles is taken into account we need the framework of PRM in order to assure the integrability conditions of the N Liouville or the N continuity equations. In Ref. 15 the important result establishing the Lorentz invariance of the generalized distribution is derived.

Starting from Eq. (14) and again taking into account Eq. (22), we obtain the Liouville equation

$$\frac{\partial f}{\partial t}+\sum_a \vec{v}_a\cdot\frac{\partial f}{\partial \vec{x}_a}+\sum_a \gamma_a^{-1}(\vec{\xi}_a)_{t_b=t}\cdot\frac{\partial f}{\partial \vec{u}_a}=0 \tag{25}$$

for the standard distribution function $f(t, \vec{x}_a, \vec{u}_a)$ on the space of the positions $\vec{x}_a$ and velocities $\vec{u}_a$.

As we have mentioned in the Introduction, we have a canonical formalism for the dynamical system described by the instantaneous three-accelerations $\mu_a^i|_{t_b=t}$ so that we can write directly the Liousville equation

$$\frac{\partial \tilde{f}}{\partial t}+\sum_a\frac{d\vec{q}_a}{dt}\cdot\frac{\partial \tilde{f}}{\partial \vec{q}_a}+\sum_a\frac{d\vec{p}_a}{dt}\cdot\frac{\partial \tilde{f}}{\partial \vec{p}_a}=0\,, \tag{26}$$

where $(\vec{q}_a, \vec{p}_a)$ are canonical coordinates and the standard distribution function, $\tilde{f}=\tilde{f}(t, \vec{q}_b, \vec{p}_b)$ is defined on phase space.

The standard distribution functions f and $\tilde{f}$ are connected by the relation

$$f=\prod_{b=1}^N (m_b)^3\frac{D(\vec{x}_a, m_a\vec{u}_a)}{D(\vec{q}_a, \vec{p}_a)}\tilde{f}\,, \tag{27}$$

where $D(\vec{x}_a, m_a\vec{u}_a)/D(\vec{q}_a, \vec{p}_a)$ is the Jacobian determinant for the coordinate transformation $(\vec{x}_a, m_a\vec{u}_a)\to(\vec{q}_a, \vec{p}_a)$. Note that the determinant is an integral of the motion since in a dilute plasma the continuity equation (14) becomes the Liouville equation (25) in the $(\vec{x}_a, \vec{u}_a)$ space. Indeed, our guess is that the value of the determinant is const $\times\exp[(\beta-\beta')H]$, where $\beta=1/kT$ and β' is a constant. [Provided that we have a Liouville equation for the distribution function $f(t, \vec{x}_a, \vec{u}_a)$, the same considerations which are used in Newtonian statistical mechanics (see e.g., Ref. 16 for the Newtonian case) lead here to $f=\text{const}\times\exp(-\beta' H)$ for an equilibrium system. On the other hand for canonical coordinates we have $\tilde{f}=\text{const}\times\exp(-\beta H)$ and therefore, from Eq. (27), it follows the expression $\text{const}\times\exp(\beta-\beta')H$ for the Jacobian determinant. Obviously in the limit of the density going to zero β' goes to β. In this case the determinant takes the value 1.] Then we have for the partition function of a dilute plasma in equilibrium

$$Z=\text{const}\times\int\exp(-\beta' H)\prod_{a=1}^N d^3\vec{x}_a d^3\vec{u}_a\,, \tag{28}$$

where it is assumed that the total momentum and the intrinsic angular momentum are zero. Compared with Eq. (8) which gives Z in the general case, this expression for Z could represent an important progress as long as the actual calculation of Z, for the dilute plasma, is concerned.

Let us now consider the definition (15) for $F^{(s)}(t_A, \vec{x}_A, \vec{u}_A)$. By integration over $\vec{x}_R, \vec{u}_R$, we obtain from (24)

$$u_A^\alpha\frac{\partial F^{(s)}}{\partial x_A^\alpha}+\sum_B \xi_{AB}^\alpha\frac{\partial F^{(s)}}{\partial u_A^\alpha}+\sum_R\int \xi_{AR}^\alpha\frac{\partial F^{(s+1)}}{\partial u_A^\alpha}d^3\vec{x}_R d^3\vec{u}_R=0\,,\quad B\neq A \tag{29}$$

where Eq. (19) has been taken into account. This hierarchy of equations corresponds to the BBGKY hierarchy for the standard reduced distribution functions of the nonrelativistic case.[17] Here, for each value of s we have not one but s equations corresponding to the different values of the label A: $1,\ldots,s$. Furthermore, when one writes the nonrelativistic BBGKY hierarchy for interactions which depend on the particle velocities, it can be seen that each equation of the hierarchy, corresponding to each value of s, involves three reduced standard distribution functions, $f^{(s)}$, $f^{(s+1)}$, and $f^{(s+2)}$, instead of only the first two functions, as is the case when the accelerations do not depend on the velocities. We see in (32) that the "relativistic hierarchy" involves only two reduced generalized distribution functions in spite of the fact that relativistic accelerations do depend on velocities.

As in the nonrelativistic case the determination of the reduced generalized distribution function $F^{(s)}$ can only be made when the hierarchy is cut off somewhere, that is, when for some value s we give $F^{(s+1)}$ as a function of the other $F^{(r)}$ functions with $r<s+1$. Let us remark that we have here restrictions on function $F^{(s+1)}$ that we do not have in the nonrelativistic case, since now $s(s-1)/2$ conditions of integrability, coming from (29), must be satisfied. To first order in the products of the charges these integrability conditions are

$$\sum_R \int \left[\xi^\alpha_{AR} \frac{\partial}{\partial u^\alpha_A}\left(u^\rho_B \frac{\partial F^{(s+1)}}{\partial x^\rho_B}\right) - \xi^\alpha_{BR}\frac{\partial}{\partial u^\alpha_B}\left(u^\rho_A \frac{\partial F^{(s+1)}}{\partial x^\rho_A}\right)\right] d^3\vec{x}_R d^3\vec{u}_R = 0 . \quad (30)$$

Of course, the integrability conditions for the whole hierarchy (29) are satisfied as long as we do not introduce any approximation. This follows from the fact that, because of the conditions (3) on the four-accelerations, the integrability conditions for Eq. (24) are satisfied.

IV. APPROXIMATE SOLUTIONS FOR THE RELATIVISTIC BBGKY HIERARCHY

In a completely ionized plasma, which is dilute enough, because of the long-range character of the interaction, the term $\sum_B \xi^\alpha_{AB}\partial F^{(s)}/\partial u^\alpha_A$ in Eq. (29) is small compared with the large collective effect represented by the integral term. Of course, this is only true as long as $s \ll N$. Then, according to what is done in the nonrelativistic case[17] we set, whatever particles 1, 2, 3 are,

$$F^{(2)}(1,2) = F^{(1)}(1)F^{(1)}(2)[1+G(1,2)] , \quad (31)$$

$$F^{(2)}(1,2,3) = F^{(1)}(1)F^{(1)}(2)F^{(1)}(3) \times [1+G(1,2)+G(1,3)+G(2,3)] , \quad (32)$$

where it is supposed that $G(1,2)\ll 1$. Equation (32), when $s=1,2$, gives for the functions $F^{(1)}$ and G the equations

$$u^\alpha_1 \frac{\partial F^{(1)}(1)}{\partial x^\alpha_1} + \frac{\partial F^{(1)}(1)}{\partial u^\alpha_1}\sum_R \int F^{(1)}(R)\xi^\alpha_{1R} d^3\vec{x}_R d^3\vec{u}_R = -\sum_R \int F^{(1)}(R)\xi^\alpha_{1R}\frac{\partial}{\partial u^\alpha_1}[F^{(1)}(1)G(1,R)]d^3\vec{x}_R d^3\vec{u}_R , \quad (33)$$

$$u^\alpha_A \frac{\partial G(1,2)}{\partial x^\alpha_A} = -\xi^\alpha_{AB}\frac{\partial}{\partial u^\alpha_A}\ln F^{(1)}(A) - G(1,2)u^\alpha_A \frac{\partial}{\partial x^\alpha_A}\ln F^{(1)}(A) - \frac{1}{F^{(1)}(A)}\frac{\partial[F^{(1)}(A)G(1,2)]}{\partial u^\alpha_A}\sum_R \int F^{(1)}(R)\xi^\alpha_{AR}d^3\vec{x}_R d^3\vec{u}_R$$
$$-\frac{\partial \ln F^{(1)}(A)}{\partial u^\alpha_A}\sum_R \int F^{(1)}(R)G(B,R)\xi^\alpha_{AR}d^3\vec{x}_R d^3\vec{u}_R , \quad B\neq A \quad (34)$$

to first order in the small quantities $\sum_B \xi^\alpha_{AB}\partial F^{(s)}/\partial u^\alpha_A$ and G. It can be seen that this is equivalent to work to lowest order in the dimensionless parameter $\epsilon=(N/V)(e^2/kT)^3$, where e is a typical charge of the system and V its volume (see Ref. 17 for the analogous nonrelativistic case). Note that the left-hand side of Eq. (33) equated to zero gives the relativistic Vlasov equation. This corresponds to considering Eq. (29) to zeroth order in ϵ.

The integrability condition of Eq. (34) is

$$u^\alpha_2 \frac{\partial}{\partial x^\alpha_2}\left[u^\beta_1 \frac{\partial G(1,2)}{\partial x^\beta_1}\right] - u^\alpha_1 \frac{\partial}{\partial x^\alpha_1}\left[u^\beta_2 \frac{\partial G(1,2)}{\partial x^\beta_2}\right] = 0 . \quad (35)$$

After some algebra we obtain in an obvious notation

$$u^\alpha_1 \frac{\partial F^{(1)}(1)}{\partial x^\alpha_1}\xi^\beta_{21}\frac{\partial F^{(1)}(2)}{\partial u^\beta_2} + \frac{\partial F^{(1)}(2)}{\partial u^\beta_2}\frac{\partial[F^{(1)}(1)\xi^\beta_{21}]}{\partial u^\alpha_1} \times \sum_R \int F^{(1)}(R)\xi^\alpha_{1R}d^3\vec{x}_R d^3\vec{u}_R - (1 \rightarrow 2) = 0 , \quad (36)$$

which is a nonlinear integrodifferential equation involving only $F^{(1)}$.

Let us consider the case of a homogeneous plasma in equilibrium. Then $F^{(1)}$ is the one-particle distribution function of an ideal gas. That is, in a frame relative to which the system is macroscopically at rest, we must set for $F^{(1)}$ the relativistic Maxwellian distribution

$$F^{(1)}(1) = [\beta m_1/4\pi V K_2(\beta m_1)]\exp(-\beta m_1\gamma_1) , \quad \beta = 1/kT \quad (37)$$

where V is the volume of the system and $K_2(\beta m_1)$ is the modified second-order Bessel function. Let us see what happens with the four equations (33), (34), and (35). First of all, taking into account Eq. (20), it can be seen that Eq. (35) is automatically satisfied and that the left-hand side of Eq. (33) is zero. Then we have

$$\sum_R \int F^{(1)}(R)\xi^\alpha_{1R}\frac{\partial}{\partial u^\alpha_1}[F^{(1)}(1)G(1,R)]d^3\vec{x}_R d^3\vec{u}_R = 0 , \quad (38)$$

while Eq. (34) becomes

$$u_A^\alpha \frac{\partial G(1,2)}{\partial x_A^\alpha} = \beta m_A \xi_{AB}^0 + \beta m_A \sum_R \int F^{(1)}(R)\, \xi_{AR}^0 G(B,R) d^3\vec{x}_R d^3\vec{u}_R\,, \tag{39}$$

with $F^{(1)}$ given by Eq. (37).

One can be convinced of the existence of solutions of Eqs. (38) and (39) by working out the low-velocity approximation. To first order in $1/c$ we obtain from Eq. (20)

$$\xi_{ab}^0 = \frac{e_a e_b}{m_a}\frac{\vec{x}_{ab}\cdot\vec{v}_a}{r_{ab}^3} + \frac{1}{c}\frac{x_{ab}^0}{r_{ab}^3}\left(\frac{3\vec{x}_{ab}\cdot\vec{v}_b}{r_{ab}^2}\vec{x}_{ab} - \vec{v}_b\right), \quad r_{ab} \equiv |\vec{x}_{ab}|\,. \tag{40}$$

Hence, taking into account Eq. (37), we obtain that the Debye-Hückel solution (see, for example, Ref. 16) for a two-component plasma

$$G(1,2) = -\beta \frac{e_1 e_2}{r_{12}} e^{-\kappa r_{12}}\,,$$
$$\kappa = (4\pi N \beta e^2/V)^{1/2}\,, \quad e^2 \equiv e_1{}^2 = e_2{}^2 \tag{41}$$

satisfies Eqs. (38) and (39) to first order in $1/c$. In Eqs. (40) and (41) the notation $\vec{x}_a - \vec{x}_b \equiv \vec{x}_{ab}$, $c(t_a - t_b) \equiv x_a^0 - x_b^0 \equiv x_{ab}^0$ has been used.

In fact it can be seen that $G(1,2)$ given by Eqs. (41) is the only physical solution for Eqs. (38) and (39), when the interaction is given by (40). So to order $1/c$, although the acceleration Eq. (40) does depend on t_{ab}, the function $G(1,2)$ given in Eq. (41) does not depend on t_{12}. That is, the relativistic corrections in $G(1,2)$ begin with $1/c^2$ terms, which incorporate the dependence on t_{12}.

V. THE CASE OF A DILUTE SLIGHTLY RELATIVISITIC PLASMA IN EQUILIBRIUM

As a test for the theory we calculate in this section the standard two-particle distribution function, $f^{(2)}(\vec{x}_{12}, \vec{u}_1, \vec{u}_2) = F^{(2)}(x_{12}^\alpha, \vec{u}_1, \vec{u}_2)_{t_1 = t_2}$, in the particular case of a dilute, slightly relativistic, homogeneous plasma in equilibrium, where only two different kinds of charges, e and $-e$, are present. The plasma is supposed to be completely ionized. It will be seen that in the appropriate limit previously known results are recovered.

Let us multiply Eq. (39) by $\gamma_A{}^{-1}$ and then sum the two equations that we obtain setting alternatively $A = 1, 2$. When the system is in equilibrium, $G(1,2)$ depends on t_1, t_2 only through its difference t_{12}. So we obtain

$$\vec{v}_1\cdot\frac{\partial G(1,2)}{\partial \vec{x}_1} - \beta m_1 \gamma_1{}^{-1}\xi_{12}^0 - \beta m_1\gamma_1{}^{-1}\sum_R \int \xi_{1R}^0 F^{(1)}(R) G(2,R) d^3\vec{x}_R d^3\vec{u}_R + \vec{v}_2\cdot\frac{\partial G(1,2)}{\partial\vec{x}_2} - \beta m_2\gamma_2{}^{-1}\xi_{21}^0 - \beta m_2 \gamma_2{}^{-1}\sum_R\int \xi_{2R}^0 F^{(1)}(R) G(1,R) d^3\vec{x}_R d^3\vec{u}_R = 0\,. \tag{42}$$

Now, $G(1,2)$ does not depend on t_R [see Eq. (18)] and the same is true for ξ_{12}^0 and ξ_{21}^0. So we can set $t_R = t_1 = t_2$ in Eq. (45). Then $G(1,2)$, ξ_{12}^0, ξ_{21}^0, $G(1,R)$, $G(2,R)$ are changed in $G(1,2)|_{t_{12}=0}$, $\xi_{12}^0|_{t_{12}=0}$, $G(1,R)|_{t_{1R}=0}$ etc. In order to simplify the notation in the remainder of this section, $G(1,2)|_{t_{12}=0}$ will be written $G(1,2)$ and so on. Hereafter Eq. (42) must be read with this new notation.

Now, in agreement with the result we want to obtain (getting $F^{(2)}|_{t_1=t_2}$ for a dilute slightly relativistic homogeneous plasma in equilibrium) let us set $t_{ab}=0$ in ξ_{ab}^α given by Eq. (20) and then expand this expression in powers of $1/c$ to second order. In this way we obtain the accelerations which can be derived from the Darwin Lagrangian[18] to first order in the products of the charges. For the time component we have

$$\gamma_a{}^{-1}\xi_{ab}^0 = \frac{e_a e_b}{m_a}\frac{\vec{x}_{ab}\cdot\vec{v}_a}{r_{ab}^3} + \frac{e_a e_b}{2m_a c^2}\frac{\vec{x}_{ab}\cdot\vec{v}_a}{r_{ab}}\left[v_b{}^2 - 3\frac{(\vec{x}_{ab}\cdot\vec{v}_b)^2}{r_{ab}^2}\right]. \tag{43}$$

As an ansatz let us write $G(1,2)$ as

$$G(1,2) = G_1(r_{12}) + G_2(r_{12})\frac{\vec{v}_1\cdot\vec{v}_2}{c^2} + G_3(r_{12})\frac{(\vec{x}_{12}\cdot\vec{v}_1)(\vec{x}_{12}\cdot\vec{v}_2)}{c^2 r_{12}^2}\,, \tag{44}$$

where $G_1(r_{12})$ is the Debye-Hückel solution given in Eq. (41). After some algebra, Eq. (42) gives

$$\vec{v}_1\cdot\frac{\partial}{\partial\vec{x}_{12}}\left[G(1,2) + \beta e_1 e_2\left(1 + \frac{1}{2}\frac{v_2{}^2}{c^2}\right)\frac{1}{r_{12}} + (\beta e_1/V)\sum_R e_R \int \frac{G_1(r_{2R})}{r_{1R}} d^3\vec{x}_R\right] - \vec{v}_2\cdot\frac{\partial}{\partial\vec{x}_{12}}\left[G(1,2) + \beta e_1 e_2\left(1+\frac{1}{2}\frac{v_2{}^2}{c^2}\right)\frac{1}{r_{12}} + (\beta e_2/V)\sum_R e_R \int \frac{G_1(r_{1R})}{r_{2R}} d^3\vec{x}_R\right] - \tfrac{3}{2}\beta e_1 e_2 \frac{(\vec{x}_{12}\cdot\vec{v}_2)^2}{c^2 r_{12}^2}\vec{v}_1\cdot\frac{\partial}{\partial\vec{x}_{12}}\frac{1}{r_{12}} + \tfrac{3}{2}\beta e_1 e_2\frac{(\vec{x}_{12}\cdot\vec{v}_1)^2}{c^2 r_{12}^2}\vec{v}_2\cdot\frac{\partial}{\partial\vec{x}_{12}}\frac{1}{r_{12}} = 0\,, \tag{45}$$

where Eqs. (43) and (44) have been taken into account. Then

$$(\vec{v}_1-\vec{v}_2)\cdot\frac{\partial}{\partial\vec{x}_{12}}\left[G_2(r_{12})\vec{v}_1\cdot\vec{v}_2+G_3(r_{12})\frac{(\vec{x}_{12}\cdot\vec{v}_1)(\vec{x}_{12}\cdot\vec{v}_2)}{r_{12}^2}\right]-\tfrac{1}{2}\beta e_1e_2v_2^2\frac{\vec{x}_{12}\cdot\vec{v}_1}{r_{12}^3}+\tfrac{1}{2}\beta e_1e_2v_1^2\frac{\vec{x}_{12}\cdot\vec{v}_2}{r_{12}^3}+\tfrac{3}{2}\beta e_1e_2\frac{(\vec{x}_{12}\cdot\vec{v}_1)^2(\vec{x}_{12}\cdot\vec{v}_1)}{r_{12}^5}-\tfrac{3}{2}\beta e_1e_2\frac{(\vec{x}_{12}\cdot\vec{v}_1)^2(\vec{x}_{12}\cdot\vec{v}_2)}{r_{12}^5}=0\,. \quad (46)$$

This equation can be separated into the following:

$$G_3(r_{12})=-\frac{\beta}{2}\frac{e_1e_2}{r_{12}},\quad \frac{dG_2(r_{12})}{dr_{12}}=\frac{\beta}{2}\frac{e_1e_2}{r_{12}^2}. \quad (47)$$

Then, $G(1,2)$ becomes

$$G(1,2)=-\frac{\beta e_1e_2}{r_{12}}\left[e^{-\kappa r_{12}}+\frac{\vec{v}_1\cdot\vec{v}_2}{2c^2}+\frac{(\vec{x}_{12}\cdot\vec{v}_1)(\vec{x}_{12}\cdot\vec{v}_2)}{2c^2r_{12}^2}\right] \quad (48)$$

and, according to Eq. (34), we obtain for the standard two-particle distribution function, $f^{(2)}(1,2)=F^{(2)}(1,2)\,|_{t_1=t_2}$,

$$f^{(2)}(1,2)=F^{(1)}(1)F^{(1)}(2)\left\{1-\frac{\beta e_1e_2}{r_{12}}\left[e^{-\kappa r_{12}}+\frac{\vec{v}_1\cdot\vec{v}_2}{2c^2}+\frac{(\vec{x}_{12}\cdot\vec{v}_1)(\vec{x}_{12}\cdot\vec{v}_2)}{2c^2r_{12}^2}\right]\right\} \quad (49)$$

with $F^{(1)}$ given by Eq. (37) (in fact only terms of $F^{(1)}$ up to $1/c^2$ must be retained). It can be verified that $G(1,2)$ given by Eq. (48), with ξ^0_{1R} given by Eq. (50) and $F^{(1)}$ by Eq. (37), satisfies identically the condition (38) for $t_{12}=0$, to first order in $1/c^2$.

The standard two-body distribution function for a dilute slightly relativistic plasma has been calculated previously by Kosachev and Trubnikov[19] starting from the Darwin Lagrangian. Our result, Eq. (50), agrees with theirs to order $1/c^2$, but not to higher-order terms. We think that these terms are meaningless unless one goes beyond the Darwin Lagrangian, which is only correct to order $1/c^2$. In a forthcoming paper we obtain the two-body distribution function to all orders in $1/c$ starting from ξ^0_{ab} given by (20).

From Eq. (51) we can calculate the energy of the plasma E,

$$E=\sum_{a=1}^{N}\sum_{b=1}^{N}\int H_{ab}f^{(2)}(a,b)d^3\vec{u}_ad^3\vec{u}_bd^3\vec{x}_ad^3\vec{x}_b\,, \quad (50)$$

H_{ab} being the Darwin Hamiltonian[18]

$$H_{ab}=\tfrac{1}{2}m_av_a^2+\tfrac{1}{2}m_bv_b^2-\frac{1}{8c^2}m_av_a^4-\frac{1}{8c^2}m_bv_b^4+\frac{e_ae_b}{r_{ab}}-\frac{e_ae_b}{2c^2r_{ab}}\left[\vec{v}_a\cdot\vec{v}_b+\frac{(\vec{x}_{ab}\cdot\vec{v}_a)(\vec{x}_{ab}\cdot\vec{v}_b)}{r_{ab}^2}\right]. \quad (51)$$

Actually, by symmetry arguments, this calculation gives modulus $O(1/c^3)$ the same result as in the Coulomb approximation.[16] To this order, our result agrees with the value obtained for E by Krizan and Havas,[20] who only have taken into account what they call the "short-range relativistic correlations."

Since the plasma energy we have obtained adds nothing to the plasma energy in the Coulomb approximation, there is no correction, to this order, to the equation of state. In order to obtain relevant results we would have to push the approximation further than to $1/c^2$, which can be done in the framework of this paper (we will give some results about it in the near future). Therefore the two-particle distribution function shown in Eq. (49) has very little interest on its own. It has only been reported here as a simple test for the goodness of the theory that has been presented in this paper.

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