FINITE CODIMENSIONAL ISOMETRIES ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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Abstract

Based on the vector-valued generalization of Holsztyński’s theorem by M. Cambern, we provide a complete description of the linear isometries of $C(X,E)$ into $C(Y,F)$ whose range has finite codimension.

1 Introduction.

Throughout this paper, $X$ and $Y$ will stand for compact Hausdorff spaces, and $E$ and $F$ for Banach spaces over the field $\mathbb{K}$ of real or complex numbers. $C(X,E)$ and $C(Y,F)$ will be the Banach spaces of continuous $E$-valued and $F$-valued functions defined on $X$ and $Y$, respectively, endowed with the supremum norm $\|\cdot\|_\infty$. If $E = F = \mathbb{K}$, then we will write $C(X)$ and $C(Y)$ instead of $C(X,E)$ and $C(Y,F)$.

The classical Banach-Stone theorem states that if there exists a linear isometry $T$ of $C(X)$ onto $C(Y)$, then there are a homeomorphism $\psi$ of $Y$ onto $X$ and a continuous map $a : Y \to \mathbb{K}$, $|a| \equiv 1$, such that $T$ can be written as a weighted composition map, that is,

$$(Tf)(y) = a(y)f(\psi(y))$$

for all $y \in Y$ and all $f \in C(X)$.

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An important generalization of the Banach-Stone theorem was given by W. Holsztyński in [13] (see also [3]) by considering non-surjective isometries. Namely, he proved that, in this case, there is a closed subset $Y_0$ of $Y$ where the isometry can still be represented as a weighted composition map.

This result of Holsztyński was used in [11] (see also [2, 4, 9, 10, 12, 14, 16]) to classify linear isometries on $C(X)$ whose range has codimension 1 as follows: Let $T : C(X) \to C(X)$ be a codimension 1 linear isometry. Then there exists a closed subset $X_0$ of $X$ such that either

1. $X_0 = X \setminus \{p\}$ where $p$ is an isolated point of $X$, or
2. $X_0 = X$,

and such that there exists a continuous map $h$ of $X_0$ onto $X$ and a function $a \in C(X_0), |a| \equiv 1$, such that $(Tf)(x) = a(x) \cdot f(h(x))$ for all $x \in X_0$ and all $f \in C(X)$.

In the context of continuous vector-valued functions, M. Jerison ([18]) investigated the vector analogue of the Banach-Stone theorem: If $X$ and $Y$ are compact Hausdorff spaces and $E$ is a strictly convex Banach space, then every linear isometry $T$ of $C(X, E)$ onto $C(Y, E)$ can be written as a weighted composition map; namely, $(Tf)(y) = \omega(y)(f(\psi(y)))$, for all $f \in C(X, E)$ and all $y \in Y$, where $\omega$ is a continuous map from $Y$ into the space of continuous linear operators from $E$ to $E$ (taking values in the subset of surjective isometries) endowed with the strong operator topology. Furthermore, $\psi$ is a homeomorphism of $Y$ onto $X$. As in the scalar-valued case, Jerison’s results have been extended in many directions (see e.g., [5], [1], [15] or [6]). In particular, M. Cambern obtained in [8] the following formulation of Holsztyński’s theorem for spaces of continuous vector-valued functions.

**Theorem 1.1** If $F$ is a strictly convex Banach space, then every linear isometry $T$ of $C(X, E)$ into $C(Y, F)$ can be written as a weighted composition map; namely,

$$(Tf)(y) = J_y(f(h(y))),$$

for all $f \in C(X, E)$ and all $y \in Y_0 \subset Y$, where $J$ is a continuous map from $Y$ into the space $L(E, F)$ of bounded operators from $E$ into $F$ endowed with the strong operator topology, with $\|J_y\| \leq 1$ for all $y \in Y$ and $\|J_y\| = 1$ for $y \in Y_0$. Furthermore, $h$ is a continuous function of $Y_0$ onto $X$. If $E$ is finite-dimensional, then $Y_0$ is a closed subset of $Y$. 
Let us recall that there are counter-examples (see [7] or [18]) which show that all the above results may not hold if the assumption of strict convexity is not observed.

In this paper we provide, based on this theorem of Cambern, a complete description of the linear isometries of $C(X, E)$ into $C(Y, F)$, $E$ and $F$ strictly convex, whose range has finite codimension $n_0$.

2 Preliminaries and main results.

Given a continuous linear operator $T : C(X, E) \rightarrow C(Y, F)$, the map

$$J : Y \rightarrow L(E, F)$$

$$y \mapsto J_y$$

given by $J_y(e) := (T\hat{e})(y)$ for all $e \in E$ (being $\hat{e}$ the function constantly equal to $e$) is well defined and continuous when, as usual, $L(E, F)$ is endowed with the strong operator topology. Furthermore, $\|J_y\| \leq \|T\|$ for all $y \in Y$.

On the other hand, we can define three subsets of $Y$ as follows:

\begin{align*}
Y_3 & := \{y \in Y : (Tf)(y) = 0 \ \forall f \in C(X, E)\}; \\
Y_1 & := \{y \in Y \setminus Y_3 : \exists x_y \in X \text{ such that } (Tf)(y) = 0 \text{ if } f(x_y) = 0, f \in C(X, E)\}; \\
Y_2 & := Y \setminus (Y_1 \cup Y_3).
\end{align*}

It is easy to see that the point $x_y \in X$ corresponding to each $y \in Y_1$ is uniquely determined, so if we define $h : Y_1 \rightarrow X$ by $h(y) := x_y$, then

$$(Tf)(y) = J_y(f(h(y)))$$

for every $f \in C(X, E)$ and $y \in Y_1$. Summing up, $Y_1$ coincides with the subset of $Y$ where $T$ can be written as a (nontrivial) weighted composition map. This implies that, given any $y_0 \in Y_1$ and a neighborhood $U$ of $\overline{h}(y_0)$ in $X$, there exists $f \in C(X, E)$ such that $f \equiv 0$ outside $U$ and $(Tf)(y_0) \neq 0$, so the set $V$ of all $y \in Y_1$ with $(Tf)(y) \neq 0$ is an open neighborhood of $y_0$ in $Y_1$. Now it is clear that $h(V_1) \subset U$, and the fact that $h$ is continuous follows easily.

Recall that a Banach space $E$ is said to be strictly convex if every element of its unit sphere is an extreme point of the closed unit ball of $E$. It is well-known that if $E$ is strictly convex and $e_1, e_2 \in E \setminus \{0\}$, then $\|e_1 + e_2\|$ =
\|e_1\| + \|e_2\| \text{ implies } e_1 = r e_2 \text{ for some positive real } r \text{ (see [19, pp. 332–336]). From this, it is straightforward to see that }

\|e_1\|, \|e_2\| < \max\{\|e_1 + e_2\|, \|e_1 - e_2\|\}

whenever \( e_1, e_2 \in E \setminus \{0\} \).

From now on, \( E \) and \( F \) will be strictly convex normed spaces (see Remark 2.1 below). Also, \( T \) will be a linear isometry of \( C(X, E) \) into \( C(Y, F) \) whose range has finite codimension \( n_0 \geq 1 \).

For a function \( f \in C(X, E) \), we will write \( c(f) \) to denote the cozero set of \( f \), that is, \( c(f) := \{x \in X : f(x) \neq 0\} \). If \( V \) is a subset of \( X \), we will write \( \text{cl} V \) to denote its closure in \( X \).

We rephrase the formulation of Holsztyński’s theorem for spaces of continuous vector-valued functions obtained by M. Cambern in [8].

**Theorem 2.1 (Cambern)**  The restriction of \( \overline{h} \) to \( Y_0 := \{y \in Y_1 : \|J_y\| = 1\} \) is a continuous function onto \( X \). Also, if \( E \) is finite-dimensional, then \( Y_0 \) is a closed subset of \( Y \).

We denote by \( h \) the restriction of \( \overline{h} \) to \( Y_0 \). We then have that \( h : Y_0 \longrightarrow X \) is continuous and surjective, and that for \( y \in Y_1 \setminus Y_0 \), the mapping \( J_y : E \longrightarrow F \) defined by

\[ J_y(e) := (T \hat{e})(y) \]

is linear and continuous and its norm is less than 1.

Points in \( Y_1 \) can be classified into two disjoint categories:

\[ Y_{10} := \{y \in Y_1 : J_y \text{ is an isometry}\} ; \]
\[ Y_{11} := \{y \in Y_1 : J_y \text{ is not an isometry}\} . \]

We shall see that \( Y_{11} \cup Y_2 \cup Y_3 \) consists of finitely many isolated points of \( Y \). Indeed, if \( F \) is assumed to be infinite-dimensional, then it will be proved that \( Y_{11} \cup Y_2 \cup Y_3 \) is empty, that is, \( Y = Y_0 = Y_{10} \).

Related to the subsets \( Y_0 \) and \( Y_1 \) and the corresponding maps \( h \) and \( \overline{h} \), we consider, for each \( x \in X \), the sets

\[ F_x := \{y \in Y_0 : h(y) = x\} \]
and
\[ G_x := \{ y \in Y_1 : \overline{h}(y) = x \}. \]

It will turn out that \( G_x \) (and consequently \( F_x \)) is finite for every \( x \in X \).

Prior to providing the description of \( T \), we still need to classify the points of \( X \) into three not necessarily disjoint classes that will be widely used in the paper:

\[
\begin{align*}
    A_0 & := \{ x \in X : \exists y \in F_x \text{ with } J_y \text{ not a surjective isometry} \}; \\
    A_1 & := \{ x \in X : x \notin A_0, \text{card } G_x = 1 \}; \\
    A_2 & := \{ x \in X : \text{card } G_x \geq 2 \}.
\end{align*}
\]

We shall prove that \( A_0 \) and \( A_2 \) are finite.

Summarizing, there exists \( J : Y \rightarrow L(E,F) \) continuous with respect to the strong operator topology and \( \overline{h} : Y_1 \rightarrow X \) continuous and surjective such that \( (Tf)(y) = J_y(f(\overline{h}(y))) \) for all \( f \in C(X,E) \) and \( y \in Y_1 \). We next state (in full) the main results, where we keep the notation above.

**Theorem 2.2** Let \( X, Y \) be compact Hausdorff spaces, \( E, F \) be strictly convex Banach spaces, and \( T : C(X,E) \rightarrow C(Y,F) \) be a linear isometry. Suppose that the range of \( T \) has finite codimension \( n_0 \geq 1 \).

If \( F \) is infinite-dimensional, then there exist a finite subset \( Y_N \) of \( Y \) and a surjective homeomorphism \( h : Y \rightarrow X \) such that
\[
(Tf)(y) = J_y(f(h(y))),
\]
for all \( f \in C(X,E) \) and all \( y \in Y \). Here, \( J_y : E \rightarrow F \) is an isometry for all \( y \in Y \), and it is surjective whenever \( y \notin Y_N \).

Moreover,
\[
\sum_{y \in Y_N} \text{codim (ran } J_y) = n_0.
\]

The finite-dimensional case turns out to be more intricate. First it is apparent that, since \( \overline{h} \) is surjective, if \( Y \) is finite, then \( X \) is also finite. Consequently, it is clear that \( n_0 = (\text{dim } F)(\text{card } Y) - (\text{dim } E)(\text{card } X) \). Next we study the case when \( Y \) is infinite.

**Theorem 2.3** Let \( X, Y \) be compact Hausdorff spaces, \( E, F \) be strictly convex Banach spaces, and \( T : C(X,E) \rightarrow C(Y,F) \) be a linear isometry. Suppose that the range of \( T \) has finite codimension \( n_0 \geq 1 \).
If \( F \) is finite-dimensional and \( Y \) is infinite, then there exists a cofinite subset \( Y_1 \) of \( Y \) and a continuous surjection \( h: Y_1 \rightarrow X \) such that
\[
(Tf)(y) = J_y(f(h(y)))
\]
for all \( f \in C(X,E) \) and \( y \in Y_1 \).

Furthermore, the set of all \( y \in Y \) for which \( J_y: E \rightarrow F \) is a surjective isometry is clopen, its complement is finite and
\[
n_0 = (\dim F) \left( \text{card}(Y \setminus Y_1) + \text{card} h^{-1}(A_2) - \text{card} A_2 \right),
\]
where \( A_2 = \{ x \in X : \text{card} h^{-1}(x) \geq 2 \} \).

**Remark 2.1** Theorem 2.3 does not hold in general if \( E \) (or \( F \)) is not strictly convex. For instance, suppose that, for \( F = \mathbb{K} \) and \( E = \mathbb{K}^2 \) endowed with the sup norm, and \( Y \) being the topological sum of two copies \( X \times \{1\}, X \times \{2\} \) of \( X \) and \( n_0 \) isolated points \( p_i \). It is easy to see that the map \( T: C(X,E) \rightarrow C(Y,F) \) defined, for each \( f \in C(X,E) \), by \( (Tf)(x,i) := \langle f(x), e_i \rangle \) (where \( \{e_1, e_2\} \) is the canonical basis in \( \mathbb{K}^2 \)), and \( (Tf)(p_j) := 0 \) for all \( j \), is a linear isometry with codimension \( n_0 \). As in [17], it can be checked that \( T \) is not a weighted composition map.

### 3 Some technical lemmas.

**Lemma 3.1** The set \( A_0 \) is finite.

**Proof.** Suppose, contrary to what we claim, that \( A_0 \) is infinite. Then we can find pairwise distinct \( x_1, x_2, \ldots, x_{n_0+1} \in A_0 \). For \( i = 1, 2, \ldots, n_0 + 1 \), we choose \( y_i \in F_{x_i} \) with \( J_{y_i} \) not a surjective isometry. Next we divide the set \( \{1, 2, \ldots, n_0 + 1\} \) into three mutually disjoint subsets. Namely,
\[
I_1 := \{ i \in \{1, 2, \ldots, n_0 + 1\} : J_{y_i} \text{ isometry} \};
\]
\[
I_2 := \{ i \in \{1, 2, \ldots, n_0 + 1\} : J_{y_i} \text{ not injective} \};
\]
\[
I_3 := \{ i \in \{1, 2, \ldots, n_0 + 1\} : J_{y_i} \text{ injective but not isometry} \}.
\]

Let \( i \in I_2 \). Then there is \( e_i \in E \) with \( \|e_i\| = 1 \) and \( J_{y_i}(e_i) = 0 \). Take \( f_i \in C(X) \) such that \( 0 \leq f_i \leq 1 \), \( f_i(x_i) = 1 \), and \( f_i(x_j) = 0 \) for \( j \neq i \). It is
clear that, if we put \( k_i := f_i e_i \in C(X, E) \), then \( \|k_i\|_\infty = 1 \) and \( (Tk_i)(y_i) = 0 \). Furthermore, for \( j \neq i, 1 \leq j \leq n_0 + 1 \), we have that
\[
  k_i(x_j) = k_i(h(y_j)) = 0.
\]
Hence, \( (Tk_i)(y_j) = 0 \).

Consequently, for each \( i \in I_2 \), the set
\[
  V_i := \left\{ y \in Y : \|Tk_i(y)\| < \frac{1}{2} \right\}
\]
is open in \( Y \) and contains \( y_j \) for all \( j \). For the same reason, if we define \( V := Y \) if \( I_2 = \emptyset \) and otherwise, then \( V \) is an open neighborhood of \( y_j \) for all \( j \in \{1, 2, \ldots, n_0 + 1\} \).

Next we consider pairwise disjoint open neighborhoods \( V'_i \) of \( y_i \) in \( Y \) for all \( i \in \{1, 2, \ldots, n_0 + 1\} \), and define
\[
  W_i := V'_i \cap V.
\]

It is clear that \( W_i \cap W_j = \emptyset \) if \( i \neq j \) and that \( y_i \in W_i \) for all \( i \).

Next we consider, for each \( i \in \{1, 2, \ldots, n_0 + 1\} \), a function \( g_i \in C(Y) \) such that \( 0 \leq g_i \leq 1 \), \( c(g_i) \subset W_i \) and \( g_i(y_i) = 1 \), and a vector \( f_i \in F \) given as follows:

1. If \( i \in I_1 \), then we choose \( f_i \notin \text{ran} J_{y_i} \) with \( \|f_i\| = 1 \).

2. If \( i \in I_2 \cup I_3 \), then we take a norm-one \( e'_i \in E \) with \( 0 < \|J_{y_i}(e'_i)\| < 1 \), and define \( f_i := J_{y_i}(e'_i) \).

As the codimension of the range of \( T \) is \( n_0 \), there exist \( a_1, \ldots, a_{n_0 + 1} \in \mathbb{K} \) such that \( g := \sum_{i=1}^{n_0 + 1} a_i g_i f_i \neq 0 \) belongs to the range of \( T \). Let us choose \( i_0 \) such that \( \|g\|_\infty = |a_{i_0}| \|f_{i_0}\| \). We claim that \( i_0 \in I_2 \) (so \( I_2 \neq \emptyset \)).

Let \( f \in C(X, E) \) with \( Tf = g \). If we fix \( i \in I_1 \), then
\[
  a_i f_i = (Tf)(y_i) = J_{y_i}(f(h(y_i))).
\]

This is to say that \( a_i f_i \) belongs to the range of \( J_{y_i} \) and, since \( i \in I_1 \), we get \( a_i = 0 \). Hence \( i_0 \notin I_1 \). Next, if \( i \in I_3 \), then \( g(y_i) = J_{y_i}(f(x_i)) \), and also
\(g(y_i) = a_i f_i = a_i J_{y_i}(e'_i)\), implying that \(|a_i| = |a_i| \|e'_i\| = \|f(x_i)\| \leq \|g\|_\infty\).

Hence \(|a_i| \|f_i\| < \|g\|_\infty\) and \(i_0 \notin I_3\), as we wanted to prove.

Since \(\|g\|_\infty = |a_{i_0}| \|f_{i_0}\| = \|J_{y_{i_0}}(f(x_{i_0}))\|\), we deduce that \(f(x_{i_0}) \neq 0\) and, since \(E\) is strictly convex, it is now clear that either
\[
\|k_{i_0}(x_{i_0}) + f(x_{i_0})\| > 1
\]
or
\[
\|k_{i_0}(x_{i_0}) - f(x_{i_0})\| > 1,
\]
that is, either \(\|k_{i_0} + f\|_\infty > 1\) or \(\|k_{i_0} - f\|_\infty > 1\).

With no loss of generality, we shall assume that \(\|g\|_\infty = \frac{1}{2}\).

We claim that \(\|Tk_i \pm g\|_\infty \leq 1\) for all \(i\). To this end, fix \(y \in Y\) and assume first that \(y \in c(g)\), so \(y \in V\). Hence \(\|(Tk_i)(y)\| < 1/2\) and, consequently, \(\|(Tk_i \pm g)(y)\| < 1\). Assume next that \(y \notin c(g)\), which is to say that \(g(y) = 0\). Then, since \(\|k_i\|_\infty = 1\), \(\|(Tk_i \pm g)(y)\| \leq 1\). Hence
\[
\|Tk_i \pm g\|_\infty \leq 1.
\]

This contradicts the isometric property of \(T\), and we are done. \(\square\)

The proof of the following lemma is immediate.

**Lemma 3.2** Let \(x \in X\) and let \(y_1, y_2 \in G_x\) with \(J_{y_1}\) injective. If \(g \in C(Y, F)\) satisfies \(g(y_1) = 0\) and \(g(y_2) \neq 0\), then \(g \notin \text{ran} T\).

**Lemma 3.3** The set \(A_2\) is finite.

**Proof.** Suppose, contrary to what we claim, that \(A_2\) is infinite. Then, since \(A_0\) is finite by Lemma 3.1, we can find pairwise distinct \(x_1, x_2, \ldots, x_{n_0+1}\) in \(A_2 \setminus A_0\). For each \(i = 1, 2, \ldots, n_0 + 1\), we choose two distinct elements \(y^1_i, y^2_i\) in \(G_x\). Since \(h\) is onto, we can assume that \(y^1_i \in F_{x_i}\) for all \(i\).

Also for each \(i\), we can choose a function \(g_i \in C(Y, F)\) such that
\begin{itemize}
  \item \(g_i(y^2_j) \neq 0\) and \(g_i(y^2_j) = 0\) for \(j \neq i\).
  \item \(g_i(y^1_j) = 0\) for all \(j = 1, 2, \ldots, n_0 + 1\).
\end{itemize}

By Lemma 3.2, no nonzero linear combination of the \(g_i\) belongs to \(\text{ran} T\), which is impossible. \(\square\)

**Lemma 3.4** For each \(x \in X\), the set \(G_x\) is finite.
Proof. Suppose, contrary to what we claim, that there is \( x_0 \in X \) such that \( G_{x_0} \) is infinite.

First, if there exists \( y_0 \in G_{x_0} \) such that \( J_{y_0} \) is injective, then we take \( y_1, y_2, \ldots, y_{n_0+1} \in G_{x_0} \) pairwise distinct and different from \( y_0 \). For each \( i \in \{1, 2, \ldots, n_0+1\} \) we choose a function \( g_i \in C(Y,F) \) such that \( g_i(y_i) \neq 0 \) and \( g_i(y_j) = 0 = g_i(y_0) \) for \( j \neq i \). Using Lemma 3.2, no nontrivial linear combination of the \( g_i \) belongs to \( \text{ran} \ T \). We conclude that, for all \( y \in G_{x_0} \), \( J_y \) is not injective.

We shall prove that this is also impossible. To this end, let us first see that

\[
G_{x_0} \cap \text{cl} (h^{-1}(X \setminus A_0)) = \emptyset.
\]

If \( y \in G_{x_0} \), then there exists \( e_y \in E, \|e_y\| = 1 \), such that \( J_y(e_y) = 0 \). On the other hand, given \( y' \in h^{-1}(X \setminus A_0) \), \( J_{y'} \) is an isometry and, consequently, \( \|J_{y'}(e_y)\| = 1 \). In other words, we have that \( (T \hat{e}_y)(y) = 0 \) and, for all \( y' \in h^{-1}(X \setminus A_0) \), \( \|T \hat{e}_y(y')\| = 1 \). This yields \( y \notin \text{cl} (h^{-1}(X \setminus A_0)) \).

Since we are assuming that \( G_{x_0} \) is infinite, we can now consider two subsets of \( G_{x_0} \), \( \{y_1, \ldots, y_{n_0+1}\} \) and \( \{y_2, \ldots, y_{n_0+1}\} \), consisting of \( 2n_0 + 2 \) pairwise distinct elements.

Let us also consider, for each \( i \in \{1, 2, \ldots, n_0+1\} \) and each \( j \in \{1, 2\} \), an open neighborhood \( U_i^j \) of \( y_i^j \) such that \( U_i^j \cap h^{-1}(X \setminus A_0) = \emptyset \). Clearly, we can assume that these \( 2n_0 + 2 \) sets are pairwise disjoint, and then take functions \( g_i^j \in C(Y,F) \) such that \( c(g_i^j) \subset U_i^j \) and \( \|g_i^j(y_i^j)\| = 1 = \|g_i^j\|_\infty \), for all \( i, j \).

Then we have two nonzero functions \( g_1 := \sum_{i=1}^{n_0+1} \alpha_i g_i^1 \) and \( g_2 := \sum_{i=1}^{n_0+1} \beta_i g_i^2 \) in the range of \( T \), that is, \( Tf_1 = g_1 \) and \( Tf_2 = g_2 \) for some \( f_1, f_2 \in C(X,E) \). Assume, without loss of generality, that \( \|g_1\|_\infty = \|g_2\|_\infty = 1 \).

Since \( g_i \equiv 0 \) on \( h^{-1}(X \setminus A_0) \) \( (i = 1, 2) \), we infer that \( f_i \equiv 0 \) on \( X \setminus A_0 \). However, if \( f_i(x_0) = 0 \), then \( g_i(y) = 0 \) for all \( y \in G_{x_0} \). Consequently, \( f_i(x_0) \neq 0 \) for \( i = 1, 2 \). As \( A_0 \) is finite and \( x_0 \in A_0 \), we deduce that \( \{x_0\} \) is an open set. Then we can write the functions \( f_i \) as

\[
f_i = f_i \chi_{(x_0)} + f_i \chi_{A_0 \setminus \{x_0\}}.
\]

As \( f_i \chi_{A_0 \setminus \{x_0\}}(x_0) = 0 \), then \( (Tf_i \chi_{A_0 \setminus \{x_0\}})(y) = 0 \) for all \( y \in G_{x_0} \), so \( (Tf_i \chi_{\{x_0\}})(y) = (Tf_i)(y) \) for all \( y \in G_{x_0} \).

Hence, since each \( \|Tf_i(y)\| = \|g_i(y)\| \) attains its maximum in \( G_{x_0} \),

\[
\|Tf_i \chi_{\{x_0\}}\|_\infty \geq \|Tf_i\|_\infty = 1,
\]

Therefore, \( f_i \) is nonzero.
implying that \( \|Tf_i\chi_{\{x_0\}}\|_\infty = 1 \). This yields \( \|f_i(x_0)\| = 1, i = 1, 2 \). As a consequence, either \( \|f_1(x_0) + f_2(x_0)\| > 1 \) or \( \|f_1(x_0) - f_2(x_0)\| > 1 \), which implies that either

\[
\|Tf_1 + Tf_2\|_\infty > 1
\]

or

\[
\|Tf_1 - Tf_2\|_\infty > 1.
\]

These inequalities contradict the fact that \( \|g_1 \pm g_2\|_\infty = \max (\|g_1\|_\infty, \|g_2\|_\infty) = 1 \).

\[\square\]

Lemma 3.5 The set \( Y_3 \) is finite.

Proof. Suppose that there exist \( n_0 + 1 \) distinct points \( y_1, \ldots, y_{n_0 + 1} \) in \( Y_3 \). Let us choose \( n_0 + 1 \) functions \( g_1, \ldots, g_{n_0 + 1} \) in \( C(Y,F) \) such that \( g_i(y_j) = 0 \) if \( i \neq j \) and \( g_i(y_i) \neq 0 \) for \( i \in \{1, \ldots, n_0 + 1\} \). It is apparent that no nonzero linear combination of \( \{g_1, \ldots, g_{n_0 + 1}\} \) belongs to the range of \( T \), which is impossible.

Lemma 3.6 The set \( Y_2 \) is finite and each point of \( Y_2 \) is isolated in \( Y \).

Proof. We first check that \( Y_2 \cap \text{cl} Y_1 = \emptyset \). Obviously, \( Y_2 \cap Y_1 = \emptyset \).

First, by Lemmas 3.1, 3.3 and 3.4, \( \overline{h}^{-1}(A_0 \cup A_2) \) is finite. Since \( X = A_0 \cup A_2 \cup A_1 \), in order to prove that \( Y_2 \cap \text{cl} Y_1 = \emptyset \), it suffices to check that

\[
Y_2 \cap \text{cl}(\overline{h}^{-1}(A_1)) = \emptyset,
\]

which, by the definition of \( A_1 \), is the same as proving \( Y_2 \cap \text{cl}(h^{-1}(A_1)) = \emptyset \).

Let \( y_0 \in \text{cl}(h^{-1}(A_1)) \) and consider, for \( f \in C(X,E) \) and \( \epsilon > 0 \), the set

\[
K(f, \epsilon) := \{ x \in X : \|f(x)\| - \|(Tf)(y_0)\| \leq \epsilon \}.
\]

Each of these is a closed subset of \( X \), which is also nonempty as a consequence of the fact that, for each \( y \in h^{-1}(A_1) \), \( \|f(h(y))\| = \|(Tf)(y)\| \). We are going to check that the family of all these sets satisfies the finite intersection property. Indeed, we shall prove that if \( f_1, \ldots, f_n \in C(X,E) \) and \( \epsilon_1, \ldots, \epsilon_n > 0 \), then

\[
\bigcap_{i=1}^n K(f_i, \epsilon_i) \neq \emptyset.
\]
The set
\[
U := \bigcap_{i=1}^{n} \left\{ y \in Y : \| (Tf_i)(y) - (Tf_i)(y_0) \| < \epsilon_i \right\}
\]
is an open neighborhood of \(y_0\) and, by assumption, there exists \(y_1 \in h^{-1}(A_1) \cap U\). Then
\[
\| (Tf_i)(y_1) \| - \| (Tf_i)(y_0) \| < \epsilon_i
\]
for \(i = 1, 2, \ldots, n\). On the other hand, for each \(i\), \((Tf_i)(y_1) = J_{y_1}(f_i(h(y_1)))\) and, as \(J_{y_1}\) is a surjective isometry, we have that \(\| (Tf_i)(y_1) \| = \| f_i(h(y_1)) \|\). Consequently,
\[
\| f_i(h(y_1)) \| - \| (Tf_i)(y_0) \| < \epsilon_i,
\]
which implies that, as was to be proved,
\[
h(y_1) \in \bigcap_{i=1}^{n} K(f_i, \epsilon_i).
\]

Hence, since \(X\) is compact, there exists
\[
x_0 \in \bigcap_{\epsilon > 0} K(f, \epsilon).
\]

By definition, we deduce that, for every \(f \in C(X, E), \| f(x_0) \| = \| (Tf)(y_0) \|\). In particular, if \(f(x_0) = 0\), then \((Tf)(y_0) = 0\), and consequently \(y_0 \notin Y_2\). This contradiction yields
\[
Y_2 \cap \text{cl} Y_1 = \emptyset.
\]
Now, as \(Y_2 = Y \setminus (Y_3 \cup \text{cl} Y_1)\) and \(Y_3\) is a finite set, we infer that \(Y_2\) is open.

Next, suppose that \(Y_2\) contains infinitely many elements. Then there exist \(n_0 + 1\) pairwise disjoint open subsets \(V_1, \ldots, V_{n_0+1}\) contained in \(Y_2\). For each \(i \in \{1, 2, \ldots, n_0 + 1\}\), we can take \(g_i \in C(Y, F)\), \(g_i \neq 0\), with \(c(g_i) \subset V_i\). From the finite codimensionality of the range of \(T\), we infer that there exists a nonzero linear combination \(g := \sum_{i=1}^{n_0+1} \alpha_i g_i\) in the range of \(T\), that is, there exists \(f \in C(X, E)\) such that \(Tf = g\). Then, it is apparent that \(g(h^{-1}(X)) \equiv 0\) and, in order to get a contradiction, it suffices to check that \(f(X) \equiv 0\). To this end, note that, by definition, if \(x \notin A_0\), then, given \(y \in F_x, J_y\) is an isometry. Hence, \(0 = (Tf)(y) = J_y(f(x))\) yields \(f(x) = 0\), which is to say that \(f \equiv 0\) on \(X\) except perhaps on a finite set \(\{x_1, \ldots, x_n\} \subset A_0\). Then we can write \(f = f \chi_{\{x_1\}} + \ldots + f \chi_{\{x_n\}}\). Also
for each \( y \in Y_1 \), there exists at most one \( i \) such that \( (Tf\chi_{\{x_i\}})(y) \neq 0 \) because in that case, necessarily, \( T(x) = x_i \). We then infer that \( T(x) \equiv 0 \) on \( Y_1 \) for all \( i \). Hence there exists \( y_1 \in Y_2 \) such that \( \| (Tf\chi_{\{x_i\}})(y_1) \| = \| Tf\chi_{\{x_i\}} \|_\infty \neq 0 \) for some \( i \in \{1, \ldots, n\} \). Since \( y_1 \in Y_2 \), we can find \( k \in C(X, E) \) such that \( k(x_i) = 0 \) and \( (Tk)(y_1) \neq 0 \). If we suppose, with no loss of generality, that \( \|k\| = \|f\chi_{\{x_i\}}\|_\infty = 1 \), then \( \| k \pm f\chi_{\{x_i\}} \|_\infty = 1 \), but either \( \| (Tf\chi_{\{x_i\}})(y_1) + (Tk)(y_1) \| > 1 \) or \( \| (Tf\chi_{\{x_i\}})(y_1) - (Tk)(y_1) \| > 1 \), which is impossible. \( \Box \)

**Lemma 3.7** The set \( Y_{11} \cup Y_2 \cup Y_3 \) is finite, and all of its points are isolated in \( Y \).

**Proof.** We already know, by Lemma 3.6, that the result is true for \( Y_2 \). On the other hand, it is apparent that

\[
Y_{11} \subset \bigcup_{x \in X \setminus A_0} (G_x \setminus F_x) \cup \bigcup_{x \in A_0} G_x.
\]

Since \( A_0, A_2 \) and \( G_x \) are finite sets (see Lemmas 3.1, 3.3 and 3.4), then we deduce that \( Y_{11} \) is finite. Also, for any \( e \in E \), \( \|e\| = 1 \), the open set \( C_e := \{ y \in Y : \|(T\hat{e})(y)\| < 1 \} \) is contained in the finite set \( Y_{11} \cup Y_2 \cup Y_3 \), which implies that \( C_e \) consists of isolated points. If \( y_0 \in Y_{11} \), then there exists \( e \in E \) such that \( \|e\| = 1 \) and \( \|(T\hat{e})(y_0)\| = \|J_{y_0}(e)\| < 1 \), which is to say that \( y_0 \in C_e \), that is, it is isolated.

A similar reasoning shows that every element of \( Y_3 \) is isolated in \( Y \). \( \Box \)

**Corollary 3.1** \( Y_1 \) is a clopen subset of \( Y \).

### 4 The infinite-dimensional case

In this section we shall assume that \( F \) is infinite-dimensional. Our first result shows that \( J_y \) is an isometry for all \( y \in Y \).

**Lemma 4.1** \( Y_{11} \cup Y_2 \cup Y_3 = \emptyset \).

**Proof.** Suppose that \( y_0 \in Y_{11} \cup Y_2 \cup Y_3 \) and consider \( n_0 + 1 \) linearly independent vectors \( g_1, \ldots, g_{n_0 + 1} \in F \). Since \( \{y_0\} \) is a clopen subset (Lemma 3.7),
then \( \chi_{\{y_0\}} g_1, \ldots, \chi_{\{y_0\}} g_{n_0+1} \) belong to \( C(Y,F) \) and are linearly independent. Then, there exists a nonzero linear combination

\[
g := \sum_{i=1}^{n_0+1} \alpha_i \chi_{\{y_0\}} g_i
\]

in the range of \( T \).

It is apparent that \( g(h^{-1}(X \setminus A_0)) \equiv 0 \). Hence, \( f := T^{-1} g \) satisfies \( f(X \setminus A_0) \equiv 0 \) and, if we write \( A_0 = \{x_1, \ldots, x_k\} \) (see Lemma 3.1), then \( f = f \chi_{\{x_1\}} + \ldots + f \chi_{\{x_k\}} \). As \( g(y_0) \neq 0 \), we infer that \( y_0 \notin Y_3 \). Hence we only have two possible cases:

1. \( y_0 \in Y_2 \)
2. \( y_0 \in Y_{11} \)

Before studying these cases, we need some preparation. With no loss of generality, we can assume that \( \|g\|_\infty = \|f\|_\infty = 1 \). Hence, there exists \( j \in \{1, \ldots, k\} \), say \( j = 1 \), such that \( \|f(x_1)\| = 1 \). Let us now check that \( f(x_2) = \cdots = f(x_k) = 0 \). To this end, we define

\[
f_1 := f \chi_{\{x_1\}} \quad \text{and} \quad f_2 := f \chi_{\{x_2, \ldots, x_k\}}.
\]

**Claim 4.1** \( Tf_1 = g \).

As \( \|f(x_1)\| = 1 \), there is \( y_1 \in Y \) with \( \|(Tf_1)(y_1)\| = 1 \). Besides, as \( f_1 \equiv 0 \) on \( X \setminus \{x_1\} \), \( y_1 \notin G_x \) for any \( x \neq x_1 \), which is to say that \( y_1 \in G_{x_1} \cup Y_2 \). Therefore, if \( y_1 \neq y_0 \), then we have

\[
\|T(f_1 - f_2)(y_1)\| = \|(Tf_1)(y_1) - (Tf)(y_1) + (Tf_1)(y_1)\| = 2\|Tf_1(y_1) - g(y_1)\| = \|2(Tf_1)(y_1)\| = 2
\]

but

\[
\|f_1 - f_2\|_\infty = \|f_1(x_1)\| \neq 1.
\]

This contradiction yields \( y_1 = y_0 \) and, consequently, \( \|(Tf_1)(y_0)\| = 1 \).

On the other hand, let us check that \( (Tf_2)(y_0) = 0 \). If this is not the case, then \( \|f_1 + f_2\|_\infty = 1 = \|f_1 - f_2\|_\infty \), but as \( F \) is strictly convex, then either

\[
\|(Tf_1)(y_0) + (Tf_2)(y_0)\| > 1
\]

13
or
\[ \| (Tf_1)(y_0) - (Tf_2)(y_0) \| > 1, \]
which is impossible since \( T \) is an isometry.

Consequently, for \( y_2 \in Y \setminus \{y_0\} \) with \( \| (Tf_2)(y_2) \| = \| Tf_2 \|_\infty \leq 1 \), we have
\( (Tf_1)(y_2) = -(Tf_2)(y_2) \). Also, if \( Tf_2 \neq 0 \), then either
\[ \| (Tf_1)(y_2) + \frac{(Tf_2)(y_2)}{\| Tf_2 \|_\infty} \| > 1 \]
or
\[ \| (Tf_1)(y_2) - \frac{(Tf_2)(y_2)}{\| Tf_2 \|_\infty} \| > 1, \]
contrary to the fact that
\[ \| f_1 \pm \frac{f_2}{\| Tf_2 \|_\infty} \|_\infty = 1. \]

This contradiction yields \( f_2 \equiv 0 \), which is to say that \( Tf_1 = g \). The proof of the claim is done.

**Case 1** If we suppose that \( y_0 \in Y_2 \), then there exists \( f_3 \in C(X,E) \) such that \( \| f_3 \|_\infty = 1 \), \( f_3(x_1) = 0 \) and \( (Tf_3)(y_0) \neq 0 \). It is clear that \( \| f_3 + f_1 \|_\infty = 1 = \| f_3 - f_1 \|_\infty \) but either
\[ \| (Tf_3 + Tf_1)(y_0) \| > 1 \]
or
\[ \| (Tf_3 - Tf_1)(y_0) \| > 1. \]
This contradiction shows that \( y_0 \not\in Y_2 \).

**Case 2** Assume finally that \( y_0 \in Y_{11} \), that is, \( J_{y_0} \) is not an isometry. Hence we know that there exists \( e \in E \), \( \| e \| = 1 \), such that \( \| J_{y_0}(e) \| < 1 \). Let us define
\[ \alpha = 1 - \| J_{y_0}(e) \| \]
and
\[ f_3 := \chi_{\{x_1\}} e. \]
It is clear that \( \| f_3 \|_\infty = 1 \) and \( \| (Tf_3)(y_0) \| = \| J_{y_0}(e) \| < 1 \). On the other hand
\[ \| (T(\alpha f_1 \pm f_3))(y_0) \| \leq \alpha \| (Tf_1)(y_0) \| + \| (Tf_3)(y_0) \| = 1. \]
Also if \( y \neq y_0 \), \((Tf_1)(y) = 0\) and \( \|(Tf_3)(y)\| \leq \|Tf_3\|_{\infty} = 1 \). Consequently
\[
\|(T(\alpha f_1 \pm f_3))\|_{\infty} \leq 1.
\]
However, either
\[
\|\alpha f_1(x_1) + f_3(x_1)\| > 1
\]
or
\[
\|\alpha f_1(x_1) - f_3(x_1)\| > 1
\]
which contradicts the isometric condition of \( T \). The lemma is proved. \( \Box \)

**Lemma 4.2** \( Y = Y_0 \) and \( h : Y \to X \) is a surjective homeomorphism. Moreover \( J_y \) is an isometry for every \( y \in Y \). Furthermore, the set \( Y_N \subset Y \) of all \( y \) such that \( J_y \) is not surjective is finite.

**Proof.** By Lemma 4.1, \( Y = Y_{10} \), so every \( J_y \) is an isometry and \( Y = Y_0 \).

Suppose next that there exists \( x_0 \in X \) with \( \operatorname{card} G_{x_0} \geq 2 \), and take \( y_1, y_2 \in G_{x_0} \), \( y_1 \neq y_2 \). Pick \( g = Tf \in C(Y, F) \) with \( g(y_1) = 0 \). By Lemma 3.2, \( g(y_2) = 0 \), which is impossible because \( \operatorname{codim} \langle \operatorname{ran} T \rangle \) is finite. We deduce that, for all \( x \in X \), \( \operatorname{card} G_x = 1 \), and consequently \( F_x = G_x \). We infer that \( h \) is injective and, since it is a continuous surjection and \( Y \) is compact, then \( h \) is a surjective homeomorphism.

Finally, let us note that, if \( h(y) \notin A_0 \), then \( J_y \) is a surjective isometry. Consequently, as \( A_0 \) is finite, so is \( Y_N \). \( \Box \)

**Proposition 4.1** Let \( g \in C(Y, F) \) be such that \( g(y) \in \operatorname{ran} J_y \) for all \( y \in Y \). Then \( g \in \operatorname{ran} T \).

**Proof.** By Lemma 4.2, given \( x \in X \),
\[
J_{h^{-1}(x)} : E \to F
\]
is a linear isometry which is also surjective except for finitely many \( x \in h(Y_N) \), being \( Y_N := \{y_1, \ldots, y_k\} \).

Fix any \( x_0 \in X \) and take an open neighborhood \( V \) of \( h^{-1}(x_0) \) such that \( V \cap h^{-1}(x_0) \subset \{h^{-1}(x_0)\} \). Hence, for all \( y \in V \setminus \{h^{-1}(x_0)\} \), we have that \( J_y \) is a surjective isometry.

**Claim 4.2** Let \( f \in \operatorname{ran} J_{h^{-1}(x_0)} \) and let \( \epsilon > 0 \). There exists an open neighborhood \( U_\epsilon \) of \( x_0 \) such that, if \( x \in U_\epsilon \), then \( f \in \operatorname{ran} J_{h^{-1}(x)} \) and
\[
\|(J_{h^{-1}(x_0)})^{-1}(f) - (J_{h^{-1}(x)})^{-1}(f)\| < \epsilon.
\]
As $f \in \text{ran} J_{h^{-1}(x_0)}$, there exists $e \in E$ with $J_{h^{-1}(x_0)}(e) = f$. Hence $(T \hat{e})(h^{-1}(x_0)) = J_{h^{-1}(x_0)}(e) = f$ and there exists an open neighborhood $V_\epsilon$ of $h^{-1}(x_0)$ such that $V_\epsilon \subset V$ and
\[
\|(T \hat{e})(y) - (T \hat{e})(h^{-1}(x_0))\| < \epsilon
\]
for all $y \in V_\epsilon$, that is,
\[
\|J_y(e) - f\| < \epsilon.
\]

On the other hand, as $f \in \text{ran} J_y$ for all $y \in V_\epsilon$, there exists $e'_y \in E$ such that $f = J_y(e'_y)$. Hence, if $y \in V_\epsilon$, then $\|J_y(e) - J_y(e'_y)\| < \epsilon$, that is,
\[
\|J_y(e - e'_y)\| < \epsilon,
\]
and, since $J_y$ is an isometry, $\|e - e'_y\| < \epsilon$. Summarizing, if $x \in U_\epsilon := h(V_\epsilon)$, then
\[
\|(J_{h^{-1}(x_0)})^{-1}(f) - (J_{h^{-1}(x)})^{-1}(f)\| < \epsilon
\]
and the proof of the claim is done.

Next, define the function $f : X \rightarrow E$ by
\[
f(x) := (J_{h^{-1}(x)})^{-1}(g(h^{-1}(x)))
\]
for all $x \in X$. Hence, if we prove that $f$ is continuous, then for $y = h^{-1}(x)$, we have
\[
(T f)(y) = J_y(f(y)) = J_y((J_y)^{-1}(g(y))) = g(y).
\]
Thus, it only remains to check the continuity of $f$ at $x_0$. To this end, fix any $\epsilon > 0$. Since $g$ is continuous, there exists an open neighborhood $W$ of $h^{-1}(x_0)$ in $Y$ such that, if $y \in W$, then
\[
\|g(y) - g(h^{-1}(x_0))\| < \frac{\epsilon}{2}.
\]
Let us define $U := h(W) \cap U_{\epsilon/2}$, where $U_{\epsilon/2}$ is given by the claim above for $f := g(h^{-1}(x_0))$. Then, by definition, if $x \in U$,
\[
\|f(x_0) - f(x)\| = \|\left( (J_{h^{-1}(x_0)})^{-1}(g(h^{-1}(x_0))) \right) - \left( (J_{h^{-1}(x)})^{-1}(g(h^{-1}(x))) \right) \| \\
\leq \| (J_{h^{-1}(x_0)})^{-1}(f) - (J_{h^{-1}(x)})^{-1}(f) \| \\
+ \| (J_{h^{-1}(x)})^{-1}(f) - (J_{h^{-1}(x)})^{-1}(g(h^{-1}(x))) \| \\
< \frac{\epsilon}{2} + \| (J_{h^{-1}(x)})^{-1}(f - g(h^{-1}(x))) \| \\
= \frac{\epsilon}{2} + \| f - g(h^{-1}(x)) \| \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

16
and the continuity of $f$ is proved. \hfill $\square$

We can now prove the main result in this section.

\textit{Proof of Theorem 2.2.} Taking into account the previous lemmas, it only remains to check that $\sum_{i=1}^{k} \text{codim} (\text{ran } J_{y_i}) = n_0$, where $Y_N = \{y_1, \ldots, y_k\}$ is the subset introduced in Lemma 4.2.

Notice first that, due to the representation of $T$,

$$\text{codim} (\text{ran } J_{y_i}) \leq \text{codim}(\text{ran } T)$$

for each $i$. Then there exist $k$ sets formed by linearly independent vectors

$$F_1 := \{f(1,1), \ldots, f(1,n_1)\},$$
$$F_2 := \{f(2,1), \ldots, f(2,n_2)\},$$
$$\vdots$$
$$F_k := \{f(k,1), \ldots, f(k,n_k)\}$$

such that

$$\text{ran } J_{y_i} + \text{span } F_i = \text{F}$$

and

$$\text{ran } J_{y_i} \cap \text{span } F_i = \{0\}$$

(1)

for each $i \in \{1, 2, \ldots, k\}$.

Contrary to what we claim, suppose first that

$$\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \text{codim} (\text{ran } J_{y_i}) > n_0.$$ Let us consider, for each $i \in \{1, 2, \ldots, k\}$, an open neighborhood $V_i$ of $y_i$ such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Let $g_i \in C(Y)$ be such that $c(g_i) \subset V_i$ and $g_i(y_i) = 1$. Define also, for each $i \in \{1, 2, \ldots, k\}$ and each $j \in \{1, 2, \ldots, n_i\}$, a function $g(i, j) := g_i f(i, j)$. Hence we have $\sum_{i=1}^{k} n_i$ linearly independent functions in $C(Y, F)$, so there exists a linear combination

$$g_0 := \sum_{i,j} \alpha(i, j) g(i, j)$$
in the range of $T$, with some $\alpha(i_0, j_0) \neq 0$. Let $f \in C(X, E)$ satisfy $Tf = g_0$. Then

$$0 \neq \sum_{j=1}^{n_{i_0}} \alpha(i_0, j)f(i_0, j) = g_0(y_{i_0}) = (Tf)(y_{i_0}) = J_{y_{i_0}}(f(h(y_{i_0}))).$$

We deduce that $\text{ran } J_{y_{i_0}} \cap \text{span } F_{i_0} \neq \{0\}$, which contradicts (1) above. Hence $\sum_{n=1}^{k} \text{codim (ran } J_{y_{n}}) \leq n_0$.

Suppose now that $\sum_{n=1}^{k} \text{codim (ran } J_{y_{n}}) < n_0$. We shall check that, given $n_0$ linearly independent functions $g_1, \ldots, g_{n_0}$ in $C(Y, F)$, there exists a nonzero linear combination in the range of $T$. This fact implies that the codimension of the range of $T$ is strictly less than $n_0$, which is impossible.

Let us define the linear mappings

$$\lambda : K^{n_0} \longrightarrow \text{span } \{g_1, \ldots, g_{n_0}\}$$

by $\lambda(\gamma_1, \ldots, \gamma_{n_0}) := \sum_{j=1}^{n_0} \gamma_j g_j$ for all $(\gamma_1, \ldots, \gamma_{n_0}) \in K^{n_0}$. Next, for $i \in \{1, 2, \ldots, k\}$, consider

$$\mu_i : C(Y, F) \longrightarrow F/ \text{ran } J_{y_i}$$

where $\mu_i(g) := g(y_i) + \text{ran } J_{y_i}$ for all $g \in C(Y, F)$, and finally let

$$\mu : C(Y, F) \longrightarrow (F/ \text{ran } J_{y_1}) \times \cdots \times (F/ \text{ran } J_{y_k}),$$

where $\mu(g) := (\mu_1(g), \ldots, \mu_k(g))$ for all $g$. As a consequence, $\mu \circ \lambda$ turns out to be a linear mapping from a $n_0$-dimensional space to a space whose dimension is $\sum_{i=1}^{k} n_i < n_0$. It is apparent that $\mu \circ \lambda$ is not injective. Thus there exists $(\gamma_1, \ldots, \gamma_{n_0}) \in K^{n_0} \setminus \{(0, \ldots, 0)\}$ such that $(\mu \circ \lambda)(\gamma_1, \ldots, \gamma_{n_0}) = 0$. This means that $(\mu_i \circ \lambda)(\gamma_1, \ldots, \gamma_{n_0}) = 0 + \text{ran } J_{y_i}$ for each $i \in \{1, \ldots, k\}$, which is to say that $\sum_{j=1}^{n_0} \gamma_j g_j(y_i) \in \text{ran } J_{y_i}$ for all $i \in \{1, \ldots, k\}$. Taking into account the definition of $Y_N$, we see by Proposition 4.1 that $\sum_{j=1}^{n_0} \gamma_j g_j \in \text{ran } T$, as was to be proved.

Contrary to what could be expected in principle, the points of $Y_N$ need not be isolated, as the following example shows.

**Example 4.1** Let $X = Y := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and let $h : Y \longrightarrow X$ be the identity map. Given $f \in C(X, \ell^2)$, we define

$$(Tf) \left( \frac{1}{n} \right) := (\lambda_n^n, \lambda_1^n, \lambda_2^n, \ldots, \lambda_{n-1}^n, \lambda_{n+1}^n, \ldots),$$

\[18\]
where $f(1/n) := (\lambda_1^n, \lambda_2^n, \ldots, \lambda_{n-1}^n, \lambda_n^n, \lambda_{n+1}^n, \ldots)$. Also, if
\[
f(0) = (\lambda_1^0, \lambda_2^0, \ldots, \lambda_{n-1}^0, \lambda_n^0, \lambda_{n+1}^0, \ldots),
\]
then define
\[
(Tf)(0) := (0, \lambda_1^0, \lambda_2^0, \ldots, \lambda_{n-1}^0, \lambda_n^0, \lambda_{n+1}^0, \ldots),
\]
so that $Tf$ belongs to $C(Y, \ell^2)$.

It is clear that $T$ is a linear isometry where
\[
J_n : \ell^2 \to \ell^2 \quad \text{turns out to be}
\]
\[
J_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n, \lambda_{n+1}, \ldots) = (\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_{n+1}, \ldots).
\]
On the other hand $J_0(e_n) = e_{n+1}$ for all $n \in \mathbb{N}$, and $J_0$ is a codimension 1 linear isometry on $\ell^2$. Consequently $T$ is a codimension 1 linear isometry, where the constant function $\hat{e}_1$ does not belong to the range of $T$. In this case, $Y_N = \{0\} \subseteq Y$, which is not isolated.

5 The finite-dimensional case.

From now on, we shall assume that $m := \dim F < \infty$.

**Lemma 5.1** Suppose that $x \in X$ and $G_x = \{y_1, \ldots, y_{n_x}\}$. Then the mapping $Q_x : E \to F^{n_x}$, defined by
\[
Q_x(e) := ((Te)(y_1), \ldots, (Te)(y_{n_x}))
\]
for all $e \in E$, is a linear isometry if $F^{n_x}$ is endowed with the sup norm
\[
\|f_1, \ldots, f_{n_x}\|_{\infty} = \max_{1 \leq i \leq n_x} \|f_i\|.
\]

**Proof.** Fix $e \in E$ with $\|e\| = 1$. Since $T$ is an isometry, $\|Q_x(e)\| \leq 1$, so we must see that there exists $i \in \{1, \ldots, n_x\}$ with $\|J_{y_i}(e)\| = 1$. Obviously, if some $y_i$ belongs to $Y_{10}$, then $J_{y_i}$ is an isometry and we are done.

Consequently, we suppose that $G_x \cap Y_{10} = \emptyset$. This implies that $x \notin \overline{h}(Y_{10})$ and, since $Y_{10}$ is compact, $x$ is isolated in $X$. Hence the characteristic function $f := \chi_{\{x\}}e$ is continuous. As $f \equiv 0$ on $X \setminus \{x\}$, it is clear that $Tf \equiv 0$ on $\overline{h}^{-1}(X) \setminus \overline{h}^{-1}(x)$, which is to say that there must exist $y \in G_x \cup Y_2$ such that $\|(Tf)(y)\| = \|Tf\|_\infty = 1$. If we suppose that $y \in Y_2$, then there exists $f' \in C(X, E)$ with $f'(x) = 0$ and $(Tf')(y) \neq 0$. Without loss of generality, we shall assume that $\|f'\|_\infty = 1$. Hence $\|f + f'\|_\infty = 1 = \|f - f'\|_\infty$. However, as $F$ is strictly convex, we have $\|(Tf)(y) + (Tf')(y)\| > 1$ or $\|(Tf)(y) - (Tf')(y)\| > 1$, which contradicts the isometric property of $T$. 19
As a consequence, $Tf$ attains its maximum in $G_x$, which is to say that there exists $i \in \{1, \ldots, n_x\}$ with $\|J_{y_i}(e)\| = \|(Tf)(y_i)\| = 1$, as we wanted to see.

□

Next we deduce the relationship between the sets $A_0$ and $A_2$ introduced in Section 2.

**Corollary 5.1** $A_0$ is contained in $A_2$.

*Proof.* Let $x_0 \in A_0$ and $y_0 \in F_{x_0}$ with $J_{y_0}$ not a surjective isometry, which, in this finite-dimensional case, means that it is not an isometry. If $x_0 \notin A_2$, then $G_{x_0} = F_{x_0} = \{y_0\}$, and Lemma 5.1 easily leads to a contradiction. □

**Proposition 5.1** Let $Y$ be infinite. Suppose that $g \in C(Y, F)$ satisfies $g(\overline{h}^{-1}(A_2)) \equiv 0$. Then there exists a unique $f \in C(X, E)$ such that $Tf \equiv g$ on $Y_1$.

*Proof.* Define the function $f \in C(X, E)$ as follows:

- $f(x) := 0$ for $x \in A_2$.
- $f(x) := (J_{\overline{h}^{-1}(x)})^{-1}(g(\overline{h}^{-1}(x)))$ if $x \notin A_2$.

We first check that $f$ is well-defined outside $A_2$, that is, $J_{\overline{h}^{-1}(x)}$ is a surjective isometry. Let $x \notin A_2$. Then $\overline{h}^{-1}(x) = h^{-1}(x)$ because $G_x = F_x$. Also, by Corollary 5.1, $x \notin A_0$, so $J_{\overline{h}^{-1}(x)} : E \to F$ is a surjective isometry.

Next we study the continuity of $f$. Let $x_0 \in X \setminus A_2$ and $\epsilon > 0$. We consider an open neighborhood $V_1$ of $h^{-1}(x_0)$ in $Y$ such that, for all $y \in V_1$,

$$\|g(y) - g(h^{-1}(x_0))\| < \frac{\epsilon}{2}.$$  

With no loss of generality, we can assume that $V_1 \subset Y_{10}$ because $h^{-1}(x_0) \in Y_{10} \setminus \overline{h}^{-1}(A_2)$ and this set is open being $Y_{10}$ clopen by Lemma 3.7. Also, since $\overline{h}^{-1}(A_2)$ is finite, $V_1$ can be taken such that $\text{cl}(V_1) \cap \overline{h}^{-1}(A_2) = \emptyset$.

We can rewrite the above inequality as

$$\|J_y(f(h(y))) - J_{\overline{h}^{-1}(x_0)}(f(x_0))\| < \frac{\epsilon}{2}.$$  

20
for all \( y \in V_1 \).

On the other hand, since \( Y_{10} \subset Y_0 \) is clopen and \( J : Y_0 \rightarrow L(E,F) \) is continuous with respect to the strong operator topology, we can take an open neighborhood \( V_2 \) of \( h^{-1}(x_0) \) with \( V_2 \subset Y_{10} \) such that

\[
\| J_y(f(x_0)) - J_{h^{-1}(x_0)}(f(x_0)) \| < \frac{\epsilon}{2}
\]

for all \( y \in V_2 \). We thus deduce that if \( y \in V_1 \cap V_2 \), then

\[
\| J_y(f(h(y))) - J_y(f(x_0)) \| < \epsilon
\]

that is,

\[
\| J_y[f(h(y)) - f(x_0)] \| < \epsilon.
\]

But as \( y \in Y_{10} \), \( J_y \) is an isometry, and consequently,

\[
\| f(h(y)) - f(x_0) \| < \epsilon
\]

(2)

for all \( y \in V_1 \cap V_2 \). Hence, in order to obtain the continuity of \( f \) at \( x_0 \in X \setminus A_2 \), it suffices to notice that sets of the form \( h(V_1 \cap V_2) \) are open neighborhoods of \( x_0 \).

Let us now study the continuity of \( f \) on \( A_2 \). To this end, fix \( x_0 \in A_2 \). Since \( A_2 \) is a finite set, there exists an open neighborhood \( U \) of \( x_0 \) such that \( U \cap A_2 = \{x_0\} \).

Suppose that \( f \) is not continuous at \( x_0 \). Then there exist \( \epsilon > 0 \) and a net \( (x_\alpha) \) in \( U \) which converges to \( x_0 \) such that \( \| f(x_\alpha) \| \geq \epsilon \) for all \( \alpha \). Since each element of the net \( x_\alpha \) belongs to \( X \setminus A_2 \), we infer that \( h^{-1}(x_\alpha) \) is a singleton in \( Y_{10} \). Furthermore, as \( Y_{10} \) is compact, there exists a subnet \( h^{-1}(x_\beta) \) convergent to a certain \( y_0 \in Y_{10} \). Since \( h \) is continuous, we deduce that \( (x_\beta) \) converges to \( h(y_0) \) and, as a consequence, that \( h(y_0) = x_0 \). This fact yields \( y_0 \in h^{-1}(A_2) \).

By hypothesis, \( g(y_0) = 0 \). However, each \( J_{h^{-1}(x_\beta)} \) is an isometry and, by the definition of \( f \),

\[
g(h^{-1}(x_\beta)) = J_{h^{-1}(x_\beta)}(f(x_\beta)).
\]

Hence \( \| g(h^{-1}(x_\beta)) \| \geq \epsilon \) for all \( \beta \). This implies that \( g \) is not continuous at \( y_0 \), a contradiction, which completes the proof of the continuity of \( f \). The rest of the proof is apparent. \( \square \)
Proof of Theorem 2.3. Put $A_2 = \{x_1, x_2, \ldots, x_k\}$ and, for each $x_i \in A_2$ (see Lemmas 3.3 and 3.4), let

$$G_{x_i} = \{y(x_i, 1), \ldots, y(x_i, n_i)\}. \tag{10}$$

By Corollary 3.1, for each $i \in \{1, 2, \ldots, k\}$ and each $j \in \{1, 2, \ldots, n_i\}$ we can consider an open neighborhood $U(i, j)$ of $y(x_i, j)$ such that $U(i, j) \subset Y_1$ and $U(i, j) \cap U(i', j') = \emptyset$ if $(i, j) \neq (i', j')$. For each pair $(i, j)$ we choose a function $g(i, j) \in C(Y)$ such that $g(i, j)(y(x_i, j)) = 1 = \|g(i, j)\|_\infty$ and $c(g(i, j)) \subset U(i, j)$.

Note that, since $Y$ is infinite, the set $Y_{10} \setminus \overline{h^{-1}(A_0)}$ is nonempty, which easily leads to $\dim E = \dim F$. Now, by Lemma 5.1, each mapping $Q_{x_i} : E \longrightarrow F^m$ is an isometry, so $m := \dim F = \dim Q_{x_i}(E)$. Hence we can find $m(n_i - 1)$ linearly independent vectors in $F^m$ of the form

$$\mathcal{Z}(i, l) := (f(i, l, 1), f(i, l, 2), \ldots, f(i, l, n_i)) \tag{3}$$

for $l = 1, \ldots, m(n_i - 1)$ such that

$$F^m = \operatorname{ran} Q_{x_i} \bigoplus \operatorname{span}\{\mathcal{Z}(i, 1), \ldots, \mathcal{Z}(i, m(n_i - 1))\}. \tag{3}$$

Next we define, for each $i \in \{1, 2, \ldots, k\}$, $m(n_i - 1)$ functions in $C(Y, F)$ related to $\mathcal{Z}(i, j)$ and $g(i, j)$ of the form

$$\mathcal{N}_{[i, l]} := \sum_{j=1}^{n_i} g(i, j)f(i, l, j) \tag{4}$$

for $l = 1, \ldots, m(n_i - 1)$.

Note that, for $i \in \{1, 2, \ldots, k\}$ and each $l \in \{1, 2, \ldots, m(n_i - 1)\}$, we have $\mathcal{N}_{[i, l]}(Y_2 \cup Y_3) \equiv 0$, and if $i' \neq i$, $i' \in \{1, 2, \ldots, k\}$, then $\mathcal{N}_{[i, l]}(G_{x_i}) \equiv 0$, and, for any $j \in \{1, 2, \ldots, n_i\}$,

$$\mathcal{N}_{[i, l]}(y(x_i, j)) = f(i, l, j). \tag{4}$$

Now assume that $Y_2 := \{z_1, \ldots, z_t\}$ and $Y_3 := \{w_1, \ldots, w_s\}$ (see Lemmas 3.5, 3.6 and 3.7). For every $i \in \{1, 2, \ldots, t\}$ and every $l \in \{1, 2, \ldots, m\}$ we can consider $\mathcal{X}_{[i, l]} := \chi(z_i) b_l \in C(Y, F)$ where $B := \{b_1, b_2, \ldots, b_m\}$ is a basis of $F$. In like manner, we can define, for every $i \in \{1, 2, \ldots, s\}$ and every $l \in \{1, 2, \ldots, m\}$, $\mathcal{Y}_{[i, l]} := \chi(w_i) b_l \in C(Y, F)$.

We now claim that the functions we have just introduced are linearly independent. To this end, suppose that

$$\sum_{i, l} \alpha(i, l)\mathcal{N}_{[i, l]} + \sum_{i, l} \beta(i, l)\mathcal{X}_{[i, l]} + \sum_{i, l} \gamma(i, l)\mathcal{Y}_{[i, l]} \equiv 0 \in C(Y, F).$$

22
If we evaluate this sum at the point $z_i \in Y_2$, then we get
\[
\sum_{l=1}^{m} \beta(i, l) b_l = 0 \in F.
\]
As $\{b_1, \ldots, b_m\}$ is a basis of $F$, we infer that each $\beta(i, l) = 0$. Similarly, by evaluating the above sum at each point of $Y_3$, we conclude that $\gamma(i, l) = 0$ for each $i \in \{1, 2, \ldots, s\}$ and $l \in \{1, 2, \ldots, m\}$.

On $G_x$, the above sum turns out to be
\[
m(n_i - 1) \sum_{l=1}^{m} \alpha(i, l) \mathcal{R}_{[i,l]} \equiv 0 \in C(Y, F).
\]
Taking into account equality (4), this means that for each $y(x_i, j)$, $1 \leq j \leq n_i$,
\[
\sum_{l=1}^{m(n_i - 1)} \alpha(i, l) f(i, l, j) \equiv 0, \tag{5}
\]
so $\sum_{l=1}^{m(n_i - 1)} \alpha(i, l) Y_{[i,l]} = 0 \in F^{m_i}$. As a consequence, all the $\alpha(i, l)$ are zero because all vectors $Y_{[i,l]}$ are linearly independent.

**Claim 5.1** The function
\[
g := \sum_{i,l} \alpha(i, l) \mathcal{R}_{[i,l]} + \sum_{i,l} \beta(i, l) \Xi_{[i,l]} + \sum_{i,l} \gamma(i, l) \Upsilon_{[i,l]}
\]
does not belong to the range of $T$, except when $g \equiv 0$.

Suppose that there exists $f \in C(X, E)$ with $Tf = g$. This yields, by the definition of $Y_3$, that each $\gamma(i, l)$ is zero. We shall check that all $\alpha(i, l)$ are zero. Fix $i \in \{1, \ldots, k\}$. Given $j \in \{1, 2, \ldots, n_i\}$, we have
\[
g(y(x_i, j)) = J_{y(x_i,j)}(f(x_i)).
\]
On the other hand, by equality (4),
\[
g(y(x_i, j)) = \sum_{l=1}^{m(n_i - 1)} \alpha(i, l) \mathcal{R}_{[i,l]}(y(x_i, j))
\]
which implies that

\[ Q_{x_i}(f(x_i)) = \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\Im(i,l) \in F^{n_i}. \]

Since \( \text{ran} Q_{x_i} \cap \text{span}\{\Im(i,1), \ldots, \Im(i,m(n_i-1))\} = \{0\}, \)
we have \( Q_{x_i}(f(x_i)) = 0 \in F^{n_i}, \) and consequently \( \alpha(i,l) \) is zero for all \( l. \)
Summarizing, \( g \equiv 0 \) on \( Y_1, \) implying that \( g \equiv 0 \) on \( Y_2. \) This completes the proof of the claim.

Gathering the information obtained so far, we deduce that the vectors

\[ \Re[i,l] + \text{ran} T, \Xi[i,l] + \text{ran} T, \Upsilon[i,l] + \text{ran} T, \]
are linearly independent in the space \( C(Y,F)/\text{ran} T. \) In order to finish the proof, it suffices to check that, given \( g \in C(Y,F), \) there exist scalars \( \alpha(i,j), \beta(i,j), \gamma(i,j) \) such that

\[ g - \sum_{i,l} \alpha(i,l)\Re[i,l] + \sum_{i,l} \beta(i,l)\Xi[i,l] + \sum_{i,l} \gamma(i,l)\Upsilon[i,l] \]
belongs to the range of \( T. \)

For each \( i \in \{1,2,\ldots,k\} \) we consider the vector

\[ N_i := (g(y(x_i,1)), g(y(x_i,2)), \ldots, g(y(x_i,n_i))) \in F^{n_i}. \]

Then, by equality (3), there exist \( e_i \in E \) and constants \( \alpha(i,1), \ldots, \alpha(i,m(n_i-1)) \) such that

\[ N_i = Q_{x_i}(e_i) + \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\Im(i,l). \]

Hence, if we fix \( j \in \{1,2,\ldots,n_i\}, \) then, by equality (4),

\[ g(y(x_i,j)) = (T e_i)(y(x_i,j)) + \sum_{l=1}^{m(n_i-1)} \alpha(i,l)f(i,l,j) + \sum_{l=1}^{m(n_i-1)} \alpha(i,l)\Re[i,l](y(x_i,j)) \in F, \]

24
where \( f_i \in C(X, E) \) with \( f_i(x_i) = e_i \) and \( f_i(x_{i'}) = 0 \) for \( i \neq i' \). If we do so for each \( i \in \{1, 2, \ldots, k\} \) and each \( j \in \{1, 2, \ldots, n_i\} \), we obtain \( k \) functions \( f_i \in C(X, E) \) such that, for \( i_0 \in \{1, 2, \ldots, k\} \) and \( j_0 \in \{1, 2, \ldots, n_{i_0}\} \),

\[
g(y(x_{i_0}, j_0)) = \sum_{i=1}^{k} (Tf_i)(y(x_{i_0}, j_0)) + \sum_{i,l} \alpha(i, l)\mathcal{N}_{i,l}(y(x_{i_0}, j_0)).
\]

Therefore, the function

\[
g_0 := g - \sum_{i=1}^{k} Tf_i - \sum_{i,l} \alpha(i, l)\mathcal{N}_{i,l}
\]

vanishes on each \( y(x_i, j) \), which is to say, on \( \overline{h^{-1}(A_2)} \). By Proposition 5.1, there exists \( f_0 \in C(X, E) \) such that \( Tf_0 \equiv g_0 \) on \( Y_1 \). Hence there exist certain constants \( \beta(i, l) \) and \( \gamma(i, l) \) such that

\[
g_0 - Tf_0 - \sum_{i,l} \beta(i, l)\Xi_{i,l} - \sum_{i,l} \gamma(i, l)\Upsilon_{i,l} \equiv 0
\]

on \( Y_2 \cup Y_3 \) and, consequently, on \( Y \). That is,

\[
g - \sum_{i=1}^{k} Tf_i - Tf_0 - \sum_{i,l} \alpha(i, l)\mathcal{N}_{i,l} - \sum_{i,l} \beta(i, l)\Xi_{i,l} - \sum_{i,l} \gamma(i, l)\Upsilon_{i,l} \equiv 0
\]

on \( Y \). We now easily complete the proof of the theorem. \( \square \)

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**References**


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