PERTURBATION CLASSES OF SEMI-FREDHOLM OPERATORS IN BANACH LATTICES

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Abstract. We prove some results giving positive answers to the perturbation classes problem for semi-Fredholm operators acting on Banach lattices satisfying certain conditions, and we show that these results can be applied to some Lorentz and Orlicz function spaces.

1. Introduction

In [19, Theorem 5.2] Kato proved that the upper semi-Fredholm operators Φ_+ are stable under additive perturbation by strictly singular operators $\mathcal{SS}$: given Banach spaces X and Y for which $\Phi_+(X,Y)$ is nonempty, the set of strictly singular operators $\mathcal{SS}(X,Y)$ is contained in the perturbation class of $\Phi_+(X,Y)$, defined as follows:

$$P\Phi_+(X,Y) := \{K \in \mathcal{L}(X,Y) : T + K \in \Phi_+ \text{ for every } T \in \Phi_+(X,Y)\}.$$

Vladimirskii [28, Corollary 1] proved that the lower semi-Fredholm operators Φ_- are stable under additive perturbation by strictly cosingular operators $\mathcal{SC}$; i.e., $\mathcal{SC}(X,Y) \subset P\Phi_-(X,Y)$ when $\Phi_-(X,Y)$ is nonempty. The question whether the equalities $\mathcal{SS}(X,Y) = P\Phi_+(X,Y)$ and $\mathcal{SC}(X,Y) = P\Phi_-(X,Y)$ are satisfied when the perturbation classes are defined was raised by Gohberg, Markus and Feldman [11, page 74] for $\mathcal{SS}$ and $P\Phi_+$, and both questions were stated in [26, 26.6.12]; see also [27, Section 3]. These questions are referred to as the *perturbation classes problem for semi-Fredholm operators*.

Some partial positive answers to the perturbation classes problem were obtained in [20, 29, 1, 2], but it was proved in [12] that the answer is negative in general: There exists a separable, reflexive Banach space Z for which $P\Phi_+(Z) \neq \mathcal{SS}(Z)$ and $P\Phi_-(Z^*) \neq \mathcal{SC}(Z^*)$. Further negative answers can be found in [10] and [13].

Although the answer to the perturbation classes problem for Φ_+ and Φ_- is negative in general, it is still interesting to find spaces X and Y for which $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ or $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$, because in these cases we have intrinsic characterizations of the operators K in the perturbation classes, i.e., characterizations involving the action of K, instead of the properties of the sums of K with all the operators in $\Phi_+(X,Y)$ or $\Phi_-(X,Y)$. Moreover, the spaces that appear in the counterexamples in [12, 10, 13] are very special: they involve finite products in which at least one of the factors is an indecomposable space. The existence of Banach spaces of this kind was only recently proved by Gowers and Maurey [17].

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Positive results showing that $\mathcal{SS}(X,Y) = P\Phi_+(X,Y)$ or $\mathcal{SC}(X,Y) = P\Phi_-(X,Y)$ holds when X or Y satisfy some conditions have been recently obtained in [16], [15] and [10]. We refer to the introduction of [10] for a description of these results.

In this paper we apply some Banach lattice techniques to obtain further positive answers to the perturbation classes problem for semi-Fredholm operators. In Section 2 we prove a result for $P\Phi_+$ for spaces satisfying some technical conditions, and we derive a result for $P\Phi_-$ from it (Theorems 7 and 8). In Section 3 we show that these two results can be applied to some Lorentz and Orlicz function spaces.

We will use standard notation. If X is a Banach space, S_X stands for its unit sphere, and $[x_1,\dots,x_n]$ is the linear span of $x_1,\dots,x_n \in X$. The class of (bounded, linear) operators between Banach spaces X and Y will be denoted by $\mathcal{L}(X,Y)$, and we will write $\mathcal{A}(X,Y)$ for $\mathcal{A}\cap\mathcal{L}(X,Y)$ for any class of operators $\mathcal{A}$. Given $T \in \mathcal{L}(X,Y)$, the conjugate operator of T is $T^* \in \mathcal{L}(Y^*,X^*)$. An operator $T \in \mathcal{L}(X,Y)$ is *upper semi-Fredholm* if its kernel $N(T)$ is finite-dimensional and its range $R(T)$ is closed; and T is *lower semi-Fredholm* if $R(T)$ is finite-codimensional and closed in Y. The classes of all upper semi-Fredholm and lower semi-Fredholm operators will be denoted by Φ_+ and Φ_-, respectively. It follows from the basic duality relations for operators that $T \in \Phi_+$ if and only if $T^* \in \Phi_-$ and $T \in \Phi_-$ if and only if $T^* \in \Phi_+$.

An operator $T \in \mathcal{L}(X,Y)$ is *strictly singular* if the restriction of T to an infinite-dimensional closed subspace E is never an isomorphism; T is *strictly cosingular* if given a closed infinite-codimensional subspace F of Y the composition $Q_F T$ is never surjective, where Q_F is the quotient operator onto Y/F. We refer to [14] for an exposition of the perturbation theory for semi-Fredholm operators. An operator $T \in \mathcal{L}(X,Y)$ is ℓ_p-singular if it is not an isomorphism when restricted to any subspace isomorphic to ℓ_p.

If X is a Banach lattice, then $T \in \mathcal{L}(X,Y)$ is disjointly strictly singular if it is not an isomorphism when restricted to any subspace spanned by a disjoint sequence, and T is AM-compact if the image of every order interval is a relatively compact set.

2. Main results

The following perturbation result is essentially known; we include it here for the convenience of the reader.

Lemma 1. *Let X, Y be Banach spaces, let M be a closed subspace of X and N be a closed subspace of Y and let $S \in \mathcal{L}(X,Y)$ be an operator such that $S|_M$ is an isomorphism and $S(M)+N$ is not closed. Then there is a compact operator $K \in \mathcal{L}(X,Y)$ such that $(S+K)(M)\cap N$ is infinite-dimensional.*

Proof. If $S(M)\cap N$ is already infinite-dimensional, the proof is finished by taking $K=0$, so we only have to deal with the case where $S(M)\cap N$ is finite-dimensional; by passing to a finite-codimensional subspace of N, we can further assume that $S(M)\cap N = 0$.

First, since $S(M)+N$ is not closed, there exist $x_1 \in S_M$ and $y_1 \in N$ such that $\|S(x_1)-y_1\| < 1/2$, and then there also exists $x_1^* \in S_{X^*}$ such that $\langle x_1^*, x_1\rangle = 1$. Define P_0 to be the identity mapping on M.

Assume now that we have $x_1, \ldots, x_n \in S_M$ and $x_1^*, \ldots, x_n^* \in X^*$ such that $\langle x_i^*, x_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ and $\|S(x_k) - y_k\| \, \|x_k^*\| < 1/2^k$ for all $k \in \{1, \ldots, n\}$. Define $M_n := M \cap \bigcap_{i=1}^n N(x_i^*)$, so that $M = M_n \oplus [x_1, \ldots, x_n]$, and let $P_n \colon M \longrightarrow M_n$ be the projection from M onto M_n with kernel $[x_1, \ldots, x_n]$. Then $S(M_n) + N$ is still not closed in Y, so there exist $x_{n+1} \in S_{M_n}$ and $y_{n+1} \in N$ such that $\|S(x_{n+1}) - y_{n+1}\| < 1/(2^{n+1}\|P_n\|)$; note that $x_{n+1} \in M_n$ implies $\langle x_i^*, x_{n+1} \rangle = 0$ for all $i \in \{1, \ldots, n\}$. Now, by the Hahn-Banach theorem, there exists $x_{n+1}^* \in S_{X^*}$ such that $\langle x_{n+1}^*, x_j \rangle = 0$ for all $j \in \{1, \ldots, n\}$ and $\langle x_{n+1}^*, x_{n+1} \rangle = \operatorname{dist}(x_{n+1}, [x_1, \ldots, x_n]) \geq \|P_n\|^{-1}$; scaling as needed, we can assume $\langle x_{n+1}^*, x_{n+1} \rangle = 1$ and $\|x_{n+1}^*\| \leq \|P_n\|$, so $\|S(x_{n+1}) - y_{n+1}\| \, \|x_{n+1}^*\| < 1/2^{n+1}$.

By induction, we now have a biorthogonal sequence $(x_n^*, x_n)_{n \in \mathbb{N}}$ in $X^* \times M$ and a sequence $(y_n)_{n \in \mathbb{N}} \subseteq N$ such that $\|x_n\| = 1$ and $\|S(x_n) - y_n\| \, \|x_n^*\| < 1/2^n$ for all $n \in \mathbb{N}$. Then the the expression $K(x) = \sum_{n=1}^\infty \langle x_n^*, x \rangle \big(y_n - S(x_n)\big)$ defines a compact operator $K \colon X \longrightarrow Y$ such that $(S+K)(M) \cap N$ contains the infinite-dimensional subspace $[(y_n)_{n \in \mathbb{N}}] = [((S+K)(x_n))_{n \in \mathbb{N}}]$.

Lemma 2. *Let X be a Banach space, let Y be a Banach space containing an isomorphic copy Z of X, let M be an infinite-dimensional complemented subspace of X and let $S \colon X \longrightarrow Y$ be an operator such that $S|_M$ is an isomorphism. Then:*

(i) if $S(M) \cap Z$ is finite-dimensional and $S(M) + Z$ is closed, then $S \notin P\Phi_+(X, Y)$;

(ii) if $S \in P\Phi_+(X, Y)$, then there exists a compact operator $K \in \mathcal{L}(X, Y)$ such that $(S+K)(M) \cap Z$ is infinite-dimensional.

Proof. (i) Let $U \colon X \longrightarrow Z$ be an isomorphism, let H be a complement of M in X, and define the operator $T \colon X = M \oplus H \longrightarrow Y$ as $T(m+h) = -S(m) + U(h)$. Then $N(T) \subseteq S^{-1}(S(M) \cap Z) + U^{-1}(S(M) \cap Z)$ is finite-dimensional and $R(T)$ is closed because $R(T) = S(M) + U(H) \subseteq S(M) + Z$ and $S(M) \cap Z$ is finite-dimensional, so $T \in \Phi_+(X, Y)$. However, $M \subseteq N(T+S)$, so $T + S \notin \Phi_+(X, Y)$ and $S \notin P\Phi_+(X, Y)$.

(ii) If $S \in P\Phi_+(X, Y)$, then either $S(M) \cap Z$ is infinite-dimensional or $S(M) + Z$ is not closed, so taking K to be either 0 or the operator provided by Lemma 1 finishes the proof.

The following result is essentially contained in [8, Proposition 2.5], although not formally stated in this form. We refer to [21, Definition 1.e.12] for the concept of cotype of a Banach space.

Proposition 3. *Let X be a Banach lattice with finite cotype, let Y be a Banach space and let $T \colon X \longrightarrow Y$ be an operator such that $(Tf_n)_{n \in \mathbb{N}}$ is relatively compact for any order-bounded sequence $(f_n)_{n \in \mathbb{N}}$ equivalent to the unit vector basis of ℓ_2. Then T is AM-compact.*

Proof. Let $x \in X_+$ and denote by E_x the closed ideal of X generated by x. Since X is q-concave for some $2 < q < \infty$ [22, Corollary 1.f.9], we have $L_q(\mu) \hookrightarrow E_x \hookrightarrow L_1(\mu)$ for a certain probability measure μ [18, p. 14].

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $[-x, x]$, which means that $(f_n)_{n \in \mathbb{N}}$ is uniformly bounded in E_x. Since the order intervals in X are weakly compact, we can assume that $(f_n)_{n \in \mathbb{N}}$ is weakly null without loss of generality. If $(f_n)_{n \in \mathbb{N}}$ is not seminormalised, it has a convergent subsequence, and so has $(Tf_n)_{n \in \mathbb{N}}$.

Otherwise, take $p > q$; then $(f_n)_{n\in\mathbb{N}}$ is also weakly null and seminormalised in $L_p(\mu)$. Moreover, using [8, Lemma 1.4] and passing to a subsequence, we can assume that $(f_n)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_2 in $L_p(\mu)$. Since a normalised disjoint sequence in $L_p(\mu)$ spans a subspace isomorphic to ℓ_p, the span of $(f_n)_{n\in\mathbb{N}}$ has to be strongly embedded in $L_p(\mu)$ [8, Proposition 1.1]. This means that the $L_p(\mu)$ and $L_1(\mu)$ topologies coincide on the span of $(f_n)_{n\in\mathbb{N}}$, and so $(f_n)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_2 in E_x, too. By hypothesis, $(Tf_n)_{n\in\mathbb{N}}$ is relatively compact in this case as well, finishing the proof.

Recall that a Banach lattice Y satisfies a lower 2-estimate if there exists a constant C such that for every choice of pairwise disjoint elements $(y_j)_{j=1}^n$ in Y, we have

$$\left\|\sum_{j=1}^n y_j\right\| \geq C^{-1}\left(\sum_{j=1}^n \|y_j\|^2\right)^{1/2}.$$

Remark 4. Proposition 3 can be used, of course, under the stronger assumption that $(Tf_n)_{n\in\mathbb{N}}$ is relatively compact for any sequence $(f_n)_{n\in\mathbb{N}}$ equivalent to the unit vector basis of ℓ_2, not just those that are order-bounded. This is equivalent to requiring that TU be compact for every isomorphic embedding $U\colon \ell_2 \longrightarrow X$, as is the case when $T\colon X \longrightarrow Y$ is ℓ_2-singular and $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$. This last condition $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$ holds, for instance, whenever Y is a Banach lattice with a lower 2-estimate [8, Proposition 2.1]; note that all Banach lattices with cotype 2 satisfy a lower 2-estimate, although the converse is not true [22, Example 1.f.19].

This leads us to the following result.

Proposition 5. *Let X be a Banach lattice with finite cotype such that every copy of ℓ_2 in X contains a complemented copy, let Y be a Banach space containing an isomorphic copy of X such that $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$ and let $S \in P\Phi_+(X, Y)$. Then S is AM-compact.*

Proof. We will first prove that S is ℓ_2-singular. Assume, to the contrary, that there exists a subspace $M \subseteq X$ isomorphic to ℓ_2 such that $S|_M$ is an isomorphism. By hypothesis, M can be assumed to be complemented. Let Z be an isomorphic copy of X in Y; then, by Lemma 2, there exists a compact operator $K \in \mathcal{L}(X, Y)$ such that $\tilde{S}(M) \cap Z$ is infinite-dimensional, where $\tilde{S} = S + K \in P\Phi_+(X, Y)$. By passing to a subspace of M, we can further assume that $\tilde{S}|_M$ is an isomorphism and that $\tilde{S}(M) \subseteq Z$. Now, again by hypothesis, $\tilde{S}(M)$ must contain a subspace complemented in Z, so it is possible to find subspaces $N \subseteq M$ and $H \subseteq Z$ such that $Z = \tilde{S}(N) \oplus H$ and H is isomorphic to X. But then Lemma 2 provides $\tilde{S} \notin P\Phi_+(X, Y)$, a contradiction, so S must indeed be ℓ_2-singular.

To finish the proof, take any isomorphism $U \in \mathcal{L}(\ell_2, X)$; then $SU \in \mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$, so Proposition 3 can be applied and S is AM-compact.

Throughout the following result we will make liberal use of this well-known fact (see, for instance, [3, Proposition 2.2.1]): If M is isomorphic to ℓ_p, for some $1 \leq p < \infty$, then every infinite-dimensional closed subspace of M contains a further subspace N such that N is isomorphic to ℓ_p and complemented in M. As a consequence, if M is a complemented subspace of X, then so is N.

Proposition 6. *Let X be a Banach lattice and let $p \in (1, \infty)$ be such that*

(i) every subspace of X spanned by a disjoint sequence contains a further subspace that is complemented in X and isomorphic to ℓ_p;

(ii) for every subspace M of X isomorphic to ℓ_p, there exist subspaces $N \subseteq M$ and $H \subseteq X$ such that N is infinite-dimensional, H is isomorphic to X, $N \cap H = 0$ and $N + H$ is closed.

Let Y be a Banach space containing an isomorphic copy of X and let $S \in P\Phi_+(X, Y)$. Then S is disjointly strictly singular.

Proof. Assume, to the contrary, that there exists a subspace $M \subseteq X$, spanned by a sequence of disjoint elements, such that $S|_M$ is an isomorphism. By (i), we can assume M to be complemented and isomorphic to ℓ_p.

Take Z to be an isomorphic copy of X in Y; then, by Lemma 2, there exists a compact operator $K \in \mathcal{L}(X, Y)$ such that $\tilde{S}(M) \cap Z$ is infinite-dimensional, where $\tilde{S} = S + K \in P\Phi_+(X, Y)$. By passing to a subspace of M, we can further assume that $\tilde{S}|_M$ is an isomorphism and that $\tilde{S}(M) \subseteq Z$, while M is still complemented in X.

Now, by hypothesis (ii), there exist subspaces $N \subseteq \tilde{S}(M)$ and $H \subseteq Z$ such that N is infinite-dimensional, H is isomorphic to X, $N \cap H = 0$ and $N + H$ is closed. Then $\tilde{S}|_M^{-1}(N) \subseteq M$, which is isomorphic to ℓ_p and complemented in X, so there is $G \subseteq S|_M^{-1}(N)$ again isomorphic to ℓ_p and complemented in X. But this means that $\tilde{S}(G) \cap H = 0$ and $\tilde{S}(G) + H$ is closed, contradicting $\tilde{S} \notin P\Phi_+(X, Y)$ by Lemma 2.

Combination of Propositions 5 and 6 brings the following.

Theorem 7. *Let X be a Banach lattice with finite cotype such that*

(i) every copy of ℓ_2 in X contains a complemented copy;

(ii) there exists $p \in (1, \infty)$ such that every subspace of X spanned by a disjoint sequence contains a further subspace that is complemented in X and isomorphic to ℓ_p;

(iii) for every subspace M of X isomorphic to ℓ_p, there exist subspaces $N \subseteq M$ and $H \subseteq X$ such that N is infinite-dimensional, H is isomorphic to X, $N \cap H = 0$ and $N + H$ is closed.

Let Y be a Banach space containing an isomorphic copy of X and such that $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$. Then $P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$.

Proof. Let $S \in P\Phi_+(X, Y)$; then, by Propositions 5 and 6, S is both AM-compact and disjointly strictly singular, so $S \in \mathcal{SS}(X, Y)$ [8, Theorem 2.4].

Theorem 7 admits an immediate dual version, when Y is reflexive, using [13, Theorem 2.3].

Theorem 8. *Let Y be a reflexive Banach lattice with finite type such that Y^* satisfies conditions (i), (ii) and (iii) in Theorem 7, and let X be a Banach space admitting a quotient isomorphic to Y and such that $\mathcal{SC}(X, \ell_2) = \mathcal{K}(X, \ell_2)$. Then $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$.*

Proof. Since $\mathcal{SC}(X, \ell_2) = \mathcal{K}(X, \ell_2)$, by [13, Lemma 2.1] we have $\mathcal{SS}(\ell_2, X^*) = \mathcal{K}(\ell_2, X^*)$, so Theorem 7 provides $P\Phi_+(Y^*, X^*) = \mathcal{SS}(Y^*, X^*)$, from which it follows $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$ [13, Theorem 2.3].

This proof relies on the fact that $\mathcal{SC}(X,\ell_2)=\mathcal{K}(X,\ell_2)$ implies $\mathcal{SS}(\ell_2,X^*)=\mathcal{K}(\ell_2,X^*)$; it is not difficult to see that, in fact, the converse also holds.

3. Applications

While the conditions described in the hypotheses of Theorem 7 are somewhat technical, they are readily satisfied by large classes of Banach spaces, such as some Lorentz and Orlicz function spaces.

We begin with a straightforward lemma that will be useful later. If X is a Köthe function space on $[0,1]$, that is, a Banach lattice which is lattice-isomorphic to a (not necessarily closed) ideal of L_1, and $0\le a<b\le 1$, we will write $X(a,b)$ for the subspace of X consisting of all functions $f\in X$ such that $f=f\chi_{[a,b]}$. We will say that a closed subspace M of a Köthe function space on $[0,1]$ is strongly embedded when the natural inclusion of M into L_1 is an isomorphic embedding.

Lemma 9. *Let X be a Köthe function space on $[0,1]$, and let M be a reflexive, strongly embedded subspace of X. Then there exist $0<a<b\le 1$ such that $M\cap X(a,b)=0$ and $M+X(a,b)$ is closed.*

Proof. Assume otherwise; then, for every $n\in\mathbb{N}$, we can find $f_n\in S_M$ and $g_n\in X(\frac{1}{n+1},\frac{1}{n})$ such that $\|f_n-g_n\|<2^{-n}$. Since M is strongly embedded in L_1, there is $C>0$ such that $\|f_n\|_1>C$ for all $n\in\mathbb{N}$; combined with $\|f_n-g_n\|_1\le\|f_n-g_n\|<2^{-n}$, this means that (a subsequence of) $(g_n)_{n\in\mathbb{N}}$ is disjointly supported and seminormalised in L_1, hence equivalent to the unit vector basis of ℓ_1, and so must be $(f_n)_{n\in\mathbb{N}}$. But this is impossible since M is reflexive.

Our first application of Theorem 7 will be to the class of $L_{p,q}$ spaces, for suitable values of p and q. Recall that, given $1<p<\infty$ and $1\le q<\infty$, the space $L_{p,q}(I)$ is the space of (equivalence classes of) measurable functions f on I such that $\|f\|_{p,q}<\infty$, where

$$\|f\|_{p,q}=\left(\int_I \frac{q}{p}f^*(t)^q\,t^{\frac{q}{p}-1}\,dt\right)^{1/q};$$

here, f^* is the decreasing rearrangment of $|f|$. We will only be concerned with $I=[0,1]$ or $I=[0,\infty)$, and will generally follow [4] when dealing with these spaces. Note that $L_{p,q}(0,1)$ and $L_{p,q}(0,\infty)$ are not isomorphic unless $p=q$ [4, Corollary 2.2], although $L_{p,q}(0,1)$ is a subspace of $L_{p,q}(0,\infty)$. Given $s\in(0,\infty)$, we will denote the restriction mapping from $L_{p,q}(0,\infty)$ to $L_{p,q}(0,s)$ by $P_s(f)=f|_{[0,s]}$.

Theorem 10. [4, Theorem 2.5] *Let $1<p<\infty$ and $1\le q<\infty$.*

(i) If M is a subspace of $L_{p,q}(0,1)$, then M contains a complemented copy of ℓ_q or M is strongly embedded in $L_{p,q}(0,1)$.

(ii) If M is a subspace of $L_{p,q}(0,\infty)$, then M contains a complemented copy of ℓ_q or there exists $s\in(0,\infty)$ such that P_s is an isomorphism on M and $P_s(M)$ is strongly embedded in $L_{p,q}(0,s)$.

Proposition 11. *Let $1<p<\infty$ and $1\le q<\infty$, and let I be either $[0,1]$ or $[0,\infty)$. Then*

(i) every copy of ℓ_2 in $L_{p,q}(I)$ contains a complemented copy;

(ii) every subspace of $L_{p,q}(I)$ spanned by a disjoint sequence contains a complemented copy of ℓ_q;

(iii) for every copy M of ℓ_q in $L_{p,q}(I)$, there exist subspaces $N \subseteq M$ and $H \subseteq L_{p,q}(I)$ such that N is isomorphic to ℓ_q, H is isomorphic to $L_{p,q}(I)$, $N \cap H = 0$ and $N + H$ is closed.

Proof. (i) Let M be a copy of ℓ_2 in $L_{p,q}(I)$. If M contains a complemented copy of ℓ_q, it must be $q = 2$ and we are done.

Otherwise, if $I = [0,1]$, M is strongly embedded in $L_{p,q}(0,1)$. In this case, take any $1 < r < p$; then M is a copy of ℓ_2 in $L_r(0,1)$ and there exist $N \subseteq M$ and $H \subseteq L_r(0,1)$ such that $L_r(0,1) = N \oplus H$ [25, Theorem 3.1], so $L_{p,q}(0,1) = N \oplus (H \cap L_{p,q}(0,1))$.

If $I = [0,\infty)$, there exists $s \in I$ such that P_s is an isomorphism on M and $P_s(M)$ is strongly embedded in $L_{p,q}(0,s)$. By the previous paragraph, there exist $N \subseteq P_s(M)$ and $H \subseteq L_{p,q}(0,s)$ such that $L_{p,q}(0,s) = N \oplus H$, so $L_{p,q}(0,\infty) = P_s|_M^{-1}(N) \oplus P_s^{-1}(H)$.

(ii) See [4, Corollary 2.4].

(iii) Let M be a copy of ℓ_q in $L_{p,q}(I)$; if M contains a complemented copy of ℓ_q, it is easy to find a further subspace of M whose complement is isomorphic to $L_{p,q}(I)$, so we need only check the case where M does not contain a complemented copy of ℓ_q.

If $I = [0,1]$, then M is strongly embedded in $L_{p,q}(0,1)$, and therefore in L_r for $1 < r < p$, hence M is reflexive. By Lemma 9, there exists $H := L_{p,q}(a,b) \subseteq L_{p,q}(0,1)$ such that $M \cap H = 0$ and $M + H$ is closed, where H is isomorphic to $L_{p,q}(0,1)$ (so no N is needed).

If $I = [0,\infty)$, there exists $s \in I$ such that P_s is an isomorphism on M and $P_s(M)$ is strongly embedded in $L_{p,q}(0,s)$. By the previous paragraph, there exists a subspace $H \subseteq L_{p,q}(0,s)$ isomorphic to $L_{p,q}(0,s)$ such that $P_s(M) \cap H = 0$ and $P_s(M) + H$ is closed. Then $P_s^{-1}(H)$ is isomorphic to $L_{p,q}(0,\infty)$, $M \cap P_s^{-1}(H) = 0$ and $M + P_s^{-1}(H)$ is closed.

It is now just one step away to prove the coincidence of the perturbation class of upper semi-Fredholm operators with the class of strictly singular operators when $X = L_{p,q}(I)$ and Y meets the criteria of the last result. Note, however, that the requirement that $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$, where Y must contain a copy of $L_{p,q}(I)$, excludes all values of p and q for which $L_{p,q}(I)$ itself contains a copy of ℓ_r for $r > 2$, so only $p, q \leq 2$ make sense.

Proposition 12. *Let $1 < p \leq 2$ and $1 \leq q \leq 2$, let I be either $[0,1]$ or $[0,\infty)$ and let Y be a Banach space containing an isomorphic copy of $L_{p,q}(I)$ such that $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$. Then $P\Phi_+(L_{p,q}(I), Y) = \mathcal{SS}(L_{p,q}(I), Y)$.*

Proof. $L_{p,q}(I)$ is a Banach lattice with finite cotype [6, Theorems 3.5 and 3.6]; apply Theorem 7 and Proposition 11.

This loss in the range of p and q is partially compensated by the fact that, for $2 < p < \infty$, the space $L_{p,q}(I)$ is strongly subprojective [15, Proposition 2.4], so a similar conclusion would follow from [15, Theorem 2.6].

This result can be applied, for instance, to the following spaces.

Corollary 13. *Let $1 \leq q < p < 2$. Then $P\Phi_+(L_{p,q}(0,\infty), Y) = \mathcal{SS}(L_{p,q}(0,\infty), Y)$ when Y is one of*

(i) L_q*;*
(ii) $L_q(\ell_s)$*, for some* $p < s \leq 2$*;*
(iii) $L_{r,s}(0,1)$*, for some* $1 < r < q$ *and* $1 \leq s \leq 2$*.*

Proof. All of the aforementioned spaces Y are Banach lattices with cotype 2, so $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$ [8, Proposition 2.1]. In order to use Proposition 12, it remains to check that they contain an isomorphic copy of $L_{p,q}(0,\infty)$. For $Y = L_q$ and $Y = L_q(\ell_s)$, this can be seen in [5, Theorem 2, Corollary 2].

For $Y = L_{r,s}(0,1)$, we have that L_q contains a copy of $L_{p,q}(0,\infty)$ [5, Corollary 2]. Take now $r < t < q$; then, in turn, L_t contains a copy of L_q [22, Corollary 2.f.5]. This copy must be strongly embedded in L_t, as it cannot contain ℓ_t, so it must also be a closed subspace of $L_{r,s}$.

Theorem 7 can also be applied to Lorentz function spaces of type $\Lambda(W,p)$. To fix our notation, we will say that any unbounded, non-increasing function W on $(0,1]$ with $\int_0^1 W(t)\,dt = 1$ and $W(1) > 0$ is a Lorentz weight function. Given $1 \leq p < \infty$ and a Lorentz weight function W, $\Lambda(W,p)$ will be the Lorentz function space of all measurable functions on $[0,1]$ such that $\|f\| = \left(\int_0^1 f^*(t)^p\, W(t)\,dt\right)^{1/p} < \infty$, where f^* is the decreasing rearrangement of $|f|$. These spaces were studied in [7].

We need the following auxiliary result.

Lemma 14. *Let* $1 \leq p < \infty$*, let* W *be a Lorentz weight function and let* $0 \leq a < b \leq 1$*. Then* $\Lambda(W,p)(a,b)$ *is isomorphic to* $\Lambda(W,p)$*.*

Proof. We will see that the expression $Tf(t) = f(a + t(b-a))$ defines a bijective isomorphism $T\colon \Lambda(W,p)(a,b) \longrightarrow \Lambda(W,p)$. It is easy to check that $(Tf)^*(t) = f^*(t(b-a)) \geq f^*(t)$, so $\|Tf\|^p = \int_0^1 ((Tf)^*(t))^p\, W(t)\,dt \geq \|f\|^p$. On the other hand, since W is non-increasing,

$$\|Tf\|^p = \int_0^1 ((Tf)^*(t))^p\, W(t)\,dt = \int_0^1 f^*(t(b-a))^p\, W(t)\,dt$$
$$= \int_0^{b-a} f^*(t)^p\, W\left(\frac{t}{b-a}\right)\frac{1}{b-a}\,dt \leq \frac{1}{b-a}\int_0^{b-a} f^*(t)^p\, W(t)\,dt = \frac{1}{b-a}\|f\|^p$$

So T is well-defined, and clearly bijective.

In the following proposition, the conditions under which $\Lambda(W,p)$ has finite cotype can be seen in [24].

Proposition 15. *Let* $1 < p < 2$ *and let* W *be a Lorentz weight function such that* $\Lambda(W,p)$ *has finite cotype. Let* Y *be a Banach space containing an isomorphic copy of* $\Lambda(W,p)$ *such that* $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$*. Then* $P\Phi_+(\Lambda(W,p), Y) = \mathcal{SS}(\Lambda(W,p), Y)$*.*

Proof. We only need to show that $\Lambda(W,p)$ meets the requirements of Theorem 7 for X.

(i) Let M be a copy of ℓ_2 in $\Lambda(W,p)$. Since M cannot contain a complemented copy of ℓ_p for $p < 2$, it must embed isomorphically into L_p [7, Remark 5.6], so there exist $N \subseteq M$ and $H \subseteq L_p$ such that $L_p = N \oplus H$ [25, Theorem 3.1], and $\Lambda(W,p) = N \oplus (H \cap \Lambda(W,p))$, where N is a complemented copy of ℓ_2.

(ii) See [7, Theorem 5.1].

(iii) Let M be a copy of ℓ_p in $\Lambda(W,p)$; then either M contains a complemented copy of ℓ_p, or M embeds isomorphically into L_p [7, Remark 5.6]. If M contains a complemented copy of ℓ_p, it must also contain a copy whose complement is $\Lambda(W,p)$. Otherwise, if M embeds isomorphically into L_p, it must be strongly embedded in L_p and, by Lemma 9, there exists $H := \Lambda(W,p)(a,b) \subseteq \Lambda(W,p)$ such that $M \cap H = 0$ and $M + H$ is closed, where H is isomorphic to $\Lambda(W,p)$ by Lemma 14.

Orlicz function spaces are also good candidates for Theorem 7. Recall that, given an Orlicz function φ, the space L_φ consists of all measurable functions f on $[0,1]$ such that $\int_0^1 \varphi(|f(t)|/\rho)\,dt < \infty$ for some $\rho > 0$, where the norm is given by $\|f\| = \inf\{\, \rho > 0 : \int_0^1 \varphi(|f(t)|/\rho)\,dt < \infty \,\}$. The complementary function of φ will be denoted by φ^*, so that $L_\varphi^* = L_{\varphi^*}$ [21, Chapter 4], and E_φ^∞ will be the set of functions $G(t)$ of the form $\lim_{n\to\infty} \varphi(ty_n)/\varphi(y_n)$, $0 \le t < 1$, for some sequence $y_n \to \infty$ [23, Section 4]. We will write $E_\varphi^\infty \equiv \{F\}$ when every function in E_φ^∞ is equivalent to a certain function F at 0.

Proposition 16. *Let φ be an Orlicz function such that $E_\varphi^\infty \equiv \{t^p\}$ for some $1 < p < 2$, and let Y be a Banach space containing an isomorphic copy of L_φ such that $\mathcal{SS}(\ell_2, Y) = \mathcal{K}(\ell_2, Y)$. Then $P\Phi_+(L_\varphi, Y) = \mathcal{SS}(L_\varphi, Y)$.*

Proof. Again, we will prove that L_φ meets the requirements of Theorem 7 for X. First of all, note that L_φ satisfies a lower 2-estimate [22, Proposition 2.b.5], so it has finite cotype.

(i) The Boyd indices for L_φ are $p_{L_\varphi} = q_{L_\varphi} = p$ [22, Proposition 2.b.5] [23, Section 4]; if we take any $1 < q < p$, then L_φ is contained in L_q [22, Proposition 2.b.3], so any copy of ℓ_2 in L_φ must contain a complemented copy by the argument of the proof of Proposition 11 (i).

(ii) Let $(f_n)_{n\in\mathbb{N}}$ be a normalised disjoint sequence in L_φ, and take $(g_n)_{n\in\mathbb{N}}$ a normalised sequence in L_φ^* such that $\operatorname{supp} f_n = \operatorname{supp} g_n$ for all $n \in \mathbb{N}$ and $\langle g_i, f_j\rangle = \delta_{ij}$ for all i, $j \in \mathbb{N}$; note that $E_\varphi^\infty \equiv \{t^p\}$ implies $E_{\varphi^*}^\infty \equiv \{t^q\}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Since $(f_n)_{n\in\mathbb{N}}$ is disjointly supported, and $E_\varphi^\infty \equiv \{t^p\}$, by passing to a subsequence we can assume that $(f_n)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_p [23, Proposition 3], and similarly that $(g_n)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_q. Then $Q(f) = \sum_{n\in\mathbb{N}} \langle g_n, f\rangle f_n$ defines a projection from L_φ onto the span of $(f_n)_{n\in\mathbb{N}}$.

(iii) Let M be a copy of ℓ_p in L_φ; then either M contains an almost disjoint sequence or M is strongly embedded in L_1 [7, Theorem 4.1]. If M contains an almost disjoint sequence, an argument similar to the previous paragraph shows that it contains a complemented copy of ℓ_p. Otherwise, if M is strongly embedded in L_1, by Lemma 9, there exists $H := L_\varphi(a,b) \subseteq L_\varphi$ such that $M \cap H = 0$ and $M + H$ is closed, where H is isomorphic to L_φ.

Many regular Orlicz function satisfy the condition $E_\varphi^\infty \equiv \{t^p\}$. For example, any φ such that $\lim_{t\to\infty} \big(t\varphi'(t)\big)/\varphi(t) = p$, such as $\varphi(t) = t^p \log^\alpha(1+t)$ for $-\infty < \alpha < \infty$ [9, Section 4].

Dual results for $P\Phi_-$. Using Theorem 8, we can derive some results for $P\Phi_-$ from Propositions 12, 15 and 16. These results are summarised below.

Proposition 17. *Let Y be one of the spaces*
(i) $L_{p,q}(0,1)$ or $L_{p,q}(0,\infty)$ with $2 \leq p, q < \infty$;
(ii) $\Lambda(W,p)$ with $2 < p < \infty$ and finite type;
(iii) L_φ with $E_\varphi^\infty \equiv \{t^p\}$ for some $2 < p < \infty$,
and let X be a Banach space satisfying $\mathcal{SC}(X,\ell_2) = \mathcal{K}(X,\ell_2)$ and admitting a quotient isomorphic to Y. Then $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$.

The proof is similar to that of Theorem 8. Note that Y is reflexive in all cases.

References

[1] P. Aiena and M. González. Inessential operators between Banach spaces. Rendiconti Circ. Mat. Palermo, Ser II Suppl. 68 (2002) 3–26.
[2] P. Aiena; M. González; A. Martinón. On the perturbation classes of continuous semi-Fredholm operators. Glasgow Math. J. 45 (2003) 91–95
[3] F. Albiac; N. J. Kalton. Topics in Banach Space Theory. Springer, 2006.
[4] N. L. Carothers; S. J. Dilworth. Subspaces of $L_{p,q}$. Proc. Amer. Math. Soc. 104 (2) (1988) 537–545
[5] N. L. Carothers; S. J. Dilworth. Some Banach space embeddings of classical function spaces. Bull. Austral. Math. Soc. 43 (1) (1991) 73–77
[6] J. Creekmore. Type and Cotype in Lorentz $L_{p,q}$ Spaces. Indag. Math. 43 (1981) 145–152.
[7] T. Figiel; W. Johnson; L. Tzafriri. On Banach Lattices and Spaces Having Local Unconditional Structure, with Applications to Lorenz Function Spaces J. Approx. Theory 13 (1975) 395–412.
[8] J. Flores; F. L. Hernández; N. J. Kalton; P. Tradacete. Characterizations of strictly singular operators on Banach lattices. J. London Math. Soc. 79 (2009) 612–630.
[9] J. Flores; F. L. Hernández; E. M. Semenov; P. Tradacete. Strictly singular and power-compact operators on Banach lattices. Israel J. Math. 188 (2012) 323–352.
[10] J. Giménez; M. González; A. Martínez-Abejón. Perturbation of semi-Fredholm operators on products of Banach spaces. J. Operator Theory 68 (2012) 501–514.
[11] I. C. Gohberg; A. S. Markus; I. A. Feldman. Normally solvable operators and ideals associated with them. Bul. Akad. Štiince RSS Moldoven 10 (76) (1960) 51–70. Translation: Amer. Math. Soc. Transl. (2) 61 (1967) 63–84.
[12] M. González. The perturbation classes problem in Fredholm theory. J. Funct. Anal. 200 (2003) 65–70.
[13] M. González. Duality results for perturbation classes of semi-Fredholm operators. Arch. Math. 97 (4) (2011) 345–352.
[14] M. González; A. Martínez-Abejón. Tauberian Operators. Birkhäuser, 2010.
[15] M. González; A. Martínez-Abejón; M. Salas-Brown. Perturbation classes for semi-Fredholm operators on subprojective and superprojective spaces. Ann. Acad. Sci. Fennicae Math. 36 (2011) 481–491.
[16] M. González; M. Salas-Brown. Perturbation classes for semi-Fredholm operators on $L_p(\mu)$-spaces. J. Math. Anal. Appl. 370 (2010) 11-17.
[17] W. T. Gowers; B. Maurey. The unconditional basic sequence problem. J. Amer. Math. Soc. 6 (1993) 851–874.
[18] W. Johnson; B. Maurey; G. Schechtman; L. Tzafriri. Symmetric structures in Banach spaces. Mem. Amer. Math. Soc. 19 (1979).
[19] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators. J. d'Analyse Math. 6 (1958) 261–322.
[20] A. Lebow; M. Schechter. Semigroups of operators and measures of noncompactness. J. Funct. Anal. 7 (1971) 1–26.
[21] J. Lindenstrauss; L. Tzafriri. Classical Banach Spaces I. Springer-Verlag, 1977.
[22] J. Lindenstrauss; L. Tzafriri. Classical Banach Spaces II. Springer-Verlag, 1979.
[23] J. Lindenstrauss; L. Tzafriri. On Orlicz sequence spaces III. Israel J. Math. 14 (1973) 368–389.

[24] S. Ya. Novikov. Cotype and type of Lorenz function spaces. Mat. Zametki 32 (1982) 213–221. Translation: Math. Notes 32 (1982) 586–590.
[25] A. Pełczyński; H. P. Rosenthal. Localization techniques in L^p spaces. Studia Math. 52 (1975) 263–289.
[26] A. Pietsch. Operator ideals. North-Holland, 1980.
[27] H.-O. Tylli. Lifting non-topological divisors of zero modulo the compact operators. J. Funct. Anal. 125 (1994) 389–415.
[28] J. I. Vladimirskii. Strictly cosingular operators. Soviet Math. Doklady 8 (1967) 739–740.
[29] L. Weis. On perturbations of Fredholm operators in $L_p(\mu)$-spaces. Proc. Amer. Math. Soc. 67 (1977) 287–292.

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