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On Tubular vs. Swung surfaces

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Abstract

We determine necessary and sufficient conditions for a tubular surface to be swung, and viceversa. From these characterizations, we derive two symbolic algorithms. The first one decides whether a given implicit equation, of a tubular surface, admits a swung parametrization and, in the affirmative case, it outputs such a parametrization. The second one decides whether a given swung surface parametrization is a tubular surface and, in the affirmative case, it outputs the implicit equation.

Keywords: Swung surface, tubular surface.

1. Introduction

In CAGD, many different families of surfaces are usually considered. For instance, we may talk about revolution, ruled, tubular, swung, swept, etc. surfaces (see [7] for a nice survey on these different families of surfaces). However, it is possible that a surface belongs to more than one of these families. For example, every revolution surface is an instance of a swung surface, that is also an example of swept surface.

But the inclusion of different families of surfaces into each other, does not hold in general. In many cases, surfaces belonging to a particular family have to verify some extra conditions in order to be, as well, members of a different family of surfaces. These extra conditions usually take an algebraic form, so the intersection of the two families of surfaces has measure zero in the space representing each family. But, when this happens, the manipulation of such a surface, belonging to several families of surfaces, can profit from the accumulated knowledge about surfaces on each concrete family. For instance, elements belonging to some families can have a simple implicit description, while those pertaining to some other families could enjoy having straightforward parametric representations. Belonging simultaneously to two families of such different

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kinds could result, for example, in an easier method for solving the symbolic implicit/parametric conversion for the given surface.

In this paper, we determine those surfaces that are simultaneously tubular and swung.

Tubular surfaces are those irreducible surfaces described by an implicit equation

$$A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0$$

where $A, B, C \in \mathbb{R}[x_3]$, $\gcd(A, B, C) = 1$ and the total degree w.r.t. $\{x_1, x_2\}$ is 2; note that in a tubular surface it cannot happen that two of the polynomials A, B, C vanish simultaneously. Notice that any surface with a pencil of rational curves is birational equivalent to a tubular surface. Algorithms to parametrize a tubular surface are described in [4], where it is also shown that many instances of the real surface parametrization problem can be reduced to the tubular case. See [2], Example 2.3 for an application in the context of swung surfaces. The importance of tubular surfaces concerning this relevant, generally unsolved, problem of parametrizing over the reals, is one of the reasons for our choice of tubular surfaces as one of the families in our double test approach.

On the other hand, swung surfaces are a generalization of the well known revolution surfaces (around the x_3-axis) in which a profile curve parametrized by $(0, \phi_1(t), \phi_2(t))$ is transported *around* a trajectory curve $(\psi_1(s), \psi_2(s), 0)$. The obtained surface is the surface parametrized by

$$(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t)) .$$

If the trajectory curve is a circle, then the swung surface is just the classical revolution surface. Swung surfaces have been subject of recent research, even considering elementary issues as the problem of implicitizing; see [6] where the authors use μ-bases to develop specific techniques for implicitation of swung surfaces, as an alternative of the well-know techniques in elimination theory.

Notice that, if the profile curve of a revolution surface is given by the graph of a rational function $x_2 = (f/g)(x_3)$, then the revolution surface has equation $g(x_3)^2x_1^2+g(x_3)^2x_2^2-f(x_3)^2 = 0$ and this is clearly a tubular surface. Conversely, a necessary condition for a swung surface to be tubular is that its intersection with the family of planes $\{x_3 = c\}$ is a pencil of conics. However, this is not sufficient in general, see Example 4.2.

Since tubular surfaces are very relevant in the real reparametrization problem and swung surfaces are very useful in CAD, in this paper we want to merge both advantages and determine necessary and sufficient conditions for a tubular surface to be swung and viceversa (Theorems 2.3 and 3.1). These characterizations provide a symbolic algorithm that passes from an implicit tubular representation to a swung parametrization whenever possible, as well as a symbolic algorithm that decides whether a swung parametrization is tubular and, if so, it computes the implicit equation. An example of the above mentioned advantages of being simultaneously in both categories is developed in Example 4.3.

Throughout the paper we assume that the implicit representations of the tubular surfaces are real polynomials, and that the swung surface parametrizations are real.

2. From tubular to swung

We show how to decide if a tubular surface is swung and, then, how to compute a swung parametrization. A naive approach could start applying parametrization algorithms to the given implicit equation of the tubular surface, expecting to obtain swung parametrization (if the surface is swung). But parametrization algorithms are not trivial and, even if a parametrization is obtained, it is not expected that it will have the structure of a swung parametrization. Thus, we need to develop some specific techniques to deal with this problem.

We start this section analyzing some special cases. We already know that two of the polynomials A, B, C can not vanish simultaneously. Let us study what happens when one of them vanishes.

Lemma 2.1. *Let $A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0$ be the implicit equation of a tubular surface. If $AB = 0$, and the surface is rational over $\mathbb{R}$, then it is swung.*

Proof. By definition of tubular surface, we know that A, B cannot be simultaneously zero. Let $A = 0$ but $B \neq 0$. Let $P(u, v) = (u, M(v), N(v))$ be a proper real parametrization of the tubular surface $B(x_3)x_2^2 + C(x_3) = 0$. We observe that M is not zero, because the surface is not a plane. Then, taking $Q(s, t) = P(M(t)s, t)$ we get $Q(s, t) = (M(t)s, M(t), N(t))$ that is a swung proper parametrization (note that $(M(t)s, t)$ is a birational map of $\mathbb{R}^2$ on $\mathbb{R}^2$) with profile curve $(0, M(t), N(t))$ and trajectory curve $(s, 1, 0)$.

Let $A \neq 0$ but $B = 0$. Then, the same reasoning works. In this case if $P(u, v) = (M(v), u, N(v))$ then the profile curve is $(0, M(t), N(t))$ and the trajectory curve is $(1, s, 0)$. □

Lemma 2.2. *Let $A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0$ be the implicit equation of a tubular surface. If $C = 0$ the surface is not swung.*

Proof. Assume that $(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$ is a swung parametrization of the surface. Then

$$\phi_1(t)^2(A(\phi_2(t))\psi_1(s)^2 + B(\phi_2(t))\psi_2(s)^2) = 0.$$

Since $\phi_1(t)$ is not zero, because otherwise the surface would degenerate to a curve, then

$$A(\phi_2(t))\psi_1(s)^2 + B(\phi_2(t))\psi_2(s)^2 = 0.$$

In addition, $AB \neq 0$. So, $A(\phi_2(t))B(\phi_2(t)) \neq 0$. On the other hand, we also have that not both rational functions $\psi_i(s)$ can be zero; say $\psi_1(s) \neq 0$. Then

$$\frac{A(\phi_2(t))}{B(\phi_2(t))} = -\frac{\psi_2(s)^2}{\psi_1(s)^2}.$$

This implies that $A(\phi_2(t)) = \lambda B(\phi_2(t))$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Thus, $A(x_3) = \lambda B(x_3)$. Moreover, since $\gcd(A, B, C) = 1$ then A and B must be constants, and the equation of the tubular surface is $\lambda x_1^2 + x_2^2 = (x_2 - \sqrt{-\lambda} x_1)(x_2 + \sqrt{-\lambda} x_1) = 0$. But this is a contradiction, because a tubular surface is irreducible. □

Taking into account the previous lemmas we will assume that $ABC \neq 0$.

Theorem 2.3. *Let $A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0$ be the implicit equation of a real tubular surface, such that $ABC \neq 0$, $\gcd(A, B, C) = 1$. Then, the surface is a swung surface if and only if:*

1. *$B(x_3)/A(x_3) = k \in \mathbb{R}$ is constant.*
2. *One of the curves (or a component of) $A(y)x^2 \pm C(y)$ is rational parametrizable over $\mathbb{R}$.*

Proof. Assume that $A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0$ is swung. Then, there exists a swung parametrization $(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$ of the surface. Hence

$$A(\phi_2(t))\phi_1(t)^2\psi_1(s)^2 + B(\phi_2(t))\phi_1(t)^2\psi_2(s)^2 + C(\phi_2(t)) = 0.$$

We observe that $\phi_2(t)$ cannot be a constant, because the surface is not a plane. Also, note that $\phi_1(t)$ cannot be zero, since otherwise the given variety would be a line. This, in particular implies that $C(\phi_2(t))B(\phi_2(t))A(\phi_2(t))\phi_1(t)$ is not zero. So, manipulating the above expression, we get that

$$\frac{\psi_1(s)^2}{\alpha(t)} + \frac{\psi_2(s)^2}{\beta(t)} - 1 = 0,$$

where

$$\alpha(t) = -\frac{C(\phi_2(t))}{A(\phi_2(t))\phi_1(t)^2} \neq 0, \ \beta(t) = -\frac{C(\phi_2(t))}{B(\phi_2(t))\phi_1(t)^2} \neq 0. \tag{1}$$

Therefore $(\psi_1(s), \psi_2(s))$ parametrizes the conic, defined over $\mathbb{R}(t)$ by $x_1^2/\alpha(t) + x_2^2/\beta(t) = 1$. However, since $(\psi_1(s), \psi_2(s))$ is over $\mathbb{R}$, its implicit equation is over $\mathbb{R}$. So, since $x_1^2/\alpha(t) + x_2^2/\beta(t) = 1$ is irreducible as conic over $\mathbb{R}(t)$, we get that both implicit equations must be equal, and hence $\alpha(t), \beta(t) \in \mathbb{R} \setminus \{0\}$. Thus, $\alpha(t)/\beta(t) = (B/A)(\phi_2(t))$ is constant. Hence, since ϕ_2 is not constant, $B(x_3)/A(x_3) = k \in \mathbb{R} \setminus \{0\}$ is constant.

Moreover, by equation (1), $(\phi_1(t), \phi_2(t))$ is a parametrization (of a component of) the curve defined by $C(y) + \alpha x^2 A(y)$ (recall that $\alpha(t) \in \mathbb{R} \setminus \{0\}$) and $(\phi_1/\sqrt{|\alpha|}, \phi_2)$ is a parametrization (of a component of) $C(y) + \operatorname{sign}(\alpha)x^2 A(y)$.

Assume now that we have a tubular surface

$$A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3)$$

such that $ABC \neq 0$, $B/A = k \in \mathbb{R}$ is constant and that $(\phi_1(t), \phi_2(t))$ is a real parametrization of (a component of) $C(y) \pm x^2 A(y)$. We want to prove that it is swung. Consider the profile curve $(0, \phi_1(t), \phi_2(t))$. We have to construct a sliding curve adapted to the tubular surface.

Consider the conic $x_1^2 + kx_2^2 = \pm 1$. This will be our trajectory curve; note that $k \neq 0$. Let $(\psi_1(s), \psi_2(s))$ be a parametrization of the conic (we will see below that the parametrization can be taken over $\mathbb{R}$), we have to prove that $(\phi_1(t)\psi_1(s),\ \phi_1(t)\psi_2(s),\ \phi_2(t))$ parametrizes the surface. But

$$A(\phi_2(t))\phi_1(t)^2\psi_1(s)^2 + B(\phi_2(t))\phi_1(t)^2\psi_2(s)^2 + C(\phi_2(t)) =$$

$$A(\phi_2(t))\phi_1(t)^2\left(\psi_1(s)^2 + k\psi_2(s)^2\right) + C(\phi_2(t)) = \pm A(\phi_2(t))\phi_1(t)^2 + C(\phi_2(t)) = 0.$$

It only remains to prove that the corresponding conic $x_1^2+kx_2^2 = \pm 1$ is real, from where it follows that the parametrization $(\psi_1(s), \psi_2(s))$ can always be taken over $\mathbb{R}$. For this purpose, we distinguish several cases. We first observe that, since $\gcd(A, B, C) = \gcd(A, C) = 1$, if $C(y) \pm x^2 A(y)$ factors then it has two factors and they are linear in x, and hence both rational. Let $\mathcal{C}^{\pm}$ be the curve defined by $C(y) \pm x^2 A(y)$. Furthermore, we note that $\mathcal{C}^+$ (resp. a component of it) is rational (over $\mathbb{C}$) iff $\mathcal{C}^-$ (resp. a component of it) is rational (over $\mathbb{C}$).

(i) Let $\mathcal{C}^+$ (or a component of it) be parametrizable over $\mathbb{R}$. Then we have to parametrize $x_1^2 + kx_2^2 = 1$ that is always real, independently of the sign of k.

(ii) Let $\mathcal{C}^+$ (nor a component of it) not be parametrizable over $\mathbb{R}$. Then, by hypothesis, $\mathcal{C}^-$ (or a component of it) is parametrizable over $\mathbb{R}$. In this case, we have to parametrize $x_1^2+kx_2^2 = -1$. We prove that $k < 0$ and, so, the conic real. Let us assume that $k > 0$. No component of $\mathcal{C}^+$ is a real curve. Therefore, the curve $\mathcal{C}^+$ cannot have a real regular point. On the other hand, the tubular surface, that is defined by $A(y)(x_1^2 + kx_2^2) + C(y)$, is a real surface. Therefore, it contains a regular real point $P = (\alpha, \beta, \gamma)$. So,

$$A(\gamma)(\alpha^2 + k\beta^2) + C(\gamma) = 0 \tag{2}$$

Observe that $A(\gamma) \neq 0$, since otherwise $C(\gamma) = 0$ and $\gcd(A, B, C) \neq 1$ which is a contradiction. Now, since P is regular, we have that either $\alpha A(\gamma) \neq 0$ or $k\beta A(\gamma) \neq 0$ or $A'(\gamma)(\alpha^2 + k\beta^2) + C'(\gamma) \neq 0$. That is (note that $k \neq 0$) either $\alpha \neq 0$ or $\beta \neq 0$ or $A'(\gamma)(\alpha^2 + k\beta^2) + C'(\gamma) \neq 0$. In addition, since $A(\gamma) \neq 0$ we have that

$$Q := \left(\pm\sqrt{-\frac{C(\gamma)}{A(\gamma)}}, \gamma\right) \in \mathcal{C}^+.$$

We analyze each case.

- Let $\alpha \neq 0$. We observe that $C(\gamma) \neq 0$ because: if $C(\gamma) = 0$, since $A(\gamma) \neq 0$, by (2), one has that $\alpha^2 + k\beta^2 = 0$ but this is impossible because $\alpha \neq 0$ and $k > 0$. But this implies that the square of the partial derivative w.r.t. x of $C(y)+x^2A(y)$ at Q is $-4C(\gamma)A(\gamma) \neq 0$. Thus Q is a regular point of $\mathcal{C}^+$. Therefore, Q cannot be real. So $C(\gamma)A(\gamma) > 0$. But, from (2), we get then that $\alpha^2 + k\beta^2 < 0$ which is impossible since $\alpha \neq 0$ and $k > 0$.

  - If $\beta \neq 0$ the reasoning is above.
  - Let $A'(\gamma)(\alpha^2 + k\beta^2) + C'(\gamma) \neq 0$. Because of the two previous cases, we can assume w.l.o.g. that $\alpha = \beta = 0$. So, by (2), we have that $C(\gamma) = 0$. So $Q = (0, \gamma)$. Then, the derivative w.r.t. y of $C(y) + x^2 A(y)$ at Q is $C'(\gamma)$. Therefore, since Q is real, we have that $C'(\gamma) = 0$ which contradicts our assumption.

□

Remark 2.4. Assume $A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0$ is a tubular surface with a swung parametrization. Then by the previous result, $B(x_3)/A(x_3) = k \in \mathbb{R}$. So, since C is not zero (see Lemma 2.2), all non-degenerated sections with the family of planes $\{x_3 = c\}$ will yield conics of the same type, either they are all ellipses or all hyperbolas. When $A = 0$ or $B = 0$, sections with $x_3 = c$ degenerate to a pair of lines.

Lemma 2.1 and Theorem 2.3 show how to check whether a rational tubular surface is swung. Moreover, their proofs provide a method to find a swung parametrization of a swung tubular surface. More precisely, one has the following algorithm.

(Parametrization) Algorithm Tubular/Swung

Input: let $A(x_3)x_1^2 + B(x_3)x_2^2 + C(x_3) = 0, C \neq 0, \gcd(A, B, C) = 1$, be the implicit equation of a rational tubular surface $\mathcal{S}$.
Output: decision on whether $\mathcal{S}$ admits a swung parametrization or not. If so, a swung parametrization of $\mathcal{S}$ is obtained.

1. If $A = 0$, compute a real parametrization (M, N) of $B(x_3)x_2^2 + C(x_3) = 0$ (see [3]), and return $(M(t)s, M(t), N(t))$ as parametrization of the surface.
2. If $B = 0$, compute a real parametrization (M, N) of $A(x_3)x_1^2 + C(x_3) = 0$ (see [3]), and return $(M(t), M(t)s, N(t))$ as a parametrization of the surface.
3. Else (i.e. $AB \neq 0$)
   (a) Compute $k = B/A$. If k is not constant then return that $\mathcal{S}$ is not swung.
   (b) [Profile curve computation] Apply algorithm in [3] to compute a real parametrization $(\psi_1(t), \psi_2(t))$ of (a factor of) one of the curves $C \pm x^2 A$; if no component is parametrizable over $\mathbb{R}$ then return that $\mathcal{S}$ is not swung. Take $\epsilon = 0$ if $(\phi_1(t), \phi_2(t))$ parametrizes (a factor of) $C + x^2 A$ and $\epsilon = 1$ if parametrizes (a factor of) $C - x^2 A$.
   (c) [Trajectory curve computation] Compute a real parametrization $(\psi_1(s), \psi_2(s))$ of the conic $x^2 + ky^2 = (-1)^\epsilon$.

3. From swung to tubular

We now work the other way around, given a swung surface, detect if it is tubular. Of course, one could just implicitize the surface and check if the implicit equation corresponds to a tubular surface, but we will provide a characterization based on the profile and trajectory curves, providing insight and not simply blind, costly, resultant or Gröbner bases implicitization algorithms. See Example 4.1.

Theorem 3.1. *Let $(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$ be a parametrization of a swung surface different from the planes $x_1 = 0$, $x_2 = 0$. The surface is tubular if and only if*

1. *The trajectory curve $(\psi_1(s), \psi_2(s))$ is either a conic in normal form $x_1^2/a + x_2^2/b - 1 = 0$ or a line of the form $x_1 = \lambda$ or $x_2 = \lambda$, with $\lambda \neq 0$.*
2. *There exists a rational function h such that $\phi_1(t)^2 = h(\phi_2(t))$.*

Proof. Assume that the surface is tubular. There exists A, B, C such that

$$A(\phi_2)\phi_1^2\psi_1^2 + B(\phi_2)\phi_1^2\psi_2^2 + C(\phi_2) = 0$$

We distinguish cases. Let $AB \neq 0$. By Theorem 2.3, the surface is tubular and swung, so $B(x_3) = kA(x_3)$. Evaluating at a value $t = t_0$, we get that $(\psi_1(s), \psi_2(s))$ parametrizes the curve $x_1^2/a + kx_2^2/a - 1 = 0$, where $a = -C(\phi_2(t_0))/(A(\phi_2(t_0))\phi_1(t_0)^2)$. Now, manipulating the equation we get

$$\begin{aligned} 0 &= A(\phi_2)\phi_1^2\psi_1^2 + B(\phi_2)\phi_1^2\psi_2^2 + C(\phi_2) = \\ &= A(\phi_2)\phi_1^2(\psi_1^2 + k\psi_2^2) + C(\phi_2) = \\ &= A(\phi_2)\phi_1^2 a + C(\phi_2) \end{aligned}$$

So, $\phi_1^2 = -C(\phi_2)/(aA(\phi_2)) = h(\phi_2)$.

Let $A = 0$, then $BC \neq 0$. Evaluating at a value $t = t_0$, we get that $B(\phi_2(t_0))\phi_1(t_0)^2\psi_2(s)^2 + C(\phi_2(t_0)) = 0$. So $\psi_2(s)$ is constant, say λ, and the trajectory curve is the line $x_2 = \lambda$; note that, since ψ_2 is real then $\lambda \in \mathbb{R}$ and since the surface is not $x_2 = 0$ then $\lambda \neq 0$. Moreover, $\phi_1^2 = -C(\phi_2)/(\lambda^2 B(\phi_2)) = h(\phi_2)$. If $B = 0$ the reasoning is similar.

Conversely, let us assume first that (ψ_1, ψ_2) parametrizes a conic $x_1^2/a + x_2^2/b - 1$ and that $\phi_1^2 = C(\phi_2)/A(\phi_2)$ for some $a, b \in \mathbb{R}$, $C, A \in \mathbb{R}[x_3]$, with $\gcd(A, C) = 1$ and $C \neq 0$ (note that ϕ_1 cannot be zero). We want to prove that the given surface is tubular. Consider the equation

$$A(x_3)x_1^2 + a/bA(x_3)x_2^2 - aC(x_3) = 0$$

Substituting the parametrization, we get

$$A(\phi_2)\phi_1^2(\psi_1^2 + a/b\psi_2^2) + aC(\phi_2) = aA(\phi_2)\phi_1^2 - aC(\phi_2) = 0$$

so the surface is tubular. Note that, by construction $\gcd(A, a/bA, C) = 1$ and the total degree w.r.t. $\{x_1, x_2\}$ is 2 because C is no zero. Now, let us assume that (ψ_1, ψ_2) parametrizes a line $x_2 = \lambda$, and that $\phi_1^2 = C(\phi_2)/B(\phi_2)$ for some

$\lambda \in \mathbb{R} \setminus \{0\}$, $C, B \in \mathbb{R}[x_3]$, with $\gcd(B, C) = 1$ and $C \neq 0$. Then, the surface is the tubular surface of equation

$$\frac{1}{\lambda^2} B(x_3) x_2^2 - C(x_3) = 0.$$

If the trajectory curve is a line of the type $x_1 = \lambda$, with $\lambda \neq 0$, the reasoning is similar.

□

Remark 3.2. In order to compute polynomials A and C such that $\phi_1^2 = C(\phi_2)/A(\phi_2)$, we may use rational decomposition techniques [1]. In particular if $n = \max\{\deg(\text{numer}(\phi_2)), \deg(\text{denom}(\phi_2))\}$, $m = \max\{\deg(\text{numer}(\phi_1)), \deg(\text{denom}(\phi_1))\}$, then the degree of A and C is bounded by $2m/n$. If $2m/n$ is not an integer then there is no solution and the surface is not tubular. If $2m/n$ is an integer, we may take A and C as polynomials of degree $2m/n$ with undetermined coefficients, then evaluate the expression $\phi_1(t)^2 C(\phi_2(t)) = A(\phi_2(t))$ at $4m/n+2$ values of t where $C(\phi_2(t)) \neq 0$ and, finally compute the coefficients of A and C by solving the resulting linear system of equations.

Using the last argument in the previous remark, we get the following corollaries of Theorem 3.1.

Corollary 3.3. *If the (implicit) profile curve of a swung surface has degree bigger than 2 w.r.t. the first variable, then it is not tubular.*

Using Theorem 4.2.1. in [5], one has the next result

Corollary 3.4. *Let $(0, \phi_1, \phi_2)$, with $\phi_2 \neq 0$, be a proper parametrization of the profile curve of a swung surface. If $\deg(\phi_2) > 2$ then the surface is not tubular.*

We finish the section with an algorithm that decides whether a swung surface is tubular and, in the affirmative case, computes the implicit equation.

<u>(Implicitization) Algorithm Swung/Tubular</u>

<u>Input</u>: Let $(\phi_1(t)\psi_1(s), \phi_1(t)\psi_2(s), \phi_2(t))$ be a parametrization of a swung surface $\mathcal{S}$ different from the planes $x_1 = 0$, $x_2 = 0$.
<u>Output</u>: decision on whether $\mathcal{S}$ is tubular or not. If $\mathcal{S}$ is tubular the implicit equation is also obtained

1. Check whether $(\psi_1(s), \psi_2(s))$ is in one of the following cases
   (i) it is a conic in normal form $x_1^2/a + x_2^2/b - 1 = 0$.
   (ii) it is a line of the form $x_1 = \lambda$, with $\lambda \neq 0$.
   (iii) it is a line of the form $x_2 = \lambda$, with $\lambda \neq 0$.

   If the answer is no, then return that $\mathcal{S}$ is not tubular.
2. Use Remark 3.2 to check whether there exists a rational function $h = C/A$ with $\gcd(A, C) = 1$, $C \neq 0$, such that $\phi_1(t)^2 = h(\phi_2(t))$. If the answer is yes, then return

(a) If in Step 1, (i) holds then return that $\mathcal{S}$ is tubular and that $A(x_3)x_1^2 + \frac{a}{b}A(x_3)x_2^2 - aC(x_3)$ is its implicit equation.

(b) If in Step 1, (ii) holds then return that $\mathcal{S}$ is tubular and that $\frac{1}{\lambda^2}A(x_3)x_1^2 - C(x_3)$ is its implicit equation.

(c) If in Step 1, (iii) holds then return that $\mathcal{S}$ is tubular and that $\frac{1}{\lambda^2}A(x_3)x_2^2 - C(x_3)$ is its implicit equation.

If the answer is no, then return that $\mathcal{S}$ is not tubular else return that $\mathcal{S}$ is tubular.

4. Examples

We illustrate our results by some examples.

Example 4.1. We consider the swung surface (see Fig. 1)

$$\left(4\,\frac{\left(t^2+t+1\right)\left(s^2-1\right)}{\left(t^3+2\right)\left(s^2+1\right)}, 18\,\frac{\left(t^2+t+1\right)s}{\left(t^3+2\right)\left(s^2+1\right)}, t\right).$$

Without any computation, it is elementary to see that the trajectory curve

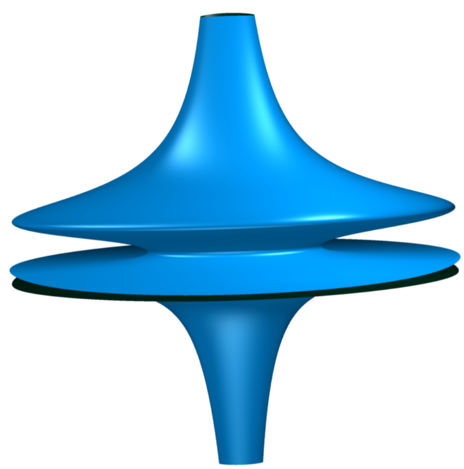

Figure 1: Tubular, swung, surface in Example 4.1

is the ellipse $(1/16)x^2 + (1/81)y^2 - 1$ (step 1.i from the above algorithm); so $a = 16, b = 81$, and $h = \left(z^2+z+1\right)^2/\left(z^3+2\right)^2$ (step 2). Therefore the surface is tubular and its implicit equation is (step 2.a)

$$\left({x_3}^3+2\right)^2{x_1}^2 + \frac{16}{81}\left({x_3}^3+2\right)^2{x_2}^2 - 16\left({x_3}^2+x_3+1\right)^2$$

Example 4.2. Consider the swung surface defined by (see Fig. 2)

$$\left(t^2\frac{s^2-1}{s^2+1}, t^2\frac{2s}{s^2+1}, t^3\right).$$

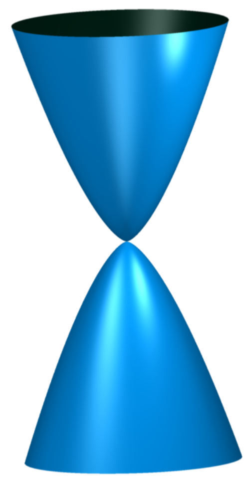

Figure 2: Non-Tubular, swung, surface in Example 4.2

Since t^4 cannot be expressed as $h(t^3)$, the surface is not tubular (step 2). Observe also that the profile curve is $(0, t^2, t^3)$ that is proper and the degree of ϕ_2 is $3 > 2$ (see Corollary 3.4). However, for any value of $t = t_0$, excluding 0, the curve

$$\left(t_0^2 \frac{s^2-1}{s^2+1}, t_0^2 \frac{2s}{s^2+1}, t_0^3\right)$$

is a circle. The implicit equation of the surface is $x_1^6+3x_1^4x_2^2+3x_1^2x_2^4+x_2^6-x_3^4 = 0$. We may express this polynomial as

$$\left(x_1^2 + x_2^2 - \sqrt[3]{x_3^4}\right)\left(x_1^2 + x_2^2 + \frac{1-i\sqrt{3}}{2}\sqrt[3]{x_3^4}\right)\left(x_1^2 + x_2^2 + \frac{1+i\sqrt{3}}{2}\sqrt[3]{x_3^4}\right)$$

and notice that the pencil of conics $\left(x_1^2 + x_2^2 - \sqrt[3]{x_3^4}\right)$ is not rational.

Example 4.3. $F \equiv (-36)x^2 + (-32z)xy + (4z^2 - 100)y^2 + (16z^2 + 144)x + (-4z^3 + 164z)y + z^4 - 82z^2 + 81 = 0$. This surface (See Fig. 3) is a pencil of conics and, can be transformed into a tubular surface. In fact, let us take the following $\mathbb{R}(z)$-change of variables $x_1 = x + (\frac{4}{9}z)y - \frac{2}{9}z^2 - 2$, $y_1 = y - 1/2z$. Then F is transformed into $F^* \equiv -36x_1^2 + (100/9z^2 - 100)y_1^2 + (-25z^2 + 225) = 0$ (see Fig. 4 left). This surface is not in the hypotheses of Theorem 2.3, so it is tubular but not swung (step 3.a of the Tubular/Swung algorithm). Now, consider the new change of variables $x_2 = 1/x_1, y_2 = y_1/x_1$, we get the surface $F^{**} = (-25z^2 + 225)x_2^2 + (100/9z^2 - 100)y_2^2 - 36$ (Fig. 4 right), where $A = (-25z^2 + 255)$, $B = (100/9z^2 - 100)$, $C = -36$. This tubular surface verifies the hypotheses of Theorem 3.1, since $A/B = -9/4 = k$ (step 3.a) and the curve $A(y)x^2 + C = -25x^2y^2 + 225x^2 - 36$ is parametrizable by the profile curve (step

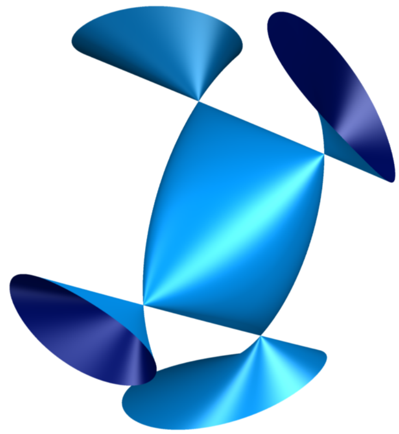

Figure 3: Surface F in Example 4.3

3.b)):

$$\left(\frac{2}{5}\frac{t^2+1}{t^2-1}, \frac{6t}{t^2+1}\right).$$

Following that theorem, the trajectory curve is $x_2^2 - 4/9y_2^2 - 1 = 0$ that we can parametrize as $\left(\frac{s^2+1}{s^2-1}, \frac{3s}{s^2-1}\right)$ (step 3.c). This provides the following parametrization of the surface

$$\begin{cases} x_2 &= \frac{2}{5}\frac{t^2+1}{t^2-1}\cdot\frac{s^2+1}{s^2-1} \\ y_2 &= \frac{2}{5}\frac{t^2+1}{t^2-1}\cdot\frac{3s}{s^2-1} \\ z &= \frac{6t}{t^2+1} \end{cases}$$

Finally, reverting the change of variables, we get the following parametrization of the original surface

$$\begin{cases} X &= -\frac{4}{9}\frac{\phi_2\psi_2}{\psi_1} + \frac{1}{\phi_1\psi_1} + 2 \\ Y &= \frac{1}{2}\phi_2 + \frac{\psi_2}{\psi_1} \\ Z &= \phi_2 \end{cases}$$

with $\phi_1 = \frac{2}{5}\frac{t^2+1}{t^2-1}$, $\phi_2 = \frac{6t}{t^2+1}$, $\psi_1 = \frac{s^2+1}{s^2-1}$, $\psi_2 = \frac{3s}{s^2-1}$.

$$\begin{cases} X &= \frac{1}{2}\frac{\cdot(3ts-t+s-3)\cdot(3ts+t-s-3)}{(s^2+1)\cdot(t^2+1)} \\ Y &= 3\frac{\cdot(t+s)\cdot(ts+1)}{(s^2+1)\cdot(t^2+1)} \\ Z &= \frac{6t}{t^2+1} \end{cases}$$

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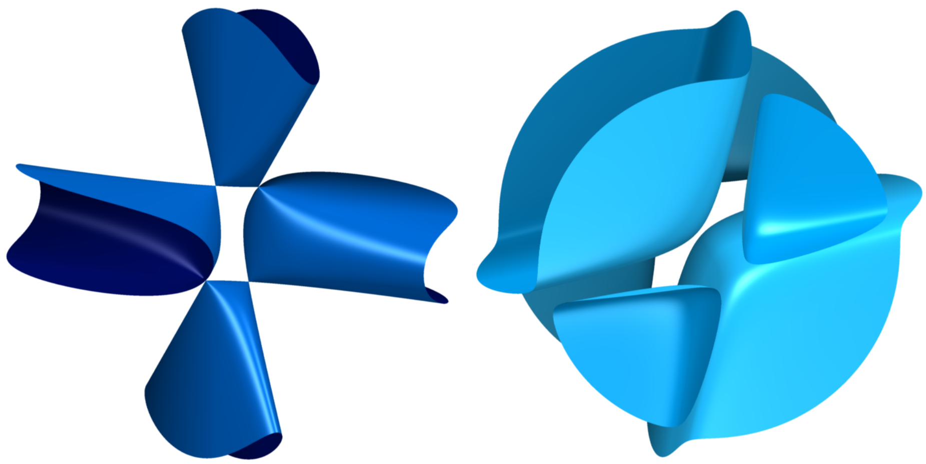

Figure 4: Left: Tubular, non-swung, surface F^* in Example 4.3. Right: Tubular, swung, surface F^{**} in Example 4.3

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