A canonical form for the continuous piecewise polynomial functions

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Abstract

We present in this paper a canonical form for the elements in the ring of continuous piecewise polynomial functions. This new representation is based on the use of a particular class of functions

$$\{C_i(P) : P \in \mathbb{Q}[x], i = 0, \dots, \deg(P)\}$$

defined by

$$C_i(P)(x) = \begin{cases} 0 & \text{if} \quad x \leq \alpha \\ P(x) & \text{if} \quad x \geq \alpha \end{cases}$$

where α is the i-th real root of the polynomial P. These functions will allow us to represent and manipulate easily every continuous piecewise polynomial function through the use of the corresponding canonical form.

It will be also shown how to produce a "rational" representation of each function $C_i(P)$ allowing its evaluation by performing only operations in $\mathbb{Q}$ and avoiding the use of any real algebraic number.

Keywords: Continuous piecewise polynomial functions; Pierce-Birkhoff conjecture; Canonical form for functions; Conversion algorithms.

1 Introduction

The aim of this paper is to give a canonical representation for the elements in the ring of the continuous piecewise polynomial functions. While general piecewise polynomial functions are interesting in general, most applications of them to CAGD require the functions to be continuous. In fact, splines are, by definition, sufficiently smooth piecewise-defined polynomial functions. This, then, includes the special cases b-splines and NURBS. Since some important families of curves are continuous piecewise defined polynomials, it seems useful to have a specifically defined representation for them that can take advantage of such continuity.

In [11] von Mohrenschildt proposed a normal form, for piecewise polynomial functions, by means of the *step* functions

$$\text{step}(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

which are discontinuous. In [4], Chicurel-Uziel used characteristic functions of semilines to introduce a very natural form, with the same discontinuity issue. Furthermore, Carette [3] has worked on a canonical form for piecewise defined functions using range partitions.

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We are interested, however, in representing canonically the continuous piecewise polynomial functions but based on a collection of continuous functions, which should prevent errors to grow out of control when evaluating on approximate numbers. Such a suitable set of continuous functions was introduced in [6] and [9]. They define, for every non-negative integer i, a mapping C_i from the set of polynomials on the set of continuous piecewise polynomial functions, such that

$$C_i(P)(x) = \begin{cases} 0 & \text{if} \quad x \leq \alpha \\ P(x) & \text{if} \quad x \geq \alpha \end{cases}$$

where α is the i-th distinct real root of the polynomial P. If i is bigger than the number of real roots of P then $C_i(P)$ is defined as 0.

In [6] and [9] the $C_i(P)$ functions were used to study the Pierce–Birkhoff conjecture. This is a well–known and classical open problem in Real Algebraic Geometry asking if every continuous and piecewise polynomial function $h\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ defined over $\mathbb{Q}$ can be represented by means of a sup–inf expression over a finite set of polynomials with rational coefficients. This conjecture has been proved in the affirmative sense only for $n = 1$ and $n = 2$ (see [6] and [9]) and remains still open for $n \geq 3$, while results in [2] and [10] lead to a proof in certain polyhedral domains.

In this paper we show that they provide a canonical representation which is easily computable from the piecewise expression of the functions. Moreover performing algebraic operations between canonical forms of continuous piecewise polynomial functions is simple and fast. In section 2 we give the complete definition of the C_i functions along with some of their properties. Section 3 is devoted to the proof of existence and uniqueness of our canonical form. In section 4 we show how to obtain easily the canonical form for the sum, product and composition of continuous piecewise polynomial functions. Section 5 shows how to produce a "rational" representation of each function $C_i(P)$ allowing its evaluation by performing only operations in $\mathbb{Q}$ and avoiding the use of any real algebraic number. Before the conclusions, Section 6 attacks some complexity aspects of the canonical form and the operations.

2 Preliminaries

Let us denote by $\mathcal{CP}(\mathbb{Q}[x])$ the set of continuous piecewise polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ defined by polynomials with rational coefficients.

In order to represent canonically the continuous piecewise polynomial functions, we will use the set of mappings

$$C_i : \mathbb{Q}[x] \to \mathcal{CP}(\mathbb{Q}[x]),$$

$i \in \mathbb{N}$, presented in [6] and [9].

Definition 2.1 *Let $P(x) \in \mathbb{Q}[x]\backslash\{0\}$, $\deg(P) = n$, $\{\alpha_1, \ldots, \alpha_r\}$ the set of real roots of P, $\alpha_0 = -\infty$, $\alpha_k = +\infty$ for every $k > r$. Then, for every $i \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}$,*

$$C_i(P)(x) = \begin{cases} 0 & \textit{if} \quad x \leq \alpha_i \\ P(x) & \textit{if} \quad x \geq \alpha_i. \end{cases}$$

For completeness, we also define $C_i(P) = 0$ when $P = 0$.

The following result can be found in [6] and [9] as basis for the proof of Pierce-Birkhoff conjecture for the case one and two dimensional. It gives a natural representation of continuous piecewise polynomial functions in terms of the C_i functions.

Proposition 2.1 *Let ϕ be a continuous piecewise polynomial function*

$$\phi(x) = \begin{cases} Q_1(x) & \textit{if} \quad x \leq \alpha_1 \\ Q_2(x) & \textit{if} \quad \alpha_1 \leq x \leq \alpha_2 \\ \vdots & \\ Q_N(x) & \textit{if} \quad \alpha_{N-1} \leq x \end{cases}$$

with $Q_i \in \mathbb{Q}[x]$, $Q_i \neq Q_{i+1}$ *for all* i, *and* $\alpha_j \in \mathbb{R}$. *Then* ϕ *can be written in the following way:*

$$\phi(x) = Q_1(x) + \sum_{i=1}^{N-1} C_{s(i)}(\Delta_i)(x)$$

where $\Delta_i = Q_{i+1} - Q_i$ *and* $s(i)$ *is the position index of* α_i *as a root of* Δ_i.

Proof For every $i \in \{1, \ldots, N-1\}$ we define the polynomial in $\mathbb{Q}[x]$ given by:

$$\Delta_i(x) = Q_{i+1}(x) - Q_i(x).$$

The continuity of ϕ implies that every α_i is a real root of $\Delta_i(x)$. Let $s(i)$ be the position index of α_i as root of $\Delta_i(x)$. In these conditions:

$$\phi = Q_1 + \sum_{i=1}^{N-1} C_{s(i)}(\Delta_i)$$

as desired. □

Example 2.1 *Let* ψ *be defined by*

$$\psi(x) = \begin{cases} x^6 + 1 & \text{if} \quad x \leq 1 \\ x^4 - \frac{1}{2}x^3 - \frac{7}{2}x^2 - x + 6 & \text{if} \quad 1 \leq x \leq \sqrt{2} \\ x^4 + x^3 - 5x^2 - 4x + 9 & \text{if} \quad \sqrt{2} \leq x \end{cases}$$

In this case:

$$\Delta_1(x) = -(x-1)\left(x^5 + x^4 + \frac{1}{2}x^2 + 4x + 5\right)$$

and

$$\Delta_2(x) = \frac{3}{2}(x-1)\left(x^2 - 2\right).$$

Since 1 *is the second real root of* $\Delta_1(x)$ *and* $\sqrt{2}$ *the third real root of* $\Delta_2(x)$, *we can write*

$$\psi(x) = x^6 + 1 + C_2(\Delta_1(x)) + C_3(\Delta_2(x)).$$

Proposition 2.1 along with the following properties of C_i functions will allow us to give a canonical representation of the elements of $\mathcal{CP}(\mathbb{Q}[x])$.

Proposition 2.2 *Let* $P(x), Q(x)$ *and* $H(x) \in \mathbb{Q}[x]$, $\alpha_1 < \ldots < \alpha_r$ *and* $\beta_1 < \ldots < \beta_s$ *the real roots of* P *and* Q *respectively. Then*

i) *If* $\alpha_i \geq \beta_j$ *then* $C_i(P)C_j(Q) = QC_i(P)$.

ii) $C_i(P^k) = P^{k-1}C_i(P)$.

iii) *If* $P = QH$ *and* $\alpha_i = \beta_k$ *then* $C_i(P) = HC_k(Q)$.

The proof of the proposition is straightforward.

Example 2.2 *Consider the function* ψ *in Example 2.1 written in the form*

$$\psi(x) = x^6 + 1 + C_2(\Delta_1(x)) + C_3(\Delta_2(x)).$$

According to Proposition 2.2,

$$C_2(\Delta_1(x)) = -\left(x^5 + x^4 + \frac{1}{2}x^2 + 4x + 5\right) C_1(x-1)$$

and

$$C_3(\Delta_2(x)) = \frac{3}{2}(x-1)\, C_2\left(x^2 - 2\right).$$

Therefore the pieceswise polynomial function can be written as

$$\psi(x) = x^6 + 1 - (x^5 + x^4 + \frac{1}{2}x^2 + 4x + 5)C_1(x-1) + \frac{3}{2}(x-1)C_2(x^2-2).$$

3 The Canonical Form

The following theorem proves the existence and uniqueness of a canonical representation for the continuous piecewise polynomial functions in terms of the functions C_i and polynomials:

Theorem 3.1 *Let ϕ: $\mathbb{R} \to \mathbb{R}$ be a continuous piecewise polynomial function defined by polynomials in $\mathbb{Q}[x]$. Then ϕ can be written uniquely in the form*

$$\phi = F_0 + \sum_{i=1}^{N} F_i C_{u_i}(P_i)$$

where $F_0 \in \mathbb{Q}[x]$, $F_i \in \mathbb{Q}[x] \setminus \{0\}$, $P_i \in \mathbb{Q}[x] \setminus \{0\}$ is monic, irreducible, with at least one real root and, for every $i \in \{1, \ldots, N\}$, the pairs (P_i, u_i) are different.

Proof Let ϕ be a continuous piecewise polynomial function

$$\phi(x) = \begin{cases} Q_1(x) & \text{if} \quad x \le \alpha_1 \\ Q_2(x) & \text{if} \quad \alpha_1 \le x \le \alpha_2 \\ & \vdots \\ Q_M(x) & \text{if} \quad \alpha_{M-1} \le x \end{cases} \tag{1}$$

with $Q_i \in \mathbb{Q}[x]$, $Q_i \neq Q_{i+1}$ for all i, and $\alpha_j \in \mathbb{R}$.

By Proposition 2.1 ϕ can be written as

$$\phi(x) = Q_1(x) + \sum_{i=1}^{M-1} C_{s(i)}(\Delta_i)(x)$$

where $\Delta_i = Q_{i+1} - Q_i$ and $s(i)$ is the position index of α_i as a root of Δ_i.

Writing the decomposition of Δ_i into irreducible factors,

$$\Delta_i = a_i f_{i_1}^{e_{i_1}} \ldots f_{i_r}^{e_{i_r}}$$

with e_{i_j} $(1 \le j \le r)$ positive integers, $a_i \in \mathbb{Q}$ and $f_{i_1}, \ldots, f_{i_r} \in \mathbb{Q}[x]$ distinct monic irreducible polynomials.

The $s(i)$–th real root of Δ_i is the u_i–th real root of f_{i_j} for some $1 \le j \le r$. By Proposition 2.2,

$$C_{s(i)}(\Delta_i) = a_i f_{i_1}^{e_{i_1}} \ldots f_{i_{j-1}}^{e_{i_{j-1}}} f_{i_{j+1}}^{e_{i_{j+1}}} \ldots f_{i_r}^{e_{i_r}} C_{u_i}(f_{i_j}^{e_{i_j}}) = a_i f_{i_1}^{e_{i_1}} \ldots f_{i_{j-1}}^{e_{i_{j-1}}} f_{i_{j+1}}^{e_{i_{j+1}}} \ldots f_{i_r}^{e_{i_r}} f_{i_j}^{e_{i_j}-1} C_{u_i}(f_{i_j})$$

Taking $N = M - 1$, $F_0 = Q_1$, $P_i = f_{i_j}$ and

$$F_i = a_i f_{i_1}^{e_{i_1}} \ldots f_{i_{j-1}}^{e_{i_{j-1}}} f_{i_{j+1}}^{e_{i_{j+1}}} \ldots f_{i_r}^{e_{i_r}} f_{i_j}^{e_{i_j}-1}$$

it is obtained

$$\phi(x) = F_0(x) + \sum_{i=1}^{N} F_i(x) C_{u_i}(P_i)(x). \tag{2}$$

If $k < l$ and $P_k = P_l$ then α_k and α_l are real roots of the same irreducible polynomial P_k. Since $\alpha_k < \alpha_l$ then $u_k < u_l$.

It remains to prove the uniqueness of this expression. Let us assume that there exist two representations of ϕ:

$$\phi = F_0 + \sum_{i=1}^{N} F_i C_{u_i}(P_i) \tag{3}$$

$$\phi = G_0 + \sum_{i=1}^{M} G_i C_{v_i}(R_i) \tag{4}$$

Let α_i be the u_i–th real root of P_i and β_i the v_i–th real root of R_i. We can assume, without loss of generality, that $\alpha_1 < \alpha_2 < \ldots < \alpha_N$ and $\beta_1 < \beta_2 < \ldots < \beta_M$. Let us denote $\theta_i = \min\{\alpha_i, \beta_i\}$.

Applying expressions (1) and (2) to $-\infty \leq x \leq \theta_1$, $F_0(x) = \phi(x) = G_0(x)$. Since F_0 and G_0 are polynomials, then

$$F_0 = G_0.$$

Let us assume now, without loss of generality, that $\alpha_1 \leq \beta_1$. If it were $\alpha_1 < \beta_1$, since $\alpha_1 < \alpha_2$, the open interval $I = (\alpha_1, \min\{\alpha_2, \beta_1\})$ would not be empty and for every $x \in I$:

$$\begin{array}{lcl} (3) & \Rightarrow & \phi(x) = F_0(x) + F_1(x)P_1(x) \\ (4) & \Rightarrow & \phi(x) = F_0(x) \end{array}$$

Thus $F_1(x)P_1(x) = 0$ but F_1 and P_1 have only finitely many roots. Therefore, $\alpha_1 = \beta_1$. Since α_1 and β_1 are roots respectively of P_1 and R_1, both monic irreducible polynomials,

$$P_1 = R_1 \text{ and } u_1 = v_1.$$

Now, if $\alpha_1 < x < \theta_2$ then

$$\begin{array}{lcl} (3) & \Rightarrow & \phi(x) = F_0(x) + F_1(x)P_1(x) \\ (4) & \Rightarrow & \phi(x) = F_0(x) + G_1(x)R_1(x). \end{array}$$

Therefore $F_1(x)P_1(x) = G_1(x)R_1(x)$. Since $P_1 = R_1$ and it has only finitely many roots, we can conclude that

$$F_1 = G_1$$

Repeating the same argument for $i = 2, \ldots, \min\{N, M\}$, it is proved that $F_i = G_i$, $P_i = H_i$, $u_i = v_i$ and $N = M$, as desired. □

We will call canonical form the expression obtained in the preceding theorem.

Example 3.1 Let us consider the following continuous piecewise polynomial function

$$\phi(x) = \begin{cases} x^4 + 4x^3 - 2x^2 & \text{if} \quad x \leq \alpha \\ x^4 + x^3 - 2x^2 - 3x - 3 & \text{if} \quad \alpha \leq x \leq \sqrt{3} \\ 2x^3 + x^2 - 6x - 3 & \text{if} \quad \sqrt{3} \leq x \end{cases}$$

where α is the unique real root of $x^3 + x + 1$. In this case

$$\Delta_1(x) = -3(x^3 + x + 1)$$

and

$$\Delta_2(x) = -x(x-1)(x^2-3).$$

The only real root of Δ_1 is α, while Δ_2 has roots $-\sqrt{3} < 0 < 1 < \sqrt{3}$ so that $\sqrt{3}$ is the fourth real root of Δ_2. According to Proposition 2.1, $\phi(x)$ is equal to

$$x^4 + 4x^3 - 2x^2 + C_1(-3(x^3 + x + 1)) + C_4(-x(x-1)(x^2-3))$$

Following the proof of Theorem 3.1 the canonical form of $\phi(x)$ is obtained:

$$x^4 + 4x^3 - 2x^2 - 3C_1(x^3 + x + 1) - x(x-1)C_2(x^2 - 3).$$

4 Using the canonical form for sums, products and compositions

In this section it is shown how to determine the canonical form for the sum, the product and the composition of continuous piecewise polynomial functions given in canonical form. We also give an explanation on how things work for differentiation and integration. Operations with piecewise polynomial functions can be useful for computer aided geometric modeling in the sense of piecewise polynomially defined movements (or deformations) of piecewise polynomially defined objects.

The canonical form of the sum is immediately obtained from the canonical form of the summands. The product needs only to take into account one of the properties appearing in Proposition 2.2. Composition requires considering certain real roots of the given polynomials and applying the corresponding rules for sums and products.

Let ϕ and ψ be continuous piecewise polynomial functions in canonical form, i.e.

$$\phi = F_0 + \sum_{i=1}^{N} F_i C_{u_i}(P_i)$$

$$\psi = G_0 + \sum_{j=1}^{M} G_j C_{v_j}(R_j)$$

4.1 Sums and products

If the pairs $(P_i, u_i),\ 1 \leq i \leq N,\ (R_j, v_j),\ 1 \leq j \leq M$ are all different then the canonical form of $\phi + \psi$ is clearly

$$(F_0 + G_0) + \sum_{i=1}^{N} F_i C_{u_i}(P_i) + \sum_{j=1}^{M} G_j C_{v_j}(R_j).$$

Otherwise, it will be enough to sum up the terms $F_i C_{u_i}(P_i), R_j C_{v_j}(R_j)$ such that $(P_i, u_i) = (R_j, v_j)$ in order to obtain the canonical form.

Concerning the product $\phi\psi$, we only need to obtain the canonical form of the terms of type

$$F_i C_{u_i}(P_i) G_j C_{v_j}(R_j) = F_i G_j C_{u_i}(P_i) C_{v_j}(R_j).$$

We can assume, without loss of generality, that the u_i–th real root of P_i is greater than or equal to the v_j–th real root of R_j. Then, by Proposition 2.2

$$F_i G_j C_{u_i}(P_i) C_{v_j}(R_j) = F_i G_j R_j C_{u_i}(P_i).$$

Example 4.1 *Let us determine the canonical form for the product of ϕ and ψ, the functions considered in examples 3.1 and 2.1 respectively. We compute the product of the canonical form of ϕ and ψ term by term, taking into account that*

$$C_1(x^3 + x + 1) C_1(x - 1) = (x^3 + x + 1) C_1(x - 1),$$

since the only real root of $x^3 + x + 1$ is smaller than 1. By the same reasoning,

$$C_1(x^3 + x + 1) C_2(x^2 - 2) = (x^3 + x + 1) C_2(x^2 - 2),$$

$$C_2(x^2 - 3) C_1(x - 1) = (x - 1) C_2(x^2 - 3)$$

and

$$C_2(x^2 - 3) C_2(x^2 - 2) = (x^2 - 2) C_2(x^2 - 3).$$

Therefore, the canonical form of $\phi\psi(x)$ is

$$H_0 + H_1 C_1(x^3 + x + 1) + H_2 C_1(x - 1) + H_3 C_2(x^2 - 2) + H_4 C_2(x^2 - 3)$$

where

$$H_0 = x^{10} + 4\,x^9 - 2\,x^8 + x^4 + 4\,x^3 - 2\,x^2,$$

$$H_1 = -3\,x^6 - 3,$$

$$H_2 = -x^9 - 2\,x^8 + x^7 + \frac{9}{2}x^6 + \frac{3}{2}x^5 - 5\,x^4 + \frac{9}{2}x^3 + \frac{47}{2}x^2 + 27\,x + 15,$$

$$H_3 = 3/2\,x^5 - 9/2\,x^3 - 3/2\,x^2 + 9/2$$

and

$$H_4 = -x^6 + 6\,x^4 - x^3 - 13\,x^2 + 9\,x.$$

4.2 Compositions

We are interested in determining the canonical form of $\phi \circ \psi$. Since we have already seen how to compute the canonical form of sums and products, the only remaining problem concerning the composition is the computation of the canonical form of the expressions

$$C_{v_j}(R_j)(\phi(x)) = \begin{cases} 0 & \text{if} \quad \phi(x) \leq \alpha_j \\ R_{v_j}(\phi(x)) & \text{if} \quad \phi(x) \geq \alpha_j \end{cases}$$

where α_j is the v_j–th real root of R_j.

First we must determine the values of x for which $\phi(x) \geq \alpha_j$. Since ϕ is a continuous function, it is enough to find the points where ϕ takes the value α_j and then give the sign of $\phi(x) - \alpha_j$ in each interval. The points where $\phi(x)$ is equal to α_j are real roots of $R_j \circ \phi$. Therefore, we can determine the real roots of $R_j \circ \phi$, which is a piecewise polynomial function, and take those roots $\gamma_1, \ldots, \gamma_s$ such that $\phi(\gamma_k) = \alpha_j$. Let $\gamma_0 = -\infty$, $\gamma_{s+1} = \infty$. Thus,

$$C_{v_j}(R_j)(\phi(x)) = \begin{cases} 0 & \text{if} \quad x \in I_1 \\ R_{v_j}(\phi(x)) & \text{if} \quad x \in I_2, \end{cases}$$

where I_1 is the union of the intervals (γ_i, γ_{i+1}) such that $\phi(x) \leq \alpha_j$ and $I_2 = \mathbb{R} \setminus I_1$.

Partitioning I_2 by the intervals of definition of ϕ, we obtain the usual form of a piecewise polynomial function and, by Theorem 3.1, the canonical form is easily computable.

Example 4.2 Let us compute the canonical form of $C_2(x^2-2)(\phi(x))$, being ϕ the function defined in Example 3.1. First we must determine the values of x for which $\phi(x) \leq \sqrt{2}$. To this purpose, we have computed the roots γ of the function

$$(x^2-2)\circ\phi(x) = \begin{cases} (x^2-2)\circ(x^4+4x^3-2x^2) & \text{if} \quad x \leq \alpha \\ (x^2-2)\circ(x^4+x^3-2x^2-3x-3) & \text{if} \quad \alpha \leq x \leq \sqrt{3} \\ (x^2-2)\circ(2x^3+x^2-6x-3) & \text{if} \quad \sqrt{3} \leq x \end{cases}$$

such that $\phi(\gamma) = \sqrt{2}$.

The roots of $(x^2-2)\circ\phi(x)$ are:

- The roots of $(x^2-2)\circ(x^4+4x^3-2x^2)$ in $(-\infty, \alpha]$,
- The roots of $(x^2-2)\circ(x^4+x^3-2x^2-3x-3)$ in $[\alpha, \sqrt{3})$, and
- The roots of $(x^2-2)\circ(2x^3+x^2-6x-3)$ in $[\sqrt{3}, \infty)$.

One can compare α and the roots of these three polynomials:

- The first and second real roots of $(x^2-2)\circ(x^4+4x^3-2x^2)$ are less than α. Applying ϕ gives $\sqrt{2}$ for the first one and $-\sqrt{2}$ for the second one.
- The third real root of $(x^2-2)\circ(x^4+x^3-2x^2-3x-3)$ is the only one greater than α and less than $\sqrt{3}$, but its image by ϕ is $-\sqrt{2}$.
- The sixth real root of $(x^2-2)\circ(2x^3+x^2-6x-3)$ is the only one greater than $\sqrt{3}$, and its image by ϕ is $\sqrt{2}$.

Therefore, γ_1, the first real root of $(x^2-2)\circ(x^4+4x^3-2x^2)$, and γ_2, the sixth real root of $(x^2-2)\circ(2x^3+x^2-6x-3)$ are the algebraic numbers we looked for. Then we compute that

$$C_2(x^2-2)(\phi(x)) = \begin{cases} (x^2-2)\circ\phi(x) & \text{if} \quad x \leq \gamma_1 \\ 0 & \text{if} \quad \gamma_1 \leq x \leq \gamma_2 \\ (x^2-2)\circ\phi(x) & \text{if} \quad \gamma_2 \leq x \end{cases}$$

Since $\gamma_1 \leq \alpha \leq \sqrt{3} \leq \gamma_2$, we have that

$$C_2(x^2-2)(\phi(x)) = \begin{cases} (x^2-2)\circ(x^4+4x^3-2x^2) & \text{if} \quad x \leq \gamma_1 \\ 0 & \text{if} \quad \gamma_1 \leq x \leq \gamma_2 \\ (x^2-2)\circ(2x^3+x^2-6x-3) & \text{if} \quad \gamma_2 \leq x \end{cases}$$

Now applying Theorem 3.1 we obtain the canonical form

$$C_2(x^2-2)(\phi(x)) = S_0 - C_1(S_0) + C_6(S_2),$$

where

$$S_0 = (x^2-2)\circ(x^4+4x^3-2x^2) = x^8+8\,x^7+12\,x^6-16\,x^5+4\,x^4-2$$

and

$$S_2 = (x^2-2)\circ(2x^3+x^2-6x-3) = 4\,x^6+4\,x^5-23\,x^4-24\,x^3+30\,x^2+36\,x+7,$$

both irreducible polynomials.

4.3 A note on differentiation and integration

One could think of integration and differentiation as interesting operations on continuous functions. While the derivative of a continuous piecewise polynomial function does, in general, not exist at the endpoints of the defining intervals, one can easily determine when this case happens.

Lemma 4.1 *Let us consider a continuous piecesewise polynomial function ϕ in its canonical form*

$$\phi = F_0 + \sum_{i=1}^{N} F_i C_{u_i}(P_i).$$

The derivative ϕ' is continuous if and only if P_i divides F_i for every $i=1,...,N$. In this case,

$$\phi' = F_0' + \sum_{i=1}^{N}\left(F_i' + \frac{F_i}{P_i}P_i'\right)C_{u_i}(P_i).$$

Proof Let us call α_i the $u_i - th$ real root of P_i. The function ϕ is differentiable at the point α_i if and only if the left and right derivatives are equal at α_i, i.e.

$$\left(F_0 + \sum_{j=1}^{i-1} F_j P_j\right)'(\alpha_i) = \left(F_0 + \sum_{j=1}^{i} F_j P_j\right)'(\alpha_i).$$

This equality holds true if and only if $(F_iP_i)'(\alpha_i) = 0$. Since $F_iP_i(\alpha_i) = 0$, we have that α_i is a multiple root of F_iP_i. Taking into account that P_i is irreducible and has α_i as a root, we conclude that ϕ is differentiable at α_i if and only if P_i divides F_i in $\mathbb{Q}[x]$. When P_i divides F_i for every $i=1,...,N$, the derivative ϕ' is clearly continuous and, noticing that

$$C_{v_i}\left((F_iP_i)'\right) = \left(F_i' + \frac{F_i}{P_i}P_i'\right)C_{u_i}(P_i),$$

where v_i is the position of α_i as real root of $(F_iP_i)'$, we have that

$$\phi' = F_0' + \sum_{i=1}^{N}\left(F_i' + \frac{F_i}{P_i}P_i'\right)C_{u_i}(P_i).$$

□

Regarding integration, it has been treated for piecewise functions in [8] and [7]. However, while the primitive of a continuous piecewise polynomial function remains continuous, the field of definition of the polynomial must, in general, be extended. Think, for example of the function $\psi = 3C_2(x^2-2)$, i.e.

$$\psi(x) = \begin{cases} 0 & \text{if} \quad x \le \sqrt{2} \\ 3x^2-6 & \text{if} \quad x \ge \sqrt{2} \end{cases}$$

Then any of its continuous primitives has the shape:

$$\phi(x) = \begin{cases} a - 4\sqrt{2} & \text{if} \quad x \le \sqrt{2}, \\ x^3-6x+a & \text{if} \quad x \ge \sqrt{2}, \end{cases}$$

where $a \in \mathbb{R}$. It is obvious that at least one of the two pieces of F is defined by a polynomial that is not in $\mathbb{Q}[x]$, so $\phi \notin \mathcal{CP}(\mathbb{Q}[x])$.

5 The evaluation of $C_i(P)$

In this section we introduce two methods of evaluation of $C_i(P)$.

5.1 The simplest alternative

Perhaps the most immediate way to evaluate $C_i(P)$ could be the following one. We have a_i and b_i from the moment of the definition of $C_i(P)$. Then:

- If $x < a_i$, then $C_i(P)(x) = 0$.
- If $x > b_i$, then $C_i(P)(x) = P(x)$.
- If $x \in [a_i, b_i]$, since P is irreducible and monic, it is easy to check that the sign of P in α_i is $\text{sign}_{a_i} = (-1)^{i+\deg(P)-1}$ and the sign in b_i is $\text{sign}_{b_i} = (-1)^{i+\deg(P)}$. So, if $\text{sign}(P(x)) = \text{sign}_{b_i}$, then $C_i(P)(x) = P(x)$. Otherwise $C_i(P(x)) = 0$.

5.2 A closed form solution

While the former method seems quite simple and fast, it is worth introducing a different one with the appeal of a closed formula and gives a constructive approach to Pierce–Birkhoff conjecture.

In this subsection we show that every $C_i(P)$ can be evaluated at any rational (or real value) with just a rational isolating interval $[a_i, b_i]$ of the i–th real root α_i of P (i.e. $\alpha_{i-1} < a_i \le \alpha_i \le b_i < \alpha_{i+1}$, $a_i, b_i \in \mathbb{Q}$). Observe that, if α_i is rational (and then P is linear), the evaluation problem is trivial, so we will suppose this is not the case and then we have an isolating interval of positive length. For that we show that each function $C_i(P)(x)$ has a rational expression in terms of the polynomial P, its absolute value $|P|$ and very simple piecewise polynomial functions in the canonical basis whose defining intervals have rational endpoints. In fact, we prove that each function $C_i(P)(x)$ is a semipolynomial over $\mathbb{Q}$, i.e. a function that can be described as the composition of polynomials in $\mathbb{Q}[x]$ and the absolute value function $|\bullet|$. This representation provides a new algorithm for evaluating $C_i(P)$ that avoids the inherent exponential complexity of the algorithm in [6] or [9] . Anyway it is still open if the algorithm presented in this section has or not a polynomial complexity while the first performed experiments show a very good practical behaviour.

The procedure to express $C_i(P)$ as a semipolynomial is based on a hint a referee gave to us: Given an isolating interval, the expression

$$C_i(P)(x) = \begin{cases} 0 & \text{if } x \le a_i \\ 0 & \text{if } a_i \le x \le b_i \text{ and } signP(x) = signP(a_i) \\ P(x) & \text{if } a_i \le x \le b_i \text{ and } signP(x) = signP(b_i) \\ P(x) & \text{if } x \ge b_i \end{cases}$$

allows to easily evaluate $C_i(P)$.

To obtain the expression of $C_i(P)$ as a semipolynomial, let us denote $n = \deg(P)$, $\{\alpha_1, \ldots, \alpha_r\} = \{\alpha \in \mathbb{R} : P(\alpha) = 0\}$, $\alpha_0 = -\infty$ and $\alpha_{r+1} = +\infty$. Firstly, we only need to study the case where P is squarefree because

$$\widetilde{P} = \frac{P}{\gcd(P, P^{(1)})} \quad \Longrightarrow \quad C_i(P) = \gcd(P, P^{(1)}) C_i(\widetilde{P})$$

Also we can assume that $i \in \{1, \ldots, r\}$ because

$$C_0(P) = P \qquad \text{and} \qquad C_{r+1}(P) = 0.$$

Now we define the following semipolynomials:

$$[P]^+ = \sup\{P, 0\} = \frac{P + |P|}{2},$$

$$[P]^- = \inf\{P, 0\} = \frac{P - |P|}{2}.$$

In the easiest case: $a_i = \alpha_i$ or $b_i = \alpha_i$ (just checked by computing $P(a_i)$ and $P(b_i)$), we know α_i, which is rational and

$$P(x) = (x - \alpha_i)Q(x).$$

In this case, $C_i(P)(x) = [x - \alpha_i]^+ Q(x)$.

In general, we have $a_i < \alpha_i < b_i$ so that we can choose any $a' \in [a_i, \alpha_i)$ and $b' \in (\alpha_i, b_i]$. Let us define the semipolynomials

$$f(x) := \left[\frac{x - a_i}{a' - a_i}\right]^+ - \left[\frac{x - a'}{a' - a_i}\right]^+ = \begin{cases} 0 & \text{if } x < a_i \\ \dfrac{x - a_i}{a' - a_i} & \text{if } a_i \le x \le a' \\ 1 & \text{if } a' < x \end{cases}$$

$$g(x) := \left[\frac{x - b'}{b_i - b'}\right]^+ - \left[\frac{x - b_i}{b_i - b'}\right]^+ = \begin{cases} 0 & \text{if } x < b' \\ \dfrac{x - b'}{b_i - b'} & \text{if } b' \le x \le b_i \\ 1 & \text{if } b_i < x \end{cases}$$

which are continuous piecewise polynomial functions with canonical forms

$$f(x) = \frac{1}{a' - a_i}\left(C_1(x - a_i) - C_1(x - a')\right)$$

$$g(x) = \frac{1}{b_i - b'}\left(C_1(x - b') - C_1(x - b_i)\right)$$

We will use $[P]^+, [P]^-, f$ and g to prove the following result:

Theorem 5.1 *Let P be a squarefree polynomial in $\mathbb{Q}[x]$ of degree n with r real roots $\alpha_1 < ... < \alpha_r$. Let a_i, a', b_i and b' be real numbers such that $\alpha_{i-1} < a_i < a' < \alpha_i < b' < b_i < \alpha_{i+1}$. Then*

$$C_i(P)(x) = \frac{1}{b_i - b'}[P]^{signP(a_i)}(x)\left(C_1(x - b') - C_1(x - b_i)\right) + \frac{1}{a' - a_i}[P]^{signP(b_i)}(x)\left(C_1(x - a_i) - C_1(x - a')\right)$$

Proof We can write the expression on the right side of the equation as $[P]^{signP(a_i)}(x)g(x)+[P]^{signP(b_i)}(x)f(x)$. Since f and g are equal in $(-\infty, a_i), (b_i, \infty)$ and $[P]^+ + [P]^- = P$, it is obvious that $[P]^{signP(a_i)}g + [P]^{signP(b_i)}f$ is 0 and P respectively in $(-\infty, a_i)$ and (b_i, ∞). It is also clearly 0 at α_i. It remains to study this function in the intervals $[a_i, \alpha_i)$ and $(\alpha_i, b_i]$. Since $signP(x) = signP(a_i)$ for every $x \in [a_i, \alpha_i)$ and $signP(x) = signP(b_i)$ for every $x \in (\alpha_i, b_i]$ we have:

$$\text{if } signP(a_i) = + \text{ then } [P]^+g + [P]^-f = \begin{cases} 0 & \text{in} & [a_i, \alpha_i) \\ P & \text{in} & (\alpha_i, b_i] \end{cases}$$

$$\text{if } signP(a_i) = - \text{ then } [P]^-g + [P]^+f = \begin{cases} 0 & \text{in} & [a_i, \alpha_i) \\ P & \text{in} & (\alpha_i, b_i] \end{cases}$$

We conclude that $[P]^{signP(a_i)}g + [P]^{signP(b_i)}f = C_i(P)$. □

Remark 5.1 *Since the isolating interval of the i-th root is computed when passing a piecewise polynomial function to canonical form, the only extra time will be choosing $a' \in [a_i, \alpha_i)$ and $b' \in (\alpha_i, b_i]$.*

Example 5.1 To compute the above expressions for the functions $\{C_i(P)\}_i$ for $P = x^3 - 3x + 1$, we first realize that $-3 < -2 < \alpha_1 < -1 < 0 < \alpha_2 < \frac{1}{2} < 1 < \alpha_3 < 2 < 3$, where the α_i are the roots of P (just by checking the sign of $P(x)$ for $x = -3, -2, -1, 0, \frac{1}{2}, 1, 2, 3)$. Then by Theorem 5.1, it is obvious that:

$$C_1(P)(x) = [P]^-(x)\left(C_1(x+1) - C_1(x)\right) + [P]^+(x) + \left(C_1(x+3) - C_1(x+2)\right),$$

$$C_2(P)(x) = 2[P]^+(x)\left(C_1\left(x - \frac{1}{2}\right) - C_1(x-1)\right) + [P]^-(x)\left(C_1(x+1) - C_1(x)\right),$$

$$C_3(P)(x) = [P]^-(x)\left(C_1(x-2) - C_1(x-3)\right) + 2[P]^+(x)\left(C_1\left(x - \frac{1}{2}\right) - C_1(x-1)\right).$$

6 On complexity

Remark 6.1 *The canonical form in this paper should be, in general, lighter than the piecewise standard one and those in [11] and [3]. The point is that the classical methods to represent piecewise polynomial functions store the polynomials that define the function in the different intervals (the Q_i in (1)) and the extremes of such intervals (the roots α_j), which could be algebraic numbers (and then their minimal polynomial must also be kept). In this new canonical form, one still has to store one polynomial for every interval (all F_i in (2)) and also one polynomial for every extreme (the P_i). However,*

$$Q_j(x) = F_0(x) + \sum_{i=1}^{j} F_i(x)P_i(x),$$

which is expected to have greater degree than F_i in general. So the representation and storage complexity of the piecewise polynomial functions when considered in canonical form (i.e. storing all F_i and all P_i with the intervals separating α_i from the other roots) should be less than that when considered in standard form, since both the bigger Q_i and the α_i, whose information include P_i and the isolating interval, must be stored.

*As an example, the piecewise polynomial function in Example 2.1, is 145 bytes long in Maple (from the command "*length*") while the canonical form in sage returns 72 for Python's command "*getsizeof*".*

When comparing with other forms, the sum with the canonical form introduced in this paper is clearly easier to manage, since we just check equalities of monic polynomials and then we either sum polynomials or add terms to a list.

Given ϕ and ψ piecewise polynomial continuous functions in canonical form with N and M summands each, their product takes $(N-1)(M-1)$ comparisons of algebraic numbers, $2NM$ products of polynomials and $MN - M - N + 1$ sums of polynomials (since, at most, the product of two piecewise functions has as many breakpoints as the two functions together: $N + M - 1$).

Composition $\phi\psi$ involves the resolution of $O(MN)$ polynomial equations (M for each summand of ϕ), a varying number (depending on the number of real roots of the polynomials, N and M) of algebraic number comparisons, up to $O(NM)$ compositions of polynomials (again M for each summand of ϕ). The complexity of the result depends more on the number of real roots of the polynomials we use than on N or M.

The first method of evaluation, when computing $\phi(x)$ needs to compare x with $N-1$ algebraic numbers and then we need to evaluate up to N polynomials and sum them up to N resulting numbers.

Finally, for the second method of evaluation, we can assume that

$$a',\ b',\ \frac{1}{b_i - b'},\ \text{and}\ \frac{1}{a_i - a'}$$

are already computed (observe that a_i and b_i are already in the description of the algebraic number α_i), since it can be done just once for all evaluations (in fact, they can be gotten when the canonical form is computed, and storing them, as rationals, is not too expensive). Then we compute $\phi(x)$ just by evaluating N rational (with the license of taking absolute values) functions and summing the results.

Remark 6.2 *If we want to preserve an order in the summands (for instance we could order them by the order of the roots u_i of the polynomials P_j), we would complicate the sum with some comparisons of algebraic numbers. On the other side, we would remove comparisons from product (reduced to $M + N - 2$ by using the order among the roots) and evaluation (reduced $O(\log_2(N))$ if we use a binary search as in [3]). Moreover, composition would also be simplified if we store the order data of the roots.*

Example 6.1 *The canonical form has been implemented in Sage as a class*[1] *and tested running on an AMD Turion 64×2 at 1,9 GHz with 2GB RAM under Ubuntu 14.04. Given the functions:*

$$\begin{aligned}
\phi(x) &= x^3 - 5 + C_1(x^2-2) + C_2(x^2-2) + (x^3 - 2x + 1)C_1(x^3 - x - 7)\\
\psi(x) &= x^6 + 1 - \left(x^5 + x^4 + \frac{1}{2}x^2 + 4x + 5\right) C_1(x-1) + \left(\frac{3}{2}x - \frac{3}{2}\right) C_2(x^2-2),
\end{aligned}$$

the computer gave the sum in 4.51 ms. It computed the product in 84.6 ms. Finally, the composition took 709 ms and the evaluation of such composition at $5/4$ took 238 µs.

[1] The source code is available at `http://www.mat.ucm.es/~jcaravan/Paquetillos/CPWPF_SAGE02.sage`

7 Conclusions

We have introduced a new canonical form for the elements in the ring of the continuous piecewise polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ defined by polynomials in $\mathbb{Q}[x]$. In algebraic terms, we have shown that this ring agrees with the $\mathbb{Q}[x]$–module generated by the functions:

$$\{C_i(P) : i \in \mathbb{N}, P \in \mathbb{Q}[x] \setminus \{0\} \text{ monic and irreducible}\}.$$

It has been also shown how to use this canonical form in order to perform sums, products and compositions and to obtain the corrresponding canonical form of the result.

We have also presented how to produce a "rational" representation of the $C_i(P)$ functions allowing its evaluation by performing only operations in $\mathbb{Q}$ and avoiding the use of real algebraic numbers.

To finish, all the obtained results and algorithms can be stated in terms of an ordered field $\mathbb{K}$ (instead of $\mathbb{Q}$) and a real–closed field $\mathbb{F}$ (instead of $\mathbb{R}$) containing $\mathbb{K}$. In this case, and when the real–closed $\mathbb{F}$ is not archimedean preventing from using isolating intervals, the manipulation of the involved real algebraic numbers in the proof of Theorem 5.1 must be performed by using Thom's codes (see [1]).

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