Facultad de Ciencias

CLASSIFICATION OF LATTICE 3-POLYTOPES WITH FEW POINTS

Trabajo Fin de Máster
para acceder al

MÁSTER EN MATEMÁTICAS Y COMPUTACIÓN

Autor: Mónica Blanco Gómez

Director: Francisco Santos Leal

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Resumen

El objetivo de este trabajo es comenzar la clasificación de politopos reticulares de dimensión 3 con n puntos reticulares, para n pequeño, módulo equivalencia unimodular. El punto de partida es la demostración de que, aunque el número de clases de equivalencia para cada valor de n es infinito, solamente una cantidad finita de ellas tienen anchura mayor que uno, y la clasificación de las de anchura uno es un problema relativamente fácil.

Para $n = 4$ estamos hablando de *tetraedros vacíos*, cuya clasificación es bastante clásica (White, Howe) y tienen todos anchura 1. Para $n = 5$ demostramos que hay exactamente 9 (clases de) politopos de anchura 2, y ninguno de anchura mayor. Para $n = 6$ demostramos que hay 74 clases de anchura 2, dos clases de anchura 3, y ninguna de anchura mayor.

Nuestra motivación proviene en parte del concepto de *politopos con sumas distintas* (distinct-pair-sum, o dps) estudiado por Reznick. Es sabido que los 3-politopos dps tienen como mucho 8 puntos reticulares. Entre las $9 + 74 + 2$ clases mencionadas más arriba exactamente $9 + 44 + 1$ son dps. Una posible continuación de este trabajo, que requeriría quizá nuevas técnicas, sería continuar la clasificación para $n = 7$ y 8, restringida al caso dps.

Contents

1. Introduction

A *lattice polytope* is the convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice). A polytope is *d-dimensional* if it contains $d+1$ affinely independent points. We call *size* of P its number $P \cap \mathbb{Z}^d$ of lattice points and *volume* of P its volume normalized to the lattice (that is, $d+1$ points form a simplex of volume one if and only if they are an affine lattice basis).

Two such polytopes P and Q are said *$\mathbb{Z}$-equivalent* or *unimodularly equivalent* if there is an affine integer unimodular transformation $f : \mathbb{Z}^d \to \mathbb{Z}^d$ with $f(P) = Q$. We call such a transformation a *$\mathbb{Z}$-equivalence.*

We are interested in the complete classification of such polytopes in dimension 3, for small size.

The same question in dimension 2 has a relatively simple, and fully algorithmic, answer. Once we fix the size n of P, Pick's Theorem implies that

$$\mathrm{vol}(P) = n - 2 + i \leq 2n - 5, \tag{1}$$

where $i \leq n-3$ is the number of lattice points in the interior of P. Also, every configuration contains a *unimodular triangle* which we may identify, without loss of generality, with the *standard unimodular triangle* $\{(0,0),(1,0),(0,1)\}$. These two things together implies $A \subset \mathrm{conv}(\{(-2n+6,-2n+6),(-2n+6,4n-11),(4n-11,-2n+6)\}$. This implies that the classification can a priori be done by simply considering all the possible subsets of n points in this finite set.

By a simple constructive approach it is easy to compute the full list of 2-dimensional polygons with up to five points, which we show in Figure 1.

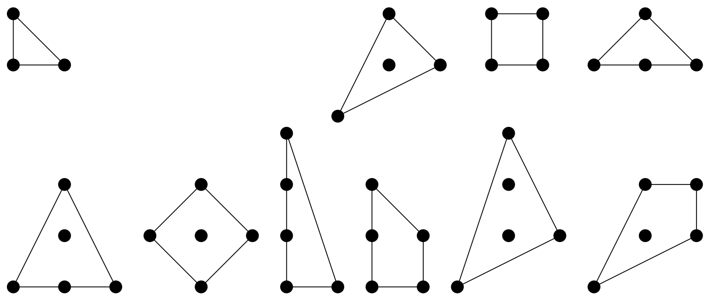


Figure 1. The 2-dimensional polygons with up to five points

In dimension 3 the situation is completely different. In particular, there are *empty tetrahedra* with arbitrarily large volume, so no analogue of Pick's Theorem is possible. Here, an *empty tetrahedron* is the same as a 3-polytope of size four: a tetrahedron with integer vertices and no other integer point.

However, the following three results are valid in arbitrary dimension. In the first one, a *hollow* lattice polytope is one with no lattice point in its interior.

Theorem 1.1 (Nill-Ziegler [7, Thm. 1.2])**.** *In each dimension d, there is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope.*

Theorem 1.2 (Hensley [4, Thm. 3.4])**.** *For fixed positive integers k and d there is a number $V(k,d)$ such that every lattice d-polytope with k interior lattice points has volume bounded above by $V(k,d)$.*

Theorem 1.3 (Lagarias-Ziegler [6, Thm. 2])**.** *A family of lattice d-polytopes (for any fixed d) with bounded volume contains only a finite number of integral equivalence classes.*

Since there is a unique hollow 2-polytope of width larger than one, we get the following in dimension 3, where the *width* of a lattice polytope is the minimum of $\max_{x\in P} f(x) - \min_{x\in P} f(x)$, among all possible (non-constant) choices of an integer linear functional $f : \mathbb{Z}^d \to \mathbb{Z}$. In particular, P has *width one* if its vertices lie in two consecutive parallel lattice hyperplanes.

Corollary 1.4. *There are finitely many lattice 3-polytopes of width greater than one for each size n.*

Proof. Once we fix n, every lattice 3-polytope P with n lattice points falls in one of the following (not mutually exclusive) categories:

- It is not hollow. In this case it has a positive, but bounded by n, number of interior points, so Theorem 1.2 implies a bound for its volume. This, in turn, implies a finite number of possibilities via Theorem 1.3.

- It is hollow, but does not project to a hollow 2-polytope. These are a finite family, by Theorem 1.1.
- It is hollow, and it projects to a 2-polytope of width 1. This implies that P itself has width 1.
- It is hollow, and it projects to a hollow 2-polytope of width larger than one. The only such 2-polytope is the second dilation of a unimodular triangle. It is easy to check that only finitely many (equivalence classes) of 3-polytopes of size n project to it:

  Let $P = \mathrm{conv}\{p_1, ..., p_n\}$ be a 3-polytope of size n that projects onto $T = \mathrm{conv}\{(0,0),(2,0),(0,2)\}$, with $T \cap \mathbb{Z}^3 = \{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$.

  We must have at least one point projecting to each vertex of T. That is: there are $p_1 = (0,0,z_1)$, $p_2 = (2,0,z_2)$ and $p_3 = (0,2,z_3)$ in P. The unimodular transformation
$$(x,y,z) \mapsto \left(x, y, z - z_1 - x\left\lfloor \frac{z_2 - z_1}{2}\right\rfloor - y\left\lfloor\frac{z_3 - z_1}{2}\right\rfloor\right)$$
  allows us to assume that $z_1, z_2, z_3 \in \{0,1\}$. This implies that $P \subset T \times [1-n, n]$, so there are a finite number of possibilities for P.

□

So, it makes sense to classify, for each size n, separately the 3-polytopes of width one and those of width larger than one. Those of width larger than one are a finite list. Those of width one are infinite, but easy to describe: they consist of two 2-polytopes of sizes n_1 and n_2 ($n_1+n_2 = n$) placed on parallel consecutive planes (without loss of generality, the planes $z = 0$ and $z = 1$). For each of the two sub configurations there is a finite number of possibilities, but infinitely many ways to "rotate" (in the integer sense, that is via an element of $SL(\mathbb{Z}, 2)$) one with respect to the other.

For example, it is a now classical result that all empty tetrahedra have width one. From this, the classification of empty tetrahedra (stated in Theorem 2.3 below) follows easily.

Theorem 1.5 (White [12, Thm. 1])**.** *Every lattice 3-polytope of size four has width one with respect to (at least) one of its three pairs of opposite edges.*

The following generalization of this fact is very useful to us:

Theorem 1.6 (Howe, see [11, Thm. 1.3])**.** *Every lattice 3-polytope with no lattice points other than its vertices has width 1. In particular, all maximal 3-polytopes with that property consist of two empty parallelograms in consecutive parallel lattice planes.*

Our motivation comes partially from the notion of *distinct pair-sum* lattice polytopes, defined as lattice polytopes in which all the pairwise sums $a+b, a, b \in P \cap \mathbb{Z}^d$ are distinct. Equivalently, they are lattice polytopes containing no three collinear lattice points nor the vertices of a non degenerate parallelogram [2, Lemma 1]. A dps d-polytope cannot have two lattice points in the same class modulo $(2\mathbb{Z})^d$. In particular, it cannot have more than 2^d lattice points. Reznick [9] asks:

- What is the range for the volume of dps polytopes of size 2^d in R^d?
- Is every dps d-polytope a subset of one of size 2^d?
- How many "inequivalent" dps polytopes of size 2^d are there in R^d?

In dimension 2 there are only two dps polytopes: a unimodular triangle, and a triangle of volume three with a unique interior point, and only the second one is maximal.

Our ultimate goal would be to answer these questions in dimension three.

The results we have achieved are:

Theorem 1.7 (Corollary 3.10)**.** *There are exactly 9 3-polytopes of size 5 and width > 1, all of width 2. They are all dps (See Table 1).*

Theorem 1.8 (Theorems 4.1–4.6)**.** *There are exactly 76 3-polytopes of size 6 and width > 1, 74 of width 2 and 2 of width 3. 44 and 1 of those, respectively, are dps (See Tables 2–10)*

In future work we hope to at least classify the dps polytopes of sizes 7 and 8. Since maximal polytopes in dimension three have 8 lattice points, this would complete the classification of dps 3-dimensional lattice polytopes, which should help us answer, among others, the question presented in [2, page 6]:

- Is every dps 3-polytope a subset of a dps 3-polytope of size 8?

Acknowledgment: We thank Bruce Reznick for useful comments and references on the topic of this work.

2. Preliminaries

2.1. **Empty simplices.** Lattices 3-polytopes of size four, that is, empty tetrahedra, are well classified (see, e.g., [11, 8]). To review the classification we take Theorem 1.5 as a starting point. That is, the fact that every empty tetrahedron has width one with respect to a functional that is constant in two opposite edges.

Let T be an empty tetrahedron, of a certain volume q. For our purposes, rather than thinking of T as having vertices in $\mathbb{Z}^3$ that span a sublattice of index q, we think of T as the standard tetrahedron, with vertices $o = (0,0,0)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$, but lying in a superlattice Λ of $\mathbb{Z}^3$ with $\Lambda : \mathbb{Z}^3 = q$. This is obtained by an affine change of coordinates. Clearly, two empty tetrahedra that produce the same lattice are $\mathbb{Z}$-equivalent.

Also, without loss of generality, suppose that T has width one with respect to the edges oe_3 and e_1e_2, that is, with respect to the functional $x + y$. In these conditions, we have that

$$\Lambda \subset \{(x, y, z) \in \mathbb{R}^3 : x + y \in \mathbb{Z}\}.$$

Let $H_i := \{(x, y, z) \in \mathbb{R}^3 : x + y = i\}$, for each $i \in \mathbb{Z}$. The different slices $\Lambda \cap H_i$ are obtained from one another by integer translation, so understanding one of them is enough to understand Λ. We look at the slice

$$\Lambda_0 := \Lambda \cap H_0 = \{(x, y, z) \in \Lambda : x + y = 0\}.$$

Λ_0 is a superlattice of $\mathbb{Z}^3 \cap H_0$ with index q. In particular, the rectangle

$$R := \mathrm{conv}\{(0,0,0), (0,0,1), (1,-1,0), (1,-1,1)\},$$

which is a fundamental rectangle (that is, of lattice area 2) with respect to $\mathbb{Z}^3 \cap H_0$ has area $2q$ with respect to Λ_0. Moreover, Λ_0 contains no non-integer points in the integer vertical and horizontal lattice lines of $\mathbb{Z}^3 \cap H_0$, (in particular, on the edges of R), since each primitive integer segment along these lines is a lattice translation of $0e_3$ or e_1e_2. Thus, by Pick's Theorem (1), R contains exactly $q - 1$ lattice points of Λ_0 in its interior. Finally, no two of these points can have one coordinate the same, because then we would have a horizontal or vertical lattice segment in Λ_0 of length smaller than one, in contradiction to the fact that oe_3 and e_1e_2 are primitive in Λ. As a conclusion:

Lemma 2.1. *For each $i = 1, \dots, q-1$, R contains exactly one point of Λ_0 with $z = i/q$ and one (which may or may not be the same) with $x = -y = i/q$.*

As a consequence, knowing the value of $p \in \{1, \dots, q-1\}$ for which $(p/q, -p/q, 1/q) \in \Lambda_0$ or the value $p' \in \{1, \dots, q-1\}$ for which $(1/q, -1/q, p'/q) \in \Lambda_0$ is enough to recover Λ_0, and Λ. More precisely we will have:

$$\Lambda = (p/q, -p/q, 1/q) + \mathbb{Z}^3 = (1/q, -1/q, p'/q) + \mathbb{Z}^3.$$

Observe that these two values are not independent. Indeed, if $(a/q, -a/q, b/q)$ lies in Λ_0 then $pb \equiv a \pmod{q}$ so, in particular, p and p' are inverses modulo q (which also implies $\gcd(p, q) = 1$, but that was already necessary in order not to have any non-integer lattice points in vertical lines).

Definition 2.2. *We call an empty lattice simplex T of type $T(p, q)$, $1 \le p < q$, if it has volume q and there is an affine transformation sending its vertices to $o = (0,0,0)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and sending the lattice $\mathbb{Z}^3$ to the lattice*

$$\Lambda(p, q) := (p/q, -p/q, 1/q) + \mathbb{Z}^3.$$

We call $R(p, q) := \mathrm{conv}\{(0,0,0), (0,0,1), (1,-1,0), (1,-1,1)\}$ the *fundamental rectangle* of $T(p, q)$. Observe that all non-integer lattice points of $\Lambda(p, q)$ lie in an integer translation of $R(p, q)$.

Theorem 2.3 (Classification of empty tetrahedra, White [12], Howe [11])**.** *Every empty lattice simplex of volume $q \ge 2$ is of type $T(p, q)$ for some $p \in \{1, q-1\}$ with $\gcd(p, q) = 1$. Moreover, if $p' \equiv p^{-1} \pmod{q}$, we have*

$$T(p, q) \cong T(p', q) \cong T(q-p, q) \cong T(q-p', q).$$

Remark 2.4. *The choice of representative chosen in [10] for tetrahedron $T(p, q)$ is:*

$$T(p, q) = \mathrm{conv}\{(0,0,0), (1,0,0), (0,0,1), (p, q, 1)\}$$

which has width one with respect to the functional z.

If the tetrahedron is unimodular, i.e., if $q = 1$, then we choose $p = 0$.

Figure 2 shows the lattice Λ_0 in the fundamental square R, for all the equivalence classes of lattice polytopes with $q < 8$. Observe how we go from $T(p, q)$ to $T(q-p, q)$ by reflection with vertical mirror, and form $T(p, q)$ to $T(p^{-1}, q)$ by reflection with diagonal mirror.

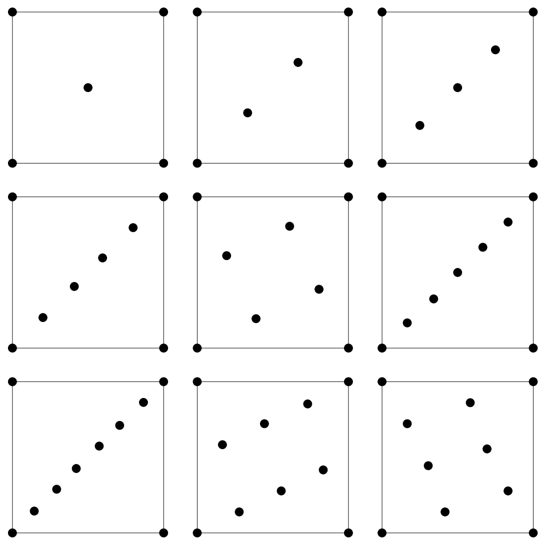


FIGURE 2. Classification of empty tetrahedra of volume up to seven. Only values of $p \leq q/2$ are shown; $T(q-p, q)$ is obtained from $T(p, q)$ by reflection with vertical mirror. Observe that $T(2, 7) \cong T(3, 7)$ by 90 degree rotation, expressing the fact that $3 \cdot 2 \equiv -1 \pmod 7$

Simplices of type $T(1, q)$ are somehow special: they have all lattice points in the diagonal of the fundamental rectangle, and they have width one with respect to two different pairs of opposite edges (with our choice of coordinates they have width one not only with respect to the functional $x + y$, but also with respect to $y + z$, that is, with respect to the edges oe_1 and e_2e_3). They are called *tetragonal* in [10].

$T(2, 1)$ is even more special: it has width one with respect to any of the three pairs of opposite edges.

For future reference we include the following statement which can be read as "no vertex of an empty tetrahedron is more special than the others".

Lemma 2.5. *Let $T = \mathrm{conv}\{o, e_1, e_2, e_3\}$ be the standard simplex. Let $\Lambda(p, q)$ be the lattice of type (p, q), for some $1 \leq p < q$. Then, for every $i \in \{1, 2, 3\}$ there is a $\mathbb{Z}$-automorphism f_i of T sending $e_i \to o$ and sending $\Lambda(p, q)$ either to itself or to $\Lambda(p', q)$, where $p' \equiv p^{-1} \pmod q$.*

Proof.

- $i = 1$: The affine transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{f_1} \begin{pmatrix} 1-(x+y+z) \\ z \\ y \end{pmatrix}$$

exchanges $(0, 0, 0) \leftrightarrow (1, 0, 0)$ and $(0, 1, 0) \leftrightarrow (0, 0, 1)$, and maps $(p/q, -p/q, 1/q)$ to $(1 - 1/q, 1/q, -p/q)$. Then

$$\begin{aligned} f_1(\Lambda(p, q)) &= f_1(\mathbb{Z}^3 + (p/q, -p/q, 1/q)) = \mathbb{Z}^3 + (1 - 1/q, 1/q, -p/q) = \\ &= \mathbb{Z}^3 + (1/q, -1/q, p/q) = \mathbb{Z}^3 + (p'/q, -p'/q, 1/q) = \Lambda(p', q). \end{aligned}$$

- $i = 2$: The affine transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{f_2} \begin{pmatrix} z \\ 1-(x+y+z) \\ x \end{pmatrix}$$

exchanges $(0,0,0) \leftrightarrow (0,1,0)$ and $(1,0,0) \leftrightarrow (0,0,1)$, and maps $(p/q, -p/q, 1/q)$ to $(1/q, 1 - 1/q, p/q)$. Then

$$\begin{aligned} f_2(\Lambda(p,q)) &= f_2(\mathbb{Z}^3 + (p/q, -p/q, 1/q)) = \mathbb{Z}^3 + (1/q, 1-1/q, p/q) = \\ &= \mathbb{Z}^3 + (1/q, -1/q, p/q) = \mathbb{Z}^3 + (p'/q, -p'/q, 1/q) = \Lambda(p', q). \end{aligned}$$

- $i = 3$: The affine transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{f_3} \begin{pmatrix} y \\ x \\ 1-(x+y+z) \end{pmatrix}$$

exchanges $(0,0,0) \leftrightarrow (0,0,1)$ and $(1,0,0) \leftrightarrow (0,1,0)$, and maps $(p/q, -p/q, 1/q)$ to $(-p/q, p/q, 1 - 1/q)$. Then

$$\begin{aligned} f_3(\Lambda(p,q)) &= f_3(\mathbb{Z}^3 + (p/q, -p/q, 1/q)) = \mathbb{Z}^3 + (-p/q, p/q, 1-1/q) = \\ &= \mathbb{Z}^3 + (p/q, -p/q, 1/q) = \Lambda(p, q). \end{aligned}$$

□

Remark 2.6. *Let us consider, as in the proof of this lemma, the standard tetrahedron $T = \{o, e_1, e_2, e_3\}$ and the lattice $\Lambda(p,q)$. The transformation f_3 in the proof (exchanging $o \leftrightarrow e_3$ and $e_1 \leftrightarrow e_2$) is the only affine map, other than the identity, sending T to itself and preserving $\Lambda(p,q)$ for every p and q. The other 22 symmetries of T are automorphisms of $\Lambda(p,q)$ only for particular values of p and q.*

This means that the sentence "no vertex of an empty tetrahedron is more special than the others" is not true if we fix a particular class $T(p,q)$ of simplices. If we want to stay within a particular class $T(p,q)$ and in this class $p \not\equiv p^{-1} \pmod q$, then vertices o and e_3 are in one orbit and e_1 and e_2 in another.

Among the methods we have used to elaborate the classification, we often need to check whether certain tetrahedra are empty and to find their type (p,q). We implement a MATLAB program to do this for us (see Appendix A).

Sometimes, we have an infinite list of tetrahedra of which we want to know whether they are empty or not. Doing it computationally is no longer a viable method. A more analytical way is to use one of the necessary conditions for a tetrahedron to be empty: its four facets have to be unimodular triangles, which is usually easy to check. Now, if this happens, then there is an affine transformation sending the first three points to $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$; let (a,b,q) be the image of the fourth point. (Note: the transformation is not unique, but q is unique up to a sign, since it equals the volume of T, and a and b are determined modulo q). Then, our problem reduces to knowing when

$$\operatorname{conv}\{(0,0,0), (1,0,0), (0,1,0), (a,b,q)\}$$

is an empty tetrahedron. The following lemma gives us the answer:

Lemma 2.7. *The lattice tetrahedron $T = \operatorname{conv}\{(0,0,0),\ (1,0,0),\ (0,1,0), (a,b,q)\}$ is empty (with respect to the integer lattice $\mathbb{Z}^3$) if, and only, if, at least one of the following happens:*

(i) $a \equiv 1 \pmod q$ *and* $\gcd(b,q) = 1$.
(ii) $b \equiv 1 \pmod q$ *and* $\gcd(a,q) = 1$.
(iii) $a + b \equiv 0 \pmod q$ *and* $\gcd(a,q) = 1$.

Proof. Assume without loss of generality that $q > 0$.

Observe that T is an empty tetrahedron if, and only if, all its edges are primitive and its width equals one with respect to one of the three pairs of edges. In our case, primitivity of edges is equivalent to

$$\gcd(a,b,q) = \gcd(a-1,b,q) = \gcd(a,b-1,q) = 1 \tag{2}$$

Let us examine the width with respect to the three pairs of edges, depending on the values of a and b:

(i) Width w.r.t. $\overrightarrow{(0,0,0)(0,1,0)}$ and $\overrightarrow{(1,0,0)(a,b,q)}$.
The parallel planes containing those segments have equations $qx + (1-a)z = 0$ and $qx + (1-a)z = q$, respectively. Width one is equivalent to the functional $qx + (1-a)z$ taking only values that are multiples of q, which in turn is equivalent to $a - 1 \equiv 0 \pmod q$. If this happens, the primitivity conditions (Lemma 2) become equivalent to

$$\gcd(b, q) = 1.$$

(ii) Width w.r.t. $\overrightarrow{(0,0,0)(1,0,0)}$ and $\overrightarrow{(0,1,0)(a,b,q)}$.
The parallel planes containing those segments have equations $-qy + (b-1)z = 0$ and $-qy + (b-1)z = -q$, respectively. Width one is equivalent to the functional $-qy + (b-1)z$ taking only values that are multiples of q, which in turn is equivalent to $b - 1 \equiv 0 \pmod q$. If this happens, the primitivity conditions (Lemma 2) become equivalent to

$$\gcd(a, q) = 1.$$

(iii) Width w.r.t. $\overrightarrow{(0,0,0)(a,b,q)}$ and $\overrightarrow{(1,0,0)(0,1,0)}$.
The parallel planes containing those segments have equations $-qx - qy + (a+b)z = 0$ and $-qx - qy + (a+b)z = -q$, respectively. Width one is equivalent to the functional $-qx - qy + (a+b)z$ taking only values that are multiples of q, which in turn is equivalent to $a + b \equiv 0 \pmod q$. If this happens, the primitivity conditions (Lemma 2) become equivalent to

$$\gcd(a, q) = 1.$$

□

2.2. **Volume vectors.** One basic property of $\mathbb{Z}$-equivalence is that it preserves volume. This makes the following definition useful for our classification:

Definition 2.8. *Let $\{p_1, p_2, \dots, p_n\}$, with $n \geq 4$, be the set of lattice points in a 3-polytope P. The* volume vector *of P is the vector*

$$\overline{w} = (w_{i,j,k,l})_{1 \leq i < j < k < l \leq n} \in \mathbb{Z}^{\binom{n}{4}}$$

where

$$w_{i,j,k,l} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_i & p_j & p_k & p_l \end{pmatrix}. \tag{3}$$

Observe that the definition of volume vector implicitly assumes a specific ordering for the n points in P.

Remark 2.9. *The volume vector encodes the unique (modulo a scalar factor) dependence among each set of 5 points $\{p_i, p_j, p_k, p_l, p_r\}$, which is:*

$$\begin{aligned} w_{j,k,l,r} \cdot p_i - w_{i,k,l,r} \cdot p_j + w_{i,j,l,r} \cdot p_k - w_{i,j,k,r} \cdot p_l + w_{i,j,k,l} \cdot p_r &= 0 \\ w_{j,k,l,r} - w_{i,k,l,r} + w_{i,j,l,r} - w_{i,j,k,r} + w_{i,j,k,l} &= 0. \end{aligned}$$

The volume preserving property of $\mathbb{Z}$-equivalences gives us the following:

Lemma 2.10. *Let $P = \operatorname{conv}\{p_1, \dots, p_n\}$ and $Q = \operatorname{conv}\{q_1, \dots, q_n\}$ have respective volume vectors $w_P = (w_j^P)_j$ and $w_Q = (w_j^Q)_j$, $j \in [\binom{n}{4}] = \{1, \cdots, \binom{n}{4}\}$.*
If P and Q are $\mathbb{Z}$ − equivalent, then there exists a permutation σ in $[\binom{n}{4}]$ such that

$$|w_P^j| = |w_Q^{\sigma(j)}|$$

for all $j \in [\binom{n}{4}]$.

Another theorem gives us a sufficient, yet not necessary, condition:

Theorem 2.11. *Let $P = \operatorname{conv}\{p_1, \dots, p_n\}$ and $Q = \operatorname{conv}\{q_1, \dots, q_n\}$ be two n-sized polytopes in $\mathbb{Z}^3$ and suppose they have the same, or opposite, volume vector*

$$\overline{w} = (w_{i,j,k,l})_{1 \leq i < j < k < l \leq n}$$

with respect to the given ordering of the points. Then:

(1) *There is an affine map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(p_i) = q_i$ for all $i = 1, \dots, n$*

(2) $\det(f) = \pm 1$
(3) *If* $\gcd(w_{i,j,k,l})_{1\le i<j<k<l\le n} = 1$ *then* f *has integer coefficients. Hence, it is a* $\mathbb{Z}$*-equivalence between* P *and* Q*.*

Proof. Let us prove parts (1) and (2) by induction on n:

- $\underline{n = 4:}$ Then their volume vector is an integer

$$w = w_{1,2,3,4} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix} = \pm \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix} \neq 0$$

  (1) If both $P = \mathrm{conv}\{p_1, p_2, p_3, p_4\}$ and $Q = \mathrm{conv}\{q_1, q_2, q_3, q_4\}$ are 3-dimensional polytopes, then their sets of lattice points are affine basis of some 3-dimensional lattice in $\mathbb{R}^3$. Then there is an invertible affine map f such that $f(p_i) = q_i$ for $i = 1, 2, 3, 4$. That is, there exist an invertible matrix $M \in \mathbb{Q}_{3\times 3}$ (basis have integer coefficients) and a translation vector $m \in \mathbb{Z}^3$ such that

$$Mp_i + m = q_i \ \ \forall i = 1, 2, 3, 4$$

  (2) Now, the previous argument implies that

$$\begin{pmatrix} 1 & 0 \\ m & M \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}$$

  and since the determinants of the two last matrices represent the volume of the tetrahedra P and Q, which is w for both, up to a -1 factor, then $|\det(f)| = |\det(M)| = 1$.

- $\underline{n > 4}$: Suppose that (1) and (2) hold true for $4 \le k < n$, and let us see it holds for n. The volume vector of P and Q is

$$\overline{w} = (w_{i,j,k,l})_{1\le i<j<k<l\le n}$$

  Without loss of generality, we may assume that $w_{1,2,3,4} \neq 0$. This means that polytopes $P' = \mathrm{conv}\{p_1, \ldots, p_{n-1}\}$ and $Q' = \mathrm{conv}\{q_1, \ldots, q_{n-1}\}$ are 3-dimensional polytopes of size $n-1$, and with volume vector (up to a -1 factor):

$$\overline{w}' = (w_{i,j,k,l})_{1\le i<j<k<l\le n-1}$$

  By induction hypothesis, there exists an affine map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $f(p_i) = q_i$ for all $i = 1, \ldots, n-1$ and with $\det(f) = \pm 1$. It remains to be checked that f maps p_n to q_n.

  Let us now consider the 5-point configurations $P_1 = \mathrm{conv}\{p_1, p_2, p_3, p_4, p_n\}$ and $Q_1 = \mathrm{conv}\{q_1, q_2, q_3, q_4, q_n\}$, and denote

$$v_i = (-1)^{i-1} \cdot w_{\{1,2,3,4,n\}\setminus\{i\}} \ \ for \ \ i = 1, 2, 3, 4, n$$

  Then, since $\overline{w}$ is the volume vector of both P and Q up to a -1 factor, we have the following dependences:

$$\sum_{i=1,\ldots,4} v_i p_i + v_n p_n = 0 \ (*)$$

$$\sum_{i=1,\ldots,4} v_i q_i + v_n q_n = 0 \ (**)$$

And with these two equations it is easy to check that f also maps p_n to q_n:

$$\begin{aligned} f(p_n) &= M \cdot p_n + m \underset{(*)}{=} M \cdot \left[\frac{1}{v_n}(-\sum_{i=1,\dots,4} v_i p_i)\right] + m = \\ &= \frac{1}{v_n}\left[-\sum_{i=1,\dots,4} v_i f(p_i) + m \sum_{i=1,\dots,4} v_i\right] + m = \\ &= \frac{1}{v_n}\left[-\sum_{i=1,\dots,4} v_i q_i - m v_n\right] + m = \\ &= -\frac{1}{v_n}\sum_{i=1,\dots,4} v_i q_i \underset{(**)}{=} \frac{1}{v_n} v_n q_n = q_n \end{aligned}$$

For part (3), let $d = \gcd(w_{i,j,k,l})_{1 \le i<j<k<l \le n}$. We have a unimodular affine map f that maps p_i to q_i for all $i = 1, \dots, n$ and want to check that f has integral coefficients, if $d = 1$.

$T = \{p_i\}_{i=1,\dots,n}$ and $T' = \{q_i\}_{i=1,\dots,n}$ respectively span 3-dimensional sublattices $\Lambda(T), \Lambda(T') \le \mathbb{Z}^3$. Since f maps T to T', then it maps $\Lambda(T)$ to $\Lambda(T')$. The index $\mathbb{Z}^3 : \Lambda(T)$ is the minimal volume (with respect to $\mathbb{Z}^3$) of a basis of $\Lambda(T)$. Thus the index divides $w_{i,j,k,l}$ for all $1 \le i < j < k < l \le n$, and therefore it divides d. In particular, if $d = 1$, then $\Lambda(T) = \mathbb{Z}^3 = \Lambda(T')$. This implies f maps $\mathbb{Z}^3$ to itself, so it has integer coefficients. □

3. Polytopes with five lattice points

The five points $A = \{p_1, p_2, p_3, p_4, p_5\}$ in a 3-polytope of size five have a unique affine dependence. The *Radon partition* of A is obtained by looking at the signs of coefficients in this dependence. We say that $P = \operatorname{conv} A$ has *signature* (i, j) if this dependence has i positive and j negative coefficients. The five possibilities for (i, j) are $(2, 1)$, $(3, 1)$, $(4, 1)$, $(2, 2)$ and $(3, 2)$. (Observe that (i, j) and (j, i) are the same signature).

In order for the volume vector of P to encode its signature, and taking into account Remark 2.9, for a size 5 polytope we modify its volume vector to be

$$(w_{2,3,4,5}, -w_{1,3,4,5}, w_{1,2,4,5}, -w_{1,2,3,5}, w_{1,2,3,4})$$

where $w_{i,j,k,l}$ is as in Equation 3.

In this way, the signature of P is just the number of positive and negative entries in the volume vector, and the sum of coordinates in the volume vector vanishes.

To classify 3-polytopes of size five, we treat separately signatures $(2, *)$ and $(*, 1)$ (the case $(2, 1)$ appears in both, but that is not a problem).

3.1. Polytopes of signature $(2, *)$.

Our starting point is that every polytope P of size five and signature $(2, *)$ have width one. The proof is based on the fact that if T is the tetrahedron of largest volume in P then P is contained in the second dilation of T.

Theorem 3.1. *If P is a lattice 3-polytope of size 5 and of one of the signatures $(3, 2)$, $(2, 2)$ or $(2, 1)$, then P has width one.*

Proof. Let $(v_1, v_2, v_3, v_4, v_5)$ be the volume vector, reordered so that $v_i \le 0 < v_4 \le v_5$, $i = 1, 2, 3$. Let $T_5 = \operatorname{conv}\{p_1, p_2, p_3, p_4\}$ which is a tetrahedron of volume v_5. Without loss of generality we can assume T_5 to have width one with respect to the pair of edges p_1p_2 and p_3p_4. Hence we can assume T_5 to be the standard simplex, with $p_4 = o$, $p_1 = e_1$, $p_2 = e_2$, $p_3 = e_3$. This change means we are considering the lattice $\Lambda(p, q)$ corresponding to the type of T_5 (with $q = v_5$).

The affine dependence

$$\sum v_i p_i = 0$$

implies that $p_5 = \frac{-1}{v_5}(v_1, v_2, v_3)$ and, since $v_i \le 0$ for $i \in \{1, 2, 3\}$, p_5 lies in the (closed) positive orthant. Also, since $v_4 \le v_5$, and considering that $\sum v_i = 0$ we have $2v_5 \ge v_4 + v_5 = -(v_1 + v_2 + v_3)$. Hence $p_5 \in \{(x, y, z) : \sum x_i \le 2$. Putting these two things together, p_5 lies in the second dilation of T_5.

Since we have the lattice $\Lambda(p,q)$, the width of $2T_5$ is two with respect to the functional $x+y$, and its lattice points are:

- $x+y=0$: o, e_3 and $2e_3$.
- $x+y=2$: $2e_1$, $2e_2$ and e_1+e_2.
- $x+y=1$: the translated fundamental rectangle with vertices $(1,0,0)$, (0,1,0), $(1,0,1)$, $(0,1,1)$ and its interior points, in particular the point $(p/q,(q-p)/q,1/q)$.

Then the width of P with respect to the functional $x+y$ is at most two, and it equals two only if p_5 is one of $(2,0,0)$, $(1,1,0)$ or $(0,2,0)$. In the first and last cases P has signature $(2,1)$ (since two of the v_i's are zero). In the second case it has signature $(2,2)$. We claim that:

- If $p_5=(2,0,0)$ then, in order for P not to contain more lattice points of $\Lambda(p,q)$, we need $p=1$. But in this case $T(p,q)$ has width one with respect to $y+z$.
- The case $p_5=(0,2,0)$ is symmetric to the previous one, exchanging the roles of x and y.
- If $p_5=(1,1,0)$ then, in order for P not to contain more lattice points we need $q=1$. That is, T_5 is unimodular and P has width one with respect to, for example, the functional z.

To prove the claims, Figure 3 shows the intersection of P with the (translated) fundamental rectangle of vertices $(1,0,0)$, (0,1,0), $(1,0,1)$ and $(0,1,1)$. In both cases the upper vertex of the shaded triangle is the mid-point of p_5 and $p_3=(0,0,1)$. The restrictions appear by considering the values of p and q so that the point $(p/q,(q-p)/q,1/q)$ is not in the shaded area.

FIGURE 3. Intersection of P with the (translated) fundamental rectangle

□

Once we know this, the classification of these configurations, though infinitely many, is relatively easy. We now go back to having $\mathbb{Z}^3$ as our lattice.

Theorem 3.2. *Let A be a set of 5 lattice points. Then:*

1. $P=\operatorname{conv}\{A\}$ *is a 3-polytope of size 5 and of signature $(2,1)$ if and only if A consists of two non-parallel edges at lattice distance one, one of them primitive and the other one with a single lattice point in the middle.*
2. $P=\operatorname{conv}\{A\}$ *is a 3-polytope of size 5 and of signature $(2,2)$ if and only if A consists of an emtpy 2-dimensional parallelogram and a fifth point at lattice distance one.*
3. $P=\operatorname{conv}\{A\}$ *is a 3-polytope of size 5 and of signature $(3,2)$ if and only if A consists of a unimodular triangle and a primitive edge at lattice distance one, and has no coplanarities.*

Proof. The "if" part is trivial. For the only if part, by Theorem 3.1 the 5 lattice points of P lie in two consecutive lattice planes. Say n_0 points are at $z=0$ and $5-n_0$ at $z=1$. There are two possibilities:

- $n_0=4$: P has four points in a lattice plane. Then these four points form one of the three 2-dimensional polytopes of size 4 displayed in the right top row of Figure 1, of signatures $(2,2)$, $(2,1)$ and $(3,1)$ respectively, and P is of size 5 and of the same signature. The case $(3,2)$ is not part of this statement, the case $(2,2)$ is as in part (2) of the statement, and the case $(2,1)$ notice that the configuration has width one also with respect to two edges, the one consisting of the three collinear points, and the one with the remaining two points.

- $n_0=3$: P has three points in the lattice plane $z=0$, and two in the next plane $z=1$. There are two possibilities:

– If the three points at $z = 0$ are collinear, without loss of generality we can assume they are $(-1, 0, 0)$, $(0, 0, 0)$ and $(1, 0, 0)$. One of the points at $z = 1$ can be assumed to be $(0, 0, 1)$ and the fifth point has coordinates $(p, q, 1)$. In order for P to be 3-dimensional, we need $q \neq 0$ and without loss of generality $q > 0$. Also we need the edge at $z = 1$ to be primitive, so $\gcd(p, q) = 1$. The volume vector is $(q, -2q, q, 0, 0)$. Notice that the previous case with volume vector $(1, -2, 1, 0, 0)$ is already considered with $p = 0$ and $q = 1$.
– If the three points at $z = 0$ are not collinear then they have to form a unimodular triangle, and without loss of generality we assume they are $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 1, 0)$. One of the points at $z = 1$ can be assumed to be $(0, 0, 1)$ and the fifth point has coordinates $(a, b, 1)$. By the same argument as before, we need $\gcd(a, b) = 1$ and by symmetries with respect to the triangle at $z = 0$ we can assume $0 < p \leq q$.
Whenever the edge at $z = 1$ is not parallel to one of the edges of the triangle at $z = 0$, the volume vector is $(a + b, -a, -b, -1, 1)$, so the signature is $(3, 2)$.
Otherwise, the signature of the configuration is $(2, 2)$, and P has width 1 with respect to this parallelogram and the remaining vertex.

□

Remark 3.3. *Observe that cases (2) and (3) of the Theorem are equivalent, respectively, to the volume vector being equal (modulo reordering) to* $(1, 1, 0, -1, -1)$ *and to*

$$\pm(a + b, 1, -1, -a, -b)$$

for some $1 \leq a \leq b$ *with* $\gcd(a, b) = 1$*. For the proof observe that, since the* $\gcd$*'s in these volume vectors are both* 1*, all configurations with those volume vectors are* $\mathbb{Z}$*-equivalent to one another (Theorem 2.11).*

In case (1) the volume vector is (modulo reordering) $\pm(q, q, 0, 0, -2q)$ *for some* $q > 0$*, but this volume vector alone does not imply* P *to be of size 5.*

Let us check that this case study agrees with the configurations listed in the first three corresponding blocks of Table 1. Signatures $(2, 2)$ and $(3, 2)$ are clear, by Remark 3.3. In signature $(2, 1)$ the only thing that is not obvious from Theorem 3.2 and Remark 3.3 is why we can assume $0 \leq p \leq q/2$ and why different values of p in the range $0 \leq p \leq q/2$ give non-equivalent configurations. For this, let q be fixed and let $p_1, p_2 \in \mathbb{Z}$, with $\gcd(p_1, q) = 1 = \gcd(p_2, q)$. Let P_1 and P_2 be two of these configurations with the values $(p_1, q, 1)$ and $(p_2, q, 1)$ for the fifth point, respectively. Again by Theorem 2.11, all the possible unimodular transformations that map P_1 to P_2 must verify that: $(0, 0, 0)$ is a fixed point, $(1, 0, 0)$ and $(-1, 0, 0)$ can be fixed or mapped into one another, and $(0, 0, 1)$ and $(p_i, q, 1)$ can again be fixed or mapped into one another. So we have four possible unimodular transformations mapping P_1 to P_2. The reader can check that these four transformations imply $p_1 \equiv \pm p_2 \pmod q$.

Another way to check whether a configuration is a lattice 3-polytope of size 5 and signature $(2, 1)$ is the following lemma:

Lemma 3.4. *In signature* $(2, 1)$*, a configuration with volume vector* $(q, q, 0, 0, -2q)$ *is a* 5*-sized polytope if and only if*

- $q = \pm 1$*, or*
- $|q| > 1$ *and, in coordinates for which the* $(2, 1)$ *circuit is* $\mathrm{conv}\{o, e_2, -e_2\}$ *and the fourth point is* e_1*, the fifth point* (a, b, q) *verifies* $a \equiv 1 \mod q$ *and* $\gcd(b, q) = 1$*.*

Proof. Let us see what happens for the different values of $|q| > 0$ (if $q = 0$, the polytope is 2-dimensional):

- **$|q| = 1$:** In this case the volume vector is $(1, 1, 0, 0, -2)$. Since the signs correspond to a $(2, 1)$ signature, and the entries 1 imply that the two tetrahedra are empty, then the result follows.

- **$|q| > 1$:** Without loss of generality assume $P = \mathrm{conv}\{p_1, p_2, p_3, p_4, p_5\}$, with
$$p_1 = (0, 0, 0), \quad p_2 = (0, 1, 0), \quad p_3 = (0, -1, 0), \quad p_4 = (1, 0, 0),$$
one of the facets that contains the $(2, 1)$ circuit, and the fifth point is $p_5 = (a, b, q)$ for some $(a, b, q) \in \mathbb{Z}^3$. Then $P = T_1 \cup T_2$ where $T_i = \mathrm{conv}\,\{C_i \cup (a, b, q)\}$:
$$C_1 = \mathrm{conv}\,\{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$$
$$C_2 = \mathrm{conv}\,\{(0, 0, 0), (1, 0, 0), (0, -1, 0)\}$$

To determine whether these two tetrahedra are empty, we use Lemma A.

– $C_1 = \Delta^2$, and so (a, b, q) must verify at least one of the following:
  (i) $a \equiv 1 \pmod q$ and $\gcd(b, q) = 1$.
  (ii) $b \equiv 1 \pmod q$ and $\gcd(a, q) = 1$.
  (iii) $a + b \equiv 0 \pmod q$ and $\gcd(a, q) = 1$.

– Δ^2 is the image of C_2 by the unimodular transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$

and we get that (a, b, q) must verify at least one of the following:
  (i') $a \equiv 1 \pmod q$ and $\gcd(b, q) = 1$.
  (ii') $b \equiv -1 \pmod q$ and $\gcd(a, q) = 1$.
  (iii') $a - b \equiv 0 \pmod q$ and $\gcd(a, q) = 1$

And the cases when at least one condition from each group holds true, can be reduce to one, which is when $a \equiv 1 \mod q$ and $\gcd(b, q) = 1$.

□

3.2. **Polytopes of signature** $(*, 1)$**.** Let $(v_1, v_2, v_3, v_4, v_5)$ be the volume vector of a polytope of signature $(*, 1)$, reordered so that $v_5 < 0 \le v_i \le v_4$, $i = 1, 2, 3$. That is, point p_5 lies in the interior of $\mathrm{conv}\{p_1, p_2, p_3, p_4\}$, and the tetrahedron $T_4 := \mathrm{conv}\{p_1, p_2, p_3, p_5\}$ has the maximum volume among the empty simplices in P. Also, without loss of generality, assume that T_4 has width one with respect to the pair of edges p_1p_2 and p_3p_5. That is, the affine change of coordinates that sends T_4 to be the standard simplex, with $p_5 = o$, $p_1 = e_1$, $p_2 = e_2$, $p_3 = e_3$, sends $\mathbb{Z}^3$ to the lattice $\Lambda(p, q)$ corresponding to the type of T_4.

The affine dependence

$$\sum v_i p_i = 0$$

implies that $p_4 = \frac{-1}{v_4}(v_1, v_2, v_3)$. Now, since $v_i \ge 0$ for $i \in \{1, 2, 3\}$, p_4 lies in the (closed) negative orthant. Also, considering that $\sum v_i = 0$, we have $v_4 = -(v_1 + v_2 + v_3 + v_5)$ and then p_4 lies in the cube $[-1, 0]^3$. There are four possibilities for p_4:

(a) If $p_4 \in \{(-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$, then P has signature $(2, 1)$. In particular, it has width one and has been analyzed in the previous section.

(b) If $p_4 \in \{(-1, -1, 0), (0, -1, -1), (-1, 0, -1)\}$, then P has signature $(3, 1)$ and volume vector $(q, q, q, 0, -3q)$.

(c) If $p_4 = (-1, -1, -1)\}$, then P has signature $(4, 1)$ and volume vector $(q, q, q, q, -4q)$.

(d) If p_4 is not a vertex of the cube $[-1, 0]^3$, then it must be an interior lattice point in the (translated) fundamental rectangle $(-1, 0, 0)$, $(0, -1, 0)$, $(-1, 0, -1)$, $(0, -1, -1)$. The signature is again $(4, 1)$ and the volume vector is $(v_1, v_2, v_3, v_4, -(v_1 + v_2 + v_3 + v_4)$ with $v_i < v_4$ for at least one of $i = 1, 2, 3$.

Cases (b) and (d) have a unifying feature that we will make use of. Case (c) will be treated separately.

Lemma 3.5. *In cases* (b) *and* (d)*, the configuration has width two. Moreover, it has width two with respect to a functional that takes the values* $1, 1, 0, 0, -1$ *(not necessarily in this order, but with these multiplicities) in our five points.*

Proof. The statement holds for the functional $x + y$ in all cases except $p_4 = (-1, -1, 0)$. Thus, we assume this to be the case for the rest of the proof. In particular, $p_0 = p_5$ is the centroid of p_1, p_2 and p_4:

$$p_0 = (p_1 + p_2 + p_4)/3$$

Since the tetrahedron $\mathrm{conv}(\{p_0, p_1, p_2, p_3\})$ has width one with respect to the pair of edges p_0p_3 and p_1p_2, we can, by an affine change of coordinates, send $\Lambda(p,q)$ to $\mathbb{Z}^3$ in such a way that

$$p_0 \to (0,0,0), \quad p_3 \to (1,0,0), \quad p_1 \to (0,1,0), \quad p_2 \to (p,1,q).$$

Moreover, without loss of generality, $p \in \{0, \dots, q-1\}$. Under this transformation we have $p_4 \to (-p,-2,-q)$. Now, Lemma 2.7 applied to the tetrahedron $\mathrm{conv}(\{p_0, p_3, p_1, p_4\})$ implies that in order for this tetrahedron to be empty we need one of the following conditions:

(i) $p = q-1$ and $\gcd(2,q) = 1$.
(ii) $-2 \equiv 1 \pmod q$ and $\gcd(p,q) = 1$.
(iii) $p = q-2$ and $\gcd(2,q) = 1$.

In case (i) we take the functional $x - z$. In case (iii) we take $x + y - z$. In case (ii) we have $q = 3$ and $p \in \{1,2\}$. If $p = 1$ this is a special case of (iii) and if $p = 2$ it is a special case of (i). ☐

In the light of Lemma 3.5, we can analyze these configurations as follows: taking the functional of the lemma to be the z-coordinate, we assume without loss of generality that the first four points are $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(p,q,1)$, and the fifth is and $(a,b,-1)$ for some $a,b \in \mathbb{Z}$. The question whether the convex hull of these five points has no other lattice point is a two-dimensional question, since all such extra points should have $z = 0$. Analyzing the possibilities for p and q we obtain:

Theorem 3.6. *In case (d), there are only the following six distinct types: configurations with* $(0,0,0)$, $(1,0,0)$, $(0,0,1)$ *and with fourth and fifth vertices*

- $(1,2,1)$ *and* $(-1,-1,-1)$, *volume vector* $(-5,1,1,1,2)$.
- $(1,3,1)$ *and* $(-1,-2,-1)$, *volume vector* $(-7,1,1,2,3)$.
- $(2,5,1)$ *and* $(-1,-2,-1)$, *volume vector* $(-11,1,3,2,5)$.
- $(2,5,1)$ *and* $(-1,-1,-1)$, *volume vector* $(-13,3,4,1,5)$.
- $(2,7,1)$ *and* $(-1,-2,-1)$, *volume vector* $(-17,3,5,2,7)$.
- $(3,7,1)$ *and* $(-2,-3,-1)$, *volume vector* $(-19,5,4,3,7)$.

Proof. Our configuration of points is $P = \mathrm{conv}\{p_1, p_2, p_3, p_4, p_5\}$, with

$$p_5 = (0,0,0), \quad p_1 = (1,0,0), \quad p_2 = (0,0,1), \quad p_3 = (p,q,1)$$

for some $0 \le p < q$, $\gcd(p,q) = 1$. And $p_4 = (a,b,-1)$.

The five points form a $(4,1)$ circuit, with either p_5 or p_1 the interior point. Without loss of generality (by Remark 2.6) let p_5 be this point. The volume vector of our configuration $P = \mathrm{conv}\,\{p_5, p_1, p_2, p_3, p_4\}$ is

$$(-(pb + q(2-a)), pb - qa, q + b, -b, q)$$

Since we assumed the normalized volume of $T_4 = \mathrm{conv}\,\{p_1, p_2, p_3, p_5\}$ to be the biggest, we have $0 < -b \le q$, $0 < q + b \le q$ and $0 < pb - qa \le q$.

Let us now consider the projection in the direction of functional z. The origin must be an interior point of the intersection of P at $z = 0$. That is, the triangle of vertices $(1,0)$, $(a/2, b/2)$ and $((a+p)/2, (b+q)/2)$ in the plane. Let $c = a + p$ and $d = b + q$.

We now have to study the values of $a,b,c,d \in \mathbb{Z}$ such that $(a/2,b/2), (c/2,d/2) \notin \mathbb{Z}$ and $(0,0) \in \mathrm{int}(\mathrm{conv}\,\{(1,0), (a/2,b/2), (c/2,d/2)\})$. In particular this implies, since $d > 0$, that $b < 0$. By symmetry we may also assume that $|d| \le |b|$. This also tells us that a and b cannot both be even, and the same for c and d.

Fixing the value of d, we can consider the values of $c \pmod d$. For $d = 1$ we take $c = 0$. For $d > 1$, if we take into account that the point $(0,1,0)$ cannot be in the convex hull of P, then $(c/2, d/2)$ cannot be in the segment symmetric to $(0,0,0)(1,0,0)$ with respect to the point $(0,1,0)$. That is, $c/2 \notin [1 - d/2, 0]$, which is equivalent to $c \notin [2-d, 0]$, and the only remaining value for $c \pmod d$ is $c = 1$. Moreover, for $d \ge 4$, the only possible values for (a,b), if there are any, are all with $d > |b|$.

Thus, we can assume that $(c,d) \in \{(0,1), (1,2), (2,3)\}$. To study the possibilities for (a,b) we look at the intersection of our configuration with the plane $z = 0$. Taking into account the constraints (i) $b < -d$, (ii) p_5 is in the interior of $\mathrm{conv}(P)$, and (iii) $\mathrm{conv}(P) \cap \{z = 0\}$ does not contain any lattice point other than p_5 and p_1, leaves a finite (and small) number of candidates for (a,b) for each of the three possible values of (c,d). More precisely, $p_2p_4 \cap \{z = 0\}$ must lie, respectively, in the white regions of the pictures in Figure 4, so the candidates for (a,b) (or, equivalently, for $p_4 = (a,b,-1)$) are in correspondence with the half-integer points in those regions, marked with crosses in the figure. This gives a priori 16 possibilities for p_4.

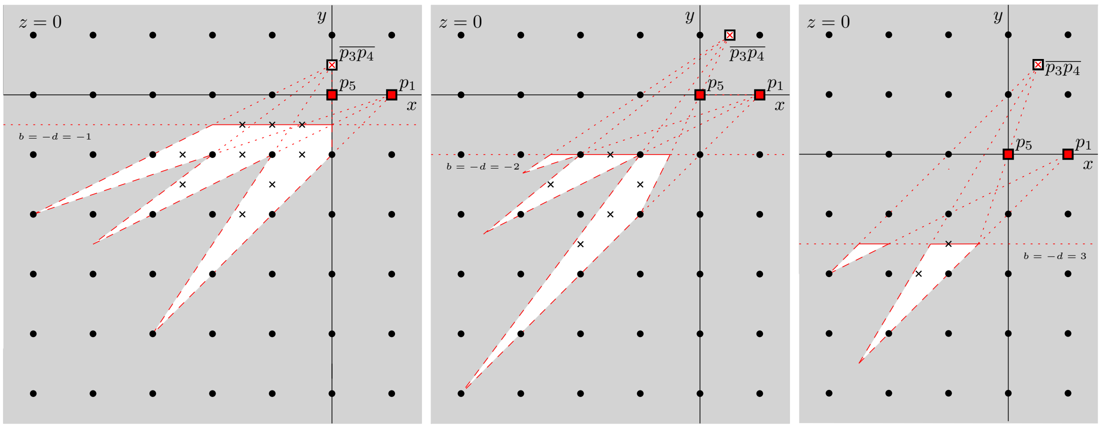


FIGURE 4. The case analysis in the proof of Theorem 3.6 for the three possibilities of (c, d). Red squares represent the points p_1 and p_5 of P in the displayed plane $z = 0$. The red crossed square is the intersection of p_3p_4 with that same plane. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge p_2p_4 and the plane $z = 0$.

Now, these 16 points reduce to only six possibilities by considering $\gcd(p, q) = 1$, $0 \le p < q$ and observing that, in order to have primitive edges, we must have

$$\gcd(a, b, 2) = \gcd(p - a, q - b, 2) = 1. \tag{4}$$

The final list is:

c	d	a	b	$p = c - a$	$q = d - b$	$p - a$	$q - b$
0	1	-1	-1	1	2	2	3
0	1	-1	-2	1	3	2	5
1	2	-2	-3	3	5	5	8
0	1	-3	-4	3	5	6	9
1	2	-4	-5	5	7	9	12
1	3	-3	-4	4	7	7	11

In order to get smaller coordinates for the configuration, whenever $p > q/2$ we apply the transformation

$$(x, y, z) \mapsto (x - y + (q - p)z, qz - y, z)$$

This fixes p_1 and p_5, and exchanges the roles of p_2 and p_3 giving $(q - p, q, 1)$ as the new p_3. The new parameters are the following, as stated:

a	b	p	q
-1	-1	1	2
-1	-2	1	3
-1	-2	2	5
-1	-1	2	5
-1	-2	2	7
-2	-3	3	7

$\square$

Theorem 3.7. *In signature* $(3, 1)$*, a configuration with volume vector* $(q, q, q, 0, -3q)$ *is a* 5*-sized polytope if and only if*

- $q = \pm 1$*, or*
- $q = \pm 3$ *and, in coordinates for which the* $(3, 1)$ *circuit is* $\operatorname{conv}\{o, e_1, e_2, -e_1 - e_2\}$*, the fifth point* (a, b, q) *verifies* $a \equiv -b \equiv \pm 1 \pmod 3$.

Proof. Without loss of generality assume $P = \operatorname{conv}\{p_1, p_2, p_3, p_4, p_5\}$, with

$$p_5 = (0,0,0), \quad p_1 = (1,0,0), \quad p_2 = (0,0,1), \quad p_3 = (p,q,1)$$

for $q \neq 0$, $\gcd(p,q) = 1$, and $p_4 = (a,b,-1)$. The value of p can be considered modulo q.

The five points form a $(3,1)$ circuit. The only possible way is that the barycenter is at $z = 0$ and the vertices are all in different z-planes. Without loss of generality (by Remark 2.6) let p_5 be the barycenter, and p_1, p_2 and p_4 the vertices of the triangle. Thus $p_4 = 2p_5 - p_1 - p_2 = (-1,0,-1)$.

Now we have the four coplanar points in the plane $y = 0$, so p_3 must have $q \neq 0$. We may assume $q > 0$, since the negative case is symmetric (and $\mathbb{Z}$-equivalent to the positive case). Now, the edges that join p_3 with the points on the plane $y = 0$ must be primitive. That is:

$$\gcd(p,q,1) = \gcd(p-1,q,1) = \gcd(p,q,0) = \gcd(p+1,q,2) = 1$$

and the previous holds if and only if q is odd. In summary, the convex hull has one point at $z = -1$, the primitive edge $(0,0,1)(p,q,1)$ at $z = 1$ and the triangle of vertices $(1,0,0)$, $(-1/2,0,0)$ and $((p-1)/2, q/2, 0)$ at $z = 0$. Figure 5 shows that the possibilities for p and q such that this triangle has no more lattice points than p_1 and p_5 are:

- $q = 1$, $p \in \mathbb{Z}$. All points at $y = 1$ give equivalent configurations by mapping $x \to x + \alpha y$ and $z \to z + \beta y$.
- $q = 3$ and $p \equiv 2 \pmod 3$. By mapping $x \to x + 3\alpha y$ and $z \to z + 3\beta y$ we get that all points $(a,3,b)$ with $a \equiv -b \equiv -1 \pmod 3$ give equivalent configurations. Also, by considering all the possible unimodular transformations in $\mathbb{Z}^3$ that are an automorphism between the points at height $z = 0$, we see that the only remaining valid points at $y = 3$ are those with $a \equiv -b \equiv 1 \pmod 3$, by reflection of the previous with respect to the plane $x = z$.

By simply applying the permutation of coordinates $y \leftrightarrow z$, the result follows.

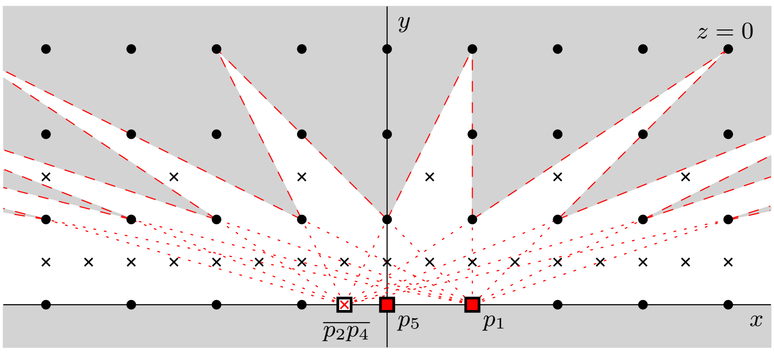


FIGURE 5. The case analysis in the proof of Theorem 3.7. Red squares represent the points p_1 and p_5 of P in the displayed plane $z = 0$. The red crossed square is the intersection of p_2p_4 with that same plane. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge p_3p_4 and the plane $z = 0$.

□

Finally, in case (c) we show that there are only two possibilities:

Theorem 3.8. *There are only the following two configurations of signature* $(4,1)$ *with volume vector* $(-4k,k,k,k,k)$*:* $(0,0,0)$*,* $(1,0,0)$*,* $(0,0,1)$ *and with fourth and fifth vertices equal to:*

- $(1,1,1)$ *and* $(-2,-1,-2)$*, volume vector* $(-4,1,1,1,1)$.
- $(2,5,1)$ *and* $(-3,-5,-2)$*, volume vector* $(-20,5,5,5,5)$.

Proof. With previous notation, we have that our configuration is $P = \operatorname{conv}\{p_1,p_2,p_3,p_4,p_5\}$ with $p_5 = (0,0,0)$, $p_i = e_i$ for $i = 1,2,3$ and $p_4 = (-1,-1,-1)$ and in the lattice $\Lambda(p,q)$, for some $0 \leq p < q$, with p prime with q. We apply the transformation that maps $\Lambda(p,q) \to \mathbb{Z}^3$ and

$$p_5 \to (0,0,0), \quad p_3 \to (1,0,0), \quad p_1 \to (0,0,1), \quad p_2 \to (p,q,1).$$

This transformation maps $p_4 = (-1,-1,-1) \to (-p-1,-q,-2)$.

Now the convex hull of the configuration consists of four thetrahedra glued together, all of normalized volume q, where the new three tetrahedra are:

- $T_1 = \{(0,0,0),(1,0,0),(0,0,1),p_4\}$
- $T_2 = \{(0,0,0),(1,0,0),(p,q,1),p_4\}$
- $T_3 = \{(0,0,0),(0,0,1),(p,q,1),p_4\}$

It remains to check whether these tetrahedra are empty. If $q = 1$ then all tetrahedra are unimodular and therefore empty. In this case, $p_3 = (1,1,1)$ and $p_4 = (-2,-1,-2)$. We consider $q > 1$ for the rest of the proof.

Denote $p_4 = (a,b,c)$. We have $T_i = \text{conv}\,(C_i \cup p_4)$, where:

- $C_1 = \{(0,0,0),(1,0,0),(0,0,1)\}$
- $C_2 = \{(0,0,0),(1,0,0),(p,q,1)\}$
- $C_3 = \{(0,0,0),(0,0,1),(p,q,1)\}$

Let us use Lemma 2.7 to evaluate for which values of a, b and c these tetrahedra are empty.

- Δ^2 is the image of C_1 by the unimodular transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ z \\ y \end{pmatrix}$$

  and so by the lemma it must happen at least one of the following
  (i) $a \equiv 1 \pmod{b}$ and $\gcd(b,c) = 1$.
  (ii) $c \equiv 1 \pmod{b}$ and $\gcd(a,b) = 1$.
  (iii) $a + c \equiv 0 \pmod{b}$ and $\gcd(a,b) = 1$.
- Δ^2 is the image of C_2 by the unimodular transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x - pz \\ z \\ y - qz \end{pmatrix}$$

  and again by the lemma at least at least one of the following must happen:
  (i') $a - pc \equiv 1 \pmod{b - qc}$ and $\gcd(b,c) = 1$.
  (ii') $c \equiv 1 \pmod{b - qc}$ and $\gcd(a - pc, b - qc) = 1$.
  (iii') $a - (p-1)c \equiv 0 \pmod{b - qc}$ and $\gcd(a - pc, b - qc) = 1$.
- To describe a unimodular transformation that maps C_3 to Δ^2, we ought to know the Bezout coefficients of p and q. Hence we evaluate directly the widths of T_3 with respect to the three pairs of edges: tetrahedron T_3 is empty if and only if:

$$\gcd(a,b,c) = \gcd(a-p, b-q, c-1) = \gcd(a,b,c-1) = 1$$

  and at least one on the following values is 1:

$$\frac{1}{\gcd(a-p, b-q)}(bp - aq)$$

$$\frac{1}{\gcd(q(c-1) - b, p(c-1) - a, bp - aq)}(bp - aq)$$

$$\frac{1}{\gcd(c, bp - aq)}(bp - aq)$$

Let us now use the known coordinates of A:

(i) $p \equiv -2 \pmod{q}$ and $\gcd(q,2) = 1$, i.e. q has to be odd and $p = q - 2$.
(ii) $3 \equiv 0 \pmod{q}$ and $\gcd(p+1, q) = 1$, i.e. $q = 3$ has to be odd and $p = 1$.
(iii) $p \equiv -3 \pmod{q}$ and $\gcd(p+1, q) = 1$, i.e. q has to be odd (if it were even, then either p or $p+1$ would have a common factor 2 with q) and $p = q - 3$.

So tetrahedron T_1 is emtpy if and only of q is odd and $p = q - 2$ ($q > 2$) or $p = q - 3$ ($q > 3$).

(i') $p \equiv 2 \pmod{q}$ and $\gcd(q,2) = 1$, i.e. q has to be odd and $p = 2$.
(ii') $3 \equiv 0 \pmod{q}$ and $\gcd(p-1, q) = 1$, i.e. $q = 3$ has to be odd and $p = 2$.

(iii') $p \equiv 3 \pmod q$ and $\gcd(p-1, q) = 1$, i.e. q has to be odd (if it were even, then either p or $p-1$ would have a common factor 2 with q) and $p = 3$.

So tetrahedron T_2 is emtpy if and only of q is odd and $p = 2$ ($q > 2$) or $p = 3$ ($q > 3$).

And then both thetraedra T_1 and T_2 are empty if and only if

$$q \text{ is odd} \wedge (p = 2 \vee p = 3) \wedge (p = q - 2 \vee p = q - 3)$$

And the previous can only happen for $q = 5$ with $p = 2$ or $p = 3$.
We put $q = 5$, and consider $p \in \{2, 3\}$. Then we must have that one of the following is 1:

$$\frac{q}{\gcd(2p+1, 2q)} = \frac{5}{\gcd(2p+1, 10)} = 1 \iff p = 2$$

$$\frac{q}{\gcd(2p-1, q)} = \frac{5}{\gcd(2p-1, 5)} = 1 \iff p = 3$$

$$\frac{q}{\gcd(2, q)} = \frac{5}{\gcd(2, 5)} = 5 \neq 1 \quad for\ p = 2, 3$$

And so in this case, the three tetrahedra are all empty only if $q = 5$, and $p \in \{2, 3\}$. The following matrix represents a $\mathbb{Z}$-equivalence between the two possible configurations:

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence there is a unique equivalence class for $q = 5$. In this case, $p_3 = (2, 5, 1)$ and $p_4 = (-3, -5, -2)$. □

Corollary 3.9. *The convex hull P of a set of 5 lattice points is a 3-polytope of size 5 and of signature $(4, 1)$ if and only if the volume vector is equal (after adequate permutation of points) to one of the eight possibilities in Theorems 3.6 and 3.8, and if the case is $(-20, 5, 5, 5, 5)$, the type of the four subtetrahedra must be $T(2, 5)$.*

Corollary 3.10. *There are exactly nine 3-polytopes of size 5 and width two and none of larger width. Eight of them have signature $(4, 1)$, and one has signature $(3, 1)$. All of them are dps.*

The following Table 1 shows a choice of representative for each equivalence class of lattice 3-polytopes of size 5. The configurations are grouped according to their signatures.

Signature	**Volume vector**	**Width**	**Representative**
$(2,2)$	$(-1,1,1,-1,0)$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$,$(0,0,1)$
$(2,1)$	$(-2q,q,0,q,0)$ $0 \le p \le \frac{q}{2}$ with $\gcd(p,q)=1$	1	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(-1,0,0)$,$(p,q,1)$
$(3,2)$*	$(a+b,-a,-b,-1,1)$ $0 < a \le b$ with $\gcd(a,b)=1$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$,$(a,b,1)$
$(3,1)$*	$(-3,1,1,1,0)$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$,$(0,0,1)$
	$(-9,3,3,3,0)$	2	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$,$(1,2,3)$
$(4,1)$*	$(-4,1,1,1,1)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,1,1)$,$(-2,-1,-2)$
	$(-5,1,1,1,2)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,2,1)$,$(-1,-1,-1)$
	$(-7,1,1,2,3)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,3,1)$,$(-1,-2,-1)$
	$(-11,1,3,2,5)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,1,5)$,$(-1,-2,-1)$
	$(-13,3,4,1,5)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-1,-1,-1)$
	$(-17,3,5,2,7)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,7,1)$,$(-1,-2,-1)$
	$(-19,5,4,3,7)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(3,7,1)$,$(-2,-3,-1)$
	$(-20,5,5,5,5)$	2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-3,-5,-2)$

TABLE 1. Complete classification of lattice 3-polytopes of size 5. Those marked with an * are dps

Let us mention that part of these results were already known:

- Configurations of signatures $(2, 2)$ and $(3, 2)$ have width 1 by Howe's Theorem (1.6).
- Configurations of signature $(4, 1)$ were classified by Reznick [8, Thm. 7] and Kasprzyk [5], who obtained exactly the same result as we do.

4. Polytopes with six lattice points

Let $A = \{p_1, \ldots, p_6\}$ be a set of six lattice points in $\mathbb{Z}^3$. The *volume vector* of A is the following vector $w \in \mathbb{Z}^{15}$:

$$\begin{aligned} w \;=\; & (w_{1234}, w_{1235}, w_{1236}, w_{1245}, w_{1246}, w_{1256}, w_{1345}, \\ & w_{1346}, w_{1356}, w_{1456}, w_{2345}, w_{2346}, w_{2356}, w_{2456}, w_{3456}) \end{aligned}$$

where

$$w_{ijkl} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_i & p_j & p_k & p_l \end{pmatrix}$$

for $1 \le i < j < k < l \le 6$.

We denote $P = \mathrm{conv}(A)$ and, for each $i = 1, \ldots, 6$, $P^i = \mathrm{conv}(A \setminus \{p_i\})$. The five components w_{jklr} such that $i \notin \{j, k, l, r\}$ give us the volume vector of P^i. For example, for $i = 6$, the volume vector of P^6 (with the sign and ordering convention of Section 3) is $(w_{2345}, -w_{1345}, w_{1245}, -w_{1235}, w_{1234})$.

If $A = \mathbb{Z}^3 \cap P$, so that P has size 6, then for every vertex p_i of P, P^i is a lattice polytope of size 5. The converse is also true: if for every vertex i of P we have that P^i has size five then P has size six.

Remember that a polytope Q of size 5 has a unique circuit, consisting of $a+b$ points (a positive and b negative ones). In what follows we call such a Q an *(a, b)-polytope*. This is not exactly the same as an *(a, b)-circuit* C which consists only of the $a+b \le 5$ points and is not required to have $C = \mathrm{conv}(C) \cap \mathbb{Z}^3$.

In order to completely classify polytopes of size 6 we repeatedly use the classification of size 5; either we start with a polytope of size 5 and look at the different ways to add a sixth point to it or we start with two polytopes Q_1 and Q_2 of size five having four points in common and check whether $Q_1 \cup Q_2$ is a polytope of size six (that is, whether no new lattice points arise in $\mathrm{conv}(Q_1 \cup Q_2)$). More precisely, we look separately at the following cases for a polytope of size 6, which clearly cover all the possibilities:

1. Polytopes containing 5 coplanar points. Here the idea is to start with the six polygons of size five (see Figure 1) and look whether there is a way to add a sixth point apart of the obvious one, which is adding it at lattice distance one from the coplanarity.
2. Polytopes containing a coplanarity of type $(3, 1)$ (and no five coplanar points). Here the main ingredient is: by the classification of polytopes of size five, the two extra points are at lattice distance ± 1 or ± 3 from the $(3, 1)$ coplanarity. This gives several cases that are treated separately.
3. Polytopes containing a coplanarity of type $(2, 2)$ (and none of the above). Now, the extra two points must be at lattice distance ± 1 from the coplanarity, which makes this an even easier case.
4. Polytopes containing a collinearity of type $(2, 1)$ (and none of the above). Polytopes of this type can be obtained by taking a polytope Q of size five, together with a pair $\{p_i, p_j\}$ of points in it and checking whether $Q \cup \{2p_i - p_j\}$ contains extra lattice points. We need to consider all the possible cases for i, j, but we can assume that Q is of signature $(2, 1)$, $(3, 2)$ or $(4, 1)$ since the other cases have been already dealt with.
5. Polytopes with no coplanarities and with at least one interior point. An oriented matroid argument shows that all these configurations have the following property: there are two vertices p_i and p_j such that P^i and P^j are $(4, 1)$-polytopes. P^i and P^j clearly have a tetrahedron in common and $P = P^i \cup P^j$. Hence, our approach is to look one by one at the (finitely many) ways of gluing to one another the eight different $(4, 1)$-polytopes obtained in Section 3.2.
6. Polytopes with no coplanarities and with no interior points. By Howe's Theorem 1.6 these polytopes have width one, which makes them easy to classify: their vertices consist of two unimodular triangles in consecutive parallel planes and the classification is basically depending upon the $SL(2, \mathbb{Z})$ motion sending one of the triangles to (a parallel copy of) the other one.

In some of the cases exhaustive searches of all the possibilities have been done with a computer. In other cases we use the idea, already used in the case of size 5, that if we know our points to lie in three parallel lattice planes then checking that their convex hull has no extra lattice points can be transformed into a two-dimensional problem. We will refer to this as the *parallel-planes* method.

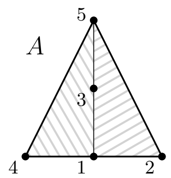
As we go along the classification we will denote the different sub-cases with capital letters A to Z. Often, but not always, this sub classification coincides with the classification by (dual) oriented matroid. In the tables at the end we give both classifications.

4.1. **Polytopes with 5 coplanar points.** P has five points in a lattice plane (for example $z = 0$) and the sixth has to be outside this plane for the polytope to be three dimensional. These five points form one of the six 2-dimensional configurations displayed in the bottom row of Figure 1. Let the sixth point be $p_6 = (a, b, q) \in \mathbb{Z}^3$ with $q \neq 0$. Without loss of generality, $q > 0$, and the values of a, b can be considered modulo q.

The main idea to deal with this case is that for each vertex p_i of the base P^6 of the pyramid we need to have P^i to be a polytope of size 5. In some cases not all the possibilities for i need to be considered.

In all cases we take, without loss of generality, The first three points are $p_1 = o := (0, 0, 0)$, $p_2 = e_1 := (1, 0, 0)$ and $p_3 = e_2 := (0, 1, 0)$.

- **Case A:** $p_4 = (-1, 0, 0)$ and $p_5 = (0, 2, 0)$.



Clearly, $P = P^2 \cup P^4$, so we only need to check whether P^2 and P^4 are of size five.

$P^2 = C \cup \{(a, b, q)\}$, with $C = \{o, -e_1, e_2, 2e_2\}$, contains a $(2, 1)$-circuit. Hence P^2 has signature $(2, 1)$. The unimodular transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -x + y - 1 \\ z \end{pmatrix}$$

sends C to $\{o, e_1, e_2, -e_2\}$ and p_6 to $(-a, -a + b - 1, q)$. Now, by Lemma 3.4, P^2 is a $(2, 1)$-polytope if and only if $q = 1$ or

$$q > 1, \quad a \equiv -1 \pmod{q}, \quad \gcd(b, q) = 1$$

Analogously, $P^4 = C \cup \{(a, b, q)\}$, with $C = \{o, e_1, e_2, 2e_2\}$, again has to be a $(2, 1)$-polytope. In this case, the transformation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ x + y - 1 \\ z \end{pmatrix},$$

p_6 is sent to $(a, a + b - 1, q)$ and P^4 has size 5 if and only if $q = 1$ or

$$q > 1, \quad a \equiv 1 \pmod{q}, \quad \gcd(b, q) = 1.$$

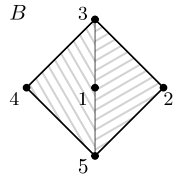
If $q = 1$ we can assume $p_6 = (0, 0, 1)$, since a and b are considered only modulo q. This gives the configuration

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ A.1 & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

If $q > 1$, then $1 \equiv a \equiv -1 \pmod{q}$ if and only if $q = 2$ and $a = 1$. Then, in order to have $\gcd(b, q) = 1$, we need $b = 1$.

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ A.2 & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

- **Case B:** $p_4 = (-1, 0, 0)$ and $p_5 = (0, -1, 0)$.



As before, $P = P^2 \cup P^4$ and both P^2 and P^4 are of signature $(2, 1)$. A similar analysis as in the previous case leads to the following two possibilities:

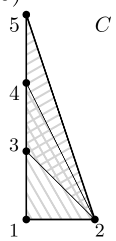
If $q = 1$, then modulo unimodular transformation we get

$$B.1 \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

If $q > 1$, then we need $a, b \equiv 1 \pmod q$ with $q = 2$. We get, modulo unimodular transformation:

$$B.2 \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

- **Case C:** $p_4 = (0, 2, 0)$ and $p_5 = (0, 3, 0)$.



As before, $P = P^1 \cup P^5$ with both P^1 and P^5 of signature $(2, 1)$.

$P^1 = C \cup \{(a, b, q)\}$, with $C = \{e_1, e_2, 2e_2, 3e_2\}$. The unimodular transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} 2x + y - 2 \\ x \\ z \end{pmatrix}$$

sends C to $\{o, e_1, e_2, -e_2\}$ and p_6 is sent to $(a, a+b-1, q)$. By the Lemma, P^1 is a $(2, 1)$-polytope if and only if $q = 1$ or

$$q > 1, \quad a \equiv 1 \pmod q, \quad \gcd(b, q) = 1$$

Analogously, $P^5 = C \cup \{(a, b, q)\}$, with $C = \{o, e_1, e_2, 2e_2\}$, again has to be a $(2, 1)$-polytope. In this case, the transformation is as for P^4 in case **A**, and P^5 has size 5 if and only if $q = 1$ or

$$q > 1, \quad a \equiv 1 \pmod q, \quad \gcd(b, q) = 1$$

– If $q = 1$, all points in this plane give equivalent configurations.

– If $q > 1$, then we get $a = 1$ and $\gcd(b, q) = 1$.

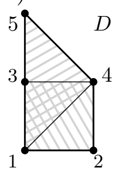
Both cases can be summarize considering $p_6 = (1, p, q)$ with $q \geq 1$ and $\gcd(p, q) = 1$. Remember that the value of p can be considered modulo q.

$$C \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 & p \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix} \end{array}$$

This gives us an infinite list of polytopes of size 6 depending on some parameters. For all values of these parameters, the minimum width is 1 with respect to the functional x. Notice that if the list had width greater than 1, then there would only be a finite list of equivalence classes, by Corollary 1.4.

This is what we will refer to as an *infinite series*, and we will perform further study on Subsection 4.7.

- **Case D:** $p_4 = (1, 1, 0)$ and $p_5 = (0, 2, 0)$.



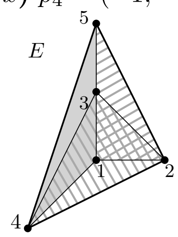
$P = P^2 \cup P^5$ with P^2 of signature $(2, 1)$ and P^5 of signature $(2, 2)$. P^5 will be a $(2, 2)$-polytope if and only if $q = 1$.

On the other hand, $P^2 = C \cup \{(a, b, q)\}$, with $C = \{o, e_2, e_1 + e_2, 2e_2\}$. And with $q = 1$, then P^2 is automatically a $(2, 1)$-polytope.

Then without loss of generality, $p_6 = e_3$:

$$\begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5 & 6\end{matrix} \\ D & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

- **Case E: (cambiar dibujo $x \to -x$)** $p_4 = (-1, -1, 0)$ and $p_5 = (0, 2, 0)$.



In this case, $P = P^2 \cup P^4 \cup P^5$, where P^2 and P^4 have signature $(2, 1)$ and P^5 is of signature $(3, 1)$.

P^5 will be a $(3, 1)$-polytope if and only if $q = 1$ or $q = 3$ and $a \equiv -b \equiv \pm 1 \pmod 3$. If $q = 1$, both P^2 and P^4 are automatically $(2, 1)$-polytopes. This gives us, modulo unimodular transformation:

$$\begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5 & 6\end{matrix} \\ E & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Now let $q = 3$ and $a \equiv -b \equiv \pm 1 \pmod 3$. $P^2 = C \cup \{(a, b, 3)\}$, with $C = \{o, e_2, 2e_2, -e_1 - e_2\}$. The unimodular transformation

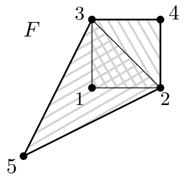
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -2x + y - 1 \\ z \end{pmatrix}$$

sends C to $\{o, e_1, e_2, -e_2\}$ and p_6 is sent to $(-a, -2a + b - 1, 3)$. By the Lemma, P^2 is a $(2, 1)$-polytope if and only if $a \equiv -1 \pmod 3$ and $\gcd(b + 1, 3) = 1$.

Analogously, $P^4 = C \cup \{(a, b, q)\}$, with $C = \{o, e_1, e_2, 2e_2\}$, again has to be a $(2, 1)$-polytope. In this case, the transformation is as for P^4 in case **A**, so P^4 has size 5 if and only if $a \equiv 1 \pmod 3$ and $\gcd(b, 3) = 1$.

But $1 \equiv a \equiv -1 \pmod 3$ is not possible, so $q = 3$ does not give configurations of size 6.

- **Case F:** $p_4 = (1, 1, 0)$ and $p_5 = (-1, -1, 0)$.



$P = P^4 \cup P^5$ with P^4 of signature $(2, 2)$ and P^5 of signature $(3, 1)$. P^4 will be a $(2, 2)$-polytope if and only if $q = 1$.

On the other hand, $P^5 = C \cup \{(a, b, q)\}$, with $C = \{o, e_1, e_2, -e_1 - e_2\}$. And with $q = 1$, then P^5 is automatically a $(3, 1)$-polytope.

Then without loss of generality, $p_6 = e_3$:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ F & \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let us now analyze the width of these configurations.

- Clearly A.1, B.1, D, E and F have width one with respect to the functional z, and C with respect to the functional x.
- On the other hand, A.2 and B.2 both have an interior point in the subconfiguration at $z = 0$, hence the only functional that could possibly give width 1 is z. But this functional gives width 2 for both cases.

In summary:

Theorem 4.1. *Among the lattice 3-polytopes of size six with 5 coplanar points, there are exactly 2 equivalence classes of width two, and none of larger width, as shown in Table 2. Both are non-dps.*

4.2. **Polytopes containing a $(3, 1)$-circuit (and no five coplanar points).** Without loss of generality, we assume they contain the standard $(3, 1)$-circuit: $p_1 = o$, $p_2 = e_1$, $p_3 = e_2$ and $p_4 = -e_1 - e_2$.

We treat separately the case of the other two points lying on the same side or on opposite sides of this circuit.

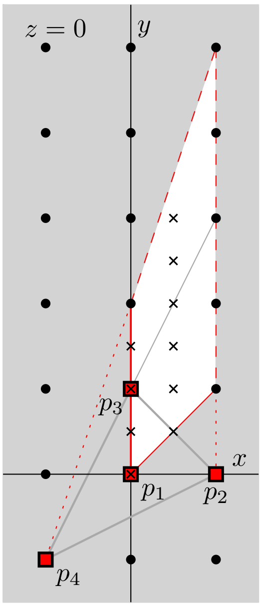
4.2.1. *The other two points lie in opposite sides of the $(3, 1)$ coplanarity.* Then both p_5 and p_6 are vertices and so P^5 and P^6 must be $(3, 1)$-polytopes. Then the fifth and sixth points are (a_i, b_i, q_i) with $q_i = \pm 1$ or $q_i = \pm 3$ and verifying $a_i \equiv -b_i \equiv \pm 1 \pmod 3$.

⋄ **Case G:** $p_5 = (0, 0, 1)$ and $p_6 = (a, b, -1)$, $a, b \in \mathbb{Z}$.
We use the *parallel-planes* method. The configuration is contained in the three planes $z \in \{-1, 0, 1\}$. There is one single point in the planes $z = \pm 1$, and at $P \cap \{z = 0\}$ is the convex hull of the $(3, 1)$-circuit and the intersection point $(a/2, b/2, 0)$ of the edge p_5p_6 with the plane $z = 0$. Without loss of generality (because of the S_3-symmetries present in P^6) the intersection point can be assumed to be in the region $0 \le x \le y$. In order for $(1, 1, 0)$ and $(0, 2, 0)$ not to be in $P \cap \{z = 0\}$, the region is bounded by $x < 1$ and $y < 3x + 2$ (non-shaded area in Figure 6). This gives ten possibilities for the pair (a, b).

All the options have automatically size 6 since no more points arise at $P \cap \{z = 0\}$ and the only points at $z = 1$ and $z = -1$ are, respectively, p_5 and p_6. They all have different volume vectors (in absolute value) as shown in Table 3, and so they give different equivalent classes.

Notice that the intersection point of the edge p_5p_6 with the plane $z = 0$ determines the (dual) oriented matroid depending on whether it is contained or not in one or more straight lines spanned by the edges in $z = 0$. The ten configurations in the table are separated according to their (dual) oriented matroid.

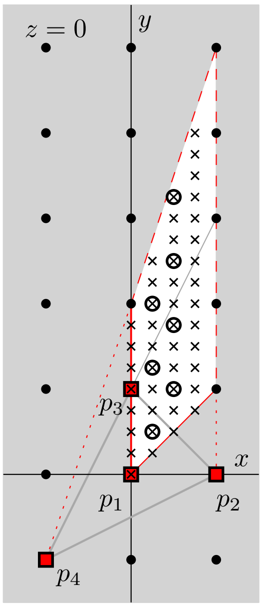
⋄ **Case H:** $p_5 = (0, 0, 1)$ and $p_6 = (a, b, -3)$ with $a \equiv -b \equiv \pm 1 \pmod 3$.
Again we use the *parallel-planes* method. The configuration is contained between the hyperplanes $z = 1$ and $z = -3$. The intersection point of the edge p_5p_6 with $z = 0$ is $(a/4, b/4, 0)$. Again without loss of generality, the intersection point can be in the region $0 \le x \le y$, and as before, bounded by $x < 1$ and



Id.	a	b
$G.1$	1	4
$G.2$	0	3
$G.3$	1	2
$G.4$	1	3
$G.5$	1	5
$G.6$	1	6
$G.7$	0	2
$G.8$	0	1
$G.9$	1	1
$G.10$	0	0

FIGURE 6. The analysis of case G. Red squares represent the points p_1, p_2, p_3 and p_4 of P in the displayed plane $z = 0$. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge p_5p_6 and the plane $z = 0$.

$y < 3x + 2$. This gives us 44 options for the pair (a, b), displayed in Figure 7 and separated according to the dual oriented matroid.



a	b
0	0
0	$\{1, 2, 3\}$
0	4
0	$\{5, 6, 7\}$
1	1
1	2
1	3
2	2
3	3
1	$\{4, 5\}$
2	$\{3, ..., 7\}$
3	$\{4, ..., 9\}$
1	6
2	8
3	10
1	$\{7, ..., 10\}$
2	$\{9, ..., 13\}$
3	$\{11, ..., 16\}$

FIGURE 7. The analysis of case H. Red squares represent the points p_1, p_2, p_3 and p_4 of P in the displayed plane $z = 0$. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge p_5p_6 and the plane $z = 0$. The circled black crosses are the remaining possibilities after demanding conditions so that P^5 is a $(3, 1)$-polytope.

After considering the restrictions on a and b so that P^6 is a $(3, 1)$-polytope ($a \equiv -b \equiv \pm 1 \pmod 3$), we get just 7 possibilities:

a	b
1	2
1	5
2	$\{4,7\}$
1	8
2	$\{10,13\}$

On the other hand, we need the edge p_5p_6 to be primitive, so we need $\gcd(a,b,4)=1$, which again eliminates two possibilities:

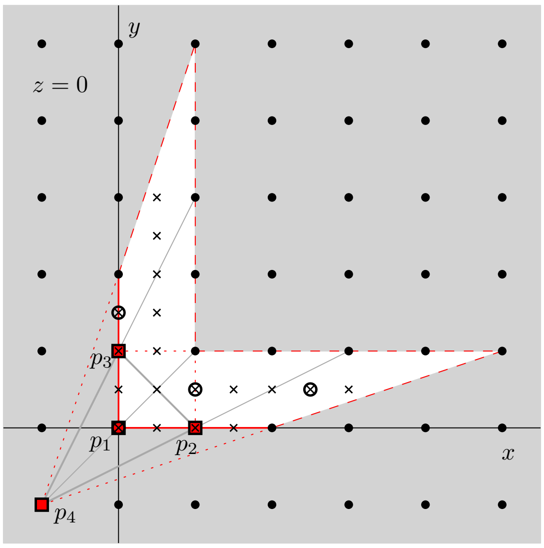
a	b
1	2
1	5
2	7
1	8
2	10

Notice that there may still be points at $z=-1$ and $z=-2$. We consider triangulations of the P, depending on the values of (a,b). When considering a tetrahedron $T_{ijkl}=\operatorname{conv}\{p_i,p_j,p_k,p_l\}$, we will use the the *empty.m* MATLAB program to check whether it is empty or not.

If $(a,b)=(1,2)$, a triangulation is $P=P^5\cup P^6$, and since both P^5 and P^6 are $(3,1)$-polytopes, then P has size 6. If $(a,b)=(1,5)$ or $(2,7)$, a triangulation is $P=P^5\cup P^6\cup T_{2356}$. T_{2356} is empty for $(1,5)$, but not for $(2,7)$. If $(a,b)=(1,8)$ or $(2,13)$, a triangulation is $P=P^5\cup P^6\cup T_{2356}\cup T_{3456}$. Both T_{2356} and T_{3456} are empty for $(1,8)$, but T_{2356} is not empty for $(2,13)$.

The three possibilities that give size 6 are displayed in Table 3.

$\diamond$ **Case I:** $p_5=(1,2,3)$ and $p_6=(a,b,-3)$ with $a\equiv -b\equiv \pm 1 \pmod 3$.
In this case the configuration is contained between the hyperplanes $z=3$ and $z=-3$, and the intersection point of the edge p_5p_6 with $z=0$ is $(a'/2,b'/2,0)=(\frac{a+1}{2},\frac{b+2}{2},0)$. In this case, P^6 doesn't have all the symmetries as before, and we can only assume without loss of generality that this intersection point lies in $x,y\geq 0$. This time, in order for $(2,0,0)$, $(1,1,0)$ and $(0,2,0)$ not to be in $P\cap\{z=0\}$, the region is divided in two: either $x<1$ and $y<3x+2$, or $y<1$ and $x<3y+2$. This gives us 15 options for the pair (a',b'), displayed in Figure 8 and separated according to the dual oriented matroid.



a'	b'
0	0
0	1
1	0
0	2
2	0
0	3
3	0
1	1
1	$\{2,3\}$
$\{2,3\}$	1
1	4
4	1
1	$\{5,6\}$
$\{5,6\}$	1

FIGURE 8. The analysis of case I. Red squares represent the points p_1, p_2, p_3 and p_4 of P in the displayed plane $z=0$. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge p_5p_6 and the plane $z=0$. The circled black crosses are the remaining possibilities after demanding conditions so that P^5 is a $(3,1)$-polytope.

After considering the restrictions on a and b so that P^6 is a $(3,1)$-polytope ($a \equiv -b \equiv \pm 1 \pmod 3$), we get just 4 possibilities:

a'	b'	a	b
0	0	-1	-2
0	3	-1	1
2	1	1	-1
5	1	4	-1

On the other hand, we need the edge p_5p_6 to be primitive, so we need $\gcd(a-1, b-2, 6) = 1$, except for the case $(a', b') = (0,0)$, in which case the edge has the middle point p_1 and we need $\gcd(a-1, b-2, 6) = 2$. This leaves us with two possibilities:

a'	b'	a	b
0	0	-1	-2
0	3	-1	1

There may still be points at $z = \pm 1$ and $z = \pm 2$.

If $(a', b') = (0,0)$, a triangulation is $P = P^5 \cup P^6$ so P has automatically size 6. If $(a', b') = (0,3)$, a triangulation is $P = P^5 \cup P^6 \cup T_{2356} \cup T_{3456}$. Both T_{2356} and T_{3456} are empty in this case.

The two possibilities are displayed in Table 3, and by the same argument as before they belong to different equivalent classes.

4.2.2. *The other two points lie in the same side of the* $(3,1)$ *coplanarity.* Observe that a configuration may contain more than one $(3,1)$-circuit. If one of them leaves the other two points in opposite sides we have already treated it, so we here assume that all the $(3,1)$ coplanarities leave the two other points at the same side.

Without loss of generality, the two extra points are in the semispace $z > 0$, and p_5 will have z-coordinate less than or equal to the z-coordinate of p_6. There are two options, either both p_5 and p_6 are vertices of the final polytope, or one of them is in the convex hull of the other five points.

$\diamond$ Suppose only one of them is a vertex, without loss of generality p_6. In this case, P^6 must be a $(3,1)$-polytope and $p_5 \in P^5$. In particular this means that p_5 has z-coordinate strictly smaller than p_6.

In order for P^6 to be a $(3,1)$-polytope, without loss of generality $p_5 = (0,0,1)$ or $p_5 = (1,2,3)$.

On the other hand $p_5 \in P^5$. Because of the rotation symmetries of the $(3,1)$-circuit, we can assume that p_5 and p_6 are so that $p_5 \in T_{1236}$. The position of the point inside of this tetrahedron can be, considering that $p_5 \notin P \cap \{z = 0\}$:

- **Case J:** In one of the edges. Without loss of generality, $p_5 \in p_2p_6$, an exterior edge of P^5, or $p_5 \in p_1p_6$, the interior edge.
  - $p_5 = (0,0,1)$ and $p_6 = 2p_5 - p_2 = (0,0,2) - (1,0,0) = (-1,0,2)$. In this case, a triangulation is $P = P^6 \cup T_{3456}$. P^6 is a $(3,1)$-polytope and T_{3456} is an empty tetrahedron for this values of p_5 and p_6, so P has size 6:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
J.1 & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}
\end{array}
$$

  - $p_5 = (1,2,3)$ and $p_6 = 2p_5 - p_2 = (2,4,6) - (1,0,0) = (1,4,6)$. The triangulation is the same as before: $P = P^6 \cup T_{3456}$, but in this case T_{3456} is not empty.
  - $p_5 = (0,0,1)$ and $p_6 = 2p_5 - p_1 = (0,0,2)$. In this case, a triangulation is $P = P^6 \cup T_{2356} \cup T_{2456} \cup T_{3456}$. P^6 is a $(3,1)$-polytope and all of T_{2356}, T_{2456}, T_{3456} are empty tetrahedra for this values of p_5 and p_6, so P has size 6:

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
J.2 & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}
\end{array}
$$

– $p_5 = (1,2,3)$ and $p_6 = 2p_5 - p_1 = (2,4,6)$. The triangulation is the same as before: $P = P^6 \cup T_{2356} \cup T_{2456} \cup T_{3456}$, and again all three tetrahedra are empty:

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ J.3 & \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 6 \end{pmatrix} \end{array}$$

- In one of the facets. Without loss of generality, $p_5 \in p_1p_2p_6$, an interior facet of P^5, or $p_5 \in p_2p_3p_6$, the exterior facet.

  In the first case, this facet contains a $(3,1)$-circuit and leaves the other two points at opposite sides. Hence it has been already considered.

  In the second case we have two options. If the points p_1, p_4, p_5 and p_6 are in the same plane, then they form a $(2,2)$-circuit, which is impossible with lattice points since edges p_1p_4 and p_5p_6 are not parallel. If the points are not coplanar, then the configuration is dps, and this case is not possible by the following result of Curcic:

  Lemma 4.2 (Curcic [3, Lemma 3.1.1])**.** *A dps polytope in* $\mathbb{R}^3$ *cannot contain two* $(3,1)$*-circuits sharing an edge or sharing the centroid and another vertex.*

- **Case K:** In the interior. We have two options: if the points p_1, p_4, p_5 and p_6 are in the same plane, then they form a $(2,2)$-circuit, which is impossible with lattice points since edges p_1p_4 and p_5p_6 are not parallel.

  If these four points are not coplanar, then P^4 must be a $(4,1)$-polytope, sharing tetrahedron T_{1235} with P^6, which is a $(3,1)$-polytope. This means that tetrahedron T_{1235} will have volume 1 or 3. We do this computationally (see Algorithm 1 in Section A). There are 2 equivalence classes in this case, displayed in Table 4.

$\diamond$ If both p_5 and p_6 are vertices, then both P^5 and P^6 have to be $(3,1)$-polytopes. This means that these two points have coordinates (a_i, b_i, q_i) with $q_i = 1$ or $q_i = 3$ and $a_i \equiv -b_i \equiv \pm 1 \pmod 3$.

- **Case L**. Both points at $z = 1$: without loss of generality $p_5 = (0,0,1)$ and $p_6 = (a,b,1)$, $a, b \in \mathbb{Z}$. Because of the S_3-symmetries present in P^6, we can assume that $0 \le a \le b$. The configuration has width 1 with respect to the functional z, so in order for P to have size 6, it suffices to have $\gcd(a,b) = 1$.

  Now, depending on whether the edge at $z = 1$ is parallel or not to an or another edge in $z = 0$, the configuration will correspond to a different dual oriented matroid:

  – If $p_6 = (1,1,1)$ or $p_6 = (0,1,1)$, then p_5p_6 is parallel to p_1p_4 and p_1p_3, respectively. Since both these edges are interior edges of the $(3,1)$-circuit, and because of the S_3-symmetries present in P^6, they will give equivalent configurations. We choose $p_6 = (0,1,1)$:

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ L.1 & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

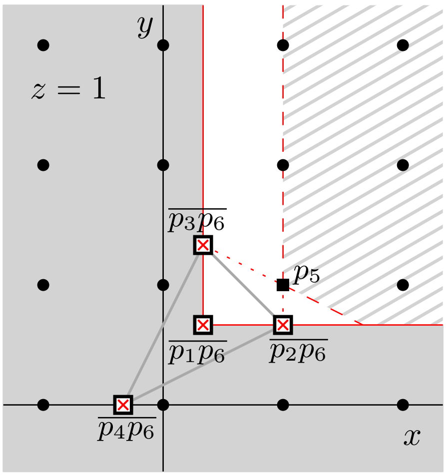
  – If $p_6 = (1,2,1)$, then p_5p_6 is parallel to p_3p_4, an exterior edge of the $(3,1)$-circuit:

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ L.2 & \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

  – If the edge at $z = 1$ is not parallel to any edge in $z = 0$, that is if $0 < a < b$, and $(a,b) \neq (1,2)$, then we get an infinite list of configurations of width 1, and further study will be done in Section 4.7.

$$L.3 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & a \\ 0 & 0 & 1 & -1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

- **Case M**. One point at each $z = 1$ and $z = 3$: $p_6 = (1, 2, 3)$ and $p_5 = (a, b, 1)$ for some $a, b \in \mathbb{Z}$.
  The configuration is contained between the hyperplanes $z = 0$ and $z = 3$ so we first use the parallel-planes method to check which coordinates for p_5 do not produce more lattice points in the plane $z = 1$. The intersection points of edges p_1p_6, p_2p_6, p_3p_6 and p_4p_6 with $z = 1$ are, respectively, $(1/3, 2/3, 1)$, $(1, 2/3, 1)$, $(1/3, 4/3, 1)$ and $(-1/3, 0, 1)$. We want to check what values of a, b leave p_5 outside the convex hull of those four intersection points. Without loss of generality, because of the rotation symmetries of P^5, we can assume that p_5 lies in $x \geq 1/3$, $y \geq 2/3$ (non-shaded area in Figure 9).
  Suppose $a > 1$ or $b > 1$. Then the convex hull of P at $z = 1$ will enclose $(1, 1, 1)$ as a seventh lattice point. Hence the only possibility is that $a = 1 = b$, and $p_5 = (1, 1, 1)$.
  It remains to check whether any lattice points at $z = 2$ lie inside P. A triangulation is $P = P^5 \cup T_{2356}$. We already know P^5 to be a $(3, 1)$-polytope, and T_{2356} is indeed empty. Hence the configuration has size 6.



$$M \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \end{array}$$

FIGURE 9. The analysis of case M. The red crossed squares are the intersection of p_1p_6, p_2p_6, p_3p_6 and p_4p_6 with the displayed plane $z = 0$. Black dots are the lattice points in the plane and black squares represent the possible lattice points for p_5.

It happens that configuration M has the same dual oriented matroid as L.3 (dual oriented matroids change with coplanarities, but not with parallelism), so the points in M are ordered according to the labels of the corresponding dual oriented matroid.

- Both points at $z = 3$: without loss of generality $p_5 = (1, 2, 3)$ and $p_6 = (a, b, 3)$ with $a \equiv -b \equiv \pm 1 \pmod 3$. In this case, all six points have the z-coordinate 0 or 3. So the volume of every tetrahedra with vertices in those six points will have a factor 3.
  On the other hand, because of the construction of the configuration, all P^2, P^3 and P^4 must be $(3, 2)$-polytopes. But for this to happen we would need some unimodular tetrahedra in P. Hence P will not have size 6.

We now analyze the width in cases G to M: in Table 3 and 4 we list functionals that give width 1, 2, or 3 for each of the configurations obtained. To see that these functionals actually give the minimum width we argue as follows:

- The only functional that can possibly give width one is the functional z: every other functional is non-constant in the plane $z = 0$. Since the subconfiguration in this plane has an interior point, every non-constant functional takes at least three values on it. That is, the only configurations giving width 1 are those in case L.

- For the rest, we have found functionals of width two for all cases except J.3, in which we give one of width three. But the volume vector of J.3 has gcd equal to three, which implies its width is a multiple of three.

In summary:

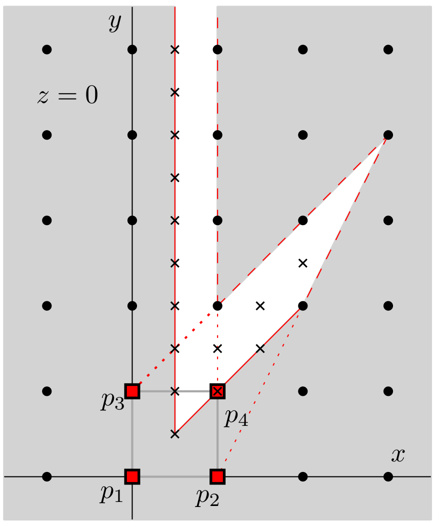
Theorem 4.3. *Among the lattice 3-polytopes of size six with no 5 coplanar points but with some $(3,1)$ coplanarity, there are exactly 20 equivalence classes of width two, 1 of width three, and none of width larger than three, as shown in Tables 3 and 4. 13 of those are dps, all of them of width two.*

4.3. **Polytopes containing a $(2,2)$-circuit (but no $(3,1)$-circuit, and no five coplanar points).** These are clearly non-dps configurations. Without loss of generality, we assume they contain the standard $(2,2)$-circuit: $p_1 = o$, $p_2 = e_1$, $p_3 = e_2$ and $p_4 = e_1 + e_2$.
Again, we treat separately the possibilities for which side of the coplanarity the other two other points lie in.

4.3.1. ***Case N:*** *The other two points lie in opposite sides of the $(2,2)$-circuit.* Then both p_5 and p_6 are vertices and so P^5 and P^6 must be $(2,1)$-polytopes. Then the fifth and sixth points must be a lattice distance one from the $(2,2)$-circuit. Without loss of generality $p_5 = e_3$, and $p_6 = (a, b, -1)$ for $a, b \in \mathbb{Z}$.

The configuration is contained in the three planes $z \in \{-1, 0, 1\}$, so we use the parallel-planes method to check when there are no more lattice points in the plane $z = 0$. There is one single point in the planes $z = \pm 1$, and at $P \cap \{z = 0\}$ is the convex hull of the $(2,2)$-circuit and the intersection point $(a/2, b/2, 0)$ of the edge p_5p_6 with the plane $z = 0$.

Without loss of generality (because of the symmetries present in P^6) the intersection point can be assumed to be in the region $1/2 \le x \le y$. In order for $(1,2,0)$ and $(2,2,0)$ not to be in $P \cap \{z = 0\}$, the region is bounded and divided in two: either $x < 1$, or $2x - 2 < y < x + 1$ (non-shaded area in Figure 10). This gives an infinite number possibilities for the pair (a, b), separated according to its dual oriented matroid in Figure 10.



a	b
1	1
1	2
2	2
2	3
3	3
3	4
4	5
1	≥ 3

FIGURE 10. The analysis of case N. Red squares represent the points p_1, p_2, p_3 and p_4 of P in the displayed plane $z = 0$. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge p_5p_6 and the plane $z = 0$.

All the options have automatically size 6 since no more points arise at $P \cap \{z = 0\}$ and the only points at $z = 1$ and $z = -1$ are, respectively, p_5 and p_6.

- ∘ If $(a, b) = (1, 1)$, then the configuration has size 6 since $P = P^5 \cup P^6$ is a polyhedral subdivision into two polytopes of size five and signature $(2, 2)$:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ N.1 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

- ∘ If $(a, b) = (2, 2)$ then (p_5p_6, p_4) form a $(2, 1)$-circuit. This case is equivalent as p_1 being the middle point of this collinearity, so instead we choose $p_6 = (0, 0, -1)$:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ N.2 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

- ∘ If $(a, b) = (1, 2)$, then (p_3p_4, p_5p_6) form a $(2, 2)$-circuit. Again the case is equivalent as having (p_1p_3, p_5p_6) a $(2, 2)$-circuit, so instead we choose $p_6 = (0, 1, -1)$:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ N.3 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

- ∘ If $(a, b) = (2, 3)$ or $(3, 3)$, then $(p_2p_5p_6, p_4)$ and $(p_1p_5p_6, p_4)$, respectively, are $(3, 1)$-circuits, and this case has already been dealt with.

- ∘ If $(a, b) = (3, 4)$ or $(4, 5)$, the two points correspond to the same dual oriented matroid, but they have different volume vectors, as shown in Table 5.

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ N.4 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ N.5 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

- ∘ If $(a, b) = (1, b)$ for $b \geq 3$ then we get an infinite list of configurations of width 1, and further study will be done in Section 4.7.

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ N.6 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$$

4.3.2. *The other two points lie on the same side of the* $(2, 2)$*-circuit.* Observe that a configuration may contain more than one $(2, 2)$-circuit. If one of them leaves the other two points in opposite sides we have already treated it, so we here assume that all the $(2, 2)$ coplanarities leave the two other points at the same side.

Without loss of generality, the two extra points are in the semispace $z > 0$, and p_5 will have z-coordinate less than or equal to the z-coordinate of p_6. There are two options, either both p_5 and p_6 are vertices of the final polytope, or one of them is in the convex hull of the other five points.

⋄ Suppose only one of them is a vertex, without loss of generality p_6. In this case, P^6 must be a $(2, 2)$-polytope and $p_5 \in P^5$. In particular this means that p_5 has z-coordinate strictly smaller than p_6.

In order for P^6 to be a $(2, 2)$-polytope, without loss of generality $p_5 = (0, 0, 1)$.

On the other hand $p_5 \in P^5$. Because of the symmetries of the $(2, 2)$-circuit, we can assume that p_5 and p_6 are so that $p_5 \in T_{1236}$. The position of the point inside of this tetrahedron can be, considering that $p_5 \notin P \cap \{z = 0\}$:

- **Case O:** In one of the edges. Without loss of generality, $p_5 \in p_1p_6$, that is, $p_6 = (0,0,2)$. Then P has automatically size 6 since it has width 1 with respect to the functional x, and there are only 2 and 4 lattice points, respectively, in the planes $x = 1$ and $x = 0$:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ O & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \end{array}$$

- In one of the facets. But then there would be a $(3,1)$-circuit, and this case has been already considered.

- **Case P:** In the interior. We have two options: if the points p_1, p_4, p_5 and p_6 are in the same plane, then they form a $(2,2)$-circuit, which is impossible with lattice points since edges p_1p_4 and p_5p_6 are not parallel.

  If these four points are not coplanar, then P^4 must be a $(4,1)$-polytope, sharing tetrahedron T_{1235} with P^6, which is a $(2,2)$-polytope. This means that tetrahedron T_{1235} has volume 1. We do this computationally (see Algorithm 2 in Section A). There are 2 equivalence classes in this case, displayed in Table 6.

$\diamond$ If both p_5 and p_6 are vertices, then both P^5 and P^6 have to be $(2,2)$-polytopes. Then without loss of generality $p_5 = (0,0,1)$ and $p_6 = (a,b,1)$, $a,b \in \mathbb{Z}$. Because of the symmetries present in P^6, we can assume that $0 \le a \le b$. The configuration has width 1 with respect to the functional z, so in order for P to have size 6, it suffices to have $\gcd(a,b) = 1$.

Now, depending on whether the edge at $z = 1$ is parallel or not to an or another edge in $z = 0$, the configuration will correspond to a different dual oriented matroid:

- If $p_6 = (0,1,1)$, then p_5p_6 is parallel to p_1p_3:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ Q.1 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

- If $p_6 = (1,1,1)$, then (p_1p_6, p_4p_5) will form a $(2,2)$-circuit with the other two points at opposite sides, but this case has already been considered.

- If the edge at $z = 1$ is not parallel to any edge in $z = 0$, that is if $0 < a < b$, then we get an infinite list of configurations of width 1, and further study will be done in Section 4.7.

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ Q.2 & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & a \\ 0 & 0 & 1 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Let us analyze the width in cases N to Q:

- Clearly configurations N.1, N.2, N.3, N.6, O and Q have width 1 with respect to the functional x, and Q with respect to z.
- In cases N.4 and N.5, the subconfiguration at $z = 0$ contains one interior point, so we use the same argument as before to state that the width is at least 2. We have found functionals giving width 2 for both cases.
- In cases P, a $(4,1)$-polytope is a subpolytope of the configuration, so they have an interior point and the width is at least 2. Again we have found functionals that give this width.

In summary:

Theorem 4.4. *Among the lattice 3-polytopes of size six with no 5 coplanar points, no $(3,1)$ coplanarity, but with some $(2,2)$ coplanarity, there are exactly 4 equivalence classes of width two, and none of larger width, as shown in Tables 5 and 6. All of them are non-dps.*

4.4. **Polytopes containing a $(2,1)$-circuit (but no other coplanarity).** Non-dps configurations consisting of a lattice 3-polytope with one primitive edge extended. This polytope will be of signature $(2,1)$, $(4,1)$ or $(3,2)$, since the other cases have already been dealt with. Suppose that $R = \operatorname{conv}\{r_1, r_2, r_3, u_1, u_2, u_3\}$ is a configuration of size 6, where $r_1r_2r_3$ is an edge with three collinear points. Then either r_1 or r_3 is a vertex. Suppose the latter holds, then $R \setminus \{r_3\}$ must be a $(2,1)$, $(4,1)$ or $(3,2)$-polytope.

- **Case R:** $R \setminus \{r_3\}$ is a $(2,1)$-polytope. We take the representative of a $(2,1)$-polytope:

$$p_1 = (0,0,0), \quad p_2 = (1,0,0), \quad p_3 = (0,0,1), \quad p_4 = (-1,0,0), \quad p_5 = (p,q,1)$$

with $0 \le p \le \frac{q}{2}$ and $\gcd(p,q) = 1$. The only way to extend an edge and not getting 5 coplanar points is to extend $\overline{p_3p_5}$. They have width 1, so by simple traslation, we can choose, without loss of generality, $p_6 = (-p,-q,1)$. Each configuration consists of two double non-parallel edges at lattice distance 1, so they have automatically size 6. Again we get an infinite list of configurations of width 1, and further study will be done in Section 4.7. Notice that in this case, both $R\setminus\{r_3\}$ and $R\setminus\{r_1\}$ are $(2,1)$-polytopes.

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ R & \begin{pmatrix} 0 & 1 & 0 & -1 & p & -p \\ 0 & 0 & 0 & 0 & q & -q \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

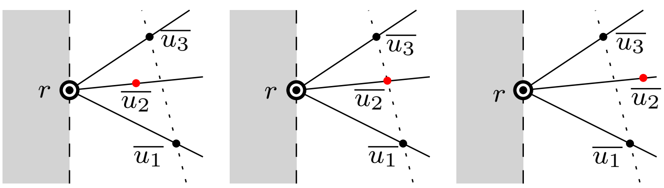
- $R \setminus \{r_3\}$ is a $(4,1)$-polytope. There are three possible ways to extend an edge of a $(4,1)$-polytope, each of them corresponding to a different dual oriented matroid. Let $r_3 = 2r_2 - r_1$:
  - ∗ **Case S:** r_1 is the interior point of the $(4,1)$-polytope, and r_2 is a vertex.
  - ∗ **Case T:** r_2 is the interior point of the $(4,1)$-polytope, and r_1 is a vertex.
  - ∗ **Case U:** both r_1 and r_2 are vertices of the $(4,1)$-polytope.

We do this computationally (see Algorithm 3 in Section A). There are 6, 6 and 5 equivalence classes, respectively, displayed in Table 7.

In this case, $R \setminus \{r_1\}$ is, respectively, a polytope of signature $(4,1)$ but size 6, and a $(3,2)$-polytope in the last two cases.

- $R \setminus \{r_3\}$ is a $(3,2)$-polytope. Now the case when $R \setminus \{r_3\}$ or $R \setminus \{r_1\}$ are $(2,1)$ or $(4,1)$-polytopes is covered. So we actually want both of them to be $(3,2)$-polytopes. It is easy to check that with this condition, r_2 is in a boundary edge, and all other 5 points must be vertices of the configuration.

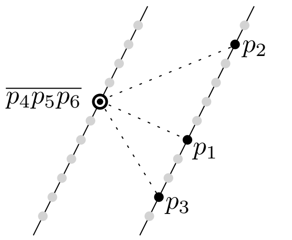
Let us consider the projection to a plane in the direction of the edge $r_1r_2r_3$, and let us denote with r the projection of this edge, and use $\overline{u_i}$ for the projection of each other point. Considering what we said above, and taking under account that we do not want 5 coplanar points, it is easy to see that $\overline{u_1}, \overline{u_2}, \overline{u_3}$ each lie in a different ray with vertex r, and that they are all contained in an open semispace defined by a hyperplane passing through r. There are three options, as shown in the figure below:



Now, $R \setminus \{u_i\}$ is a $(2,1)$-polytope for each $i = 1,2,3$. This condition is verified if and only if $r_1r_2r_3$ and u_iu_j are contained in parallel consecutive lattice planes for $i,j \in \{1,2,3\}$, $i \ne j$. In the projection, this means that each $\overline{u_iu_j}$ must span a straight line at lattice distance 1 from r. Let us see each case separately:

  - ∗ In the first case, it is clear that $\overline{u_1u_3}$ is not at lattice distance 1 from r, since $\overline{u_2}$ is in an intermediate parallel line.

- **Case V:** In the second case, $\overline{u_1}$, $\overline{u_2}$ and $\overline{u_3}$ must all lie in the same straight line at lattice distance 1 from r, as shown in Figure 11. This means that they lie in a plane parallel to the three collinear points $r_1r_2r_3$. In order for R to have size 6, $u_1u_2u_3$ must be a unimodular triangle. Without loss of generality, these parallel planes are $z = 0, 1$, and the unimodular triangle is $p_1 = (0, 0, 0)$, $p_2 = (1, 0, 0)$ and $p_3 = (0, 1, 0)$. The fourth point can be, also without loss of generality, $p_4 = (0, 0, 1)$, and because of the symmetries of T_{1234}, $p_5 = (a, b, 1)$ with $0 < a \leq b$ ($a > 0$ so no new coplanarity arises) and $p_6 = (-a, -b, 1)$. In order for $P = \operatorname{conv}\{p_i\}$ to have size 6, we just need the edge to be double, that is, to have $\gcd(a, b) = 1$.



$$
V \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}
$$
$$
V = \begin{pmatrix} 0 & 1 & 0 & 0 & a & -a \\ 0 & 0 & 1 & 0 & b & -b \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}
$$

FIGURE 11. The analysis of case V. The projection onto a plane in the direction of edge $p_4p_5p_6$ is displayed. The two straight lines represent the parallel lines in which the six points of P lie in the projection. Black dots correspond to the projections of points of P, and grey dots indicate the places where the presence of other lattice points in the projection is not excluded. The circled dot represents the projection of the three collinear points

Notice that all the different values of (a, b) correspond to the same dual oriented matroid. Further study on this infinite series is done in Section 4.7.

- **Case W:** In the third case, in particular we need to have lattice distance 1 from r to the line spanned by $\overline{u_1}$ and $\overline{u_3}$. Besides, if the edge $\overline{u_1u_3}$ were not primitive in the projection, then either $\overline{u_1u_2}$ or $\overline{u_2u_3}$ would not be at lattice distance 1 from r. In particular this means that the projection of $u_1u_2u_3$ must be a unimodular triangle. And in that case $u_1u_2u_3$ itself is a unimodular triangle. Without loss of generality we can now assume that the double edge r is $p_1 = (0, 0, 0)$, $p_2 = (1, 0, 0)$, $p_3 = (-1, 0, 0)$ (proyection in the direction of functional x) and u_1 and u_3 are, respectively, $p_4 = (0, 1, 0)$ and $p_5 = (0, 0, 1)$. The sixth point u_2 will be $p_6 = (a, b, c)$. Since its projection is (b, c), then we will need $b, c > 0$. And since P^6, with $P = \operatorname{conv}\{p_i\}$, is symmetric with respect to the plane $y = z$, so we may also assume $b \geq c$ (non-shaded area in Figure 12). P^6 is also symmetric with respect to $x = 0$, and since we do not want more coplanarities, we may also assume $a > 0$.

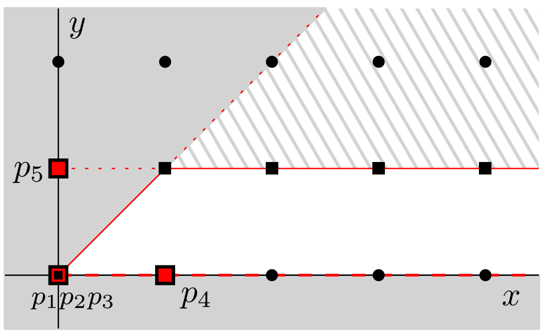


FIGURE 12. The analysis of case W. The projection onto the plane $x = 0$ in the direction of edge $p_1p_2p_3$ is displayed. Red squares represent the points p_1, p_4 and p_5 of P in the displayed plane $x = 0$. And the double red square represents the projection of edge $p_1p_2p_3$. Black dots correspond to the projection of lattice points and black squares represent the possible projections of the point p_6 onto $x = 0$.

Suppose now that $c > 1$. Then the points $(x, 1, 1)$ lie in an intermediate plane parallel to edges p_5p_6 and p_2p_3, so P^4 does not have width 1 with respect to this pair of edges and it is not a $(2, 1)$-polytope. Hence we can only have $c = 1$.

In this case, the configuration has width 1 with respect to the functional z. At $z = 0$ there are only four points $p_1, \ldots, p_4$, and at $z = 1$, the edge p_5p_6. In order for this edge to be primitive, we need $\gcd(a, b) = 1$. Finally, we want to avoid $(2, 2)$-circuits. The only possibility for that is if $a = 1 = b$, and this case is excluded if we consider two separate possibilites: either $0 < a < b$, or $0 < b < a$.

This is as well useful since it happens that for each of this conditions, the configuration gives a different dual oriented matroid:

$$W.1 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}, \qquad 0 < a < b \text{ and } \gcd(a, b) = 1$$

$$W.2 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} a & 0 & 0 & 0 & -1 & 1 \\ b & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad 0 < b < a \text{ and } \gcd(a, b) = 1$$

Further study will be done in Section 4.7.

Let us analyze the width in cases R to W.

- Clearly configurations R, V and W have width 1 with respect to the functional z.
- In cases S, T and U, since a $(4, 1)$-polytope is a subpolytope of the configuration, they all have an interior point and the width is at least 2. We were able to find functionals giving width 2 to all of them.

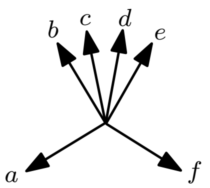
In summary:

Theorem 4.5. *Among the lattice 3-polytopes of size six with some $(2, 1)$ coplanarity and no other coplanarity, there are exactly 17 equivalence classes of width two, and none of larger width, as shown in Table 7. All of them are non-dps.*

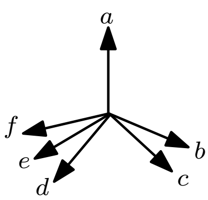
4.5. **Polytopes with no coplanarities and with at least one interior point.** Since there are no coplanarities, in particular these configurations are dps. Apart from that, they must have unimodular triangles as facets and, since they have at least one interior point, they will have width > 1.

We explore the possible dual oriented matroids for a configuration with no coplanarities and at least one interior point, and we see that there are exactly 2, modulo symmetries and rotation, displayed below, along with the 6 circuits present in the configuration:

$(bcde, a)$
$(abcd, f)$
(abc, ef)
$(abcd, f)$
(def, ab)
$(bcde, f)$



6.1*



6.2*

$(bcde, a)$
(bcd, af)
(aef, bc)
(def, ab)
$(cdef, a)$
(def, bc)

So in any such polytope P there are two vertices p_i and p_j such that both P^i and P^j are $(4, 1)$-polytopes. Hence, the full classification of them can be done by an exhaustive exploration of all the possible ways to glue together two of the eight $(4, 1)$-polytopes from Table 1:

• **Case X:** If the $(4, 1)$-polytopes share the same interior point, then the final configuration has 1 interior point. We do this computationally (see Algorithm 4 in Section A). There are 20 equivalence classes, displayed in Table 8.

• **Case Y:** If they do not share the same interior point, then the interior point of one is a vertex of the other and viceversa, and the final configuration has 2 interior points. We do this computationally (see Algorithm 5 in Section A). There are 12 equivalence classes, displayed in Table 9.

Since they all contain an interior point, the width is at least 2. All of them have exactly width 2 except for the one labeled $Y.12$. We leave to the reader to check that all sets of three parallel planes containing the points of P are not consecutive planes, which means that the width is at least 3. And indeed we find a functional that gives width 3.

In summary:

Theorem 4.6. *Among the lattice 3-polytopes of size six with no coplanarities and at least one interior point, there are exactly 31 equivalence classes of width two, 1 of width three, and none of larger width, as shown in Tables 8 and 9. All of them are dps.*

4.6. **Polytopes with no coplanarities and with no interior points. Case Z**

As before, these configurations are dps. Let P be a lattice 3-polytope of size 6, with no interior points and with no coplanarities. This implies that every lattice point in P must be a vertex. By Howe's Theorem 1.6, P has width 1. Since there are no coplanarities, it consists of two unimodular triangles in consecutive lattice planes, with no two parallel edges.

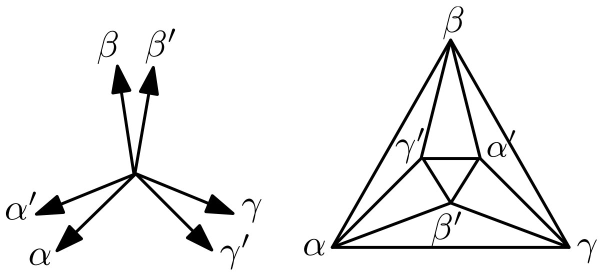
There are two possible dual oriented matroids for such a configuration, namely (the duals of) 6.3 and 6.4 of Figure 13. We are going to prove that 6.3 cannot be realized by a lattice point configuration without extra lattice points in the convex hull

Lemma 4.7. *Let P be a lattice 3-polytope of size 6, consisting of two unimodular triangles in consecutive lattice planes, with no two parallel edges. Then its dual oriented matroid is 6.4 of Figure 13.*

Proof. Let $T = \operatorname{conv}\{p_1, p_2, p_3\}$ be a triangle with primitive edges in the plane $z = 0$. For $i = 1, 2, 3$, let t_i be the outward normal vector of the edge p_jp_k, where $\{i, j, k\} = \{1, 2, 3\}$, normalized so that its endpoints are lattice points and so that it is primitive. (Each t_i is the 90 degree rotation of the corresponding edge vector). Clearly for each pair in $i, j \in \{1, 2, 3\}$, $|\det(t_i, t_j)|$ equals the normalized volume of T.

Let now $S = \operatorname{conv}\{q_i\}$ be another triangle with primitive edges in the parallel plane $z = 1$, with no edges parallel to those in T. Let s_i be the corresponding normal vectors as before. We now have six distinct normal vectors.

We claim that if the (dual) oriented matroid of $P = \operatorname{conv}(T \cup S)$ was 6.3, then the sequence of the six normal vectors, cyclically ordered according to their angle, should alternate one vector from each triangle. For this, the following figure shows the dual oriented matroid 6.3 and the Schlegel diagram of its dual (an octahedron):

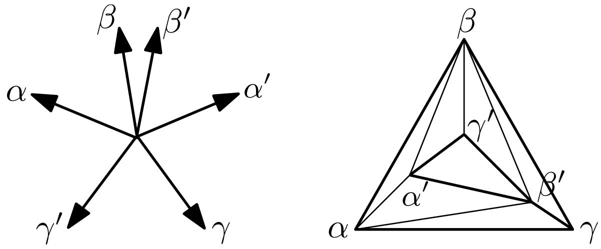


The six triangles in the region between the triangles $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ in the Schlegel diagram correspond to the six normal vectors. The fact that the triangles alternate between using two vertices from S and from T implies that the normal vectors alternate (to see this, observe that two points p_i and p_j form a triangle with a point q_k if and only if the normal vector of p_ip_j belongs to the normal cone of q_k).

Suppose now that our triangles are unimodular and let us see that such an alternation of normal vectors is impossible. We can now assume without loss of generality that the normal vectors for T are $t_1 = (1, 0)$, $t_2 = (0, 1)$ and $t_3 = (-1, -1)$ and that each s_i is between t_j and t_k where $\{i, j, k\} = \{1, 2, 3\}$. By symmetry, we assume that $s_2 = (a, -b)$ with $a > 0$, $b > 0$ (if $s_2 = -t_2$, a coplanarity arises). And clearly $s_3 = (c, d)$ with $c, d > 0$.

Then in order for S to be unimodular we need $\det(s_2, s_3) = \pm 1$. But $\det(s_2, s_3) = ad + bc \geq 2$ since $a, b, c, d > 0$. And the result follows. $\square$

Now we know that our configuration will have dual oriented matroid 6.4, and that the sequence of the normal vectors must not alternate between T and S. Then this sequence contains two consecutive s's and two consecutive t's. The next figure shows the dual matroid and its Schlegel diagram.



In terms of the vertex cones of the triangles, we can see that in T, the vertex cone in γ contains two edge vectors of S, the vertex cone in α contains one edge vector of S, and the vertex cone in β does not contain any edge vector. And respectively with the vertex cones of γ', α' and β' in S.

We will now describe our configuration so that the vertex cone in p_1 contains two edge vectors, and so that the vertex cone in p_3 contains one.

Without loss of generality, T is the standard unimodular triangle $p_1 = (0,0,0)$, $p_2 = (1,0,0)$ and $p_3 = (0,1,0)$. Again without loss of generality, the unimodular triangle S is $p_4 = q_1 = (0,0,1)$, $p_5 = q_2 = (a,b,1)$ and $p_6 = q_3 = (c,d,1)$, with $ad - bc = 1$. Since two edge vectors of S must be contained in the vertex cone of p_1, then $a,b,c,d > 0$. And since the vertex cone in p_3 must contain the remaining edge vector, then $c + d > a + b$:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ Z & \begin{pmatrix} 0 & 1 & 0 & 0 & a & c \\ 0 & 0 & 1 & 0 & b & d \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Further study will be done in Section 4.7.

4.7. **Infinite series.**

In the previous discussion we have found several infinite lists of lattice 3-polytopes of size 6, expressed depending on some parameters. The lists are disjoint pairwise, since the sections in which they were derived were explicitly disjoint. On the other hand, different parameters within each case may give equivalent configurations. In this section we work on each list and try to reduce the range of the parameters so that each choice of parameters gives a different equivalence class.

Case C: The polytopes on the list have points

$$p_1 = (0,0,0), \quad p_3 = (0,1,0), \quad p_4 = (0,2,0), \quad p_5 = (0,3,0)$$

$$p_2 = (1,0,0), \quad p_6 = (1,p,q)$$

for some $q \geq 1$, $\gcd(p,q) = 1$. We already now that the value of p can be considered modulo q since it gives equivalent configurations. So we assume $0 \leq p < q$.

Now, the volume vector of this configuration is

$$\begin{pmatrix} 0, & 0, & q, & 0, & 2q, & 3q, & 0, & 0, & 0, & 0, & 0, & -q, & -2q, & -q, & 0 \end{pmatrix}$$

so clearly different values of q give non-equivalent configurations.

Let $P = \{p_i\}_i$ and $P' = \{p'_i\}_i$ where $p'_i = p_i$ except $p'_6 = (1, p', q)$ for some $0 \leq p' < q$ with $\gcd(p', q) = 1$.

Now, in P and P', $p_1p_3p_4p_5$ are four collinear points and p_2p_6 form and edge at lattice distance 1. Then any unimodular transformation that maps P to P' will fix the following sets of points:

$$\{p_1, p_5\}, \quad \{p_3, p_4\}, \quad \{p_2, p_6\}$$

and p_1 is fixed if and only if p_3 is fixed. That gives us four possible unimodular transformations:

- All points fixed (it is implied $p_6 \mapsto p_6'$):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{p'-p}{q} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- $p_2 \mapsto p_6'$ and $p_6 \mapsto p_2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ p' & 1 & -\frac{p+p'}{q} \\ q & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- $p_1 \leftrightarrow p_5$ and $p_3 \leftrightarrow p_4$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & \frac{p+p'}{q} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

- $p_1 \leftrightarrow p_5$, $p_3 \leftrightarrow p_4$, $p_2 \mapsto p_6'$ and $p_6 \mapsto p_2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ p'-3 & -1 & \frac{p-p'}{q} \\ q & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

So p and p' give equivalent configurations if and only if $p' \equiv \pm p \pmod q$. This reduces the range of the parameters to $0 \leq p \leq \frac{q}{2}$, and $q \geq 1$.

Case L.3: The polytopes on the list have points

$$p_1 = (0,0,0), \quad p_2 = (1,0,0), \quad p_3 = (0,1,0), \quad p_4 = (-1,-1,0)$$

$$p_5 = (0,0,1), \quad p_6 = (a,b,1)$$

for some $0 < a < b$, $\gcd(a,b) = 1$ and $(a,b) \neq (1,2)$.

Now, the volume vector of this configuration is

$$\begin{pmatrix} 0, & 1, & 1, & -1, & -1, & -b, & 1, & 1, & a, & -a+b, & 3, & 3, & a+b, & -a+2b, & -2a+b \end{pmatrix}$$

Let $P = \{p_i\}_i$ and $P' = \{p_i'\}_i$ where $p_i' = p_i$ except $p_6' = (a', b', 1)$ for some $0 < a' < b'$, $\gcd(a', b') = 1$ and $(a', b') \neq (1,2)$.

Now, in P and P', $p_1p_2p_3p_4$ form a $(3,1)$-circuit (with p_1 the barycenter point) and p_5p_6 form and edge at lattice distance 1. Then any unimodular transformation that maps P to P' will fix the following sets of points:

$$\{p_1\}, \quad \{p_2, p_3, p_4\}, \quad \{p_5, p_6\}$$

That gives us twelve possible unimodular transformations: the six unimodular transformations of the $(3,1)$-circuit (permutation of vertices), and for each of them, the possibility of fixing p_5 or mapping it to p_6'.

Suppose we fix $p_5 = (0,0,1)$. The only transformation that maps (a,b) to a point in the region $0 < x < y$ is the one that fixes all points in the $(3,1)$-circuit. And this transformation is the identity and maps $(a,b,1)$ to itself.

Suppose now $p_5 = (0,0,1)$ is mapped to $p_6' = (a', b', 1)$. Then $(a,b,1)$ ought to be mapped to $(1,0,0)$. Let us consider all six transformations of the circuit:

- $p_5 \mapsto p_6'$ and $p_6 \mapsto p_5$:

$$\begin{pmatrix} 1 & 0 & a' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $p_1 \leftrightarrow p_2$, $p_5 \mapsto p_6'$ and $p_6 \mapsto p_5$:

$$\begin{pmatrix} 0 & 1 & a' \\ 1 & 0 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} b+a' \\ a+b' \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $p_1 \leftrightarrow p_3$, $p_5 \mapsto p_6'$ and $p_6 \mapsto p_5$:

$$\begin{pmatrix} -1 & 0 & a' \\ -1 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -a+a' \\ -a+b+b' \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $p_2 \leftrightarrow p_3$, $p_5 \mapsto p_6'$ and $p_6 \mapsto p_5$:

$$\begin{pmatrix} 1 & -1 & a' \\ 0 & -1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} a-b+a' \\ -b+b' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \leftrightarrow\ b=b',\ a'=b-a$$

- $p_1 \mapsto p_2 \mapsto p_3 \mapsto p_1$, $p_5 \mapsto p_6'$ and $p_6 \mapsto p_5$:

$$\begin{pmatrix} 0 & -1 & a' \\ 1 & -1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -b+a' \\ a-b+b' \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $p_1 \mapsto p_3 \mapsto p_2 \mapsto p_1$, $p_5 \mapsto p_6'$ and $p_6 \mapsto p_5$:

$$\begin{pmatrix} -1 & 1 & a' \\ -1 & 0 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} -a+b+a' \\ -a+b' \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So (a,b) and (a',b') with $\gcd(a,b) = \gcd(a',b') = 1$ and $0 < a < b$, $0 < a' < b'$ give equivalent configurations if and only if $b = b'$ and $a' = b - a$. This reduces the range of the parameters to $0 < 2a \leq b$, but since $(2,1)$ is not a valid point, then $0 < 2a < b$.

Case N.6:
The volume vector of this configuration is

$$\begin{pmatrix} 0, & 1, & -1, & 1, & -1, & -a, & -1, & 1, & 1, & 1-a, & -1, & 1, & a-1, & -1, & -a+2 \end{pmatrix}$$

so clearly each value of $a > 2$ gives a different equivalence class.

Case Q.2: The volume vector of this configuration is

$$\begin{pmatrix} 0, & 1, & 1, & 1, & 1, & -b, & -1, & -1, & a, & a-b, & -1, & -1, & a+b, & a, & -b \end{pmatrix}$$

so clearly each pair (a,b) with $0 < a < b$ and $\gcd(a,b) = 1$ gives a different equivalence class.

Case R:
The polytopes on the list have points

$$q_1 = (1,0,0), \quad q_2 = (0,0,0), \quad q_3 = (-1,0,0)$$

$$q_4 = (p,q,1), \quad q_5 = (0,0,1), \quad q_6 = (-p,-q,1)$$

for some $0 \leq p \leq \frac{q}{2}$, $\gcd(p,q) = 1$.

Now, the volume vector of this configuration (for the original ordering of the points) is

$$\begin{pmatrix} 0, & -q, & q, & 0, & 0, & 2q, & -q, & q, & 0, & -2q, & -2q, & 2q, & 0, & -4q, & 0 \end{pmatrix}$$

so clearly different values of q give non-equivalent configurations.

Let $P = \{q_i\}_i$ and $P' = \{q_i'\}_i$ where $q_i' = q_i$ except $q_4' = (p',q,1)$ and $q_6' = (-p',-q,1)$ for some $0 \leq p' \leq \frac{q}{2}$ with $\gcd(p',q) = 1$.

Now, in P and P', $q_1q_2q_3$ and $q_4q_5q_6$ are two sets of three collinear points at lattice distance 1. Then any unimodular transformation that maps P to P' will fix the following sets of points:

$$\{q_1, q_3, q_4, q_6\}, \quad \{q_2, q_5\}$$

and q_2 is fixed if and only if $\{q_1, q_3\}$ is fixed. That gives us eight possible unimodular transformations:

- All points fixed (it is implied $q_5 \mapsto q_5'$ and $q_6 \mapsto q_6'$):

$$\begin{pmatrix} 1 & \frac{p'-p}{q} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- $q_1q_2q_3 \mapsto q_3q_2q_1$:

$$\begin{pmatrix} -1 & \frac{p'+p}{q} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- $q_4q_5q_6 \mapsto q_6'q_5q_4'$:

$$\begin{pmatrix} 1 & -\frac{p'+p}{q} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- $q_1q_2q_3 \mapsto q_3q_2q_1$ and $q_4q_5q_6 \mapsto q_6'q_5q_4'$:

$$\begin{pmatrix} -1 & -\frac{p'-p}{q} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- $q_1q_2q_3 \mapsto q_4'q_5q_6'$ and $q_4q_5q_6 \mapsto q_1q_2q_3$:

$$\begin{pmatrix} p' & \frac{1-pp'}{q} & 0 \\ q & -p & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $q_1q_2q_3 \mapsto q_6'q_5q_4'$ and $q_4q_5q_6 \mapsto q_1q_2q_3$:

$$\begin{pmatrix} -p' & \frac{1+pp'}{q} & 0 \\ -q & p & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $q_1q_2q_3 \mapsto q_4'q_5q_6'$ and $q_4q_5q_6 \mapsto q_3q_2q_1$:

$$\begin{pmatrix} p' & -\frac{1+pp'}{q} & 0 \\ q & -p & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $q_1q_2q_3 \mapsto q_6'q_5q_4'$ and $q_4q_5q_6 \mapsto q_3q_2q_1$:

$$\begin{pmatrix} -p' & \frac{pp'-1}{q} & 0 \\ -q & p & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So p and p' give equivalent configurations if and only if $p' \equiv \pm p^{\pm 1} \pmod q$. In order to remove redundancies, we change p to the minimum of $\{\pm p, \pm p^{-1}\}$, if needed.

Case V:

The volume vector of this configuration is

$$\begin{pmatrix} 1, & 1, & 1, & -b, & b, & 2b, & a, & -a, & -2a, & 0, & a+b, & -a-b, & -2a-2b, & 0, & 0 \end{pmatrix}$$

so clearly each pair (a, b) with $0 < a \leq b$ and $\gcd(a, b) = 1$ gives a different equivalence class.

Case W.1:

The polytopes on the list have points

$$q_1 = (1,0,0), \quad q_2 = (0,0,0), \quad q_3 = (-1,0,0)$$
$$q_4 = (0,1,0), \quad q_5 = (0,0,1), \quad q_6 = (a,b,1)$$

with $0 < a < b$ and $\gcd(a,b) = 1$.

Now, the volume vector of this configuration (for the original ordering of the points) is

$$\begin{pmatrix} 0, & 0, & 0, & 1, & 1, & -b, & -1, & -1, & b, & a, & -2, & -2, & 2b, & a+b, & a-b \end{pmatrix}$$

so clearly different values of b give non-equivalent configurations.

Let $P = \{q_i\}_i$ and $P' = \{q'_i\}_i$ where $q'_i = q_i$ except $q'_6 = (a', b, 1)$ for some $0 < a' < b$ with $\gcd(a', b) = 1$.

Now, in P and P', considering their dual oriented matroid, any unimodular transformation that maps P to P' will fix the following sets of points:

$$\{q_1\}, \quad \{q_2\}, \quad \{q_3\}, \quad \{q_6\}, \quad \{q_4, q_5\}$$

Then two possible transformations are the identity and the one that exchanges q_4 and q_5. The latter one corresponds to the map $y \leftrightarrow z$. The image of $q_6 = (a,b,1)$ is $(a,1,b)$ and $q_6 = (a',b,1) = (a,1,b)$ if and only if $b = 1$ and $a = a'$. But this implies $b < a$ and we are in case $W.2$.

Case W.2:

The polytopes on the list have points and volume vector as the previous case $W.1$. Following the previous argument, two configurations $P = \{q_i\}$ and $P' = \{q'_i\}$ are equivalent if and only if $(a', b') = (a, b)$. So each pair (a,b) with $0 < a < b$ and $\gcd(a,b) = 1$ gives a different equivalence class.

Case Z:

The polytopes on the list have points

$$p_1 = (0,0,0), \quad p_2 = (1,0,0), \quad p_3 = (0,1,0),$$

$$p_4 = (0,0,1), \quad p_5 = (a,b,1), \quad p_6 = (c,d,1),$$

with $a, b, c, d > 0$, $ad - bc = 1$ and $c + d > a + b$.

The volume vector of this configuration is

$$\begin{pmatrix} 1, & 1, & 1, & -b, & -d, & b-d, & a, & c, & -a+c, & 1, & a+b, & c+d, & -a-b+c+d, & 1, & 1 \end{pmatrix}$$

All these configurations have the same dual oriented matroid, the one we labeled 6.4. The only redundancy in this classification comes from the fact that this oriented matroid has a symmetry, (exchanging the triangle $p_1p_2p_3$ with the the triangle $p_5p_4p_6$, with vertices in this order). But it is not easy to give an irredundant system of values for the parameters.

4.8. Classification tables.

We here put together all the results of Section 4. Tables 2–10 collect all the lattice 3-polytopes of size 6, with the same classification of the previous sections.

Each row in each table represents one configuration (except for Table 10 which collects infinite lists of configurations). In each row we first give the identification of the dual oriented matroid (OM), according to the list of Figure 13. The dual oriented matroid alone says whether the configuration is dps; those which are dps are marked with an asterisk in the tables. Next there is the identification letter plus number, referring to the classification in the previous sections, the volume vector and a functional giving the width of the configuration.

After each table we list representatives for each configuration (given as matrices having with points as columns). The volume vector and width functional in the table are both given according to these representatives.

OM	**Id.**	**Volume vector**	**Width (functional)**
3.2	A.1	0 0 1 0 0 2 0 1 0 −2 0 2 −1 −4 −1	1 (z)
	A.2	0 0 2 0 0 4 0 2 0 −4 0 4 −2 −8 −2	2 (z)
3.3	B.1	0 0 1 0 0 −1 0 1 0 1 0 2 2 2 2	1 (z)
	B.2	0 0 2 0 0 −2 0 2 0 2 0 4 4 4 4	2 (z)
4.3	D	0 0 1 0 1 2 0 −1 0 2 0 −1 −1 1 1	1 (z)
4.4	E	0 0 1 0 −1 2 0 1 0 −2 0 3 −1 −5 −1	1 (z)
4.5	F	0 0 1 0 −1 1 0 1 −1 0 0 3 −1 −2 2	1 (z)

$$A.1\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$A.2\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

$$B.1\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$B.2\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

$$D\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$E\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$F\quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

TABLE 2. Lattice 3-polytopes of size 6 containing 5 coplanar points. They are all non-dps

OM	Id.	Volume vector															Width (functional)	
3.8	G.7	0	1	−1	−1	1	−2	1	−1	0	2	3	−3	0	6	0	2	(z)
3.9	G.9	0	1	−1	−1	1	−1	1	−1	1	0	3	−3	0	3	−3	2	(z)
3.13	G.10	0	1	−1	−1	1	0	1	−1	0	0	3	−3	−2	2	−2	2	(z)
	I.2	0	3	−3	−3	3	0	3	−3	0	0	9	−9	−6	6	−6	2	(x)
4.13*	G.1	0	1	−1	−1	1	−4	1	−1	1	3	3	−3	3	9	0	2	(z)
4.17*	G.2	0	1	−1	−1	1	−3	1	−1	0	3	3	−3	1	8	1	2	(z)
	I.1	0	3	−3	−3	3	−9	3	−3	0	9	9	−9	3	24	3	2	(x)
4.18	G.8	0	1	−1	−1	1	−1	1	−1	0	1	3	−3	−1	4	−1	2	(z)
5.10*	G.5	0	1	−1	−1	1	−5	1	−1	1	4	3	−3	4	11	1	2	(z)
	G.6	0	1	−1	−1	1	−6	1	−1	1	5	3	−3	5	13	2	2	(z)
	H.1	0	1	−3	−1	3	−8	1	−3	1	7	3	−9	5	19	2	2	(x)
5.11*	H.2	0	1	−3	−1	3	−2	1	−3	1	1	3	−9	−1	7	−4	2	(x)
5.12*	G.3	0	1	−1	−1	1	−2	1	−1	1	1	3	−3	1	5	−2	2	(z)
	G.4	0	1	−1	−1	1	−3	1	−1	1	2	3	−3	2	7	−1	2	(z)
	H.3	0	1	−3	−1	3	−5	1	−3	1	4	3	−9	2	13	−1	2	(x)

$$G.1\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&4\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad G.6\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&6\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad H.1\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&8\\0&0&0&0&1&-3\end{pmatrix}\end{array}$$

$$G.2\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&3\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad G.7\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&2\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad H.2\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&2\\0&0&0&0&1&-3\end{pmatrix}\end{array}$$

$$G.3\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&2\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad G.8\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&1\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad H.3\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&5\\0&0&0&0&1&-3\end{pmatrix}\end{array}$$

$$G.4\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&3\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad G.9\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&1\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad I.1\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&1&-1\\0&0&1&-1&2&1\\0&0&0&0&3&-3\end{pmatrix}\end{array}$$

$$G.5\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&5\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad G.10\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&0\\0&0&0&0&1&-1\end{pmatrix}\end{array} \qquad I.2\ \begin{array}{c}\begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&1&-1\\0&0&1&-1&2&-2\\0&0&0&0&3&-3\end{pmatrix}\end{array}$$

TABLE 3. Lattice 3-polytopes of size 6: configurations containing a $(3,1)$ coplanarity, with the other two points on opposite sides; the dps ones are marked with * in the first column (since dps is an oriented matroid property)

OM	Id.	Volume vector															Width (functional)	
3.6	J.1	0	1	2	−1	−2	0	1	2	−1	1	3	6	0	0	3	2	(x)
3.11	J.2	0	1	2	−1	−2	0	1	2	0	0	3	6	1	−1	1	2	(x)
	J.3	0	3	6	−3	−6	0	3	6	0	0	9	18	3	−3	3	3	(x)
4.7	L.2	0	1	1	−1	−1	−2	1	1	1	1	3	3	3	3	0	1	(z)
4.16	L.1	0	1	1	−1	−1	−1	1	1	0	1	3	3	1	2	1	1	(z)
5.4*	K.1	0	1	5	−1	−5	1	1	5	−2	1	3	15	1	−4	7	2	(y)
	K.2	0	1	7	−1	−7	1	1	7	−2	1	3	21	3	−6	9	2	(y)
5.6*	M	0	1	3	−1	−3	−2	1	3	1	1	3	9	5	1	2	2	(x)

$$J.1\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&-1\\0&0&1&-1&0&0\\0&0&0&0&1&2\end{pmatrix}\end{array}$$

$$L.1\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&1\\0&0&0&0&1&1\end{pmatrix}\end{array}$$

$$K.1\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&-2\\0&0&1&-1&0&-1\\0&0&0&0&1&5\end{pmatrix}\end{array}$$

$$J.2\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&0\\0&0&0&0&1&2\end{pmatrix}\end{array}$$

$$L.2\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&1\\0&0&1&-1&0&2\\0&0&0&0&1&1\end{pmatrix}\end{array}$$

$$K.2\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&-2\\0&0&1&-1&0&-1\\0&0&0&0&1&7\end{pmatrix}\end{array}$$

$$J.3\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&0&2\\0&0&1&-1&0&4\\0&0&0&0&1&6\end{pmatrix}\end{array}$$

$$M\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&-1&1&1&1\\0&1&-1&0&1&2\\0&0&0&0&1&3\end{pmatrix}\end{array}$$

Table 4. Lattice 3-polytopes of size 6: configurations containing a $(3,1)$ coplanarity, with all $(3,1)$ circuits with the other two points on the same side, and containing no five coplanar points; the dps ones are marked with * in the first column

OM	Id.	Volume vector															Width (functional)	
3.10	N.2	0	1	−1	1	−1	0	−1	1	0	0	−1	1	−2	−2	2	1	(x)
3.12	N.1	0	1	−1	1	−1	−1	−1	1	1	0	−1	1	0	−1	1	1	(x)
4.14	N.3	0	1	−1	1	−1	−1	−1	1	0	−1	−1	1	−1	−2	1	1	(x)
5.13	N.4	0	1	−1	1	−1	−4	−1	1	3	−1	−1	1	5	1	−2	2	(z)
	N.5	0	1	−1	1	−1	−5	−1	1	4	−1	−1	1	7	2	−3	2	(z)

$$\text{N.1}\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&1&0&1\\0&0&1&1&0&1\\0&0&0&0&1&-1\end{pmatrix}\end{array}$$

$$\text{N.3}\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&1&0&0\\0&0&1&1&0&1\\0&0&0&0&1&-1\end{pmatrix}\end{array}$$

$$\text{N.4}\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&1&0&3\\0&0&1&1&0&4\\0&0&0&0&1&-1\end{pmatrix}\end{array}$$

$$\text{N.2}\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&1&0&0\\0&0&1&1&0&0\\0&0&0&0&1&-1\end{pmatrix}\end{array}$$

$$\text{N.5}\quad \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&1&0&4\\0&0&1&1&0&5\\0&0&0&0&1&-1\end{pmatrix}\end{array}$$

Table 5. 3-polytopes of size 6 containing a $(2,2)$ coplanarity, with the other two points on opposite sides, and containing no $(3,1)$ coplanarity nor five coplanar points. They are all non-dps

OM	Id.	Volume vector															Width (functional)	
3.4	Q.1	0	1	1	1	1	−1	−1	−1	0	−1	−1	−1	1	0	−1	1	(x)
3.5	O	0	1	2	1	2	0	−1	−2	0	0	−1	−2	1	1	−1	1	(x)
5.5	P.1	0	1	5	1	5	1	−1	−5	−2	−1	−1	−5	1	2	−3	2	(y)
	P.2	0	1	7	1	7	2	−1	−7	−3	−1	−1	−7	1	3	−4	2	$(x-y)$

$$O \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \end{array}$$

$$P.1 \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \end{array}$$

$$Q.1 \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$P.2 \quad \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \end{array}$$

TABLE 6. 3-polytopes of size 6 containing some $(2,2)$ coplanarity, with all $(2,2)$ circuits with the other two points on the same side, and containing no $(3,1)$ coplanarity nor five coplanar points; all non-dps

OM	Id.	Volume vector															Width (functional)	
4.21	S.1	1	−1	−2	1	2	0	−1	−2	0	0	−4	−7	−1	1	−1	2	(y)
	S.2	1	−2	−4	1	2	0	−1	−2	0	0	−5	−9	−2	1	−1	2	(z)
	S.3	2	−1	−2	1	2	0	−1	−2	0	0	−5	−8	−1	1	−1	2	$(x-z)$
	S.4	1	−3	−6	2	4	0	−1	−2	0	0	−7	−13	−3	2	−1	2	(z)
	S.5	3	−2	−4	1	2	0	−1	−2	0	0	−7	−11	−2	1	−1	2	$(x-z)$
	S.6	5	−3	−6	2	4	0	−1	−2	0	0	−11	−17	−3	2	−1	2	$(x-z)$
4.22	T.1	1	−1	1	1	−1	0	−1	1	0	0	−4	2	2	−2	2	2	(y)
	T.2	1	−2	2	1	−1	0	−1	1	0	0	−5	3	4	−2	2	2	(z)
	T.3	2	−1	1	1	−1	0	−1	1	0	0	−5	1	2	−2	2	2	(z)
	T.4	1	−3	3	2	−2	0	−1	1	0	0	−7	5	6	−4	2	2	(z)
	T.5	3	−2	2	1	−1	0	−1	1	0	0	−7	1	4	−2	2	2	(z)
	T.6	5	−3	3	2	−2	0	−1	1	0	0	−11	1	6	−4	2	2	(z)
4.11	U.1	1	−1	−3	1	2	1	−1	−2	−1	0	−4	−8	−4	0	0	2	(y)
	U.2	1	−1	−3	2	4	2	−1	−2	−1	0	−5	−10	−5	0	0	2	(z)
	U.3	2	−1	−4	1	2	1	−1	−2	−1	0	−5	−10	−5	0	0	2	$(x-z)$
	U.4	2	−1	−4	3	6	3	−1	−2	−1	0	−7	−14	−7	0	0	2	$(x-z)$
	U.5	3	−1	−5	2	4	2	−1	−2	−1	0	−7	−14	−7	0	0	2	$(x-z)$

$$S.1\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-2&1&2\\0&1&0&-1&0&0\\0&1&1&-2&0&0\end{pmatrix}\end{array} \qquad T.1\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-2&1&-1\\0&1&0&-1&0&0\\0&1&1&-2&0&0\end{pmatrix}\end{array} \qquad U.1\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-2&1&4\\0&1&0&-1&0&1\\0&1&1&-2&0&2\end{pmatrix}\end{array}$$

$$S.2\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&1&2\\0&2&0&-1&0&0\\0&1&1&-1&0&0\end{pmatrix}\end{array} \qquad T.2\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&0&-1&1&-1\\0&2&0&-1&0&0\\0&1&1&-1&0&0\end{pmatrix}\end{array} \qquad U.2\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&-1&1&1&1\\0&0&-1&2&0&-2\\0&1&-1&1&0&-1\end{pmatrix}\end{array}$$

$$S.3\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&1&-1&-2\\0&0&0&2&-1&-2\\0&1&0&1&-1&-2\end{pmatrix}\end{array} \qquad T.3\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&1&-1&1\\0&0&0&2&-1&1\\0&1&0&1&-1&1\end{pmatrix}\end{array} \qquad U.3\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&1&-1&-3\\0&0&0&2&-1&-4\\0&1&0&1&-1&-3\end{pmatrix}\end{array}$$

$$S.4\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&-1&1&2\\0&0&3&-2&0&0\\0&1&1&-1&0&0\end{pmatrix}\end{array} \qquad T.4\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&-1&1&-1\\0&0&3&-2&0&0\\0&1&1&-1&0&0\end{pmatrix}\end{array} \qquad U.4\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&-1&0&1&2\\0&0&-2&0&3&6\\0&0&-1&1&1&1\end{pmatrix}\end{array}$$

$$S.5\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&1&-1&-2\\0&0&0&3&-2&-4\\0&1&0&1&-1&-2\end{pmatrix}\end{array} \qquad T.5\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&0&1&1&-1&1\\0&0&0&3&-2&2\\0&1&0&1&-1&1\end{pmatrix}\end{array} \qquad U.5\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&1&0&-1&-2\\0&0&3&0&-2&-4\\0&0&1&1&-1&-3\end{pmatrix}\end{array}$$

$$S.6\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&2&0&-1&-2\\0&0&5&0&-2&-4\\0&0&1&1&-1&-2\end{pmatrix}\end{array} \qquad T.6\ \begin{array}{c} \begin{matrix}1&2&3&4&5&6\end{matrix}\\ \begin{pmatrix}0&1&2&0&-1&1\\0&0&5&0&-2&2\\0&0&1&1&-1&1\end{pmatrix}\end{array}$$

TABLE 7. 3-polytopes of size 6 containing a $(2,1)$ coplanarity (but no other coplanarity). All are non-dpsa

OM	Id.	Volume vector															Width (functional)	
6.2*	X.1	1	−1	−1	1	3	−2	−1	−2	1	1	−4	−7	3	5	−1	2	(y)
	X.2	1	−1	−3	1	5	−2	−1	−4	1	1	−4	−13	1	7	−3	2	(y)
	X.3	1	−1	−1	1	2	−1	−2	−3	1	1	−5	−7	2	3	−1	2	(z)
	X.4	1	−1	−2	1	3	−1	−2	−5	1	1	−5	−11	1	4	−3	2	(z)
	X.5	1	−2	−5	1	4	−3	−1	−3	1	1	−5	−13	1	7	−2	2	(x)
	X.6	2	−1	−1	1	5	−2	−1	−3	1	1	−5	−11	3	7	−2	2	$(x-z)$
	X.7	2	−1	−3	1	7	−2	−1	−5	1	1	−5	−17	1	9	−4	2	$(x-z)$
	X.8	1	−2	−3	1	2	−1	−3	−5	1	1	−7	−11	1	3	−2	2	(z)
	X.9	2	−1	−1	1	3	−1	−3	−5	1	2	−7	−11	2	5	−1	2	(z)
	X.10	2	−3	−7	1	5	−4	−1	−3	1	1	−7	−17	1	9	−2	2	(z)
	X.11	3	−2	−1	1	5	−3	−1	−2	1	1	−7	−11	5	8	−1	2	(z)
	X.12	3	−1	−1	1	4	−1	−2	−5	1	1	−7	−13	2	5	−3	2	$(x-z)$
	X.13	3	−1	−2	1	5	−1	−2	−7	1	1	−7	−17	1	6	−5	2	$(x-z)$
	X.14	3	−2	−5	1	7	−3	−1	−4	1	1	−7	−19	1	10	−3	2	$(x-z)$
	X.15	5	−2	−1	1	3	−1	−3	−4	1	1	−11	−13	3	4	−1	2	(z)
	X.16	5	−2	−3	1	4	−1	−3	−7	1	1	−11	−19	1	5	−4	2	$(x-z)$
	X.17	5	−3	−5	1	5	−2	−2	−5	1	1	−11	−20	1	7	−3	2	$(x-z)$
	X.18	3	−4	−5	1	2	−1	−5	−7	1	1	−13	−17	1	3	−2	2	(z)
	X.19	4	−5	−7	1	3	−2	−3	−5	1	1	−13	−19	1	5	−2	2	(z)
	X.20	5	−3	−4	1	3	−1	−4	−7	1	1	−13	−19	1	4	−3	2	$(x-z)$

$$X.1 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&1&-2&0&1&4\\0&1&-1&0&0&1\\0&1&-2&1&0&3\end{pmatrix}\end{array}$$

$$X.2 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&1&-2&0&1&6\\0&1&-1&0&0&1\\0&1&-2&1&0&3\end{pmatrix}\end{array}$$

$$X.3 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&-1&1&0&1&1\\0&-1&2&0&0&-1\\0&-1&1&1&0&0\end{pmatrix}\end{array}$$

$$X.4 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&-1&0&1&1&3\\0&-1&0&2&0&1\\0&-1&1&1&0&0\end{pmatrix}\end{array}$$

$$X.5 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&-1&1&1\\0&0&2&-1&0&-3\\0&1&1&-1&0&-2\end{pmatrix}\end{array}$$

$$X.6 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&1&-1&-3\\0&0&2&0&-1&-5\\0&1&1&0&-1&-4\end{pmatrix}\end{array}$$

$$X.7 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&1&-1&-5\\0&0&2&0&-1&-7\\0&1&1&0&-1&-6\end{pmatrix}\end{array}$$

$$X.8 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&-1&0&1&1&2\\0&-2&0&3&0&1\\0&-1&1&1&0&0\end{pmatrix}\end{array}$$

$$X.9 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&-1&0&1&1&2\\0&-2&0&0&3&5\\0&-1&1&0&1&1\end{pmatrix}\end{array}$$

$$X.10 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&1&0&-1&1&2\\0&0&0&-2&3&7\\0&0&1&-1&1&1\end{pmatrix}\end{array}$$

$$X.11 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&1&-1&-2\\0&0&0&3&-2&-1\\0&1&0&1&-1&-1\end{pmatrix}\end{array}$$

$$X.12 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&1&1&0&-1&-3\\0&3&0&0&-2&-5\\0&1&0&1&-1&-2\end{pmatrix}\end{array}$$

$$X.13 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&1&1&0&-1&-4\\0&3&0&0&-2&-7\\0&1&0&1&-1&-3\end{pmatrix}\end{array}$$

$$X.14 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&1&-1&-4\\0&0&0&3&-2&-5\\0&1&0&1&-1&-3\end{pmatrix}\end{array}$$

$$X.15 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&2&-1&-1\\0&0&0&5&-2&-1\\0&1&0&1&-1&-1\end{pmatrix}\end{array}$$

$$X.16 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&1&2&-1&-2\\0&0&0&5&-2&-3\\0&1&0&1&-1&-2\end{pmatrix}\end{array}$$

$$X.17 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&2&1&0&-1&-3\\0&5&0&0&-2&-5\\0&1&0&1&-1&-2\end{pmatrix}\end{array}$$

$$X.18 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&-1&2&0&1&1\\0&-1&5&0&0&-1\\0&-1&1&1&0&0\end{pmatrix}\end{array}$$

$$X.19 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&1&2&-1&0&-1\\0&0&5&-1&0&-2\\0&0&1&-1&1&1\end{pmatrix}\end{array}$$

$$X.20 \quad \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{pmatrix}0&0&2&1&-1&-2\\0&0&5&0&-1&-3\\0&1&1&0&-1&-2\end{pmatrix}\end{array}$$

TABLE 8. Lattice 3-polytopes of size 6 with no coplanarities and 1 interior point. All dps

OM	Id.	Volume vector															Width (functional)	
6.1*	Y.1	1	−1	5	1	1	−6	−1	−2	7	−1	−4	1	19	5	−9	2	$(x-2y)$
	Y.2	1	−1	7	1	1	−8	−1	−2	9	−1	−4	3	25	7	−11	2	$(x-z)$
	Y.3	1	−2	5	1	1	−7	−1	−2	9	−1	−5	1	23	6	−11	2	$(x-y)$
	Y.4	1	−2	7	1	1	−9	−1	−2	11	−1	−5	3	29	8	−13	2	$(x-z)$
	Y.5	2	−1	7	1	1	−4	−1	−3	5	−1	−5	1	17	3	−8	2	$(x-z)$
	Y.6	2	−1	11	1	1	−6	−1	−3	7	−1	−5	5	25	5	−10	2	$(x-z)$
	Y.7	2	−3	7	1	1	−5	−1	−3	8	−1	−7	1	23	4	−11	2	$(x-z)$
	Y.8	2	−3	11	1	1	−7	−1	−3	10	−1	−7	5	31	6	−13	2	$(x-z)$
	Y.9	3	−1	7	2	1	−5	−1	−2	3	−1	−7	1	16	3	−5	2	(x)
	Y.10	3	−2	13	1	1	−5	−1	−4	7	−1	−7	5	27	4	−11	2	$(x-z)$
	Y.11	3	−5	11	2	1	−9	−1	−2	7	−1	−11	5	32	7	−9	2	$(x-z)$
	Y.12	5	−2	11	3	1	−7	−1	−2	3	−1	−11	3	23	4	−5	3	$(x-z)$

$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.1 & \begin{pmatrix}0&1&0&-2&1&-12\\0&1&0&-1&0&-7\\0&1&1&-2&0&-13\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.5 & \begin{pmatrix}0&0&1&1&-1&3\\0&0&0&2&-1&7\\0&1&0&1&-1&2\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.9 & \begin{pmatrix}0&0&1&1&-1&2\\0&0&3&0&-2&-1\\0&1&1&0&-1&-1\end{pmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.2 & \begin{pmatrix}0&0&1&-2&1&-15\\0&0&1&-1&0&-8\\0&1&1&-2&0&-17\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.6 & \begin{pmatrix}0&1&0&1&-1&4\\0&0&0&2&-1&11\\0&0&1&1&-1&5\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.10 & \begin{pmatrix}0&0&1&1&-1&4\\0&0&0&3&-2&13\\0&1&0&1&-1&3\end{pmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.3 & \begin{pmatrix}0&1&0&-1&1&-7\\0&2&0&-1&0&-9\\0&1&1&-1&0&-8\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.7 & \begin{pmatrix}0&0&1&-1&1&-4\\0&0&0&-2&3&-7\\0&1&0&-1&1&-5\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.11 & \begin{pmatrix}0&1&2&-1&0&-5\\0&0&5&-2&0&-9\\0&0&1&-1&1&-4\end{pmatrix}\end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.4 & \begin{pmatrix}0&0&1&-1&1&-8\\0&0&2&-1&0&-9\\0&1&1&-1&0&-10\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.8 & \begin{pmatrix}0&1&0&-1&1&-7\\0&0&0&-2&3&-11\\0&0&1&-1&1&-6\end{pmatrix}\end{array} \quad \begin{array}{cc} & \begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ Y.12 & \begin{pmatrix}0&1&0&2&-1&4\\0&0&0&5&-2&11\\0&0&1&1&-1&2\end{pmatrix}\end{array}$$

TABLE 9. 3-polytopes of size 6 with no coplanarities and 2 interior points. All dps

OM	Id.	Volume vector	Parameters	F.
4.1	C	$\begin{pmatrix} 0 & 0 & q & 0 & 2q & 3q & 0 & 0 & 0 & 0 & 0 & -q & -2q & -q & 0 \end{pmatrix}$	$0 \le p \le \frac{q}{2}$, $\gcd(p,q)=1$	x
5.6*	L.3	$\begin{pmatrix} 1 & a & b \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$	$0 < 2a < b$, $\gcd(a,b)=1$	z
5.15	N.6	$\begin{pmatrix} 1 & a \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$	$a > 2$	x
5.8	Q.2	$\begin{pmatrix} 1 & a & b \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$	$0 < a < b$, $\gcd(a,b)=1$	z
2.1	R	$\begin{pmatrix} 0 & -q & q & 0 & 0 & 2q & -q & q & 0 & -2q & -2q & 2q & 0 & -4q & 0 \end{pmatrix}$	$0 \le p < q$, $\gcd(p,q)=1$, $p = min\{\pm p^{\pm 1} \mod q\}$	z
4.9	W.1	$\begin{pmatrix} 1 & a & b \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & -1 \end{pmatrix}$	$0 < a < b$, $\gcd(a,b)=1$	z
4.15	V	$\begin{pmatrix} 1 & a & b \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 0 & 1 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 0 & 0 \end{pmatrix}$	$0 < a \le b$, $\gcd(a,b)=1$	z
4.15	W.2	$\begin{pmatrix} 1 & a & b \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 0 & 1 & -1 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$0 < b < a$, $\gcd(a,b)=1$	z
6.4*	Z	$\begin{pmatrix} 1 & a & b & c & d \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$	$a,b,c,d > 0$, $ad - bc = 1$, $c + d > a + b$	z

$$C \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 & p \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix} \end{array} \qquad Q.2 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & a \\ 0 & 0 & 1 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad W.1 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$L.3 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & a \\ 0 & 0 & 1 & -1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad R \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & -1 & p & -p \\ 0 & 0 & 0 & 0 & q & -q \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array} \qquad W.2 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} a & 0 & 0 & 0 & -1 & 1 \\ b & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$N.6 \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array} \qquad V \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & a & -a \\ 0 & 0 & 1 & 0 & b & -b \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad Z \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & a & c \\ 0 & 0 & 1 & 0 & b & d \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

TABLE 10. Infinite series of lattice 3-polytopes of size 6 and width 1; the dps ones are marked with * next to their dual oriented matroid. To save space and improve readability, the volume vector is sometimes expressed as a product of a parameter vector and a matrix

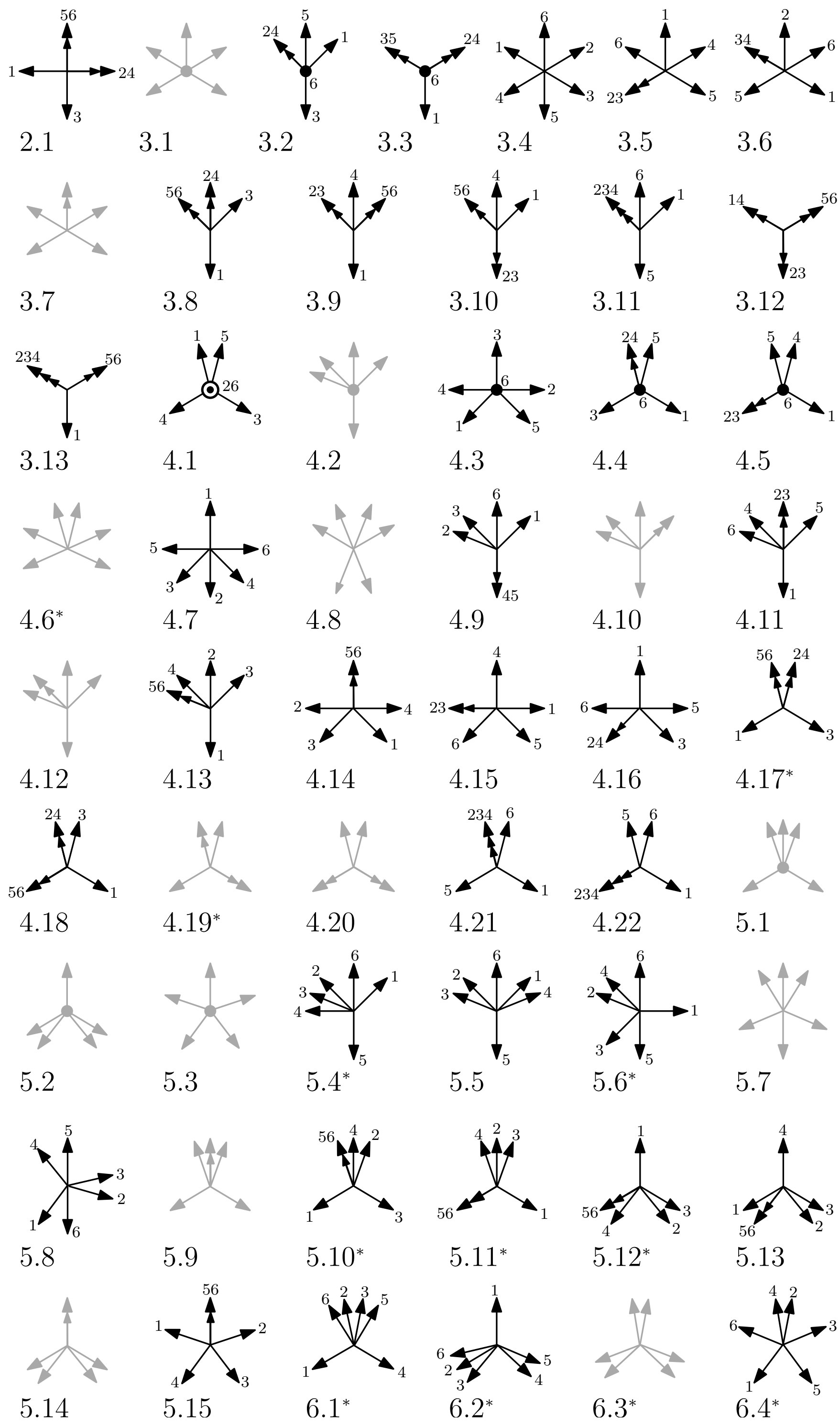


FIGURE 13. Oriented matroids dual to configurations of six different points in $\mathbb{R}^3$. Those non-realizable with lattice polytopes of size 6 are in gray. Those that are dps are marked with * next to their identification number. The first digit in the identification number equals the number of cocircuits in the dual oriented matroid, that is, the number of circuits in the point configuration

4.9. **Polytopes of size 6 and width 1.** As a complement to the classification, we rework in this section the classification of lattice 3-polytopes of width one. This includes all the infinite lists discussed in Section 4.7, plus some sporadic configurations from the other cases.

The classification is based on the fact that the 6 lattice points of P lie in two consecutive lattice planes, which we assume to be $z = 0$ and $z = 1$. We denote P_0 and P_1 the corresponding 2-dimensional configurations in each plane. There are three possibilities depending on the sizes of P_0 and P_1.

- $|P_0| = 5$, $|P_1| = 1$. P_0 is one of the six 2-dimensional polytopes of size 5 displayed in Figure 14.

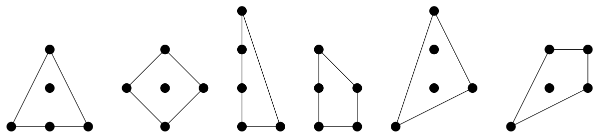


FIGURE 14. 2-dimensional polytopes of size 5

The equivalence class of P depends only on P_0, and not on the choice of the sixth point at $z = 1$. The six choices of P_0 produce, respectively, configurations **A.1**, **B.1**, **C** (with $q = 1$), **D**, **E** and **F**. All these polytopes are non dps, since 5 points in the same plane cannot be dps

- $|P_0| = 4$, $|P_1| = 2$. P_0 is one of the three 2-dimensional polytopes of size 4 displayed in Figure 15, or it consists of four collinear points. We look at these four cases one by one.

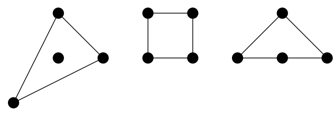
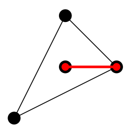
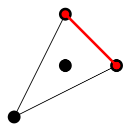


FIGURE 15. 2-dimensional polytopes of size 4

- ◦ P_0 is a $(3, 1)$ circuit. Let P_0 consist of 0, e_1, e_2 and $-e_1 - e_2$. The edge at $z = 1$ can be:
  - Parallel to an interior edge of P_0, for example $\{o, e_1\}$: $P_1 = \{e_3, (0, 1, 1)\}$, which gives case **L.2**.

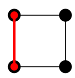
    

  - Parallel to an exterior edge of P_0, for example $\{e_1, e_2\}$: $P_1 = \{(1, 0, 1), (0, 1, 1)\}$, which gives case **L.3**.

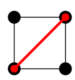
    

  - Not parallel to any edge of P_0: without loss of generality $P_1 = \{e_3, (a, b, 1)\}$ with $\gcd(a, b) = 1$, $a \neq 0$ and $b/a \notin \{-1, 0, 1/2, 1, 2\}$. This gives the infinite series **L.1**.

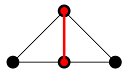
- ◦ P_0 is a $(2, 2)$ circuit. Let P_0 consist of 0, e_1, e_2 and $e_1 + e_2$. The edge at $z = 1$ can be:
  - Parallel to an exterior edge of P_0, for example $\{o, e_2\}$: $P_1 = \{e_3, (0, 1, 1)\}$, which gives case **Q.1**.

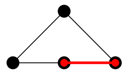
  - Parallel to an interior edge of P_0, for example $\{o, e_1 + e_2\}$: $P_1 = \{e_3, (0, 1, 1)\}$, which gives case **N.3**.

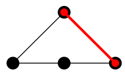
  - Non-parallel to any edge of P_0: $P_1 = \{(0,0,1),(a,b,1)\}$ with $\gcd(a,b) = 1$, $a \neq 0$ and $b/a \notin \{-1,0,1\}$. This gives the infinite series **Q.2**.
- ◦ P_0 is a $(2,1)$ circuit. Let P_0 consist of 0, e_1, $-e_1$ and e_2. The edge at $z = 1$ can be:
  - Parallel to an interior edge of P_0, for example $\{o, e_2\}$: $P_1 = \{e_3, (0,1,1)\}$, which gives case **N.2**.



  - Parallel to an edge of the $(2,1)$ circuit in P_0, for example $\{o, e_1\}$: $P_1 = \{e_3, (1,0,1)\}$, which gives case **D**.

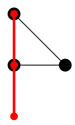


  - Parallel to an exterior edge in P_0, not in the $(2,1)$ circuit, for example $\{e_1, e_2\}$: $P_1 = \{(1,0,1),(0,1,1)\}$, which gives case **O**.



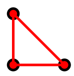
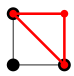
  - Non-parallel to any edge of P_0: $P_1 = \{e_3, (a,b,1)\}$ with $\gcd(a,b) = 1$, $a \neq 0$ and $b/a \notin \{-1,0,1\}$. This gives infinite series **W.1** for $|b| > |a|$ and **W.2** for $|b| < |a|$.
- ◦ The four points are collinear. Let P_0 consist of 0, e_2, $2e_2$ and $3e_2$. The edge at $z = 1$ must be non-parallel to this collinearity: $P_1 = \{e_3, (a,b,1)\}$ with $\gcd(a,b) = 1$ and $a \neq 0$. This gives the infinite series **C**.

- $|P_0| = 3$, $|P_1| = 3$. Either P_0 is the 2-dimensional polytope of size 3 (unimodular triangle), or the points in P_0 are collinear.
  - ◦ If both sets of three points are collinear: without loss of generality $P_0 = \{o, e_1, -e_1\}$ and $P_1 = \{e_3, (a,b,1), (-a,-b,1)\}$ with $\gcd(a,b) = 1$ and $b \neq 0$, which gives the infinite series **R**.
  - ◦ If the points at $z = 0$ are not collinear, then they have to form a unimodular triangle: without loss of generality $P_0 = \{o, e_1, e_2\}$. We have two possibilities.
    - The points at $z = 1$ are collinear: this double edge can be:
      - ∗ Parallel to an edge of P_0, for example $\{o, e_2\}$: $P_1 = \{e_3, (0,1,1), (0,-1,1)\}$, which gives again case **D**.

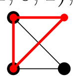


      - ∗ Non-parallel to any edge of P_0: $P_1 = \{e_3, (a,b,1), (-a,-b,1)\}$ with $\gcd(a,b) = 1$, $a \neq 0$ and $b/a \notin \{-1,0\}$. This gives the infinite series **V**.

    - The points at $z = 1$ are not collinear: they form a unimodular triangle. This triangle can have collinear edges with the one in $z = 0$:
      - ∗ 3 collinear edges: $P_1 = \{e_3, (1,0,1), (0,1,1)\}$ gives case **Q.1** and $P_1 = \{(1,0,1), (0,1,1), (1,1,1)\}$ gives case **N.1**.

      - ∗ 2 collinear edges: $P_1 = \{e_3, (1,0,1), (1,1,1)\}$, which gives case **N.3**.



      - ∗ 1 collinear edge: $P_1 = \{e_3, (0,1,1), (-1,b,1)\}$, with $b \notin \{0,1,2\}$, gives the infinite series **N.6** and $P_1 = \{e_3, (0,1,1), (1,b,1)\}$, with $b \notin \{-1,0,1\}$, gives the infinite series **Q.2** for $a = 1$.

      - ∗ No collinear edges: $P_1 = \{e_3, (a,b,1), (c,d,1)\}$ with $ad - bc = \pm 1$, $a, c, a-c \neq 0$ and $\frac{b}{a}, \frac{d}{c}, \frac{b-d}{a-c} \notin \{-1,0\}$. This gives infinite series **Z**.

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Appendix A.
Computer programs

We include here the computer programs that have been used in the classification, all written in MATLAB.

A.1. **Common subroutines.** There are two basic subroutines that we need to use over and over in several parts:

- Test whether four given lattice points form an empty tetrahedron (file `empty.m`)
- Given n lattice points, output their volume vector (file `volumevectors.m`).

A.1.1. *Test for the emptiness of a given tetrahedron.*

The algorithm works as follows:
Our input is a generic lattice tetrahedron

$$T = \{p_1, p_2, p_3, p_4\}$$

We want to know whether this tetrahedron is empty, and the values of p and q such that T is $\mathbb{Z}$-equivalent to $T(p, q)$.

We may assume the tetrahedron to have p_1 at the origin, so we first perform the translation that sends p_1 to the origin:

$$T = \{o, p_2 - p_1, p_3 - p_1, p_4 - p_1\}$$

Whether T is empty or not is relatively easy to check. It suffices to see that:

- All its edges are primitive segments (the greatest common divisor of the coordinates of vector must be one).
- Has minimum width 1. Since in empty tetrahedra the minimum width is always achieved with respect to a pair of edges, we check the width with respect to all three pairs and the tetrahedron will be empty if and only if (at least) one of the widths is 1.

Our outputs are:

- q: the normalized volume of the tetrahedron, which is computed from the vertex set. If this volume is 1, then it automatically is empty.
- p is set to 0 as default.
- A variable k which is set to be:
  $k = -1$: If some edge is not a primitive segment, in which case the algorithm does not evaluate widths.
  $k = 0$: If all edges are primitive segments, but none of the widths is 1.
  $k = 1$: Both properties hold, in which case T is empty.

In the case of $k = 1$ (and $q > 1$), the algorithm proceeds to calculate (one of) the values $0 < p \leq \frac{q}{2}$ such that T is equivalent to $T(p, q)$.

Suppose the widths are computed, and one of them is 1. Looking back at our $T(p, q)$ tetrahedron we have that: all three widths are 1 if and only if $q = 2$ and $p = 1$; exactly two widths are 1 if and only if $p = 1$ and $q > 2$; only one of the widths (with respect to the pair $\overline{(0,0,0)(1,0,0)}$ and $\overline{(0,0,1)(p,q,1)}$) is 1 if and only if $q > 3$ and $p > 1$. Hence we have two cases:

- At least two widths are 1: set $p = 1$.
- Only one of the widths is 1. In this case we need to work further to find p.

We may assume the tetrahedron to have width 1 with respect to the edges p_1p_4 and p_2p_3 (use a permutation of the order of the vertices if necessary).

Then we apply an affine transformation A mapping $T = \{p_1, p_2, p_3, p_4\}$ to $\{o, e_1, e_2, e_3\}$, in the specified order. This map will be linear since $p_1 = o$. We are now in the superlattice $\Lambda(p, q)$ of $\mathbb{Z}^3$, where p is still unknown.

But it is easy to find p since the only point $S_0 = (x_0, y_0, z_0) \in \Lambda(p, q)$ satisfying:

$$0 < x_0, y_0 < 1$$

$$z_0 = 1/q$$

will be $S_0 = (p/q, 1-p/q, 1/q)$.

Let us now look at a generic point S at height $z = 1/q$. Then the functional $f(x, y, z) = z$ has value $1/q$ at this point. We also have that $f(o) = f(e_1) = f(e_2) = 0$ and $f(e_3) = 1$.

With respect to T and the integer lattice, let g be the primitive functional that is constant at p_1, p_2 and p_3. Then $g(p_4) - q(p_i) = q$ for $i = 1, 2, 3$ (in an empty tetrahedron, all widths with respect to a facet and a vertex are equal to its normalize volume). And hence $S' = A(S)$ will be a point satisfying $g(S') = g(p_1) + 1 = 1$ (since p_1 is the origin).

This functional $g(x, y, z) = ax + by + cz$ with $\gcd(a, b, c) = 1$ can be computed from the points p_i. And then finding a point $S' = (x', y', z') \in \mathbb{Z}^3$ such that $g(S') = 1$ is easy by solving the double Bezout identity:

$$ax + by + cz = 1$$

Let now S' be a solution of the previous equation. S can now be recovered by applying A^{-1} to S'.

Remember that S is a point in $\Lambda(p, q)$ with $z = 1/q$. It suffices to take the positive fractional part of coordinates x and y to have the initial point $S_0 = (p/q, 1-p/q, 1/q)$. And then it is inmediate to recover p.

```
function [k,p,q] = empty(p1,p2,p3,p4)
%El programa tiene como input los cuatro vertices (vectores fila) en Z^3 de
%un tetraedro reticular.
%Primero trasladamos p1 al origen:
p2=p2-p1;
p3=p3-p1;
p4=p4-p1;
p1=[0,0,0];
%Queremos que el programa nos diga si el tetraedro es vacio, y que nos de
%los valores de p y q para los cuales el tetraedro es equivalente a T(p,q).

p=0;
q=abs(det ([1,p1;
    1,p2;
    1,p3;
    1,p4]));
q=round(q);

% k sera un marcador que valdra 1 cuando el tetraedro sea vacio, y 0 o -1
%si no lo es
k=-1;

%La primera condicion para que sea un tetraedro vacio es que sus aristas
%sean primitivas. Si no lo son, k=-1. Si si que lo son, cambiamos a k=0.
if gcd(gcd(p1(1)-p2(1),p1(2)-p2(2)),p1(3)-p2(3))==1
    if gcd(gcd(p1(1)-p3(1),p1(2)-p3(2)),p1(3)-p3(3))==1
        if gcd(gcd(p1(1)-p4(1),p1(2)-p4(2)),p1(3)-p4(3))==1
            if gcd(gcd(p3(1)-p2(1),p3(2)-p2(2)),p3(3)-p2(3))==1
                if gcd(gcd(p4(1)-p2(1),p4(2)-p2(2)),p4(3)-p2(3))==1
                    if gcd(gcd(p3(1)-p4(1),p3(2)-p4(2)),p3(3)-p4(3))==1

                        k=0;

                    end
                end
            end
        end
    end
end

%Si el tetraedro es de volumen 1, entonces es vacio y no hace falta seguir
%calculando nada
if q==1;
```

```
    %p=0 por defecto
    k=1;
end

%A partir de ahora, analizaremos si q>1.

%Calculamos las anchuras respecto de los tres pares de aristas. El
%tetraedro T(p,q) tiene anchuras de la siguiente manera:

%- Par de aristas (000)(100) / (001)(pq1)  : anchura siempre 1
%- Par de aristas (000)(001) / (100)(pq1)  : anchura 1 sii p=1
%- Par de aristas (000)(pq1) / (100)(001)  : anchura 1 sii p=q-1
%inicializamos a 0 las 3 variables en las que guardaremos las anchuras.

a1=0;%Anchura respecto de (p1p2)(p3p4)
a2=0;%Anchura respecto de (p1p3)(p2p4)
a3=0;%Anchura respecto de (p1p4)(p2p3)

%Luego pondremos ai=1 si esa anchura es 1, y ai=0 si es mayor que 1.
if k==0 %El tetraedro tiene aristas primitivas y tiene volumen q>1.

    %Anchura respecto de (p1p2)(p3p4)
    x=p1;
    X=p1-p2;
    y=p3;
    Y=p3-p4;
    A=X(2)*Y(3)-X(3)*Y(2);
    B=-(X(1)*Y(3)-X(3)*Y(1));
    C=X(1)*Y(2)-X(2)*Y(1);

    if abs((A*(x(1)-y(1))+B*(x(2)-y(2))+C*(x(3)-y(3)))/gcd(gcd(A,B),C))==1
        k=1;
        a1=1;
    end
    clearvars x X y Y A B C

    %Anchura respecto de (p1p3)(p2p4)
    x=p1;
    X=p1-p3;
    y=p2;
    Y=p2-p4;
    A=X(2)*Y(3)-X(3)*Y(2);
    B=-(X(1)*Y(3)-X(3)*Y(1));
    C=X(1)*Y(2)-X(2)*Y(1);

    if abs((A*(x(1)-y(1))+B*(x(2)-y(2))+C*(x(3)-y(3)))/gcd(gcd(A,B),C))==1
        k=1;
        a2=1;
    end
    clearvars x X y Y A B C

    %Anchura respecto de (p1p4)(p2p3)
    x=p1;
    X=p1-p4;
    y=p2;
    Y=p2-p3;
    A=X(2)*Y(3)-X(3)*Y(2);
    B=-(X(1)*Y(3)-X(3)*Y(1));
    C=X(1)*Y(2)-X(2)*Y(1);

    if abs((A*(x(1)-y(1))+B*(x(2)-y(2))+C*(x(3)-y(3)))/gcd(gcd(A,B),C))==1
        k=1;
        a3=1;
    end
    clearvars x X y Y A B C

    %Ahora podemos encontrarnos en 4 situaciones:
    % a1+a2+a3=0 : el tetraedro NO es vacio,
```

```
% a1+a2+a3=1 : el tetraedro es vacio, q>3 y 1<p<q-1
% a1+a2+a3=2 : el tetraedro es vacio, q>2 y p=1 o p=q-1
% a1+a2+a3=3 : el tetraedro es vacio, q=2 y p=1

if a1+a2+a3>1
    k=1;

    %En este caso puede ser p=1 o p=q-1, pero nos interesa el menor
    %valor de p:
    p=1;
elseif a1+a2+a3==1
    %Ahora ya sabemos que el tetraedro es vacio, pero aun queda por
    %calcular el valor de p. Queremos ordenar los puntos del tetraedro
    %de manera que p1 sea el origen y que la anchura 1 sea respecto de
    %el par de aristas p1p4 / p2p3
    k=1;
    if a1==1 %Anchura 1 respecto de p1p2/p3p4
        %Realizamos la permutacion que lleva p2 a p4
        aux=p2;
        p2=p4;
        p4=aux;

    elseif a2==1 %Anchura 1 respecto de p1p3/p2p4
        %Realizamos la permutacion que lleva p3 a p4
        aux=p3;
            p3=p4;
        p4=aux;
    end
    %Ahora vamos a llevar nuestro tetraedro al tetraedro unimodular
    %estandar. La aplicacion lineal que lleva el tetraedro de la base
    %(o,e1,e2,e3) a (p1,p2,p3,p4) en ese orden, es:

    A=[p2',p3',p4'];

    %Y calculamos la inversa que es la que nos interesa.

    %B=inv(A);

    %Sea g=ax+by+cz el funcional primitivo que es constante en la
    %cara {p1,p2,p3}. El valor del funcional sera 0 (pues p1 es el
    %origen) y sera q en p4.

    a=p2(2)*p3(3)-p2(3)*p3(2);
    b=-p2(1)*p3(3)+p2(3)*p3(1);
    c=p2(1)*p3(2)-p2(2)*p3(1);

    %Queremos que el funcional sea primitivo, asi que dividimos entre
    %el gcd de los coeficientes.
    m=gcd(gcd(a,b),c)*sign(a*p4(1)+b*p4(2)+c*p4(3));

    a=a/m;
    b=b/m;
    c=c/m;


    %Ahora queremos encontrar un punto que al aplicarle B, nos de
    %coordenada z=1/q. Para ello nos valdra con encontrar un punto
    %(x,y,z) tal que g(x,y,z)=g(p1)+1=1;
    %Es decir, resolver el siguiente sistema: ax+by+cz=1.
    %Puesto que 1=gcd(a,b,c), entonces esto corresponde a la identidad
    %de Bezout.

    [gab,ai,bi]=gcd(a,b);
    %Esto significa que gab=gcd(a,b)=a*ai+b*bi

    [r,gi,ci]=gcd(gab,c);
    %Esto significa que 1=gcd(c,gab)=gab*gi+c*ci
```

```
        %Es decir,
        %   1=gab*gi+c*ci=(a*ai+b*bi)*gi+c*ci=(ai*gi)*a+(bi*gi)*b+ci*c
        % y una solucion entera de nuestra ecuacion es (ai*gi,bi*gi,ci)

        S=[ai*gi,bi*gi,ci];

        %Aplicamos B=inv(A), que nos da el punto S con coordenada z=1/q;
        S=A\(S');

        %Este punto sera de la forma (x0 +p/q, y0 -p/q, 1/q) con x0, y0
        %enteros. Y de aqui sacamos el valor de p:
        p=S(1);
        p=round(q*(p-floor(p)));
    end


    %Puesto que T(p,q) es equivalente a T(q-p,q), nos quedaremos con el
    %menor valor de entre p y q-p:
    if p > q/2
        p=q-p;
    end
end
```

A.1.2. *Compute the volume vector of a set of n lattice points in dimension* 3.

Our input is a set of n lattice points $p_1, ..., p_n$ in dimension 3, given by a matrix $A \in \mathbb{Z}_{3\times n}$ with the points as columns:

$$A = \begin{pmatrix} p_1 & ... & p_n \end{pmatrix}$$

The algorithm computes the volume vector of $P = \text{conv}\{p_i\}$ as defined in Definition 2.8:

$$\overline{v} = (v_{i,j,k,l})_{1\leq i<j<k<l\leq n} \in \mathbb{Z}^{\binom{n}{4}}$$

where

$$v_{i,j,k,l} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_i & p_j & p_k & p_l \end{pmatrix}$$

Whenever $n = 5$, the output is a small variation of the volume vector, modified for this particular case as in Section 3.

The second output of the algorithm is va, which consists of the absolute values of the coordinates of volume vector v, ordered from smallest to largest.

By Lemma 2.10, to have the volume vector (both v and va) of a configuration will allow us in many cases to determine whether two configurations can or not belong to equivalence classes.

```
function [v,va] = volumevectors(A,n)
% A es una matriz de 3 filas y n columnas, donde cada columna son las
% coordenadas de cada uno de los n puntos.
% La idea es calcular el vector de volumenes tal y como esta definido en la
% definicion general. Es decir.
% v= ( w_{i,j,k,l}) donde el orden de los indices es lexicografico:
%                    1 =< i < j < k < l =< n
% w_{i,j,k,l}=det (A_i,A_j,A_k,A_l)

%Primero la matriz A tiene que tener una nueva fila de unos:
A=[ones(1,n);A];

%M sera una matriz en la que iremos guardando filas para calcular los
%determinantes 4x4:
M=zeros(4);

%v sera el vector de volumenes. En va iremos guardando las coordenadas de v
%en su valor absoluto.
v=[];
va=[];
```

```
for i=1:(n-3)
    M(:,1)=A(:,i);
    for j=(i+1):(n-2)
        M(:,2)=A(:,j);
        for k=(j+1):(n-1)
            M(:,3)=A(:,k);
            for l=(k+1):n
                M(:,4)=A(:,l);
                v=[v,round(det(M))];
                va=[va,abs(round(det(M)))];
            end
        end
    end
end

%Ahora ordenamos va de menor a mayor.
va=sort(va);

%Ahora, en el caso de 5 puntos, queremos modificar el vector: si antes era
%v=(v1234,v1235,v1245,v1345,v2345), ahora queremos que sea:

%v=(v2345,-v1345,v1245,-v1235,v1234)

if n==5
    vv=zeros(1,5);
    for i=1:5
        vv(i)=v(6-i);
    end
    v=vv;
    v(2)=-v(2);
    v(4)=-v(4);
end


end
```

A.2. **Algorithms specific to parts of the classification.**

Parts of the classification of lattice 3-polytopes of size 6 have been completed by computer. This has always happened when a configuration of size 6 can be obtained as the union of two configurations of size 5 with a tetrahedron in common, and one of the configurations being of signature $(4,1)$. As a preparation for the gluing we first have a program that computes the (p,q)-types of all the empty subtetrahedra of the $(4,1)$-polytopes and asignes a number $t \in \{1,...,7\}$ corresponding to one of the 7 non-equivalent (p,q)-types that arise in the eight configurations of signature $(4,1)$ (file `tipos.m`).

After that, there are five groups of files devoted to gluing different types of configuration:

- Algorithm 1: gluing a $(4,1)$ with a $(3,1)$ (files `alg1.m`, `eval1.m`, `trans1.m`, `reord1.m`).
- Algorithm 2: gluing a $(4,1)$ with $(2,2)$ (files `alg2.m`, `eval2.m`, `trans2.m`, `reord2.m`).
- Algorithm 3: gluing a $(4,1)$ with $(2,1)$ (files `alg3.m`, `eval31.m`, `eval32.m`, `reord312.m`, `eval33.m`, `reord33.m`.
- Algorithm 4: gluing two $(4,1)$ sharing the same interior point (files `alg4.m`, `eval4.m`, `trans4.m`, `reord4.m`).
- Algorithm 5: gluing two $(4,1)$, not sharing the same interior point (files `alg5.m`, `eval5.m`, `trans5.m`, `reord5.m`).

A.2.1. *Types* (p, q) *of tetrahedra in* $(4, 1)$*-polytopes of size 5.* The output of the algorithm are: A an array containing all the points of the eight $(4, 1)$-polytopes; arrays P, Q and T containing the values of p, q and t, respectively, of each subtetrahedron.

```
function [T,A,P,Q]= tipos

%A guarda los puntos de cada politopo (4,1)
A=zeros(8,5,3);
%Q(k,i), P(k,i) y T(k,i) son los valores q,p y t del tetraedro
%correspondiente a eliminar el vertice i en la configuracion k, donde cada
%valor de t representa un par (p,q) distinto.
Q=zeros(8,4);
P=zeros(8,4);
T=zeros(8,4);

A(1,:,:)=[1,0,0;0,0,1;1,1,1;-2,-1,-2;0,0,0];  %V(:,1,1)=[1;1;1;1;-4];
A(2,:,:)=[1,0,0;0,0,1;1,2,1;-1,-1,-1;0,0,0];  %V(:,1,2)=[1;1;1;2;-5];
A(3,:,:)=[1,0,0;0,0,1;1,3,1;-1,-2,-1;0,0,0];  %V(:,1,3)=[1;1;2;3;-7];
A(4,:,:)=[1,0,0;0,0,1;2,5,1;-1,-2,-1;0,0,0];  %V(:,1,4)=[1;3;2;5;-11];
A(5,:,:)=[1,0,0;0,0,1;2,5,1;-1,-1,-1;0,0,0];  %V(:,1,5)=[3;4;1;5;-13];
A(6,:,:)=[1,0,0;0,0,1;2,7,1;-1,-2,-1;0,0,0];  %V(:,1,6)=[3;5;2;7;-17];
A(7,:,:)=[1,0,0;0,0,1;3,7,1;-2,-3,-1;0,0,0];  %V(:,1,7)=[5;4;3;7;-19];
A(8,:,:)=[1,0,0;0,0,1;2,5,1;-3,-5,-2;0,0,0];  %V(:,1,8)=[5;5;5;5;-20];

for k=1:8
    for i=1:4
        PP=zeros(7,3);
        for j=1:(i-1)
            PP(j,:)=A(k,j,:);
        end
        for j=(i+1):4
            PP(j-1,:)=A(k,j,:);
        end
        PP(5,:)=A(k,i,:);
        PP(7,:)=[k,i,0];
        %Ahora tenemos la configuracion (4,1) con P(4) el punto
        %interior y el vertice escogido P(5)

        [r1,p,q] = empty(PP(1,:),PP(2,:),PP(3,:),PP(4,:));
        Q(k,i)=q; P(k,i)=p;

        %Puesto que los valores de q en estos tetraedros van de 1 hasta 7,
        %hay un numero finito de posibilidades para (p,q), y las ordenamos
        %segun el siguiente convenio:

        %T1= T(0,1)      T2= T(1,2)      T3= T(1,3)     T4= T(1,4)
        %T5= T(1,5)      T6= T(2,5)      T7= T(2,7)=T(3,7)
        %El caso T(1,7) no se da

        if q==1 || q==2 || q==3 || q==4
            T(k,i)=q;
        elseif q==5
            if p==1 %or p==4
                T(k,i)=5;
            else % p=2,3
                T(k,i)=6;
            end
        elseif q==7
            if p==2 || p==3
                T(k,i)=7;
            end
        end
    end
end
```

Once we have the full lists of configurations of each type that have exactly size 6, the specific ordering of the points allows us to compare them pairwise to see how many of them are non-equivalent.

```
function [PP1,VV1,VV1a] = clases(PP)

%En PP tenemos la lista completa de todas las configuraciones de tamaño 6
%ordenadas univocamente
n=size(PP,3);
m=size(PP,1);
%En VV guardamos los vectores de volumenes de todas esas congifuraciones. Y
%en VVa los vectores ordenados y en valor absoluto.
VV=zeros(1,15,n);
VVa=zeros(1,15,n);

for i=1:n

    B=PP(1:6,:,i)';
    [v,va]=volumevectors(B,6);
    %if v(1)<0
    %    v=-v;
    %end
    VV(:,:,i)=v;
    VVa(:,:,i)=va;
end
clear i v B

%En PP1, VV1 y VV1a iremos guardando los datos de las
%configuraciones no repetidas de PP, VV y VVa, respectivamente
PP1=zeros(m,3,n);
VV1=zeros(1,15,n);
VV1a=zeros(1,15,n);
%s1 sera el contador que vaya añadiendo elementos a estas listas.
s1=0;

%La lista CC es una lista con todas las entradas 1. Cuando una
%configuracion es comparada a una inferior (menor indice en PP) y son
%iguales, este contador se pone a 0 para dicha configuracion. Al final
%CC sera una lista de 0's y 1's de manera que las posiciones de los 1's
%seran los primeros representantes de cada clase que aparecen.
CC=ones(1,n);

for i=1:n
    if CC(i)==1
        s1=s1+1;
        PP1(:,:,s1)=PP(:,:,i);
        VV1(:,:,s1)=VV(:,:,i);
        VV1a(:,:,s1)=VVa(:,:,i);

        for j=(i+1):n
            if CC(j)==1
                if VV(:,:,i)==VV(:,:,j)
                    CC(j)=0;
                end
            end
        end
    end
end
PP1=PP1(:,:,1:s1);
VV1=VV1(:,:,1:s1);
VV1a=VV1a(:,:,1:s1);
```

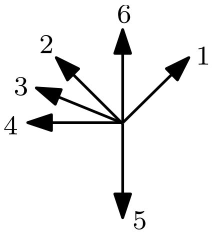
A.2.2. *Algorithm 1.* **Case K**

This configuration consists of a $(4,1)$-polytope with a sixth point that forms a $(3,1)$-circuit with one of its facets.

The dual oriented matroid of this configuration is the one we labeled 5.4*. Since this oriented matroid has no symmetries, this provides a unique ordering for the points. We will organize the points $1 := p_1$ to $6 := p_6$ so that:

- 1 is the baricenter of triangle 234.
- 5 is the interior point of the base $(4,1)$-polytope, namely that with vertex set 1236. This leaves 4 to be the point so that $5 \in int(P^4)$.
- 2 is such that the configuration contains the circuit $(126, 45)$. This leaves 3 to be such that the configuration contains the circuit $(345, 16)$.

5.4*



$(234, 1)$
$(1236, 5)$
$(126, 45)$
$(345, 16)$
$(2346, 5)$

With that ordering of the points, a triangulation of P is $P = P^4 \cup T_{1246} \cup T_{1346}$. Since we chose P^4 to be a $(4,1)$-polytope, P will have size 6 if and only if T_{1246} and T_{1346} are empty tetrahedra.

```
function [PP]= alg1

A=zeros(7,5,3);
V=zeros(7,5);
A(1,:,:)=[1,0,0;0,0,1;1,1,1;-2,-1,-2;0,0,0];  V(1,:)=[1;1;1;1;-4];
A(2,:,:)=[1,0,0;0,0,1;1,2,1;-1,-1,-1;0,0,0];  V(2,:)=[1;1;1;2;-5];
A(3,:,:)=[1,0,0;0,0,1;1,3,1;-1,-2,-1;0,0,0];  V(3,:)=[1;1;2;3;-7];
A(4,:,:)=[1,0,0;0,0,1;2,5,1;-1,-2,-1;0,0,0];  V(4,:)=[1;3;2;5;-11];
A(5,:,:)=[1,0,0;0,0,1;2,5,1;-1,-1,-1;0,0,0];  V(5,:)=[3;4;1;5;-13];
A(6,:,:)=[1,0,0;0,0,1;2,7,1;-1,-2,-1;0,0,0];  V(6,:)=[3;5;2;7;-17];
A(7,:,:)=[1,0,0;0,0,1;3,7,1;-2,-3,-1;0,0,0];  V(7,:)=[5;4;3;7;-19];
%A(8,:,:)=[1,0,0;0,0,1;2,5,1;-3,-5,-2;0,0,0];  V(8,:)=[5;5;5;5;-20];

s=0;
PP=zeros(7,3,48);
for k=1:7
    for i=1:4
        v=abs(V(k,i));
        if v==1 || v==3
            P=zeros(7,3);
            for j=1:(i-1)
                P(j,:)=A(k,j,:);
            end
            for j=(i+1):4
                P(j-1,:)=A(k,j,:);
            end
            P(4,:)=[-1,-1,0];
            %P(5,:)=[0,0,0];
            P(6,:)=A(k,i,:);
            P(7,:)=[k,i,0];

            %Ahora tenemos una configuracion (4,1) con P(5) el punto
            %interior, el tetraedro P(1)P(2)P(3)P(5) de volumen 1 o 3, y el
            %P(6) el vertice escogido del (4,1)

            %Ahora la cara p1p2p3 va a ser la que extendamos a un circuito
            %(3,1). Hay tres posibles maneras de hacer esto, que se
            %corresponde con escoger el baricentro de dicho circuito.
            %Para cada una de estas permutaciones, calcularemos la
            %transformacion unimodular que lleva el tetraedro
```

```
            % P(1)P(2)P(3)P(5) a (000),(100),(010),(001)/(123),
            %en ese orden. Los 6 puntos de nuestra configuracion seran la
            %imagen de p1, p2, p3, p5 y p6, junto con p4  (inalterado).
            %Luego ordenaremos los puntos de acuerdo a la matroide
            %orientada, y evaluaremos si los dos tetraedros 1246 y 1346 son
            %vacios.

            %P(1):
            [PP,s]=eval1(1,PP,s,P,v,1,2,3);

            %P(2):
            [PP,s]=eval1(2,PP,s,P,v,2,1,3);

            %P(3):
            [PP,s]=eval1(3,PP,s,P,v,3,2,1);

        end
    end
end
PP=PP(:,:,1:s);
```

```
function [PP,s]= eval1(M,PP,s,P,v,i,j,k)
%Permutamos los vertices segun los valores de i,j,k:
p1=P(i,:);
p2=P(j,:);
p3=P(k,:);

P(1,:)=p1;
P(2,:)=p2;
P(3,:)=p3;
P(7,3)=M;

%Ahora calculamos la transformacion unimodular y obtenemos los 6 puntos:
[P]=trans1(P,v);

%Ordenamos los puntos segun la matroide orientada:
[P]=reord1(P);

%Evaluamos si los tetraedros son vacios.
k1=empty(P(1,:),P(6,:),P(2,:),P(4,:));
k2=empty(P(1,:),P(6,:),P(3,:),P(4,:));
if k1+k2==2
    %Si lo son, la configuracion es valida
    s=s+1;
    PP(:,:,s)=P;
end
```

```
function [P]= trans1(P,v)
%pi=P(i,:) son vectores fila:

%Queremos la transformacion (afin) unimodular que nos lleve
%p1-->000
%p2-->100
%p3-->010
%p5-->001 si v=1 o 123 si v=3

%Hacemos la traslacion que nos lleva el p1 al 000

for i=2:6
    if i~=4
        P(i,:)=P(i,:)-P(1,:);
    end
end
P(1,:)=[0,0,0];
```

```
if v==1
    %La inversa de dicha matriz es
    A=zeros(3,3);
    A(:,1)=P(2,:)';
    A(:,2)=P(3,:)';
    A(:,3)=P(5,:)';
    P(1,:)=round(A\P(1,:)')';
    P(2,:)=round(A\P(2,:)')';
    P(3,:)=round(A\P(3,:)')';
    P(5,:)=round(A\P(5,:)')';
    P(6,:)=round(A\P(6,:)')';


elseif v==3

    %La matriz inversa sera una matriz A 4x4 con:
    q2=P(2,:)';
    q3=P(3,:)';

    A=zeros(3,3);
    A(:,1)=q2;
    A(:,2)=q3;
    %Y la ultima columna habra de ser acorde para que el determinante de A
    %sea 1

    %Entonces si la ultima columna es (x,y,z), se tiene que verificar que
    %ax+by+cz=1, donde
    a=q2(2)*q3(3)-q2(3)*q3(2);
    b=-q2(1)*q3(3)+q2(3)*q3(1);
    c=q2(1)*q3(2)-q2(2)*q3(1);

    %Para que haya solucion entera a esta ecuacion se ha de verificar que
    %gcd(a,b,c)=1. Pero de momento lo vamos a utilizar en casos en los que
    %si se verifica esta condicion.

    [gab,ai,bi]=gcd(a,b);
    %Esto significa que gab=gcd(a,b)=a*ai+b*bi

    [~,gi,ci]=gcd(gab,c);
    %Esto significa que 1=gcd(c,gab)=gab*gi+c*ci

    %Es decir,
    %    1=gab*gi+c*ci=(a*ai+b*bi)*gi+c*ci=(ai*gi)*a+(bi*gi)*b+ci*c
    % y una solucion entera de nuestra ecuacion es (ai*gi,bi*gi,ci)

    A(:,3)=[ai*gi;bi*gi;ci];

    P(5,:)=round(A\P(5,:)')';

    %Queremos que p5 y p6 esten en el semiplano z>0
    if P(5,3)<0
        A(3,:)=-A(3,:);
    end

    P(1,:)=round(A\P(1,:)')';
    P(2,:)=round(A\P(2,:)')';
    P(3,:)=round(A\P(3,:)')';
    P(5,:)=round(A\P(5,:)')';
    P(6,:)=round(A\P(6,:)')';

    %Ahora p5=(a,b,3) verificando a = -b = \pm 1 mod 3. Realizamos la
    %transformacion unimodular que deja fijo el plano z=0 y lleva p5 al
    %(1,2,3)
    if mod(P(5,1),3)==1
        A=eye(3);
        A(1,3)=(P(5,1)-1)/3;
        A(2,3)=(P(5,2)-2)/3;
```

```
    elseif mod(P(5,1),3)==2
        A=zeros(3,3);
        A(2,1)=1;
        A(1,2)=1;
        A(3,3)=1;
        A(1,3)=round((P(5,1)-2)/3);
        A(2,3)=round((P(5,2)-1)/3);

    end

    P(1,:)=round(A\P(1,:)')';
    P(2,:)=round(A\P(2,:)')';
    P(3,:)=round(A\P(3,:)')';
    P(5,:)=round(A\P(5,:)')';
    P(6,:)=round(A\P(6,:)')';

end
```

```
function [P]= reord1(P)
%Guardamos en B los puntos 12456, en ese orden.
B=zeros(5,3);
B(1:2,:)=P(1:2,:);
B(3:5,:)=P(4:6,:);
v=volumevectors(B',5);

%Ahora queremos que el vector tenga mas entradas (o igual) positivas que
%negativas
if v(1)+v(2)+v(3)+v(4)+v(5)<0
    v=-v;
end



vv=zeros(1,5);
for i=1:5
    vv(i)=sign(v(i));
end

%Para que P contenga el circuito (126,45), vv=++--+
%Si no lo contiene, entonces se intercambian los puntos 2 y 3
if vv(1)~=1 || vv(2)~=1 || vv(3)~=-1 || vv(4)~=-1 || vv(5)~=1
    aux=P(2,:);
    P(2,:)=P(3,:);
    P(3,:)=aux;
end
```

A.2.3. *Algorithm 2.* **Case P**

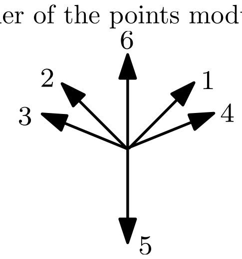
This configuration consists of a $(4,1)$-polytope with a sixth point that forms a $(2,2)$-circuit with one of its facets.

The dual oriented matroid of this configuration is the one we labeled 5.5. This oriented matroid has one symmetry. We will organize the points $1 := p_1$ to $6 := p_6$ so that:

- $(14, 23)$ forms a $(2,2)$-circuit.
- 5 is the interior point of the base $(4,1)$ polytope, namely that with vertex set 1236. The point 2 is such that 5 is also the interior point of the $(4,1)$-polytope with vertex set 1246.
- Points 3 and 4 (and consequently 1 and 2) are ordered so that the volume of the corresponding $(4,1)$-polytopes P^3 and P^4 are ordered so that the one of P^3 is smaller or equal than that of P^4.

The ambiguity comes from the fact that the roles of 2 and 3 can be exchanged, at the same time, with those of 1 and 4, respectively. In terms of equivalence, the fact that the volume of P^4 is bigger than that of P^3, gives a unique order of the points modulo equivalence.

5.5



$(23, 14)$
$(1236, 5)$
$(1246, 5)$
$(146, 35)$
$(236, 45)$

With that ordering of the points, a triangulation of P is $P = P^4 \cup T_{2346}$. Since we chose P^4 to be a $(4,1)$-polytope, P will have size 6 if and only if T_{2346} is an empty tetrahedron.

```
function [PP]= alg2

A=zeros(4,5,3);
V=zeros(4,5);
A(1,:,:)=[1,0,0;0,0,1;1,1,1;-2,-1,-2;0,0,0];  V(1,:)=[1;1;1;1;-4];
A(2,:,:)=[1,0,0;0,0,1;1,2,1;-1,-1,-1;0,0,0];  V(2,:)=[1;1;1;2;-5];
A(3,:,:)=[1,0,0;0,0,1;1,3,1;-1,-2,-1;0,0,0];  V(3,:)=[1;1;2;3;-7];
A(4,:,:)=[1,0,0;0,0,1;2,5,1;-1,-2,-1;0,0,0];  V(4,:)=[1;3;2;5;-11];
A(5,:,:)=[1,0,0;0,0,1;2,5,1;-1,-1,-1;0,0,0];  V(5,:)=[3;4;1;5;-13];
%A(6,:,:)=[1,0,0;0,0,1;2,7,1;-1,-2,-1;0,0,0];  V(6,:)=[3;5;2;7;-17];
%A(7,:,:)=[1,0,0;0,0,1;3,7,1;-2,-3,-1;0,0,0];  V(7,:)=[5;4;3;7;-19];
%A(8,:,:)=[1,0,0;0,0,1;2,5,1;-3,-5,-2;0,0,0];  V(8,:)=[5;5;5;5;-20];

s=0;
PP=zeros(7,3,33);
for k=1:4
    for i=1:4
        if abs(V(k,i))==1
            P=zeros(7,3);
            for j=1:(i-1)
                P(j,:)=A(k,j,:);
            end
            for j=(i+1):4
                P(j-1,:)=A(k,j,:);
            end
            P(4,:)=[1,1,0];
            %P(5,:)=[0,0,0];
            P(6,:)=A(k,i,:);
            P(7,:)=[k,i,0];
            %Ahora tenemos una configuracion (4,1) con P(5) el punto
            %interior, el tetraedro P(1)P(2)P(3)P(5) de volumen 1, y el
            %P(6) el vertice escogido del (4,1)

            %Ahora la cara p1p2p3 va a ser la que extendamos a un circuito
            %(2,2). Hay tres posibles maneras de hacer esto, que se
            %corresponde con escoger cual sera la diagonal del circuito.
```

```
            %Para cada una de estas permutaciones, calcularemos la
            %transformacion unimodular que lleva el tetraedro
            % P(1)P(2)P(3)P(5) a (000),(100),(010),(001),
            %en ese orden. Los 6 puntos de nuestra configuracion seran la
            %imagen de p1, p2, p3, p5 y p6, junto con p4 (inalterado).
            %Luego ordenaremos los puntos de acuerdo a la matroide
            %orientada, guardando la posible ambiguedad explicada con la
            % matroide, y evaluaremos si el tetraedro 2346 es vacio.

            %P(1):
            [PP,s]= eval2(1,PP,s,P,1,2,3);

            %P(2):
            [PP,s]= eval2(2,PP,s,P,2,1,3);

            %P(3):
            [PP,s]= eval2(3,PP,s,P,3,2,1);

        end
    end
end
PP=PP(:,:,1:s);
```

```
function [PP,s]= eval2(M,PP,s,P,i,j,k)
%Permutamos los vertices segun los valores de i,j,k:
p1=P(i,:);
p2=P(j,:);
p3=P(k,:);

P(1,:)=p1;
P(2,:)=p2;
P(3,:)=p3;
P(7,3)=M;

%Ahora calculamos la transformacion unimodular y obtenemos los 6 puntos:
[P]=trans2(P);

%Ordenamos los puntos segun la matroide orientada, y segun volumen:
[P]=reord2(P);

%Evaluamos si el tetraedro es vacios.
k1=empty(P(2,:),P(3,:),P(4,:),P(6,:));
if k1==1
    %Si lo es, la configuracion es valida
    s=s+1;
    PP(:,:,s)=P;
end
```

```
function [P]= trans2(P)
%pi=P(i,:) son vectores fila:


%Queremos la transformacion (afin) unimodular que nos lleve
%p1-->000
%p2-->100
%p3-->010
%p5-->001

%Hacemos la traslacion que nos lleva el p1 al 000

for i=2:6
    if i~=4
        P(i,:)=P(i,:)-P(1,:);
    end
```

```
end
P(1,:)=[0,0,0];

%La inversa de dicha matriz es
A=zeros(3,3);
A(:,1)=P(2,:)';
A(:,2)=P(3,:)';
A(:,3)=P(5,:)';
P(1,:)=round(A\P(1,:)')';
P(2,:)=round(A\P(2,:)')';
P(3,:)=round(A\P(3,:)')';
P(5,:)=round(A\P(5,:)')';
P(6,:)=round(A\P(6,:)')';
```

```
function [P]= reord2(P)
%Guardamos en B los puntos 12456, en ese orden.
B=zeros(5,3);
B(1:2,:)=P(1:2,:);
B(3:5,:)=P(4:6,:);
v=volumevectors(B',5);

%Ahora queremos que el vector tenga mas entradas (o igual) positivas que
%negativas
if v(1)+v(2)+v(3)+v(4)+v(5)<0
    v=-v;
end



vv=zeros(1,5);
for i=1:5
    vv(i)=sign(v(i));
end

%Para que P contenga el circuito (1246,5), vv=+++-+
%Si no es asi, entonces se intercambian los puntos 2 y 3
if vv(1)~=1 || vv(2)~=1 || vv(3)~=1 || vv(4)~=-1 || vv(5)~=1
    aux=P(2,:);
    P(2,:)=P(3,:);
    P(3,:)=aux;
end






%Ahora queremos que el (4,1) que corresponde a eliminar el vertice 4 tenga
%mayor o igual volumen al de eliminar el vertice 3:


%Guardamos en B los puntos 12456, en ese orden.
B3=zeros(5,3);
B3(1:2,:)=P(1:2,:);
B3(3:5,:)=P(4:6,:);
v3=volumevectors(B3',5);
%Ahora queremos que el vector tenga mas entradas (o igual) positivas que
%negativas
if v3(1)+v3(2)+v3(3)+v3(4)+v3(5)<0
    v3=-v3;
end

VOL3=abs(v3(4));

%Guardamos en B los puntos 12356, en ese orden.
B4=zeros(5,3);
```

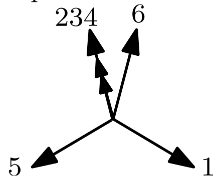
```
B4(1:3,:)=P(1:3,:);
B4(4:5,:)=P(5:6,:);
v4=volumevectors(B4',5);
%Ahora queremos que el vector tenga mas entradas (o igual) positivas que
%negativas
if v4(1)+v4(2)+v4(3)+v4(4)+v4(5)<0
    v4=-v4;
end

VOL4=abs(v4(4));


if VOL3>VOL4
    aux=P(2,:);
    P(2,:)=P(3,:);
    P(3,:)=aux;
    aux1=P(1,:);
    P(1,:)=P(4,:);
    P(4,:)=aux1;
end
```

A.2.4. *Algorithm 3.* These configurations consists of a $(4,1)$-polytope with a sixth point that forms a $(2,1)$-circuit with one of its edges (interior or exterior).

- **Case S**. The dual oriented matroid of this configuration is the one we labeled 4.21. We will organize the points $1 := p_1$ to $6 := p_6$ so that:
  - $(5, 16)$ forms a $(2,1)$-circuit.
  - 1 is the interior point of the base $(4,1)$-polytope, namely the one with vertex set 2345.
  - 2, 3 and 4 are chosen in increasing order of the absolute value of the volume of P^i.

  In terms of equivalence, the previous order is unique.

  4.21

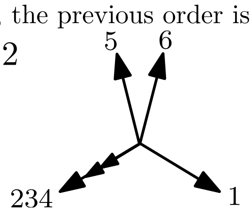
  $(2345, 1)$
  $(2346, 1)$
  $(16, 5)$
  $(2346, 5)$

  With that ordering of the points, a triangulation of P is $P = P^6 \cup T_{2356} \cup T_{2456} \cup T_{3456}$. Since we chose P^6 to be a $(4,1)$-polytope, P will have size 6 if and only if T_{2356},T_{2456} and T_{3456} are empty tetrahedra.

- **Case T**. The dual oriented matroid of this configuration is the one we labeled 4.22. We will organize the points $1 := p_1$ to $6 := p_6$ so that:
  - $(1, 56)$ forms a $(2,1)$-circuit.
  - 1 is the interior point of the base $(4,1)$-polytope, namely the one with vertex set 2345.
  - 2, 3 and 4 are chosen in increasing order of the absolute value of the volume of P^i.

  In terms of equivalence, the previous order is unique.

  4.22

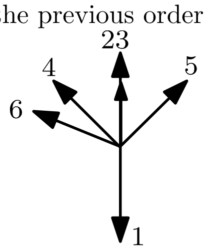
  $(2345, 1)$
  $(234, 16)$
  $(56, 1)$
  $(234, 56)$

  With that ordering of the points, a triangulation of P is $P = P^6 \cup T_{2346}$. Since we chose P^6 to be a $(4,1)$-polytope, P will have size 6 if and only if T_{2346} is an empty tetrahedron.

- **Case U**. The dual oriented matroid of this configuration is the one we labeled 4.11. We will organize the points $1 := p_1$ to $6 := p_6$ so that:
  - $(5, 46)$ forms a $(2,1)$-circuit.
  - 1 is the interior point of the base $(4,1)$-polytope, namely the one with vertex set 2345.
  - 2 and 3 are chosen in increasing order of the absolute value of the volume of P^i.

  In terms of equivalence, the previous order is unique.

  4.11

  $(2345, 1)$
  $(2346, 1)$
  $(235, 16)$
  $(46, 5)$

  With that ordering of the points, a triangulation of P is $P = P^6 \cup T_{2356}$. Since we chose P^6 to be a $(4,1)$-polytope, P will have size 6 if and only if T_{2356} is an empty tetrahedron.

```
function [PP1,PP2,PP3]= alg3

A=zeros(8,5,3);
V=zeros(8,5);
```

```
A(1,:,:)=[1,0,0;0,0,1;1,1,1;-2,-1,-2;0,0,0];  V(1,:)=[1;1;1;1;-4];
A(2,:,:)=[1,0,0;0,0,1;1,2,1;-1,-1,-1;0,0,0];  V(2,:)=[1;1;1;2;-5];
A(3,:,:)=[1,0,0;0,0,1;1,3,1;-1,-2,-1;0,0,0];  V(3,:)=[1;1;2;3;-7];
A(4,:,:)=[1,0,0;0,0,1;2,5,1;-1,-2,-1;0,0,0];  V(4,:)=[1;3;2;5;-11];
A(5,:,:)=[1,0,0;0,0,1;2,5,1;-1,-1,-1;0,0,0];  V(5,:)=[3;4;1;5;-13];
A(6,:,:)=[1,0,0;0,0,1;2,7,1;-1,-2,-1;0,0,0];  V(6,:)=[3;5;2;7;-17];
A(7,:,:)=[1,0,0;0,0,1;3,7,1;-2,-3,-1;0,0,0];  V(7,:)=[5;4;3;7;-19];
A(8,:,:)=[1,0,0;0,0,1;2,5,1;-3,-5,-2;0,0,0];  V(8,:)=[5;5;5;5;-20];

%PP1 almacenara todas las configuraciones donde extendemos la arista que
%une el punto interior con uno de los vertices, hacia afuera
s1=0;
PP1=zeros(7,3,32);

%PP1 almacenara todas las configuraciones donde extendemos la arista que
%une el punto interior con uno de los vertices, hacia adentro
s2=0;
PP2=zeros(7,3,32);

%PP1 almacenara todas las configuraciones donde extendemos la arista que
%une dos de los vertices
s3=0;
PP3=zeros(7,3,96);

for k=1:8
    for i=1:4
        P=zeros(7,3);
        v=zeros(1,4);
        for j=1:(i-1)
            P(j+1,:)=A(k,j,:);
            v(j+1)=V(k,j);
        end
        for j=(i+1):4
            P(j,:)=A(k,j,:);
            v(j)=V(k,j);
        end
        %P(1,:)=[0,0,0];
        P(5,:)=A(k,i,:);
        P(7,:)=[k,i,0];
        %Ahora tenemos la configuracion (4,1) con P(1) el punto
        %interior y el vertice escogido P(5)
        %v es un vector que contiene los volumenes asociados a los puntos
        %p2, p3 y p4 (tiene 4 coordenadas, para simplificar)


        %Ahora extendemos la arista correspondiente.
        %Luego ordenaremos los puntos de acuerdo a la matroide
        %orientada correspondiente, y evaluaremos los tetraedros que sea
        %necesario

        %Circuito (5,16)
        [PP1,s1]= eval31(PP1,s1,P,v);

        %------------------------

        %Circuito (1,56)
        [PP2,s2]= eval32(PP2,s2,P,v);

        %------------------------

        %Circuito (5,i6), para i=2,3,4

        [PP3,s3]= eval33(1,PP3,s3,P,v,2,3,4);

        [PP3,s3]= eval33(2,PP3,s3,P,v,2,4,3);

        [PP3,s3]= eval33(3,PP3,s3,P,v,4,3,2);
```

```
    end
end
PP1=PP1(:,:,1:s1);
PP2=PP2(:,:,1:s2);
PP3=PP3(:,:,1:s3);
```

```
function [PP1,s1]= eval31(PP1,s1,P,v)

P(6,:)=2*P(5,:)-P(1,:);

%Comprobamos si los tetraedros son vacios
k11=empty(P(2,:),P(3,:),P(5,:),P(6,:));
k12=empty(P(2,:),P(4,:),P(5,:),P(6,:));
k13=empty(P(3,:),P(4,:),P(5,:),P(6,:));
if k11+k12+k13==3
    s1=s1+1;
    [P]=reord312(P,v); %ordenamos 234 segun el volumen
    PP1(:,:,s1)=P;
end
```

```
function [PP2,s2]= eval32(PP2,s2,P,v)

P(6,:)=2*P(1,:)-P(5,:);

%Comprobamos si el tetraedro es vacio
k2=empty(P(2,:),P(3,:),P(4,:),P(6,:));
if k2==1
    s2=s2+1;
    [P]=reord312(P,v);%ordenamos 234 segun el volumen
    PP2(:,:,s2)=P;
end
```

```
function [P]=reord312(P,v)

%Queremos ordenar los puntos p2,p3,p4 de manera que los volumenes asociados
%v2,v3,v4 esten ordenados de menor a mayor.

p2=P(2,:); v2=v(2);
p3=P(3,:); v3=v(3);
p4=P(4,:); v4=v(4);

if v2 < v3
    if v2 < v4
        P(2,:)=p2; v(2)=v2;
        if v3 < v4 % v2<v3<v4
            P(3,:)=p3; v(3)=v3;
            P(4,:)=p4; v(4)=v4;
        else % v2<v4<v3
            P(3,:)=p4; v(3)=v4;
            P(4,:)=p3; v(4)=v3;
        end
    else %v4<v2<v3
        P(2,:)=p4; v(2)=v4;
        P(3,:)=p2; v(3)=v2;
        P(4,:)=p3; v(4)=v3;
    end
else %v2>v3
    if v2 > v4
        P(4,:)=p2; v(4)=v2;
        if v3 < v4 % v3<v4<v2
            P(2,:)=p3; v(2)=v3;
            P(3,:)=p4; v(3)=v4;
```

```
        else % v4<v3<v2
            P(2,:)=p4; v(2)=v4;
            P(3,:)=p3; v(3)=v3;
        end
    else %v3<v2<v4
        P(2,:)=p3; v(2)=v3;
        P(3,:)=p2; v(3)=v2;
        P(4,:)=p4; v(4)=v4;
    end
end
```

```
function [PP3,s3]= eval33(M,PP3,s3,P,v,i,j,k)
%Permutamos los vertices segun los valores de i,j,k:
p2=P(i,:); v2=v(i);
p3=P(j,:); v3=v(j);
p4=P(k,:); v4=v(k);

P(2,:)=p2; v(2)=v2;
P(3,:)=p3; v(3)=v3;
P(4,:)=p4; v(4)=v4;
P(7,3)=M;

P(6,:)=2*P(5,:)-P(4,:);
%Comprobamos si el tetraedro es vacio
k3=empty(P(2,:),P(3,:),P(5,:),P(6,:));
if k3==1
    s3=s3+1;
    [P]=reord33(P,v);%ordenamos 23 segun el volumen
    PP3(:,:,s3)=P;
end
```

```
function [P]=reord33(P,v)

%Queremos ordenar los puntos p2,p3 de manera que los volumenes asociados
%v2,v3 esten ordenados de menor a mayor.

p2=P(2,:); v2=v(2);
p3=P(3,:); v3=v(3);

if v2 < v3
    P(2,:)=p2; v(2)=v2;
    P(3,:)=p3; v(3)=v3;

else %v2>v3
    P(2,:)=p3; v(2)=v3;
    P(3,:)=p2; v(3)=v2;

end
```

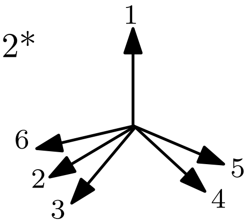
A.2.5. *Algorithm 4.* **Case X**

The dual oriented matroid of this configuration is the one we labeled 6.2*. We will organize the points $1 := p_1$ to $6 := p_6$ so that:

- 5 and 6 are the points so that P^5 and P^6 are $(4,1)$-polytopes.
- 1 is the interior point of those $(4,1)$-polytopes.
- 4 is such that the configuration contains the circuit $(236, 45)$.
- 2 is such that the configuration contains the circuit $(126, 45)$. This leaves 3 to be such that the configuration contains the circuit $(345, 16)$.

In terms of equivalence, the previous order is unique.

6.2*



$(2345, 1)$
$(2346, 1)$
$(236, 15)$
$(126, 45)$
$(345, 16)$
$(236, 45)$

With that ordering of the points, a triangulation of P is $P = P^5 \cup T_{2356}$. Since we chose P^5 to be a $(4,1)$-polytope, P will have size 6 if and only if T_{2356} is an empty tetrahedron.

```
function [PP]=alg4


[T,A,Px,Qx]=tipos;


%En esas tablas tenemos guardados los datos de todos los 32 subtetraedros
% T(p,q) de las 8 configuraciones (4,1) (4 tetraedros por cada
%configuracion):
%A: puntos
%Px: p
%Qx: q
%T: tipo del 1 al 7

%Entre dichos 32 tetraedros hay 7 tipos.
%Si hay n tetraedros de un tipo, entonces podremos pegar esos n tetraedros
%de (n sobre 2) + n maneras: formas de pegar dos de ellos + formas de pegar
%cada uno consigo mismo.
%Hay 32 maneras de pegar cada uno consigo mismo, una por cada tetraedro.

%Numero de tetraedros de cada tipo: 11,4,5,2,1,7,2, en total las formas de
%pegar dos distintos de cada tipo son, respectivamente: 55, 6, 10, 1, 0, 21
%y 1 = 94

%En total hay 32 + 94= 126 maneras de pegar dos (4,1) por un tetraedro del
%mismo tipo.

%Ahora, para cada una de estas maneras, tenemos que "pegar" dos tetraedros
%T(p,q) que son equivalentes. Es decir, existe un automorfismo que envia
%uno en el otro. En realidad existen 24 automorfismos, aunque solo algunos
%de ellos tendran coeficientes enteros. Ademas, en esta configuracion
%el punto interior de los dos (4,1) que pegamos es el mismo, asi que en
%realidad son 6 las posibilidades. Calcularemos estos automorfismos y
%comprobaremos si es entero en la funcion trans4.m

%Ahora vamos entonces primero a escoger la pareja de tetraedros que vamos a
%pegar.


%PP almacenara todas las configuraciones

s=0;
```

```
PP=zeros(8,3,756);

for k=1:8
    for i=1:4
        P=zeros(8,3);
        for j=1:(i-1)
            P(j+1,:)=A(k,j,:);
        end
        for j=(i+1):4
            P(j,:)=A(k,j,:);
        end
        %P(1,:)=[0,0,0];
        P(5,:)=A(k,i,:);
        P(7,:)=[k,i,0];
        %Ahora tenemos la configuracion (4,1) con P(1) el punto
        %interior y el vertice escogido P(5)


        q=Qx(k,i);

        for k1=1:8
            for i1=1:4
                t=0;
                if T(k1,i1)==T(k,i)
                    if k1==k && i1>i
                        t=1;
                    elseif k1==k && i1==i
                        t=1;
                    elseif k1>k
                        t=1;
                    end
                end

                if t==1

                    Q=zeros(5,3);
                    for j=1:(i1-1)
                        Q(j+1,:)=A(k1,j,:);
                    end
                    for j=(i1+1):4
                        Q(j,:)=A(k1,j,:);
                    end
                    %Q(1,:)=[0,0,0];
                    Q(5,:)=A(k1,i1,:);
                    P(8,:)=[k1,i1,0];


                %Ahora tenemos la configuracion (4,1) con Q(1) el punto
                %interior y el vertice escogido Q(5)

        %Ahora tenemos dos conf (4,1) P y Q que queremos pegar por los
        %tetraedros P 1,2,3,4 y Q 1,2,3,4


        %Consideramos las transformaciones unimodulares que envian Q(1) en
        %P(1), es decir, que mantienen fijo el origen.
        %Tenemos 6 posibilidades, que son las 6 posibles permutaciones de
        %234.
        %Para cada una de estas permutaciones, calcularemos la
        %transformacion unimodular que lleva el tetraedro q1q2q3q4 al
        %p1p2p3p4, en ese orden, y comprobamos que tenga coeficientes
        %enteros.
        %Luego ordenaremos los puntos de acuerdo a la matroide orientada,
        %y evaluaremos si el tetraedro 2356 es vacio.

                    %Id

                    [PP,s]= eval4(1,PP,s,P,Q,q,2,3,4);
```

```
                    %(23)
                    [PP,s]= eval4(2,PP,s,P,Q,q,3,2,4);

                    %(24)
                    [PP,s]= eval4(3,PP,s,P,Q,q,4,3,2);

                    %(34)
                    [PP,s]= eval4(4,PP,s,P,Q,q,2,4,3);

                    %(234)
                    [PP,s]= eval4(5,PP,s,P,Q,q,3,4,2);

                    %(243)
                    [PP,s]= eval4(6,PP,s,P,Q,q,4,2,3);

                end
            end
        end
    end
end
PP=PP(:,:,1:s);
```

```
function [PP,s]= eval4(M,PP,s,P,Q,q,i,j,k)
%Permutamos los vertices segun los valores de i,j,k:
q2=Q(i,:);
q3=Q(j,:);
q4=Q(k,:);

Q(2,:)=q2;
Q(3,:)=q3;
Q(4,:)=q4;
P(8,3)=M;

%Ahora calculamos la transformacion unimodular y obtenemos los 6 puntos:

[P,k0]= trans4(q,P,Q);
%k0=1 me dice que no hay ninguna coplanaridad y si la transformacion es
%entera
if k0==1
    %Ordenamos los puntos segun la matroide orientada, y segun volumen:
    [P]=reord4(P);
    %Evaluamos si el tetraedro es vacio.
    k2=empty(P(2,:),P(3,:),P(5,:),P(6,:));
    if k2==1
        s=s+1;
        PP(:,:,s)=P;
    end
end
```

```
function [P,k0]= trans4(q,P,Q)

k0=1;
%pi=P(i,:),qi=Q(i,:) son vectores fila

%Queremos la transformacion unimodular que lleve qi-->pi para cada
%i=1,2,3,4 (deja fijo el origen p1=q1)

%El output sera P, donde P(6,:) sera la imagen de q5 por esta
%transformacion unimodular

%A*(q_i)=(p_i)
%A1=(p_i)
%A2=(q_i)
```

```
%A=A1*(A2)^(-1)

A1=zeros(3,3);
A1(:,1)=P(2,:)';
A1(:,2)=P(3,:)';
A1(:,3)=P(4,:)';

A2=zeros(3,3);
A2(:,1)=Q(2,:)';
A2(:,2)=Q(3,:)';
A2(:,3)=Q(4,:)';

A=A1*inv(A2);

%Por el teorema del vector de volumenes, el indice de la aplicacion
%dividira al gcd de los volumenes. Si el tetraedro tiene volumen q,
%entonces A*q tiene numeros enteros. Queremos ver si tambien
%lo es A

A3=round(A.*q);

d=gcd(A3(1,:),gcd(A3(2,:),A3(3,:)));
d=gcd(d(1),gcd(d(2),d(3)));

if d<q
    k0=0;
else
    A=round(A);
end


P(6,:)=(A*Q(5,:)')';

%Ahora vamos a calcular el vector de volumenes de la nueva configuracion
%para comprobar que no haya puntos repetidos o coplanaridades.
    B=P(1:6,:)';

    [v,va]=volumevectors(B,6);
if k0==1  %Si la transformacion no es entera, no nos sirve de nada seguir
    if va(1)==0
        k0=0;
    end
end
```

```
function [P]= reord4(P)

%P es una matriz de 8x3 entradas. No nos interesan las dos ultimas dos para
%nada (son meros indicativos para saber de que par de tetraedros se han
%pegado).
%Queremos coger los cinco vertices de P (filas de 2 a 6) y ordenarlos
%en el orden especificado de su matroide orientada dual.



R=P(2:6,:);

%Sabemos que el primer punto es el punto interior. Los otros 6 puntos p2 a
%p6 pasaremos a llamarles r1 a r5 para trabajar con ellos. Y luego les
%daremos su orden adecuado.
%Entonces ahora los puntos r1 a r5 (llamemos r0 al punto interior) estan
%organizados de manera que tanto r0r1r2r3r4 como r0r1r2r3r5 son
%configuraciones (4,1).
%Por otro lado, r1...r5 es una configuracion (3,2) no vacia. Primero
%queremos averiguar que puntos se encuentran el la 3-parte y cuales en la
%2-parte.
```

```
B=R';
v=volumevectors(B,5);

vv=sort(v); %ordenamos para que quede --+++ o ---++
if vv(3)<0 %si tenemos ---++, queremos v con --+++ asi que cambiamos el
           %signo
    v=-v;
end

%Ahora ya tenemos que v es un vector con 3 entradas positivas y 2
%negativas. Una cosa que sabemos es que las dos entradas negativas seran un
%punto en (4,5) y otro en (1,2,3).

%Queremos reordenar la matriz de manera que tengamos ++--+ con los puntos
%(4,5) (no comunes a los tetraedros) aun en sus respectivas posiciones, y
%lo mismo con los puntos (1,2,3) (comunes a ambos

%De esta manera, el segmento r3r4 corta el triangulo de vertices r1,r2,r5
%Y el plano que contiene a este triangulo deja r3 y r0 a un lado, y
%r4 al otro. Es decir, que para que la configuracion solo tenga 6 puntos
%enteros, bastara con que el tetraedro r1,r2,r4,r5 sea vacio, lo cual
%comprobaremos luego (correspondera al p2,p3,p5,p6)


%Primero vamos a permutar los primeros 3 puntos para que sean ++- o --+

%Ahora mismo solo uno de 1,2,3 es negativo

r1=R(1,:);
r2=R(2,:);
r3=R(3,:);
r4=R(4,:);
r5=R(5,:);


if v(1)<0
    R(3,:)=r1;
    if v(3)<v(2)
        R(1,:)=r3;
        R(2,:)=r2;
    else
        if v(3)==v(2)
            n=1;
        end

        R(1,:)=r2;
        R(2,:)=r3;
    end

elseif v(2)<0
    R(3,:)=r2;
    if v(3)<v(1)
        R(1,:)=r3;
        R(2,:)=r1;
    else
        if v(3)==v(1)
            n=1;
        end

        R(1,:)=r1;
        R(2,:)=r3;
    end

elseif v(3)<0
    R(3,:)=r3;
    if v(2)<v(1)
        R(1,:)=r2;
        R(2,:)=r1;
```

```
    else
        if v(1)==v(2)
            n=1;
        end

        R(1,:)=r1;
        R(2,:)=r2;
    end

end



%Ahora mismo solo uno de 4,5 es negativo. Lo ponemos en r4

if v(4)<0
    R(5,:)=r5;
    R(4,:)=r4;
elseif v(5)<0
    R(5,:)=r4;
    R(4,:)=r5;
end

B=R';
v=volumevectors(B,5);
%Ahora nuestro vector deberia ser ++--+ o --++-. Queremos lo primero
vv=sort(v);
if vv(3)<0
    v=-v;
end

P(2:6,:)=R;





%Ahora vamos a escoger el orden de 2 y 3, de manera que haya el circuito
%(126,45)
%Guardamos en R los puntos 12456, en ese orden.
B=zeros(5,3);
B(1:2,:)=P(1:2,:);
B(3:5,:)=P(4:6,:);
v=volumevectors(B',5);

%Ahora queremos que el vector tenga mas entradas (o igual) positivas que
%negativas
if v(1)+v(2)+v(3)+v(4)+v(5)<0
    v=-v;
end



vv=sort(v);
for i=1:5
    vv(i)=sign(v(i));
end

%Para que P contenga el circuito (126,45), vv=++--+
%Si no lo contiene, entonces se intercambian los puntos 2 y 3
if vv(1)~=1 || vv(2)~=1 || vv(3)~=-1 || vv(4)~=-1 || vv(5)~=1
    aux=P(2,:);
    P(2,:)=P(3,:);
    P(3,:)=aux;
end
```

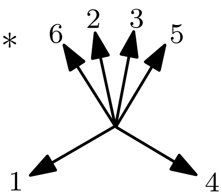
A.2.6. *Algorithm5.* **Case Y**

The dual oriented matroid of this configuration is the one we labeled 6.1*. The oriented matroid has one symmetry. We will organize the points $1 := p_1$ to $6 := p_6$ so that:

- 5 and 6 are the points so that P^5 and P^6 are $(4,1)$-polytopes, and the volume of P^6 is smaller or equal to that of P^5.
- 1 and 4 are the interior points of the $(4,1)$-polytopes P^6 and P^5, respectively.
- 2 is such that the configuration contains the circuit $(126, 45)$. This leaves 3 to be such that the configuration contains the circuit $(345, 16)$.

The ambiguity comes from the fact that the roles of 1, 2 and 6 can be exchanged, at the same time, with those of 4, 3 and 5, respectively. In terms of equivalence, the fact that the volume of P^5 is bigger than that of P^6, gives a unique order of the points modulo equivalence.

6.1*



$(2345, 1)$
$(1236, 4)$
$(2356, 1)$
$(126, 45)$
$(345, 16)$
$(2356, 4)$

With that ordering of the points, a triangulation of P is:

$$P = P^6 \cup T_{1256} \cup T_{1356} \cup T_{1456} \cup T_{2456} \cup T_{3456}$$

Since we chose P^6 to be a $(4,1)$-polytope, P will have size 6 if and only if those five tetrahedra are empty.

```
function [PP]=alg5


[T,A,Px,Qx]=tipos;


%En esas tablas tenemos guardados los datos de todos los 32 subtetraedros
% T(p,q) de las 8 configuraciones (4,1) (4 tetraedros por cada
%configuracion)
%A: puntos
%P: p
%Qx: q
%T: tipo del 1 al 7

%Entre dichos 32 tetraedros hay 7 tipos.
%Si hay n tetraedros de un tipo, entonces podremos pegar esos n tetraedros
%de (n sobre 2) + n maneras: formas de pegar dos de ellos + formas de pegar
%cada uno consigo mismo.
%Hay 32 maneras de pegar cada uno consigo mismo, una por cada tetraedro.

%Numero de tetraedros de cada tipo: 11,4,5,2,1,7,2, en total las formas de
%pegar dos distintos de cada tipo son, respectivamente: 55, 6, 10, 1, 0, 21
%y 1 = 94

%En total hay 32 + 94= 126 maneras de pegar dos (4,1) por un tetraedro del
%mismo tipo.

%Ahora, para cada una de estas maneras, tenemos que "pegar" dos tetraedros
%T(p,q) que son equivalentes. Es decir, existe un automorfismo que envia
%uno en el otro. En realidad existen 24 automorfismos, aunque solo algunos
%de ellos tendran coeficientes enteros.
%Vamos a pegar dos configuraciones (4,1) de manera que el punto interior de
%una sea un vertice de la otra y viceversa. Es decir, que de los 24
%automorfismos nos quedamos con 18. Calcularemos estos automorfismos y
%comprobaremos si es entero en la funcion trans5.m
```

```
%PP almacenara todas las configuraciones
s=0;
PP=zeros(8,3,2268);

for k=1:8
    for i=1:4
        P=zeros(8,3);
        for j=1:(i-1)
            P(j+1,:)=A(k,j,:);
        end
        for j=(i+1):4
            P(j,:)=A(k,j,:);
        end
        %P(1,:)=[0,0,0];
        P(5,:)=A(k,i,:);
        P(7,:)=[k,i,0];
        %Ahora tenemos la configuracion (4,1) con P(1) el punto
        %interior y el vertice escogido P(5)


        q=Qx(k,i);
        for k1=1:8
            for i1=1:4
                t=0;
                if T(k1,i1)==T(k,i)
                    if k1==k && i1>i
                        t=1;
                    elseif k1==k && i1==i
                        t=1;
                    elseif k1>k
                        t=1;
                    end
                end

                if t==1


                    Q=zeros(5,3);
                    for j=1:(i1-1)
                        Q(j,:)=A(k1,j,:);
                    end
                    for j=(i1+1):4
                        Q(j-1,:)=A(k1,j,:);
                    end
                    %Q(4,:)=[0,0,0];
                    Q(5,:)=A(k1,i1,:);
                    P(8,:)=[k1,i1,0];


                %Ahora tenemos la configuracion (4,1) con Q(4) el punto
                %interior y el vertice escogido Q(5)

        %Ahora tenemos dos conf (4,1) P y Q que queremos pegar por los
        %tetraedros P 1,2,3,4 y Q 1,2,3,4

        %Empezamos por considerar las transformaciones unimodulares que
        %envian Q(4) (el origen) en P(4), es decir, p4 es el punto interior
        %del segundo (4,1), y p1 es el punto interior del primero.

        %Tenemos 6 posibilidades, que son las 6 posibles permutaciones de
        %123.
        %Para cada una de estas permutaciones, calcularemos la
        %transformacion unimodular que lleva el tetraedro q1q2q3q4 al
        %p1p2p3p4, en ese orden, y comprobamos que tenga coeficientes
        %enteros.
```

```
%Luego ordenaremos los puntos de acuerdo a la matroide orientada,
%y evaluaremos si los cinco tetraedros son vacios.

            %RId
            %QId
            [PP,s]= eval5(1,PP,s,q,P,2,3,4,Q,1,2,3);

            %Q(12)
            [PP,s]= eval5(2,PP,s,q,P,2,3,4,Q,2,1,3);

            %Q(13)
            [PP,s]= eval5(3,PP,s,q,P,2,3,4,Q,3,2,1);

            %Q(23)
            [PP,s]= eval5(4,PP,s,q,P,2,3,4,Q,1,3,2);

            %Q(123)
            [PP,s]= eval5(5,PP,s,q,P,2,3,4,Q,2,3,1);

            %Q(132)
            [PP,s]= eval5(6,PP,s,q,P,2,3,4,Q,3,1,2);


%Ahora cambiaremos p2 por p4 y repetimos el proceso:

            %R(24)
            %QId
            [PP,s]= eval5(7,PP,s,q,P,4,3,2,Q,1,2,3);

            %Q(12)
            [PP,s]= eval5(8,PP,s,q,P,4,3,2,Q,2,1,3);

            %Q(13)
            [PP,s]= eval5(9,PP,s,q,P,4,3,2,Q,3,2,1);

            %Q(23)
            [PP,s]= eval5(10,PP,s,q,P,4,3,2,Q,1,3,2);

            %Q(123)
            [PP,s]= eval5(11,PP,s,q,P,4,3,2,Q,2,3,1);

            %Q(132)
            [PP,s]= eval5(12,PP,s,q,P,4,3,2,Q,3,1,2);


%Ahora cambiaremos p3 por p4 y repetimos el proceso:

            %R(34)
            %QId
            [PP,s]= eval5(13,PP,s,q,P,2,4,3,Q,1,2,3);

            %Q(12)
            [PP,s]= eval5(14,PP,s,q,P,2,4,3,Q,2,1,3);

            %Q(13)
            [PP,s]= eval5(15,PP,s,q,P,2,4,3,Q,3,2,1);

            %Q(23)
            [PP,s]= eval5(16,PP,s,q,P,2,4,3,Q,1,3,2);

            %Q(123)
            [PP,s]= eval5(17,PP,s,q,P,2,4,3,Q,2,3,1);

            %Q(132)
            [PP,s]= eval5(18,PP,s,q,P,2,4,3,Q,3,1,2);
```

```
                end
            end
        end
    end
end
PP=PP(:,:,1:s);
```

```
function [PP,s]= eval5(M,PP,s,q,P,xp,yp,zp,Q,xq,yq,zq)
%Permutamos los puntos de Q segun los valores xq,yq,zq:

q1=Q(xq,:);
q2=Q(yq,:);
q3=Q(zq,:);

Q(1,:)=q1;
Q(2,:)=q2;
Q(3,:)=q3;


%Permutamos los puntos de P segun los valores xp,yp,zp:

p2=P(xp,:);
p3=P(yp,:);
p4=P(zp,:);

P(2,:)=p2;
P(3,:)=p3;
P(4,:)=p4;

P(8,3)=M;

%Ahora calculamos la transformacion unimodular y obtenemos los 6 puntos:

[P,k0]=trans5(q,P,Q);
%k0=1 me dice que no hay ninguna coplanaridad y si la transformacion es
%entera
if k0==1
    P=reord5(P);
    k21=empty(P(1,:),P(2,:),P(5,:),P(6,:));
    k22=empty(P(1,:),P(3,:),P(5,:),P(6,:));
    k23=empty(P(1,:),P(4,:),P(5,:),P(6,:));
    k24=empty(P(2,:),P(4,:),P(5,:),P(6,:));
    k25=empty(P(3,:),P(4,:),P(5,:),P(6,:));
    if k21+k22+k23+k24+k25==5
        s=s+1;
        PP(:,:,s)=P;
    end
end
```

```
function [P,k0]= trans5(q,P,Q)

k0=1;
%pi=P(i,:),qi=Q(i,:) son vectores fila

%Queremos la transformacion unimodular que lleve qi-->pi para cada
%i=1,2,3,4 donde p1 y q4 son el origen

%El output sera P, donde p6 sera la imagen de q5 por esta transformacion
%unimodular

%Hacemos entonces primero la traslacion que me lleva el politopo Q de
%manera que q1 va al origen (p1).

Q(2,:)=Q(2,:)-Q(1,:);
```

```
Q(3,:)=Q(3,:)-Q(1,:);
Q(4,:)=Q(4,:)-Q(1,:);
Q(5,:)=Q(5,:)-Q(1,:);
Q(1,:)=Q(1,:)-Q(1,:);

%Ahora la aplicacion fija el origen y envia
%q1-->p1 (origen)
%q2-->p2
%q3-->p3
%q4-->p4


%A*(q_i)=(p_i)
%A1=(p_i)
%A2=(q_i)
%A=A1*(A2)^(-1)

A1=zeros(3,3);
A1(:,1)=P(2,:)';
A1(:,2)=P(3,:)';
A1(:,3)=P(4,:)';

A2=zeros(3,3);
A2(:,1)=Q(2,:)';
A2(:,2)=Q(3,:)';
A2(:,3)=Q(4,:)';

A=A1*inv(A2);

%Por el teorema del vector de volumenes, el indice de la aplicacion
%dividira al gcd de los volumenes. Si el tetraedro tiene volumen q,
%entonces AA*q tiene numeros enteros. Queremos ver si tambien
%lo es A

A3=round(A.*q);

d=gcd(A3(1,:),gcd(A3(2,:),A3(3,:)));
d=gcd(d(1),gcd(d(2),d(3)));

if d<q
    k0=0;
else
    A=round(A);
end


P(6,:)=(A*Q(5,:)')';

%Ahora vamos a calcular el vector de volumenes de la nueva configuracion
%para comprobar que no haya puntos repetidos o coplanaridades.
    B=P(1:6,:)';

    [v,va]=volumevectors(B,6);
if k0==1
    if va(1)==0
        k0=0;
    end
end
```

```
function [P]= reord5(P)

%P es una matriz de 8x3 entradas. No nos interesan las dos ultimas dos para
%nada
%Los puntos p1 a p6 estan organizados de manera que:
%p1,p2,p3,p4 son los puntos comunes a los dos (4,1)

%p5 y p6 son los puntos no comunes

%p4 es el punto interior asociado a p6, y p1 el punto interior asociado a
%p5

%p1 y p4 estaran ordenados de manera que el volumen total del (4,1) de p1
%sea menor o igual que el de p4.
R=P(1:6,:);


%Solo queda por decidir el orden entre p2 y p3. Utilizando las matroides
%orientadas, se tienen que verificar que

%(45,a16)

%y

%(16,b45)

%son circuitos (3,2), donde a y b son p2 y p3, en algun orden.
%Vamos a escoger p2=a.

%Calculamos el vector de volumenes de p4,p5,p2,p1,p6:
B=zeros(3,5);
B(:,1)=R(4,:)';
B(:,2)=R(5,:)';
B(:,3)=R(2,:)';
B(:,4)=R(1,:)';
B(:,5)=R(6,:)';

v=volumevectors(B,5);

%Ahora queremos que el vector tenga mas entradas (o igual) positivas que
%negativas
if v(1)+v(2)+v(3)+v(4)+v(5)<0
    v=-v;
end

%Ahora ya tenemos que v es un vector con 3 entradas positivas y 2
%negativas. Para que a=p2, tiene que ocurrir que este vector tiene signos
%--+++. SI esto no ocurre, entonces a=p3, y tenemos que permutar estos dos
%puntos.
vv=zeros(1,5);
for i=1:5
    vv(i)=sign(v(i));
end


if vv(1)==-1 && vv(2)==-1 && vv(3)==1 && vv(4)==1 && vv(5)==1
    P(2:3,:)=R(2:3,:);
else
    P(2,:)=R(3,:);
    P(3,:)=R(2,:);
end
```