SINGULAR TROPICAL HYPERSURFACES

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ABSTRACT. We study the notion of singular tropical hypersurfaces of any dimension. We characterize the singular points in terms of tropical Euler derivatives and we give an algorithm to compute all singular points. We also describe non-transversal intersection points of planar tropical curves.

1. INTRODUCTION

The concept of a singular point of a tropical variety is not well established yet. A natural definition is the following. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 with a valuation $val : \mathbb{K}^* \to \mathbb{R}$. We say that a point q in a tropical variety $V \subset \mathbb{R}^d$ is singular if there exists a singular algebraic subvariety of the torus $(\mathbb{K}^*)^d$, with tropicalization V, with a singular point of valuation q (see Definition 2.1). This definition of singularity in terms of the tropicalization of classical algebraic varieties has been considered in [10] in the case $d = 2$ of planar curves, and indirectly in [4] and [13], in the general hypersurface case. Thus, in principle, one should study all the *preimages* of V under the valuation map to decide whether V is singular. We present an equivalent formulation when V is a hypersurface defined by a tropical polynomial with prescribed support A and the residue field of $\mathbb{K}$ has also characteristic 0 (but our approach can be extended if this hypothesis is relaxed).

Recall that given a finite set $A \subseteq \mathbb{Z}^d$, Gel'fand, Kapranov and Zelevinsky [7] defined and studied the main properties of the A-discriminant Δ_A associated to the family of hypersurfaces with support A. Let ∇_0 be the variety of Laurent polynomials F with coefficients in $\mathbb{K}$ and support in A which define a singular hypersurface $\{F = 0\}$ in the torus $(\mathbb{K}^*)^d$. If ∇_0 has codimension one, then there exists a unique (up to sign) polynomial $\Delta_A \in \mathbb{Z}[a_i | i \in A]$ such that if $F = \sum_{i \in A} a_i x^i$ has a singular point in $(\mathbb{K}^*)^d$ then $\Delta_A((a_i)_{i \in A}) = 0$. This polynomial is called the *A-discriminant* and its locus coincides with the dual variety X_A^* of the equivariantly embedded toric variety X_A rationally parametrized by the monomials with exponents in A. If $\mathrm{codim}(\nabla_0) > 1$ (in which case X_A is said to be *defective*), the polynomial Δ_A is defined to be the constant polynomial 1, and we call in this case A-discriminant the ideal of X_A^*. The varieties X_A and X_A^*, as well as the A-discriminant, are affine invariants of the configuration A.

A tropical polynomial $f = \bigoplus_{i \in A} p_i \odot w^i$ with coefficients in $\mathbb{R}$ defines a singular tropical hypersurface precisely when its vector of coefficients p lies in the tropicalization $\mathcal{T}(X_A^*)$ of the A-discriminant. The concept of singularities of tropical

varieties, as well as the concept of tropical tangency, can thus be addressed via the tropicalization of the A-discriminant described in [4]. We call $\widetilde{A} \in \mathbb{Z}^{d \times A}$ the integer matrix with columns $(1, i), i \in A$. Theorem 1.1 in [4] states that the tropical discriminant equals the Minkowski sum of the co-Bergman fan $\mathcal{B}^*(\widetilde{A})$ and the row space of the matrix $\widetilde{A}$. The co-Bergman fan $\mathcal{B}^*(\widetilde{A})$ is the tropicalization of the kernel of $\widetilde{A}$, or equivalently, the tropicalization of the space of affine relations among the vectors $i \in A$. Following the notations in Section 2 (cf. also the discussion in Section 4), we can give an equivalent appealing definition of singular point of the tropical hypersurface defined by $f = \bigoplus_{i \in A} p_i \odot w^i$. Let $\phi(x) = \sum_{i \in A} x_i$ be the linear form with all coefficients equal to 1, and denote by Φ the tropical hypersurface $\Phi := \mathcal{T}(Trop(\phi))$ consisting of those vectors $v \in \mathbb{R}^A$ for which the minimum of the coordinates of v is attained at least twice. Clearly, $\mathcal{B}^*(\widehat{A}) \subseteq \Phi$. We then have

$$\mathcal{T}(f) = \{w \in \mathbb{R}^d : w \cdot A + p \in \Phi\},$$

and the singularities $\text{sing}(\mathcal{T}(f)) \subseteq \mathcal{T}(f)$ are described by

$$\text{sing}(\mathcal{T}(f)) = \{q \in \mathbb{R}^d : q \cdot A + p \in \mathcal{B}^*(\widehat{A})\}.$$

Deciding whether a given tropical polynomial defines a singular tropical hypersurface, amounts with this approach to finding a way of writing its vector of coefficients p as the sum of an element in $\mathcal{B}^*(\widetilde{A})$ plus an element in the rowspan of $\widetilde{A}$. This is particularly involved when X_A is defective, and there was no algorithm known in the general case (cf. [13], where an algorithm is presented under some geometric assumptions, or the arguments in the proof of [10, Lemma 3.12]).

Instead, we give in Theorem 2.9 a direct characterization of tropical singular points in terms of analogs of Euler derivatives of tropical polynomials, which allows us to recover Theorem 1.1 in [4]. Our tropical approach translates into an algorithm to decide whether the tropical variety associated to f is singular and to detect all the singular points.

Note that, given a Laurent polynomial $F = \sum_{i \in A} a_i x^i$, if the vector of valuations $q = (val(a_i))_{i \in A}$ defines a non singular tropical hypersurface, we get a "certificate" that F defines a non singular hypersurface in the torus $(\mathbb{K}^*)^d$. However, it is not possible to find a simple combinatorial formula to describe all singular points because the situation is not, as one could expect, completely local (cf. Proposition 3.6, and the concept of Δ-equivalence by Gel'fand, Kapranov and Zelevinsky). We give several combinatorial conditions which characterize singular tropical hypersurfaces.

In [11] or [3], tropical smooth curves are defined in terms of coherent subdivisions where all points in A are marked, and which define primitive triangulations of the convex hull $N(A)$ of A (cf. also the concept of singular tropical curves of maximal dimensional type in [10]). If the dual subdivision of $N(A)$ induced by a tropical curve $\mathcal{T}(f)$ is a primitive triangulation, $\mathcal{T}(f)$ will always be a smooth curve in our sense too. But our definition allows for certain non primitive triangulations which correspond to smooth tropical hypersurfaces, that is, tropical hypersurfaces that cannot be the tropicalization of an algebraic hypersurface with a singularity in the algebraic torus $(\mathbb{K}^*)^2$ (see Examples 3.2, 3.3 and Proposition 3.5). When A does not admit a unimodular triangulation or if not all the lattice points in the convex hull of A are marked, we can have smooth points at facets where the weight (as it is currently defined, see [12]) is > 1. Note that we concentrate on *affine* singular points, that is points in the "torus" $\mathbb{R}$ of the tropical semifield.

Our study includes all coherent subdivisions of A. Thus, we refine the definition in [11] and we explore the whole A-discriminant. Our method also generalizes trivially to hypersurfaces in arbitrary dimension. In the last section, we apply our tools to define and study the non-transversal intersections of two tropical curves with fixed monomial support, that is, the tropicalization of mixed discriminants of bivariate polynomials.

2. Tropical singularities through Euler derivatives

We fix throughout the text a finite lattice set A in $\mathbb{Z}^d$ of cardinality n. We will assume without loss of generality that the $\mathbb{Z}$-linear span of A equals $\mathbb{Z}^d$. The $\mathbb{R}$-*affine* span of a subset S in $\mathbb{R}^d$ will be denoted by $\langle S \rangle$.

We consider the tropical semifield $(\mathbb{T}, \oplus, \otimes)$, where $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ and the tropical operations are defined by $w \oplus w' = \min\{w, w'\}$, $w \odot w' = w + w'$. Our object of study are *tropical polynomials* $f = \bigoplus_{i \in A} p_i \odot w^{\odot i} \in \mathbb{T}[w_1, \ldots, w_d]$ with *support* A, that is $p_i \in \mathbb{R}$ for *all* i. To simplify the notation, we will write $w^{\odot j} = w^j = \langle j, w \rangle$.

The *tropical hypersurface* defined by a non zero tropical polynomial f with support A, is the set

(1) $\quad \mathcal{T}(f) = \{w \in \mathbb{R}^d : \exists i_1 \neq i_2 \in A \text{ such that } f(w) = \langle i_1, w \rangle + p_{i_1} = \langle i_2, w \rangle + p_{i_2}\}.$

Any tropical hypersurface is a rational polyhedral complex. For any $q \in \mathcal{T}(f)$, its associated cell σ^* is the closure of all the points $q' \in \mathcal{T}(f)$ for which $f(q)$ and $f(q')$ are attained at the same subset σ of A. Each cell σ^* comes with a marking, given by the subset σ. So, a tropical hypersurface associated with a tropical polynomial with fixed support A will be a *marked rational polyhedral complex*. This marking will be transparent in the notation. We refer to the beginning of section 4 for further details.

We will also work with Laurent polynomials F with support A and coefficients in an algebraically closed field $\mathbb{K}$ of characteristic 0, that is

(2) $$F(x) = \sum_{i \in A} a_i x^i \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}].$$

We will assume that the field $\mathbb{K}$ is provided with a rank-one non-archimedean valuation $val : \mathbb{K} \to \mathbb{R}$, and that the residue field $\mathbf{k}$ of $\mathbb{K}$ is also of characteristic zero. The *tropicalization* of a non zero polynomial F as in (2) is the tropical polynomial

(3) $$f = Trop(F) = \bigoplus_{i \in A} val(a_i) \odot w^i.$$

When the valuation group is not the whole of $\mathbb{R}$, we will suppose that the coefficients p_i of a tropical polynomial $f = \bigoplus_{i \in A} p_i \odot w^{\odot i}$ or a tropical point q that we want to lift *lie in the image* of the valuation map. To accompany our notions in the classical and tropical settings, the elements of $\mathbb{K}, \mathbb{K}^d$ and $\mathbb{K}^n$ will be denoted systematically by the letters a, b, c, x, y, z and the elements of $\mathbb{T}, \mathbb{T}^d$ and $\mathbb{T}^n$ by the letters p, q, w, v, l. We will denote by t a fixed element of $\mathbb{K}$ of valuation one. The elements of $A \subset \mathbb{Z}^d$ with be denoted by the letter i.

We introduce now the notion of singular point of a tropical hypersurface.

Definition 2.1. Let $A \subseteq \mathbb{Z}^d$ as before. Let $f = \bigoplus_{i \in A} p_i \odot w^i \in \mathbb{R}[w_1, \ldots, w_d]$ be a tropical polynomial. Let q be a point in the tropical hypersurface $\mathcal{T}(f)$. Then, q is a *singular point* of $\mathcal{T}(f)$ if there is a polynomial $F = \sum_{i \in A} a_i x^i \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

and a point $b \in (\mathbb{K}^*)^d$ such that $val(a_i) = p_i$, $val(b) = q$ and b is a singular point of the algebraic hypersurface $V(F)$ defined by F. If $\mathcal{T}(f)$ has a singular point, we call it a *singular tropical hypersurface*.

For instance, if $A = \{0, \ldots, m\} \in \mathbb{Z}$, with $m \geq 2$, and $f = \bigoplus_{j=0}^{m} 0 \odot w^j$, then $q = 0$ is always a singular point of $\mathcal{T}(f)$ since for all m there exist univariate polynomials of degree m with coefficients of valuation 0 and multiple roots with valuation 0 (just consider $F = (x-1)^m$ which has a multiple root at 1).

Let $L(w)$ be an *integral affine function* on $\mathbb{R}^d$,

$$(4) \qquad\qquad L(w) = j_1 w_1 + \ldots + j_d w_d + \beta,$$

where $(j_1, \ldots, j_d) \in \mathbb{Z}^d$ and $\beta \in \mathbb{Z}$. The Euler derivative of a tropical polynomial f with support in A with respect to L is defined as follows.

Definition 2.2. Let $f = \bigoplus_{i \in A} p_i \odot w^i$ and $L = j_1 w_1 + \ldots + j_d w_d + \beta$ be an integral affine function. The *Euler derivative* of f with respect to L is the tropical polynomial

$$\frac{\partial f}{\partial L} = \bigoplus_{i \in A, L(i) \neq 0} p_i \odot w^i.$$

We also have the standard Euler derivative of classical Laurent polynomials.

Definition 2.3. Let $F = \sum_{i \in A} a_i x^i \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ and $L = j_1 w_1 + \ldots + j_d w_d + \beta$ be an integral affine function. We associate to L the Euler vector field $L_\Theta = j_1 \Theta_1 + \ldots + j_d \Theta_d + \beta$, where $\Theta_j = x_j \frac{\partial}{\partial x_j}$ for all $j = 1, \ldots, d$. The *Euler derivative* of F with respect to L is the polynomial

$$\frac{\partial F}{\partial L} := L_\Theta(F) = j_1 x_1 \frac{\partial F}{\partial x_1} + \ldots + j_d x_d \frac{\partial F}{\partial x_d} + \beta F.$$

It is clear that for any singular point $b \in (\mathbb{K}^*)^d$ of $V(F)$, it holds that $\frac{\partial F}{\partial L}(b) = 0$ for all integral affine functions L. Note that if L is the constant function 1, then $\frac{\partial F}{\partial L} = F$.

We relate the derivative of F with respect to L with the derivative with respect to L of its tropicalization.

Lemma 2.4. *Given a tropical polynomial f with support A and an integral affine function L, the equality*

$$\frac{\partial f}{\partial L} = Trop\left(\frac{\partial F}{\partial L}\right)$$

holds for any polynomial F with support A such that $Trop(F) = f$.

Proof. Take any F with $Trop(F) = f$. Note that the Euler derivative of F with respect to L equals $\frac{\partial F}{\partial L} = \sum_{i \in A} L(i) a_i x^i$. From our assumption that the residue field of $\mathbb{K}$ is also of characteristic zero, it follows that $val(L(i)) = 0$ whenever $L(i) \neq 0$ and $val(L(i)) = \infty$ otherwise. The result is then a direct consequence of Definition 2.2 of the Euler derivative with respect to L in the tropical context. $\square$

As A is finite, the set $\left\{ \frac{\partial f}{\partial L} | L \right\}$, with L ranging over all possible integer affine linear functions, is finite for any f with support A.

Example 2.5. Consider the tropical conic $f = 1 \oplus 0 \odot w_1 \oplus 0 \odot w_2 \oplus 0 \odot w_1 \odot w_2 \oplus 1 \odot w_1^2 \oplus 1 \odot w_2^2$. Let $L_1 = w_1, L_2 = w_2$ and let $F = a_{(0,0)} + a_{(1,0)}x + a_{(0,1)}y + a_{(1,1)}xy + a_{(2,0)}x^2 + a_{(0,2)}y^2$ be any polynomial with tropicalization f. The associated A discriminant Δ_A equals $1/2$ of the determinant of the matrix

$$(5) \qquad \begin{pmatrix} 2a_{(2,0)} & a_{(1,1)} & a_{(1,0)} \\ a_{(1,1)} & 2a_{(0,2)} & a_{(0,1)} \\ a_{(1,0)} & a_{(0,1)} & 2a_{(0,0)} \end{pmatrix},$$

which is non zero since $2a_{(1,0)}a_{(1,1)}a_{(0,1)}$ is the only term in the expansion of the determinant with lowest valuation 0. Thus, as one could expect, $\mathcal{T}(f)$ is a non singular tropical hypersurface according to Definition 2.1. It is straightforward to verify that $\frac{\partial F}{\partial L_1} = x\frac{\partial F}{\partial x} = a_{(1,0)}x + a_{(1,1)}xy + 2a_{(2,0)}x^2$, $\frac{\partial F}{\partial L_2} = y\frac{\partial F}{\partial y} = a_{(0,1)}y + a_{(1,1)}xy + 2a_{(0,2)}y^2$, and $\frac{\partial f}{\partial L_1} = 0 \odot w_1 \oplus 0 \odot w_1 \odot w_2 \oplus 1 \odot w_1^2$, $\frac{\partial f}{\partial L_2} = 0 \odot w_2 \oplus 0 \odot w_1 \odot w_2 \oplus 1 \odot w_2^2$, correspond to the standard partial derivatives. Note that $q = (0,0) \in \mathcal{T}(f)$ is non singular but it also lies in the intersection of the tropical hypersurfaces $\mathcal{T}(\frac{\partial f}{\partial L_1})$ and $\mathcal{T}(\frac{\partial f}{\partial L_2})$. Consider now the affine form $L_3 = w_1 - 1$. Then, $\frac{\partial f}{\partial L_3} = 1 \oplus 0 \odot w_2 \oplus 1 \odot w_1^2 \oplus 1 \odot w_2^2$ and $q = (0,0)$ does not lie in the associated tropical hypersurface $\mathcal{T}(\frac{\partial f}{\partial L_3})$ since the minimum of the linear forms associated to the 4 terms is attained only once at q.

The main result in this section is Theorem 2.9, which characterizes singular tropical hypersurfaces (with a given support) in terms of tropical Euler derivatives. As we saw in Example 2.5, it is not enough to consider the d Euler derivatives corresponding to the coordinate axes. It is not difficult to solve this problem by appealing to the notion of a tropical basis [2], which we now recall.

Definition 2.6. Let $I \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ be an ideal. Then, $Trop(I)$ consists of all those weights $w \in \mathbb{R}^d$ which satisfy the following: $w \in \mathcal{T}(Trop(F))$ for every nonzero $F \in I$. By [5, 14], $Trop(I)$ coincides with $Trop(V(I))$, that is with (the closure of) the image under the valuation map of the zeros $V_{\mathbb{K}^*}(I)$ of I in the torus $(\mathbb{K}^*)^d$. A tropical basis of I is a finite set of polynomials $F_1, \ldots, F_r$ generating I such that $Trop(I) = \cap_{i=1}^r \mathcal{T}(Trop(F_i))$.

Given a finite lattice set $A \subseteq \mathbb{Z}^d$ with n elements, we will identify in what follows the space of polynomials with coefficients in $\mathbb{K}$ and support A with $(\mathbb{K}^*)^n$. Denote by $\bar{1}$ the point $(1, \ldots, 1) \in (\mathbb{K}^*)^d$. The subvariety

$$H_1 = \{F \in (\mathbb{K}^*)^n | F \text{ is singular at } \bar{1}\}$$

of polynomials with support A and a singularity at $\bar{1}$ is a linear space. Its closure in $\mathbb{P}^{n-1}(K)$ equals the dual space to the tangent space at the point $\bar{1}$ of X_A. See the discussion of this space and the following results in Section 4.

Denote by $\mathcal{L}$ the set of integer affine functions $L = j_1 w_1 + \ldots + j_d w_d + \beta$ such that $\gcd(j_1, \ldots, j_d) = 1$ and $\dim\langle \{L = 0\} \cap A \rangle = d - 1$.

Proposition 2.7. *Let $(v_1, \ldots, v_n)$ be variables. The finite set of tropical linear polynomials*

$$P_1 := \left\{ \bigoplus_{i \in A - \{L=0\}} 0 \odot v_i \mid L \in \mathcal{L} \right\}$$

is a tropical basis of $Trop(H_1)$.

Proof. Let $F = \sum_{i \in A} y_i x^i \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}, y_i (i \in A)]$ be the generic polynomial with support A. Note that as L runs over all integer affine functions, the Euler derivatives $\frac{\partial F}{\partial L}$ are precisely *all* integral linear combination of $F, x_1 \frac{\partial F}{\partial x_1}, \ldots, x_d \frac{\partial F}{\partial x_d}$. H_1 is the linear space (in the variables $(y_1, \ldots, y_n)$) defined by the linear equations $F(x = \bar{1})$, $x_j \frac{\partial F}{\partial x_j}(x = \bar{1}) = 0$, $1 \le j \le d$. We know that the linear forms vanishing on H_1 form a tropical basis of H_1 [14], and it is enough to consider linear forms with rational (and a fortiori, integer) entries. Now, $Trop(\frac{\partial F}{\partial L}(x = \bar{1})) = \bigoplus_{i \in A - \{L=0\}} 0 \odot v_i$. Moreover, by [2], the set of linear forms in H_1 that have minimal support define a tropical basis of $Trop(H_1)$. This set corresponds to the affine functions L such that $\{L = 0\} \cap A$ spans an affine space of maximal dimension $d - 1$. $\square$

We have defined a tropical basis of the set of polynomials with a singularity at $\bar{1}$. If we have another point $a \in (\mathbb{K}^*)^d$, we can easily provide a tropical basis of the variety H_a of hypersurfaces with a singular point at a by considering a diagonal change of coordinates. Explicitly, if $F = \sum_{i \in A} a_i x^i$ is a Laurent polynomial with coefficients in $(\mathbb{K})^*$ with a singularity at $\bar{1}$ and such that $val(a_i) = p_i \odot q_1^{i_1} \odot \ldots \odot q_d^{i_d} = p_i + \langle i, q \rangle$, then the polynomial $F_1 = \sum_{i \in A} a_i t^{-q_1 i_1 - \cdots - q_d i_d} x^i$ has a singularity at $(t^{q_1}, \ldots, t^{q_d})$ and $val(a_i t^{-q_1 i_1 - \cdots - q_d i_d}) = p_i$. We can easily deduce the following.

Proposition 2.8. *Let $A \subseteq \mathbb{Z}^d$ with $\mathbb{Z}$-linear span $\mathbb{Z}^d$. As before, identify $(\mathbb{K}^*)^n$ with the space of polynomials with support A. Consider the incidence variety $H = \{(F, u) \in (\mathbb{K}^*)^n \times (\mathbb{K}^*)^d | F \text{ is singular at } u\}$. Let $F = \sum_{i \in A} y_i x^i$ be the generic polynomial with support A, where $(x_1, \ldots, x_d)$ and $(y_i)_{i \in A}$ are variables. Then the finite set*

$$P' = \{Trop(\frac{\partial F}{\partial L}) \text{ with } \langle \{L = 0\} \cap A \rangle \text{ of maximal dimension } d - 1\},$$

is a tropical basis of $Trop(H)$.

We have now the tools to prove the following tropical characterization of singular tropical hypersurfaces with fixed support.

Theorem 2.9. *Let $f = \bigoplus_{i \in A} p_i \odot w^i$ be a tropical polynomial with support A. Let $q \in \mathcal{T}(f)$ be a point in the hypersurface defined by f. Then, q is a singular point of $\mathcal{T}(f)$ if and only if $q \in \mathcal{T}(\frac{\partial f}{\partial L})$ for all L.*
Thus, f defines a singular tropical hypersurface if and only if

$$\bigcap_L \mathcal{T}(\frac{\partial f}{\partial L}) \ne \emptyset.$$

This intersection is given by a finite number of Euler derivatives of f; for instance, we can take only the affine linear functions $L \in \mathcal{L}$ defined before Proposition 2.7.

Proof. One implication is trivial. If q is a singular point of $\mathcal{T}(f)$ there exists a polynomial $F = \sum_{i \in A} a_i x^i$, $val(a_i) = p_i$ with a singularity at a point b with $val(b) = q$. Then, $\frac{\partial F}{\partial L}(b) = 0$ for all L, and so $val(b) = q \in \mathcal{T}(\frac{\partial f}{\partial L})$ for all L.

For the converse, let q be a point in $\bigcap_L \mathcal{T}(\frac{\partial f}{\partial L})$. In particular, $q \in \mathcal{T}(f)$. Then, for any integer affine function L, the minimum $\min_{i \in A, L(i) \ne 0} \{p_i + \langle q, i \rangle\}$ is attained at least twice. This happens if and only if for all L the point $(p, q) \in \mathcal{T}(\bigoplus_{i \in A - \{L=0\}} v_i \odot w^i)$. It follows from Proposition 2.8 that (p, q) belongs to the incidence variety $Trop(H)$. So, by Kapranov's theorem [5], there is a point
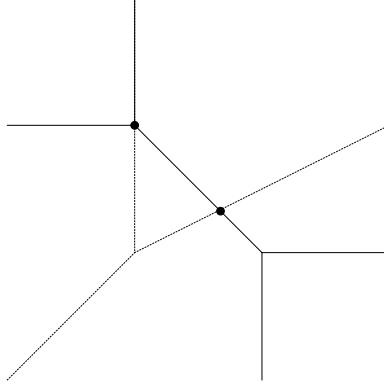
FIGURE 1. The curve f (bold) and $\frac{\partial f}{\partial(w_1+w_2)}$ (pointed) in Example 2.10

$(F, b) \in V(H)$ such that F is an algebraic polynomial with support in A and a singularity at b such that $Trop(F) = f$ and $Trop(b) = q$. $\qquad\square$

We present two examples that illustrate Theorem 2.9.

Example 2.10. Let $A = \{(0,0), (1,0), (2,0), (1,1), (2,2), (0,2)\}$. Consider the tropical polynomial $f = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_1^2 \oplus 0 \odot w_1 \odot w_2 \oplus 0 \odot w_1^2 \odot w_2^2 \oplus 6 \odot w_2^2$. Let us compute its singular points. Let $L_1 = w_2 - 2$, $\frac{\partial f}{\partial L_1} = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_1^2 \oplus 0 \odot w_1 \odot w_2$, $L_2 = w_2$, $\frac{\partial f}{\partial L_2} = 0 \odot w_1 \odot w_2 \oplus 0 \odot w_1^2 \odot w_2^2 \oplus 6 \odot w_2^2$. The intersection set of these three curves is the segment S whose ends are $(0,0), (3,-3)$. Consider now $L_3 = w_1 - w_2$ that contains all the monomials *dual* to S (cf. the beginning of Section 4 for a more precise explanation of this duality). Then, $\frac{\partial f}{\partial L_3} = 0 \odot w_1 \oplus 0 \odot w_1^2 \oplus 6 \odot w_2^2$. The intersection of this tropical curve with the segment S is the set of points $\{(0,0), (2,-2)\}$ (See Figure 1). Let us check that these two points are valid singular points. The polynomial $F_1 := -1 + 4x + (-2 + t^6)x^2 + (-2 - 2t^6)xy + x^2y^2 + t^6y^2$ has support A. It defines a curve with a singularity at $(1,1)$ and $Trop(F_1) = f$. On the other side, $F_2 = (1 - t^2 + t^4) + (2 - 2t^2)x + x^2 + (-2 - 2t^2)xy + x^2y^2 + t^6y^2$ is a polynomial with support A. F_2 defines a curve with a singularity at (t^2, t^{-2}) and such that also $Trop(F_2) = f$. It is not difficult to see that it is not possible to find a single polynomial F with two singular points with valuations $(0,0)$ and $(2,-2)$.

Example 2.11. Let $A = \{(0,0), (1,0), (2,0), (1,1), (2,2), (0,2)\}$ be as in Example 2.10 and let $A' = A \cup \{(2,10), (0,1), (1,2)\}$, so that A' are all the lattice points in the convex hull of A. Consider the tropical polynomial $f' = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_1^2 \oplus 0 \odot w_1 \odot w_2 \oplus 0 \odot w_1^2 \odot w_2^2 \oplus 6 \odot w_2^2 \oplus 0 \odot w_1^2 w_2 \oplus 3 \odot w_2 \oplus 3 \odot w_1 w_2^2$ with support in A'. That is, the coefficients of the 3 new points are given by interpolation of the linear functions defining the subdivision associated to the polynomial f in the previous example. Note that all points in A' are thus marked. It is easy to check that all points in $\mathcal{T}(f')$ are singular. Indeed, in this case f' can be lifted to the polynomial $F = (1 + x + xy + t^3y)^2$.

We now present Theorem 2.9 into action in a defective example, where the tropical A-discriminant can be explicitly computed.

Example 2.12. Let $A \subset \mathbb{Z}^3$ be the configuration $A = \{\alpha_1 = (0,0,0), \alpha_2 = (1,0,0), \alpha_3 = (2,0,0), \alpha_4 = (0,0,1), \alpha_5 = (0,1,1), \alpha_6 = (0,2,1)\}$. Thus, A is the union of two one dimensional circuits and the convex hull of A is the lattice tetrahedron with vertices $\{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$. Note that A does not contain any circuit of full dimension 3. The zero set of any integer affine function L such that the affine span of $\{L = 0\} \cap A$ is equal to 2, consists of one of the circuits plus one more point. Consider a tropical polynomial $f = \oplus_{\ell=1}^{6} p_{\alpha_i} \odot w^{\alpha_i}$ with support in A. Thus, there exists a singular point $q \in \mathcal{T}(f)$ if and only if

$$(6) \qquad\qquad 2p_{\alpha_2} = p_{\alpha_1} + p_{\alpha_3}, \quad 2p_{\alpha_5} = p_{\alpha_4} + p_{\alpha_6}.$$

This corresponds to the fact that this configuration is self-dual; indeed, the dual variety $X_A^* \subset \mathbb{P}^5(\mathbb{K})$ has (projective) dimension 3, it is isomorphic to the toric variety X_A and it is cut out by the binomials $y_2^2 - 4y_1 y_3 = 0$, $y_5^2 = 4y_4 y_6$, where $(y_1 : \cdots : y_6)$ are homogeneous coordinates in $\mathbb{P}^5(K)$. The tropicalization of this binomial ideal is the rowspan of the associated matrix A in $\mathbb{R}^6$, which is defined by the equations (6).

3. Marked tropical hypersurfaces and tropical singularities

Given a tropical polynomial $f = \oplus_{i \in A} p_i \odot w^i$ with support A, most of the (finite) Euler derivatives $\frac{\partial f}{\partial L}$ do not provide relevant information to detect singular points of $\mathcal{T}(f)$. In this section we give further conditions and characterizations to detect singular points.

We need to recall the following duality [7]. The vector of coefficients $p = (p_i)_{i \in A}$ of f defines a *coherent marked subdivision* Π_p of the convex hull $N(A)$ of A. That is, p defines a collection of subsets of A (called *marked cells*) which are in one-to-one correspondence with the domains of linearity of the affine function cutting the faces of the lower convex hull of the set of lifted points $\{(i, p_i), i \in A\}$ in $\mathbb{R}^{d+1}$. Assume that a lower face Γ_φ equals the graph of an affine function $\varphi(w_1, \ldots, w_n) = \langle q_\varphi, w \rangle + \beta_\varphi$. The corresponding marked cell σ_φ of the subdivision of $N(A)$ is the subset of A of all those i for which $p_i = \varphi(i)$.

The marked subdivision Π_p is combinatorially dual to the *marked* tropical variety $\mathcal{T}(f)$. As we saw, this is a polyhedral complex which is a union of dual cells σ_φ^*, where we also record the information of the dual cell σ_φ, and not only of the geometric information of the vertices of σ_φ. More explicitly, the dual cell σ^* in $\mathcal{T}(f)$ of a given cell σ of Π_p equals the closure of the union of the points q_φ for all ways of writing $\sigma = \sigma_\varphi$, and we also record the information of all the points in σ, that is, of *all the points at which the minimum $f(q_\varphi)$ is attained* for any point q_φ in the relative interior of σ^* (which is the *marking* of the cell). The sum of the dimensions of a pair of dual cells is d. In particular, vertices of $\mathcal{T}(f)$ correspond to marked cells of Π_p of maximal dimension d.

We now prove that Π_p is a (coherent) triangulation, then the tropical hypersurface associated to f is non singular, as expected. As we will see, the converse to this statement is not true and involves a complicated combinatorial study. Recall that a point configuration is a pyramid, if all but one of its points lie in an affine hyperplane.

Lemma 3.1. *Let $q \in \mathcal{T}(f)$ lying in the relative interior of a cell σ^* such that the dual cell σ in Π_p is a pyramid. Then, q is non singular. In particular, if Π_p is a coherent triangulation, then the tropical hypersurface $\mathcal{T}(f)$ is non singular.*
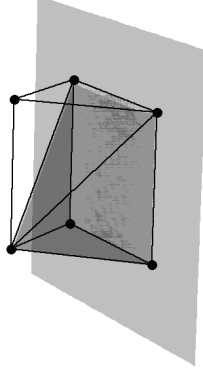
FIGURE 2. The subdivision of Example 3.3

Proof. If σ is a pyramid, let L be a linear functional such that $\{L = 0\}$ intersects σ in a facet and leaves out one point. This means that the minimum of $\frac{\partial f}{\partial L}$ at q is attained at exactly one monomial. Hence, $q \notin \mathcal{T}(\frac{\partial f}{\partial L})$ and q is not a singular point. $\qquad\square$

Example 3.2 (Example 2.5, continued)**.** Consider the same configuration A of Example 2.5, that is, the six lattice points of the 2-simplex in the plane and the tropical conic $g = 0 \oplus 1 \odot w_1 \oplus 1 \odot w_2 \oplus 1 \odot w_1 \odot w_2 \oplus 0 \odot w_1^2 \oplus 0 \odot w_2^2$. Then the associated marked subdivision has only one cell $\sigma = \{(0,0), (2,0), (0,2)\}$ and it is not singular by Lemma 3.1, even if there are points in A that do not occur in the subdivision and this single cell has lattice volume bigger than one. Also, it is straightforward to check that any polynomial $G = a_{(0,0)} + a_{(1,0)}x + a_{(0,1)}y + a_{(1,1)}xy + a_{(2,0)}x^2 + a_{(0,2)}y^2 \in \mathbb{K}[x, y]$ with the given valuations is non singular, because this time the only term in the expansion of the determinant of the matrix (5) with smallest valuation 0, is the diagonal term $8a_{(2,0)}a_{(0,2)}a_{(0,0)}$.

Example 3.3. Consider the tropical polynomial $f = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_2 \oplus 1 \odot w_3 \oplus 0 \odot w_1 \odot w_3 \oplus 0 \odot w_2 \odot w_3$. We read the support A from f. The coefficients of f induce the coherent marked subdivision depicted in Figure 2, which has two cells of dimension 3 and it is not a triangulation. One of these top dimensional cells is a unimodular 3-simplex. The second top dimensional cell contains a circuit $Z = \{(1,0,0), (0,1,0), (1,0,1), (0,1,1)\}$ of dimension $d - 1 = 2$ and it is a pyramid over the point $(0,0,0)$. The affine integer function $L = 1 - w_1 - w_2$ verifies that $Z \subset \{L = 0\}$. Computing $\frac{\partial f}{\partial L}$, we check that f defines a *non singular* tropical surface.

We now analyze some further conditions that a point $q \in \mathcal{T}(f)$ must satisfy in order to be a tropical singular point.

Theorem 3.4. *Let $f = \bigoplus_{i \in A} p_i \odot w^i$ be a tropical polynomial and $q \in \mathcal{T}(f)$ lying in the interior of a cell σ^*. Then, q is a singular point if and only if the dual cell σ is not a pyramid and we have that $q \in \mathcal{T}(\frac{\partial f}{\partial L})$ for all affine linear functions L such that $\dim\langle\{L = 0\} \cap A\rangle = d - 1$ and $\sigma \subseteq \{L = 0\}$. So, in the particular case of a vertex q of $\mathcal{T}(f)$, q is singular if and only if σ is not a pyramid.*

Proof. If q is a singular point, σ is not a pyramid by Lemma 3.1. As $q \in \mathcal{T}(\frac{\partial f}{\partial L})$ for any L by definition, in particular this happens for those L of the form described in the hypotheses. Suppose now that q is not a singular point and let L' be an affine integer function such that $q \notin \mathcal{T}(\frac{\partial f}{\partial L'})$. Let $i \in A - \{L' = 0\}$ be the unique point of A at which $\mathcal{T}(\frac{\partial f}{\partial L'})(q)$ is attained. Then, if σ is not contained in $\{L' = 0\}$, we have that $i \in \sigma$ and it is the unique point of σ outside $\{L' = 0\}$, and so σ is a pyramid. Otherwise, take any integer hyperplane $\{L = 0\}$ such that $A \cap \{L' = 0\} \subseteq A \cap \{L = 0\}$, $A \cap \{L = 0\}$ spans an affine space of dimension $d - 1$ and $i \notin L$. For any such L we have that $q \notin \mathcal{T}(\frac{\partial f}{\partial L})$, as wanted. $\square$

As a consequence, we can easily describe the polynomials that define singular hypersurfaces in the case of 1 and 2 variables. Recall that, if A is not defective, then $\mathcal{T}(Trop(\Delta(A)))$ is a subfan of the secondary fan of A. In the simplest case of one variable, $A \subseteq \mathbb{Z}$, it holds that $\mathcal{T}(Trop(\Delta_A))$ equals the union of the non top dimensional cones in the secondary fan (since the only proper faces of A are vertices). Hence, a univariate polynomial is singular if and only if the induced marked subdivision is not a triangulation. With our notation, this is a simple case of Theorem 3.4, because all circuits of A are of maximal dimension 1.

The following result, in the smooth case, appears in [7, Prop. 3.9, Ch. 11].

Corollary 3.5. *Let $A \subseteq \mathbb{Z}^2$ with n elements. Suppose $p \in \mathbb{R}^n$ induces a coherent marked subdivision Π_p in A that is not a triangulation. Then p is in $\mathcal{T}(Trop(\Delta_A))$ (equivalently, the polynomial $f = \oplus_{i \in A} p_i \odot w^i$ defines a singular tropical hypersurface) in exactly the following situations:*

   i) *There exists a marked cell of Π_p which contains a circuit of dimension 2.*
   ii) *All circuits contained in a cell of Π_p have affine dimension 1 and there exists a marked cell σ of Π_p of dimension 1 and cardinality $|\sigma| \geq 3$ with the following property: Let L be an integer affine function such that $\sigma \subset \{L = 0\}$. Then, $\sigma^* \cap \mathcal{T}(\frac{\partial f}{\partial L}) \neq \emptyset$.*

The first item is contained in Theorem 3.4. With respect to the second item, note that σ contains a circuit Z (of dimension 1) and for any integral affine function L, $Z \subset \{L = 0\}$ if and only if $\sigma \subset \{L = 0\}$. The result follows again by Theorem 3.4.

In case item ii) of Proposition 3.5 holds, $\sigma^* \cap \mathcal{T}(\frac{\partial f}{\partial L}) \neq \emptyset$ if and only if there is a cell σ' of dimension 2 containing σ, such that $\sigma' \cap \{L \neq 0\} = \{i_1\}$ consists of a single point $i_1 \in A$ and, assuming $L(i_1) > 0$, there exists another point $i_2 \in A - \sigma'$ with $L(i_2) < L(i_1)$. This is a particular case of the following more general result. Recall that we always assume that the convex hull of our exponent set A is full dimensional.

Proposition 3.6. *Let $A \subseteq \mathbb{Z}^d$. Let $p \in \mathbb{R}^n$ such that Π_p contains a top dimensional cell σ' which contains a circuit Z of dimension $d - 1$ and it is a pyramid over a point i_1. Let L be an affine integer function such that $Z \subset \sigma' \cap \{L = 0\}$ and $L(i_1) > 0$. Then, there exists a singular point $q \in \mathcal{T}(\oplus_{i \in A} p_i \odot w^i) \cap \{(\sigma' \cap \{L = 0\})^*\}$ with $\langle q, i_1 \rangle > 0$ if and only if there exists another point $i_2 \in A$ not in σ' such that $L(i_2) < L(i_1)$. In particular, if Z intersects the interior of $N(A)$, then $\mathcal{T}(\oplus_{i \in A} p_i \odot w^i)$ is singular.*

Proof. We can assume that $L(w) = j_1 w_1 + \cdots + j_d w_d + \beta$, with $j_1, \ldots, j_d$ coprime. To make the notation easier, we apply an invertible affine linear transformation to our configuration A so that $L(w) = w_1$. Denote by $\varphi(w) = \varphi_1 w_1 + \cdots + \varphi_d w_d$

the linear form which interpolates p over the cell σ', that is $\varphi(i) = p_i$ for all $i \in \sigma'$ and $\varphi(i) < p_i$ for all $i \notin \sigma'$. Thus, $p' \in \mathbb{R}^n$ defined by $p'_i := p_i - \varphi(i)$ defines the same marked subdivision. So we can assume that $p_i = 0$ for all $i \in \sigma'$ and $p_i > 0$ otherwise. Therefore, $q = (0, \ldots, 0)$ is the vertex of $\mathcal{T}$ dual to σ', which is not singular since it does not lie in $\mathcal{T}(\frac{\partial f}{\partial L})$. There will be a singular point $q = (q_1, 0, \ldots, 0)$ in $(\sigma' \cap \{L = 0\})^*$ with $\langle q, i_1 \rangle > 0$ if and only if there exists $q_1 > 0$ and two points i_2, i_3 in A such that

$$q_1 L(i_2) + p_{i_2} = q_1 L(i_3) + p_{i_3} \leq q_1 L(i) + p_i,$$

for all $i \in A$ with $L(i) \neq 0$. Note that as σ' is a pyramid over i_1, for any point i_2 in A for which $L(i_1) = L(i_2)$ it holds that i_2 is not in σ', or equivalently, $p_{i_2} > 0$. Assume first that there is a point i in $A' := A - \sigma'$ with $L(i) < L(i_1)$ and let i_2 with these properties and such that moreover $\frac{p_{i_2}}{L(i_1) - L(i_2)} = \min_{i \in A'} \frac{p_i}{L(i_1) - L(i)}$. Then, it is enough to take $q_1 = \frac{p_{i_2}}{L(i_1) - L(i_2)}$ and $i_3 = i_1$. Reciprocally, assume there exists a singular point $q = (q_1, 0, \ldots, 0)$ with $q_1 > 0$. As $q \in \mathcal{T}(\frac{\partial f}{\partial L})$, there exist two points $i_2 \neq i_3$ such that $q_1 L(i_2) + p_{i_2} = q_1 L(i_3) + p_{i_3} \leq q_1 L(i_1)$. Assume $i_2 \neq i_1$. Then, $0 < p_{i_2} \leq q_1(L(i_1) - L(i_2))$. Therefore, $L(i_2) < L(i_1)$, as wanted. The condition that Z intersects the interior of $N(A)$ guarantees the existence of a point $i_2 \in A - \sigma'$ with $L(i_2) < L(i_1)$. $\square$

Note that the point i_2 in the statement of Proposition 3.6 does not need to belong to any cell in Π_p.

4. Weight classes and the co-Bergman fan of $\widetilde{A}$

In this section, we relate our definitions to the results and definitions in [1, 4, 10]. As before, $f = \bigoplus_{i \in A} p_i \odot w^{\odot i} \in \mathbb{R}[w_1, \ldots, w_d]$ denotes a tropical polynomial with support A.

Definition 4.1. Let q be in the interior of a cell $\sigma^* \subseteq \mathcal{T}(f)$. We define the *flag* of f with respect to q as the flag of subsets $\mathcal{F}(q)$ of A defined inductively by: $F_0(q) = \sigma \subsetneq F_1(q) \subsetneq \ldots \subsetneq F_r(q)$, $dim\langle F_r(q) \rangle = d$, and for any ℓ: $F_{\ell+1}(q) - F_\ell(q)$ is the subset of $A - \langle F_\ell(q) \rangle$ where the tropical polynomial $\bigoplus_{i \in A - \langle F_\ell(q) \rangle} p_i \odot w^i$ attains its minimum at q. The *weight class* of the flag $\mathcal{F}(q)$ are all the points $q' \in \mathcal{T}(f)$ for which $\mathcal{F}(q) = \mathcal{F}(q')$

Theorems 2.9 and 3.4 provide an algorithm to decide if $q \in \mathcal{T}(f)$ is singular or not, which is similar to the method presented in [13] but which works without any restrictive hypothesis on A. The algorithm returns an L such that $q \notin \mathcal{T}(\frac{\partial f}{\partial L})$ or "q is a singular point". First, we compute $F_0(q) = \sigma$. If σ is a pyramid, there exists $i \in F_0(q)$ such that $i \notin \langle F_0(q) - \{i\} \rangle$ and we can compute an L defining the facet $\langle F_0(q) - \{i\} \rangle$ of $F_0(q)$, which verifies $q \notin \mathcal{T}(\frac{\partial f}{\partial L})$. If this is not the case and the dimension of $\langle F_0(q) \rangle < d$, we compute $F_1(q)$ and we iterate the procedure. We stop when we find an L that certifies that q is not singular or when F_ℓ spans an affine dimension d, in which case q is singular.

Number the elements $i_1 \ldots, i_n$ of A and call $A \in \mathbb{Z}^{d \times n}$ matrix with these columns. Let $\widetilde{A}$, as in the Introduction, be the integer matrix with columns $(1, i_k), k = 1, \ldots, n$. Thus, the vector $\bar{1} = (1, \ldots, 1)$ lies in the row span of $\widetilde{A}$. In fact, as the A-discriminant is an affine invariant of the configuration A, we could assume without loss of generality that A has this property, but we prefer to point out the

fact the we are interested in affine properties of the configuration A, equivalent to linear properties of $\widetilde{A}$.

Let $L(w) = j_1 w_1 + \ldots + j_d w_d + \beta$ be an affine linear function. We can associate to L the linear form $\ell_L(x_1, \ldots, x_n) = \sum_{k=1}^{n} L(i_k) x_k$. Then, the support of ℓ_L is precisely $A - \{L = 0\}$. Moreover, the coefficient vector $(L(i_k))_{k=1}^{n}$ lies in the row span of $\widetilde{A}$, as it is obtained as the product $(\beta, j_1, \ldots, j_d) \cdot \widetilde{A}$, and all linear forms in the row span of $\widetilde{A}$ (which span the ideal of $\ker(\widetilde{A})$), are of this form. Let $\mathcal{B} \subset \mathbb{Z}^c$ be a Gale dual configuration of $\widetilde{A}$. The linear forms in the row span of $\widetilde{A}$ with minimal support correspond to the circuits in $\mathcal{B}$ and to the affine linear forms L such that $\dim\langle A \cap \{L = 0\}\rangle = d - 1$.

Denote by $v_1, \ldots, v_n$ (tropical) variables. The tropicalization $Trop(\ell_L)$ equals:

$$Trop(\ell_L)(v) \;=\; \bigoplus_{L(i_k) \neq 0} 0 \odot v_k.$$

We recover the fact that H_1 is an incarnation of $\ker(\widetilde{A})$ and so $Trop(H_1)$ equals the co-Bergman fan $\mathcal{B}^*(\widetilde{A})$ (cf. Proposition 2.7). The flag of sets $\mathcal{F}(q)$ and the weight classes in Definition 4.1 coincide for instance with those occurring in [1, Page 3].

Our previous algorithm can be modified to decide whether $f = \bigoplus_{i \in A} p_i \odot w^i$ contains a singular point, that is, to decide whether p lies in the tropical A-discriminant, and in this case, to compute all the singular points. Just notice that, as weight classes induce a fine subdivision on the co-Bergman fan $\mathcal{B}^*(\widehat{A})$, they also induce a finer polyhedral subdivision of $\mathcal{T}(f)$. Two points q and $q' \in \mathcal{T}(f)$ belong to the relative interior of the same cell of the fine subdivision if and only if q, q' belong to the same weight class. If σ is a cell of the fine subdivision of $\mathcal{T}(f)$, then either every point of σ is singular or all points are regular. Since the number of cells in this subdivision is finite and computable, we can derive an algorithm to compute all singular points of $\mathcal{T}(f)$ that uses this information.

Proposition 4.2. *The (finitely many) weight classes associated to a tropical polynomial $f = \bigoplus_{i \in A} p_i \odot w^i$ with support A, are relatively open polyhedral cells which refine the polyhedral structure of $\mathcal{T}(f)$ dual to the marked coherent subdivision Π_p. If C is a cell in this new subdivision, then all points in C are singular or all of them are regular. The previous algorithm applied to any of the points in C, allows us to decide if C is a set of singular or regular points.*

We can thus reprove [4, Theorem 1.1]: a point $p = (p_i)_{i \in A}$ lies in the tropicalization $Trop(X_A^*)$ of the A-discriminant if and only if there exists a singular point $q \in \mathcal{T}(\bigoplus_{i \in A} p_i \odot w^i)$. This happens if and only if the n-th dimensional vector $v = (v_i)_{i \in A} = p + \langle q, \cdot \rangle$ defined by the equalities

$$v_i = p_i + \langle q, i \rangle, \quad i \in A,$$

lies in $\mathcal{B}^*(\widehat{A})$, by the characterization given in Theorem 2.9 (expressed by the previous algorithm). Equivalently, if and only if

$$(7) \hspace{5cm} p = v + \langle -q, \cdot \rangle$$

with v in $\mathcal{B}^*(\widehat{A})$ and $q \in \mathbb{R}^d$. That is, if and only if p lies in the Minkowski sum of $\mathcal{B}^*(\widehat{A})$ and the row span of A, which equals the Minkowski sum of $\mathcal{B}^*(\widehat{A})$ and the row span of $\widetilde{A}$, since the vector $\overline{1} \in \mathcal{B}^*(\widehat{A})$. It follows that if $p \in \mathcal{T}(X_A^*)$, the
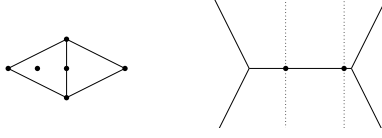
FIGURE 3. Newton subdivision and singular curve of Example 4.3
(in bold) and $\frac{\partial f}{\partial(w_1-2)}$ (pointed)

singular points of $\mathcal{T}(\bigoplus_{i\in A} p_i \odot w^i)$ are those q which occur in a decomposition of the form (7).

We end this section with some examples that exhibit different interesting features of these objects. The next example shows a tropical polynomial f whose coefficients lie in a codimension one cone of the secondary fan of A, for which $\mathcal{T}(f)$ has two singular points.

Example 4.3. Let $f = 0 \odot w_1^2 \oplus 0 \odot w_1^2 \odot w_2 \oplus 0 \odot w_1^2 \odot w_2^2 \oplus 7 \odot w_2 \oplus 4 \odot w_1 \odot w_2 \oplus 7 \odot w_1^4 \odot w_2$. The subdivision induced by f in its Newton Polygon is a triangulation except for the circuit of exponents in $\{w_1^2, w_1^2 w_2, w_1^2 w_2^2\}$. The A-discriminant of the support of f is (with the obvious meaning of the variables a_{ij})

$\underline{256a_{01}^2 a_{21}^8} - 192a_{11}^4 a_{20} a_{21}^4 a_{22} - 4096a_{01}^2 a_{20} a_{21}^6 a_{22} - 6144a_{01} a_{11}^2 a_{20} a_{21}^3 a_{22} + \underline{24576a_{01}^2 a_{20} a_{21}^4 a_{22}} - 1024a_{11}^4 a_{20}^3$
$a_{22}^3 + 8192a_{01} a_{11}^2 a_{20}^3 a_{21} a_{22}^3 - \underline{65536a_{01}^2 a_{20}^4 a_{21}^2 a_{22}^3} + \underline{65536a_{01}^2 a_{20}^4 a_{22}^4} + 216a_{11}^6 a_{21}^3 a_{41} - 2016a_{01} a_{11}^4 a_{21}^4 a_{41} +$
$5632\, a_{01}^2 a_{11}^2 a_{21}^5 a_{41} - 4096a_{01}^3\, a_{21}^6\, a_{41} + 2592\, a_{11}^6\, a_{20} a_{21} a_{22} a_{41} - 20736a_{01} a_{11}^4 a_{20} a_{21}^2 a_{22} a_{41} + 28672a_{01}^2 a_{11}^2 a_{20}$
$a_{21}^3\, a_{22} a_{41} + 16384a_{01}^3\, a_{20} a_{21}^4 a_{22} a_{41} + 4608a_{01} a_{11}^4 a_{20}^2 a_{22}^2 a_{41} - 204800a_{01}^2 a_{11}^2 a_{20}^2\, a_{21}\, a_{22}^2 a_{41} + 65536a_{01}^3\, a_{20}^2$
$a_{21}^2 a_{22}^2 a_{41} - 262144a_{01}^3 a_{20}^3 a_{22}^3 a_{41} + 729a_{11}^8 a_{41}^2 - 7776\, a_{01}\, a_{11}^6 a_{21} a_{41}^2 + 27648a_{01}^2\, a_{11}^4\, a_{21}^2\, a_{41}^2 - 38912a_{01}^3 a_{11}^2$
$a_{21}^3 a_{41}^2 + 24576a_{01}^4 a_{21}^4 a_{41}^2 - 55296\, a_{01}^2 a_{11}^4 a_{20} a_{22} a_{41}^2 + 122880\, a_{01}^3\, a_{11}^2 a_{20} a_{21} a_{22} a_{41}^2 + 65536\, a_{01}^4 a_{20} a_{21}^2 a_{22} a_{41}^2$
$+ 393216a_{01}^4 a_{20}^2 a_{22}^2 a_{41}^2 - 13824a_{01}^3\, a_{11}^4 a_{41}^3 + 73728a_{01}^4 a_{11}^2 a_{21} a_{41}^3 - 65536a_{01}^5\, a_{21}\, a_{41}^3 - 262144a_{01}^5 a_{20} a_{22} a_{41}^3 +$
$65536\, a_{01}^6\, a_{41}^4 + 768a_{11}^4 a_{20} a_{21}^2 a_{22}^2 + 16a_{11}^4 a_{21}^6 - 128a_{01} a_{11}^2 a_{21}^7 = 1536a_{01} a_{11}^2 a_{20} a_{21}^5 a_{22}.$

The minimum valuation of the terms in the A-discriminant is attained for any choice of coefficients a_{ij} with valuations prescribed by the coefficients of f, in the five underlined monomials of the A-discriminant. Three of these monomials $a_{01}^2 a_{20}^2 a_{21}^4 a_{22}^2, a_{01}^2 a_{20} a_{21}^6 a_{22}, a_{01}^2 a_{20}^3 a_{21}^2 a_{22}^3$ lie in the convex hull of the other two $a_{01}^2 a_{21}^8, a_{01}^2 a_{20}^4 a_{22}^4$. Hence, the exponents of the monomials of the A-discriminant where the minimum is attained lie on an edge, and the vector of coefficients of f belongs to a maximal cell of the tropicalization of the A-discriminantal variety. The singular points of this curve are $(3,0),(-1,0)$ (See Figure 3). Two lifts of the curve and the singular point are: $t^7 x^4 y + x^2 y^2 + (-3t^{13} + t - 2)x^2 y + x^2 + (2t^{16} - 2t^4)xy + t^7 y$ with a singularity at $(t^3, 1)$, and $t^7 x^4 y + x^2 y^2 + (t^9 - 3t^5 - 2)x^2 y + x^2 + (-2t^8 + 2t^4)xy + t^7 y$ with singularity at $(1/t, 1)$.

Our next example shows that for two vectors of coefficients inducing the same coherent subdivision of A, the associated flags need not coincide.

Example 4.4. Let $A = \{\alpha_1 = (0,0), \alpha_2 = (0,1), \alpha_3 = (0,2), \alpha_4 = (2,0), \alpha_5 = (1,2), \alpha_6 = (-2,0)\}$ and $p_v = (0,0,0,0,v_1,v_2)$, with $v = (v_1, v_2) \in R_{>0}^2$ arbitrary. In this case, p_v defines the curve given by $f_v = 0 \oplus 0 \odot w_2 \oplus 0 \odot w_2^2 \oplus 0 \odot w_1^2 \oplus v_1 \odot w_1 \odot w_2^2 \oplus v_2 \odot w_1^{-2}$. The marked subdivision Π induced by any p_v contains three maximal cells: $\sigma_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \sigma_2 = \{\alpha_3, \alpha_4, \alpha_5\}, \sigma_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_6\}$. We claim that all these curves are singular, with a singular point in the cell dual to

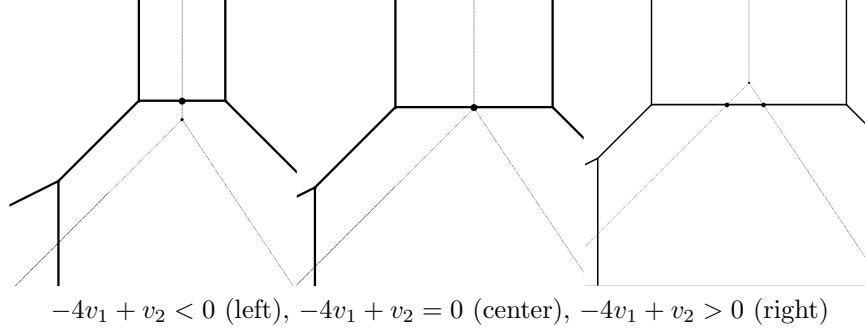$-4v_1 + v_2 < 0$ (left), $-4v_1 + v_2 = 0$ (center), $-4v_1 + v_2 > 0$ (right)

FIGURE 4. Cases of Example 4.4

the marked edge $\{\alpha_1, \alpha_2, \alpha_3\}$. However, as we will see, the number and locations of the singular points vary.

Cell σ_1 is dual to the point $(0,0)$ in the curve and σ_3 is dual to the point $(v_1/2, 0)$. By Theorem 3.5 there is a singularity if there is a point in the segment $[(0,0), (v_2/2, 0)]$ that also belongs to the partial derivative $g_v = \frac{\partial f_v}{\partial (w_1 = 0)} = 0 \odot w_1^2 \oplus v_1 \odot w_1 \odot w_2^2 \oplus v_2 \odot w_1^{-2}$. In the segment, g_v attains its minimum at $(0,0)$ on the linear form associated to α_4 and g_v attains its minimum at $(v_2/2, 0)$ on the linear form associated to α_6. Since g_v is a continuous function, there must be a point $(q, 0)$ where the minimum of g_v is attained twice, so this point will be a singularity of f (cf. [10]). This reasoning works for any hypersurface in dimension d with a circuit in the interior of A of dimension $d - 1$.

- If $-4v_1 + v_2 < 0$ there is a singular point at $q = (v_2/4, 0)$, the flag with respect to q is: $\{\alpha_1, \alpha_2, \alpha_3\} \subsetneq \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\} \subsetneq A$.
- If $-4v_1 + v_2 = 0$ there is a singular point at $q = (v_2/4, 0)$, the flag with respect to q is: $\{\alpha_1, \alpha_2, \alpha_3\} \subsetneq A$
- If $-4v_1 + v_2 > 0$ we get two different singular points:
  - $q = (v_1, 0)$ with flag with respect to q:
    $\{\alpha_1, \alpha_2, \alpha_3\} \subsetneq \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subsetneq A$.
  - $q = ((v_2 - v_1)/3, 0)$ with flag with respect to q:
    $\{\alpha_1, \alpha_2, \alpha_3\} \subsetneq \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\} \subsetneq A$.

Thus, we can take different values of $v, v' \in \mathbb{R}^2_{>0}$ such that, while keeping $\Pi = \Pi_{p_v} = \Pi_{p_{v'}}$ invariant, it is not possible to find singular points $q_v, q_{v'}$ in $\mathcal{T}(f_v), \mathcal{T}(f_{v'})$ for which the flags coincide $F_\ell(q_v) = F_\ell(q_{v'})$ for all ℓ.

It is worthwhile to note that the set of singular points in a hypersurface is not, in general, a tropical variety.

Example 4.5. Let $f = 0 \oplus 0 \odot w_2 \oplus 0 \odot w_2^2 \oplus 0 \odot w_1 \oplus 0 \odot w_1 \odot w_2 \oplus 1 \odot w_1^2$ represents a tropical conic. So, if it is singular, it is a pair of lines. It happens that this conic is the union of the lines $0 \oplus 0 \odot w_1 \oplus 0 \odot w_2$ and $0 \oplus 1 \odot w_1 \oplus 0 \odot w_2$. The intersection of these two lines is the ray $(0,0) + p(1,0)$, $p \geq 0$. This is not a tropical variety. However, any of its points is a valid singular point of the conic. We can take any point q in the intersection set and lift the whole configuration, as it is an acyclic configuration (see [15]). In fact, for the point $(p,0)$, $p \geq 0$ we can take the lift $F = (1 + x - (1 + t^p)y)(1 + tx - (1 + t^{p+1})y) = 1 + (1+t)x + (-2 - t^p -$

$t^{p+1})y + tx^2 + (-1 - t - t^{p+1})xy + (1 + t^p + t^{p+1} + t^{2p+1})y^2$ that has a singularity at $(t^p, 1)$.

5. Tropical curves with non transversal intersection

We fix two finite subsets A_1, A_2 of $\mathbb{Z}^2$ with $|A_1|, |A_2| \geq 2$ and such that $\mathbb{Z}A_1 + \mathbb{Z}A_2 = \mathbb{Z}^2$. In this section we define and study non transversal intersections of two tropical curves associated to tropical polynomials with respective supports A_1, A_2. As in Definition 2.1, we will say that the intersection is non transverse when it comes from the tropicalization of a classical non transverse intersection of two curves. In this case, we will see that the standard definitions do not have a straightforward translation to the tropical setting.

Definition 5.1. Let $f = \bigoplus_{i \in A_1} p_i^1 \odot w^i$, $g = \bigoplus_{i \in A_2} p_i^2 \odot w^i$ be two tropical polynomials in $\mathbb{R}[w_1, w_2]$. Let q be a point in the intersection of the tropical curves $\mathcal{T}(f) \cap \mathcal{T}(g)$. Then q is a *non-transversal (or non-smooth) intersection point* of $\mathcal{T}(f)$ and $\mathcal{T}(g)$ if there exists two Laurent polynomials $F = \sum_{i \in A_1} a_i^1 x^i, G = \sum_{i \in A_2} a_i^2 x^i$ in $\mathbb{K}[x_1, x_2]$, with respective supports A_1, A_2 and a point $b \in (\mathbb{K}^*)^2$ which is a non-transversal intersection of $\{F = 0\}$ and $\{G = 0\}$ such that $Trop(F) = f$ (that is, $val(a_i^1) = p_i^1$ for all $i \in A_1$), $Trop(G) = g$ and $val(b) = q$.

Recall that $b \in \{F = 0\} \cap \{G = 0\}$ is a non-transversal intersection point if moreover the Jacobian $J_{F,G}$ vanishes at b. This Jacobian is the determinant of the Jacobian matrix (or the matrix of the differential of the map (F, G)), and also of its transpose $M_{F,G} = \begin{pmatrix} F_{x_1}(b) & G_{x_1}(b) \\ F_{x_2}(b) & G_{x_2}(b) \end{pmatrix}$. The condition that $J_{F,G}(b) = 0$ is obtained from elimination of variables from the following equivalent fact: $\{F = 0\}$ and $\{G = 0\}$ intersect non transversally at b if and only if their tangent lines coincide, or equivalently, the matrix $M_{F,G}$ has a non trivial kernel, that is, there exists a non trivial vector (y_1, y_2) which is a solution of the system

$$(8) \qquad F_{x_1}(b)y_1 + G_{x_1}(b)y_2 = F_{x_2}(b)y_1 + G_{x_2}(b)y_2 = 0.$$

Given two Laurent polynomials $F = \sum_{i \in A_1} a_i^1 x^i, G = \sum_{i \in A_2} a_i^2 x^i$ in $\mathbb{K}[x_1, x_2]$, with respective supports $A_1, A_2 \subset \mathbb{Z}^2$, the *mixed discriminant* of F and G is the A-discriminant associated to the polynomial $y_1 F + y_2 G \in \mathbb{K}[x_1, x_2, y_1, y_2]$ with support in the Cayley configuration (cf. [7])

$$e_1 \times A_1 \cup e_2 \times A_2 \subset \mathbb{Z}^4.$$

In fact, this is a three dimensional configuration lying in the plane defined by the sum of the two first coordinates equal to 1. This mixed discriminant vanishes at the vectors of coefficients $((a_i^1)_{i \in A_1}, (a_i^2)_{i \in A_2})$ whenever F and G have a non-transversal intersection at a point $b \in (K^*)^2$ for which the system (8) has a solution $(y_1, y_2) \in (K^*)^2$. In particular, note that horizontal and vertical tangents are not necessarily reflected in the mixed discriminant (cf. [6, Section 3] for a more general definition of discriminants which takes into account different supports). Hence, we do not take cover these extremal cases and will only describe the non-transversal intersection points for which the system (8) has a solution $(y_1, y_2) \in (K^*)^2$.

Lemma 5.2. *Let* $f, g \in \mathbb{R}[w_1, w_2]$ *be two tropical bivariate polynomials with respective supports* A_1, A_2. *The tropical plane curves that* f *and* g *define intersect non-transversally at an intersection point* $q = (q_1, q_2) \in \mathbb{R}^2$ *if and only if there*

exists $l \in \mathbb{R}$ such that $\overline{q} = (q_1, q_2, l)$ belongs to the tropical discriminant associated to the polynomial $f \oplus w_3 \odot g \in \mathbb{R}[w_1, w_2, w_3]$ with support in the configuration $\mathcal{C}(A_1, A_2) = A_1 \times \{0\} \cup A_2 \times \{1\} \subset \mathbb{Z}^3$, that is if $\overline{q}$ is a singular point of $\mathcal{T}(f \oplus w_3 \odot g)$.

Note that the configurations $\mathcal{C}(A_1, A_2) \subset \mathbb{Z}^3$ and the Cayley configuration $e_1 \times A_1 \cup e_2 \times A_2 \subset \mathbb{Z}^4$ are affinely equivalent. Therefore, the associated sparse discriminants coincide (up to the names of the variables).

Proof. Suppose that there is an element $l \in \mathbb{R}$ such that (q_1, q_2, l) belongs to the tropical discriminant of $f \oplus w_3 \odot g$. So, $(f \oplus w_3 \odot g, (q_1, q_2, l))$ belongs to the incidence variety $Trop(H)$ associated to $A = \mathcal{C}(A_1, A_2)$. Then, there are algebraic polynomials $F, G \in \mathbb{K}[x_1, x_2]$ and a point $(b_1, b_2, b_3) \in \mathbb{K}^3$ such that $(F(x) + x_3 G(x), (b_1, b_2, b_3))$ lies in the incidence variety H, $Trop(F) = f$, $Trop(G) = g$, $val(b_1, b_2, b_3) = (q_1, q_2, l)$. Hence (b_1, b_2, b_3) is a singular point of $F + x_3 G$, and so the partial derivatives must vanish:

$$F_{x_1}(b_1, b_2) + b_3 G_{x_1}(b_1, b_2) = 0, \ \ F_{x_2}(b_1, b_2) + b_3 G_{x_2}(b_1, b_2) = 0, G(b_1, b_2) = 0.$$

It follows that $F(b_1, b_2) = 0$ and $b = (b_1, b_2)$ is a non-transversal intersection point of $\{F = 0\}$ and $\{G = 0\}$. To prove the converse, suppose (q_1, q_2) is a non transversal intersection point of f and g. There exists $F, G \in \mathbb{K}[x_1, x_2]$ with respective supports A_1, A_2 such that $Trop(F) = f, Trop(G) = g$ and $b = (b_1, b_2)$ as in Definition 5.1. Then $F(b) = 0$, $G(b) = 0$, and let (y_1, y_2) be as in (8). It follows that $(b_1, b_2, y_2/y_1)$ is a singular point of the surface defined by $F + x_3 G$, so $(F + x_3 G, (b_1, b_2, y_2/y_1)) \in H$ and $(f \oplus w_3 \odot g, (q_1, q_2, val(y_2/y_1))) \in Trop(H)$. Therefore, $(q_1, q_2, val(y_2/y_1))$ is a non-transversal intersection point of $f \oplus w_3 \odot g$. $\qquad\square$

Example 5.3. Let $f = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_2$ and $g = 1 \oplus 0 \odot w_1 \oplus 0 \odot w_2$ be two tropical lines. These lines intersect at an infinite number of points (all the points in the ray $\{w_1 = w_2 \le 0\}$). However, two algebraic lines intersect non-transversally if and only if they are the same, and we expect that this also happens in the tropical setting. Since $f \ne g$, let us check that they intersect transversally according to our definition. Consider the surface defined by $f \oplus w_3 \odot g = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_2 \oplus 1 \odot w_3 \oplus 0 \odot w_1 \odot w_3 \oplus 0 \odot w_2 \odot w_3$. Both lines have the same support $A_1 = A_2 = \{(0,0), (1,0), (0,1)\}$. The associated mixed subdivision of $\mathcal{C}(A_1, A_2) = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1), (0,1,1)\}$ is precisely the marked subdivision occurring in Example 3.3 and depicted in Figure 2, so $f \oplus w_3 \odot g$ is indeed non singular according to Lemma 5.2.

We now revisit Example 2.12.

Example 5.4. Let $A_1 = \{(0,0), (1,0), (2,0)\}$, $A_2 = \{(0,0), (0,1), (0,2)\}$. So, f and g are in fact univariate polynomials in different variables. The associated configuration $\mathcal{C}(A_1, A_2)$ is just the configuration A occurring in Example 2.12. In this case, the mixed discriminant has codimension bigger than 1 and a point of intersection $q = (q_1, q_2)$ is non-transversal if and only if q_1 is a singular point of $\mathcal{T}(p_1 \oplus p_2 \odot w_1 \oplus p_3 \odot w_1^2)$ and q_2 is a singular point of $\mathcal{T}(p_4 \oplus p_5 \odot w_2 \oplus p_6 \odot w_2^2)$, which is the translation of the fact that q is singular if and only if it lies in the rowspan of A.

Since we are looking for the singular points (q_1, q_2, l) in $\mathcal{T}(f \oplus w_3 \odot g)$, we could use the known tropical basis of the discriminant of this surface to compute them, as in the discussion above. However, we would like to obtain a method that involves

only f and g and no new variables. Just checking if the Jacobian matrix is singular will not work.

Example 5.5. Let $f = 0 \oplus 0 \odot w_1 \oplus 0 \odot w_2 \oplus 0 \odot w_1 \odot w_2 \oplus 0 \odot w_2^2 \oplus 1 \odot w_1^2$, $g = 0 \oplus 1 \odot w_1 \oplus 0 \odot w_2$. Consider the intersection point $q = (0,0)$. If we would like to use a kind of tropical Jacobian matrix, a natural choice would be the matrix

$$\begin{pmatrix} \frac{\partial f}{\partial\{w_1=0\}} & \frac{\partial f}{\partial\{w_2=0\}} \\ \frac{\partial g}{\partial\{w_1=0\}} & \frac{\partial g}{\partial\{w_2=0\}} \end{pmatrix} = \begin{pmatrix} 0 \odot w_1 \oplus 0 \odot w_1 \odot w_2 \oplus 1 \odot w_1^2 & 0 \odot w_2 \oplus 0 \odot w_1 \odot w_2 \oplus 0 \odot w_2^2 \\ 1 \odot w_1 & 0 \odot w_2 \end{pmatrix}.$$

When we evaluate at q, we get the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which is nonsingular because $0 \odot 0 \neq 0 \odot 1$. However, take $F = (1-t)+2x+2y+2xy+y^2+x^2t$, $G = (1+t)+tx+y$, $b = (-1,-1)$ is an intersection point and $Trop(F) = f$, $Trop(G) = g$, $val(b) = (0,0)$ and

$$(1,2)\begin{pmatrix} F_x(b) & F_y(b) \\ G_x(b) & G_y(b) \end{pmatrix} = (1,2)\begin{pmatrix} -2t & -2 \\ t & 1 \end{pmatrix} = (0,0)$$

So the tropical point $(0,0)$ is a non-transversal intersection point. The problem is that in general, given a polynomial $J \in \mathbb{K}[x_1,\ldots,x_n]$ and a point $b \in (\mathbb{K}^*)^n$, $Trop(J)(val(b)) \neq val(J(b))$. In our case J is the Jacobian $J_{FG} = 2 - 2t + y(2 - 2t)$ and $b = (-1,-1)$. We have that $Trop(J_{FG})(val(b)) = (0 \oplus 0 \odot w_2)(0,0) = 0$, while $val(J_{FG})(b) = \infty$.

We now show another phenomenon that occurs.

Example 5.6. Let $F = G = 1 + x_1 + x_2$. $f = Trop(F)$ and $g = Trop(G)$ equal $0 \oplus 0 \odot w_1 \oplus 0 \odot w_2$ and intersect non transversally at any point of $\mathcal{T}(f) = \mathcal{T}(g)$. The toric Jacobian $x_1 x_2 J_{FG}$ is the determinant of the 2×2 matrix M with columns the Euler derivatives of F and G with respect to $L = w_1, w_2$, and so it is identically zero. If we consider the matrix $Trop(M)$ obtained by taking the tropicalization of each of the entries of M, we get the 2×2 matrix with two equal rows $[0 \odot w_1 \; 0 \odot w_2]$. The tropical determinant of $Trop(M)$ equals the permanent $(0 \odot w_1) \odot (0 \odot w_2) \oplus (0 \odot w_1) \odot (0 \odot w_2)$. If we forget the fact that we have *twice* the term $(0 \odot w_1) \odot (0 \odot w_2) = 0 \odot w_1 \odot w_2$, we lose information. We just get a monomial, which defines an empty tropical curve.

We deal now with two easy cases.

Proposition 5.7. *Let f, g be two tropical bivariate polynomials with respective supports A_1, A_2. Let $q = (q_1, q_2)$ be a tropical point that is a vertex in both $\mathcal{T}(f)$ and $\mathcal{T}(g)$. Then q is a non-transversal intersection point of f and g.*

Proof. Set $l := f(q) - g(q) \in \mathbb{R}$. We claim that $\bar{q} = (q_1, q_2, l)$ is a singular point of the tropical surface $\mathcal{T}(f \oplus w_3 \odot g)$. To see this, note that the points in $\mathcal{C}(A_1, A_2)$ where the minimum in $\bar{q}$ is attained are of the form $(i,0) \in A_1 \times 0$, for all i such that $f(q)$ attains its minimum and those of the form $(i,1) \in A_2 \times 1$, for all i where $g(q)$ attains its minimum. Since q is a vertex of both $\mathcal{T}(f)$ and $\mathcal{T}(g)$, the minimum of $f(q)$ is attained in a 2-dimensional cell σ_1 in A_1 and the minimum of $g(q)$ is attained in a 2-dimensional cell σ_2 in A_2. Let $L = j_1 w_1 + j_2 w_2 + j_3 w_3 + \beta$ be any integer affine function in three variables, $L \neq j_3 w_3 + \beta$. Then $\{L = 0\}$ cannot contain σ_1 nor σ_2, so there are at least two different points of $\mathcal{C}(A_1, A_2)$ where the minimum of $f \oplus w_3 \odot g$ is attained at $\bar{q}$. It follows that $\bar{q} \in \mathcal{T}\left(\frac{\partial(f \oplus w_3 \odot g)}{\partial L}\right)$. In case L is of the form $j_3 w_3 + \beta$, $\{L = 0\}$ is disjoint from $\mathcal{C}(A_1, A_2)$ unless $\beta = -j_3 - 1, 0$, and in

this case either all the monomials corresponding to the points in $A_1 \times 0$ or $A_2 \times 1$ occur in the Euler derivative with respect to L of $f + w_3 \odot g$. So, by Theorem 2.9, $\bar{q}$ is a singular point of $f \oplus w_3 \odot g$ and q is a nontransversal intersection point of f and g. $\square$

Proposition 5.8. *Let f, g be tropical polynomials with respective supports A_1, A_2. Suppose that q is a vertex of $\mathcal{T}(f)$ and lies on a segment or ray of $\mathcal{T}(g)$. Moreover, suppose that, locally, p is the only local intersection point in $\mathcal{T}(f) \cap \mathcal{T}(g)$. Then q is a non-transversal intersection point of f and g.*

Proof. The minimum of f at q is attained in at least three non-collinear monomials associated to the points $\{\alpha_1, \alpha_2, \alpha_3\}$ and the minimum in g is attained in at least 2 monomials associated to the points $\{\beta_1, \beta_2\}$. Take again $l := f(q) - g(q)$. Then, (q_1, q_2, l) of $f \oplus w_3 \odot g$ is attained at the monomials corresponding to $C = \{(\alpha_1, 0), (\alpha_2, 0), (\alpha_3, 0), (\beta_1, 1), (\beta_2, 1)\}$. By construction, $\{(\alpha_1, 0), (\alpha_2, 0), (\alpha_3, 0), (\beta_1, 1)\}$ is a simplex. Since the intersection of $\mathcal{T}(f)$ and $\mathcal{T}(q)$ around q is just the point q, it follows that the line through $(\beta_1, 1)$ and $(\beta_2, 1)$ is not coplanar with any line generated by two different points among $\{(\alpha_1, 0), (\alpha_2, 0), (\alpha_3, 0)\}$. In other words, C is a circuit of dimension 3. By Theorem 3.4, (q_1, q_2, l) is a singular point of $f \oplus w_3 \odot g$ and thus q is a non-transversal intersection point of f and g. $\square$

In order to deal with the general case, we introduce some notation. Let $L' = j_1 w_1 + j_2 w_2 + c w_3 + \beta$ be an integer affine function in 3 variables. Call $L = j_1 w_1 + j_2 w_2 + \beta \in \mathbb{R}[w_2, w_2]$, so that

$$(9) \qquad\qquad L' = L + c w_3.$$

We clearly have:

$$(10) \qquad\qquad \frac{\partial(f \oplus w_3 \odot g)}{\partial L'} = \frac{\partial f}{\partial L} \oplus w_3 \odot \frac{\partial g}{\partial(L+c)}.$$

Lemma 5.9. *Let $q \in \mathbb{R}^2$ be an intersection point of $\mathcal{T}(f)$ and $\mathcal{T}(g)$ and let L' be an integer affine function in 3 variables. There exists $l \in \mathbb{R}$ such that $\bar{q} = (q_1, q_2, l)$ is a point in the tropical surface $\mathcal{T}(\frac{\partial(f \oplus w_3 \odot g)}{\partial L'})$.*

Proof. Define L as in (9). It is enough to take $l = \frac{\partial f}{\partial L}(q) - \frac{\partial g}{\partial(L+c)}(q)$. $\square$

In fact, keeping the notations of the previous Lemma, we can make a finer classification of the affine linear forms L' into 4 types and describe all possible choices of l in each case. We say that L' is of type 1 if the minimum of $\frac{\partial f}{\partial L}$ at q is attained at least twice and exactly once in $\frac{\partial g}{\partial(L+c)}$. In this case, we can take any $l \geq \frac{\partial f}{\partial L}(q) - \frac{\partial g}{\partial(L+c)}(q)$. We say that L' is of type 2 if instead the minimum at q is attained at least twice in $\frac{\partial g}{\partial L+c}$ and once in $\frac{\partial f}{\partial L}$. In this case, we can take any $l \leq \frac{\partial f}{\partial L}(q) - \frac{\partial g}{\partial(L+c)}(q)$. The integer affine linear function L' is said to be of type 3 if the minimum is attained at least twice in both $\frac{\partial f}{\partial L}$ and $\frac{\partial g}{\partial(L+c)}$. In this case, we can take any $l \in \mathbb{R}$. Finally, we say that L' is of type 4 if the minimum is attained once in both $\frac{\partial f}{\partial L}$ and $\frac{\partial g}{\partial(L+c)}$. In this case, the only possible choice is $l = \frac{\partial f}{\partial L}(q) - \frac{\partial g}{\partial(L+c)}(q)$.

We now present the main result in this section.

Theorem 5.10. *Let f, g be two tropical bivariate polynomials with respective supports A_1, A_2 and q an intersection point of $\mathcal{T}(f)$ and $\mathcal{T}(g)$. Consider the set $\mathcal{A}(q) = \{(L_1, L_2, c_1, c_2)\}$ of all 4-tuples where L_1, L_2 are integer affine functions in two variables and $c_1, c_2 \in \mathbb{R}$, for which q belongs to at least one of the tropical curves associated to $\frac{\partial f}{\partial L_1}$, $\frac{\partial f}{\partial L_2}$, $\frac{\partial g}{\partial(L_1+c_1)}$, $\frac{\partial g}{\partial(L_2+c_2)}$. Let $\mathcal{B}(q) = \{(L_1, L_2, c_1, c_2)\}$ be the set of 4-tuples for which q does* not *belong to any of these four tropical curves.*

   Then, q is a non-transversal intersection point of f and g if and only if

- *For all $(L_1, L_2, c_1, c_2) \in \mathcal{A}(q)$, q belongs to the tropical curve associated to the tropical polynomial*

$$\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2 + c_2)} \oplus \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1 + c_1)}.$$

- *The following equalities hold for all $(L_1, L_2, c_1, c_2) \in \mathcal{B}(q)$:*

$$\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2 + c_2)}(q) = \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1 + c_1)}(q).$$

It is enough to check these conditions for a finite number of 4-tuples.

Proof. By Theorem 2.9, a point $q \in \mathbb{R}^2$ is a non-transversal intersection point if and only if there is an l such that (q_1, q_2, l) belongs to all the partial derivatives $\frac{\partial(f \oplus w_3 \odot g)}{\partial L'}$. Assume first that $q \in \mathbb{R}^2$ is a non-transversal intersection point and let $l \in \mathbb{R}$ such that (q_1, q_2, l) belongs to all the partial derivatives $\frac{\partial(f \oplus w_3 \odot g)}{\partial L}$. Let L_1, L_2 be integer affine functions of 2 variables and $c_1, c_2 \in \mathbb{R}$. Call $L_1' = L_1 + c_1 w_3, L_2' = L_2 + c_2 w_3$. Then, by (10), we have that the minimum at $\frac{\partial f}{\partial L_i}(q) \oplus l \odot \frac{\partial g}{\partial(L_i+c_i)}(q)$ is attained twice for $i = 1, 2$. It is not difficult to see that the two conditions in the statement of the Theorem hold, separating the arguments for the different types of functions L'. Reciprocally, assume that these conditions hold for all 4-tuples of the form (L_1, L_2, c_1, c_2) and let q be an intersection point. Then, any (q_1, q_2, l) belongs to the tropical surfaces defined by $f \oplus w_3 \odot g$ and $\frac{\partial(f \oplus w_3 \odot g)}{\partial(w_3 + \beta)}$ for all constants β. Hence, it remains to check the Euler derivatives when L' for which $L \neq 0$.

   Let l_1 be the maximum of $\frac{\partial f}{\partial L}(q) - \frac{\partial g}{\partial(L+c)}(q)$ for all $L' = L + w_3 c$ of type 1. If there are no L' of type 1 set $l_1 = -\infty$. Let l_2 be the minimum of $\frac{\partial f}{\partial L}(q) - \frac{\partial g}{\partial(L+c)}(q)$ for all $L' = L + w_3 c$ of type 2. If there are no L' of type 2, set $l_2 = \infty$. Finally, let $L_1', \ldots, L_r'$ be the integral affine functions of type 4 whose supports represent all possible supports with this type, and write $L_i' = L_i + c_i$. Call $l_4(L_i') = \frac{\partial f}{\partial L_i}(q) - \frac{\partial g}{\partial(L_i+c_i)}(q)$. Then, it is clear that there exists an l such that (q_1, q_2, l) belongs to all partial derivatives if and only if

$$l_1 \leq l_4(L_1') = l_4(L_2') = \ldots = l_4(L_r') \leq l_2$$

We have to translate these conditions into Jacobian like equations.

   Let $L_1' = L_1 + w_3 c_1$, $L_2' = L_2 + w_3 c_2$ be two integral affine linear functions. If L_1 and L_2 are of the same type 1 or 2, or if one of the types is 3, then it always happens that q is a point in the curve $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)}(q) \oplus \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}(q)$, because we are adding two polynomials for which q is a "zero". If L_1' is of type 1 and L_2' is of type 2 then: $\frac{\partial f}{\partial L_1}$ and $\frac{\partial g}{\partial(L_2+c_2)}$ attain its minimum at least twice in q, and $\frac{\partial f}{\partial L_2}$ and $\frac{\partial g}{\partial(L_1+c_1)}$ attains its minimum once in q. Hence q is a point in $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)} \oplus \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}$ if and only if $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)}(q) \leq \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}(q)$.

This is equivalent to $\frac{\partial f}{\partial L_1}(q) + \frac{\partial g}{\partial(L_2+c_2)}(q) \leq \frac{\partial f}{\partial L_2}(q) + \frac{\partial g}{\partial(L_1+c_1)}(q)$. So, $\frac{\partial f}{\partial L_1}(q) - \frac{\partial g}{\partial(L_1+c_1)}(q) \leq \frac{\partial f}{\partial L_2}(q) - \frac{\partial g}{\partial(L_2+c_2)}(q)$. This happens for every pair of integral affine functions L_1 of type 1 and L_2 of type 2 if and only if $l_1 \leq l_2$. Let L_1' be of type 1 and L_2' of type 4. Then q is a point in $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)} \oplus \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}$ for all L_1' of type 1 if and only if $l_1 \leq l_4(L_2')$. Let L_1' be of type 2 and L_2' of type 4. Then q is a point in $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)} \oplus \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}$ for all L_1' of type 2 if and only if $l_4(L_2') \leq l_2$. Finally, if both equations are of type 4, then the condition $l_4(L_1') = l_4(L_2')$ translates into: $\frac{\partial f}{\partial L_1}(q) - \frac{\partial g}{\partial(L_1+c_1)}(q) = \frac{\partial f}{\partial L_2}(q) - \frac{\partial g}{\partial(L_2+c_2)}(q)$. We deduce that $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)}(q) = \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}(q)$, which ends the proof. $\square$

Note that in the last equality of the above proof, there is only one monomial on each side of the equality where the minimum is attained. If this monomial happens to be the same on both sides, we cannot ensure that q is on the variety $\frac{\partial f}{\partial L_1} \odot \frac{\partial g}{\partial(L_2+c_2)} \oplus \frac{\partial f}{\partial L_2} \odot \frac{\partial g}{\partial(L_1+c_1)}$. Both items in the statement of Theorem 5.10 are similar but the first one concerns tropical varieties, while the second one needs to deal with equalities of the values taken by two tropical polynomials, that do not represent tropical varieties. A more homogeneous approach can be given if we use the supertropical algebra introduced by Izhakian and then developed by Izhakian and Rowen, to unify both conditions.

Recall that one can consider an *extended tropical semiring* $(\mathbb{T}', \oplus', \odot')$ constructed from our tropical ring $(\mathbb{T}, \oplus, \odot)$ [8]. This semiring structure has a partial idempotent addition that distinguishes between sums of similar elements and sums of different elements. Set theoretically, this bigger semiring $\mathbb{T}'$ is composed from the disjoint union of two copies of $\mathbb{R}$, denoted $\mathbb{R}$ and $\mathbb{R}^\nu$, plus the neutral element for the sum ∞. There is a natural bijection $\nu : \mathbb{R} \to \mathbb{R}^\nu$ and the operation $\oplus'$ verifies $a \oplus' a = \nu(a)$, for all $a \in \mathbb{R}$. The elements in $\mathbb{R}^\nu$ are called *ghosts*, so $a \oplus' a$ is a ghost element. This terminology reflects the idea that in a field $\mathbb{K}$ with a valuation, if two elements have the same valuation, we cannot predict in general the valuation of their sum. We refer to [8, 9] for further details, in particular for the full definition of the operations in this supertropical algebra.

An element $q \in \mathbb{T}'^d$ lies in the variety $\mathcal{T}'(h)$ defined by a supertropical polynomial $h = \bigoplus'_{i \in A} p_i \odot' w^i$ with a finite support set $A \in \mathbb{Z}^d$, when $h(q)$ is a ghost element. Given A_1, A_2 as before and two supertropical polynomials f, g with respective supports in A_1, A_2 and coefficients in $\mathbb{R}$ (the so called tangible elements of $\mathbb{T}'$), we can mimic our previous definitions. Thus, a point $q \in \mathbb{R}^2$ is said to be a non-transversal intersection of f and g if there exists $l \in \mathbb{R}$ such that $(\frac{\partial(f \oplus' w_3 \odot' g)}{\partial L'})(q_1, q_2, l)$ is a ghost element for any integer affine linear form L' in 3 variables.

We can translate Theorem 5.10 in the following terms:

Theorem 5.11. *Let f, g be two supertropical bivariate polynomials with respective supports A_1, A_2 and let $q \in \mathcal{T}'(f) \cap \mathcal{T}'(g)$. Then q is a non-transversal intersection point if and only if for all 4-tuples (L_1, L_2, c_1, c_2), it holds that q lies in the variety defined by the supertropical polynomial $\frac{\partial f}{\partial L_1} \odot' \frac{\partial g}{\partial(L_2+c_2)} \oplus' \frac{\partial f}{\partial L_2} \odot' \frac{\partial g}{\partial(L_1+c_1)}$; that is, the value of this polynomial at q is a ghost.*

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