Exact synchronization bound for coupled time-delay systems

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We obtain an exact bound for synchronization in coupled time-delay systems using the generalized Halanay inequality for the general case of time-dependent delay, coupling, and coefficients. Furthermore, we show that the same analysis is applicable to both uni- and bidirectionally coupled time-delay systems with an appropriate evolution equation for their synchronization manifold, which can also be defined for different types of synchronization. The exact synchronization bound assures an exponential stabilization of the synchronization manifold which is crucial for applications. The analytical synchronization bound is independent of the nature of the modulation and can be applied to any time-delay system satisfying a Lipschitz condition. The analytical results are corroborated numerically using the Ikeda system.

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Synchronization is a fundamental nonlinear phenomenon observed in diverse natural, engineering, and social systems [1,2]. Investigations on synchronization in coupled nonlinear time-delay systems have received an immense amount of attention in current research due to their dynamical complexity [3–14] and their ease of experimental realization [15–18]. Synchronization in systems with delay has potential applications in enhancement of output power in lasers, generating high-precision random numbers, high-speed information transfer due to broad bandwidth, optimization of nonlinear systems parameters, pattern recognition, (cf. [3–18]), among many others. Furthermore, time delay gives rise to a plethora of novel collective dynamical behaviors and explains mechanisms behind several natural phenomena. Examples include the key role of coupling and delay in resting brain fluctuations, a temporal coherent state [19], zero time lag neuronal synchrony despite long conduction delays [20], the coexistence of coherent, and incoherent states with nonlocal delayed coupling [21].

Analytical frameworks have been employed in deducing conditions for stable synchronization in coupled time-delay systems. In particular, the Krasovskii-Lyapunov functional has been widely used to for this [3–12]. Recently, the Krasovskii-Lyapunov theory is applied to time-delay systems with time-dependent coefficients [9]. However, all these works require the derivative of the positive definite Lyapunov functional to be negative and are applied only to unidirectionally coupled time-delay systems.

Here, we provide an exact bound for the synchronization of coupled time-delay systems with time-dependent delay, time-dependent coupling, and time-dependent coefficients using the generalized Halanay inequality [22,23]. The bound is independent of conditions on the derivative of the parameters and is less restrictive than the Lyapunov functional approach. Furthermore, this exact synchronization bound assures an exponentially stable synchronization compared to asymptotic synchronization resulting from the latter. The analytical synchronization bound is also independent of the delay, the nonlinear function, and the modulation in time-dependent parameters. A few recent investigations have used the Halanay inequality along with matrix measures to demonstrate synchronization but only in unidirectionally coupled time-delay systems; e.g., in systems with constant coefficients and fixed time delay [24] or variable but not chaotic delay [25]. But here, we will study a very general case and obtain a less restrictive (conservative) condition than others.

Interestingly, we will show that the same analysis is applicable to both uni- and bidirectionally coupled time-delay systems with an appropriate evolution equation for their synchronization manifold. Furthermore, one can include a large class of synchronization manifolds including complete synchronization (CS) and generalized synchronization (GS) under the same analysis. The synchronization manifold of GS is constructed using the framework of the auxiliary system approach [26] originally introduced for unidirectional coupling. In addition, similar to GS, we also find an exact synchronization bound for lag, anticipatory, and other subclasses of GS using the auxiliary system approach. To the best of our knowledge, stability analysis for GS in bidirectionally coupled time-delay systems has not yet been carried out. Even for CS, there may rarely exist any analytical investigations for bidirectionally coupled time-delay systems.

The paper is organized as follows: First, we consider the simplest case where only the delay is time dependent and the coupling and the coefficients are constant to infer an exact synchronization bound for both uni- and bidirectionally coupled time-delay systems. Next, we extend our analyses for both the delay and the coupling parameters depending on time. Finally, we yield an exact synchronization bound for the more general case by including time-dependent coefficients using the generalized Halanay inequality. We corroborate the analytical synchronization bounds for all three cases by numerical analysis using the well-known Ikeda system.

We study the following coupled scalar time-delay system:

$$\dot{x}(t) = -a(t)x(t) + b(t)f(x(t-\tau(t))) + k_x(t)(y(t) - x(t)), \quad (1a)$$

$$\dot{y}(t) = -a(t)y(t) + b(t)f(y(t-\tau(t))) + k_y(t)(x(t) - y(t)), \quad (1b)$$

where $a(t)$ and $b(t)$ are coefficients, $\tau(t)$ is the delay time, and $k_{x,y}(t)$ are couplings between the drive and the response systems, and all are time dependent. $f(x)$ is a nonlinear function, which is assumed to be Lipschitz; that is, $|f(y) - f(x)| \leqslant L|y - x|$ for some $L > 0$. Here, $a(t)$, $b(t)$, $\tau(t)$, and $k(t)$ are all continuous, such that $a(t) + k(t) > 0$, $b(t) > 0$, and $0 \leqslant \tau(t) \leqslant \tau_M$. The coupling is unidirectional when $k_x(t) = 0$ and $k(t) = k_y(t)$ but otherwise bidirectional [$k(t) = k_x(t) = k_y(t)$].

The time evolution of the state variable, $\Delta(t) = y(t) - x(t)$, corresponding to the CS manifold of unidirectionally coupled time-delay systems, i.e., the synchronization error, satisfies the equation

$$\frac{d\Delta^2}{dt} = -2[a(t) + k(t)]\Delta^2(t) + 2b(t)\Delta(t) \times [f(y(t-\tau(t))) - f(x(t-\tau(t)))], \quad (2)$$

and that of bidirectionally coupled time-delay systems satisfies

$$\frac{d\Delta^2}{dt} = -2[a(t) + 2k(t)]\Delta^2(t) + 2b(t)\Delta(t) \times [f(y(t-\tau(t))) - f(x(t-\tau(t)))]. \quad (3)$$

Using the Lipschitz condition, we get

$$\frac{d\Delta^2}{dt} \leqslant -2[a(t) + k(t)]\Delta^2(t) + 2b(t)L|\Delta(t)\Delta(t-\tau(t))|, \quad (4)$$

$$\leqslant -2[a(t) + k(t)]\Delta^2(t) + 2b(t)LM^2(t), \quad (5)$$

where $M(t) = \sup\{|\Delta(s)|; -\tau_M + t \leqslant s \leqslant t\}$ for unidirectional coupling and $2k(t)$ replaces $k(t)$ for bidirectional coupling in the entire manuscript.

For GS, an auxiliary system [26] $y'(t)$ identical to the response system $y(t)$ is appended to Eq. (1) and the evolution equation of the CS manifold $\Delta(t) = y'(t) - y(t)$ between $y'(t)$ and $y(t)$, identical to Eq. (3), characterizes GS between the drive and response systems. For bidirectional coupling, an auxiliary system $x'(t)$ identical to system $x(t)$ is also analyzed in addition to $y'(t)$. Now, the CS manifolds $\Delta_y(t) = y'(t) - y(t)$ and $\Delta_x(t) = x'(t) - x(t)$, identical to Eq. (3), characterizes GS between bidirectionally coupled systems. The same analysis can also be applied to lag, anticipatory, projective synchronizations and their variants including all subclasses of GS. The following results hold when the evolution equation for $\Delta(t)$ is given by Eq. (3) irrespective of the synchronization manifold:

An exact bound can be obtained for the synchronization error $|\Delta(t)|$ by using the generalized Halanay inequality [22,23]. This inequality is proven for a function $\omega(t)$ that satisfies the following equation:

$$\frac{d\omega}{dt} \leqslant -p(t)\omega(t) + q(t)\Omega(t), \quad (6)$$

where $\Omega(t) = \sup\{\omega(s); -\tau_M + t \leqslant s \leqslant t\}$ for $t \geqslant t_0 = 0$ with $\tau_M \geqslant 0$. The equation for $\Delta(t)^2$ given by Eq. (5) is a particular case of this equation. Here $p(t)$ and $q(t)$ are continuous with $p(t) \geqslant p_0 > 0$, and $\varepsilon p(t) \geqslant q(t) > 0 \, \forall \, t \geqslant t_0$ with $0 \leqslant \varepsilon < 1$. Note that ε can be chosen very close to 1. Then the following generalized Halanay inequality is derived [23]:

$$\omega(t) \leqslant \Omega(t_0)\exp[-\mu_m(t - t_0)] \quad \text{for} \quad t \geqslant t_0, \quad (7)$$

where $\mu_m > 0$ is given by

$$\mu_m = \inf\{\mu(t) : p(t) - \mu(t) - q(t)\exp[\mu(t)\tau_M] = 0;\ t \geqslant t_0\}. \quad (8)$$

First, we study the case with time-dependent delay but constant coefficients, $a(t) = a$ and $b(t) = b$, and constant coupling $k(t) = k$ (and $2k$ for bidirectional coupling). Using Eq. (7), one gets the exact bound for the synchronization error $|\Delta(t)|$ as

$$|\Delta(t)| \leqslant M(t_0)\exp(-\lambda t), \quad (9)$$

when $\varepsilon(a + k) > bL > 0$, $0 \leqslant \varepsilon < 1$. This condition is satisfied when $(a + k) > bL > 0$, by choosing ε such that $1 > \varepsilon > bL/(a + k)$. The exponent λ comes from

$$(a + k) - \lambda - bL\exp(2\lambda\tau_M) = 0, \quad (10)$$

whose solution satisfies $0 < \lambda < (a + k - bL)$.

Now, we check Eq. (9) by using numerical simulations. To this end, we consider the nonlinear system (1) with $f(x) = \cos^2(x + \phi)$ corresponding to the Ikeda system [27], where $\phi = 0.1$ is the phase shift. A chaotic modulation in the delay time is used, given by the x_r component of the chaotic Rössler system represented by $\dot{x_r} = -y_r - z_r$, $\dot{y_r} = x_r + 0.17y_r$, $\dot{z_r} = 0.2 + z_r(x_r - 10.0)$. The chaotic delay time is $\tau(t) = \tau_c + hx_r(t)$, where we shift the minimum of $x_r(t)$ to zero such that the modulation is always positive and $h = 0.1$ is the scaling factor. We have fixed $\tau_c = 2.5$, hence $\tau_M = 5.808$, and $L = 1$ throughout the paper. We have chosen $a = 1$, $b = 6$ and the coupling strength $k = 6$ satisfying $1 > \varepsilon > bL/(a + k)$. The synchronization bound obtained from Eq. (9) is depicted in Figs. 1(a) and 1(b) for uni- and bidirectional couplings, respectively, along with the numerical synchronization error for three different initial conditions. The uncoupled as well as synchronized systems exhibit chaotic oscillations for the above parameters. It is clear from this figure that the synchronization error estimated numerically from the coupled Ikeda system (1) for the above parameter values always lies well below the analytical bound for the synchronization error given by Eq. (9). The insets display the exponential decay of the synchronization error, confirming exponential synchronization.

Next, we extend our analysis to time-dependent coupling and time-dependent delay keeping the coefficients as constant. Using Eq. (7), one obtains the same synchronization bound as in Eq. (9) but with the condition $\varepsilon(a + k(t)) \geqslant bL > 0$, where $0 \leqslant \varepsilon < 1$. Now, the exponent λ in the bound can be estimated from

$$\lambda = \inf\{\lambda(t) : a + k(t) - \lambda(t) - bL\exp[2\lambda(t)\tau_M] = 0;\ t \geqslant t_0\}. \quad (11)$$

It is important to emphasize that previous results for synchronization conditions [9–11] are based on the linearized equation

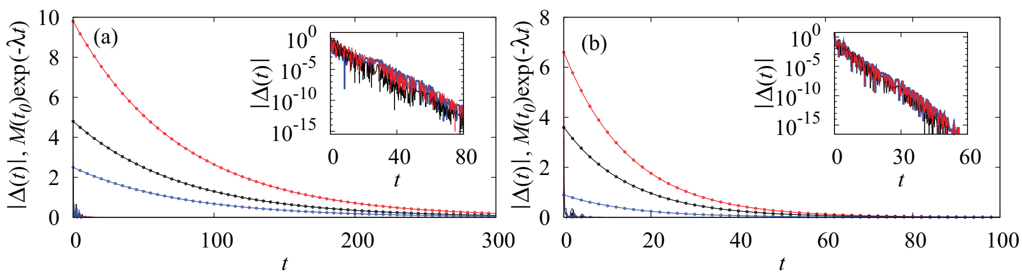

FIG. 1. (Color online) Simulations with time-dependent delay but constant coefficients and coupling strength. The synchronization bound $M(t_0)\exp(-\lambda t)$ (lines connected with solid circles) and its synchronization error $|\Delta(t)|$ for three different initial conditions with the inset illustrating exponential decay of $|\Delta(t)|$ in a semilog plot. Panel (a) is for unidirectional coupling and panel (b) is for bidirectional coupling.

for $\Delta(t)$. These approaches require the derivative of Lyapunov functionals on the evolution equation corresponding to the synchronization manifold to be negative. In particular, a sufficient condition for synchronization $a + k(t) > |bf'(y(t-\tau))|$ was obtained using the Krasovskii-Lyapunov theory for the case of time-dependent coupling and constant delay in Ref. [9], but only when the derivative of the coupling function $\dot{k}(t) \leqslant 0$. For the Ikeda system this condition is satisfied when $a + k(t) > b$, i.e., the condition obtained by using the generalized Halanay inequality, which is independent of the sign of the derivative of the coupling. Another condition $\int_{t_0}^{\infty} -[a + k(t) - b]dt = -\infty$ is yielded in Ref. [10] for the same case. A bound for the synchronization error was also obtained [11] for the case of constant coupling and time-dependent delay, but only when the additional condition $\dot{\tau}(t) < 1$ is satisfied. Thus the synchronization bound obtained using Eq. (7) is less restrictive than the Krasovskii-Lyapunov theory and further assures an exponential synchronization compared to asymptotic synchronization obtain from the latter.

We have fixed the same parameters and the x_r component of the chaotic Rössler system for modulation in both delay time and coupling $k(t) = k_c + hx_r(t)$ as discussed above with $k_c = 6.5$. The analytical synchronization bound and the numerical synchronization error of the coupled Ikeda systems (1) for three different initial conditions are shown in Figs. 2(a) and 2(b) for uni- and bidirectional couplings [now $k(t) = 2k_c + hx_r(t)$], respectively. We find that the numerical synchronization error always lies below the synchronization bound and the insets confirm the exponential synchronization of the synchronization error $\Delta(t)$.

Finally, we consider the more general case where all the coefficients, coupling, and delay are time dependent. Again one gets the synchronization bound as in Eq. (9) with the condition on the parameters as $\varepsilon(a(t) + k(t)) \geqslant b(t) > 0$, where $0 \leqslant \varepsilon < 1$, using Eq. (7). The exponent in the synchronization bound is obtained from

$$\lambda = \inf\{\lambda(t) : a(t) + k(t) - \lambda(t) - b(t)L\exp[2\lambda(t)\tau_M] = 0\}. \tag{12}$$

It is to be noted that we have obtained an exact bound for a much more general case than investigated in Refs. [9,10] by including a time-dependent delay and without any approximation [9,10]. Again, we have fixed the x_r component of the chaotic Rössler system as modulations in $a(t) = a_c + hx_r(t)$ and $b(t) = b_c + hx_r(t)$ with $a_c = 1.0$ and $b_c = 6.0$, while the modulations in τ and coupling strength are retained as discussed above. The synchronization bound and the synchronization error for three different initial condition is depicted in Figs. 3(a) and 3(b), for uni- and bidirectionally [$k(t) = 2k_c + hx_r(t)$] coupled time-delay systems, respectively. It is evident that the synchronization error always lies below the analytical bound (9) and the exponential decay of the synchronization error in the insets corroborates an exponential synchronization of the coupled time-delay systems.

To summarize, the exact bound for the synchronization in a coupled time-delay system with time-dependent parameters is obtained using the generalized Halanay inequality. We have also shown that the same analysis is applicable to both uni- and bidirectionally coupled time-delay systems and also covers different types of synchronization. First, we have found an exact bound for the complete synchronization manifold for time-dependent delay. Next, we have incorporated time-dependent coupling and finally time-dependent coefficients of the state variables and obtained an exact

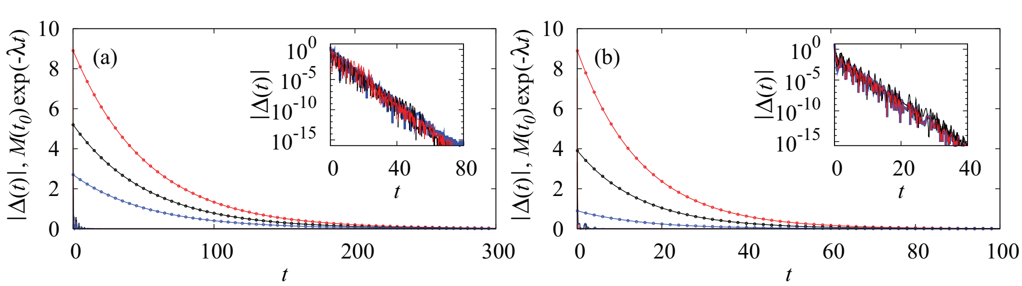

FIG. 2. (Color online) Same as Fig. 1, except for simulations with time-dependent delay and time-dependent coupling strength but with constant coefficients.

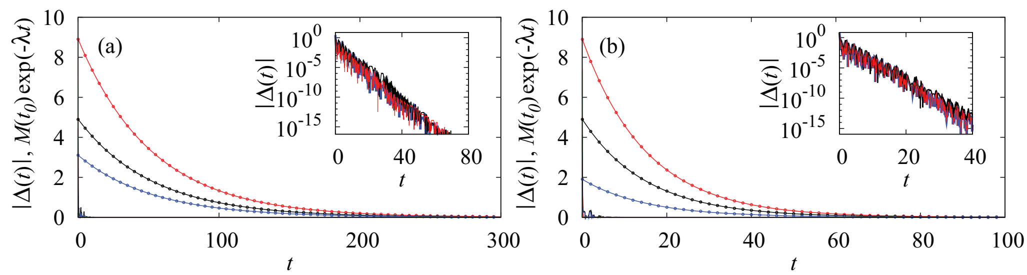

FIG. 3. (Color online) Same as Fig. 1 for time-dependent delay, coefficients, and coupling strength.

bound for the synchronization error using the generalized Halanay inequality. The analytical synchronization bound is independent of the conditions on the derivatives of parameters and without any approximation as in the Lyapunov functional approach. Furthermore, it is also independent of the type of modulation in the time-dependent parameters and the results are general for time-delay systems satisfying the Lipschitz condition and are not system specific. We have corroborated the analytical synchronization bound numerically by using the Ikeda system. It is found that the numerical synchronization error always lies well below the analytical synchronization bound. Furthermore, the synchronization bound leads to exponential synchronization of the coupled time-delay systems with time-dependent parameters, which is crucial for many applications.

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