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Research paper

Traveling gravity-capillary waves with odd viscosity

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ABSTRACT

In this note, we study the existence of traveling waves of a surface model in a non-newtonian fluid with odd viscosity. The proof relies on nonlinear bifurcation techniques.

1. Introduction and main result

The motion of an incompressible fluid is described by the following non-linear system

$$\begin{aligned} \rho\left(u_t + (u\cdot\nabla)u\right) &= \nabla\cdot\mathcal{T}, \\ \rho_t + \nabla\cdot(u\rho) &= 0, \\ \nabla\cdot u &= 0, w \end{aligned} \tag{1}$$

where $u(t,x) : [0,T]\times\Omega\to\mathbb{R}^d$, $\rho(t,x) : [0,T]\times\Omega\to\mathbb{R}$ represent the velocity field and density of the fluid respectively and $d = 2, 3$. Moreover, $\mathcal{T}$ denotes the stress tensor that varies depending on the different properties of the fluid. For inviscid and newtonian fluids, the stress tensor is given by

$$\mathcal{T}^i_j = -p\delta^i_j,$$

and system (1) reduces to the well-known Euler equations. For viscous newtonian fluids, the stress tensor has even symmetry and takes the form

$$\mathcal{T}^i_j = -p\delta^i_j + \left(\partial_{x_j}u^i + \partial_{x_i}u^j\right).$$

However, in stark contrast to the classical scenario for viscous or inviscid newtonian fluids, there are classes of fluids with broken microscopic time-reversal symmetry and parity, namely quantum fluids (magnetized plasmas or electron fluids) or classical fluid systems (polyatomic gases). We refer the interested reader to [1] for a more detailed explanation and discussion. In two-dimensional fluid systems where microscopic time reversal and parity are violated, the viscosity tensor includes a skew-symmetric component often referred to as odd viscosity given by

$$\mathcal{T}^i_j = -p\delta^i_j + \left(\partial_{x_i}\left(u^j\right)^{\perp} + \left(\partial_{x_i}\right)^{\perp}u^j\right). \tag{2}$$

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In two spatial dimensions, given a vector $v = (v_1(x_1,x_2), v_2(x_1,x_2))$ or the differential operator $\partial_{x_i} = (\partial_{x_1}, \partial_{x_2})$, we denote by $v^\perp = (-v_2, v_1)$ and $(\partial_{x_i})^\perp = (-\partial_{x_2}, \partial_{x_1})$ the counterclockwise rotation of v or ∂_{x_i} by an angle of $\pi/2$. Although in three dimensions, terms in the viscosity tensor with odd symmetry were known in the context of anisotropic fluids [2], Avron noticed that in two dimensions, odd viscosity and isotropy can hold at the same time, [3]. Despite the recently increasing interest of the mathematical and physical community in fluids with odd viscosity effects, there are not so many mathematical works considering this setting.

Recently, in [4], the authors establish a well-posedness theory in Sobolev spaces for a system of incompressible non-homogeneous fluids with odd viscosity given by (2). A well-posedness theory in Besov spaces was later proved in [5]. Remarkably, in this last paper the authors manage to prove the solution is asymptotically global in the sense that the lifespan grows as the density tends to homogeneity.

Recently, in [6], the authors obtained three new models for capillary–gravity surface waves with odd viscosity through a multi-scale expansion in the steepness of the wave. The multi-scale expansion approach (cf. [7,8]) reduces the full system to a cascade of linear equations which can be closed up to some order of precision. The derived models in [6] consider effects of both gravity and surface tension forces generalizing those in [9,10]. One of the asymptotic models studies the unidirectional surface waves, given by the dispersive equation

$$2f_t + \alpha_0 \Lambda[f_t] = \frac{1}{\varepsilon}\left\{ f_x + \mathcal{H}[f] + (\alpha_0 - \beta)\mathcal{H}[f_{xx}] \right\} + \mathcal{H}[(\Lambda f)^2] - [\![\mathcal{H}, f]\!][\Lambda f] + (\alpha_0 - \beta)[\![\mathcal{H}, f]\!]\Lambda^3 f, \tag{3}$$

where $f : [0,T] \times \mathbb{T} \to \mathbb{R}$. Here ε is known as the steepness parameter and measures the ratio between the amplitude and the wavelength of the wave, α_0 is linked to the Reynolds number and represents the ratio between gravity and odd viscosity forces. In this paper we assume this parameter to be strictly positive. Finally, β is the Bond number comparing the gravity and capillary forces. Notice that (3) conserves the total mass of water for periodic domains and for waves that decay fast enough at infinity. Besides the derivation of the model (3) the authors in [6] showed the problem is locally well-posed in $H^3(\mathbb{R})$ for $\alpha_0 > 0$, without any assumption on β. Furthermore, for $0 < \alpha_0 = \beta$, the problem admits a distributional solution in $H^{1.5}(\mathbb{R})$.

The main result provided is this manuscript shows the existence of traveling waves for Eq. (3) and reads as follows:

Theorem 1.1. *Let $0 < \alpha_0 \neq \beta$ and $m \geq 1$. Then there exists an open interval $I \subset \mathbb{R}$ containing 0, and a one-dimensional curve $s \mapsto (c_s, \varphi_s)$ with $s \in I$, such that*

$$f_0(x) = \varphi_s(x)$$

is an m-fold traveling wave solution to (3) with constant speed c_s.

Remark 1.2. For $0 < \alpha_0 = \beta$, one can mimic the proof of Theorem 1.1 and recover the same result. Notice that for $0 < \alpha_0 = \beta$, the singular commutator term $[\![\mathcal{H}, f]\!]\Lambda^3 f$ is not present, and there is no need to invoke the commutator Lemma 2.1 to show the analogue of Proposition 3.1.

The literature regarding the study of permanent progressive waves, as solitary and traveling waves, is a key area of interest. These waves, also known as steady waves, propagate without changing their shape over time. Given the significant complexity of the classical water wave problem, numerous approximate models have been studied since the early years. These models are formally derived through various scaling limits. Perhaps the canonical example is the so called Korteweg–de Vries (KdV) equation

$$u_t + 3(u^2)_x + u_{xxx} = 0,$$

to model propagation of surface water waves with small amplitudes and long wavelengths in a channel, [11,12]. The KdV equation includes the essential effects of nonlinearity and dispersion. The mathematical theory for the KdV equation is well-known, featuring a theory of well-posedness and a thorough understanding of the stability properties of solitary and traveling waves, [13–16]. Similarly, there have been other successful models where the existence of traveling waves have been extensively studied such as the Fornberg–Whitam equation [17]

$$u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}u u_{xxx} - \frac{3}{2}u u_x + u_x = 0,$$

or the Camassa–Holm equation [18,19]

$$u_t - u_{txx} + 3u u_x + 2u_x = 2u_x u_{xx} + u u_{xxx}.$$

For the former, traveling wave solutions of kink-like and antikink-like type were recently investigated in [20] and the references therein. The latter, has been deeply analyze and all types of traveling waves solutions are classified such as peakons, cuspons, stumpons, and composite waves, cf. [21–23].

To the best of the author's knowledge, Theorem 1.1 seems to be the first rigorous result regarding the existence of traveling waves solutions for fluids with odd viscosity effects.

Plan of the paper

In Section 2, we present the notation used throughout the article as well as some auxiliary results. In particular, we provide a commutator estimate for the Hilbert transform in Hölder spaces and recall basic tools in bifurcation theory. In Section 3 we introduce the formulation of the problem as well as the function spaces that will be used in order to implement the Crandall–Rabinowitz theorem. In Section 3.1 we study the spectral properties of the linearized operator and check such linear operator is a Fredholm operator of zero index. Finally, we also study the kernel and the range to verify the transversality condition. In Section 3.2, gathering the different results provided previously, we invoke the Crandall–Rabinowitz theorem to show the proof of Theorem 1.1.

2. Notation and auxiliary results

For a function $f \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with values in $\mathbb{R}$, we define the Hölder norms as

$$\|f\|_{C^0(\mathbb{T})} = \sup_{x\in\mathbb{T}}|f|,\ \|f\|_{C^k(\mathbb{T})} = \|f\|_{C^0(\mathbb{T})} + \sum_{\ell=1}^{k} \left\|\partial_x^\ell f\right\|_{C^0(\mathbb{T})},\ k\in\mathbb{N},$$

$$\|f\|_{C^\alpha(\mathbb{T})} = \|f\|_{C^0(\mathbb{T})} + \sup_{x_1,x_2\in\mathbb{T},x_1\neq x_2} \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\alpha},\ 0<\alpha<1,$$

$$\|f\|_{C^{k,\alpha}(\mathbb{T})} = \|f\|_{C^{k-1}(\mathbb{T})} + \left\|\partial_x^k f\right\|_{C^\alpha(\mathbb{T})},\ k\in\mathbb{N},\ 0<\alpha<1.$$

The Banach space of continuous functions for which the above norms are finite will be denoted $C^k(\mathbb{T};\mathbb{R})$ and $C^{k,\gamma}(\mathbb{T};\mathbb{R})$. The linear operator $\mathcal{H}$ refers to the Hilbert transform in the periodic setting and is given by

$$\mathcal{H}[f](x) = \frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} \frac{f(y)}{\tan\left(\frac{x-y}{2}\right)} dy, \tag{4}$$

and $\Lambda = \mathcal{H}\partial_x$ the Zygmund operator is defined as

$$\Lambda[f](x) = \frac{1}{4\pi} p.v. \int_{-\pi}^{\pi} \frac{f(x)-f(x-y)}{\sin^2(\frac{y}{2})} dy. \tag{5}$$

Moreover, given an operator T, we define the commutator as $[\![\mathsf{T}, f]\!][g] = \mathsf{T}(fg) - f\mathsf{T}(g)$.

We will denote with C a positive generic constant that depends only on fixed parameters. Note also that this constant might differ from line to line.

Next, we show a commutator estimate for the Hilbert transform $\mathcal{H}$ in Hölder spaces. Similar commutator estimates to the one provided in this article can be found in [24, Lemma B.1] in the context of water waves or [25, Lemma 2.2] for the interface Stokes flow problem. However, to the best of the authors knowledge, the estimate provided here does not follow from the previous results.

Lemma 2.1. *Let $\alpha \in (0,1)$, $a \in C^{2,\alpha}(\mathbb{T})$, $b \in C^{1,\alpha}(\mathbb{T})$. Then, we have that*

$$\left\|[\![\mathcal{H}, a]\!][b']\right\|_{C^{1,\alpha}(\mathbb{T})} \le C\, \|a\|_{C^{2,\alpha}(\mathbb{T})} \|b\|_{C^{1,\alpha}(\mathbb{T})}. \tag{6}$$

Proof. In order to ease the notation, we denote by $\Theta(x) := [\![\mathcal{H}, a]\!][b'](x)$. Using the definition of $\mathcal{H}$ in (4) we readily check that

$$\Theta(x) = \frac{1}{2\pi}\text{p.v.} \int_{-\pi}^{\pi} \frac{(a(y)-a(x))}{\tan\left(\frac{x-y}{2}\right)} b'(y) dy = -\frac{1}{2\pi}\text{p.v.} \int_{-\pi}^{\pi} \frac{(a(y)-a(x))}{\tan\left(\frac{x-y}{2}\right)} (b(x)-b(y))'\, dy.$$

Using integration by parts, we have that

$$\Theta(x) = \frac{1}{2\pi}\text{p.v.} \int_{-\pi}^{\pi} \left(\frac{a'(y)}{\tan(\frac{x-y}{2})} + \frac{(a(y)-a(x))}{2\sin^2(\frac{x-y}{2})} \right) (b(x)-b(y))\, dy.$$

Using the change of variable $\tilde{y} = x-y$ and further manipulation yields

$$\Theta(x) = \frac{1}{4\pi}\text{p.v.} \int_{-\pi}^{\pi} \left(\frac{2a'(x-\tilde{y})\sin(\frac{\tilde{y}}{2})\cos(\frac{\tilde{y}}{2}) - (a(x)-a(x-\tilde{y}))}{\sin^2(\frac{\tilde{y}}{2})} \right) (b(x)-b(x-\tilde{y}))\, d\tilde{y}.$$

From now on, we will just write y instead of $\tilde{y}$. Taking into account that

$$\frac{1}{\sin^2(y)} \le g(|y|)\frac{1}{y^2}, \quad 0<|y|<\pi, \quad |\sin(y)-y| \le \frac{1}{6}|y|^3, \quad y\in\mathbb{R}, \tag{7}$$

where $g(|y|) : (0,\pi) \to [0,\infty)$ is a bounded function, we find that

$$|\Theta(x)| \le C\text{p.v.} \int_{-\pi}^{\pi} \left(\frac{\|a\|_{C^1(\mathbb{T})}|y| + |(a(x)-a(x-y))|}{y^2} \right) |(b(x)-b(x-y))|g(|y|)dy + \text{l.o.t}$$

Therefore, since $g(|y|)$ is a bounded function and

$$|a(x)-a(x-y)| \le |y|\, \|a\|_{C^1(\mathbb{T})}, \quad |b(x)-b(x-y)| \le |y|^\alpha \|b\|_{C^\alpha(\mathbb{T})}, \tag{8}$$

we have that

$$\|\Theta\|_{C^0(\mathbb{T})} \le C\,\|a\|_{C^1(\mathbb{T})}\,\|b\|_{C^\alpha(\mathbb{T})}\,\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{|y|^{1-\alpha}}g(|y|)dy \le C\,\|a\|_{C^1(\mathbb{T})}\,\|b\|_{C^\alpha(\mathbb{T})}\,. \tag{9}$$

To compute the higher order norm, we first notice that

$$\Theta'(x) = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\left(\frac{2a''(x-\tilde y)\sin(\frac{\tilde y}{2})\cos(\frac{\tilde y}{2}) - \left(a'(x)-a'(x-\tilde y)\right)}{\sin^2(\frac{\tilde y}{2})}\right)(b(x)-b(x-\tilde y))\,d\tilde y$$
$$+\frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\left(\frac{2a'(x-\tilde y)\sin(\frac{\tilde y}{2})\cos(\frac{\tilde y}{2}) - (a(x)-a(x-\tilde y))}{\sin^2(\frac{\tilde y}{2})}\right)\left(b'(x)-b'(x-\tilde y)\right)d\tilde y.$$

To compute the C^α norm for $\Theta'(x)$, it is convenient to introduce the difference notation

$$\Delta_y f(x+h) := f(x+h) - f(x+h-y), \quad \Delta_y f(x) := f(x) - f(x-y).$$

Hence, calculating the Hölder difference for $h > 0$ yields

$$\begin{aligned}\Theta'(x+h)-\Theta'(x) = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\Big[&\Delta_y b(x+h)\left(2a''(x+h-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a'(x+h)\right)\\ &- \Delta_y b(x)\left(2a''(x-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a'(x)\right)\Big]dy\\ + \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\Big[&\Delta_y b'(x+h)\left(2a'(x+h-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a(x+h)\right)\\ &- \Delta_y b'(x)\left(2a'(x-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a(x)\right)\Big]dy = \mathrm{I}_1 + \mathrm{I}_2.\end{aligned} \tag{10}$$

Adding and subtracting $\Delta_y b(x+h)\left(2a''(x-y)\sin(\frac{y}{2})\cos(\frac{y}{2}) - \Delta_y a'(x)\right)$ in I_1 we find that $\mathrm{I}_1 = \mathrm{I}_{11} + \mathrm{I}_{12}$ where

$$\begin{aligned}\mathrm{I}_{11} = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\Delta_y b(x+h)\Big[&\left(2a''(x+h-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a'(x+h)\right)\\ &- \left(2a''(x-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a'(x)\right)\Big]dy,\end{aligned}$$
$$\mathrm{I}_{12} = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\left[2a''(x-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a'(x)\right]\left(\Delta_y b(x+h) - \Delta_y b(x)\right)dy.$$

Similarly, adding and subtracting $\Delta_y b'(x+h)\left(2a'(x-y)\sin(\frac{y}{2})\cos(\frac{y}{2}) - \Delta_y a(x)\right)$ in I_2 we find that $\mathrm{I}_1 = \mathrm{I}_{21} + \mathrm{I}_{22}$ where

$$\begin{aligned}\mathrm{I}_{21} = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\Delta_y b'(x+h)\Big[&\left(2a'(x+h-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a(x+h)\right)\\ &- \left(2a'(x-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a(x)\right)\Big]dy,\end{aligned}$$
$$\mathrm{I}_{22} = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\left[2a'(x-y)\sin(\tfrac{y}{2})\cos(\tfrac{y}{2}) - \Delta_y a(x)\right]\left(\Delta_y b'(x+h) - \Delta_y b'(x)\right)dy.$$

Thus

$$\Theta'(x+h) - \Theta'(x) = \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{21} + \mathrm{I}_{22}.$$

We will just bound the first two integrals $\mathrm{I}_{11}, \mathrm{I}_{12}$. The remaining terms $\mathrm{I}_{21}, \mathrm{I}_{22}$ can be estimated in a similar fashion. Let us start with I_{12}. We write

$$\mathrm{I}_{12} = \frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\left[a''(x-y)y\cos(\tfrac{y}{2}) - \Delta_y a'(x)\right]\left(\Delta_y b(x+h) - \Delta_y b(x)\right)dy$$
$$-\frac{1}{4\pi}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{\sin^2(\frac{y}{2})}\left[a''(x-y)\left(\sin(\tfrac{y}{2}) - \tfrac{y}{2}\right)\cos(\tfrac{y}{2}) - \Delta_y a'(x)\right]\left(\Delta_y b(x+h) - \Delta_y b(x)\right)dy.$$

We just bound the first integral above, since using (7) the later is even easier to control and the same estimate follows. To that purpose we first notice that

$$\Delta_y a' = y\int_0^1 a''(\lambda x + (1-\lambda)(x-y))\,d\lambda,$$

and hence

$$|a''(x-y)y\cos(\tfrac{y}{2}) - \Delta_y a'(x)| \le |(1-\cos(\tfrac{y}{2}))||y|\int_0^1 |\left(a''(\lambda x + (1-\lambda)(x-y) - a''(x-y))\right)|\,d\lambda$$

$$\leq C|y|^{1+\alpha}\left\|a'\right\|_{C^{1,\alpha}(\mathbb{T})}\leq C|y|^{1+\alpha}\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}. \tag{11}$$

Thus, combining (7),(8) together with (11) we infer that

$$\mathrm{I}_{12}\leq C\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}\left\|b\right\|_{C^{\alpha}(\mathbb{T})}h^{\alpha}\text{p.v.}\int_{-\pi}^{\pi}\frac{1}{|y|^{1-\alpha}}\,dy\leq C\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}\left\|b\right\|_{C^{\alpha}(\mathbb{T})}h^{\alpha}. \tag{12}$$

Next, let us bound I_{11}. We split the integral as

$$\mathrm{I}_{11}=\frac{1}{4\pi}\text{p.v.}\int_{-2|h|}^{2|h|}[\ldots]\,dy+\frac{1}{4\pi}\text{p.v.}\int_{(-\pi,-2|h|)\cup(2|h|,\pi)}[\ldots]\,dy:=\mathrm{J}_{in}+\mathrm{J}_{out}.$$

Again we can add and subtract $\frac{y}{2}$ and write

$$\begin{aligned}\mathrm{J}_{in}=&\frac{1}{4\pi}\text{p.v.}\int_{-2|h|}^{2|h|}\frac{1}{\sin^2(\frac{y}{2})}\Delta_y b(x+h)\Big[\Big(a''(x+h-y)y\cos(\frac{y}{2})-\Delta_y a'(x+h)\Big)\\&\qquad-\Big(a''(x-y)y\cos(\frac{y}{2})-\Delta_y a'(x)\Big)\Big]dy\\&+\frac{1}{4\pi}\text{p.v.}\int_{-2|h|}^{2|h|}\frac{1}{\sin^2(\frac{y}{2})}\Delta_y b(x+h)\Big[\Big(a''(x+h-y)\Big(\sin(\frac{y}{2})-y\Big)\cos(\frac{y}{2})-\Delta_y a'(x+h)\Big)\\&\qquad-\Big(a''(x-y)\Big(\sin(\frac{y}{2})-y\Big)\cos(\frac{y}{2})-\Delta_y a'(x)\Big)\Big]dy.\end{aligned}$$

Similarly as before, using (7) the second integral above less singular easier to control. Therefore, using (7)–(8) and writing again

$$\Delta_y a'(x)=y\int_0^1 a''(\lambda x+(1-\lambda)(x-y))\,d\lambda,\quad \Delta_y a'(x+h)=y\int_0^1 a''(\lambda(x+h)+(1-\lambda)(x+h-y))\,d\lambda, \tag{13}$$

we readily check that

$$\begin{aligned}\mathrm{J}_{in}&\leq C\left\|b\right\|_{C^{\alpha}(\mathbb{T})}\left\|a\right\|_{C^{2}(\mathbb{T})}\text{p.v.}\int_{-2|h|}^{2|h|}\frac{|y|^{1+\alpha}}{y^2}\,dy\leq C\left\|b\right\|_{C^{\alpha}(\mathbb{T})}\left\|a\right\|_{C^{2}(\mathbb{T})}\text{p.v.}\int_{-2|h|}^{2|h|}\frac{1}{y^{1-\alpha}}\,dy\\&\leq C\left\|b\right\|_{C^{\alpha}(\mathbb{T})}\left\|a\right\|_{C^{2}(\mathbb{T})}h^{\alpha}.\end{aligned} \tag{14}$$

To bound the outer part J_{out}, we perform a similar estimate. First, we add and subtract $\frac{y}{2}$ and write

$$\begin{aligned}\mathrm{J}_{out}=&\frac{1}{4\pi}\int_{(-\pi,-2|h|)\cup(2|h|,\pi)}\frac{1}{\sin^2(\frac{y}{2})}\Delta_y b(x+h)\Big[\Big(a''(x+h-y)y\cos(\frac{y}{2})-\Delta_y a'(x+h)\Big)\\&\qquad-\Big(a''(x-y)y\cos(\frac{y}{2})-\Delta_y a'(x)\Big)\Big]dy\\&+\frac{1}{4\pi}\int_{(-\pi,-2|h|)\cup(2|h|,\pi)}\frac{1}{\sin^2(\frac{y}{2})}\Delta_y b(x+h)\Big[\Big(a''(x+h-y)\Big(\sin(\frac{y}{2})-y\Big)\cos(\frac{y}{2})-\Delta_y a'(x+h)\Big)\\&\qquad-\Big(a''(x-y)\Big(\sin(\frac{y}{2})-y\Big)\cos(\frac{y}{2})-\Delta_y a'(x)\Big)\Big]dy.\end{aligned}$$

Once again we just bound the former integral, being the later less singular. On the one hand, we find that

$$|a''(x+h-y)y\cos(\frac{y}{2})-a''(x-y)y\cos(\frac{y}{2})|\leq\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}|y|h^{\alpha}. \tag{15}$$

On the other hand, recalling (13), we can write

$$\Delta_y a'(x)-\Delta_y a'(x+h)=y\int_0^1\Big(a''(\lambda x+(1-\lambda)(x-y))-a''(\lambda(x+h)+(1-\lambda)(x+h-y))\Big)\,d\lambda,$$

and hence

$$|\Delta_y a'(x)-\Delta_y a'(x+h)|\leq|y|\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}h^{\alpha}. \tag{16}$$

Combining (15)–(16) together with (7)–(8) we conclude that

$$\mathrm{J}_{out}\leq C\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}\left\|b\right\|_{C^{\alpha}(\mathbb{T})}h^{\alpha}\int_{(-\pi,-2|h|)\cup(2|h|,\pi)}\frac{1}{|y|^{1-\alpha}}\,dy\leq C\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}\left\|b\right\|_{C^{\alpha}(\mathbb{T})}h^{\alpha}. \tag{17}$$

Therefore, collecting estimates (12), (14) and (17) we have shown that

$$|\mathrm{I}_{11}+\mathrm{I}_{12}|\leq C\left\|a\right\|_{C^{2,\alpha}(\mathbb{T})}\left\|b\right\|_{C^{\alpha}(\mathbb{T})}h^{\alpha}. \tag{18}$$

Repeating the same estimates for $\mathrm{I}_{21}+\mathrm{I}_{22}$ one can readily check that

$$|\mathrm{I}_{21}+\mathrm{I}_{22}|\leq C\left\|a\right\|_{C^{1,\alpha}(\mathbb{T})}\left\|b\right\|_{C^{1,\alpha}(\mathbb{T})}h^{\alpha}. \tag{19}$$

Therefore, this concludes that

$$|\Theta'(x+h)-\Theta'(x)| \le C\, \|a\|_{C^{2,\alpha}(\mathbb{T})}\, \|b\|_{C^{1,\alpha}(\mathbb{T})}\, h^{\alpha},$$

and thus together with (9) we have shown that

$$\|\Theta\|_{C^{1,\alpha}(\mathbb{T})} \le \|a\|_{C^{2,\alpha}(\mathbb{T})}\, \|b\|_{C^{1,\alpha}(\mathbb{T})},$$

proving the desired result. □

Remark 2.2 (*Sharpened Commutator Estimate*)**.** The commutator bound in Lemma 2.1 can in fact be sharpened. Indeed, for $0<\alpha<1$ and any $a,b\in C^{1,\alpha}(\mathbb{T})$ one has

$$\big\|[\![\mathcal{H},a]\!][b']\big\|_{C^{1,\alpha}(\mathbb{T})} \;\le\; C\; \|a\|_{C^{1,\alpha}(\mathbb{T})}\; \|b\|_{C^{1,\alpha}(\mathbb{T})}\,. \tag{20}$$

The key observation is that, by differentiating under the integral in the periodic representation of the commutator and then integrating by parts, one arrives at

$$\Theta'(x) = -\,a'(x)\,\mathcal{H}b'(x) + b'(x)\,\mathcal{H}a'(x) - \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} \frac{a(x-y)-a(x)}{2\sin^2(y/2)}\big(b'(x-y)-b'(x)\big)\,dy. \tag{21}$$

The first two terms are controlled using the algebra property of Hölder spaces together with the boundedness of $\mathcal{H}$ on C^{α}, which yields

$$\big\|a'(x)\,\mathcal{H}b'(x) + b'(x)\,\mathcal{H}a'(x)\big\|_{C^{\alpha}} \;\le\; C\; \|a\|_{C^{1,\alpha}}\, \|b\|_{C^{1,\alpha}}\,.$$

For the singular integral term in (21), one may rewrite it as

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a(x-y)-a(x)-a'(x)\,y}{2\sin^2(y/2)}\,\big(b'(x-y)-b'(x)\big)\,dy \;-\; \frac{a'(x)}{2\pi}\int_{-\pi}^{\pi} \frac{y}{2\sin^2(y/2)}\,\big(b'(x-y)-b'(x)\big)\,dy,$$

and then follow the same lines of the proof of Lemma 2.1 to obtain the improved commutator estimate (20).

We refrain from stating this sharper bound as a separate lemma, since the weaker version in Lemma 2.1 already suffices for our purposes, and keeping the stronger estimate in the form of a remark helps maintain the clarity and flow of the exposition.

To conclude this section, we recall the Crandall–Rabinowitz Theorem which is a fundamental tool in bifurcation theory that will be used to provide the main result of this article. To that purpose, let us first recall the following definition:

Definition 2.3 (*Fredholm Operator*)**.** Let X and Y be two Banach spaces. A continuous linear mapping $T : X \to Y$ is a Fredholm operator if it fulfills the following properties,

(1) $\dim \operatorname{Ker} T < \infty$,
(2) $\operatorname{Im} T$ is closed in Y,
(3) $\operatorname{codim} \operatorname{Im} T < \infty$.

The integer $\dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Im} T$ is called the Fredholm index of T.

Next, we shall discuss the index persistence through compact perturbations, cf. [26,27].

Proposition 2.4. *The index of a Fredholm operator remains unchanged under compact perturbations.*

Now, we recall the classical Crandall–Rabinowitz Theorem whose proof can be found in [28].

Theorem 2.5 (*Crandall–Rabinowitz Theorem*)**.** *Let X, Y be two Banach spaces, V be a neighborhood of 0 in X and $F : \mathbb{R}\times V \to Y$ be a function with the properties,*

(1) $F(\lambda,0)=0$ for all $\lambda\in\mathbb{R}$.
(2) The partial derivatives $\partial_\lambda F$, $\partial_f F$ and $\partial_\lambda\partial_f F$ exist and are continuous.
(3) The operator $\partial_f F(\lambda_0,0)$ is Fredholm of zero index and Ker$(F_f(\lambda_0,0)) = \langle f_0\rangle$ is one-dimensional.
(4) Transversality assumption: $\partial_\lambda\partial_f F(\lambda_0,0) f_0 \notin$ Im$(\partial_f F(\lambda_0,0))$.

If Z is any complement of Ker$(\partial_f F(\lambda_0,0))$ in X, then there is a neighborhood U of $(\lambda_0,0)$ in $\mathbb{R}\times X$, an interval $(-a,a)$, and two continuous functions $\Phi : (-a,a)\to\mathbb{R}$, $\beta : (-a,a)\to Z$ such that $\Phi(0)=\lambda_0$ and $\beta(0)=0$ and

$$F^{-1}(0)\cap U = \{(\Phi(s), s f_0 + s\beta(s)) : |s|<a\} \cup \{(\lambda,0) : (\lambda,0)\in U\}.$$

In this context, we will say that λ_0 is an eigenvalue of F.

3. Formulation of the problem

We shall look for traveling waves for f and hence find φ such that

$$f(t,x)=\varphi(x-ct),$$

for some speed $c\in\mathbb{R}$. Hence, the equation reduces to

$$F[c,\varphi](\xi)=0,\quad x\in[-\pi,\pi],$$

where

$$\begin{aligned}F[c,\varphi](\xi)&=2c\varphi'(\xi)+c\alpha_0\Lambda[\varphi'](\xi)+\frac{1}{\varepsilon}\left\{\varphi'(\xi)+\mathcal{H}[\varphi](\xi)+(\alpha_0-\beta)\mathcal{H}[\varphi''](\xi)\right\}+\mathcal{H}[(\Lambda\varphi)^2](\xi)\\&\quad-[\![\mathcal{H},\varphi]\!][\Lambda\varphi](\xi)+(\alpha_0-\beta)[\![\mathcal{H},\varphi]\!][\Lambda^3\varphi](\xi)\\&=2c\varphi'(\xi)+(c\alpha_0+\frac{\alpha_0-\beta}{\varepsilon})\mathcal{H}[\varphi''](\xi)+\frac{1}{\varepsilon}\left\{\varphi'(\xi)+\mathcal{H}[\varphi](\xi)\right\}+\mathcal{H}[(\mathcal{H}\varphi')^2](\xi)\\&\quad-[\![\mathcal{H},\varphi]\!][\mathcal{H}\varphi'](\xi)-(\alpha_0-\beta)[\![\mathcal{H},\varphi]\!][\mathcal{H}\varphi'''](\xi).\end{aligned}\tag{22}$$

Hence, note that we have the following line of trivial solutions:

$$F[c,0]=0,\quad\text{more generally}\quad F[c,a]=0,\forall a\in\mathbb{R}.$$

Define also the functional spaces

$$X:=\left\{f\in C^{3,\alpha}([0,2\pi],\mathbb{R}),\quad f(\xi)=\sum_{k\geq1}f_k\cos(k\xi)\text{ with norm }\|f\|_X=\|f\|_{C^{3,\alpha}}\right\},$$
$$Y:=\left\{f\in C^{1,\alpha}([0,2\pi],\mathbb{R}),\quad f(\xi)=\sum_{k\geq1}f_k\sin(k\xi)\text{ with norm }\|f\|_Y=\|f\|_{C^{1,\alpha}}\right\}.$$

3.1. The linearized operator: spectral properties and transversality condition

The first result shows that the operator F defined in (22) is well-defined and has the desired regularity. More precisely:

Proposition 3.1. *The operator $F:\mathbb{R}\times X\to Y$ given in (22) is well-defined and $\mathscr{C}^1(\mathbb{R}\times X\to Y)$.*

Proof. Let us start checking that F is well-defined. First of all, let us check the symmetry in the spaces. That is, if $\varphi(-\xi)=\varphi(\xi)$, then

$$F[c,\varphi](-\xi)=-F[c,\varphi](\xi).$$

Indeed, it is straightforward to check that

$$\varphi'(-\xi)=-\varphi'(\xi),\quad\varphi''(-\xi)=\varphi''(\xi),\quad\varphi'''(-\xi)=-\varphi'''(\xi).$$

Furthermore, note that

$$\mathcal{H}[\varphi''](-\xi)=\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi}\frac{\varphi''(y)}{\tan\left(\frac{-\xi-y}{2}\right)}dy=-\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi}\frac{\varphi''(-y)}{\tan\left(\frac{\xi-y}{2}\right)}dy=-\mathcal{H}[\varphi''](\xi).$$

We can check, in a similar way, that the symmetry property is satisfied by the remaining integral terms.

Let us move next to the regularity properties for the operator F. We first notice that

$$\begin{aligned}\mathcal{H}[h](\xi)&=\sum_{k\geq1}h_k\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi}\frac{\cos(ky)}{\tan\left(\frac{\xi-y}{2}\right)}dy\\&=-\sum_{k\geq1}h_k\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi}\frac{\cos(k(\xi-z))}{\tan\left(\frac{z}{2}\right)}dz\\&=\sum_{k\geq1}h_k\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi}\frac{\sin(k\xi)\sin(kz)-\cos(k\xi)\cos(kz)}{\tan\left(\frac{z}{2}\right)}dz\\&=\sum_{k\geq1}h_k\sin(k\xi)\frac{1}{2\pi}p.v.\int_{-\pi}^{\pi}\frac{\sin(ky)}{\tan\left(\frac{y}{2}\right)}dy\\&=\sum_{k\geq1}h_k\sin(k\xi),\end{aligned}$$

and as a consequence we have in particular that $\mathcal{H} : X \to Y$. Furthermore, it is easy to check that the first three terms in (22) are easily bounded by

$$\|2c\varphi'(\xi) + (c\alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon})\mathcal{H}[\varphi''](\xi) + \frac{1}{\varepsilon}\left\{\varphi'(\xi) + \mathcal{H}[\varphi](\xi)\right\})\|_Y \le C\|\varphi\|_X. \tag{23}$$

Similarly, using the fact that $\mathcal{H} : X \to Y$ and the Banach Algebra property for Y yields

$$\|\mathcal{H}[(\mathcal{H}\varphi')^2](\xi) + [\![\mathcal{H}, \varphi]\!][\mathcal{H}\varphi'](\xi)\|_Y \le C\|\varphi\|_X^2. \tag{24}$$

In order to bound the commutator term $(\alpha_0 - \beta)[\![\mathcal{H}, \varphi]\!][\mathcal{H}\varphi'''](\xi)$ we make use of estimate (6) derived in Lemma 2.1. Indeed, taking $a = \varphi$ and $b = \mathcal{H}\varphi''$ in Lemma 2.1 we have that

$$\left\|[\![\mathcal{H}, \varphi]\!][\mathcal{H}\varphi'''](\xi)\right\|_{C^{1,\alpha}} \le C\,\|\varphi\|_{C^{2,\alpha}}\,\left\|\mathcal{H}\varphi''\right\|_{C^{1,\alpha}} \le C\,\|\varphi\|_X^2\,. \tag{25}$$

Altogether, we have show as claimed that $F : \mathbb{R}\times X \to Y$ given in (22) is well-defined. Next, let us demonstrate that $F \in \mathscr{C}^1(\mathbb{R}\times X \to Y)$. To do so, it is enough to observe that

$$\|\partial_\varphi F[c, \varphi_1]h - \partial_\varphi F[c, \varphi_2]h\|_Y \le C\|h\|_X\|\varphi_1 - \varphi_2\|_X, \tag{26}$$

where

$$\begin{aligned}\partial_\varphi F[c, \varphi]h = {}& 2ch' + (c\alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon})\mathcal{H}[h''] + \frac{1}{\varepsilon}\left\{h' + \mathcal{H}[h]\right\} + 2\mathcal{H}[\mathcal{H}\varphi'\mathcal{H}h'] - [\![\mathcal{H}, h]\!][\mathcal{H}\varphi'] \\ & - [\![\mathcal{H}, \varphi]\!][\mathcal{H}h'] - (\alpha_0 - \beta)[\![\mathcal{H}, h]\!][\mathcal{H}\varphi'''] - (\alpha_0 - \beta)[\![\mathcal{H}, \varphi]\!][\mathcal{H}h'''],\end{aligned}$$

denotes the Gateaux derivative. Indeed, using again the Banach Algebra property we find that

$$\|2\mathcal{H}[\mathcal{H}(\varphi_1 - \varphi_2)'\mathcal{H}h']\|_Y \le C\left\|\mathcal{H}(\varphi_1 - \varphi_2)'\right\|_Y\left\|\mathcal{H}h'\right\|_Y \le C\left\|\varphi_1 - \varphi_2\right\|_X\|h\|_X\,.$$

Similarly, it easy to check that

$$\|[\![\mathcal{H}, h]\!][\mathcal{H}(\varphi_1 - \varphi_2)']\|_Y + \|[\![\mathcal{H}, (\varphi_1 - \varphi_2)]\!][\mathcal{H}h']\|_Y \le C\left\|\varphi_1 - \varphi_2\right\|_X\|h\|_X\,.$$

To conclude, we invoke Lemma 2.1 with $a = h$ and $b = \mathcal{H}(\varphi_1 - \varphi_2)''$ and $a = \varphi_1 - \varphi_2$ and $b = \mathcal{H}h''$ respectively to find that

$$\|[\![\mathcal{H}, h]\!][\mathcal{H}(\varphi_1 - \varphi_2)''']\|_Y \le C\,\|h\|_{C^{2,\alpha}}\left\|\mathcal{H}(\varphi_1 - \varphi_2)''\right\|_{C^{1,\alpha}} \le C\,\|h\|_X\left\|\varphi_1 - \varphi_2\right\|_X\,,$$

$$\|[\![\mathcal{H}, (\varphi_1 - \varphi_2)]\!][\mathcal{H}h''']\|_Y \le C\left\|\varphi_1 - \varphi_2\right\|_{C^{2,\alpha}}\left\|\mathcal{H}h''\right\|_{C^{1,\alpha}} \le C\,\|h\|_X\left\|\varphi_1 - \varphi_2\right\|_X\,,$$

which shows estimate (26). Hence, we can conclude that the Gateaux derivative is continuous (indeed, it is Lipschitz) and then we can ensure the existence and continuity of the Fréchet derivative. □

In the following, we analyze the linearized operator at the trivial solution $(c, 0)$ given by

$$\partial_\varphi F[c, 0]h(\xi) = 2ch'(\xi) + (c\alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon})\mathcal{H}[h''](\xi) + \frac{1}{\varepsilon}\left\{h'(\xi) + \mathcal{H}[h](\xi)\right\}. \tag{27}$$

More precisely, we study the Fredholm index of the operator (27).

Proposition 3.2. *For $c \neq 0$, the operator $\partial_\varphi F[c, 0]$ is Fredholm of zero index.*

Proof. Since the coefficient $c\alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon} \neq 0$ we have that

$$\partial_\varphi F[c, 0]h(\xi) = \mathcal{L}h(\xi) + \mathcal{K}h(\xi),$$

where

$$\mathcal{L}h(\xi) = (c\alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon})\mathcal{H}[h''](\xi), \quad \mathcal{K}h(\xi) = 2ch' + \frac{1}{\varepsilon}\left\{h'(\xi) + \mathcal{H}[h]\right\}.$$

The principal part of the linear operator $\mathcal{L}h$ is an isomorphism from X to Y and thus has zero index. Indeed, this follows by noticing that for $h \in X$ we have that

$$\mathcal{L}h(\xi) = -(c\alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon})\sum_{k\ge1} h_k k^2 \sin(kx).$$

Moreover, for

$$Z := \left\{ f \in C^{2,\alpha}([0, 2\pi], \mathbb{R}), \quad f(\xi) = \sum_{k\ge1} f_k \sin(k\xi) \text{ with norm } \|f\|_Z = \|f\|_{C^{2,\alpha}} \right\},$$

the operator $\mathcal{K}h : X \to Z$ is continuous. Therefore, the embedding $Z \hookrightarrow Y$ is compact thus by Proposition 2.4, we conclude that (27) is Fredholm of zero index. □

The following result describes the kernel and range of the linearized operator.

Proposition 3.3. *If* $h(x) = \sum_{k\geq 1} h_k \cos(kx)$*, we have that*

$$\partial_\varphi F[c,0]h(x) = \sum_{k\geq 1} h_k \sin(kx)\left\{-(2c+\frac{1}{\varepsilon})k + \frac{1}{\varepsilon} - (c\alpha_0 + \frac{\alpha_0-\beta}{\varepsilon})k^2\right\}. \tag{28}$$

Hence, for

$$c_k = \frac{1}{\varepsilon}\left(\frac{1-k-(\alpha_0-\beta)k^2}{k(2+\alpha_0 k)}\right), \quad k\geq 1$$

we have that the kernel and the range of the linearized operator can be described as follows

$$Ker[\partial_\varphi F[c_k,0]] = \langle\cos(kx)\rangle,$$
$$Y/Img[\partial_\varphi F[c_k,0]] = \langle\sin(kx)\rangle.$$

Moreover, the transversal condition is satisfied, i.e. for $h_0 \in Ker[\partial_\varphi F[c_k,0]]$*, we find that*

$$\partial_c\partial_\varphi F[c_k,0]h_0 \notin Im[\partial_\varphi F[c_k,0]].$$

Proof. Let us first show how to obtain expression (28). For $h(x) = \sum_{k\geq 1} h_k\cos(kx)$ we find that

$$h'(\xi) = -\sum_{k\geq 1} h_k k\sin(kx),\ \mathcal{H}[h''](\xi) = -\sum_{k\geq 1} h_k k^2\sin(kx),\ \mathcal{H}[h](\xi) = \sum_{k\geq 1} h_k\sin(k\xi).$$

Thus, recalling (27) and the previous identities we infer that (28) holds. From the expression of the linearized operator in Fourier series (28), it is clear that the kernel of $\partial_\varphi F[c_k,0]$ is generated by

$$\langle\cos(kx)\rangle.$$

Moreover, since the linearized operator is Fredholm of zero index, one has that the codimension of the range is one dimensional and thus we can ensure that

$$Y/\text{Img}[\partial_\varphi F[c_k,0]] = \langle\sin(kx)\rangle.$$

Finally, to check the transversal condition we have to differentiate the linear operator with respect to the parameter c obtaining

$$\partial^2_{(\varphi,c)}F[c,0]h(x) = \sum_{k\geq 1} h_k\sin(kx)\left\{-2k-\alpha_0 k^2\right\}.$$

Next, we evaluate it at the generator of the kernel:

$$\partial^2_{(\varphi,c)}F[c_{k_\star},0]\cos(k_\star x) = \sin(k_\star x)\left\{-2k_\star - \alpha_0 k_\star^2\right\}.$$

for $k_\star \geq 1$. Since $\alpha_0 > 0$ we find that

$$\partial^2_{(\varphi,c)}F[c_{k_\star},0]\cos(k_\star x) \notin \text{Img}[\partial_\varphi F[c_{k_\star},0]],$$

and hence the transversal condition is satisfied. □

3.2. Proof of Theorem 1.1

Fix $m\geq 1$. In order to prove Theorem 1.1, let us introduce the symmetry m in the spaces. For that, let us define

$$X_m := \left\{ f\in C^{3,\alpha}([0,2\pi],\mathbb{R}),\quad f(\xi) = \sum_{k\geq 1} f_k\cos(mk\xi) \text{ with norm } \|f\|_{X_m} = \|f\|_{C^{3,\alpha}}\right\},$$
$$Y_m := \left\{ f\in C^{1,\alpha}([0,2\pi],\mathbb{R}),\quad f(\xi) = \sum_{k\geq 1} f_k\sin(mk\xi) \text{ with norm } \|f\|_{Y_m} = \|f\|_{C^{1,\alpha}}\right\},$$

for any $m\geq 1$. In order to check that

$$F : \mathbb{R}\times X_m \to Y_m,$$

is well-defined we can use Proposition 3.1 but it remains to check the m-fold symmetry property. For that purpose, we have to check that if

$$\varphi(\xi + \frac{2\pi}{m}) = \varphi(\xi),$$

then

$$F[c,\varphi](\xi + \frac{2\pi}{m}) = F[c,\varphi](\xi).$$

Note that is φ has the m-fold symmetry property, then all the derivatives also enjoy the same symmetry. Now, let us check the Hilbert term:

$$\mathcal{H}[\varphi''](\xi + 2\pi/m) = \frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} \frac{\varphi''(y)}{\tan\left(\frac{\xi - y + 2\pi/m}{2}\right)} dy = \frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} \frac{\varphi''(y + 2\pi/m)}{\tan\left(\frac{\xi - y + 2\pi/m - 2\pi/m}{2}\right)} dy$$
$$= \mathcal{H}[\varphi''](\xi).$$

Similar argument works for the other integral terms. Following Proposition 3.2 the linear operator is a Fredholm operator of zero index, and Proposition 3.3 gives us the expression of the linear operator in Fourier series:

$$\partial_\varphi F[c_m, 0]h(x) = \sum_{k \geq 1} h_k \sin(mkx) \left\{ -(2c_m + \frac{1}{\varepsilon})km + \frac{1}{\varepsilon} - (c_m \alpha_0 + \frac{\alpha_0 - \beta}{\varepsilon})(km)^2 \right\}.$$

Hence Proposition 3.3 gives us the one dimensionality of the kernel, which is now generated by $k = 1$:

$$\langle \cos(mx) \rangle,$$

as well as the one co-dimensionality of the range. Finally, the transversal condition is satisfied in Proposition 3.3. Hence, Crandall–Rabinowitz theorem can be applied obtaining the main result of this paper.

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Data availability

No data was used for the research described in the article.

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