The chimera of the competency-based approach to teaching mathematics: a study of carpentry purchases for home projects

José Manuel Diego-Mantecón
Universidad de Cantabria
Departamento de Matemáticas, Estadística y Computación
Facultad de Ciencias.
Avd. Los Castros 44, 39005. Santander (Cantabria). Spain.
Phone: +34 655 45 71 07 Fax: 942 20 14 02
diegojm@unican.es
ORCID: 0000-0002-4427-2724

Elena Haro
Colegio Torreanaz
Barrio La Iglesia 39, 39718 Anaz, Cantabria. Spain.
lhmaestro91@gmail.com

Teresa F. Blanco
Universidad de Santiago de Compostela
Departamento de Didácticas Aplicadas. Facultad de Ciencias de la Educación, Campus Norte. Avda. Xoán XXIII s/n. Santiago de Compostela, 15784, Spain.
teref.blanco@usc.es
ORCID: 0000-0003-4215-8677

Avenilde Romo-Vázquez
CICATA-Instituto Politécnico Nacional
Legaria 694. Alcaldía Miguel Hidalgo.
Ciudad de México 11500, Mexico.
aromov@ipn.mx
ORCID: 0000-0002-1364-5997

Abstract
The competency-based approach conceives mathematics as a necessary tool for dealing with daily-life tasks. Many studies have focused on examining the low math-competency people show when solving problems in real-life contexts, but rarely characterize the type of mathematics needed in these contexts and how people use this mathematics. The current study was designed to analyze the mathematics utilized by 312 customers when purchasing carpentry products in a store specialized in home projects. Aspects of the Anthropological Theory of the Didactic, especially the Extended Praxeological Model, were employed to undertake the analysis. While the approach is primarily qualitative, quantitative aspects were also considered to elucidate the nature of the identified mathematics tasks, the techniques that customers employed to solve them, and the difficulties associated with the use of these techniques. The study reveals that having solid mathematical knowledge is insufficient when it comes to solving everyday tasks, because related contextual knowledge is also required. The nature of the tasks identified in this study, and the didactic way in which the clerk guided the customers through the projects, suggested that there are no only complex relationships between school mathematics and outside-school mathematics, but there exist also different didactics specific to the contexts. For elaborating the home carpentry projects, the customers needed to handle a set of carpentry knowledge and techniques as well as carpentry-related mathematics that are not necessarily taught at school.

Keywords: mathematical competency, everyday tasks, mathematical knowledge, contextual knowledge, school mathematics, anthropological theory of didactic

There is a general interest in evaluating citizens' math-competency and understanding their difficulties when tackling real-life problems. Several international reports, like the Programme for the International Assessment of Adult Competencies (PIAAC) and the Programme for International Student Assessment (PISA), suggest that both adults and students worldwide show low math competency when solving problems in context (OECD, 2013, 2014, 2016), but the reasons for this low competency are not well-identified. Studies analyzing everyday situations reveal that what PIAAC and PISA call low math-competency relates to many factors, distinct from those that characterize school instruction, including the motivation to solve a problem and the life context in which it occurs (Lave, 1988; Lave & Gomes, 2019). Authors like Millroy (1992), Noss et al. (2007), as well as FitzSimons and Boistrup (2017), claim that the mathematics required in 'work settings' often demands forms that are not taught at school. Nurses, pilots and bankers, for example, need to adapt mathematics for solving tasks in their contexts (Noss et al., 2000). This suggests that applying school mathematics in real-contexts is not a straightforward procedure. To better understand the factors affecting math competency, we concur with Niss and Højgaard (2019) that it is necessary to identify the mathematics arising in real-life contexts and to analyze how people use such mathematics. For this study, we have reviewed the meaning of the competency-based approach to teaching mathematics and the claims of existing work on this focus. In the same vein, we have analyzed the approaches used to assess mathematics in real contexts, aiming to identify relationships between school mathematics and outside-school mathematics. To identify such relationships, we have analyzed a real-life context in which mathematics is required. We intensively searched for a store that would allow us to observe people employing mathematics in real-life— not simulated —contexts. In particular, we selected a store specialized in home projects, that provided us the opportunity to recognize and analyze the mathematics used by customers when purchasing carpentry products. To undertake the analysis, we selected the Anthropological Theory of the Didactic (ATD) as a framework (Chevallard, 1999, 2019).

1. Competency-based approach to teaching mathematics

During the Industrial Revolution, schools were concerned primarily with increasing literacy to enable citizens to perform jobs that required skills like reading, writing, and executing basic arithmetical operations. Today, in contrast, with easy access to information through technological media, the objective is to learn how to apply knowledge to the diverse situations that real-life presents. This change led to the emergence in schools of the so-called Competency-based Approach (Halász & Michel, 2011), designed to prepare citizens for the modern society. Math competency has been conceptualized in various ways, usually through broad definitions. PISA, for instance, refers to "The personal capacity to formulate, employ and interpret mathematics in distinct contexts, including mathematical reasoning and the utilization of mathematics concepts, procedures, data and tools to describe, explain and predict phenomena" (OECD, 2016, p. 28). More recently, Niss and Højgaard (2019), after a thorough review of the literature, concluded that mathematical competency is "someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations" (p. 4). Curricula across the world have adopted these general definitions to evaluate math competency through national and international assessments (European Union Council, 2018). Most of the definitions adopted by curricula describe math competency by referring specifically to the capacity of solving problems in everyday life situations. International assessments, including PISA and PIAAC, have reported low math competency in both students and adults worldwide (OECD, 2013, 2016), suggesting that various educational systems are not preparing citizens for solving problems in everyday life. These evaluations are even claiming that teaching traditions and textbooks may be preventing students from engaging with real problems.

2. Mathematics in real contexts

Various researchers have attempted to clarify the relationship between school mathematics and outside-school mathematics (Akkerman & Baker, 2012; Bakker, 2014; Covián & Romo, 2014; Evans et al., 2012; Lave, 1988; Nunes et al., 1993, Swanson & Williams, 2014). Evans et al. (2012) establish two perspectives when analyzing this relationship: utilitarian and situational. The utilitarian perspective of mathematics refers to whether adults have the necessary skills for dealing with everyday life or career situations. It corresponds to the idea of education as a transfer of knowledge from school problems to outside-school problems (Mestre, 2002); programs like PISA o PIAAC assess such transfer of knowledge. The situational perspective explores the relationship between school knowledge and the knowledge used in real-life or in a workplace. This perspective, arising as a reaction to the transfer of knowledge standpoint, is called situated cognition (Lave, 1988). The situated cognition perspective is based on the idea that knowledge, thinking, and learning are constrained by the situation in which they emerge. This implies that individuals' knowledge is specific to each context or practice, and therefore the transfer of knowledge from school to external environments is not likely to be as successful as expected. As a result, when considering studies of everyday contexts, it is no longer sufficient to say whether the action is right or wrong compared to school practice (Lave, 1988). The studies of Grando (1988), Schliemann (1984), Lave et al. (1984), Murtaugh (1985), and Noss et al. (2000) confirm that school mathematics differs from real-life mathematics. Nunes et al. (1993) postulate that school-mathematics knowledge can be applied outside-school and outside-school knowledge can also be brought into classroom. In this sense, Swanson and Williams (2014) suggest a unified view of mathematical authenticity. For them, mathematical authenticity is about solving concrete tasks, at either school or workplaces, that turn out to be meaningful for individuals in their social practices. Both school and workplaces are, however, potential spaces where mathematical authenticity can be obstructed by the conditions and restrictions of each context. The above shows that establishing relationships between school mathematics and outside-school mathematics is rather complex. It requires considering contextual logics, different from those at school, which force adaptations on mathematical knowledge to operate. It becomes, therefore, necessary to analyze real-life contexts and how mathematics is useful to perform tasks in such contexts. We chose the anthropological theory of didactic because it is suited for this investigation, as we expand in the next section.

3. Anthropological Theory of the Didactic

The Anthropological Theory of the Didactic (ATD) allows analyzing human activity in any setting, including academic or workplace settings using mathematics (Chevallard, 1999, 2019). The ATD rests upon two fundamental notions: institution and praxeology. An *institution* is a stable organization that provides subjects with the material and intellectual resources required to efficiently perform certain tasks (Castela & Romo, 2011). Many types of institutions exist, including warehouses, schools, and classrooms. Institutions can be classified in three categories according to their relation to knowledge: production (P_i), teaching (T_i), and using (U_i). Production institutions include disciplines that generate knowledge (e.g., mathematics, physics); teaching institutions are those responsible for transmitting knowledge (e.g., schools, universities); and using institutions are in charge of utilizing knowledge (e.g., factories, stores). *Praxeology,* in turn, refers to the minimal unit of analysis of any human activity (e.g., having a coffee, walking along a street). Every praxeology is made up of four components: type of tasks (T), technique (τ), technology (θ), and theory (Θ). Type of tasks defines what is done; technique refers to how it is done; technology (traditionally known as knowledge) involves discourses that produce, justify, and explain the technique; and theory includes broader discourses producing, justifying, and explaining technology (Chevallard, 1999, 2019). The praxeological model is thus represented by the quadruple: $[T, \tau, \theta, \Theta]$.

Praxeologies can circulate from one institution to another, but suffer modifications along the way that are called transposition processes. When a praxeology takes place in a second institution, it may contain elements from both its original institution and the one to which it is transposed. Praxeologies containing elements from two or more institutions are referred to as mixed-praxeologies (Vázquez et al., 2016); for instance, the praxeology 'driving a car in Spain' needs to be transposed into the English context to safely drive in England. For analyzing mixed-praxeologies authors have often used Castela and Romo's (2011) Extended Praxeological Model (EPM). This model has been refined though several studies, including Peters et al. (2017), Solares et al. (2016), and Chaachoua et al. (2019). Considering a mathematics teaching institution T_i (M) and any using institution U_i (A), the model can be represented as follows:

$$\begin{bmatrix} T^a & \tau^m \ \theta^m \ \Theta^m \\ & \tau^a \ \theta^a \ \Theta^a \end{bmatrix} \begin{matrix} \leftarrow T_i\,(M) \\ \leftarrow U_i\,(A) \end{matrix}$$

where T^a is the type of tasks of the using institution; $[\tau^m, \theta^m, \Theta^m]$ are respectively the technique, technology and theory of mathematics teaching institution; and $[\tau^a, \theta^a, \Theta^a]$ are, respectively, the technique, technology and theory of any using institution. The latter praxeological elements can include transposed mathematics from the teaching institution; i.e., this model permits analyzing the mathematics employed in the using institution.

4. Research Questions and Methods

As suggested above, this study aims to identify the mathematics arising in real-life contexts and to analyze how people use such mathematics. In particular, we identify and analyze the mathematics used by customers when purchasing carpentry products for elaborating home projects. To achieve these objectives, we address the following research questions:

- What school and non-school mathematics (techniques and technologies) are employed in this carpentry context?
- What processes took place to adapt school mathematics (techniques and technologies) into this carpentry context?

To answer these questions, we selected a store that would allow us to observe people employing mathematics in real-life — not simulated — contexts.

Sample selection and description

The sample was selected by a purposive strategy (Bryman, 2015). Although the subjects selected are not representative of the whole population, the context in which they interact is ideal for tackling our research questions. A store in northern Spain that sells carpentry products was chosen, because it has a sales policy requiring customers to make measurements and calculations *before* asking clerks for products. That is, customers must tackle carpentry tasks that involve mathematics to develop their home projects. Clerks can intervene if customers present erroneous data or calculations, but are prohibited from making the initial procedures. This policy allowed us observing 312 customers and one clerk while dealing with carpentry tasks that required mathematics to be solved. The clerk was a man with over 25 years' experience in carpentry projects and broad knowledge of all the products and their installation. He has also received constant training to guide customers from the beginning to end of their projects. All the costumers were men, aged 30-50, with no less than the school secondary education level that is mandatory in Spain. According to the clerk, the customers had some knowledge of carpentry based on earlier experience or they had watched video tutorials produced by the factories' websites. Many also interacted on the Internet to share ideas about their projects and participated in courses offered by the store itself. This store has three sales areas where the study was conducted: surface coatings, door and window installation, and furniture building and installation. Typical tasks in the surface coatings area include calculating the amount of material (e.g., wood or paint) needed to cover a certain surface or perimeter. The door and window area demands mainly comparative measurements, while in the furniture construction and installation section customers often need to visualize the pieces required to build and install furniture in limited spaces.

Research strategy for data analysis

To analyze the 312 home projects from the three sales areas, we adapted the extended praxeological model described in Section 3, identifying the existing institutions and the praxeological elements of these institutions, as well as their relationships. This generated the following model:

$$\left[T^c, \quad \begin{matrix} \tau^m \ \theta^m \ \Theta^m \\ \tau^c \ \ \theta^c \ \ \Theta^c \end{matrix} \quad \right] \begin{matrix} \leftarrow Sc_i(M) \\ \leftarrow St_i(C) \end{matrix}$$

$Sc_i(M)$ denotes the School Mathematics institution while $St_i(C)$ represents the Store Carpentry institution. The former refers to school mathematics praxeologies and the latter to carpentry praxeologies in which mathematics is used for developing home projects. $Sc_i(M)$ and $St_i(C)$ are thus considered a teaching and using institution, respectively. This model allows us analyzing the home projects as mixed-praxeologies because they contain elements from both institutions: carpentry tasks (T^c), mathematics and carpentry techniques $[\tau^m \ \tau^c]$, mathematics and carpentry technologies $[\theta^m, \theta^c]$, and mathematics and carpentry theories $[\Theta^m, \Theta^c]$. That is, this model facilitates analyzing the techniques and technologies employed by the customers and the clerk during the development of the carpentry tasks.

The research questions entail three phases: 1) analyzing the 312 mixed-praxeologies (home projects); 2) evaluating how the mathematical techniques (τ^m) were used to tackle the carpentry tasks (T^c); and 3) evaluating the carpentry and mathematical technologies $[\theta^c, \theta^m]$ that supported the mathematical techniques employed (τ^m). The analysis of the mathematics and carpentry theories were not considered because the 312 projects represent different mixed-praxeologies; data from different customers tackling the same type of project would be necessary to make consistent judgments. To carry out the three above phases, we mainly adopted a qualitative approach (Creswell, 2014). Though qualitative, the analysis also included figures representing the percentages of identified tasks, inadequate solutions, and issues related to the praxeological elements. The qualitative analysis helped elucidate how the mathematical techniques were used to deal with the carpentry tasks, and the interaction between the customers and the clerk that guided the development process of the projects.

Methods and procedure

For data collection we employed observations and interviews. The observations helped gather information on the mixed-praxeologies, specifically the mathematical techniques (τ^m) and carpentry and mathematical technologies $[\theta^c, \theta^m]$ used for solving the carpentry tasks. Observations were conducted by one of the authors on regular working days in April-May, 2017. This observer acted as an in-training clerk who took notes from the interaction between the customers and the official clerk, but adopted a passive role. The interviews with the official clerk were designed to gain a deeper understanding of the praxeological elements utilized by the customers and the clerk himself. Interviews were held after each product sale and during the writing of this article to corroborate our analyses and findings. The customers' notes (e.g., sketches, calculations) were reproduced by the observer during or right after purchases. Original notes were not gathered because the customers needed them. Participants were willing to provide personal data for statistical purpose (e.g., age, educational level), but observations were not video-recorded because store policy prohibits this practice. Two of the four authors made a first analysis of the whole data set, categorizing the praxeological elements emerging in each of the 312 projects. This first analysis was revised by the other two authors. The disagreements observed between the two analyses were discussed by the four authors together; during this process the clerk was contacted to expand and verify information. In the following section, we present two mixed-praxeologies resulting from our analysis of the customers' activity in the store that illustrate their work.

5. Two mixed-praxeologies

To exemplify our analysis, we present two mixed-praxeologies: renovating the floor of a house— from the surface coatings area —and building a closet to fit in a niche, from the furniture building and installation area. These two examples were selected because we found them to be representative of the way in which customers use mathematics in this institution. In the first project, the clerk provides carpentry technologies so the customer can adapt his mathematical techniques, whereas in the second, he introduces the customer to the mathematical techniques and technologies shaped in the carpentry institution.

Mixed-praxeology 1. Renovating the floor of a house

General context and specific carpentry tasks

The task emerged in the context of a customer who arrived at the store with his house plan in hand (Figure 1) to purchase materials for renovating the floor. Flooring is sold in packages of $2.47\ m^2$ and costs $6.95€/m^2$. The renovation entailed replacing the baseboard and fire-retardant material that covers the surface. The baseboard comes in $2.4\ m$ lengths costing $3.83€/m$, while the retardant is sold in rolls of 12, 15 or $24\ m^2$ with unit prices of 13.5, 22, and $27.6€$, respectively. The customer sought the most economical option for renovating the floor.

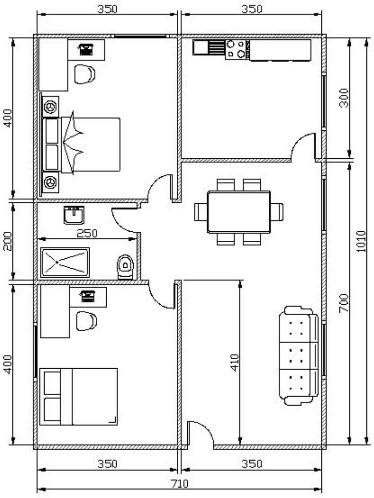

Figure 1. House floor plan in *cm* brought by the customer

Techniques and technologies employed

The technique involved five steps: (1) determining the area to be renovated; (2) determining the number of packages of flooring required; (3) determining the number of baseboard pieces needed; (4) calculating the cost of the flooring and baseboard; (5) calculating the number of rolls of fireproofing and the most economical option.

(1) Determining the area to be renovated required interpreting the house plan and the elements it contains: symbols, partial dimensions, etc. The customer identified the plan as a 7 *m* x 10 *m* rectangle by converting units from cm to m. He then calculated its area 70 m^2. The clerk intervened: "70 m^2 is the total area of the house. We shouldn't include areas without flooring, like the bathroom and kitchen; let's just add up the areas of the rooms." Consequently, the clerk extracted three figures representing the two bedrooms and living room with the measurements of each one (Figure 2).

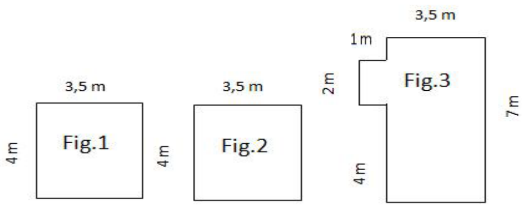

Figure 2. Figures extracted by the clerk, representing the bedrooms and living room. This is a reproduction of the clerk's drawings.

After extracting the figures, the customer calculated and added up the partial areas, given a result of 54.5 m^2 instead of 70 m^2 (Figure 3). According to the clerk, and our own observations, the customer did not interpret the floor plan correctly because of the high number of elements it contained. These elements acted as distractors, hampering the customer's process of adequately calculating the area to be renovated.

Fig.1: $3.5\ m\ x\ 4\ m = 14\ m^2$
Fig.2: $3.5\ m\ x\ 4\ m = 14\ m^2$
Fig.3: $3.5\ m\ x\ 7\ m + 2\ m\ x\ 1\ m = 26.5\ m^2$
Total area: $14\ m^2 + 14\ m^2 + 26.5\ m^2 = 54.5 m^2$

Figure 3. Calculation of the total area to be renovated; this is a faithful reproduction of the customer's notes

(2) To calculate the packages of flooring required, the customer divided the total area $54.5\ m^2$ by the $2.47\ m^2$ of one package, obtaining 22.06 packages that he rounded-off to 23. The clerk agreed with the calculations, but stated: "We should add an extra 10% to account for cuts and defects. This is a general rule to avoid running out of material before a project is complete." The clerk calculated the extra 10% and rounded-off the result to 25 packages, two more than the solution obtained by the customer. The additional 10% of material reflects the carpentry technology acquired through experience. (3) To determine the number of pieces of baseboard needed, the costumer calculated the perimeter of the bedrooms and living room, obtaining 53 m. Then, he divided the 53 m by the length of each piece (2.4 m) for a total of 23 pieces. At that stage, the clerk pointed out: "I think the perimeter is smaller; we should subtract the doorways, and, as before, add the extra 10% for cuts and defects." Considering these carpentry technologies, the clerk made new calculations, and obtained a perimeter of 50.6 m that, divided by the length of the pieces, gave a total of 22— one piece less than the customer had calculated.

(4) Having determined the amount of flooring and baseboard, the customer proceeded to calculate the cost. He multiplied the number of packages of flooring (25) by the unit price (6.95€), and the number of baseboard pieces (22) by the price per linear meter (3.83€). His results were 173.75€ and 84.26€, respectively. Because the customer confused the unit of measurement with the unit price of packaging for each material, we took this as an incorrect association of the technique with the mathematical technology. The customer did not identify units and handle the amounts associated with the concept of direct proportionality. The clerk intervened once more to correctly multiply the unit cost by the number of units in each package of flooring, and each piece of baseboard, to calculate the total price. His results were 429.16€ and 202.22€, respectively, which exceeded the customer calculation by 373.37€. During the interviews, the clerk explained that this is a common confusion because "materials are not sold by exact units of measurement, but, rather, by measurements that are easy for manufactures to store and transport."

(5) Finally, the customer calculated the rolls of fireproofing required to cover 60 m^2 in relation to the three sizes available: 12, 15, and 24 m^2. He did not divide the 60 m^2 by the size of the rolls to identify the number of them needed per option, and thus the cost. Instead, he calculated multiples of 12 (12, 24, 36, 48, 60), 15 (15, 30, 45, 60) and 24 (24, 48, 72), but not the total price of each option. He rejected the option three rolls of 24 m^2 because, in contrast to the other two, it would produce leftover material. At that stage, the clerk realized the costumer was assuming that the price per m^2 was uniform for all three type of rolls because — as he stated in a subsequent interview — "customers tend to choose the option that produce least leftover material when deciding among different options with equal costs." That is, costumers do not like to take home more material than they require.

Of the other two options, the costumer chose the one requiring the least number of rolls to cover the 60 m^2; that is, four rolls of 15 m^2. He then multiplied that by 22€, obtaining a cost of 88€. This was not, however, the most economical solution, confirming that the customer assumed a uniform price for the three types of rolls. The clerk commented during the interviews that "customers often choose the smallest number of units when deciding among different options with equal costs. This is due to a logistic reason because, normally, small number of units are easier to transport and store." In this case, the customer used an inappropriate mathematical technique as he calculated the smallest number of rolls that cover 60 m^2. Having calculated the price per roll, he should have realized the most economical option was five rolls of 12 m^2 at a cost of 67.5€, saving so 20.5€. Obeying the store rules, the clerk did not intervene as this error did not affect accomplishing the project, only the cost involved. Regarding the ATD, this a restriction of this using institution.

Mixed-praxeology 2. Building a closet for a niche

General context and specific carpentry task

This mixed-praxeology appeared in the context of a customer who wanted to purchase wood to build a closet for a niche of 42 cm wide × 82 cm high × 58 cm deep. The closet needed a central shelf to divide the interior into two equal parts. Wood was sold in pieces with standard dimensions of 244 × 122 × 1.6 cm that could be bought in quarters (Figure 3). A complete sheet costs 27€; the cost of a quarter-sheet is proportional to the total value. To make the required calculations and to represent the pieces of wood needed, the store offered the customer a sheet of graph paper (Figure 4). The graph paper is on scale 1 to 5 cm, with dimensions of 240 x 120 cm.

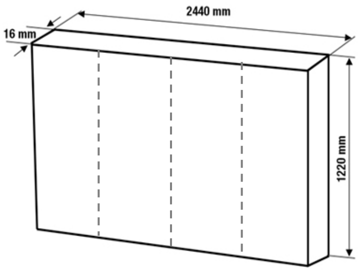

Figure 3. Unit of wood sold by the store

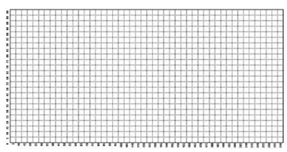

Figure 4. Graph paper of 120 x 240 cm provided by the store on scale 1 to 5 cm

Techniques and technologies employed

This customer interpreted that the closet measurements would correspond exactly to the niche dimensions (42 × 82 × 58 cm); as a consequence, he drew the closet in Figure 5 as the one he thought to build. The clerk explained that: "In reality an object built for a niche must be slightly smaller than the space available, because elements like friction, asymmetrical surfaces, and defects of materials or construction must be considered." Consequently, the clerk suggested reducing the height, width, and depth by 2 cm (40 × 80 × 56 cm) to ensure that the finished closet would fit in the niche. The aforementioned considerations belong to the carpentry technology.

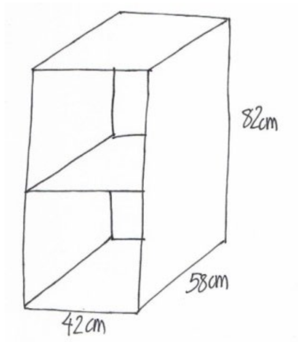

Figure 5. Photo of the customer's original drawing

Following the clerk's suggestions for constructing the closet, the customer considered 2 side boards, 3 horizontal boards (top, shelf, base), and 1 backboard, measuring 56×80 cm, 40×56 cm, and 40×80 cm, respectively. The clerk stated: "We now need to take into account thickness of the pieces and how they will be assembled", and explained that "The pieces chosen allowed two possible assemblies, but neither one would fit into the niche." The first option would maintain the width (40 cm) by placing side pieces on the base and the upper piece on top of them, but this assembly would increase the thickness of the base and top to a height of 83.2 cm (1.6 + 1.6 cm), so the closet would not fit (Figure 6). The shelf would not fit either because it is 40 cm long, while the inside of the closet would measure only 36.8 cm (40 - (2 × 1.6) = 36.8 cm).

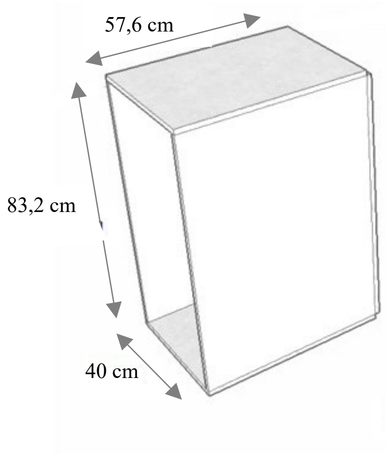

Figure 6. Closet with height greater than the niche. This is an authors' picture that represents the clerk's explanations.

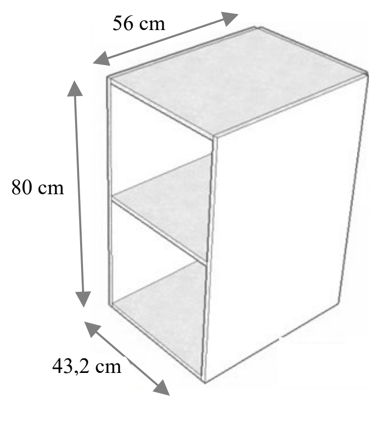

Figure 7. Closet with width greater than the niche. This is an authors' picture that represents the clerk's explanations.

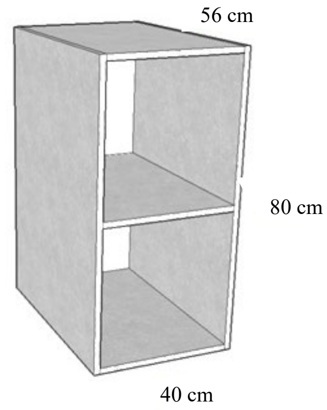

Figure 8. A closet that fits the niche. This is an authors' picture that represents the clerk's explanations.

The second version would maintain the height of 80 cm by inserting the horizontal pieces (top, shelf, base) between the sides, but this would increase the width to 43.2 cm because of the thickness of the sides. Once again, the closet would not fit into the space available (Figure 7). After the above considerations, the clerk clarified the adequate technique to build a closet measuring 40 × 80 × 56 cm (Figure 8): "The most stable structure would have the three horizontal pieces (top, shelf, base) inserted into the side pieces and fastened with screws, with the backboard screwed to the frame to give a better appearance." We interpreted that the clerk's explanations about the assemblage relate to carpentry technology. In Figure 9, we reproduce the required dimensions of the pieces as described by the clerk; this corresponds to a carpentry technique.

2 side pieces of 56 x 80 cm

1 top and 1 base of 36.8 × 56 *cm; 36.8 results from subtracting the thickness of the two sides (1,6 + 1,6 cm) from the niche width (40 cm).*

1 shelf of 36.8 × 54.4 *cm; 54.4 results from subtracting the backboard thickness (1,6 cm) from the niche depth 56 cm.*

1 backboard of 36.8 × 76.8 *cm; 76.8 results from subtracting the thickness of the top and base (1,6 + 1,6 cm) from the niche height.*

Figure 9. Required dimensions of the pieces; this is a faithful reproduction of the clerk's explanations and notes

Once the adequate dimensions of the pieces were identified, the customer calculated the minimum number of quarters of wood-sheet required (optimization task), because it is sold by quarters. He drew the pieces on a layout as shown in Figure 10, and considered that all four quarters of the sheet were required.

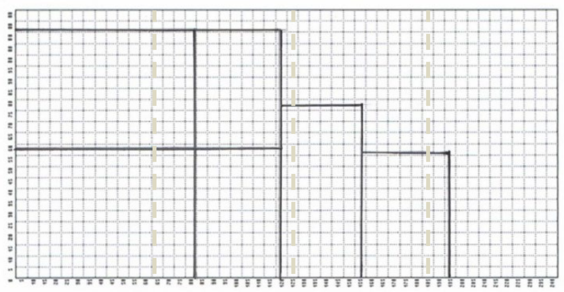

Figure 10. Customer's distribution of the pieces. This is a faithful reproduction of the costumer's drawing.

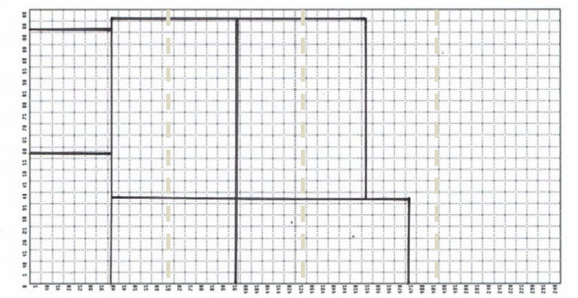

Figure 11. Optimal distribution of the pieces made by the clerk. This is a faithful reproduction of the clerk's drawing.

The clerk, working in parallel, observed that a single sheet would be enough to extract all the pieces. He knew a technique that makes it possible to quickly determine whether costumers need more than one sheet of wood: "If you subtract the total area of the required pieces from the area of the sheet and obtain a positive result, then one sheet would suffice for all required pieces." This technique works as long as no dimension of any piece exceeds those of the sheet. Figure 12 reproduces the clerk's calculations.

Area of the sheet: $1.2\ m \times 2.4\ m = 2.88\ m^2$

Area of all pieces: $2 \times (0.56\ m \times 0.8\ m) + 2 \times (0.368\ m \times 0.56\ m) + 0.368\ m \times 0.544\ m + 0.368\ m \times 0.768m = 1.79\ m^2$

Difference between the areas: $2.88\ m^2 - 1.79\ m^2 = 1.09\ m^2$

Figure 12. Strategy to identify whether one sheet of wood is enough for extracting all the pieces needed. This is a faithful reproduction of the clerk's notes.

To verify whether the customer's solution was the optimal, the clerk utilized another technique. He calculated the percentage of use by relating the percentage of the area of the pieces to that of the sheet: $\frac{1.79}{2.88} \times 100 = 62.15\%$. In this case, the percentage was below 75%, suggesting that all the pieces could be arranged on ¾ of a sheet, rather than the full sheet indicated by the customer. The clerk used his experience to easily distribute the pieces on ¾ of a layout (Figure 11), and thus validating his technique. In this way, he helped the customer save 6.75€. The total cost of the closet was thus 20.25€. To optimize the area, the customer used an overlay technique not supported by any mathematical technology, whereas the clerk used a carpentry technology (calculating the percentage of use) before drawing the pieces on a layout.

The qualitative analysis of the two mixed-praxeologies described above revealed the nature of the tasks involved in the home projects, as well as the mathematical and carpentry techniques employed by the customers and the clerk. Those techniques included calculations (amount and cost of material), identification of 2D shapes (drawing pieces of wood), and decomposition/composition of 3D figures (cutting and assembling wood pieces). These techniques are associated with mathematics and carpentry technologies; for example, knowledge related to units, direct proportionality, optimization, renovating floors, and constructing furniture, among others. To categorize the elements contained in the 312 mixed-praxeologies analyzed, the following section presents a quantitative analysis.

6. Findings of the whole sample of 312 mixed-praxeologies

Table 1 displays the percentages related to the mixed-praxeologies elements. Fifty-five percent of the customers did not adapt their mathematical techniques to obtain optimal solutions: in some cases, their calculations were incorrect, did not minimize the cost, or did not indicate the optimal amount of material. Regarding the three sales areas, 24%, 70%, and 45% of the customers obtained optimal solutions for the tasks in the surface coating, door/window installation, and furniture building/installation, respectively. Non-optimal solutions were categorized into three mixed-praxeologies elements: inadequate adaptation of mathematical techniques to the carpentry tasks; incorrect implementation of mathematical techniques; and inadequate solutions related to a lack of carpentry technology. The inadequate adaptation of the mathematical techniques to the carpentry tasks and the lack of carpentry technology encompassed about 60% of the non-optimal solutions. For the category incorrect implementation of mathematical techniques, we identified two sub-categories: errors related to mathematical technologies and errors related to mathematical techniques. The former was identified only in about 22% of the home projects and related to an incorrect use of mathematical concepts and ideas. The latter was detected in about 32% of the sample, and was associated mainly with errors in performing calculations.

Table 1
Number of home projects per area and percentages of the mixed-praxeologies elements.

Areas of the sales department	# of mixed-praxeologies (Home projects)	Non-optimal solutions to the home projects	Issues related to praxeological elements			
			Inadequate adaptation of mathematical techniques	Incorrect implementation of mathematical techniques		Inadequate solutions related to lack of carpentry technology
				Errors related to mathematical technologies	Errors related to mathematical techniques	
Surface coatings	100	76%	74%	21%	37%	58%
Installing doors/windows	80	30%	67%	17%	33%	50%
Building/installing furniture	132	55%	50%	28%	28%	78%
TOTAL	312	55%				

The analyses showed that a high percentage of customers (around 60%) adapted their mathematical techniques inadequately to the carpentry tasks, while the percentage of errors related to the implementation of such techniques was lower (figures between 22% and 32%). Importantly, the analyses also revealed a high percentage (approximately 60%) of inadequate solutions related to a lack of carpentry technologies. Below, we present some factors affecting customers' inadequate implementation of mathematical techniques and elucidate how the clerk helped them adapt, control, and verify those techniques in the carpentry context. We also illustrate how the customer implemented their mathematical techniques. Finally, we synthetize the carpentry technologies the clerk provided to allow the customers to complete their projects.

Inadequate adaptation of mathematical techniques

Several customers experienced difficulties in adapting mathematical techniques to solving carpentry tasks: for example, calculating the perimeter of rooms without subtracting doorways. The customers in general knew the concept of perimeter and the mathematical technique to calculate it, but did not adapt this technique correctly as they lacked of the required carpentry technologies. To obtain a more accurate solution, they should have known that doorways must be considered when calculating the meters of baseboard needed. Similarly, being aware of carpentry technologies— such as interpreting plans and furniture sketch designs, or knowing about material defects, thickness, friction and assembling —would have allowed customers to achieve more accurate solutions concerning the amount of material needed: to renovate a certain area, to construct a piece of furniture, or to install doors and windows.

Incorrect implementation of mathematical techniques

Most costumers (more than 70%) managed well mathematical techniques and concepts like area, perimeter, volume, symmetry, proportionality, rounding-off, changing units of measurement, and percentages. They also succeeded in converting dimensions in the International System of Units and associating 2D with 3D shapes. At a lower percentage than in the other two categories they implemented some incorrect mathematical techniques (see Table 1). We saw, for instance, customers using incorrect techniques related to proportionality and optimization. Concerning proportionality, they often confused unit cost with unit of measurement when calculating packages of material, as such packages were not sold by exact units of measurement. Regarding optimization, they usually calculated optimal areas, costs, dimensions, and amount of material using techniques that did not support optimization technologies. With respect to calculations, most customers obtained correct results; some errors were observed when they performed operations mentally, but we believe, on the basis of our observations, that they were due to nervousness caused by time pressure and fatigue.

Inadequate solutions related to a lack of carpentry technologies

The analyses revealed inadequate solutions attributable to a lack of familiarity with carpentry technologies. For instance, most customers arrived at inadequate solutions when calculating the amount of material needed for a task, because they did not know the need to add an extra 10% to account for cuts and defects associated with assembling material. As described in the qualitative analyses, this is a general rule to avoid running out of material (e.g., wood, tiles) before a project is complete. In general, the customers did not know how to determine the most stable structures for building furniture, so they required help to calculate the number of pieces needed and the best way to assemble them. In addition to the aforementioned carpentry technologies, in mixed-praxeology 2 we identified that to optimize material using a graph paper, customers learned from the clerk what we call a mathematical technology shaped by the carpentry institution. The clerk taught them a quick way, learned from experience, to identify the proportion of material needed from a standard sheet of wood sold by quarters.

7. Discussion and conclusion

This research contributes to identify the mathematics arising in real-life contexts and to analyze how people use mathematics out of the school context. We examined how customers with middle-high school levels purchased carpentry products to elaborate home projects. The analyses, undertaken through a refined version of the extended praxeological model, demonstrated the complexity of adapting mathematics techniques to solve real-life tasks, even when a professional (in this case the clerk) is revealing step by step the necessary carpentry technologies to solve the tasks. This suggests that despite individuals having relatively well-established mathematical techniques (as the customers showed to have) adapting such techniques in a context distinct from the school setting is a difficult endeavor. Our customers had to learn the particularities of the using institution (the carpentry store), its rules, and rational to establish relations with their school mathematics-knowledge, and adapting it in such specific context. This concurs with Millroy's (1992) deductions that professionals do not use the mathematics taught in school, but a transposed mathematical knowledge that has its own contextual logic.

Our study confirms that people's difficulties when using school mathematics in real-contexts is not an issue related to mastering mathematics knowledge itself, but, rather, a problem caused by their unfamiliarity with basic elements of the using institutions; a fact already suggested by the situated cognition perspective (Lave, 1988). In other words, raising high competent citizens by giving a sound grounding in mathematics is insufficient because solving everyday tasks, that involve mathematics, requires also learning specific technologies related to the contexts. It is for this reason that we believe the current competency-based approach promoted in various educational systems is a chimera. As observed by Diego-Mantecón et al. (2021), to generate an effective approach for developing life-long competences, it is required to reproduce real project conditions. Our outcomes suggest that although several real project tasks are achievable in the school-context, some project development conditions specific to the using institutions — like the way of working, the existing resources, or the didactic processes — are rather difficult to be transposed into the school context.

This matches with Wijaya et al. (2015) and Diego-Mantecón et al. (2019b) when reporting that school math-problems containing illustrations from real life, or referring to everyday phenomena, in no way ensures that students will work on real aspects that emerge in daily-life. Contextual elements of using institutions are normally oversimplified, which confines the acquisition of knowledge to ideal-type mathematics. It is unusual, for example, to find mathematics textbooks or lesson plans containing problems related to a carpentry context where students must consider elements like the thickness of wood, assembling pieces, and waste material, all of which may be required in any carpentry project to obtain adequate solutions. Many initiatives focused on modelling tasks to connect mathematics with real-life situations (e.g., García et al., 2006; Diego-Mantecón et al., 2019a), however these initiatives do not include an analysis of the utilized contexts. This entire idea was emphasized by Jablonka and Gellert (2007) when stressing that schools normally propose mathematization processes that not capture real settings. It is hardly surprising, then, to see that the literature (e.g., OECD, 2013, 2016) so often speaks of citizens' low math competency in real life, leading to criticisms of schools or teachers, when their aim is not focused on training students in real-life contexts, but just on contextualized problems.

According to the above information and the findings of our study, aiming to train mathematical competent citizens in a variety of real-life situations is certainly an illusion. The nature of the tasks identified in this study, and the didactic way in which the clerk guided the customers through the projects, reveal that there are no only complex relationships between school mathematics and outside-school mathematics (as suggested by Lave, 1988; Nunes et al., 1993; Covián & Romo, 2014; and Solares et al., 2016), but there exist also different didactics specific to the contexts. This is a relevant outcome for the mathematics education community because it implies recalling that school teaching is not the only source of mathematics learning, but instead there are many didactics through which human beings have continuously developed contextual and mathematical knowledge. We believe that further research is needed not only for identifying new mathematics related to the contexts, but also the new didactics used in these settings. Hopefully, this future research will open a wider spectrum from which to approach the learning and teaching of school mathematics.

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