The Tukey and the random Tukey depths characterize discrete distributions*

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Abstract

Through this paper it is shown that if the Tukey depths of two probabilities, P and Q, coincide and one of those distributions is discrete, then $P = Q$. The same is proved if the random Tukey depths coincide.

Key words and phrases: Tukey depth, random Tukey depth, one-dimensional projections, multidimensional data, characterization, discrete distributions.
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1 Introduction

In recent times multivariate depths have received some attention from statistical researchers (see, as an example, the recent book [11]). Given a probability distribution P defined on $\mathbb{R}^p, p \geq 1$, a depth aims to order the points in $\mathbb{R}^p$, from the center of P outwards.

If $p = 1$, it seems to be reasonable to use the order induced by the function

$$x \to D_1(x, P) := \min\{P(-\infty, x], P[x, \infty)\}.$$

Thus, the points are ranked following the decreasing order of the absolute values of the differences between their quantiles and .5, the deepest points being the medians of P.

Out of the multidimensional depths, we are here concerned with the *Tukey (or half-space) depth*, which was introduced in [12]. Given $x \in \mathbb{R}^p$, the Tukey depth of x with

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respect to P, $D_T(x, P)$, is the minimal probability which can be attained in the closed halfspaces containing x.

An equivalent definition of $D_T(x, P)$ is the following. Given $v \in \mathbb{R}^p$, let Π_v be the projection of $\mathbb{R}^p$ on the one dimensional subspace generated by v. Thus, $P \circ \Pi_v^{-1}$ is the marginal of P on this subspace, and it is obvious that, if $x \in \mathbb{R}^p$, then

$$D_T(x, P) = \inf\{D_1(\Pi_v(x), P \circ \Pi_v^{-1}) : v \in \mathbb{R}^p\}. \tag{1}$$

Therefore, $D_T(x, P)$ is the infimum of all possible one-dimensional depths of the projections of x, where those depths are computed with respect to the corresponding (one-dimensional) marginals of P.

According to [13], the Tukey depth enjoys very good properties when compared with other depths. Perhaps the main problem of this depth is that it requires an enormous computational effort in order to be computed. As a way to make this problem easier in [14] it is proposed that the infimum in (1) be approximated by the minimum on a randomly chosen finite set of vectors. This idea has been developed in [5] where the *random Tukey depth* is defined as follows:

Definition 1.1 *Let $x \in \mathbb{R}^p$, $k \in I\!N$ and let μ be an absolutely continuous distribution on $\mathbb{R}^p$. The random Tukey depth of x with respect to P based on k random vectors chosen with μ is*

$$D_{T,k,\mu}(x, P) = \min\{D_1(\Pi_{v_i}(x), P \circ \Pi_{v_i}^{-1}) : i = 1, ..., k\}, \ x \in \mathbb{R}^p,$$

where $v_1, ..., v_k$ are independent and identically distributed random vectors with distribution μ.

Obviously, the random Tukey depth is a random quantity. However, if k is big enough, the randomness will be negligible or overcome by other sources of uncertainty. Some hints about how to choose k correctly are given in [5].

It is worth mentioning that paper [4] (extended to Banach spaces in [7]), which shows that a randomly chosen projection allows us to distinguish between two distributions if one of them satisfies a condition on their moments, has triggered some interest in random projections and their applications to several statistical problems (see, for instance, [1, 2, 3, 6, 7, 8]).

It is curious that, in spite of the great interest around the depths in general and the Tukey depth in particular, the authors are not aware of many results proving that a depth determines its corresponding distribution.

In fact, with respect to the Tukey depth, we only know one result by Koshevoy (see [10]) where the author proves that if P and Q are two distributions defined on $\mathbb{R}^p$, both of them with finite support and their Tukey depths coincide, then $P = Q$. An alternative proof to the result of Koshevoy can be found in [9].

In this paper we generalize this result and prove that the Tukey depth characterizes discrete distributions. To be more precise, Theorem 2.6 states that if P is a discrete distribution (with finite or denumerable support) defined on $\mathbb{R}^p$ and Q is a Borel distribution on $\mathbb{R}^p$ such that the functions $D_T(\cdot, P)$ and $D_T(\cdot, Q)$ coincide, then $P = Q$. Thus, this result is slightly more general than Koshevoy's theorem in the sense that it only requires that one distribution be discrete and also includes denumerably supported distributions.

The result is proved, at first, for the random Tukey depth (Theorem 2.3) and then, a simple extension allows the Tukey depth to be covered. Notice that the independence assumption in Definition 1.1 is not required in Theorem 2.3.

We are not aware of any result generalizing this characterization to the continuous case although we conjecture that it should remain valid.

2 Main Results

We will employ the following notation. The unit sphere in $\mathbb{R}^p$ will be denoted by $\mathbb{S}^{p-1}$, σ_{p-1} will be the geometrical measure on $\mathbb{S}^{p-1}$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^p$. Given a set B, B^o will be its topological interior and B^c its complementary.

If $x \in \mathbb{R}^p$, $v \in \mathbb{S}^{p-1}$ and P is a Borel probability distribution on $\mathbb{R}^p$, let

$$\begin{aligned} H_{x,v} &:= \{y \in \mathbb{R}^p : \langle y - x, v \rangle = 0\} \\ A_x^P &:= \{v \in \mathbb{S}^{p-1} : P(H_{x,v}) > P(x)\} \\ c_{x,v}^P &:= \begin{cases} 1 & \text{if } P(\{y \in \mathbb{R}^p : \langle y - x, v \rangle \geq 0\}) \leq P(\{y \in \mathbb{R}^p : \langle y - x, v \rangle \leq 0\}) \\ -1 & \text{otherwise} \end{cases} \\ S_{x,v}^P &:= \{y \in \mathbb{R}^p : c_{x,v}^P \langle y - x, v \rangle \geq 0\}. \end{aligned}$$

To simplify, if there is no risk of confusion, the super-index P will be omitted. With this notation, we have $D_1(\Pi_v(x), P \circ \Pi_v^{-1}) = P(S_{x,v})$.

Given $V \subset \mathbb{S}^{p-1}$ and $x \in \mathbb{R}^p$, let us define $D_V(x, P) := \inf_{v \in V} D_1(\Pi_v(x), P \circ \Pi_v^{-1})$. Thus, if P is a discrete distribution and Z is its support, we have that

$$D_V(x, P) := \inf_{v \in V} \min \left(\sum_{z \in Z : \Pi_v(z) \leq \Pi_v(x)} P(z), \sum_{z \in Z : \Pi_v(z) \geq \Pi_v(x)} P(z) \right).$$

In the case that V contains a single element v, we will write $D_v(x, P)$. Notice that, if V is a set composed by k independent and indentically distributed randomly chosen vectors using the distribution μ, then $D_V(x, P) = D_{T,k,\mu}(x, P)$. In addition, the Tukey depth of a point x with respect to P coincides with $D_{\mathbb{S}^{p-1}}(x, P)$.

Two auxiliary results follow. They require no assumption on P.

Proposition 2.1 *If P is a Borel distribution on $\mathbb{R}^p$ and $x \in \mathbb{R}^p$, then $\sigma_{p-1}(A_x) = 0$.*

PROOF.- If $p = 1$ the result is trivial because $H_{x,v} = \{x\}$ for every $v \in \mathbb{S}^{p-1}$. Thus, let us assume that $p > 1$ and, also, that, on the contrary, $\sigma_{p-1}(A_x) > 0$. Thus, there exists $\alpha > 0$ such that if we denote

$$A_x^* := \{v \in \mathbb{S}^{p-1} : P(H_{x,v}) > P(x) + \alpha\},$$

then $\sigma_{p-1}(A_x^*) > 0$. Every sequence $\{v_n\} \subset A_x^*$ contains at least a couple of elements v_{i_1}, v_{i_2} such that $P(H_{x,v_{i_1}} \cap H_{x,v_{i_2}}) > P(x)$ because, if not, we would have

$$P(\cup_n H_{x,v_n}) = P(x) + P(\cup_n (H_{x,v_n} - \{x\})) = P(x) + \sum_n P(H_{x,v_n} - \{x\}) = \infty.$$

From here, the proof is ready if $p = 2$ because we can choose a sequence in A_x^* such that all their components are pairwise linearly independent and, then $H_{x,v_{i_1}} \cap H_{x,v_{i_2}} = \{x\}$.

Thus, let us assume that $p > 2$. Let us fix an hyperplane $H \subset \mathbb{R}^p$ such that $0 \in H$. Let Π_H be the projection map from $\mathbb{R}^p$ on H and S^H be the unit sphere in H. Given $h \in S^H$, let $A_{x,h}^* = \{v \in A_x^* : \Pi_H(v) = \lambda h, \text{ for some } \lambda \in \mathbb{R}^+\}$. By Fubini's theorem we have that

$$0 < \sigma_{p-1}(A_x^*) = \int_{S^H} \sigma_1(A_{x,h}^*) \sigma_{p-2}(dh).$$

Therefore, we have that $\sigma_{p-2}\{h \in S^H : \sigma_1(A_{x,h}^*) > 0\} > 0$, and, there exists $H^* \subset S^H$ with $\sigma_{p-2}(H^*) > 0$ such that for every $h \in H^*$ there exists a sequence $\{v_n^h\}_{n \in \mathbb{N}} \subset A_{x,h}^*$ composed of pairwise linearly independent vectors. Since $A_{x,h}^* \subset A_x^*$, each of those sequences contains a pair of vectors $v_{n_1}^h$ and $v_{n_2}^h$ such that

$$P\left(H_{x,v_{n_1}^h} \cap H_{x,v_{n_2}^h}\right) > P(x).$$

Thus, there exists $\beta > 0$ such that if we denote

$$H^\beta := \left\{h \in H^* : P\left(H_{x,v_{n_1}^h} \cap H_{x,v_{n_2}^h}\right) > P(x) + \beta\right\},$$

then, $\sigma_{p-2}\left(H^\beta\right) > 0$.

Now, repeating the same reasoning as above, we have that for every sequence of $\{h_k\}_{k \in \mathbb{N}} \subset H^\beta$ there exists, at least, a couple h, h^* such that

$$P\left[\left(H_{x,v_{n_1}^h} \cap H_{x,v_{n_2}^h}\right) \cap \left(H_{x,v_{n_1}^{h^*}} \cap H_{x,v_{n_2}^{h^*}}\right)\right] > P(x).$$

Moreover, by the construction of the sequences, it turns out that the dimension of $H_{x,v_{n_1}^h} \cap H_{x,v_{n_2}^h}$ is $p-2$ and if we choose h and h^* linearly independent, then the dimension of $\left(H_{x,v_{n_1}^h} \cap H_{x,v_{n_2}^h}\right) \cap \left(H_{x,v_{n_1}^{h^*}} \cap H_{x,v_{n_2}^{h^*}}\right)$ is $p-3$. Thus the problem is solved if $p = 3$.

If $p > 3$, we only have to apply the previous reasoning to H^β and the problem will be solved if $p = 4$. If not, we will obtain a new set, whose dimension is a unit less. Thus, since the dimension is finite, we only need to repeat the process a finite number of times to get a contradiction. •

Lemma 2.2 *Let $x \in \mathbb{R}^p$ and $\{x_n\} \subset \mathbb{R}^p$ a sequence such that $x_n \neq x$ for all $n \in \mathbb{N}$ and $\lim_n x_n = x$. Then, if $V \subset A_x^c$, we have*

$$\liminf_n D_V(x_n, P) \geq D_V(x, P) - P(x). \tag{2}$$

PROOF.- Let $\{u_n\} \subset V$ be such that $\lim_n (D_{u_n}(x_n, P) - D_V(x_n, P)) = 0$. By the definition of A_x^c we have that

$$P(H_{x,u_n}) = P(x), \text{ for all } n \in I\!N. \tag{3}$$

To obtain the result it is sufficient to show that every subsequence $\{x_{n_k}\}$ contains a further subsequence which satisfies (2). To do this, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$.

Let $z, z' \in \mathbb{R}^p$, $z \neq z'$ and $u \in \mathbb{S}^{p-1}$. It is impossible that $S_{z,u} \cap S_{z',u} = H_{z,u}$ by the definition of those sets. Thus we get that $S_{z,u} \subset S_{z',u}$; $S_{z',u} \subsetneq S_{z,u}$ or $S_{z,u} \cap S_{z',u} = \emptyset$. Therefore the subsequence $\{x_{n_k}\}$ contains a subsequence $\{x_{n_k^*}\}$ which satisfies one of the following statements

A $S_{x,u_{n_k^*}} \subset S_{x_{n_k^*},u_{n_k^*}}$, for every $k \in I\!N$.

B $S_{x_{n_k^*},u_{n_k^*}} \subsetneq S_{x,u_{n_k^*}}$, for every $k \in I\!N$.

C $S_{x_{n_k^*},u_{n_k^*}} \cap S_{x,u_{n_k^*}} = \emptyset$, for every $k \in I\!N$.

If (**A**) is satisfied, then $S_{x_{n_k^*},u_{n_k^*}} = S_{x,u_{n_k^*}} \cup (S^c_{x,u_{n_k^*}} \cap S_{x_{n_k^*},u_{n_k^*}})$. The fact that $\lim_k x_{n_k^*} = x$ implies that $\lim_k P\left(S^c_{x,u_{n_k^*}} \cap S_{x_{n_k^*},u_{n_k^*}}\right) = 0$. Thus, $\liminf_k D_{u_{n_k^*}}(x_{n_k^*}, P) = \liminf_k D_{u_{n_k^*}}(x, P)$. And, from the definition of $\{u_n\}$, and the definition of depth we deduce that

$$\liminf_k D_V(x_{n_{k^*}}, P) \geq D_V(x, P) \geq D_V(x, P) - P(x).$$

In the case that (**B**) holds, we have that $\lim_k P\left(S^o_{x,u_{n_k^*}} \cap S^c_{x_{n_k^*},u_{n_k^*}}\right) = 0$ and then,

$$\begin{aligned}\liminf_k D_V(x_{n_k^*}, P) &= \liminf_k \left(D_{u_{n_k^*}}(x, P) - P(H_{x,u_{n_k^*}}, P)\right) \\ &= \liminf_k D_{u_{n_k^*}}(x, P) - P(x) \geq D_V(x, P) - P(x),\end{aligned}$$

where the second equality is due to (3).

If the subsequence verifies (**C**) we have that $\lim_k P\left(S^c_{x,u_{n_k^*}} \cap S^c_{x_{n_k^*},u_{n_k^*}}\right) = 0$ and then

$$\begin{aligned}\liminf_k D_V(x_{n_k^*}, P) &= \liminf_k D_{u_{n_k^*}}(x_{n_k^*}, P) = \liminf_k P(S^c_{x,u_{n_k^*}}) \\ &\geq \liminf_k \left(P(S_{x,u_{n_k^*}}) - P(H_{x,u_{n_k^*}})\right) \geq D_V(x, P) - P(x),\end{aligned}$$

where the first inequality is due to the definition of $S_{x,u_{n_k^*}}$, while the second one comes from the definition of depth and (3). •

Now, we are in position to prove the characterization result for random depths.

Theorem 2.3 *Let P and Q be two probability measures. Assume that the support of P is at most denumerable. Let V be a set at most denumerable of identically distributed* *random vectors $v : \Omega \to \mathbb{S}^{p-1}$ with distribution μ, absolutely continuous with respect to σ_{p-1}, defined on the probability space (Ω, σ, κ). Let*

$$\Omega_0 := \{\omega \in \Omega : D_{V(\omega)}(x, P) = D_{V(\omega)}(x, Q), \quad \textit{for every } x \in \mathbb{R}^p\}.$$

Then $\kappa(\Omega_0) \in \{0, 1\}$, and $\kappa(\Omega_0) = 1$ if and only if $P = Q$.

PROOF.- Obviously, if $P = Q$, then $\Omega_0 = \Omega$. Thus, the result will be proved if we show that $\kappa(\Omega_0) > 0$ implies that $P = Q$. Therefore, let us assume that $\kappa(\Omega_0) > 0$.

Let Z be the support of P. To prove the theorem, it is enough to check that $P(z) = Q(z)$ for every $z \in Z$. The proof will be based on the following lemma.

Lemma 2.4 *Let us assume the hypothesis of Theorem 2.3. Let $z \in Z$. Let us define $\Omega_z^P = \{\omega \in \Omega : V(\omega) \cap A_z^P = \emptyset\}$ and similarly for Q. If $\Omega_0 \cap \Omega_z^P \cap \Omega_z^Q \neq \emptyset$, then*

$$P(z) \leq Q(z).$$

If we assume the claim and $z \in Z$, since μ is absolutely continuous with respect to σ_{p-1} and V is at most denumerable, from Proposition 2.1, we have that $\kappa(\Omega_z^P) = \kappa(\Omega_z^Q) = 1$.

Thus, $\Omega_0 \cap \Omega_z^P \cap \Omega_z^Q \neq \emptyset$, and, from the claim, we obtain $P(z) \leq Q(z)$, which implies $P = Q$ because if there were a $z \in Z$ such that the inequality were strict, we would have the contradiction

$$1 = \sum_{z \in Z} P(z) < \sum_{z \in Z} Q(z) \leq 1.$$

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PROOF OF THE LEMMA 2.4. Let $z \in Z$ and $\omega \in \Omega$. Let $\{v_n(\omega)\} \subset V(\omega)$ be such that

$$\lim_n D_{v_n(\omega)}(z, P) = D_{V(\omega)}(z, P). \tag{4}$$

In what follows the symbol ω will be omitted in the notation.

As $\mathbb{S}^{p-1}$ is a compact set and $V \subset \mathbb{S}^{p-1}$, there exists a subsequence $\{v_{n_k}\}$, $v_z \in \mathbb{S}^{p-1}$ and $c \in \{-1, 1\}$ such that $\lim_n v_{n_k} = v_z$ and that $c^P_{z, v_{n_k}} = c$, for every $k \in I\!N$. Without loss of generality we can identify $\{v_{n_k}\}$ with $\{v_n\}$.

Let $\{z_n\} \subset (S^P_{z, v_z})^o$ such that $\lim_n z_n = z$. As $z_n \notin H_{z, v_z}$, it happens that $S^P_{z_n, v_z} \subsetneq S^P_{z, v_z}$, for every $n \in I\!N$ and so, that

$$P(S^P_{z_n, v_z}) \leq P(S^P_{z, v_z}) - P(H_{z, v_z}), \text{ for all } n \in I\!N.$$

Then,

$$\limsup_n D_{v_z}(z_n, P) \leq D_{v_z}(z, P) - P(H_{z, v_z}). \tag{5}$$

Denoting $S := \{y \in \mathbb{R}^p : c\langle y-z, v_z\rangle \geq 0\}$ and taking into account that $P(S) \geq P(S^P_{z,v_z})$ and (4), we get

$$\begin{aligned} D_{v_z}(z,P) - D_V(z,P) &\leq \lim_n (P(S) - P(S^P_{z,v_n})) &(6)\\ &= \lim_n \left(P(S^o \cap (S^P_{z,v_n})^c) + P(H_{z,v_z} \cap (S^P_{z,v_n})^c) - P(S^P_{z,v_n} \cap S^c)\right). \end{aligned}$$

Note that

$$\begin{aligned} \lim_n P(S^P_{z,v_n} \cap S^c) &= \lim_n P(\{y \in \mathbb{R}^p\colon\ c\langle y-z, v_n\rangle \geq 0,\ c\langle y-z, v_z\rangle < 0\})\\ &= P(\{y \in \mathbb{R}^p :\ 0 > c\langle y-z, v_z\rangle \geq 0\}) = 0. \end{aligned}$$

Analogously it is shown that $\lim_n P(S^o \cap (S^P_{z,v_n})^c) = 0$.

As $z \notin (S^P_{z,v_n})^c$, then $P((S^P_{z,v_n})^c \cap H_{z,v_z}) \leq P(H_{z,v_z}) - P(z)$. From here and (6),

$$D_{v_z}(z,P) - P(H_{z,v_z}) \leq D_V(z,P) - P(z). \tag{7}$$

Because of the definition of depth $D_V(z_n,P) \leq D_{v_z}(z_n,P)$. This, (5) and (7) imply that

$$\limsup_n D_V(z_n,P) \leq D_V(z,P) - P(z). \tag{8}$$

Remember that $V = V(\omega)$ is a random set and that (8) holds for every $\omega \in \Omega$. Now let us take $\omega \in \Omega_0 \cap \Omega^P_z \cap \Omega^Q_z$. Thus by Lemma 2.2, it happens that

$$\begin{aligned} \lim_n D_V(z_n,P) &= D_V(z,P) - P(z) &(9)\\ \liminf_n D_V(z_n,Q) &\geq D_V(z,Q) - Q(z). &(10) \end{aligned}$$

By the definition of Ω_0, $D_V(z_n,P) = D_V(z_n,Q)$ for all $n \in I\!N$ and $D_V(z,P) = D_V(z,Q)$. From here, (9) and (10), we obtain that $P(z) \leq Q(z)$.

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We end the paper with a result which generalizes the main result in [10]. Its proof follows closely the one given for Theorem 2.3 after the following technical lemma.

Lemma 2.5 *Let P be a probability distribution and let $x \in \mathbb{R}^p$. If $V \subset (A_x)^c$ is a dense set in $\mathbb{S}^{p-1}$, then*

$$D_V(x,P) = D_{\mathbb{S}^{p-1}}(x,P).$$

PROOF.- Let $v_0 \in A_x$. By definition of A_x we have that $P(H_{x,v_0}) > P(x)$. Let $w_1, ..., w_{p-1} \in \mathbb{R}^p$ such that $v_0, w_1, ..., w_{p-1}$ is an orthogonal basis. Since

$$H_{x,v_0} = \{y \in H_{x,v_0} : c_{x,v_0}\langle y-x, w_1\rangle \geq 0\} \cup \{y \in H_{x,v_0} : c_{x,v_0}\langle y-x, w_1\rangle \leq 0\},$$

we have that

$$P\{y \in H_{x,v_0} : c_{x,v_0}\langle y-x, w_1\rangle \geq 0\} > P(x)$$

or, else,

$$P\{y \in H_{x,v_0} : c_{x,v_0}\langle y - x, w_1\rangle \leq 0\} > P(x).$$

Without loss of generality, we can assume that the second inequality holds. Repeating the same reasoning, we can also assume that

$$P\left[\cap_{i=1}^{p-1}\{y \in H_{x,v_0} : c_{x,v_0}\langle y - x, w_i\rangle \leq 0\} - \{x\}\right] > 0.$$

On the other hand, the set $W^- := \{v \in \mathbb{R}^p : \langle v, w_i\rangle < 0, i = 1, ..., p-1\}$ is open in $\mathbb{R}^p$. Since V is a dense set and v_0 belongs to the topological boundary of W^-, there exists $\{v_n\} \subset W^-$ which converges to v_0. This sequence satisfies that

$$\begin{aligned} D_{\mathbb{S}^{p-1}}(x, P) &\leq \liminf_n D_{v_n}(x, P) \leq \liminf_n P(\{y \in \mathbb{R}^p : c_{x,v_0}\langle y - x, v_n\rangle \geq 0\}) \\ &\leq P\left[S_{x,v_0} - \left(\cap_{i=1}^{p-1}\{y \in H_{x,v_0} : c_{x,v_0}\langle y - x, w_i\rangle \leq 0\} - \{x\}\right)\right] \\ &< P(S_{x,v_0}) = D_{v_0}(x, P), \end{aligned}$$

the result following on from here. •

Theorem 2.6 *Let P and Q be two probability measures such that P is discrete and that for any $x \in \mathbb{R}^p$, $D_{\mathbb{S}^{p-1}}(x, P) = D_{\mathbb{S}^{p-1}}(x, Q)$. Then $P = Q$.*

PROOF.- Let $z \in Z$, with Z being the support of P. From Proposition 2.1, it is obvious that $V_{P,Q} = (A_z^P)^c \cap (A_z^Q)^c$ is a dense subset of $\mathbb{S}^{p-1}$, which satisfies Lemma 2.5 for P and Q.

Moreover, we can consider that the set $V_{P,Q}$ is composed by a family of (non identically distributed) random vectors with constant values equal to each element in this set. Let us denote (Ω, σ, κ) to the probability space in which those random vectors are defined.

Obviously, $\kappa\{\omega \in \Omega : V_{P,Q}(\omega) \cap A_z^P = \emptyset\} = \kappa\{\omega \in \Omega : V_{P,Q}(\omega) \cap A_z^Q = \emptyset\} = 1$, and, from this point on, we can repeat the proof of Theorem 2.3 to obtain the result because, in the proof of this theorem we only required that the set V from there be denumerable and of identically distributed random vectors in order to guarantee that $\kappa(\Omega_z^P) = \kappa(\Omega_z^Q) = 1$.•

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