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ESTRUCTURAS GEOMÉTRICAS EN GEOMETRÍA GENERALIZADA

PHD THESIS

GEOMETRIC STRUCTURES IN GENERALIZED GEOMETRY

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A mis padres, mi hermana e Inés.

Yo era como el geómetra empeñado
en mesurar el círculo, que piensa
una vez y otra vez sin resolverlo,
y así, delante de una visión nueva,
quería ver el modo en que la imagen
cabía y se encajaba en aquel círculo.
Pero mis propias alas no bastaban.
Y entonces un fulgor golpeó mi mente
y dio satisfacción a mi deseo.

DANTE ALIGHIERI, Comedia: Paraíso, canto XXXIII

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Global abstract

This PhD thesis lies within the field of generalized geometry. This area studies the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M \to M$ of a smooth manifold, defined as the Whitney sum of its tangent and cotangent bundles.

The first investigations into generalized geometry focused on generalized almost complex structures $\mathcal{J}$, which assign to each section of the generalized tangent bundle another section such that $\mathcal{J}^2 = -\mathcal{I}d$. These morphisms allowed for a reinterpretation and unification of classical almost complex and almost symplectic structures, in the sense that structures of either type on the base manifold M induce generalized almost complex structures on the vector bundle $\mathbb{T}M \to M$. Later, other authors proposed similar morphisms, such as generalized almost paracomplex structures. All of these objects are referred to as generalized polynomial structures. In their original formulations, these morphisms were required to be compatible with the canonical metric on $\mathbb{T}M$, which associates a function on the base manifold M to each pair of sections of the bundle. However, when this compatibility condition is relaxed, new structures arise that broaden the scope of generalized geometry.

In the first part of this thesis, we make a comprehensive study of various generalized geometric structures, aiming to organize those introduced previously by other authors and to present new structures. To distinguish between those that are compatible with the canonical metric and those that are not, we use the terms "strong" and "weak", respectively. This approach reveals that generalized geometry not only unifies classical almost complex and almost symplectic geometry into a single framework but also encompasses a wider range of classical geometries: metrics, almost symplectic structures, and polynomial structures induce multiple morphisms on the generalized tangent bundle. We also characterize certain relevant structures, such as weak generalized metrics and weak generalized symplectic structures, through injective endomorphisms of $\mathbb{T}M$. In addition, we examine the compatibility of different types of structures, such as the commutation or anti-commutation of weak generalized polynomial structures, or the compatibility of these structures with metrics on $\mathbb{T}M$ that are different from the canonical one.

During the literature review, we identified a general lack of explicit examples, as most studies in the field introduce structures without referencing any particular manifold. Consequently, the second part of the thesis focuses on constructing explicit examples for specific smooth manifolds. We study the integrability of these examples, which requires the use of the Dorfman bracket defined on $\mathbb{T}M$ and the analysis of the involutivity of certain subbundles associated with each example.

The first case study concerns the six-dimensional sphere. It is well known that the only spheres that admit classical almost complex structures are the two-dimensional sphere $\mathbb{S}^2$ and the six-dimensional sphere $\mathbb{S}^6$. Although an integrable complex structure is known to exist on $\mathbb{S}^2$, it

remains unknown whether $\mathbb{S}^6$ admits such a structure. This question is known as the “Hopf problem”. In this dissertation, we formulate an analogous version of the Hopf problem in the context of generalized geometry. Following the spirit of partial answers offered in the classical case, we prove that there is not any integrable spherical combination of the generalized almost complex structures directly induced by the pure octonion product on $\mathbb{R}^7$.

The second scenario focuses on the existence of strong generalized complex structures on eight-dimensional nilmanifolds. The integrability of such structures can be approached through the Lie algebras associated with these homogeneous spaces. In dimension eight, the two-step nilpotent Lie algebras have been recently classified. Based on this classification and inspired by other works on six-dimensional nilmanifolds, we construct a large number of strong generalized complex structures on eight-dimensional nilmanifolds associated with these Lie algebras. Among other results, we prove that every two-step eight-dimensional nilmanifold admits either a complex structure or a symplectic structure. Moreover, we construct strong generalized complex structures that are not induced by either a complex or symplectic structure on the base manifold.

Resumen global

La presente tesis se enmarca en el área de la geometría generalizada. Este campo toma como objeto de estudio el fibrado tangente generalizado $\mathbb{T}M := TM \oplus T^*M \to M$ de una variedad diferenciable, definido como la suma de Whitney del fibrado tangente y el cotangente de la variedad.

Las primeras investigaciones en geometría generalizada se centraron en las estructuras casi complejas generalizadas $\mathcal{J}$, que hacen corresponder a cada sección del fibrado tangente generalizado otra sección de modo que $\mathcal{J}^2 = -\mathcal{I}d$. Estos morfismos permitieron reinterpretar y unificar las estructuras casi complejas y casi simplécticas clásicas, en el sentido de que las estructuras de ambos tipos en la variedad base M definen estructuras casi complejas generalizadas en el fibrado $\mathbb{T}M \to M$. Posteriormente, otros autores propusieron otros morfismos similares, tales como las estructuras casi paracomplejas generalizadas. A todos estos objetos se les llama estructuras polinómicas generalizadas. En sus definiciones originales, a todos estos morfismos se les pidió ser compatibles con la métrica canónica existente en $\mathbb{T}M$, que asocia a cada par de secciones del fibrado una función en la variedad base M. Sin embargo, si se omite esta condición de compatibilidad, aparecen estructuras nuevas que permiten ampliar el alcance de la geometría generalizada.

En la primera parte de esta tesis hacemos un amplio estudio de diferentes estructuras geométricas generalizadas, tratando de organizar las introducidas por otros autores y presentando nuevas estructuras. Para diferenciar aquellas que son compatibles con la métrica canónica de las que no lo son, utilizamos los adjetivos “fuerte” y “débil”, respectivamente. Trabajar desde este punto de vista nos permite comprobar que la geometría generalizada no solo tiene la capacidad para unificar la geometría casi compleja y la casi simpléctica en un único tipo de estructura, sino que admite muchas más geometrías clásicas: las métricas, las estructuras casi simplécticas y las estructuras polinómicas inducen una gran variedad de morfismos en el fibrado tangente generalizado, tanto ellas aisladas como cuando se combinan entre sí. También caracterizamos algunas estructuras relevantes, tales como las métricas generalizadas débiles y las estructuras simplécticas generalizadas, a través de ciertos endomorfismos inyectivos de $\mathbb{T}M$. Además, estudiamos la compatibilidad entre las diferentes estructuras introducidas, como puede ser la conmutación o anti-conmutación de algunas estructuras polinómicas generalizadas débiles, o la compatibilidad de las estructuras polinómicas generalizadas débiles con otras métricas diferentes a la canónica sobre $\mathbb{T}M$.

Durante el proceso de revisión de literatura, se ha detectado una falta general de ejemplos, ya que la mayoría de las estructuras que normalmente se trabajan se introducen sin hacer referencia a ninguna variedad en particular. Por ello, en la segunda parte de la tesis hacemos énfasis en la búsqueda de ejemplos explícitos para ciertas variedades diferenciables. La búsqueda de estos ejemplos se enmarca dentro del contexto de la integrabilidad de estructuras, para lo cual se requiere utilizar

el corchete de Dorfman definido sobre $\mathbb{T}M$ y trabajar la involutividad de ciertos subfibrados asociados a cada ejemplo.

El primer ejemplo trabajado está relacionado con la esfera de dimensión seis. Es un hecho consabido que las únicas esferas que admiten estructuras casi complejas clásicas son la esfera de dimensión dos $\mathbb{S}^2$ y la de dimensión seis $\mathbb{S}^6$. Si bien se conoce una estructura compleja integrable para $\mathbb{S}^2$, todavía no se sabe si $\mathbb{S}^6$ admite este tipo de estructuras. A este problema se le conoce como "problema de Hopf". Durante la investigación, se ha planteado una versión análoga a este problema en geometría generalizada. Siguiendo la línea de aquellos resultados que ofrecen respuestas parciales al problema de Hopf, demostramos que no existe ninguna combinación esférica de las estructuras casi complejas generalizadas que se inducen de manera directa a partir del producto de los octoniones puros en $\mathbb{R}^7$ de modo que dicha combinación sea integrable.

El otro ejemplo trabajado es la existencia de estructuras complejas generalizadas fuertes en nilvariedades de dimensión ocho. La integrabilidad de este tipo de estructuras se puede trabajar a través de las álgebras de Lie asociadas a estos espacios homogéneos. En dimensión ocho, recientemente han sido clasificadas las álgebras de Lie con paso dos. Siguiendo esta clasificación, e inspirados en el trabajo de otros autores para nilvariedades de dimensión seis, construimos un gran número de estructuras complejas generalizadas fuertes con las que dotar a las nilvariedades de dimensión ocho asociadas a estas álgebras de Lie. Entre otras cosas, demostramos que cualquier nilvariedad de dimensión ocho y paso dos puede ser dotada de una estructura compleja o una estructura simpléctica. Además, construimos estructuras complejas generalizadas fuertes que no están inducidas por estructuras complejas o simplécticas en la variedad.

Chapter 1

Introduction

1.1 General framework

Generalized geometry is a branch of differential geometry that focuses on the study of the generalized tangent bundle of a manifold. This vector bundle is defined as the Whitney sum of its tangent bundle and cotangent bundle, that is, $\mathbb{T}M := TM \oplus T^*M \to M$. One of the referring authors in the study of this mathematical object is T. J. Courant. In 1990 (see [16]), he checked that for any vector space V there is a natural symmetric pairing defined on $V \oplus V^*$ as

$$\langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle := \frac{1}{2} \left(\alpha_1(v_2) + \alpha_2(v_1) \right),$$

for $v_1 + \alpha_1, v_2 + \alpha_2 \in V \oplus V^*$. As the fibers $\mathbb{T}_pM$ of $\mathbb{T}M$ at each point $p \in M$ are equal to the direct sum of its tangent and cotangent spaces at p (that is, $\mathbb{T}_pM = T_pM \oplus T_p^*M$), it is easily surmised that this pairing can be translated into the generalized tangent bundle of a manifold in the form of a symmetric nondegenerate pairing $\mathcal{G}_0 \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \mathfrak{F}(M)$, which associates to each pair of sections of $\mathbb{T}M$ a smooth function on M.

Although it is true that Courant studied the generalized tangent bundle of a manifold before, the first to introduce the term "generalized geometry" was N. Hitchin in 2003 (see [33]). Then, his disciples G. Cavalcanti and M. Gualtieri further developed the foundations of this area in their respective PhD thesis ([11] and [29]).

Generalized almost complex structures were the first subject of study in generalized geometry. These geometric objects were initially defined as endomorphisms $\mathcal{J} \colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ such that, in a fashion similar to almost complex structures defined on a manifold, they must meet the equality $\mathcal{J}^2 = -\mathcal{I}d$. However, there is an additional requirement that draws a distinction between almost complex structures on a manifold and generalized almost complex structures: the latter were required to be compatible with the pairing $\mathcal{G}_0$. For every $u, v \in \Gamma(\mathbb{T}M)$, this compatibility condition is

$$\mathcal{G}_0(\mathcal{J}u, v) = -\mathcal{G}_0(u, \mathcal{J}v),$$

that is, $\mathcal{J}$ was requested to be skew-symmetric with respect to $\mathcal{G}_0$.

One point of interest of these structures on $\mathbb{T}M$ is that both almost complex structures and almost symplectic structures on the manifold M induce generalized almost complex structures. If $J\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an almost complex structure on a manifold M, then it generates a generalized almost complex structure $\mathcal{J}_J\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ in the following way: if we write an element of $\Gamma(\mathbb{T}M)$ as a sum of a vector field $X \in \mathfrak{X}(M)$ and a 1-form $\xi \in \Omega^1(M)$, then $\mathcal{J}_J$ is defined as

$$\mathcal{J}_J(X + \xi) = JX - J^*\xi,$$

where $J^*\colon \Omega^1(M) \to \Omega^1(M)$ denotes the dual structure of J, such that for every $Y \in \mathfrak{X}(M)$ we have $(J^*\xi)Y = \xi(JY)$. Observe that there is an essential difference between this notion and that of the lift of an almost complex structure J on M to its tangent or cotangent bundles (see [61]), because in this last case the lifted almost complex structure is an almost complex structure on the manifold TM (resp. T^*M).

On the other hand, for an almost symplectic structure $\omega \in \Omega^2(M)$ one can induce the generalized almost complex structure $\mathcal{J}_\omega\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined as

$$\mathcal{J}_\omega(X + \xi) = -\sharp_\omega\xi + \flat_\omega X.$$

Here, the morphism $\flat_\omega\colon \mathfrak{X}(M) \to \Omega^1(M)$ is given by $(\flat_\omega X)Y = \omega(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$, and $\sharp_\omega = \flat_\omega^{-1}$. In this way, almost complex and almost symplectic structures can be thought of as similar objects when working on the generalized tangent bundle.

The study of this area has been useful in the development of different scientific fields. For example, generalized geometry has been used in physics as a mathematical framework for some recent advances in string theory and supergravity. In particular, it has been shown to be useful for describing supersymmetric flux compactifications and the supersymmetric embedding of D-branes (see, for example, [37, 41]). This bundle has also appeared under different names in the literature, including "big tangent bundle" as introduced by I. Vaisman in 2015 (see [59]), and "Pontryagin bundle" as used by other researchers (see, for example, [53]).

After the first contributions by Gualtieri, similar structures on $\mathbb{T}M$ were proposed by other scholars. For example, A. Wade defined in [60] the concept of generalized almost paracomplex structures as endomorphisms $\mathcal{F}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ so that $\mathcal{F}^2 = \mathcal{I}d$. These structures were also requested to be compatible with the pairing $\mathcal{G}_0$ in the same way as generalized complex structures, that is, $\mathcal{G}_0(\mathcal{F}u, v) = -\mathcal{G}_0(u, \mathcal{F}v)$. Similarly, she showed that both almost product structures and almost symplectic structures induce interesting examples of generalized almost paracomplex structures. More explicitly, an almost product structure $F\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ induces the generalized almost paracomplex structure $\mathcal{F}_F\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ given by

$$\mathcal{F}_F(X + \xi) = FX - F^*\xi,$$

while an almost symplectic structure $\omega \in \Omega^2(M)$ induces the generalized almost paracomplex structure $\mathcal{F}_\omega \colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ given by

$$\mathcal{F}_\omega(X + \xi) = \sharp_\omega \xi + \flat_\omega X.$$

Both generalized almost complex and almost paracomplex structures have been widely studied since then (e.g., by E. A. Fernández-Culma, Y. Godoy and M. Salvai in [25]; and by A. Fino and F. Paradiso in [26]). In other recent studies, such as [4] by M. Aldi and D. Grandini, more diverse endomorphisms of sections of the generalized tangent bundle have been examined, which are also required to be compatible with $\mathcal{G}_0$.

However, a different line of thought was proposed by other researchers, such as A. Nannicini (see, for example, her works [48–52]), or C. Ida and A. Manea (see [38]). This position sustains that the compatibility condition with $\mathcal{G}_0$ may be omitted in order to study a wider range of interesting geometric structures. This line of reasoning agrees with other texts specialized in vector bundles, such as [54] by W. A. Poor, or [47] by J. W. Milnor and J. D. Stasheff. In these books, diverse geometric objects are defined given any vector bundle over a manifold $E \to M$, such that when the tangent bundle $E = TM$ is considered, we recover the well-known geometric structures on the manifold. For example, an almost complex structure on a vector bundle E is defined as a bundle endomorphism $J \colon E \to E$ such that $J^2 = -Id$ ([54, Definition 1.58]) and the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{J} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow[id]{} & M \end{array} .$$

Then, it can also be seen as a morphism of sections $J \colon \Gamma(E) \to \Gamma(E)$. If we take $E = TM$, the usual concept of an almost complex structure on a manifold is obtained; but if we consider $E = \mathbb{T}M$, we should also add the compatibility condition with $\mathcal{G}_0$ in order to recover the original definition of a generalized almost complex structure given by Gualtieri.

This argumentation also allows one to introduce other interesting structures in a natural way, such as generalized polynomial structures. A polynomial structure on a given vector bundle E is defined as a bundle endomorphism over the identity $J \colon E \to E$ together with a minimal polynomial P such that $P(J) = 0$. Aldi and Grandini considered the same definition of generalized polynomial structure adding the compatibility condition with respect to $\mathcal{G}_0$.

1.2 Our point of view

In our case, we are interested in studying as many generalized geometric structures as possible. However, it is not our intention here to minimize the importance of the canonical pairing $\mathcal{G}_0$. Therefore, we will distinguish between generalized structures that are compatible with $\mathcal{G}_0$ and those that are not compatible with $\mathcal{G}_0$ using the adjectives “strong” and “weak”, respectively. In this way, we will be able to construct more interesting generalized geometric structures. For example, we will show that any Riemannian metric on a manifold induces both a weak generalized almost complex structure and a weak generalized almost paracomplex one.

It is worth remarking that the use of the "weak" adjective throughout this document is not the same as in other publications; for example, in [4] a different notion of weakness is introduced for any generalized polynomial structure.

The attentive reader may have noticed that the word "almost" has been used not only to describe geometric structures on the tangent bundle, but also for generalized geometric structures. This is because one can define a notion of integrability in the generalized tangent bundle. To do this, we need to describe a bracket product $[\![\cdot,\cdot]\!]$ for sections of the generalized tangent bundle, in a similar way to the Lie bracket $[\cdot,\cdot]$ defined for sections of the tangent bundle (i.e., vector fields). Such a product was first proposed by I. Y. Dorfman in [18]. Working with sections of the generalized tangent bundle, which can be written as sums of vector fields in $\mathfrak{X}(M)$ and 1-forms in $\Omega^1(M)$, the Dorfman bracket is defined as

$$[\![X+\xi, Y+\eta]\!]_D := [X,Y] + (\mathcal{L}_X\eta - \iota_Y d\xi)\,,$$

for $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$. In this equation, $\mathcal{L}_X\eta$ denotes the Lie derivative of η with respect to X, such that $(\mathcal{L}_X\eta)Z = X(\eta(Z)) - \eta([X,Z])$; and $\iota_Y d\xi$ is the interior product of $d\xi$ with respect to Y, such that $(\iota_Y d\xi)Z = d\xi(Y,Z)$.

A great difference between the Lie bracket and the one proposed by Dorfman is that the last one is not skew-symmetric. However, there are some similarities between $[\cdot,\cdot]$ and $[\![\cdot,\cdot]\!]_D$, such as a Jacobi-type identity.

Even though the definition of the Dorfman bracket may look contrived, it can be explained from the point of view of exact Courant algebroids (for more information, see the letters by P. Ševera to A. Weinstein in [58]). The definition of this kind of algebroid also shows that the pairing $\mathcal{G}_0$ and the bracket $[\![\cdot,\cdot]\!]_D$ are closely related, as will be discussed in Chapter 5. Then, the integrability of a generalized polynomial structure can be defined in terms of the Dorfman bracket, and it can be characterized using an analogue to the Nijenhuis tensor, called the generalized Nijenhuis map. For a given generalized polynomial structure $\mathcal{K}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ and $u, v \in \Gamma(\mathbb{T}M)$ sections of $\mathbb{T}M$, this map is defined as follows:

$$\mathcal{N}_{\mathcal{K}}(u,v) := [\![\mathcal{K}u, \mathcal{K}v]\!]_D - \mathcal{K}([\![\mathcal{K}u, v]\!]_D + [\![u, \mathcal{K}v]\!]_D) + \mathcal{K}^2[\![u,v]\!]_D.$$

Nonetheless, the Dorfman bracket is not the only possible election for a bracket product on $\mathbb{T}M$. Other suitable choice is the skew-symmetrization of $[\![\cdot,\cdot]\!]_D$, know as the Courant bracket:

$$[\![X+\xi, Y+\eta]\!]_C := [X,Y] + \left(\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d\,(\eta(X) - \xi(Y))\right).$$

Although $[\![\cdot,\cdot]\!]_C$ is skew-symmetric, it does not have the same properties as the Dorfman bracket. Nannicini proposed another option in [49], using the Levi-Civita connection ∇ of a Riemannian manifold (M, g). This bracket product is described as

$$[\![X + \xi, Y + \eta]\!]_\nabla := [X, Y] + (\nabla_X \eta - \nabla_Y \xi)\,.$$

Each bracket exhibits distinct properties, which are outlined in Table 1.1. In light of its favorable characteristics, in this thesis we adopt the Dorfman bracket to study the integrability of generalized polynomial structures. Also, some relations can be found between the integrability of generalized structures using different bracket products (see, for example, [38]).

Bracket	Jacobi	Skew-symmetry	Leibniz	Invariance of $\mathcal{G}_0$
Dorfman	✓	✗	✓	✓
Courant	✗	✓	✗	✗
Connection	≈	✓	✓	✗

TABLE 1.1: Properties of the Dorfman, Courant and connection brackets.

In addition to generalized polynomial structures, we can analyze other geometric structures defined on $\mathbb{T}M$, such as generalized metrics. The first description of a generalized metric given by Gualtieri focused on splittings $\mathbb{T}M = E_+ \oplus E_-$ such that $\mathcal{G}_0|_{\Gamma(E_+)\times\Gamma(E_+)}$ was positive-definite and $\mathcal{G}_0|_{\Gamma(E_-)\times\Gamma(E_-)}$ negative-definite. By doing this, every $u \in \Gamma(\mathbb{T}M)$ shall be written as $u = u_+ + u_-$, with $u_+ \in \Gamma(E_+)$ and $u_- \in \Gamma(E_-)$. Then, the morphism $\mathcal{G}\colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \mathfrak{F}(M)$ defined as

$$\mathcal{G}(u, v) = \mathcal{G}_0(u_+, v_+) - \mathcal{G}_0(u_-, v_-),$$

with $u, v \in \Gamma(\mathbb{T}M)$, is a symmetric and positive-definite morphism, that is, a Riemannian metric on the vector bundle $\mathbb{T}M$.

Although this definition can be justified using the theory of structure groups of vector bundles, it may seem a bit artificial. Also, the definition is quite restrictive, as it only considers positive-definite metrics, not taking into account the pseudo-Riemannian ones. Therefore, we can adopt a similar point of view to generalized polynomial structures: we will define weak generalized metrics as those morphisms $\mathcal{G}\colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \mathfrak{F}(M)$ that are bilinear, symmetric, nondegenerate, and not necessarily compatible with the pairing $\mathcal{G}_0$; and strong generalized metrics as those that meet the compatibility condition with $\mathcal{G}_0$ as previously described.

1.3 Our results

From the information that has been presented up to now, it can be surmised that generalized geometry is a fruitful branch of differential geometry, with numerous productive researchers obtaining diverse interesting results. However, the different approaches to this field have produced multiple examples of generalized geometric structures in a disorganized way, each of them introduced for different purposes. Therefore, in order to make a satisfactory analysis, all these generalized structures should be properly organized.

This dissertation is structured in two main parts. The first part, encompassing Chapters 2 to 4, focuses on the study of geometric structures on the generalized tangent bundle $\mathbb{T}M$ of an arbitrary manifold M. The second part explores specific examples of such structures on the generalized tangent bundle of the six-dimensional sphere (Chapter 5) and on eight-dimensional nilmanifolds (Chapter 6).

The first part of this dissertation gathers a wide study of diverse generalized geometric structures, giving insight into the differences between the structures defined on TM and the ones defined on $\mathbb{T}M$. Our aim is to organize the generalized structures introduced in the scientific literature so far and present new interesting structures as well. We also reflect on what happens when some restrictions are relaxed. This part is structured as follows:

In Chapter 2, we provide the framework for the rest of the thesis. First, given any vector bundle $E \to M$ we define multiple geometric structures, such as metrics, symplectic structures, and polynomial structures. Apart from describing these structures in isolation, we show how they can interact between them: an almost complex/product structure that is symmetric or skew-symmetric with respect to a metric generates what we call an (α, ε)-metric structure (see [22]); and two almost complex/product structures that commute or anti-commute conform a triple structure (see [24]). Next, we introduce the generalized tangent bundle of a manifold, $\mathbb{T}M := TM \oplus T^*M \to M$. We establish the notation that will be used throughout the rest of the document and present the canonical structures that can be found on $\mathbb{T}M$, which are the canonical generalized metric $\mathcal{G}_0$, a canonical generalized symplectic structure Ω_0 and a canonical generalized paracomplex structure $\mathcal{F}_0$. We also study the generalized tangent bundle of a manifold from a topological point of view. In particular, we comment on the non-functorial character of the generalized tangent bundle, a great difference between $\mathbb{T}M$ and TM or T^*M. Also, we analyze the triviality of $\mathbb{T}M$ and compare it with the triviality of the tangent bundle of the manifold, proving that there are examples of manifolds with non-trivial tangent bundle, but whose generalized tangent bundle is trivial. One such manifold is the Möbius band:

Proposition 1.1. *The generalized tangent bundle $\mathbb{T}M$ of the Möbius band, which is defined as the quotient*

$$M = \mathbb{R} \times (-1, 1)/\sim, \quad (u, v) \sim (u + m, (-1)^m v), \quad m \in \mathbb{Z},$$

is trivial.

Chapter 3 is devoted to the study of weak generalized polynomial structures, that is, polynomial structures $\mathcal{K} \colon \mathbb{T}M \to \mathbb{T}M$ on the generalized tangent bundle that are not necessarily compatible with the canonical generalized metric $\mathcal{G}_0$. We show how different geometric structures on the manifold, which can be defined as tensors on M, induce diverse generalized polynomial structures. In particular, we prove how to construct weak generalized polynomial structures from metrics, almost symplectic structures, and polynomial structures on the manifold. Apart from the well-known examples introduced by Gualtieri and Wade, we analyze different structures proposed by other authors and some original non-trivial examples. After that, we do a similar study concerning weak generalized triple structures, that is, triple structures on the generalized tangent bundle. Here, we completely characterize the weak generalized triple structures in which $\mathcal{F}_0$ can participate, and we study various generalized triple structures that are induced from an (α, ε)-metric manifold.

Finally, in Chapter 4, we analyze the interaction between generalized polynomial structures and the canonical metric $\mathcal{G}_0$. We will say that a generalized polynomial structure $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$ that satisfies the compatibility condition proposed by Gualtieri is a strong polynomial structure. On the other hand, it may happen that the structure satisfies the opposite compatibility condition, that is, $\mathcal{G}_0(\mathcal{K}u, v) = \mathcal{G}_0(u, \mathcal{K}v)$. In that case, we will just say that the generalized polynomial structure is symmetric with respect to $\mathcal{G}_0$. Based on the polynomial structures introduced in Chapter 3, we determine which of them are skew-symmetric with respect to the canonical pairing, thereby constituting strong generalized polynomial structures, and which of them are symmetric with respect to $\mathcal{G}_0$. Subsequently, we dive into generalized metrics and generalized symplectic structures. In an analogous way to weak generalized polynomial structures, we define weak generalized metrics as the sections $\mathcal{G} \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ that are bilinear, symmetric, nondegenerate, and the signature of each $\mathcal{G}_p$ does not depend on p. Analogously, a generalized symplectic structure is a section $\Omega \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ that is bilinear, skew-symmetric and nondegenerate. We provide a complete characterization of all metrics that can be defined on the generalized tangent bundle, as well as of the generalized symplectic structures. In particular, we show that weak generalized metrics and generalized symplectic structures are equivalent to endomorphisms of $\mathbb{T}M$ satisfying some conditions. Finally, we focus on a specific generalized metric that is induced in a Riemannian or pseudo-Riemannian manifold (M, g). This generalized metric is given by $\mathcal{G}_g(X + \xi, Y + \eta) = g(X, Y) + g(\sharp_g \xi, \sharp_g \eta)$. After studying some of its properties, we check the compatibility between $\mathcal{G}_g$ and various weak generalized polynomial structures.

As demonstrated in the preceding chapters, classical geometric structures defined on a smooth manifold M give rise to various structures on its generalized tangent bundle $\mathbb{T}M$. These relationships are summarized in the following figure, where each arrow corresponds to a structure on $\mathbb{T}M$ induced by a structure on TM.

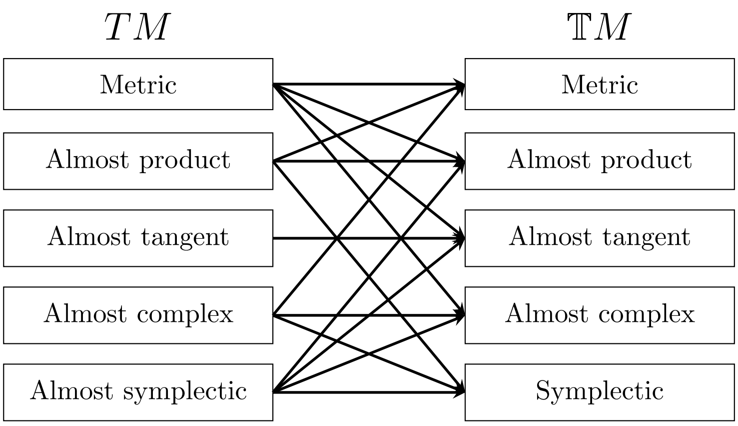


FIGURE 1.1: Scheme of the structures on $\mathbb{T}M$ that can be induced from geometric structures on the manifold M.

During the thorough revision of the literature related to this area, we could not help but notice that the majority of generalized geometric structures introduced in different studies were described in general terms, without focusing on any particular manifold. We believe that, in general, there is a significant lack of research that presents concrete examples of generalized almost complex structures on specific differentiable manifolds. Therefore, the second part of this dissertation emphasizes the search for explicit examples of generalized geometric structures for certain differentiable manifolds of interest. Specifically, we focus on the search for integrable generalized complex structures, using the definition of integrability given above.

The second part is structured as follows:

In Chapter 5, we address the question of whether there exist integrable generalized complex structures on the six-dimensional sphere. The problem of finding integrable complex structures on even-dimensional spheres $\mathbb{S}^{2n}$ was first presented by H. Hopf in [34]. It is widely known that the only spheres that admit almost complex structures are $\mathbb{S}^2$ and $\mathbb{S}^6$ (see, for example, [9]). While $\mathbb{S}^2$ is known to admit an integrable complex structure, it remains to determine whether the sphere $\mathbb{S}^6$ can be endowed with such a structure. This question is commonly referred to as the "Hopf problem". In this chapter, we study a version of the Hopf problem in generalized geometry. After a detailed description of the properties of the Dorfman bracket and the notion of integrability in the generalized tangent bundle of any manifold, we use local coordinates to derive a necessary condition that a weak generalized almost complex structure must satisfy in order to be integrable. We then turn to the nearly-Kähler structure $(\mathbb{S}^6, J, g)$ on the six-dimensional sphere, inherited from the pure octonion product in $\mathbb{R}^7$, and show that there is no straightforward way to induce a generalized complex structure from it. In particular, we prove the following result:

Theorem 1.2. *There are no weak generalized complex structures on $\mathbb{S}^6$ that can be written as a spherical combination of the weak generalized almost complex structures $\mathcal{J}_1$, $\mathcal{J}_2$, described as $\mathcal{J}_1(X+\xi) = JX + J^*\xi$, $\mathcal{J}_2(X+\xi) = -\sharp_g\xi + \flat_g X$, and the strong generalized almost complex structure $\mathcal{J}_3$, defined as $\mathcal{J}_3(X+\xi) = -\sharp_\omega\xi + \flat_\omega X$, where $(\mathbb{S}^6, J, g)$ is the nearly Kähler structure on $\mathbb{S}^6$ inherited from the pure octonion product and $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ its fundamental almost symplectic structure.*

Finally, Chapter 6 is intended to study integrable strong generalized complex structures in the context of nilmanifolds. In [12], Cavalcanti and Gualtieri searched for explicit examples of integrable generalized complex structures within the framework of six-dimensional nilmanifolds. The research in this kind of smooth manifolds is closely related to the analysis of nilpotent Lie algebras over the real numbers $\mathbb{R}$, which have been extensively studied and classified in dimension six by various authors, including L. Magnin (see [44]). Inspired by the work of Cavalcanti and Gualtieri, this chapter investigates integrable generalized complex structures on eight-dimensional nilmanifolds. Specifically, we focus on nilmanifolds described by eight-dimensional nilpotent Lie algebras over $\mathbb{R}$ that are two-step, that is, on eight-dimensional Lie algebras $\mathfrak{g}$ such that $[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}] = \{0\}$, where $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{g}$. These Lie algebras have been recently classified by M. Borovoi, B. A. Dina and W. A. de Graaf in [10]. After introducing the basic concepts about nilmanifolds and the description of integrable generalized complex structures in this context, we provide a wide range of generalized complex structures that arise on each of the nilmanifolds considered. All the propositions that are proved in this chapter can be summarized in the following theorem:

Theorem 1.3. *Every two-step nilmanifold of dimension eight admits a left-invariant strong generalized complex structure. In particular, every two-step nilmanifold of dimension eight admits a left-invariant complex structure or a left-invariant symplectic structure. Also, every such nilmanifold admits a left-invariant strong generalized complex structure that is not induced by either a left-invariant complex or a symplectic structures on the manifold.*

As a result of the investigation carried out throughout this doctoral thesis, we have published three scientific articles, which are [19–21]. The first two articles correspond to the majority of the content covered in the first part of the thesis. Specifically, [19] addresses most of the material in Chapter 3, while [20] focuses on certain results from Chapter 3 and the entirety of Chapter 4. On the other hand, [21] is devoted to the proof of Theorem 1.2, and thus its content is fully included in Chapter 5. Finally, Chapter 6 presents the research carried out in collaboration with G. Bazzoni during a predoctoral research stay. The results of this work are expected to be published in the near future.

It is worth noting that, although most of the results presented in this document can be found in the previously cited articles, many of them have been rewritten or expanded here with the aim of producing a consistent and self-contained PhD thesis. Furthermore, a significant number of non-published results are also included, such as Proposition 1.1, Propositions 6.19-6.28 or Tables 6.1-6.8.

Chapter 2

Geometry of the generalized tangent bundle

The first step towards understanding generalized geometric structures is to introduce the theory of vector bundles, that is, fiber bundles $\pi\colon E \to M$ such that, at any $p \in M$, the fiber $E_p := \pi^{-1}(p)$ is diffeomorphic to the m-dimensional vector space $\mathbb{R}^m$, being m equal for every point. Also, there must be a bundle atlas such that any two bundle charts of E are $\mathrm{GL}(m, \mathbb{R})$-compatible. The main reference that has been used to understand and explain these concepts is [54]. If we can handle geometric structures in any vector bundle, it will be easier to understand the differences and relations between $\mathbb{T}M$, TM and T^*M.

The following notation is used throughout this document: M denotes a smooth n-dimensional real manifold, Hausdorff and second countable; $\mathfrak{F}(M)$ is the ring of smooth functions from M to $\mathbb{R}$ (that is, they are $\mathcal{C}^\infty$); $\mathfrak{X}(M)$ denotes the $\mathfrak{F}(M)$-module of smooth vector fields on the manifold; and $\Omega^1(M)$ is the $\mathfrak{F}(M)$-module of smooth 1-forms on the manifold.

2.1 Geometric structures on vector bundles

We are interested in handling not only geometric structures that are defined on the tangent bundle of a manifold, but also geometric structures defined on the generalized tangent bundle. Therefore, it is convenient to give definitions of some geometric structures on any vector bundle and then specify these concepts for the generalized tangent bundle.

We work with any vector bundle $E \to M$. The fiber of E at the point $p \in M$ is denoted as E_p, the dual bundle of E is named E^*, the $\mathfrak{F}(M)$-module of smooth sections of E is denoted as $\Gamma(E)$, and $\mathrm{End}(E) \cong E \otimes E^* \to M$ is the bundle of linear endomorphisms $E \to E$ over the identity map.

Definition 2.1 ([54, Definition 3.1])**.** A *metric* on a vector bundle E is a section $g \in \Gamma(E^* \otimes E^*)$ for which, at each point $p \in M$, the morphism $g_p\colon E_p \times E_p \to \mathbb{R}$ is bilinear, symmetric and nondegenerate, and the signature (r, s) of g_p does not depend on p. If each morphism g_p is positive-definite, g is called a *Riemannian metric*; else, it is a *pseudo-Riemannian metric* of signature (r, s). When $r = s$, it is called a *neutral metric*.

A metric can also be understood as a morphism $g\colon \Gamma(E) \times \Gamma(E) \to \mathfrak{F}(M)$ carrying pairs of sections of E to smooth functions. This morphism must fulfill the conditions given in Definition 2.1. When the vector bundle is equal to the tangent bundle of the manifold (that is, when $E = TM$), the usual concepts of Riemannian manifolds and pseudo-Riemannian manifolds (M, g) are recovered.

Definition 2.2 ([54, Definition 8.4])**.** A *symplectic structure* on a vector bundle E is defined as a section $\omega \in \Gamma(E^* \otimes E^*)$ for which, at each point $p \in M$, the morphism $\omega_p\colon E_p \times E_p \to \mathbb{R}$ is bilinear, skew-symmetric and nondegenerate.

As in the case of metrics, symplectic structures can also be considered as section morphisms $\omega\colon \Gamma(E) \times \Gamma(E) \to \mathfrak{F}(M)$ that fulfill the requisites given in Definition 2.2. It is immediate to see that the rank of a vector bundle (that is, the dimension as a vector space of each fiber E_p) admitting a symplectic structure must be even. When the particular case $E = TM$ is considered, we retrieve the concept of almost symplectic manifolds (M, ω). We recall that an almost symplectic manifold (M, ω) is called a symplectic manifold when the 2-form ω is closed (i.e., when $d\omega = 0$).

Since metrics and symplectic structures are nondegenerate morphisms, they induce isomorphisms between E and its dual vector bundle E^*. For a metric g, these isomorphisms are obtained as follows.

Definition 2.3 ([54, Definition 3.4])**.** The *flat isomorphism* associated to g, $\flat_g\colon E \to E^*$, is the bundle morphism that transforms each $X \in E_p$ into $\flat_g X \in E_p^*$, which is $(\flat_g X)Y = g_p(X, Y)$ for every $Y \in E_p$. The *sharp isomorphism* associated to g, $\sharp_g\colon E^* \to E$, is defined as the inverse of $\flat_g$, i.e., $\sharp_g = \flat_g^{-1}$.

Given any $\xi \in E_p^*$, it can be easily checked that $g(\sharp_g \xi, X) = \xi(X)$ for each $X \in E_p$. From the definition, it is clear that both $\flat_g, \sharp_g$ are bundle morphisms over the identity map $id\colon M \to M$; in other words, the following diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{\flat_g} & E^* \\ \downarrow & & \downarrow \\ M & \xrightarrow[id]{} & M \end{array}, \qquad \begin{array}{ccc} E^* & \xrightarrow{\sharp_g} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow[id]{} & M \end{array}.$$

These isomorphisms can also be thought of as morphisms between sections of E and E^*, that is to say, $\flat_g\colon \Gamma(E) \to \Gamma(E^*)$ and $\sharp_g\colon \Gamma(E^*) \to \Gamma(E)$ that fulfill the conditions given in Definition 2.3.

Given the fact that every vector bundle admits a Riemannian metric (see [54, Proposition 3.3]), it is clear that E and E^* will always be isomorphic vector bundles. However, it is important to remark that the isomorphism between E and E^* is not canonical: it depends on the metric g that one chooses on the bundle.

The musical isomorphisms associated to a symplectic structure ω are defined in an analogous way: its flat isomorphism is $(\flat_\omega X)Y = \omega_p(X, Y)$, and its sharp isomorphism is $\sharp_\omega = \flat_\omega^{-1}$.

The main focus of our investigation is to study a kind of geometric structure that is defined as vector bundle endomorphisms. We now introduce the notion of polynomial structures.

Definition 2.4. A *polynomial structure* on a vector bundle E is a section $J \in \Gamma(\mathrm{End}(E))$ together with a minimal polynomial P such that $P(J) = 0$. When $P = x^2 + 1$ (that is, $J^2 = -Id$), the morphism J is called an *almost complex structure* [54, Definition 1.58]; when $P = x^2 - 1$ (that is, $J^2 = Id$), J is an *almost product structure*; and when $P = x^2$ (that is, $J^2 = 0$) and the rank of J is half the rank of E, then it is named an *almost tangent structure*.

From the above definition, it is clear that the following diagram commutes for a given polynomial structure $J\colon E \to E$:

$$\begin{array}{ccc} E & \xrightarrow{J} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow[id]{} & M \end{array} .$$

A polynomial structure can also be defined as an endomorphism $J\colon \Gamma(E) \to \Gamma(E)$ with a minimal polynomial. If we work with the tangent bundle $E = TM$, we recover the usual concepts of almost complex, almost product and almost tangent manifolds (M, J).

Depending on the minimal polynomial of J and its eigenvalues, J will present different properties. It is worthwhile to study separately the eigenbundles associated to an almost complex, almost product and almost tangent structure defined on a vector bundle E:

- To properly describe the eigenbundles of an almost complex structure J, it is required to introduce the *complexification* of a vector bundle E as follows:

$$E_{\mathbb{C}} := E \otimes \mathbb{C} = \{X + iY \colon X, Y \in E_p,\ p \in M\}.$$

  The morphism J can be extended to $E_{\mathbb{C}}$ by defining $J(iX) = iJX$ for $X \in E$. Then, since the minimal polynomial of an almost complex structure is $P = x^2 + 1$, its eigenvalues are i and $-i$. If their eigenbundles are denoted as $L_J^{1,0}, L_J^{0,1} \subset E_{\mathbb{C}}$ respectively, we have

$$\begin{aligned} L_J^{1,0} &:= \{Y \in E_{\mathbb{C}} \colon JY = iY\} = \{X - iJX \in E_{\mathbb{C}} \colon X \in E\}, \\ L_J^{0,1} &:= \{Y \in E_{\mathbb{C}} \colon JY = -iY\} = \{X + iJX \in E_{\mathbb{C}} \colon X \in E\}. \end{aligned} \tag{2.1}$$

  The projections onto these eigenbundles, $P_J^{1,0}, P_J^{0,1} \colon E_{\mathbb{C}} \to E_{\mathbb{C}}$, are obtained as

$$P_J^{1,0} X = \frac{1}{2}(X - iJX), \quad P_J^{0,1} X = \frac{1}{2}(X + iJX). \tag{2.2}$$

  As the real dimension of E is equal to the complex dimension of $E_{\mathbb{C}}$, and both $L_J^{1,0}, L_J^{0,1}$ have the same complex dimension, it is immediate to see that the rank of a vector bundle that admits an almost complex structure must be even.

- In the case of an almost product structure F, the eigenvalues of its associated minimal polynomial $P = x^2 - 1$ are 1 and -1. Thus, their respective eigenbundles L_F^+, L_F^- are the following ones:

$$\begin{aligned} L_F^+ &:= \{Y \in E \colon FY = Y\} = \{X + FX \in E \colon X \in E\}, \\ L_F^- &:= \{Y \in E \colon FY = -Y\} = \{X - FX \in E \colon X \in E\}. \end{aligned} \tag{2.3}$$

The projections $P_F^+, P_F^- \colon E \to E$ onto each eigenbundle are given by

$$P_F^+ X = \frac{1}{2}(X + FX), \quad P_F^- X = \frac{1}{2}(X - FX). \tag{2.4}$$

In contrast to almost complex structures, the eigenbundles of an almost product structure do not need to have the same dimension. When $\dim L_F^+ = \dim L_F^-$, the endomorphism F will be named an *almost paracomplex structure* on E.

- Finally, it is straightforward to see that the rank of a vector bundle E endowed with an almost tangent structure S must be even and

$$\mathrm{im}\, S = \ker S.$$

Given any bundle endomorphism $J \colon E \to E$, one can define an endomorphism $J^* \colon E^* \to E^*$ on the dual bundle which is directly related to J. This dual endomorphism is described as follows.

Definition 2.5. The *dual endomorphism* of $J \colon E \to E$ is defined as the morphism $J^* \colon E^* \to E^*$ such that for each $\xi \in E_p^*$ the element $J^*\xi \in E_p^*$ is defined as $(J^*\xi)X = \xi(JX)$ for every $X \in E_p$.

It is straightforward to check that, when the endomorphism J is a polynomial structure with minimal polynomial P, then its dual endomorphism J^* is a polynomial structure on E^* with the same minimal polynomial P.

Until now, each type of geometric structure has been described in isolation from each other. We now ask ourselves how two geometric structures can interact between them. Specifically, we study the combination of a metric g with a polynomial structure J whose minimal polynomial is $P = x^2 \pm 1$ (that is, an almost complex or almost product structure). The most natural interaction between these two kind of structures is J being an isometry or anti-isometry, as it is described below.

Definition 2.6. An (α, ε)*-metric structure* on a vector bundle E, with $\alpha, \varepsilon \in \{1, -1\}$, is defined as a structure (E, J, g) which comprises a polynomial structure J and a metric g on the bundle such that

$$J^2 = \alpha Id, \quad g(JX, JY) = \varepsilon g(X, Y), \tag{2.5}$$

for every $X, Y \in E$ in the same fiber.

Depending on the values of α, ε, an (α, ε)-metric structure will be named differently:

- If $\alpha = 1$, $\varepsilon = 1$, (E, J, g) is called an *almost product (pseudo-)Riemannian structure*, depending on whether g is Riemannian or pseudo-Riemannian. If g is a Riemannian metric and J is almost paracomplex, (E, J, g) is named an *almost para-Norden structure*.
- If $\alpha = 1$, $\varepsilon = -1$, (E, J, g) is called an *almost para-Hermitian structure*. The compatibility condition between J and g forces the metric to be neutral.
- If $\alpha = -1$, $\varepsilon = 1$, (E, J, g) is called an *almost (pseudo-)Hermitian structure*, depending on whether g is Riemannian or pseudo-Riemannian.

- If $\alpha = -1$, $\varepsilon = -1$, (E, J, g) is called an *almost Norden structure*. As in the case of almost para-Hermitian structures, g must be a pseudo-Riemannian metric with neutral signature.

This type of structure has been studied, regardless of the specific values taken by (α, ε), in the case of the tangent bundle $E = TM$ in articles such as [22, 23].

Due to the compatibility condition between the metric g and the endomorphism J, it is possible to define a new nondegenerate morphism $\varphi \in \Gamma(E^* \otimes E^*)$ associated to (E, J, g). This morphism, called the *fundamental tensor* of the (α, ε)- metric structure, is defined as

$$\varphi(X, Y) = g(JX, Y), \tag{2.6}$$

for every $X, Y \in E$ in the same fiber. It is immediate to check that the behavior of φ will be determined by the values of (α, ε):

$$\varphi(Y, X) = g(JY, X) = \varepsilon g(J^2 Y, JX) = \alpha\varepsilon g(JX, Y) = \alpha\varepsilon\varphi(X, Y).$$

Therefore, if $\alpha\varepsilon = 1$ the fundamental tensor is a metric (also called the *twin metric*), whereas if $\alpha\varepsilon = -1$ the morphism φ is a symplectic structure defined on the vector bundle (also called the *fundamental symplectic structure*).

The fundamental tensor of an (α, ε)-metric structure is a metric or a symplectic structure on E, and hence nondegenerate. Therefore, it makes sense to ask whether its musical isomorphisms are related in some way to J and g. The following result answers this question.

Proposition 2.7. *Let (E, J, g) be an (α, ε)-metric structure on a vector bundle E with fundamental tensor φ. Then, the following equations hold true:*

$$\begin{aligned} \flat_\varphi &= \flat_g J = \alpha\varepsilon J^* \flat_g, \\ \sharp_\varphi &= \varepsilon \sharp_g J^* = \alpha J \sharp_g. \end{aligned} \tag{2.7}$$

Proof. If we take $X, Y \in E$ in the same fiber, from the definition of the flat isomorphism we have

$$(\flat_\varphi X)Y = \varphi(X, Y) = g(JX, Y) = (\flat_g JX)Y.$$

This is true for each Y, hence $\flat_\varphi X = \flat_g JX$ for every $X \in E$. With respect to the second equality,

$$(\flat_g JX)Y = g(JX, Y) = \alpha\varepsilon g(X, JY) = \alpha\varepsilon(\flat_g X)(JY) = \alpha\varepsilon(J^* \flat_g X)Y,$$

for every $X, Y \in E$ in the same fiber. Therefore, the first two equalities hold true.

For the other two equations, taking any $\xi \in E^*$ and $Y \in E$ based on the same point, then

$$\begin{aligned} g(\sharp_\varphi \xi, Y) &= \varepsilon g(J\sharp_\varphi \xi, JY) = \varepsilon\varphi(\sharp_\varphi \xi, JY) = \varepsilon\xi(JY) = \varepsilon(J^*\xi)Y = g(\varepsilon\sharp_g J^*\xi, Y), \\ g(\sharp_g J^*\xi, Y) &= (J^*\xi)(Y) = \xi(JY) = g(\sharp_g \xi, JY) = \varepsilon g(J\sharp_g \xi, J^2 Y) = g(\alpha\varepsilon J\sharp_g \xi, Y). \end{aligned}$$

As the metric g is nondegenerate, the last relations in Eq. (2.7) hold true. □

The last kind of geometric structure that we describe in this section arises when we work with two compatible polynomial structures. The compatibility condition between these two endomorphisms is related to the commutation or anti-commutation between them.

Definition 2.8. A *triple structure* on a vector bundle E is defined as a structure (E, J_1, J_2, J_3) which comprises three polynomial structures $J_1, J_2, J_3 \colon E \to E$ with minimal polynomials $P_i = x^2 \pm 1$ for $i = 1, 2, 3$ (that is, they are almost complex or almost product structures, not necessarily all the same type), they commute or anti-commute, and $J_3 = J_1 J_2$.

According to Definition 2.8, there are four types of triple structures:

- *Almost hypercomplex structures*: In this case, J_1, J_2, J_3 are almost complex structures that anti-commute, that is, $J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -Id$.
- *Almost bicomplex structures*: J_1, J_2 are almost complex structures and J_3 is an almost product structure, such that J_1, J_2, J_3 commute; in other words, $J_1^2 = J_2^2 = -J_3^2 = -J_1 J_2 J_3 = -Id$.
- *Almost biparacomplex structures*: In this case, J_1, J_2 are both almost product structures and J_3 is an almost complex structure, such that J_1, J_2, J_3 anti-commute; this condition is equivalent to $J_1^2 = J_2^2 = -J_3^2 = -J_1 J_2 J_3 = Id$.
- *Almost hyperproduct structures*: J_1, J_2, J_3 are almost product structures that commute, that is, $J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = Id$.

It is no accident that almost biparacomplex structures are called this way and not "almost biproduct structures": it is straightforward to check that the almost complex structure J_3 of an almost biparacomplex structure transforms the 1-eigenbundle of J_1 into its -1-eigenbundle; in other words, $J_3(L^+_{J_1}) = L^-_{J_1}$. In consequence, it must be $\dim L^+_{J_1} = \dim L^-_{J_1}$ and $\dim L^+_{J_2} = \dim L^-_{J_2}$, that is, both J_1, J_2 are almost paracomplex structures.

Naturally, the endomorphisms in a triple structure can be compatible with a metric g. The next statement, which is straightforward to prove, reveals the compatibility of the endomorphisms in a triple structure with a metric.

Proposition 2.9. *Let (E, J_1, g) be an $(\alpha_1, \varepsilon_1)$-metric structure and (E, J_2, g) an $(\alpha_2, \varepsilon_2)$-metric structure on a vector bundle E. If $J_1 J_2 = \lambda J_2 J_1$ with $\lambda \in \{1, -1\}$, then $J_3 = J_1 J_2$ conforms a $(\lambda\alpha_1\alpha_2, \varepsilon_1\varepsilon_2)$-metric structure with respect to g.*

Proof. We just compute J_3^2 and $g(J_3 X, J_3 Y)$ for any $X, Y \in \Gamma(E)$:

$$J_3^2 = (J_1 J_2)(J_1 J_2) = \lambda J_1 J_2^2 J_1 = \lambda \alpha_2 J_1^2 = \lambda \alpha_1 \alpha_2,$$
$$g(J_3 X, J_3 Y) = g(J_1(J_2 X), J_1(J_2 Y)) = \varepsilon_1 g(J_2 X, J_2 Y) = \varepsilon_1 \varepsilon_2 g(X, Y). \qquad \square$$

These geometric structures have previously been studied in the case when the vector bundle is the tangent bundle of the manifold $E = TM$ (for example, in [24, 35]) .

2.2 The generalized tangent bundle

We now focus on the vector bundle that we will mainly handle throughout the document. This vector bundle, closely related to the tangent bundle of a manifold TM as well as the cotangent bundle T^*M, is described as follows.

Definition 2.10. The *generalized tangent bundle* of a smooth manifold M is the Whitney sum of its tangent and cotangent bundle, namely $\mathbb{T}M := TM \oplus T^*M \to M$.

At each point $p \in M$, the fiber of this vector bundle is the direct sum of the tangent and cotangent space at p, that is, $\mathbb{T}_pM := (\mathbb{T}M)_p = T_pM \oplus T_p^*M$. Therefore, the rank of $\mathbb{T}M$ is equal to $2n$ with $n = \dim M$. As every $u_p \in \mathbb{T}_pM$ will be written as $u_p = X_p + \xi_p$ with $X_p \in T_pM$ and $\xi_p \in T_p^*M$, it will be possible to write every smooth section of this vector bundle as a sum of a vector field and a 1-form on M. In other words, we have $\Gamma(\mathbb{T}M) = \mathfrak{X}(M) \oplus \Omega^1(M)$, and for each $u \in \Gamma(\mathbb{T}M)$ there are some unique $X \in \mathfrak{X}(M)$ and $\xi \in \Omega^1(M)$ such that $u = X + \xi$.

It is necessary to introduce the matrix notation that will be used throughout this document to describe any bundle endomorphisms $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$. As $\mathcal{K}$ does not need to take elements from the tangent bundle TM to itself, we can write

$$\mathcal{K} = \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix}, \tag{2.8}$$

for some morphisms $H\colon TM \to TM$, $\sigma\colon T^*M \to TM$, $\tau\colon TM \to T^*M$ and $K\colon T^*M \to T^*M$. This means that if we take any $X + \xi \in \mathbb{T}M$, then the following expression can be written:

$$\begin{aligned} \mathcal{K}(X+\xi) = \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} &= \begin{pmatrix} HX + \sigma\xi \\ \tau X + K\xi \end{pmatrix} \\ &= (HX + \sigma\xi) + (\tau X + K\xi) \in TM \oplus T^*M. \end{aligned}$$

The use of this notation becomes useful when studying the behavior of an endomorphism and looking for restrictions. For example, if we want to check whether $\mathcal{K}^2 = 0$, using matrix notation we obtain

$$\mathcal{K}^2 = \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix} \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix} = \begin{pmatrix} H^2 + \sigma\tau & H\sigma + \sigma K \\ \tau H + K\tau & \tau\sigma + K^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where 0 denotes the corresponding null morphism. Therefore, we have the following four equations that must be fulfilled in order to have $\mathcal{K}^2 = 0$:

$$\begin{cases} H^2 + \sigma\tau = 0, \\ H\sigma + \sigma K = 0, \\ \tau H + K\tau = 0, \\ \tau\sigma + K^2 = 0. \end{cases}$$

The geometric structures defined for any vector bundle E in Section 2.1 can be considered for the generalized tangent bundle $E = \mathbb{T}M$. Then, we obtain geometric structures on $\mathbb{T}M$ which are analogous to the ones defined in classical differential geometry for $E = TM$. These definitions will be thoroughly examined in Chapters 3 and 4.

The morphisms that we present now arise on the generalized tangent bundle without adding any classical geometric structure to the base manifold. This means that these morphisms are independent of the considered manifold, in the same way as the Liouville field $L \in \mathfrak{X}(TM)$ on the tangent bundle of any manifold, which is locally defined as $L = v^i \frac{\partial}{\partial v^i}$ for any natural chart (x^i, v^i) of TM; or the canonical 1-form $\theta \in \Omega^1(T^*M)$ on the cotangent bundle of any manifold, which in natural local coordinates (x^i, w_i) is given by $\theta = w_i dx^i$. The first canonical generalized geometric structure that we present was first introduced by Courant in [16].

Definition 2.11 ([16, Section 1.1])**.** The *canonical generalized metric* or *canonical pairing* is the morphism $\mathcal{G}_0 \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ such that it is defined as

$$\mathcal{G}_0(X + \xi, Y + \eta) = \frac{1}{2}(\xi(Y) + \eta(X)), \tag{2.9}$$

for each $X, Y \in T_pM$ and $\xi, \eta \in T_p^*M$.

The canonical pairing $\mathcal{G}_0$ is a pseudo-Riemannian metric with neutral signature. This can be easily checked using local coordinates $(U, (x^1, \ldots, x^n))$ with $U \subset M$. Then, taking the coordinate vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ and 1-forms $\{dx^1, \ldots, dx^n\}$, we can form a basis of $\Gamma(E)$ on U:

$$\left\{ \frac{\partial}{\partial x^1} + dx^1, \ldots, \frac{\partial}{\partial x^n} + dx^n, \frac{\partial}{\partial x^1} - dx^1, \ldots, \frac{\partial}{\partial x^n} - dx^n \right\}. \tag{2.10}$$

It can be seen that this basis is orthonormal with respect to $\mathcal{G}_0$: for any $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned}
\mathcal{G}_0\left(\frac{\partial}{\partial x^i} + dx^i, \frac{\partial}{\partial x^j} + dx^j\right) &= \frac{1}{2}\left(dx^i\left(\frac{\partial}{\partial x^j}\right) + dx^j\left(\frac{\partial}{\partial x^i}\right)\right) = \delta^i_j, \\
\mathcal{G}_0\left(\frac{\partial}{\partial x^i} + dx^i, \frac{\partial}{\partial x^j} - dx^j\right) &= \frac{1}{2}\left(dx^i\left(\frac{\partial}{\partial x^j}\right) - dx^j\left(\frac{\partial}{\partial x^i}\right)\right) = 0, \\
\mathcal{G}_0\left(\frac{\partial}{\partial x^i} - dx^i, \frac{\partial}{\partial x^j} - dx^j\right) &= -\frac{1}{2}\left(dx^i\left(\frac{\partial}{\partial x^j}\right) + dx^j\left(\frac{\partial}{\partial x^i}\right)\right) = -\delta^i_j.
\end{aligned}$$

The morphism $\mathcal{G}_0$ can also be understood from the point of view of principal fiber bundles (to see a more detailed explanation of principal fiber bundles and G-structures, see [39, 40]). We define a *linear frame* of a vector bundle $E \to M$ of rank k at the point $p \in M$ as an ordered basis $(X_1, \ldots, X_k)$ of the vector space E_p. If we denote the set of linear frames at p as F_pE, then the disjoint union of all these sets, $FE := \bigsqcup_{p\in M} F_pE$, is a principal fiber bundle $FE \to M$ with the general linear group $\mathrm{GL}(k, \mathbb{R})$ as structural group. This fiber bundle is known as the *frame bundle* of E.

In this sense, it has been studied (for example, in [14]) that any geometric structure defined on the tangent bundle $E = TM$ is associated to a reduction of the the structural group $\mathrm{GL}(n, \mathbb{R})$ to a subgroup G, i.e., to an embedding $Q \hookrightarrow F(TM)$ such that Q is a principal fiber bundle with structural group $G \subseteq \mathrm{GL}(n, \mathbb{R})$. In this sense, such a reduction is called a *G-structure*.

In particular, a Riemannian metric $g \in \Gamma(T^*M \otimes T^*M)$ is equivalent to a reduction of the structural group $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$, that is, to a $\mathrm{O}(n)$-structure. Analogously, a pseudo-Riemannian metric with signature (r, s) is equivalent to a reduction of $\mathrm{GL}(n, \mathbb{R})$ to the indefinite orthogonal group $\mathrm{O}(r, s)$, which can be represented as

$$\mathrm{O}(r, s) = \{M \in \mathrm{GL}(n, \mathbb{R})\colon M\eta M^T = \eta, \quad \eta = \mathrm{diag}(1, \overset{r}{\cdots}, 1, -1, \overset{s}{\cdots}, -1)\}.$$

The same reasoning can be followed for any vector bundle E, not just TM: any geometric structure defined on E can be identified with a reduction of $\mathrm{GL}(k, \mathbb{R})$ to a subgroup $G \subseteq \mathrm{GL}(k, \mathbb{R})$ and a principal fiber bundle embedding $Q \hookrightarrow FE$ such that G is the structural group of Q.

Returning to the generalized tangent bundle, a linear frame of $\mathbb{T}M$ at a given point $p \in M$ can be written as $(X_1 + \xi_1, \ldots, X_{2n} + \xi_{2n})$ with $X_1, \ldots, X_{2n} \in T_pM$ and $\xi_1, \ldots, \xi_{2n} \in T_p^*M$. As the canonical metric $\mathcal{G}_0$ is a neutral metric, it is related to a reduction of the principal fiber bundle $Q \hookrightarrow F(\mathbb{T}M)$ to the indefinite orthonormal group $\mathrm{O}(n, n) \subset \mathrm{GL}(2n, \mathbb{R})$. Using the previous calculations, we can choose a linear frame $(X_1, \ldots, X_n) \in F_p(TM)$ and its dual frame $(X^1, \ldots, X^n) \in F_p(T^*M)$ (i.e., $X^i(X_j) = \delta^i_j$). By doing this, we can construct the following linear frame of $\mathbb{T}M$, which is orthonormal with respect to the canonical generalized metric $\mathcal{G}_0$:

$$u_p = (X_1 + X^1, \ldots, X_n + X^n, X_1 - X^1, \ldots, X_n - X^n). \tag{2.11}$$

Therefore, it is straightforward to see that the fibers Q_p at each point $p \in M$ corresponding to the reduction $Q \hookrightarrow F(\mathbb{T}M)$ are equal to

$$Q_p = \{u_p g \in F(\mathbb{T}M)\colon g \in \mathrm{O}(n, n)\}.$$

It might be tempting to say that every linear frame in Q_p has the same form as u_p given in Eq. (2.11), that is to say, that for every $v_p \in Q_p$ there is a linear frame $(Y_1, \ldots, Y_n) \in F_p(TM)$ such that we can write $v_p = (Y_1 + Y^1, \ldots, Y_n + Y^n, Y_1 - Y^1, \ldots, Y_n - Y^n)$. However, since this would imply that $\mathrm{GL}(n, \mathbb{R}) \cong Q_p \cong \mathrm{O}(n, n)$, it cannot be true. To find a counterexample, we work with $M = \mathbb{R}^2$ and take the linear frame $\left(\frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\Big|_p \right) \in F_p(T_p\mathbb{R}^2)$ at each point. Then, the following linear frame of $\mathbb{T}M$ is orthonormal with respect to $\mathcal{G}_0$:

$$\left(\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)_p + dx_p, \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)_p - dy_p, \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)_p - dy_p, \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)_p - dx_p \right).$$

By merely changing one sign in Eq. (2.9), we obtain a different nondegenerate structure on $\mathbb{T}M$, namely, a generalized symplectic structure. This morphism is described as follows.

Definition 2.12 ([16, Section 1.1])**.** The *canonical generalized symplectic structure* is the morphism $\Omega_0 \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ such that it is defined as

$$\Omega_0(X + \xi, Y + \eta) = \frac{1}{2}(\xi(Y) - \eta(X)), \tag{2.12}$$

for each $X, Y \in T_pM$ and $\xi, \eta \in T_p^*M$.

Following a similar analysis to the one carried out for $\mathcal{G}_0$, the canonical generalized symplectic structure Ω_0 will be related to a reduction of the linear frame bundle $Q \hookrightarrow F(\mathbb{T}M)$ to the symplectic group $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{GL}(2n, \mathbb{R})$. Using the local basis from Eq. (2.10), we infer that

$$\begin{aligned}
\Omega_0\left(\frac{\partial}{\partial x^i}+dx^i, \frac{\partial}{\partial x^j}+dx^j\right) &= \frac{1}{2}\left(dx^i\left(\frac{\partial}{\partial x^j}\right)-dx^j\left(\frac{\partial}{\partial x^i}\right)\right)=0,\\
\Omega_0\left(\frac{\partial}{\partial x^i}+dx^i, \frac{\partial}{\partial x^j}-dx^j\right) &= \frac{1}{2}\left(dx^i\left(\frac{\partial}{\partial x^j}\right)+dx^j\left(\frac{\partial}{\partial x^i}\right)\right)=\delta^i_j,\\
\Omega_0\left(\frac{\partial}{\partial x^i}-dx^i, \frac{\partial}{\partial x^j}-dx^j\right) &= -\frac{1}{2}\left(dx^i\left(\frac{\partial}{\partial x^j}\right)-dx^j\left(\frac{\partial}{\partial x^i}\right)\right)=0.
\end{aligned}$$

Therefore, we can take a linear frame $(X_1, \dots, X_n) \in F_p(TM)$ and construct the linear frame $u_p \in F(\mathbb{T}M)$ given in Eq. (2.11). Using this linear frame, it is immediate to see that the fibers Q_p corresponding to the reduction $Q \hookrightarrow F(\mathbb{T}M)$ with structural group $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{GL}(2n, \mathbb{R})$ are equal to

$$Q_p = \{u_p g \in F(\mathbb{T}M) \colon g \in \mathrm{Sp}(2n, \mathbb{R})\}.$$

An interesting consequence of the existence of both $\mathcal{G}_0$ and Ω_0 arises when we study their musical isomorphisms. These isomorphisms can be seen as bundle morphisms between the generalized tangent bundle $\mathbb{T}M$ and its dual bundle $(\mathbb{T}M)^*$, that is to say, $\flat_{\mathcal{G}_0}, \flat_{\Omega_0} \colon \mathbb{T}M \to (\mathbb{T}M)^*$ and $\sharp_{\mathcal{G}_0}, \sharp_{\Omega_0} \colon (\mathbb{T}M)^* \to \mathbb{T}M$. Then, taking into account that $\mathcal{G}_0$ and Ω_0 always exist, independently of the manifold studied, the following result is directly inferred.

Proposition 2.13. *The generalized tangent bundle $\mathbb{T}M$ is canonically isomorphic to its dual bundle $(\mathbb{T}M)^*$.*

This is an enormous difference with respect to the tangent bundle TM: it is known that TM is isomorphic to the cotangent bundle T^*M, but not canonically (it is necessary to define a nondegenerate structure on the tangent bundle, such as a metric).

There is another canonical structure that can be studied on the generalized tangent bundle. This morphism, introduced by Wade in [60], can be understood as a paracomplex structure on $\mathbb{T}M$.

Definition 2.14 ([60, Example 1])**.** The *canonical generalized paracomplex structure* is the endomorphism $\mathcal{F}_0 \colon \mathbb{T}M \to \mathbb{T}M$ such that $\mathcal{F}_0(X+\xi) = -X+\xi$ for every $X+\xi \in \mathbb{T}M$.

Using matrix notation, the structure $\mathcal{F}_0$ is written as

$$\mathcal{F}_0 = \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix}. \tag{2.13}$$

From Eqs. (2.3) and (2.4), the eigenbundles $\mathbb{L}^+_{\mathcal{F}_0}, \mathbb{L}^-_{\mathcal{F}_0} \subset \mathbb{T}M$ associated to the $1, -1$-eigenvalues of the structure $\mathcal{F}_0$ are, respectively,

$$\mathbb{L}^+_{\mathcal{F}_0} = T^*M, \quad \mathbb{L}^-_{\mathcal{F}_0} = TM,$$

and the projections onto these subbundles are the expected ones:

$$\mathcal{P}^+_{\mathcal{F}_0}(X+\xi)=\xi, \quad \mathcal{P}^-_{\mathcal{F}_0}(X+\xi)=X.$$

From the point of view of principal fiber bundles, an almost paracomplex structure on a vector bundle E with even rank $2k$ is equivalent to a reduction of the structural group $\mathrm{GL}(2k,\mathbb{R})$ to $\mathrm{GL}(k,\mathbb{R})\times\mathrm{GL}(k,\mathbb{R})$ (see [17]). In this case, the canonical generalized paracomplex structure $\mathcal{F}_0$ is equivalent to the reduction $\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R})\subset\mathrm{GL}(2n,\mathbb{R})$ such that the associated principal fiber bundle $Q\hookrightarrow F(\mathbb{T}M)$ has the following fibers:

$$Q_p=\{(X_1,\dots,X_n,\xi_1,\dots,\xi_n)\colon (X_1,\dots,X_n)\in F_p(TM),(\xi_1,\dots,\xi_n)\in F_p(T^*M)\}.$$

It is immediate to see that the three canonical structures $\mathcal{G}_0,\Omega_0,\mathcal{F}_0$ are related between them: they conform what could be called a canonical generalized para-Hermitian structure, as is stated in the following statement.

Proposition 2.15. *The three canonical generalized structures are related between them by the expression*

$$\Omega_0(X+\xi,Y+\eta)=\mathcal{G}_0(\mathcal{F}_0(X+\xi),Y+\eta),$$

for every $X+\xi,Y+\eta\in\mathbb{T}M$ in the same fiber.

Using Proposition 2.7, the following corollary can be inferred.

Corollary 2.16. *The canonical generalized paracomplex structure can be obtained from the canonical pairing $\mathcal{G}_0$ and the canonical generalized symplectic structure Ω_0 as*

$$\mathcal{F}_0=\sharp_{\mathcal{G}_0}\flat_{\Omega_0}=\sharp_{\Omega_0}\flat_{\mathcal{G}_0}.$$

2.3 Topological properties of the generalized tangent bundle

2.3.1 Functoriality

Before moving on to defining various geometric structures on the generalized tangent bundle, it is interesting to analyze some topological properties of $\mathbb{T}M$. First, we make a brief comment about the non-functorial nature of the generalized tangent bundle.

It is widely known that the construction of the tangent bundle of a manifold induces a covariant functor from the category of smooth manifolds to the category of smooth real vector bundles. This functor assigns each smooth manifold M to its tangent bundle TM, and associates each smooth map $\varphi\colon M\to N$ to its differential $\varphi_*\colon TM\to TN$, such that $(\varphi_*X)f:=X(f\circ\varphi)$ for every $X\in TM$ and $f\colon N\to\mathbb{R}$.

The construction of the cotangent bundle of a manifold also induces a functor: a contravariant functor from the category of smooth manifolds to the category of smooth real vector bundles. In this case, it associates each smooth manifold M with its cotangent bundle T^*M, and each smooth map

$\varphi\colon M \to N$ to its pullback $\varphi^*\colon T^*N \to T^*M$, such that $(\varphi^*\xi)(X) := \xi(\varphi_* X)$ for $X \in T_pM$ and $\xi \in T^*_{\varphi(p)}M$. In a similar way, there is a contravariant functor between the categories of smooth manifolds and rings. In this case, the functor assigns each smooth manifold M to the ring of smooth functions on M, $\mathfrak{F}(M)$, and every smooth map $\varphi\colon M \to N$ to $\varphi^*\colon \mathfrak{F}(N) \to \mathfrak{F}(M)$ such that $\varphi^* f = f \circ \varphi$ for $f\colon N \to \mathbb{R}$.

However, the difference between the functorial nature of TM and T^*M causes the construction of the generalized tangent bundle of a manifold $\mathbb{T}M$ to not induce any obvious covariant or contravariant functor from the category of smooth manifolds to the category of smooth real vector bundles. In other words, in general, a map $\varphi\colon M \to N$ does not induce a map $\Phi\colon \mathbb{T}M \to \mathbb{T}N$ or $\Phi\colon \mathbb{T}N \to \mathbb{T}M$.

2.3.2 Triviality

We now study the triviality of the generalized tangent bundle, comparing it with the triviality of the tangent and cotangent bundles of a manifold. The concept of a trivial vector bundle is set forth below.

Definition 2.17. A vector bundle $E \to M$ of rank k is said to be *trivial* if it is diffeomorphic to the trivial vector bundle $M \times \mathbb{R}^k \to M$.

Such a diffeomorphism $f\colon E \to M \times \mathbb{R}^k$ must take each fiber E_p to the corresponding fiber of $M \times \mathbb{R}^k$, which is $\{p\} \times \mathbb{R}^k$. Therefore, if $\pi\colon E \to M$ is the projection associated to the vector bundle E and $pr_1\colon M \times \mathbb{R}^k \to M$ is the projection onto the first factor, then the following diagram must commute:

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & M \times \mathbb{R}^k \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle pr_1} \\ M & \xrightarrow[id]{} & M \end{array} \quad .$$

As it is not always easy to find an explicit diffeomorphism, the following result provides a characterization of the triviality of a vector bundle using linearly independent sections of E. A proof can be checked, for example, in [32, Chapter 1].

Proposition 2.18. *A vector bundle $E \to M$ of rank k is trivial if and only if there exist k sections which are linearly independent at any point, that is to say, $X^1, \dots, X^k \in \Gamma(E)$ such that the vectors $X^1_p, \dots, X^k_p \in E_p$ are linearly independent for every $p \in M$.*

According to the above proposition, the tangent bundle TM of a manifold will be trivial if and only if there are n vector fields $X^1, \dots, X^n \in \mathfrak{X}(M)$ such that, at each point $p \in M$, the vectors $X^1_p, \dots, X^n_p \in T_pM$ are linearly independent. On the other hand, when working with the generalized tangent bundle $\mathbb{T}M$, we have that $\mathbb{T}M$ is trivial if and only if there are $2n$ vector fields $X^1, \dots, X^{2n} \in \mathfrak{X}(M)$ and $2n$ differential forms $\xi^1, \dots, \xi^{2n} \in \Omega^1(M)$ such that, at each point $p \in M$, the vectors $X^1_p + \xi^1_p, \dots, X^{2n}_p + \xi^{2n}_p \in \mathbb{T}_pM$ are linearly independent.

The mere definition of the generalized tangent bundle of a manifold reveals that the properties of $\mathbb{T}M$ may be related to the properties of the tangent bundle of the manifold. While the triviality of TM implies the triviality of $\mathbb{T}M$, the converse implication is not true. We check both statements.

Proposition 2.19. *If the tangent bundle TM of a manifold is trivial, then its generalized tangent bundle $\mathbb{T}M$ is trivial.*

Proof. Because TM is a trivial bundle, according to Proposition 2.18 there are n vector fields $X^1, \dots, X^n \in \mathfrak{X}(M)$ which are linearly independent at each point of the manifold. Then, we can transform these vector fields into 1-forms using the flat isomorphism of any metric g on M (every smooth manifold which is Hausdorff and second countable admits a Riemannian metric). The 1-forms $\flat_g X^1, \dots, \flat_g X^n \in \Omega^1(M)$ are also linearly independent. In this manner, the $2n$ vectors $X^1_p, \dots, X^n_p, \flat_g X^1_p, \dots, \flat_g X^n_p \in T_pM \oplus T^*_pM$ are linearly independent regardless of the point $p \in M$ and thus the result is proved. □

Proposition 2.20. *The generalized tangent bundle $\mathbb{T}M$ of the Möbius band, which is defined as the quotient*

$$M = \mathbb{R} \times (-1,1)/\sim, \quad (u,v) \sim (u+m, (-1)^m v), \quad m \in \mathbb{Z},$$

is trivial.

Proof. The Möbius band is endowed with a structure of smooth manifold taking the open subsets $U = M \setminus \{[(0,v)] \in M\}$ and $V = M \setminus \{[(1/2,v)] \in M\}$. The coordinate charts associated to these subsets are, respectively, $x \colon U \longrightarrow \mathbb{R}^2$ and $y \colon V \longrightarrow \mathbb{R}^2$, which are defined as

$$x([(u,v)]) = (u,v) \text{ with } 0 < u < 1, \qquad y([(u,v)]) = \begin{cases} (u,v) & \text{if } 0 \leq u < 1/2, \\ (u-1,-v) & \text{if } 1/2 < u < 1. \end{cases}$$

Then, considering that

$$U \cap V = \{[(u,v)] \in M \colon 0 < u < 1/2 \text{ or } 1/2 < u < 1\},$$

the transition map $y \circ x^{-1} \colon x(U \cap V) \to y(U \cap V)$ is

$$(y \circ x^{-1})(u,v) = \begin{cases} (u,v) & \text{if } 0 < u < 1/2, \\ (u-1,-v) & \text{if } 1/2 < u < 1. \end{cases} \tag{2.14}$$

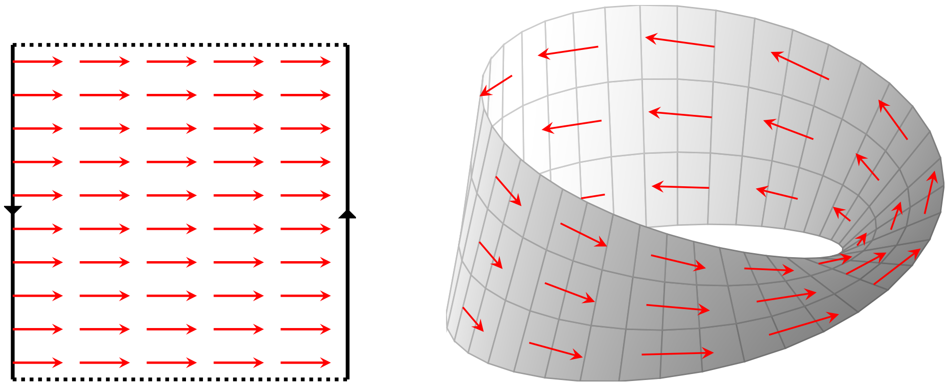
It is clear that M is a non-orientable manifold an thus TM is not trivial. From Eq. (2.14), it is immediate to see that the local coordinate fields associated to these charts fulfill the following relations for any $[(u,v)] \in U \cap V$:

$$\left.\frac{\partial}{\partial y^1}\right|_{[(u,v)]} = \left.\frac{\partial}{\partial x^1}\right|_{[(u,v)]}, \qquad \left.\frac{\partial}{\partial y^2}\right|_{[(u,v)]} = \begin{cases} \left.\dfrac{\partial}{\partial x^2}\right|_{[(u,v)]} & \text{if } 0 < u < 1/2, \\ -\left.\dfrac{\partial}{\partial x^2}\right|_{[(u,v)]} & \text{if } 1/2 < u < 1. \end{cases}$$

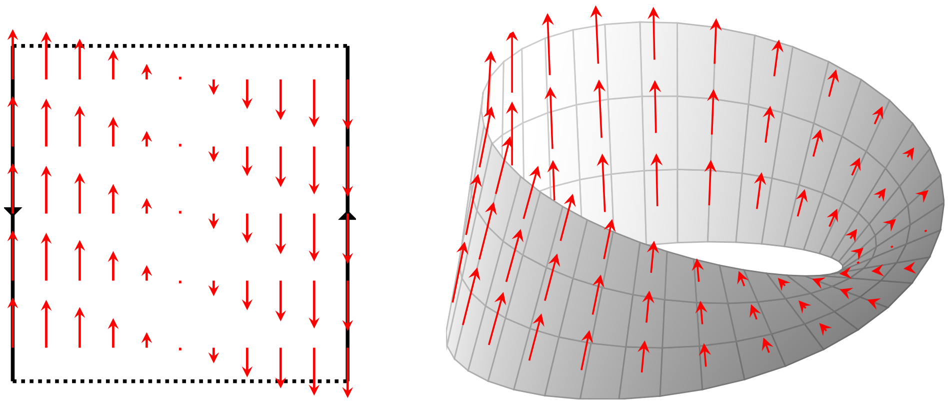
We can define the following vector fields on $\mathbb{R}^2$ using standard coordinates (u,v):

$$X = \frac{\partial}{\partial u}, \quad Y = \cos(\pi u)\frac{\partial}{\partial v}, \quad Z = \sin(\pi u)\frac{\partial}{\partial v}.$$

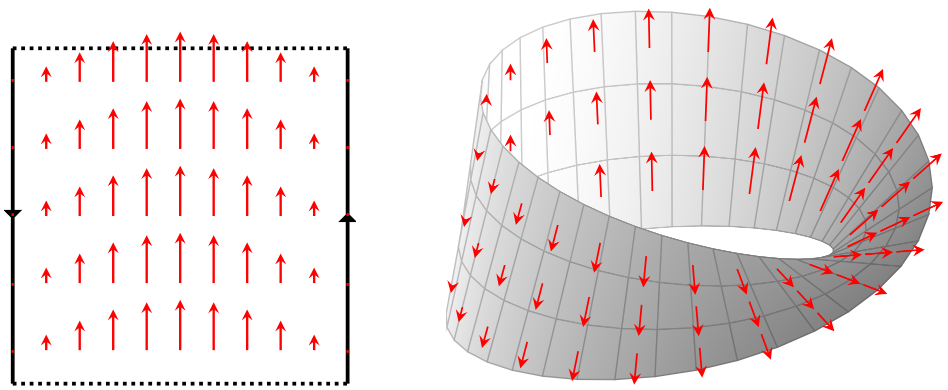
It is clear that these vector fields are smooth. If we restrict these vector fields to $\mathbb{R} \times (-1,1)$ and use the quotient map $q\colon \mathbb{R} \times (-1,1) \to M$, we must check that $q_*(X)$, $q_*(Y)$ and $q_*(Z)$ are smooth vector fields on M.



(A) Vector field $q_*(X)$.



(B) Vector field $q_*(Y)$.



(C) Vector field $q_*(Z)$.

FIGURE 2.1: Smooth vector fields defined on the Möbius band M.

To see that they are globally defined, it is necessary to check the behavior of $q_*(X)$, $q_*(Y)$, $q_*(Z)$ at the points $[(0,v)]$. As $\cos(0) = -\cos(\pi) = 1$ and $\sin(0) = -\sin(\pi) = 0$, then

$$q_*(X_{(0,y)}) = q_*(X_{(1,-y)}), \quad q_*(Y_{(0,y)}) = q_*(Y_{(1,-y)}), \quad q_*(Z_{(0,y)}) = q_*(Z_{(1,-y)}).$$

Therefore, $q_*(X), q_*(Y), q_*(Z)$ are well-defined.

We also need to check that they are smooth. Using the coordinate chart (U, x), the fields $q_*(X)$, $q_*(Y)$, $q_*(Z)$ are locally described as

$$q_*(X) = \frac{\partial}{\partial x^1}, \quad q_*(Y) = \cos(\pi x^1)\frac{\partial}{\partial x^2}, \quad q_*(Z) = \sin(\pi x^1)\frac{\partial}{\partial x^2}.$$

The description in the coordinate chart (V, x) is analogous:

$$q_*(X) = \frac{\partial}{\partial y^1}, \quad q_*(Y) = \cos(\pi y^1)\frac{\partial}{\partial y^2}, \quad q_*(Z) = \sin(\pi y^1)\frac{\partial}{\partial y^2}.$$

As $\cos(\pi(x-1)) = -\cos(\pi x)$ and $\sin(\pi(x-1)) = -\sin(\pi x)$, these coordinate representations are compatible between them. Therefore, $q_*(X)$, $q_*(Y)$, $q_*(Z)$ are smooth vector fields.

In order to ease the notation, from now on the vector fields $q_*(X)$, $q_*(Y)$, $q_*(Z)$ will be denoted as X, Y, Z respectively. These three vector fields are represented in Figure 2.1. Then, using any auxiliary Riemannian metric g on the Möbius band, we can obtain the following sections $w_1, w_2, w_3, w_4 \in \Gamma(\mathbb{T}M)$ of the generalized tangent bundle:

$$w_1 = X, \quad w_2 = \flat_g X, \quad w_3 = Y - \flat_g Z, \quad w_4 = Z + \flat_g Y.$$

It must be checked that these sections are linearly independent at every point in M. It is clear that it suffices to check the linear independence of w_3 and w_4. At any point $[(0, v)]$, using the coordinate chart y we have that $Y_{[(0,v)]} = \frac{\partial}{\partial y^2}\Big|_{[(0,v)]}$ and $Z_{[(0,v)]} = 0$, hence the linear independence of $w_{3[(0,v)]}$ and $w_{4[(0,v)]}$ ($w_{3[(0,v)]}$ is in TM, whereas $w_{4[(0,v)]}$ is in T^*M). At any $[(1/2, v)]$, with the coordinate chart x it is immediate to see that $Y_{[(1/2,v)]} = 0$ and $Z_{[(1/2,v)]} = \frac{\partial}{\partial x^2}\big|_{[(1/2,v)]}$, therefore $w_{3[(1/2,v)]}$ is in T^*M and $w_{4[(1/2,v)]}$ is in TM. Finally, for $[(u, v)]$ with $u \neq 0, 1/2$, using the coordinate chart x it is

$$\begin{aligned} w_{3[(u,v)]} &= \cos(\pi u)\, \frac{\partial}{\partial x^2}\bigg|_{[(u,v)]} - \sin(\pi u)\flat_g\, \frac{\partial}{\partial x^2}\bigg|_{[(u,v)]}, \\ w_{4[(u,v)]} &= \sin(\pi u)\, \frac{\partial}{\partial x^2}\bigg|_{[(u,v)]} + \cos(\pi u)\flat_g\, \frac{\partial}{\partial x^2}\bigg|_{[(u,v)]}. \end{aligned}$$

If $0 < u < 1/2$, then $\cos(\pi u) > 0$ and $\sin(\pi u) > 0$. Thus, there will not exist any non-null $\lambda \in \mathbb{R}$ such that $\lambda X_{3[(u,v)]} = X_{4[(u,v)]}$. Analogously, if $1/2 < u < 1$, then $\cos(\pi u) < 0$ and $\sin(\pi x) > 0$, hence the linear independence of $\lambda X_{3[(u,v)]}$ and $X_{4[(x,y)]}$.

All these calculations have shown that w_1, w_2, w_3, w_4 are four sections of the generalized tangent bundle of the Möbius band that are linearly independent. Therefore, $\mathbb{T}M$ is a trivial vector bundle, as we wanted to check. $\square$

We can also find an example of a compact manifold whose tangent bundle is not trivial but with trivial generalized tangent bundle. This manifold is the Klein bottle, defined as the quotient

$$K = \mathbb{R} \times [-1,1]/\sim, \quad (u,-1) \sim (u,1), \quad (u,v) \sim (u+m,(-1)^m v), \quad m \in \mathbb{Z}.$$

Following an analogous procedure to the previous one, it can be seen that the Klein bottle admits four sections in $\Gamma(\mathbb{T}K)$ which are linearly independent at every point of K.

Chapter 3

Weak generalized structures

So far, we have introduced the notions of several geometric structures that can be defined on any vector bundle $E \to M$, without focusing on any specific choice for E. In this chapter, we qualify some of those definitions for the generalized tangent bundle $E = \mathbb{T}M$.

The first examples by Gualtieri in [29] and Wade in [60] showed that almost complex manifolds (M, J) induce a generalized almost complex structure; almost product manifolds (M, F) induce a generalized almost paracomplex structure; and almost symplectic manifolds (M, ω) induce generalized almost complex structures and almost paracomplex ones. Using matrix notation, these generalized structures are, respectively,

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{F}_F = \begin{pmatrix} F & 0 \\ 0 & -F^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\sharp_\omega \\ \flat_\omega & 0 \end{pmatrix}, \quad \mathcal{F}_\omega = \begin{pmatrix} 0 & \sharp_\omega \\ \flat_\omega & 0 \end{pmatrix}.$$

Therefore, there is a connection between classical geometric structures and those that can be defined on the generalized tangent bundle. The first such relationships, represented by the examples by Gualtieri and Wade, are shown in Figure 3.1.

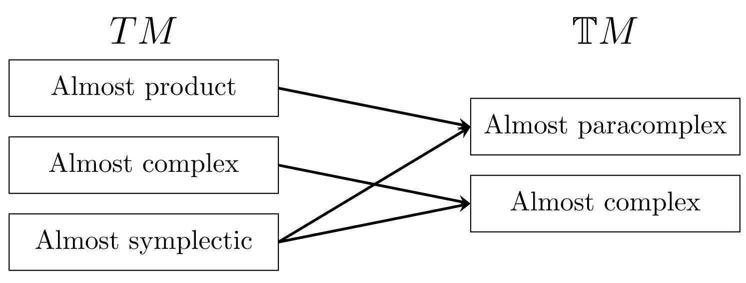


FIGURE 3.1: First generalized polynomial structures induced from geometric structures on the manifold. The arrows represent when a geometric structure on the tangent bundle induces a generalized geometric structure.

As is discussed in Chapter 1, the search for these generalized structures was strongly constrained by the canonical pairing, as each endomorphism was required to be skew-symmetric with respect to $\mathcal{G}_0$. Our main focus in this chapter is to relax this compatibility condition in order to introduce new interesting generalized polynomial structures. We will use the word "weak" as a means to refer to any kind of generalized polynomial structure. Also, we will reorganize within our scheme the generalized polynomial structures that have previously been introduced by other authors.

3.1 Weak generalized polynomial structures

The description of a generalized polynomial structure defined on a vector bundle $E \to M$ is specified in Definition 2.4. Therefore, it should be straightforward to translate this definition to the particular case when $E = \mathbb{T}M$. However, the original definition of generalized almost complex structures given by Gualtieri was more restrictive: they were required to be compatible with the canonical pairing $\mathcal{G}_0$. Therefore, in order to differentiate between the definition given for any vector bundle and the original one, we will use the adjective "weak" to refer to the first case.

Definition 3.1. A polynomial structure on the generalized tangent bundle (that is to say, an endomorphism $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$ that satisfies the requirements given in Definition 2.4) is called a *weak generalized polynomial structure*.

If $\mathcal{K}^2 = -\mathcal{I}d$, it is called a *weak generalized almost complex structure*; if $\mathcal{K}^2 = \mathcal{I}d$, it is called a *weak generalized almost product structure*; if $\mathcal{K}^2 = \mathcal{I}d$ and the ± 1-eigenbundles associated to $\mathcal{K}$ have the same dimension, $\mathcal{K}$ is called a *weak generalized almost paracomplex structure*; and if $\mathcal{K}^2 = 0$ and $\dim(\operatorname{im}\mathcal{K}) = \dim M = n$, $\mathcal{K}$ is called a *weak generalized almost tangent structure*.

When the base manifold M is endowed with certain geometric structures defined as tensor fields on the manifold, various weak generalized polynomial structures are induced on $\mathbb{T}M$. Our objective in this section is to describe those structures. The generalized structures are introduced separately according to the geometry of the base manifold.

3.1.1 Induced from a metric

First, we study weak generalized polynomial structures that arise on a base manifold endowed with a (pseudo-)Riemannian metric (M, g). In this case, the musical isomorphisms associated to the metric (see Definition 2.3) can be used to induce different endomorphisms on the generalized tangent bundle. These morphisms are two weak generalized almost tangent structures, a weak generalized almost complex structure, and a weak generalized almost paracomplex structure. These morphisms are described in the following results.

Proposition 3.2. *Let (M, g) be a (pseudo-)Riemannian manifold. Then, the endomorphisms of the generalized tangent bundle $\mathcal{S}_{g,\flat}, \mathcal{S}_{g,\sharp}\colon \mathbb{T}M \to \mathbb{T}M$, which are defined as $\mathcal{S}_{g,\flat}(X+\xi) = \flat_g X$ and $\mathcal{S}_{g,\sharp}(X+\xi) = \sharp_g \xi$, are weak generalized almost tangent structures. In matrix notation, these structures are*

$$\mathcal{S}_{g,\flat} = \begin{pmatrix} 0 & 0 \\ \flat_g & 0 \end{pmatrix}, \quad \mathcal{S}_{g,\sharp} = \begin{pmatrix} 0 & \sharp_g \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

Proof. It is immediate to see that $\mathcal{S}_{g,\flat}^2 = \mathcal{S}_{g,\sharp}^2 = 0$. Also, it is straightforward to check that the image and kernel of each of these morphisms are given by

$$\operatorname{im}\mathcal{S}_{g,\flat} = \ker\mathcal{S}_{g,\flat} = T^*M, \quad \operatorname{im}\mathcal{S}_{g,\sharp} = \ker\mathcal{S}_{g,\sharp} = TM.$$

Therefore, the rank of both $\mathcal{S}_{g,\flat}$ and $\mathcal{S}_{g,\sharp}$ is half of the rank of $\mathbb{T}M$. □

Proposition 3.3 ([50, Section 4]). *Let (M, g) be a (pseudo-)Riemannian manifold. Then, the endomorphism $\mathcal{J}_g\colon \mathbb{T}M \to \mathbb{T}M$ described as $\mathcal{J}_g(X+\xi) = -\sharp_g\xi + \flat_g X$ is a weak generalized almost complex structure. In matrix notation, it is*

$$\mathcal{J}_g = \begin{pmatrix} 0 & -\sharp_g \\ \flat_g & 0 \end{pmatrix}. \tag{3.2}$$

The $\pm i$-eigenbundles of $\mathcal{J}_g$ are

$$\mathbb{L}^{1,0}_{\mathcal{J}_g} = \{X - i\flat_g X \in \mathbb{T}M_{\mathbb{C}}\colon X \in TM_{\mathbb{C}}\}, \quad \mathbb{L}^{0,1}_{\mathcal{J}_g} = \{X + i\flat_g X \in \mathbb{T}M_{\mathbb{C}}\colon X \in TM_{\mathbb{C}}\},$$

and the projections onto each subbundle are

$$\mathcal{P}^{1,0}_{\mathcal{J}_g}(X+\xi) = \frac{1}{2}\left((X + i\sharp_g\xi) + (\xi - i\flat_g X)\right), \quad \mathcal{P}^{0,1}_{\mathcal{J}_g}(X+\xi) = \frac{1}{2}\left((X - i\sharp_g\xi) + (\xi + i\flat_g X)\right).$$

Proof. Using matrix notation, it is easy to see that $\mathcal{J}_g^2 = -\mathcal{I}d$:

$$\mathcal{J}_g^2 = \begin{pmatrix} 0 & -\sharp_g \\ \flat_g & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sharp_g \\ \flat_g & 0 \end{pmatrix} = \begin{pmatrix} -\sharp_g\flat_g & 0 \\ 0 & -\flat_g\sharp_g \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & -Id \end{pmatrix}.$$

With respect to the i-eigenbundle of $\mathcal{J}_g$, since $X - i\mathcal{J}_g X = X - i\flat_g X$ for every $X \in TM_{\mathbb{C}}$ it is clear that $\{X - i\flat_g X \in \mathbb{T}M_{\mathbb{C}}\colon X \in TM_{\mathbb{C}}\} \subseteq \mathbb{L}^{1,0}_{\mathcal{J}_g}$. To see the opposite inclusion, we must check that for every $\xi \in T^*M$ there is a $Y \in TM_{\mathbb{C}}$ such that $Y - i\flat_g Y = \xi - i\mathcal{J}_g\xi = \xi + i\sharp_g\xi$. This is immediate to prove, simply by taking $Y = i\sharp_g\xi$.

The verification for $-i$ is analogous: $X + i\mathcal{J}_g X = X + i\flat_g X$ for each $X \in TM_{\mathbb{C}}$ and thus $\{X + i\flat_g X \in \mathbb{T}M_{\mathbb{C}}\colon X \in TM_{\mathbb{C}}\} \subseteq \mathbb{L}^{0,1}_{\mathcal{J}_g}$; and given any $\xi \in T^*M$ there is a $Y' \in TM_{\mathbb{C}}$ such that $Y' + i\flat_g Y' = \xi + i\mathcal{J}_g\xi = \xi - i\sharp_g\xi$, which is given by $Y' = -i\sharp_g\xi$.

Finally, it is straightforward to check the expressions for $\mathcal{P}^{1,0}_{\mathcal{J}_g}$ and $\mathcal{P}^{0,1}_{\mathcal{J}_g}$, using the expressions given in Eq. (2.2). □

Proposition 3.4 ([38, Example 3.1]). *Let (M, g) be a (pseudo-)Riemannian manifold. Then, the endomorphism $\mathcal{F}_g\colon \mathbb{T}M \to \mathbb{T}M$, which is described as $\mathcal{F}_g(X+\xi) = \sharp_g\xi + \flat_g X$, is a weak generalized almost paracomplex structure. In matrix notation, it is*

$$\mathcal{F}_g = \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix}. \tag{3.3}$$

The ± 1-eigenbundles of $\mathcal{F}_g$ are

$$\mathbb{L}^{+}_{\mathcal{F}_g} = \{X + \flat_g X \in \mathbb{T}M\colon X \in TM\}, \quad \mathbb{L}^{-}_{\mathcal{F}_g} = \{X - \flat_g X \in \mathbb{T}M\colon X \in TM\},$$

and their associated projections are

$$\mathcal{P}^{+}_{\mathcal{F}_g}(X+\xi) = \frac{1}{2}\left((X + \sharp_g\xi) + (\xi + \flat_g X)\right), \quad \mathcal{P}^{-}_{\mathcal{F}_g}(X+\xi) = \frac{1}{2}\left((X - \sharp_g\xi) + (\xi - \flat_g X)\right).$$

Proof. It is immediate to check that $\mathcal{F}_g^2 = -\mathcal{I}d$:

$$\mathcal{F}_g^2 = \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix} \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix} = \begin{pmatrix} \sharp_g \flat_g & 0 \\ 0 & \flat_g \sharp_g \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}.$$

With respect to the eigenbundles of $\mathcal{F}_g$, as $X + \mathcal{F}_g X = X + \flat_g X$ for every $X \in TM$ we have $\{X + \flat_g X \in \mathbb{T}M \colon X \in TM\} \subseteq \mathbb{L}^+_{\mathcal{F}_g}$. To check the converse inclusion, given any $\xi \in T^*M$ we must find a $Y \in TM$ such that $Y + \flat_g Y = \xi + \mathcal{F}_g \xi = \xi + \sharp_g \xi$. This is immediate to see, taking $Y = \sharp_g \xi$.

The verification for the -1-eigenbundle is similar: $X - \mathcal{F}_g X = X - \flat_g X$ for each $X \in TM$ and hence $\{X - \flat_g X \in \mathbb{T}M \colon X \in TM\} \subseteq \mathbb{L}^-_{\mathcal{F}_g}$; and for each $\xi \in T^*M$ there is a $Y' \in TM$ such that $Y' - \flat_g Y' = \xi - \mathcal{F}_g \xi = \xi - \sharp_g \xi$, being $Y' = -\sharp_g \xi$.

Finally, it is immediate to check the expressions for $\mathcal{P}^+_{\mathcal{F}_g}$ and $\mathcal{P}^-_{\mathcal{F}_g}$, just by using the expressions given in Eq. (2.4). □

3.1.2 Induced from an almost symplectic structure

One might have noticed that the generalized structures induced on a (pseudo-)Riemannian manifold (M, g) rely only on the nondegeneracy of g and not on the properties related to its symmetry. Then, the arguments used for a metric can also be applied when the manifold is almost symplectic: using the musical isomorphisms of an almost symplectic structure ω, one can induce two weak generalized almost tangent structures, a weak generalized almost complex structure and a weak generalized almost paracomplex structure. These structures are gathered in the following three propositions. As the proofs of these results are analogous to the ones of Propositions 3.2, 3.3 and 3.4, they will be omitted.

Proposition 3.5. *Let (M, ω) be an almost symplectic manifold. Then, the endomorphisms of the generalized tangent bundle $\mathcal{S}_{\omega,\flat}, \mathcal{S}_{\omega,\sharp} \colon \mathbb{T}M \to \mathbb{T}M$, which are defined as $\mathcal{S}_{\omega,\flat}(X + \xi) = \flat_\omega X$ and $\mathcal{S}_{\omega,\sharp}(X + \xi) = \sharp_\omega \xi$, are weak generalized almost tangent structures. In matrix notation, these structures are*

$$\mathcal{S}_{\omega,\flat} = \begin{pmatrix} 0 & 0 \\ \flat_\omega & 0 \end{pmatrix}, \quad \mathcal{S}_{\omega,\sharp} = \begin{pmatrix} 0 & \sharp_\omega \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

Proposition 3.6 ([29, Example 4.10])**.** *Let (M, ω) be an almost symplectic manifold. Then, the endomorphism $\mathcal{J}_\omega \colon \mathbb{T}M \to \mathbb{T}M$ described as $\mathcal{J}_\omega(X + \xi) = -\sharp_\omega \xi + \flat_\omega X$ is a weak generalized almost complex structure. In matrix notation, it is*

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\sharp_\omega \\ \flat_\omega & 0 \end{pmatrix}. \tag{3.5}$$

The $\pm i$-eigenbundles of $\mathcal{J}_\omega$ are

$$\mathbb{L}^{1,0}_{\mathcal{J}_\omega} = \{X - i\flat_\omega X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\}, \quad \mathbb{L}^{0,1}_{\mathcal{J}_\omega} = \{X + i\flat_\omega X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\},$$

and the projections onto each subbundle are

$$\mathcal{P}^{1,0}_{\mathcal{J}_\omega}(X+\xi) = \frac{1}{2}\left((X - i\flat_\omega X) + (\xi + i\sharp_\omega \xi)\right), \quad \mathcal{P}^{0,1}_{\mathcal{J}_\omega}(X+\xi) = \frac{1}{2}\left((X + i\flat_\omega X) + (\xi - i\sharp_\omega \xi)\right).$$

Proposition 3.7 ([60, Example 2])**.** *Let (M,ω) be an almost symplectic manifold. Then, the endomorphism $\mathcal{F}_\omega\colon \mathbb{T}M \to \mathbb{T}M$, which is described as $\mathcal{F}_\omega(X+\xi) = \sharp_\omega\xi + \flat_\omega X$, is a weak generalized almost paracomplex structure. In matrix notation, it is*

$$\mathcal{F}_\omega = \begin{pmatrix} 0 & \sharp_\omega \\ \flat_\omega & 0 \end{pmatrix}. \tag{3.6}$$

The ± 1-eigenbundles of $\mathcal{F}_\omega$ are

$$\mathbb{L}^+_{\mathcal{F}_\omega} = \{X + \flat_\omega X \in \mathbb{T}M \colon X \in TM\}, \quad \mathbb{L}^-_{\mathcal{F}_\omega} = \{X - \flat_\omega X \in \mathbb{T}M \colon X \in TM\},$$

and their associated projections are

$$\mathcal{P}^+_{\mathcal{F}_\omega}(X+\xi) = \frac{1}{2}\left((X + \sharp_\omega\xi) + (\xi + \flat_\omega X)\right), \quad \mathcal{P}^-_{\mathcal{F}_\omega}(X+\xi) = \frac{1}{2}\left((X - \sharp_\omega\xi) + (\xi - \flat_\omega X)\right).$$

3.1.3 Induced from a polynomial structure

When there is a polynomial structure $K\colon TM \to TM$ defined on the manifold with minimal polynomial equal to P, it is possible to induce multiple weak generalized polynomial structures with P as their minimal polynomial. This generalization can be made using diagonal matrices. The following three results gather the weak generalized polynomial structures that are induced from an almost complex, almost product or almost tangent structure on the base manifold, respectively.

Proposition 3.8. *Let (M,S) be an almost tangent manifold and $\lambda \in \mathbb{R}\backslash\{0\}$. Then, the endomorphism $\mathcal{S}_{\lambda,S}\colon \mathbb{T}M \to \mathbb{T}M$ that is described as $\mathcal{S}_{\lambda,S}(X+\xi) = SX + \lambda S^*\xi$ is a weak generalized almost tangent structure. In matrix notation, it is*

$$\mathcal{S}_{\lambda,S} = \begin{pmatrix} S & 0 \\ 0 & \lambda S^* \end{pmatrix}. \tag{3.7}$$

Regardless of the value of λ, the image and the kernel of $\mathcal{S}_{\lambda,S}$ are given by

$$\operatorname{im} \mathcal{S}_{\lambda,S} = \ker \mathcal{S}_{\lambda,S} = \ker S \oplus \ker S^*.$$

Proof. It is clear that $\mathcal{S}^2_{\lambda,S} = 0$. The only thing that remains is to describe the image and kernel of $\mathcal{S}_{\lambda,S}$. As S and S^* are almost tangent structures and $\lambda \neq 0$, we have that, regardless of the value of λ,

$$\ker \mathcal{S}_{\lambda,S} = \ker S \oplus \ker S^* = \operatorname{im} S \oplus \operatorname{im} S^* = \operatorname{im} \mathcal{S}_{\lambda,S}.$$

Therefore, the rank of $\mathcal{S}_{\lambda,S}$ is half of the rank of $\mathbb{T}M$. □

The morphism $\mathcal{S}_{-1,S}$ has previously been studied by other authors, such as A. M. Blaga and M. Crasmareanu (see [7, Example 3.3]).

Proposition 3.9. *Let (M,J) be an almost complex manifold and $\lambda \in \{1,-1\}$. Then, the endomorphism $\mathcal{J}_{\lambda,J}\colon \mathbb{T}M \to \mathbb{T}M$ that is described as $\mathcal{J}_{\lambda,J}(X+\xi) = JX + \lambda J^*\xi$ is a weak generalized almost complex structure. In matrix notation, it is*

$$\mathcal{J}_{\lambda,J} = \begin{pmatrix} J & 0 \\ 0 & \lambda J^* \end{pmatrix}. \tag{3.8}$$

For $\lambda = 1$, the $\pm i$-eigenbundles $\mathbb{L}^{1,0}_{\mathcal{J}_{1,J}}, \mathbb{L}^{0,1}_{\mathcal{J}_{1,J}} \subset \mathbb{T}M_{\mathbb{C}}$ associated to $\mathcal{J}_{1,J}$ are

$$\mathbb{L}^{1,0}_{\mathcal{J}_{1,J}} = L^{1,0}_J \oplus L^{1,0}_{J^*}, \quad \mathbb{L}^{0,1}_{\mathcal{J}_{1,J}} = L^{0,1}_J \oplus L^{0,1}_{J^*},$$

where $L^{1,0}_J, L^{0,1}_J \subset TM_{\mathbb{C}}$ are the $\pm i$-eigenbundles of J, and $L^{1,0}_{J^*}, L^{0,1}_{J^*} \subset T^*M_{\mathbb{C}}$ are the $\pm i$-eigenbundles of J^*. The projections onto each subbundle are, respectively,

$$\mathcal{P}^{1,0}_{\mathcal{J}_{1,J}}(X+\xi) = P^{1,0}_J X + P^{1,0}_{J^*}\xi, \quad \mathcal{P}^{0,1}_{\mathcal{J}_{1,J}}(X+\xi) = P^{0,1}_J X + P^{0,1}_{J^*}\xi,$$

where $P^{1,0}_J$ and $P^{0,1}_J$ are the projections onto $L^{1,0}_J$ and $L^{0,1}_J$, respectively; and $P^{1,0}_{J^*}, P^{0,1}_{J^*}$ are the corresponding ones onto $L^{1,0}_{J^*}, L^{0,1}_{J^*}$. Likewise, for $\lambda = -1$ these subbundles are

$$\mathbb{L}^{1,0}_{\mathcal{J}_{-1,J}} = L^{1,0}_J \oplus L^{0,1}_{J^*}, \quad \mathbb{L}^{0,1}_{\mathcal{J}_{-1,J}} = L^{0,1}_J \oplus L^{1,0}_{J^*},$$

and the projections to each subbundle are

$$\mathcal{P}^{1,0}_{\mathcal{J}_{-1,J}}(X+\xi) = P^{1,0}_J X + P^{0,1}_{J^*}\xi, \quad \mathcal{P}^{0,1}_{\mathcal{J}_{-1,J}}(X+\xi) = P^{0,1}_J X + P^{1,0}_{J^*}\xi.$$

Proof. It is straightforward to see that $\mathcal{J}^2_{\lambda,J}(X+\xi) = J^2X + \lambda^2(J^*)^2\xi = -(X+\xi)$. Moreover, taking into account that $(X+\xi) \mp i\mathcal{J}_{\lambda,J}(X+\xi) = (X \mp iJX) + (\xi \mp i\lambda J^*\xi)$ we obtain the expressions for all the eigenbundles and projections. □

The morphism $\mathcal{J}_{-1,J}$ corresponds to the one presented by Gualtieri in [29, Example 4.10]. Even though $\mathcal{J}_{1,J}$ is not skew-symmetric with respect to the metric $\mathcal{G}_0$, following our terminology, both morphisms $\mathcal{J}_{-1,J}$ and $\mathcal{J}_{1,J}$ are weak generalized almost complex structures.

Proposition 3.10 ([38, Section 3.2])**.** *Let (M,F) be an almost product manifold and $\lambda \in \{1,-1\}$. Then, the endomorphism $\mathcal{F}_{\lambda,F}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ that is described as $\mathcal{F}_{F,\lambda}(X+\xi) = FX + \lambda F^*\xi$ is a weak generalized almost product structure. In matrix notation, it is*

$$\mathcal{F}_{\lambda,F} = \begin{pmatrix} F & 0 \\ 0 & \lambda F^* \end{pmatrix}. \tag{3.9}$$

For $\lambda = 1$, the ± 1-eigenbundles associated to $\mathcal{F}_{1,F}$ are, respectively,

$$\mathbb{L}^{+}_{\mathcal{F}_{1,F}} = L^{+}_F \oplus L^{+}_{F^*}, \quad \mathbb{L}^{-}_{\mathcal{F}_{1,F}} = L^{-}_F \oplus L^{-}_{F^*},$$

and the projections onto each of these subbundles are

$$\mathcal{P}^+_{\mathcal{F}_{+1,F}}(X+\xi) = P^+_F X + P^+_{F^*}\xi, \quad \mathcal{P}^-_{\mathcal{F}_{+1,F}}(X+\xi) = P^-_F X + P^-_{F^*}\xi.$$

where P^+_F and P^-_F are the projections onto L^+_F and L^-_F, respectively; and $P^+_{F^*}, P^-_{F^*}$ are the corresponding ones onto $L^+_{F^*}, L^-_{F^*}$. Likewise, for $\lambda = -1$ the ± 1-subbundles associated to $\mathcal{F}_{-1,F}$ are

$$\mathbb{L}^+_{\mathcal{F}_{-1,F}} = L^+_F \oplus L^-_{F^*}, \quad \mathbb{L}^-_{\mathcal{F}_{-1,F}} = L^-_F \oplus L^+_{F^*},$$

with respective projections

$$\mathcal{P}^+_{\mathcal{F}_{-1,F}}(X+\xi) = P^+_F X + P^-_{F^*}\xi, \quad \mathcal{P}^-_{\mathcal{F}_{-1,F}}(X+\xi) = P^-_F X + P^+_{F^*}\xi.$$

Proof. It is a direct computation to check that $\mathcal{F}^2_{\lambda,F}(X+\xi) = F^2X + \lambda^2(F^*)^2\xi = X+\xi$. Moreover, as $(X+\xi)\pm\mathcal{F}_{\lambda,F}(X+\xi) = (X\pm FX)+(\xi\pm\lambda F^*\xi)$, we obtain the expressions for all the eigenbundles and projections. □

Analyzing the dimensions of each subbundle, as $\dim L^+_F = \dim L^+_{F^*}$ then the morphism $\mathcal{F}_{1,F}$ is a weak generalized almost paracomplex structure if and only if F is an almost paracomplex structure, while $\mathcal{F}_{-1,F}$ is always a generalized almost paracomplex structure, regardless of whether F is almost paracomplex or just almost product.

The morphism $\mathcal{F}_{-1,F}$ corresponds to the one presented by Wade in [60, Example 4]. Similarly to almost complex structures, even though $\mathcal{F}_{1,F}$ is not skew-symmetric with respect to $\mathcal{G}_0$, from our point of view both $\mathcal{F}_{-1,F}$ and $\mathcal{F}_{1,F}$ are weak generalized almost product structures.

3.1.4 Induced from an (α,ε)-metric structure

The generalized structures studied so far are derived from basic geometries on the base manifold. Our aim now is to induce weak generalized polynomial structures using more complicated geometric structures on the manifold. Specifically, we work with (α,ε)-metric manifolds, that is, with a manifold endowed with a polynomial structure and a compatible metric (see Definition 2.6).

If we denote the metric of the (α,ε)-metric structure as g and its fundamental tensor (which is described in Eq. (2.6) and can be a metric or an almost symplectic structure) as φ, these two nondegenerate structures induce four weak generalized almost tangent structures. These structures are $\mathcal{S}_{g,\flat}$, $\mathcal{S}_{g,\sharp}$, $\mathcal{S}_{\varphi,\flat}$, $\mathcal{S}_{\varphi,\sharp}$, introduced in Eqs. (3.1) and (3.4). Also, they induce the weak generalized almost complex structures $\mathcal{J}_g$, $\mathcal{J}_\varphi$ from Eqs. (3.2) and (3.5), and the weak generalized almost paracomplex structures $\mathcal{F}_g$, $\mathcal{F}_\varphi$ from Eqs. (3.3) and (3.6). Furthermore, when $\alpha = -1$ the almost complex structure J of the (α,ε)-metric manifold induces the weak generalized almost complex structures $\mathcal{J}_{\lambda,J}$ defined in Eq. (3.8), while for $\alpha = 1$ the almost product/paracomplex structure F of the (α,ε)-metric manifold induces the generalized almost product or paracomplex structures $\mathcal{F}_{\lambda,F}$ from Eq. (3.9).

Moreover, using both the polynomial structure and the metric of an (α,ε)-metric manifold at the same time, we can induce other interesting generalized structures. For example, Nannicini introduced in [50, Section 4] a weak generalized almost complex structure from an almost Norden manifold (M, J, g) as follows:

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ \flat_g & -J^* \end{pmatrix}.$$

In a similar fashion, other "triangular" generalized structures can be induced when the manifold is almost Hermitian. The following result gathers all these weak generalized polynomial structures.

Proposition 3.11. *Let (M, J, g) be an (α, ε)-metric manifold with $\alpha = -1$ (that is, $J^2 = -Id$). Then, the endomorphisms $\mathcal{J}_{J,g,\flat}, \mathcal{J}_{J,g,\sharp} \colon \mathbb{T}M \to \mathbb{T}M$ defined as $\mathcal{J}_{J,g,\flat}(X+\xi) = JX + (\flat_g X + \varepsilon J^*\xi)$ and $\mathcal{J}_{J,g,\sharp}(X+\xi) = (JX + \sharp_g \xi) + \varepsilon J^*\xi$ are weak generalized almost complex structures. These structures are represented by the following matrices:*

$$\mathcal{J}_{J,g,\flat} = \begin{pmatrix} J & 0 \\ \flat_g & \varepsilon J^* \end{pmatrix}, \quad \mathcal{J}_{J,g,\sharp} = \begin{pmatrix} J & \sharp_g \\ 0 & \varepsilon J^* \end{pmatrix}. \tag{3.10}$$

The $\pm i$-eigenbundles associated to each of these structures are, respectively,

$$\begin{aligned}
\mathbb{L}^{1,0}_{\mathcal{J}_{J,g,\flat}} &= \{(X - iJX) - i\flat_g X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\}, \\
\mathbb{L}^{0,1}_{\mathcal{J}_{J,g,\flat}} &= \{(X + iJX) + i\flat_g X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\}, \\
\mathbb{L}^{1,0}_{\mathcal{J}_{J,g,\sharp}} &= \{-i\sharp_g \xi + (\xi - i\varepsilon J^*\xi) \in \mathbb{T}M_{\mathbb{C}} \colon \xi \in T^*M_{\mathbb{C}}\}, \\
\mathbb{L}^{0,1}_{\mathcal{J}_{J,g,\sharp}} &= \{i\sharp_g \xi + (\xi + i\varepsilon J^*\xi) \in \mathbb{T}M_{\mathbb{C}} \colon \xi \in T^*M_{\mathbb{C}}\}.
\end{aligned}$$

The projections onto each of these subbundles of $\mathbb{T}M_{\mathbb{C}}$ are

$$\begin{aligned}
\mathcal{P}^{1,0}_{\mathcal{J}_{J,g,\flat}}(X+\xi) &= P^{1,0}_J X + \left(P^{1,0}_{\varepsilon J^*}\xi - \frac{i}{2}\flat_g X\right), \\
\mathcal{P}^{0,1}_{\mathcal{J}_{J,g,\flat}}(X+\xi) &= P^{0,1}_J X + \left(P^{0,1}_{\varepsilon J^*}\xi + \frac{i}{2}\flat_g X\right), \\
\mathcal{P}^{1,0}_{\mathcal{J}_{J,g,\sharp}}(X+\xi) &= \left(P^{1,0}_J X - \frac{i}{2}\sharp_g \xi\right) + P^{1,0}_{\varepsilon J^*}\xi, \\
\mathcal{P}^{0,1}_{\mathcal{J}_{J,g,\sharp}}(X+\xi) &= \left(P^{0,1}_J X + \frac{i}{2}\sharp_g \xi\right) + P^{0,1}_{\varepsilon J^*}\xi,
\end{aligned}$$

where $P^{1,0}_J$ is the projection onto the eigenbundle $L^{1,0}_J$ of J, $P^{0,1}_J$ is the projection onto the eigenbundle $L^{0,1}_J$ of J, $P^{1,0}_{\varepsilon J^*}$ is the projection onto the eigenbundle $L^{1,0}_{\varepsilon J^*}$ of εJ^*, and $P^{0,1}_{\varepsilon J^*}$ is the projection onto the eigenbundle $L^{0,1}_{\varepsilon J^*}$ of εJ^*.

Proof. Using Proposition 2.7, it is straightforward to see that both $\mathcal{J}_{J,g,\flat}$ and $\mathcal{J}_{J,g,\sharp}$ are weak generalized almost complex structures:

$$\begin{pmatrix} J & 0 \\ \flat_g & \varepsilon J^* \end{pmatrix}\begin{pmatrix} J & 0 \\ \flat_g & \varepsilon J^* \end{pmatrix} = \begin{pmatrix} J^2 & 0 \\ \flat_g J + \varepsilon J^* \flat_g & \varepsilon^2 (J^*)^2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & -Id \end{pmatrix},$$
$$\begin{pmatrix} J & \sharp_g \\ 0 & \varepsilon J^* \end{pmatrix}\begin{pmatrix} J & \sharp_g \\ 0 & \varepsilon J^* \end{pmatrix} = \begin{pmatrix} J^2 & J\sharp_g + \varepsilon \sharp_g J^* \\ 0 & \varepsilon^2 (J^*)^2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & -Id \end{pmatrix}.$$

We verify now that the expressions given for the eigenbundles are correct. Since the proofs are similar for $\mathcal{J}_{J,g,\flat}$ and $\mathcal{J}_{J,g,\sharp}$, we will just check the expressions for $\mathbb{L}^{1,0}_{\mathcal{J}_{J,g,\sharp}}$ and $\mathbb{L}^{0,1}_{\mathcal{J}_{J,g,\sharp}}$. Firstly, since the equality $\xi - i\mathcal{J}_{J,g,\sharp}\xi = -i\sharp_g\xi + (\xi - i\varepsilon J^*\xi)$ is true for each $\xi \in T^*M$, we have the inclusion $\{-i\sharp_g\xi + (\xi - i\varepsilon J^*\xi) \in \mathbb{T}M_{\mathbb{C}} \colon \xi \in T^*M_{\mathbb{C}}\} \subseteq \mathbb{L}^{1,0}_{\mathcal{J}_{J,g,\sharp}}$. Likewise, as $\xi + i\mathcal{J}_{J,g,\sharp}\xi = i\sharp_g\xi + (\xi + i\varepsilon J^*\xi)$ for every $\xi \in T^*M$ then $\{i\sharp_g\xi + (\xi + i\varepsilon J^*\xi) \in \mathbb{T}M_{\mathbb{C}} \colon \xi \in T^*M_{\mathbb{C}}\} \subseteq \mathbb{L}^{0,1}_{\mathcal{J}_{J,g,\sharp}}$.

To test the converse inclusions, it must be checked that given any $X \in TM$ one can find an $\eta \in T^*M_{\mathbb{C}}$ such that $-i\sharp_g\eta + \eta - i\varepsilon J^*\eta = X - i\mathcal{J}_{J,g,\sharp}X = X - iJX$; and analogously, that for each $X \in TM$ exists a $\eta' \in T^*M_{\mathbb{C}}$ with $i\sharp_g\eta' + \eta' + i\varepsilon J^*\eta' = X + i\mathcal{J}_{J,g,\sharp}X = X + iJX$. For the eigenbundle $\mathbb{L}^{1,0}_{\mathcal{J}_{J,g,\sharp}}$, our proposal is $\eta = i\flat_g(X - iJX)$; for the eigenbundle $\mathbb{L}^{0,1}_{\mathcal{J}_{J,g,\sharp}}$, we take $\eta' = -i\flat_g(X + iJX)$. We use Eq. (2.7) to make the calculations:

$$\begin{aligned}
-i\sharp_g\eta + \eta - i\varepsilon J^*\eta &= \sharp_g\flat_g(X - iJX) + i\flat_g(X - iJX) + \varepsilon J^*\flat_g(X - iJX)\\
&= X - iJX + i\flat_g X + \flat_g JX - \flat_g JX + i\flat_g J^2X\\
&= X - iJX + i\flat_g X - i\flat_g X\\
&= X - iJX,
\end{aligned}$$

$$\begin{aligned}
i\sharp_g\eta' + \eta' + i\varepsilon J^*\eta' &= \sharp_g\flat_g(X + iJX) - i\flat_g(X + iJX) + \varepsilon J^*\flat_g(X + iJX)\\
&= X + iJX - i\flat_g X + \flat_g JX - \flat_g JX - i\flat_g J^2X\\
&= X + iJX - i\flat_g X + i\flat_g X\\
&= X + iJX.
\end{aligned}$$

Therefore, $\mathbb{L}^{1,0}_{\mathcal{J}_{J,g,\sharp}}$ and $\mathbb{L}^{0,1}_{\mathcal{J}_{J,g,\sharp}}$ are the proposed ones. Finally, we check the description of the projections $\mathcal{P}^{1,0}_{\mathcal{J}_{J,g,\sharp}}$ and $\mathcal{P}^{0,1}_{\mathcal{J}_{J,g,\sharp}}$ using Eq. (2.2):

$$\begin{aligned}
\mathcal{P}^{1,0}_{\mathcal{J}_{J,g,\sharp}}(X + \xi) &= \frac{1}{2}\left((X + \xi) - i\mathcal{J}_{J,g,\sharp}(X + \xi)\right) = \frac{1}{2}(X - iJX) + \frac{1}{2}(\xi - i\varepsilon J^*\xi) - \frac{i}{2}\sharp_g\xi\\
&= \left(P^{1,0}_J X - \frac{i}{2}\sharp_g\xi\right) + P^{1,0}_{\varepsilon J^*}\xi,
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}^{0,1}_{\mathcal{J}_{J,g,\sharp}}(X + \xi) &= \frac{1}{2}\left((X + \xi) + i\mathcal{J}_{J,g,\sharp}(X + \xi)\right) = \frac{1}{2}(X + iJX) + \frac{1}{2}(\xi + i\varepsilon J^*\xi) + \frac{i}{2}\sharp_g\xi\\
&= \left(P^{0,1}_J X + \frac{i}{2}\sharp_g\xi\right) + P^{0,1}_{\varepsilon J^*}\xi.
\end{aligned}$$

□

Similar triangular structures can also be examined for (α, ε)-metric manifolds with $\alpha = 1$. In [38], Ida and Manea proposed the following weak generalized almost paracomplex structure, induced by an almost para-Hermitian manifold (M, F, g) with F almost paracomplex:

$$\mathcal{F} = \begin{pmatrix} F & 0 \\ \flat_g & F^* \end{pmatrix}.$$

Analogously to Proposition 3.11, we present a result that gathers different triangular structures that are induced by an (α, ε)-metric manifold with $\alpha = 1$.

Proposition 3.12. *Let (M, F, g) be an (α, ε)-metric manifold with $\alpha = 1$ (that is, $F^2 = Id$). Then, the endomorphisms $\mathcal{F}_{F,g,\flat}, \mathcal{F}_{F,g,\sharp} \colon \mathbb{T}M \to \mathbb{T}M$ defined as $\mathcal{F}_{F,g,\flat}(X + \xi) = FX + (\flat_g X - \varepsilon F^* \xi)$ and $\mathcal{F}_{F,g,\sharp}(X + \xi) = (FX + \sharp_g \xi) - \varepsilon F^* \xi$ are weak generalized almost product structures. These structures are represented by the following matrices:*

$$\mathcal{F}_{F,g,\flat} = \begin{pmatrix} F & 0 \\ \flat_g & -\varepsilon F^* \end{pmatrix}, \quad \mathcal{F}_{F,g,\sharp} = \begin{pmatrix} F & \sharp_g \\ 0 & -\varepsilon F^* \end{pmatrix}. \tag{3.11}$$

The ± 1-eigenbundles associated to each of these structures are, respectively,

$$\begin{aligned} \mathbb{L}^+_{\mathcal{F}_{F,g,\flat}} &= \{(X + FX) + \flat_g X \in \mathbb{T}M \colon X \in TM\}, \\ \mathbb{L}^-_{\mathcal{F}_{F,g,\flat}} &= \{(X - FX) - \flat_g X \in \mathbb{T}M \colon X \in TM\}, \\ \mathbb{L}^+_{\mathcal{F}_{F,g,\sharp}} &= \{\sharp_g \xi + (\xi - \varepsilon F^* \xi) \in \mathbb{T}M \colon \xi \in T^*M\}, \\ \mathbb{L}^-_{\mathcal{F}_{F,g,\sharp}} &= \{-\sharp_g \xi + (\xi + \varepsilon F^* \xi) \in \mathbb{T}M \colon \xi \in T^*M\}, \end{aligned}$$

and $\dim \mathbb{L}^+_{\mathcal{F}_{F,g,\flat}} = \dim \mathbb{L}^-_{\mathcal{F}_{F,g,\flat}}$ (resp. $\dim \mathbb{L}^+_{\mathcal{F}_{F,g,\sharp}} = \dim \mathbb{L}^-_{\mathcal{F}_{F,g,\sharp}}$) if and only if F is almost paracomplex. The projections onto each of these subbundles are

$$\begin{aligned} \mathcal{P}^+_{\mathcal{F}_{F,g,\flat}}(X + \xi) &= P^+_F X + \left(P^-_{\varepsilon F^*} \xi + \frac{1}{2} \flat_g X\right), \\ \mathcal{P}^-_{\mathcal{F}_{F,g,\flat}}(X + \xi) &= P^-_F X + \left(P^+_{\varepsilon F^*} \xi - \frac{1}{2} \flat_g X\right), \\ \mathcal{P}^+_{\mathcal{F}_{F,g,\sharp}}(X + \xi) &= \left(P^+_F X + \frac{1}{2} \sharp_g \xi\right) + P^-_{\varepsilon F^*} \xi, \\ \mathcal{P}^-_{\mathcal{F}_{F,g,\sharp}}(X + \xi) &= \left(P^-_F X - \frac{1}{2} \sharp_g \xi\right) + P^+_{\varepsilon F^*} \xi, \end{aligned}$$

where P^+_F is the projection onto the eigenbundle L^+_F of F, P^-_F is the projection onto the eigenbundle L^-_F of F, $P^+_{\varepsilon F^*}$ is the projection onto the eigenbundle $L^+_{\varepsilon J^*}$ of εF^*, and $P^-_{\varepsilon F^*}$ is the projection onto the eigenbundle $L^-_{\varepsilon F^*}$ of εF^*.

Proof. Firstly, we use Proposition 2.7 to see that $\mathcal{F}_{F,g,\flat}$ and $\mathcal{F}_{F,g,\sharp}$ are weak generalized almost product structures:

$$\begin{pmatrix} F & 0 \\ \flat_g & -\varepsilon F^* \end{pmatrix} \begin{pmatrix} F & 0 \\ \flat_g & -\varepsilon F^* \end{pmatrix} = \begin{pmatrix} F^2 & 0 \\ \flat_g F - \varepsilon F^* \flat_g & \varepsilon^2 (F^*)^2 \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix},$$

$$\begin{pmatrix} F & \sharp_g \\ 0 & -\varepsilon F^* \end{pmatrix} \begin{pmatrix} F & \sharp_g \\ 0 & -\varepsilon F^* \end{pmatrix} = \begin{pmatrix} F^2 & F\sharp_g - \varepsilon \sharp_g F^* \\ 0 & \varepsilon^2 (F^*)^2 \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}.$$

We check now the expressions given for the eigenbundles. We will only verify the expressions for $\mathbb{L}^+_{\mathcal{F}_{F,g,\flat}}$ and $\mathbb{L}^-_{\mathcal{F}_{F,g,\flat}}$ because the proofs are similar for $\mathcal{F}_{F,g,\flat}$ and $\mathcal{F}_{F,g,\sharp}$. First, given any $X \in TM$ we have $X + \mathcal{F}_{F,g,\flat} = (X + FX) + \flat_g X$ and thus $\{(X + FX) + \flat_g X \in \mathbb{T}M \colon X \in TM\} \subseteq \mathbb{L}^+_{\mathcal{F}_{F,g,\flat}}$. Likewise, the equality $X - \mathcal{F}_{F,g,\flat} = (X - FX) - \flat_g X$ is true for every $X \in TM$ and hence $\{(X - FX) - \flat_g X \in \mathbb{T}M \colon X \in TM\} \subseteq \mathbb{L}^-_{\mathcal{F}_{F,g,\flat}}$.

To see that the converse inclusions are also true, we must verify that for every $\xi \in T^*M$ there is an element $Y \in TM$ such that $Y + FY + \flat_g Y = \xi + \mathcal{F}_{F,g,\flat}\xi = \xi - \varepsilon F^*\xi$; and analogously, that given any $\xi \in T^*M$ there exists a $Y' \in TM$ with $Y' - FY' - \flat_g Y' = \xi - \mathcal{F}_{F,g,\flat}\xi = \xi + \varepsilon F^*\xi$. For the eigenbundle $\mathbb{L}^+_{\mathcal{F}_{F,g,\flat}}$, we propose $Y = \sharp_g(\xi - \varepsilon F^*\xi)$; for the eigenbundle $\mathbb{L}^-_{\mathcal{F}_{F,g,\flat}}$, we take $Y' = -\sharp_g(\xi + \varepsilon F^*\xi)$. We use Eq. (2.7) to make the calculations:

$$\begin{aligned} Y + FY + \flat_g Y &= \sharp_g(\xi - \varepsilon F^*\xi) + F\sharp_g(\xi - \varepsilon F^*\xi) + \flat_g\sharp_g(\xi - \varepsilon F^*\xi) \\ &= \sharp_g\xi - F\sharp_g\xi + F\sharp_g\xi - F^2\sharp_g\xi + \xi - \varepsilon F^*\xi \\ &= \sharp_g\xi - \sharp_g\xi + \xi - \varepsilon F^*\xi \\ &= \xi - \varepsilon F^*\xi, \end{aligned}$$

$$\begin{aligned} Y - FY - \flat_g Y &= -\sharp_g(\xi + \varepsilon F^*\xi) + F\sharp_g(\xi + \varepsilon F^*\xi) + \flat_g\sharp_g(\xi + \varepsilon F^*\xi) \\ &= -\sharp_g\xi - F\sharp_g\xi + F\sharp_g\xi + F^2\sharp_g\xi + \xi + \varepsilon F^*\xi \\ &= -\sharp_g\xi + \sharp_g\xi + \xi + \varepsilon F^*\xi \\ &= \xi + \varepsilon F^*\xi. \end{aligned}$$

Therefore, $\mathbb{L}^+_{\mathcal{F}_{F,g,\flat}}$ and $\mathbb{L}^-_{\mathcal{F}_{F,g,\flat}}$ are the given ones. Directly from the description of these eigenbundles, it is immediate to see that $\dim \mathbb{L}^+_{\mathcal{F}_{F,g,\flat}} = \dim L^+_F$ and $\dim \mathbb{L}^-_{\mathcal{F}_{F,g,\flat}} = \dim L^-_F$, where L^+_F and L^-_F are the subbundles associated to the ± 1-eigenvalues of F. Therefore, $\mathcal{F}_{F,g,\flat}$ is a weak generalized almost paracomplex structure if and only if F is an almost paracomplex structure. Finally, we check the description of the projections $\mathcal{P}^+_{\mathcal{F}_{F,g,\flat}}$ and $\mathcal{P}^-_{\mathcal{F}_{F,g,\flat}}$ using Eq. (2.4):

$$\begin{aligned} \mathcal{P}^+_{\mathcal{F}_{F,g,\flat}}(X+\xi) &= \frac{1}{2}\left((X+\xi) + \mathcal{F}_{F,g,\flat}(X+\xi)\right) = \frac{1}{2}(X + FX) + \frac{1}{2}(\xi - \varepsilon F^*\xi) + \frac{1}{2}\flat_g X \\ &= P^+_F X + \left(P^-_{\varepsilon F^*}\xi + \frac{1}{2}\flat_g X\right), \end{aligned}$$

$$\begin{aligned} \mathcal{P}^-_{\mathcal{F}_{F,g,\flat}}(X+\xi) &= \frac{1}{2}\left((X+\xi) - \mathcal{F}_{F,g,\flat}(X+\xi)\right) = \frac{1}{2}(X - FX) + \frac{1}{2}(\xi + \varepsilon F^*\xi) - \frac{1}{2}\flat_g X \\ &= P^-_F X + \left(P^+_{\varepsilon F^*}\xi - \frac{1}{2}\flat_g X\right). \end{aligned}$$ □

Until now, we have induced weak generalized almost complex structures using (α,ε)-metric manifolds with $\alpha = -1$, and generalized almost product and almost paracomplex structures using (α,ε)-metric manifolds with $\alpha = 1$. Now, it is our aim to construct richer generalized structures.

We first consider an (α,ε)-metric manifold (M, J, g) such that $\alpha = -1$. Our aim is to find a weak generalized almost paracomplex structure $\mathcal{F}$ (that is, $\mathcal{F}^2 = \mathcal{I}d$) that can be written as

$$\mathcal{F} = \begin{pmatrix} \lambda_1 J & \lambda_2 \sharp_g \\ \lambda_3 \flat_g & \lambda_4 J^* \end{pmatrix}, \tag{3.12}$$

with $\lambda_i \in \mathfrak{F}(M)$ for $i = 1, 2, 3, 4$. Using matrix notation and Eq. (2.7), we compute $\mathcal{F}^2$:

$$\begin{pmatrix} \lambda_1 J & \lambda_2 \sharp_g \\ \lambda_3 \flat_g & \lambda_4 J^* \end{pmatrix} \begin{pmatrix} \lambda_1 J & \lambda_2 \sharp_g \\ \lambda_3 \flat_g & \lambda_4 J^* \end{pmatrix} = \begin{pmatrix} \lambda_1^2 J^2 + \lambda_2 \lambda_3 \sharp_g \flat_g & \lambda_2(\lambda_1 J \sharp_g + \lambda_4 \sharp_g J^*) \\ \lambda_3(\lambda_1 \flat_g J + \lambda_4 J^* \flat_g) & \lambda_2 \lambda_3 \flat_g \sharp_g + \lambda_4^2 (J^*)^2 \end{pmatrix}$$
$$= \begin{pmatrix} (\lambda_2\lambda_3 - \lambda_1^2) Id & \lambda_2(\lambda_1 - \lambda_4 \varepsilon) J \sharp_g \\ \lambda_3(\lambda_1 - \lambda_4 \varepsilon) \flat_g J & (\lambda_2 \lambda_3 - \lambda_4^2) Id \end{pmatrix}.$$

If we constrain each λ_i to be a constant function and fix $\lambda_1 = 1$, then it must be $\lambda_2\lambda_3 - 1 = 1$ and $1 - \lambda_4 \varepsilon = 0$. It is immediate to see that $\lambda_4 = \varepsilon$. In addition, if we decide to take $\lambda_2 = \lambda_3 = \sqrt{2}$, then we have a weak generalized almost paracomplex structure whose form agrees with Eq. (3.12). This structure is described in the following statement.

Proposition 3.13. *Let (M, J, g) be an (α, ε)-metric manifold with $\alpha = -1$. Then, the endomorphism $\mathcal{F}_{J,g} \colon \mathbb{T}M \to \mathbb{T}M$ with $\mathcal{F}_{J,g}(X + \xi) = (JX + \sqrt{2}\, \sharp_g \xi) + (\sqrt{2}\, \flat_g X + \varepsilon J^* \xi)$ is a weak generalized almost paracomplex structure. In matrix notation, it is written as*

$$\mathcal{F}_{J,g} = \begin{pmatrix} J & \sqrt{2}\, \sharp_g \\ \sqrt{2}\, \flat_g & \varepsilon J^* \end{pmatrix}. \tag{3.13}$$

The ± 1-eigenbundles associated to $\mathcal{F}_{J,g}$ are, respectively,

$$\mathbb{L}^+_{\mathcal{F}_{J,g}} = \{(X + JX) + \sqrt{2}\, \flat_g X \in \mathbb{T}M \colon X \in TM\},$$
$$\mathbb{L}^-_{\mathcal{F}_{J,g}} = \{(X - JX) - \sqrt{2}\, \flat_g X \in \mathbb{T}M \colon X \in TM\}.$$

The projections onto each subbundle are the following:

$$\mathcal{P}^+_{\mathcal{F}_{J,g}}(X + \xi) = \frac{1}{2}\Big((X + JX + \sqrt{2}\, \sharp_g \xi) + (\xi + \varepsilon J^* \xi + \sqrt{2}\, \flat_g X)\Big),$$
$$\mathcal{P}^-_{\mathcal{F}_{J,g}}(X + \xi) = \frac{1}{2}\Big((X - JX - \sqrt{2}\, \sharp_g \xi) + (\xi - \varepsilon J^* \xi - \sqrt{2}\, \flat_g X)\Big).$$

Proof. It has already been checked that $\mathcal{F}^2_{J,g} = \mathcal{I}d$. We verify now that the ± 1-eigenbundles of these endomorphisms are the ones previously proposed. Firstly, as $X + \mathcal{F}_{J,g} X = (X + JX) + \sqrt{2}\, \flat_g X$ with $X \in TM$, according to Eq. (2.3) we have $\{(X + JX) + \sqrt{2}\, \flat_g X \in \mathbb{T}M \colon X \in TM\} \subseteq \mathbb{L}^+_{\mathcal{F}_{J,g}}$. Equally, $X - \mathcal{F}_{J,g} X = (X - JX) - \sqrt{2}\, \flat_g X$ and $\{(X - JX) - \sqrt{2}\, \flat_g X \in \mathbb{T}M \colon X \in TM\} \subseteq \mathbb{L}^-_{\mathcal{F}_{J,g}}$.

To check the opposite inclusions, we must verify that for every element $\xi \in T^*M$ there is a $Y \in TM$ such that $Y + JY + \sqrt{2}\, \flat_g Y = \xi + \mathcal{F}_{J,g} \xi = \sqrt{2}\, \sharp_g \xi + \xi + \varepsilon J^* \xi$; and analogously, that for each $\xi \in T^*M$ there is a $Y' \in TM$ such that $Y' - JY' - \sqrt{2}\, \flat_g Y' = \xi - \mathcal{F}_{J,g} \xi = -\sqrt{2}\, \sharp_g \xi + \xi - \varepsilon J^* \xi$. For the eigenbundle $\mathbb{L}^+_{\mathcal{F}_{J,g}}$, we propose $Y = \frac{1}{\sqrt{2}} \sharp_g (\xi + \varepsilon J^* \xi)$; for the eigenbundle $\mathbb{L}^-_{\mathcal{F}_{J,g}}$, we take $Y' = -\frac{1}{\sqrt{2}} \sharp_g (\xi - \varepsilon J^* \xi)$. Both cases are computed, taking into account Eq. (2.7):

$$\begin{aligned} Y + JY + \sqrt{2}\, \flat_g Y &= \frac{1}{\sqrt{2}} \sharp_g (\xi + \varepsilon J^* \xi) + \frac{1}{\sqrt{2}} J \sharp_g (\xi + \varepsilon J^* \xi) + \flat_g \sharp_g (\xi + \varepsilon J^* \xi) \\ &= \frac{1}{\sqrt{2}} \sharp_g \xi - \frac{1}{\sqrt{2}} J \sharp_g \xi + \frac{1}{\sqrt{2}} J \sharp_g \xi - \frac{1}{\sqrt{2}} J^2 \sharp_g \xi + \xi + \varepsilon J^* \xi \\ &= \sqrt{2}\, \sharp_g \xi + \xi + \varepsilon J^* \xi, \end{aligned}$$

$$\begin{aligned} Y' - JY' - \sqrt{2}\,\flat_g Y' &= -\frac{1}{\sqrt{2}}\sharp_g(\xi - \varepsilon J^*\xi) + \frac{1}{\sqrt{2}} J\sharp_g(\xi - \varepsilon J^*\xi) + \flat_g\sharp_g(\xi - \varepsilon J^*\xi) \\ &= -\frac{1}{\sqrt{2}}\sharp_g\xi - \frac{1}{\sqrt{2}} J\sharp_g\xi + \frac{1}{\sqrt{2}} J\sharp_g\xi + \frac{1}{\sqrt{2}} J^2\sharp_g\xi + \xi - \varepsilon J^*\xi \\ &= -\sqrt{2}\,\sharp_g\xi + \xi - \varepsilon J^*\xi. \end{aligned}$$

Hence, $\mathbb{L}^+_{\mathcal{F}_{J,g}}$ and $\mathbb{L}^-_{\mathcal{F}_{J,g}}$ are the proposed ones. It is clear that $\dim \mathbb{L}^+_{\mathcal{F}_{J,g}} = \dim \mathbb{L}^-_{\mathcal{F}_{J,g}}$ and thus $\mathcal{F}_{J,g}$ is a weak generalized almost paracomplex structure. Finally, the description of the projections is immediate to check. □

Following the same reasoning, an analogous result can be checked for an (α, ε)-metric manifold such that $\alpha = 1$.

Proposition 3.14. *Let (M, F, g) be an (α, ε)-metric manifold with $\alpha = 1$. Then, the endomorphism $\mathcal{J}_{F,g} \colon \mathbb{T}M \to \mathbb{T}M$ defined as $\mathcal{J}_{F,g}(X + \xi) = (FX - \sqrt{2}\,\sharp_g\xi) + (\sqrt{2}\,\flat_g X - \varepsilon F^*\xi)$ is a weak generalized almost complex structure. In matrix notation, it is written as*

$$\mathcal{J}_{F,g} = \begin{pmatrix} F & -\sqrt{2}\,\sharp_g \\ \sqrt{2}\,\flat_g & -\varepsilon F^* \end{pmatrix}. \tag{3.14}$$

The $\pm i$-eigenbundles associated to $\mathcal{J}_{F,g}$ are, respectively,

$$\begin{aligned} \mathbb{L}^{1,0}_{\mathcal{J}_{F,g}} &= \{(X - iFX) - i\sqrt{2}\,\flat_g X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\}, \\ \mathbb{L}^{0,1}_{\mathcal{J}_{F,g}} &= \{(X + iFX) + i\sqrt{2}\,\flat_g X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\}. \end{aligned}$$

The projections onto each subbundle are the following:

$$\begin{aligned} \mathcal{P}^{1,0}_{\mathcal{J}_{F,g}}(X + \xi) &= \frac{1}{2}\Big((X - iFX + i\sqrt{2}\,\sharp_g\xi) + (\xi + i\varepsilon F^*\xi - i\sqrt{2}\,\flat_g X)\Big), \\ \mathcal{P}^{0,1}_{\mathcal{J}_{F,g}}(X + \xi) &= \frac{1}{2}\Big((X + iFX - i\sqrt{2}\,\sharp_g\xi) + (\xi - i\varepsilon F^*\xi + i\sqrt{2}\,\flat_g X)\Big). \end{aligned}$$

Proof. We must check first that $\mathcal{J}^2_{F,g} = -\mathcal{I}d$:

$$\begin{aligned} \begin{pmatrix} F & -\sqrt{2}\,\sharp_g \\ \sqrt{2}\,\flat_g & -\varepsilon F^* \end{pmatrix} \begin{pmatrix} F & -\sqrt{2}\,\sharp_g \\ \sqrt{2}\,\flat_g & -\varepsilon F^* \end{pmatrix} &= \begin{pmatrix} F^2 - 2\sharp_g\flat_g & -\sqrt{2}(F\sharp_g - \varepsilon\sharp_g F^*) \\ \sqrt{2}(\flat_g F - \varepsilon F^*\flat_g) & -2\flat_g\sharp_g + (F^*)^2 \end{pmatrix} \\ &= \begin{pmatrix} -Id & 0 \\ 0 & -Id \end{pmatrix}. \end{aligned}$$

We must see now that the $\pm i$-eigenbundles of the morphism $\mathcal{J}_{F,g}$ are the ones previously given. First, as we have $X - i\mathcal{J}_{F,g}X = X - iFX - i\sqrt{2}\,\flat_g X$ with $X \in TM_{\mathbb{C}}$, using Eq. (2.1) it is clear that $\{(X - iFX) - i\sqrt{2}\,\flat_g X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\} \subseteq \mathbb{L}^{1,0}_{\mathcal{J}_{F,g}}$ is true. Analogously, for every $X \in TM_{\mathbb{C}}$ it is also true that $X + i\mathcal{J}_{F,g}X = X + iFX + i\sqrt{2}\,\flat_g X$ and hence we have the inclusion $\{(X + iFX) + i\sqrt{2}\,\flat_g X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_C\} \subseteq \mathbb{L}^{0,1}_{\mathcal{J}_{F,g}}$.

To prove the converse inclusions, for every element $\xi \in T^*M$ we must find a $Y \in TM_{\mathbb{C}}$ such that the equality $Y - iFY - i\sqrt{2}\,\flat_g Y = \xi - i\mathcal{J}_{F,g}\xi = i\sqrt{2}\,\sharp_g\xi + \xi + i\varepsilon F^*\xi$ holds true. Analogously, there must be a $Y' \in TM_{\mathbb{C}}$ such that $Y' + iFY' + i\sqrt{2}\,\flat_g Y' = \xi + i\mathcal{J}_{F,g}\xi = -i\sqrt{2}\,\sharp_g\xi + \xi - i\varepsilon F^*\xi$. For the eigenbundle $\mathbb{L}^{1,0}_{\mathcal{J}_{F,g}}$, we propose $Y = \frac{i}{\sqrt{2}}\sharp_g(\xi + i\varepsilon F^*\xi)$; for the eigenbundle $\mathbb{L}^{0,1}_{\mathcal{J}_{F,g}}$, we take $Y' = -\frac{i}{\sqrt{2}}\sharp_g(\xi - i\varepsilon F^*\xi)$. Both cases are computed, taking into account Eq. (2.7):

$$\begin{aligned} Y - iFY - i\sqrt{2}\,\flat_g Y &= \frac{i}{\sqrt{2}}\sharp_g(\xi + i\varepsilon F^*\xi) + \frac{1}{\sqrt{2}}F\sharp_g(\xi + i\varepsilon F^*\xi) + \flat_g\sharp_g(\xi + i\varepsilon F^*\xi) \\ &= \frac{i}{\sqrt{2}}\sharp_g\xi - \frac{1}{\sqrt{2}}F\sharp_g\xi + \frac{1}{\sqrt{2}}F\sharp_g\xi + \frac{i}{\sqrt{2}}F^2\sharp_g\xi + \xi + i\varepsilon F^*\xi \\ &= i\sqrt{2}\,\sharp_g\xi + \xi + i\varepsilon F^*\xi, \end{aligned}$$

$$\begin{aligned} Y' + iFY' + i\sqrt{2}\,\flat_g Y' &= -\frac{i}{\sqrt{2}}\sharp_g(\xi - i\varepsilon F^*\xi) + \frac{1}{\sqrt{2}}F\sharp_g(\xi - i\varepsilon F^*\xi) + \flat_g\sharp_g(\xi - i\varepsilon F^*\xi) \\ &= -\frac{i}{\sqrt{2}}\sharp_g\xi - \frac{1}{\sqrt{2}}F\sharp_g\xi + \frac{1}{\sqrt{2}}F\sharp_g\xi - \frac{i}{\sqrt{2}}F^2\sharp_g\xi + \xi - i\varepsilon F^*\xi \\ &= -i\sqrt{2}\,\sharp_g\xi + \xi - i\varepsilon F^*\xi, \end{aligned}$$

Hence, $\mathbb{L}^{1,0}_{\mathcal{J}_{F,g}}$ and $\mathbb{L}^{0,1}_{\mathcal{J}_{F,g}}$ are the proposed ones. It is immediate to check the expressions given for the projections onto each eigenbundle. □

After having established the connections between the geometric structures on a manifold and the induced weak generalized polynomial structures, we return to the diagram presented in Figure 3.1. In this section, we have seen that each metric induces two weak generalized almost tangent structures, one weak generalized almost complex structure, and one weak generalized almost paracomplex structure; each almost symplectic structure induces two weak generalized almost tangent structures, one weak generalized almost complex structure, and one weak generalized almost paracomplex structure; and each polynomial structure on the manifold induces weak generalized polynomial structures of the same type. Therefore, it has been shown that, when the compatibility condition with the canonical pairing $\mathcal{G}_0$ is removed, the number of generalized geometric structures worthy of study increases. This expansion is represented in Figure 3.2. It is worth noting that this diagram will be extended with more relations in Chapter 4, where metrics and symplectic structures on the generalized tangent bundle are examined.

3.2 Weak generalized triple structures

Thus far, we have studied polynomial structures defined on the generalized tangent bundle. As shown in Section 2.1, when two polynomial structures defined on a vector bundle present compatibility conditions in the form of commutation or anti-commutation, they conform what we have called triple structures. These geometric structures will be especially relevant in Chapter 5, when exploring integrable generalized complex structures on the six-dimensional sphere. If the compatibility condition between the polynomial structures and the canonical pairing $\mathcal{G}_0$ is not considered, we will call these structures “weak”, as is stated below.

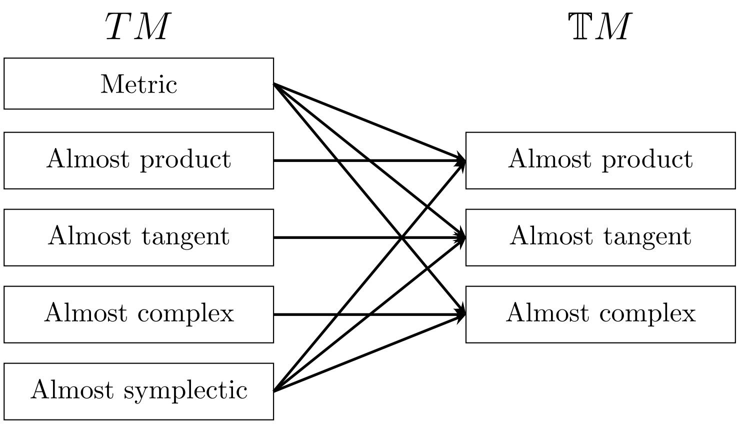


FIGURE 3.2: Scheme of the weak generalized polynomial structures that can be induced from geometric structures on the manifold. The arrows represent when a geometric structure on the tangent bundle induces a generalized geometric structure.

Definition 3.15. A triple structure on the generalized tangent bundle (that is, a set of three generalized polynomial structures $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \colon \mathbb{T}M \to \mathbb{T}M$ that satisfy the requirements detailed in Definition 2.8) is called a *weak generalized triple structure*.

According to Definition 2.8, there are four types of weak generalized triple structures: *weak generalized almost hypercomplex structures*, *weak generalized almost bicomplex structures*, *weak generalized almost biparacomplex structures* and *weak generalized almost hyperproduct structures*.

3.2.1 Regarding $\mathcal{F}_0$

To our knowledge, the only generalized polynomial structure that can be found on the generalized tangent bundle of any smooth manifold is the canonical generalized paracomplex structure $\mathcal{F}_0$, described in Eq. (2.13). Therefore, it is interesting to ask in which generalized triple structures can $\mathcal{F}_0$ come into play. Because $\mathcal{F}_0^2 = \mathcal{I}d$, this endomorphism can only be part of weak generalized almost bicomplex, almost biparacomplex and almost hyperproduct structures. In order to describe every weak generalized triple structure in which $\mathcal{F}_0$ can participate, the following result will be of great importance.

Proposition 3.16. *Let $\mathcal{K} \colon \mathbb{T}M \to \mathbb{T}M$ be a weak generalized polynomial structure such that, in matrix notation, is written as*

$$\mathcal{K} = \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix}.$$

Then, $\mathcal{K}$ and $\mathcal{F}_0$ commute if and only if $\tau = 0$ and $\sigma = 0$. On the other hand, $\mathcal{K}$ and $\mathcal{F}_0$ anti-commute if and only if $H = 0$ and $K = 0$.

Proof. The result is easily inferred using matrix notation to compute $\mathcal{F}_0\mathcal{K}$ and $\mathcal{K}\mathcal{F}_0$:

$$\mathcal{F}_0\mathcal{K} = \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix} = \begin{pmatrix} -H & -\sigma \\ \tau & K \end{pmatrix},$$

$$\mathcal{K}\mathcal{F}_0 = \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix} \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} -H & \sigma \\ -\tau & K \end{pmatrix}.$$

Then, comparing both expressions, it is clear that $\mathcal{F}_0\mathcal{K} = \mathcal{K}\mathcal{F}_0$ if and only if $\tau = 0$ and $\sigma = 0$, and $\mathcal{F}_0\mathcal{K} = -\mathcal{K}\mathcal{F}_0$ if and only if $H = 0$ and $K = 0$. □

This proposition will be used in order to find all the weak generalized triple structures that involve $\mathcal{F}_0$.

Corollary 3.17. *If the canonical generalized paracomplex structure $\mathcal{F}_0$ is part of a weak generalized triple structure, then the type of the triple structure can be one of the following three:*

(i) Generalized almost bicomplex $(\mathcal{J}_1, \mathcal{J}_2, \mathcal{F}_0)$. Here, the endomorphisms $\mathcal{J}_1, \mathcal{J}_2 \colon \mathbb{T}M \to \mathbb{T}M$ are weak generalized almost complex structures that, in matrix notation, are written as

$$\mathcal{J}_1 = \begin{pmatrix} J_1 & 0 \\ 0 & J_2^* \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} J_1 & 0 \\ 0 & -J_2^* \end{pmatrix},$$

where $J_1, J_2 \colon TM \to TM$ are almost complex structures on the manifold.

(ii) Generalized almost biparacomplex $(\mathcal{F}_0, \mathcal{F}, \mathcal{J})$. Here, the endomorphisms $\mathcal{F}, \mathcal{J} \colon \mathbb{T}M \to \mathbb{T}M$ are a weak generalized almost paracomplex structure and a weak generalized almost complex structure that, in matrix notation, are written as

$$\mathcal{F} = \begin{pmatrix} 0 & \tau^{-1} \\ \tau & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -\tau^{-1} \\ \tau & 0 \end{pmatrix},$$

where $\tau \colon TM \to T^*M$ is an isomorphism.

(iii) Generalized almost hyperproduct $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$. Here, the endomorphisms $\mathcal{F}_1, \mathcal{F}_2 \colon \mathbb{T}M \to \mathbb{T}M$ are weak generalized almost product structures that, in matrix notation, are written as

$$\mathcal{F}_1 = \begin{pmatrix} F_1 & 0 \\ 0 & F_2^* \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} -F_1 & 0 \\ 0 & F_2^* \end{pmatrix},$$

where $F_1, F_2 \colon TM \to TM$ are almost product structures on the manifold.

Proof. As stated earlier, $\mathcal{F}_0$ cannot be part of a generalized almost hypercomplex structure. Therefore, we study the other three cases separately. If the weak generalized triple structure is almost bicomplex, $\mathcal{F}_0$ must commute with two weak generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$. Then, according to Proposition 3.16, the representation of $\mathcal{J}_1$ and $\mathcal{J}_2$ in matrix notation must be diagonal. We use now that $\mathcal{J}_1^2 = \mathcal{J}_2^2 = -\mathcal{I}d$ and $\mathcal{J}_1\mathcal{J}_2 = \mathcal{F}_0$:

$$\mathcal{J}_i^2 = \begin{pmatrix} H_i^2 & 0 \\ 0 & K_i^2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & -Id \end{pmatrix}, \quad \mathcal{J}_1\mathcal{J}_2 = \begin{pmatrix} H_1H_2 & 0 \\ 0 & K_1K_2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix}.$$

Therefore, it must be $H_1 = H_2 = J_1$ and $K_1 = -K_2 = J_2^*$, where $J_1, J_2 \colon TM \to TM$ are almost complex structures on the manifold.

We suppose now that $\mathcal{F}_0$ is in a weak generalized almost biparacomplex structure. Then, $\mathcal{F}_0$ must anti-commute with a weak generalized almost paracomplex structure $\mathcal{F}$ and a weak generalized almost complex structure $\mathcal{J}$. Using Proposition 3.16, the morphisms in the diagonal of both $\mathcal{F}$ and $\mathcal{J}$ must be null. We use now that $\mathcal{F}^2 = -\mathcal{J}^2 = \mathcal{I}d$ and $\mathcal{F}\mathcal{J} = \mathcal{F}_0$:

$$\mathcal{F}^2 = \begin{pmatrix} \sigma_1\tau_1 & 0 \\ 0 & \tau_1\sigma_1 \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}, \quad \mathcal{J}^2 = \begin{pmatrix} \sigma_2\tau_2 & 0 \\ 0 & \tau_2\sigma_2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & -Id \end{pmatrix},$$

$$\mathcal{F}\mathcal{J} = \begin{pmatrix} \sigma_1\tau_2 & 0 \\ 0 & \tau_1\sigma_2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix}.$$

Therefore, it must be $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \tau^{-1}$, where $\tau\colon TM \to T^*M$ is an isomorphism between the vector bundles and 1-forms.

Finally, if the weak generalized triple structure is almost hyperproduct, $\mathcal{F}_0$ must commute with two weak generalized almost paracomplex structures $\mathcal{F}_1, \mathcal{F}_2$. Then, using Proposition 3.16, $\mathcal{F}_1$ and $\mathcal{F}_2$ can be represented by diagonal matrices. Now, we use the fact that $\mathcal{F}_1^2 = \mathcal{F}_2^2 = \mathcal{I}d$ and $\mathcal{F}_1\mathcal{F}_2 = \mathcal{F}_0$:

$$\mathcal{F}_i^2 = \begin{pmatrix} H_i^2 & 0 \\ 0 & K_i^2 \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}, \quad \mathcal{F}_1\mathcal{F}_2 = \begin{pmatrix} H_1H_2 & 0 \\ 0 & K_1K_2 \end{pmatrix} = \begin{pmatrix} -Id & 0 \\ 0 & Id \end{pmatrix}.$$

Then, it must be $H_1 = -H_2 = F_1$ and $K_1 = K_2 = F_2^*$, where $F_1, F_2\colon TM \to TM$ are almost product structures on the manifold. □

In particular, it is easy to see that an almost complex manifold (M, J) admits the weak generalized almost bicomplex structure $(\mathcal{J}_{1,J}, \mathcal{J}_{-1,J}, \mathcal{F}_0)$, where the morphisms $\mathcal{J}_{1,J}$ and $\mathcal{J}_{-1,J}$ are those introduced in Eq. (3.8). Similarly, an almost product manifold (M, F) admits the weak generalized almost hyperproduct structure $(\mathcal{F}_0, \mathcal{F}_{1,F}, -\mathcal{F}_{-1,J})$, where the structures $\mathcal{F}_{1,J}$ and $\mathcal{F}_{-1,J}$ are described in Eq. (3.9). Finally, if the base manifold is (pseudo-)Riemannian (M, g) (resp. almost symplectic (M, ω)) it admits the weak generalized almost paracomplex structure $(\mathcal{F}_0, \mathcal{F}_g, \mathcal{J}_g)$ (resp. $(\mathcal{F}_0, \mathcal{F}_\omega, \mathcal{J}_\omega)$), where the structures $\mathcal{J}_g$ and $\mathcal{F}_g$ (resp. $\mathcal{J}_\omega$ and $\mathcal{F}_\omega$) are shown in Eqs. (3.2) and (3.3) (resp. Eqs. (3.5) and (3.6)).

3.2.2 Induced from an (α, ε)-metric structure

We now consider a richer geometry within the base manifold. In particular, we take an (α, ε)-metric manifold (M, J, g) with $\alpha = -1$ (i.e., $J^2 = -Id$). Both the metric g and the almost complex structure J induce many generalized polynomial structures, as well as the ones induced by the fundamental tensor $\varphi(\cdot, \cdot) = g(J\cdot, \cdot)$. We first check how the generalized polynomial structures $\mathcal{J}_{\lambda,J}$ from Eq. (3.8) interact with $\mathcal{J}_g$ from Eq. (3.2), keeping in mind Proposition 2.7:

$$\mathcal{J}_{\lambda,J}\mathcal{J}_g = \begin{pmatrix} J & 0 \\ 0 & \lambda J^* \end{pmatrix} \begin{pmatrix} 0 & -\sharp_g \\ \flat_g & 0 \end{pmatrix} = \begin{pmatrix} 0 & -J\sharp_g \\ \lambda J^*\flat_g & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sharp_\varphi \\ -\lambda\varepsilon\flat_\varphi & 0 \end{pmatrix},$$

$$\mathcal{J}_g\mathcal{J}_{\lambda,J} = \begin{pmatrix} 0 & -\sharp_g \\ \flat_g & 0 \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & \lambda J^* \end{pmatrix} = \begin{pmatrix} 0 & -\lambda\sharp_g J^* \\ \flat_g J & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda\varepsilon\sharp_\varphi \\ \flat_\varphi & 0 \end{pmatrix}.$$

The same calculations are done for $\mathcal{J}_{\lambda,J}$ and $\mathcal{F}_g$:

$$\mathcal{J}_{\lambda,J}\mathcal{F}_g = \begin{pmatrix} J & 0 \\ 0 & \lambda J^* \end{pmatrix}\begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix} = \begin{pmatrix} 0 & J\sharp_g \\ \lambda J^*\flat_g & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sharp_\varphi \\ -\lambda\varepsilon\flat_\varphi & 0 \end{pmatrix},$$

$$\mathcal{F}_g\mathcal{J}_{\lambda,J} = \begin{pmatrix} 0 & \sharp_g \\ \flat_g & 0 \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & \lambda J^* \end{pmatrix} = \begin{pmatrix} 0 & \lambda\sharp_g J^* \\ \flat_g J & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda\varepsilon\sharp_\varphi \\ \flat_\varphi & 0 \end{pmatrix}.$$

Then, the following result, which combines all these calculations, is inferred.

Proposition 3.18. *Let (M, J, g) be an (α, ε)-metric manifold with $J^2 = -Id$ (that is, $\alpha = -1$), and let φ be its fundamental tensor. Then, the following weak generalized triple structures are induced: a weak generalized almost hypercomplex structure $(\mathcal{J}_g, \mathcal{J}_{\varepsilon,J}, \mathcal{J}_\varphi)$; two weak generalized almost bicomplex structures, namely $(\mathcal{J}_g, \mathcal{J}_{-\varepsilon,J}, \mathcal{F}_\varphi)$ and $(\mathcal{J}_\varphi, \mathcal{J}_{-\varepsilon,J}, -\mathcal{F}_g)$; and a weak generalized almost biparacomplex structure $(\mathcal{F}_g, \mathcal{F}_\varphi, \mathcal{J}_{\varepsilon,J})$.*

It is interesting to comment on the relation between the weak generalized almost biparacomplex structure $(\mathcal{F}_g, \mathcal{F}_\varphi, \mathcal{J}_{\varepsilon,J})$ and the induced structure $\mathcal{F}_{J,g}$ from Eq. (3.13). As all the structures in $(\mathcal{F}_g, \mathcal{F}_\varphi, \mathcal{J}_{\varepsilon,J})$ anti-commute, every combination in the form $a\mathcal{F}_g + b\mathcal{F}_\varphi + c\mathcal{J}_{\varepsilon,J}$ for some differentiable functions $a, b, c \in \mathfrak{F}(M)$ such that $-a^2 - b^2 + c^2 = 1$ is a weak generalized almost complex structure, whilst if $a^2 + b^2 - c^2 = 1$ then the combination is a weak generalized almost product structure. It can be simply noticed that $\mathcal{F}_{J,g}$ is such a combination of $\mathcal{F}_g, \mathcal{F}_\varphi, \mathcal{J}_{\varepsilon,J}$ for the values $a = \sqrt{2}, b = 0, c = 1$; in other words, $\mathcal{F}_{J,g} = \sqrt{2}\mathcal{F}_g + \mathcal{J}_{\varepsilon,J}$.

Analogous calculations can be performed for an (α, ε)-metric manifold (M, F, g) with $\alpha = 1$ (that is, $F^2 = Id$). Those calculations are summarized in the following proposition.

Proposition 3.19. *Let (M, F, g) be an (α, ε)-metric manifold with $F^2 = Id$ (that is, $\alpha = 1$), and let φ be its fundamental tensor. Then, the following weak generalized triple structures are induced: a weak generalized almost hyperproduct structure $(\mathcal{F}_g, \mathcal{F}_{\varepsilon,F}, \mathcal{F}_\varphi)$; two weak generalized almost biparacomplex structures, namely $(\mathcal{F}_g, \mathcal{F}_{-\varepsilon,F}, \mathcal{J}_\varphi)$ and $(\mathcal{F}_\varphi, \mathcal{F}_{-\varepsilon,F}, \mathcal{J}_g)$; and a weak generalized almost bicomplex structure $(\mathcal{J}_g, \mathcal{J}_\varphi, -\mathcal{F}_{\varepsilon,F})$.*

As was the case for the structure $\mathcal{F}_{J,g}$, the essence of the weak generalized almost complex structure $\mathcal{J}_{F,g}$ from Eq. (3.14) is determined by the weak generalized almost biparacomplex structure $(\mathcal{F}_\varphi, \mathcal{F}_{-\varepsilon,F}, \mathcal{J}_g)$. These three morphisms anti-commute, so a linear combination $a\mathcal{F}_\varphi + b\mathcal{F}_{-\varepsilon,F} + c\mathcal{J}_g$ with $a, b, c \in \mathfrak{F}(M)$ behaves as a weak generalized almost product structure if $a^2 + b^2 - c^2 = 1$, and as a weak generalized almost complex structure if $-a^2 - b^2 + c^2 = 1$. Then, it is immediate to see that the morphism $\mathcal{J}_{F,g}$ can be expressed as a combination of $\mathcal{F}_\varphi, \mathcal{F}_{-\varepsilon,F}, \mathcal{J}_g$ for $a = 0, b = 1, c = \sqrt{2}$; that is, $\mathcal{J}_{F,g} = \sqrt{2}\mathcal{J}_g + \mathcal{F}_{-\varepsilon,F}$.

Chapter 4

Strong generalized structures

In the previous chapter, we introduced a series of endomorphisms that can be understood as polynomial structures on the generalized tangent bundle of a manifold (see Propositions 3.2-3.14). Those generalized structures are induced from classical geometric structures on the manifold. Furthermore, we examined how these structures interact with the canonical generalized paracomplex structure $\mathcal{F}_0$ through generalized triple structures (see Proposition 3.16 and Corollary 3.17).

This chapter is devoted to the study of metrics on the generalized tangent bundle. In his PhD thesis [29], Gualtieri showed that the structures $\mathcal{J}_\omega$ and $\mathcal{J}_{-1,J}$, introduced in Eqs. (3.5) and (3.8), are skew-symmetric with respect to the canonical pairing $\mathcal{G}_0$:

$$\begin{aligned}\mathcal{G}_0(\mathcal{J}_\omega(X+\xi), Y+\eta) &= \mathcal{G}_0(-\sharp_\omega\xi + \flat_\omega X, Y+\eta) = \frac{1}{2}((\flat_\omega X)Y - \eta(\sharp_\omega\xi))\\ &= \frac{1}{2}(\omega(X,Y) - \omega(\sharp_\omega\eta, \sharp_\omega\xi)) = -\frac{1}{2}(\omega(Y,X) - \omega(\sharp_\omega\xi, \sharp_\omega\eta))\\ &= -\frac{1}{2}((\flat_\omega Y)X - \xi(\sharp_\omega\eta)) = -\mathcal{G}_0(X+\xi, -\sharp_\omega\eta + \flat_\omega Y)\\ &= -\mathcal{G}_0(X+\xi, \mathcal{J}_\omega(Y+\eta)),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_0(\mathcal{J}_{-1,J}(X+\xi), Y+\eta) &= \mathcal{G}_0(JX - J^*\xi, Y+\eta) = \frac{1}{2}((-J^*\xi)Y + \eta(JX))\\ &= -\frac{1}{2}((-J^*\eta)X + \xi(JY)) = -\mathcal{G}_0(X+\xi, JY - J^*\eta)\\ &= -\mathcal{G}_0(X+\xi, \mathcal{J}_{-1,J}(Y+\eta)).\end{aligned}$$

The structures $\mathcal{F}_\omega$ and $\mathcal{F}_{-1,F}$ from Eqs. (3.6) and (3.9), introduced by Wade in [60], follow the same skew-symmetry relation. Throughout this chapter, our aim is to explore the interactions between weak structures on $\mathbb{T}M$ and the canonical ones $\mathcal{G}_0$ and Ω_0. Based on these relations, we will study other metrics of interest within the field of generalized geometry and show that they are inevitably linked to $\mathcal{G}_0$.

4.1 Strong generalized polynomial structures

In Definition 3.1 we described weak generalized polynomial structures. Those structures do not take into account the canonical generalized metric $\mathcal{G}_0$ presented in Definition 2.11. Now, it is time to consider what happens when a generalized polynomial structure $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$ is compatible with $\mathcal{G}_0$. We will use the adjective "strong" to refer to this condition.

Definition 4.1. A polynomial structure on the generalized tangent bundle $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$ is called a *strong generalized polynomial structure* if it satisfies the requirements in Definition 3.1 and it is skew-symmetric with respect to the canonical pairing $\mathcal{G}_0$; in other words, if we have

$$\mathcal{G}_0(\mathcal{K}(X+\xi), Y+\eta) = -\mathcal{G}_0(X+\xi, \mathcal{K}(Y+\eta)). \tag{4.1}$$

for every $X+\xi, Y+\eta \in \mathbb{T}M$ based on the same point.

The attentive reader may ask what happens when a structure satisfies the "opposite" condition:

$$\mathcal{G}_0(\mathcal{K}(X+\xi), Y+\eta) = \mathcal{G}_0(X+\xi, \mathcal{K}(Y+\eta)). \tag{4.2}$$

In this case, we will not use the nomenclature "strong": we will just say that a weak generalized polynomial structure that fulfills Eq. (4.2) is *symmetric* with respect to $\mathcal{G}_0$.

Since $\mathcal{G}_0$ is nondegenerate, when a generalized polynomial structure $\mathcal{K}$ is injective and compatible with respect to $\mathcal{G}_0$, whether symmetric or skew-symmetric, it induces a nondegenerate morphism $\Phi \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$:

$$\Phi(X+\xi, Y+\eta) = \mathcal{G}_0(\mathcal{K}(X+\xi), Y+\eta).$$

The behavior of Φ will be further discussed in Section 4.2.

We can carry out an analysis similar to that performed in Chapter 3, studying when the weak generalized polynomial structures induced by the geometry on the base manifold are compatible with the metric $\mathcal{G}_0$. Some of those structures will be symmetric with respect to $\mathcal{G}_0$, and others will be skew-symmetric, that is to say, they will be strong generalized polynomial structures. As in Section 3.1, we introduce these results according to the geometry of the base manifold.

4.1.1 Induced from a metric

Following the same pattern as the one followed in Section 3.1, we first study whether the weak generalized polynomial structures induced on a (pseudo-)Riemannian manifold (M, g) are compatible or not with the canonical pairing $\mathcal{G}_0$. The following results show that the induced generalized structures are all symmetric with respect to $\mathcal{G}_0$, that is, they satisfy Eq. (4.2).

Proposition 4.2. *Let (M, g) be a (pseudo-)Riemannian manifold. Then, the induced weak generalized almost tangent structures $\mathcal{S}_{g,\flat}, \mathcal{S}_{g,\sharp}\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.1)*, are symmetric with respect to $\mathcal{G}_0$.*

Proof. It is straightforward to check: for every $X+\xi, Y+\eta \in \mathbb{T}M$ in the same fiber, we see that

$$\begin{aligned}\mathcal{G}_0(\mathcal{S}_{g,\flat}(X+\xi), Y+\eta) &= \mathcal{G}_0(\flat_g X, Y+\eta) = \frac{1}{2}(\flat_g X)Y = \frac{1}{2}g(X,Y)\\ &= \frac{1}{2}(\flat_g Y)X = \mathcal{G}_0(X+\xi, \flat_g Y)\\ &= \mathcal{G}_0(X+\xi, \mathcal{S}_{g,\flat}(Y+\eta)),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_0(\mathcal{S}_{g,\sharp}(X+\xi), Y+\eta) &= \mathcal{G}_0(\sharp_g \xi, Y+\eta) = \frac{1}{2}\eta(\sharp_g\xi) = \frac{1}{2}g(\sharp_g\xi, \sharp_g\eta)\\ &= \frac{1}{2}\xi(\sharp_g\eta) = \mathcal{G}_0(X+\xi, \sharp_g\eta)\\ &= \mathcal{G}_0(X+\xi, \mathcal{S}_{g,\sharp}(Y+\eta)).\end{aligned}$$

□

Proposition 4.3 ([51, Section 2])**.** *Let (M,g) be a (pseudo-)Riemannian manifold. Then, the induced weak generalized almost complex structure $\mathcal{J}_g\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.2)*, is symmetric with respect to $\mathcal{G}_0$. Furthermore, it induces the following symmetric nondegenerate operator:*

$$\Phi_{\mathcal{J}_g}(X+\xi, Y+\eta) = \frac{1}{2}(g(X,Y) - g(\sharp_g\xi, \sharp_g\eta)). \tag{4.3}$$

Proof. It is immediate to see: if we take a pair $X+\xi, Y+\eta \in \mathbb{T}M$ in the same fiber, then

$$\begin{aligned}\mathcal{G}_0(\mathcal{J}_g(X+\xi), Y+\eta) &= \mathcal{G}_0(-\sharp_g\xi + \flat_g X, Y+\eta) = \frac{1}{2}((\flat_g X)Y - \eta(\sharp_g\xi))\\ &= \frac{1}{2}(g(X,Y) - g(\sharp_g\eta, \sharp_g\xi)) = \frac{1}{2}(g(Y,X) - g(\sharp_g\xi, \sharp_g\eta))\\ &= \frac{1}{2}((\flat_g Y)X - \xi(\sharp_g\eta)) = \mathcal{G}_0(X+\xi, -\sharp_g\eta + \flat_g Y)\\ &= \mathcal{G}_0(X+\xi, \mathcal{J}_g(Y+\eta)).\end{aligned}$$

From this calculation it can be seen that $\mathcal{J}_g$ is symmetric with respect to $\mathcal{G}_0$. In addition, we deduce the expression for $\Phi_{\mathcal{J}_g}(X+\xi, Y+\eta)$ in Eq. (4.3) from the previous calculations. □

Proposition 4.4 ([38, Example 3.1])**.** *Let (M,g) be a (pseudo-)Riemannian manifold. Then, the induced weak generalized almost paracomplex structure $\mathcal{F}_g\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.3)*, is symmetric with respect to $\mathcal{G}_0$. Additionally, it induces the following symmetric nondegenerate operator:*

$$\Phi_{\mathcal{F}_g}(X+\xi, Y+\eta) = \frac{1}{2}(g(X,Y) + g(\sharp_g\xi, \sharp_g\eta)). \tag{4.4}$$

Proof. Just by taking a pair $X+\xi, Y+\eta \in \mathbb{T}M$ in the same fiber, then

$$\begin{aligned}\mathcal{G}_0(\mathcal{F}_g(X+\xi), Y+\eta) &= \mathcal{G}_0(\sharp_g\xi + \flat_g X, Y+\eta) = \frac{1}{2}((\flat_g X)Y + \eta(\sharp_g\xi))\\ &= \frac{1}{2}(g(X,Y) + g(\sharp_g\eta, \sharp_g\xi)) = \frac{1}{2}(g(Y,X) + g(\sharp_g\xi, \sharp_g\eta))\\ &= \frac{1}{2}((\flat_g Y)X + \xi(\sharp_g\eta)) = \mathcal{G}_0(X+\xi, \sharp_g\eta + \flat_g Y)\\ &= \mathcal{G}_0(X+\xi, \mathcal{F}_g(Y+\eta)).\end{aligned}$$

Therefore, $\mathcal{F}_g$ is symmetric with respect to $\mathcal{G}_0$. The identity in Eq. (4.4) is derived from the previous calculations. □

4.1.2 Induced from an almost symplectic structure

In contrast to Subsection 3.1.2, where the only relevant property in order to induce weak generalized polynomial structures was the nondegeneracy of an almost symplectic structure, the differences between a (pseudo-)Riemannian metric g and an almost symplectic structure ω are crucial in order to determine whether a generalized structure induced by ω is symmetric or skew-symmetric with respect to $\mathcal{G}_0$. In the following propositions, we see that every generalized polynomial structure induced by an almost symplectic structure is strong; that is, they satisfy the skew-symmetry condition given in Eq. (4.1).

Proposition 4.5. *Let (M,ω) be an almost symplectic manifold. Then, the induced endomorphisms $\mathcal{S}_{\omega,\flat}, \mathcal{S}_{\omega,\sharp}\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.4)*, are strong generalized almost tangent structures.*

Proof. For every $X+\xi, Y+\eta \in \mathbb{T}M$ in the same fiber, it is immediately seen that

$$\begin{aligned}
\mathcal{G}_0(\mathcal{S}_{\omega,\flat}(X+\xi), Y+\eta) &= \mathcal{G}_0(\flat_\omega X, Y+\eta) = \frac{1}{2}(\flat_\omega X)Y = \frac{1}{2}\omega(X,Y) \\
&= -\frac{1}{2}(\flat_\omega Y)X = -\mathcal{G}_0(X+\xi, \flat_\omega Y) \\
&= -\mathcal{G}_0(X+\xi, \mathcal{S}_{\omega,\flat}(Y+\eta)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_0(\mathcal{S}_{\omega,\sharp}(X+\xi), Y+\eta) &= \mathcal{G}_0(\sharp_\omega \xi, Y+\eta) = \frac{1}{2}\eta(\sharp_\omega \xi) = \frac{1}{2}g(\sharp_\omega \eta, \sharp_\omega \xi) \\
&= -\frac{1}{2}\xi(\sharp_\omega \eta) = -\mathcal{G}_0(X+\xi, \sharp_\omega \eta) \\
&= -\mathcal{G}_0(X+\xi, \mathcal{S}_{\omega,\sharp}(Y+\eta)).
\end{aligned}$$

□

The following two results correspond to the original structures presented by Gualtieri in [29] and by Wade in [60]. Some of the calculations for these structures are shown at the beginning of the chapter.

Proposition 4.6 ([29, Example 4.10])**.** *Let (M,ω) be an almost symplectic manifold. Then, the induced endomorphism $\mathcal{J}_\omega\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.5)*, is a strong generalized almost complex structure. Furthermore, it induces the following skew-symmetric nondegenerate operator:*

$$\Phi_{\mathcal{J}_\omega}(X+\xi, Y+\eta) = \frac{1}{2}(\omega(X,Y) + \omega(\sharp_\omega \xi, \sharp_\omega \eta)). \tag{4.5}$$

Proposition 4.7 ([60, Example 2])**.** *Let (M,ω) be an almost symplectic manifold. Then, the induced endomorphism $\mathcal{F}_\omega\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.6)*, is a strong generalized almost paracomplex structure. Furthermore, it induces the following skew-symmetric nondegenerate operator:*

$$\Phi_{\mathcal{F}_\omega}(X+\xi, Y+\eta) = \frac{1}{2}(\omega(X,Y) - \omega(\sharp_\omega \xi, \sharp_\omega \eta)). \tag{4.6}$$

4.1.3 Induced from a polynomial structure

Now, we analyze whether the weak generalized polynomial structures that are induced by a polynomial structure defined on a manifold, introduced in Eqs. (3.7), (3.8) and (3.9), are compatible with $\mathcal{G}_0$ or not. As will be seen in the following statements, they will not always be compatible. Some of these structures have previously been analyzed by other authors, and some of the calculations have already been displayed at the beginning of the chapter.

Proposition 4.8. *Let (M, S) be an almost tangent manifold manifold. Then, the weak generalized almost tangent structure $\mathcal{S}_{\lambda,S}\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.8) *for $\lambda \neq 0$, is strong when $\lambda = -1$, and symmetric with respect to $\mathcal{G}_0$ when $\lambda = 1$. For any other values of λ, $\mathcal{S}_{\lambda,S}$ is not compatible with the canonical pairing $\mathcal{G}_0$.*

Proof. It is easy to see just by doing a simple calculation:

$$\begin{aligned}\mathcal{G}_0(\mathcal{S}_{\lambda,S}(X+\xi), Y+\eta) &= \frac{1}{2}((\lambda S^*\xi)Y + \eta(SX)) = \frac{\lambda}{2}(\xi(SY) + (\lambda^{-1}S^*\eta)X) \\ &= \lambda\mathcal{G}_0(X+\xi, \mathcal{S}_{\lambda^{-1},S}(Y+\eta)).\end{aligned}$$

Therefore, $\mathcal{S}_{\lambda,S}$ will only be compatible with $\mathcal{G}_0$ if and only if $\lambda = \lambda^{-1}$, that is, if and only if $\lambda \in \{1, -1\}$. □

Proposition 4.9 ([29, Section 4.1])**.** *Let (M, J) be an almost complex manifold. Then, the weak generalized almost complex structure $\mathcal{J}_{\lambda,J}\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.8)*, is strong when $\lambda = -1$; and symmetric with respect to $\mathcal{G}_0$ when $\lambda = 1$. Furthermore, depending on the value of λ it induces the following nondegenerate operator:*

$$\begin{aligned}\Phi_{\mathcal{J}_{1,J}}(X+\xi, Y+\eta) &= \mathcal{G}_0(JX+\xi, JY+\eta), \\ \Phi_{\mathcal{J}_{-1,J}}(X+\xi, Y+\eta) &= -\Omega_0(JX+\xi, JY+\eta).\end{aligned} \tag{4.7}$$

Proposition 4.10 ([38, Section 3.2])**.** *Let (M, F) be an almost product manifold. Then, the weak generalized almost product structure $\mathcal{F}_{\lambda,F}\colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.9)*, is a strong generalized almost paracomplex structure when $\lambda = -1$; and symmetric with respect to $\mathcal{G}_0$ when $\lambda = 1$. Furthermore, depending on the value of λ it induces the following nondegenerate operator:*

$$\begin{aligned}\Phi_{\mathcal{F}_{1,F}}(X+\xi, Y+\eta) &= \mathcal{G}_0(FX+\xi, FY+\eta), \\ \Phi_{\mathcal{F}_{-1,F}}(X+\xi, Y+\eta) &= -\Omega_0(FX+\xi, FY+\eta).\end{aligned} \tag{4.8}$$

4.1.4 Induced from an (α, ε)-metric structure

We now work with richer geometric structures on the manifold. Specifically, we consider what happens when the manifold is endowed with an (α, ε)-metric structure (M, J, g) such that $\alpha = -1$ (that is, $J^2 = -Id$). In this case, as we have seen in Section 3.1, the geometry on the manifold induces the weak generalized almost complex structures $\mathcal{J}_{J,g,\flat}, \mathcal{J}_{J,g,\sharp}$ given in Eq. (3.10), represented by triangular matrices. In addition, they induce a weak generalized almost paracomplex structure $\mathcal{F}_{J,g}$

introduced in Eq. (3.13). We now see that these generalized polynomial structures are compatible with $\mathcal{G}_0$ only for a certain value of ε.

Proposition 4.11. *Let (M, J, g) be an (α, ε)-metric manifold with $\alpha = -1$. Then, the weak generalized almost complex structures $\mathcal{J}_{J,g,\flat}, \mathcal{J}_{J,g,\sharp} \colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.10)*, are symmetric with respect to $\mathcal{G}_0$ if and only if $\varepsilon = 1$; that is, if and only if (M, J, g) is an almost Hermitian manifold or almost pseudo-Hermitian manifold (depending on g). Additionally, each one generates a symmetric nondegenerate operator:*

$$\begin{aligned} \Phi_{\mathcal{J}_{J,g,\flat}}(X+\xi, Y+\eta) &= \mathcal{G}_0(JX+\xi, JY+\eta) + \frac{1}{2} g(X,Y), \\ \Phi_{\mathcal{J}_{J,g,\sharp}}(X+\xi, Y+\eta) &= \mathcal{G}_0(JX+\xi, JY+\eta) + \frac{1}{2} g(\sharp_g\xi, \sharp_g\eta). \end{aligned} \tag{4.9}$$

Proof. We show here the calculations for both polynomial structures, based on Proposition 2.7:

$$\begin{aligned} \mathcal{G}_0(\mathcal{J}_{J,g,\flat}(X+\xi), Y+\eta) &= \mathcal{G}_0(JX + \flat_g X + \varepsilon J^*\xi, Y+\eta) \\ &= \frac{1}{2}((\flat_g X + \varepsilon J^*\xi)Y + \eta(JX)) \\ &= \frac{1}{2}(\varepsilon\xi(JY) + \eta(JX)) + \frac{1}{2} g(X,Y) \\ &= \frac{1}{2}(\xi(\varepsilon JY) + (\flat_g Y + J^*\eta)X) \\ &= \mathcal{G}_0(X+\xi, \varepsilon JY + \flat_g Y + J^*\eta), \end{aligned}$$

$$\begin{aligned} \mathcal{G}_0(\mathcal{J}_{J,g,\sharp}(X+\xi), Y+\eta) &= \mathcal{G}_0(JX + \sharp_g\xi + \varepsilon J^*\xi, Y+\eta) \\ &= \frac{1}{2}((\varepsilon J^*\xi)Y + \eta(JX + \sharp_g\xi)) \\ &= \frac{1}{2}(\varepsilon\xi(JY) + \eta(JX)) + \frac{1}{2} g(\sharp_g\xi, \sharp_g\eta) \\ &= \frac{1}{2}(\xi(\varepsilon JY + \sharp_g\eta) + (J^*\eta)X) \\ &= \mathcal{G}_0(X+\xi, \varepsilon JY + \sharp_g\eta + J^*\eta). \end{aligned}$$

Therefore, it is clear that both $\mathcal{J}_{J,g,\flat}, \mathcal{J}_{J,g,\sharp}$ are compatible with $\mathcal{G}_0$ if and only if $\varepsilon = 1$. The expressions for the nondegenerate structures $\Phi_{\mathcal{J}_{J,g,\flat}}$ and $\Phi_{\mathcal{J}_{J,g,\sharp}}$ proposed in Eq. (4.9) can be derived from the previous calculations. □

Proposition 4.12. *Let (M, J, g) be an (α, ε)-metric manifold with $\alpha = -1$. Then, the weak generalized almost paracomplex structure $\mathcal{F}_{J,g} \colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.13)*, is symmetric with respect to $\mathcal{G}_0$ if and only if $\varepsilon = 1$; that is, if and only if (M, J, g) is an almost Hermitian or pseudo-Hermitian manifold (depending on g). Moreover, it induces the following symmetric nondegenerate operator:*

$$\Phi_{\mathcal{F}_{J,g}}(X+\xi, Y+\eta) = \mathcal{G}_0(JX+\xi, JY+\eta) + \frac{\sqrt{2}}{2}(g(X,Y) + g(\sharp_g\xi, \sharp_g\eta)). \tag{4.10}$$

Proof. We use Eq. (2.7) to check the compatibility condition with $\mathcal{G}_0$:

$$\begin{aligned}\mathcal{G}_0(\mathcal{F}_{J,g}(X+\xi),Y+\eta) &= \mathcal{G}_0(JX+\sqrt{2}\,\sharp_g\xi+\sqrt{2}\,\flat_g X+\varepsilon J^*\xi,Y+\eta)\\ &= \frac{1}{2}((\sqrt{2}\,\flat_g X+\varepsilon J^*\xi)Y+\eta(JX+\sqrt{2}\,\sharp_g\xi))\\ &= \frac{1}{2}(\varepsilon\xi(JY)+\eta(JX))+\frac{\sqrt{2}}{2}(g(X,Y)+g(\sharp_g\xi,\sharp_g\eta))\\ &= \frac{1}{2}(\xi(\varepsilon JY+\sqrt{2}\,\sharp_g\eta)+(\sqrt{2}\,\flat_g Y+J^*\eta)X)\\ &= \mathcal{G}_0(X+\xi,\varepsilon JY+\sqrt{2}\,\sharp_g\eta+\sqrt{2}\,\flat_g Y+J^*\eta).\end{aligned}$$

Therefore, since $\varepsilon JY+\sqrt{2}\,\sharp_g\eta+\sqrt{2}\,\flat_g Y+J^*\eta=\mathcal{F}_{J,g}(Y+\eta)$ if and only if $\varepsilon=1$, it is clear that $\mathcal{F}_{J,g}$ is compatible with respect to the canonical pairing if and only if (M,J,g) is an almost Hermitian or pseudo-Hermitian manifold. The expression for the structure given in Eq. (4.10) is surmised from the previous calculation. □

A similar analysis allows us to understand what happens when the manifold is endowed with an (α,ε)-metric structure (M,F,g) such that $\alpha=1$ (that is, $F^2=Id$).

Proposition 4.13. *Let (M,F,g) be an (α,ε)-metric structure on M with $\alpha=1$. Then, the weak generalized almost paracomplex structures $\mathcal{F}_{F,g,\flat},\mathcal{F}_{F,g,\sharp}\colon\mathbb{T}M\to\mathbb{T}M$, introduced in Eq.* (3.11)*, are symmetric with respect to $\mathcal{G}_0$ if and only if $\varepsilon=-1$; that is, if and only if (M,F,g) is an almost para-Hermitian manifold. Additionally, each one generates a symmetric nondegenerate operator:*

$$\begin{aligned}\Phi_{\mathcal{F}_{F,g,\flat}}(X+\xi,Y+\eta) &= \mathcal{G}_0(FX+\xi,FY+\eta)+\frac{1}{2}g(X,Y),\\ \Phi_{\mathcal{F}_{F,g,\sharp}}(X+\xi,Y+\eta) &= \mathcal{G}_0(FX+\xi,FY+\eta)+\frac{1}{2}g(\sharp_g\xi,\sharp_g\eta).\end{aligned} \tag{4.11}$$

Proof. We use Proposition 2.7 to check if $\mathcal{F}_{F,g,\flat}$ and $\mathcal{F}_{F,g,\sharp}$ are compatible with $\mathcal{G}_0$:

$$\begin{aligned}\mathcal{G}_0(\mathcal{F}_{F,g,\flat}(X+\xi),Y+\eta) &= \mathcal{G}_0(FX+\flat_g X-\varepsilon F^*\xi,Y+\eta)\\ &= \frac{1}{2}((\flat_g X-\varepsilon F^*\xi)Y+\eta(FX))\\ &= \frac{1}{2}(-\varepsilon\xi(FY)+\eta(FX))+\frac{1}{2}g(X,Y)\\ &= \frac{1}{2}(\xi(-\varepsilon FY)+(\flat_g Y+F^*\eta)X)\\ &= \mathcal{G}_0(X+\xi,-\varepsilon FY+\flat_g Y+F^*\eta),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_0(\mathcal{F}_{F,g,\sharp}(X+\xi),Y+\eta) &= \mathcal{G}_0(FX+\sharp_g\xi-\varepsilon F^*\xi,Y+\eta)\\ &= \frac{1}{2}((-\varepsilon F^*\xi)Y+\eta(FX+\sharp_g\xi))\\ &= \frac{1}{2}(-\varepsilon\xi(FY)+\eta(FX))+\frac{1}{2}g(\sharp_g\xi,\sharp_g\eta)\\ &= \frac{1}{2}(\xi(-\varepsilon FY+\sharp_g\eta)+(F^*\eta)X)\\ &= \mathcal{G}_0(X+\xi,-\varepsilon FY+\sharp_g\eta+F^*\eta).\end{aligned}$$

Then, we see that both $\mathcal{F}_{F,g,\flat}$ and $\mathcal{F}_{F,g,\sharp}$ are compatible with the canonical pairing if and only if $\varepsilon = -1$. Using these calculations, it is immediate to see that the expressions for the nondegenerate structures $\Phi_{\mathcal{F}_{F,g,\flat}}, \Phi_{\mathcal{F}_{F,g,\sharp}}$ given in Eq. (4.11) hold true. □

Proposition 4.14. *Let (M, F, g) be an (α, ε)-metric manifold with $\alpha = 1$. Then, the weak generalized almost complex structure $\mathcal{J}_{F,g} \colon \mathbb{T}M \to \mathbb{T}M$, introduced in Eq.* (3.14)*, is symmetric with respect to $\mathcal{G}_0$ if and only if $\varepsilon = -1$; in other words, if and only if (M, F, g) is an almost para-Hermitian manifold. Moreover, it induces the following symmetric nondegenerate operator:*

$$\Phi_{\mathcal{J}_{F,g}}(X+\xi, Y+\eta) = \mathcal{G}_0(FX+\xi, FY+\eta) + \frac{\sqrt{2}}{2}(g(X,Y) - g(\sharp_g\xi, \sharp_g\eta)). \tag{4.12}$$

Proof. We use the identities given in Eq. (2.7) to see that the structure is symmetric with respect to $\mathcal{G}_0$:

$$\begin{aligned}
\mathcal{G}_0(\mathcal{J}_{F,g}(X+\xi), Y+\eta) &= \mathcal{G}_0(FX - \sqrt{2}\,\sharp_g\xi + \sqrt{2}\,\flat_g X - \varepsilon F^*\xi, Y+\eta) \\
&= \frac{1}{2}((\sqrt{2}\,\flat_g X - \varepsilon F^*\xi)Y + \eta(FX - \sqrt{2}\,\sharp_g\xi)) \\
&= \frac{1}{2}(-\varepsilon\xi(FY) + \eta(FX)) + \frac{\sqrt{2}}{2}(g(X,Y) - g(\sharp_g\xi, \sharp_g\eta)) \\
&= \frac{1}{2}(\xi(-\varepsilon FY - \sqrt{2}\,\sharp_g\eta) + (\sqrt{2}\,\flat_g Y + F^*\eta)X) \\
&= \mathcal{G}_0(X+\xi, -\varepsilon FY - \sqrt{2}\,\sharp_g\eta + \sqrt{2}\,\flat_g Y + F^*\eta).
\end{aligned}$$

Therefore, as $-\varepsilon JY - \sqrt{2}\,\sharp_g\eta + \sqrt{2}\,\flat_g Y + J^*\eta = \mathcal{J}_{F,g}(Y+\eta)$ if and only if $\varepsilon = -1$, it is proved that $\mathcal{J}_{F,g}$ is symmetric with respect to $\mathcal{G}_0$ if and only if (M, J, g) is an almost para-Hermitian manifold. The identity given in Eq. (4.12) can be deduced from the previous computation. □

All the above propositions that have been proved in this section show that, among all the induced generalized polynomial structures presented in Section 3.1, the only ones that are strong are those induced by an almost symplectic structure on the manifold and certain generalized structures induced by polynomial structures on the manifold.

4.2 Weak generalized metrics and generalized symplectic structures

Until now, we have only worked with almost polynomial structures defined on the generalized tangent bundle. If we want to introduce the concept of generalized (α, ε)-metric structures, first it will be necessary to talk about metrics defined on the generalized tangent bundle. In a similar manner to the generalized almost complex structures, Gualtieri first defined a generalized metric in [29] in an apparently unnatural way, adding several restrictions. Therefore, we will introduce the adjective "weak" in order to broaden the scope of admissible metrics on $\mathbb{T}M$.

Definition 4.15. A metric on the generalized tangent bundle (i.e., a section $\mathcal{G} \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ that satisfies the conditions given in Definition 2.1 for $E = \mathbb{T}M$) is called a *weak generalized metric*.

The most important example of a metric on $\mathbb{T}M$ is given by the canonical generalized metric $\mathcal{G}_0$. Hence, it is interesting to ask whether any generalized metric is related to $\mathcal{G}_0$. In the following result, we answer this question.

Proposition 4.16. *Any weak generalized metric $\mathcal{G} \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ can be constructed from the natural generalized metric $\mathcal{G}_0$ and an injective endomorphism $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$ such that*

$$\mathcal{G}(X+\xi, Y+\eta) = \mathcal{G}_0(\mathcal{K}(X+\xi), Y+\eta), \tag{4.13}$$

for all $X+\xi, Y+\eta \in \mathbb{T}M$ in the same fiber. In matrix notation, $\mathcal{K}$ is written as

$$\mathcal{K} = \begin{pmatrix} H & \sigma \\ \tau & H^* \end{pmatrix},$$

with $(\tau X)Y = (\tau Y)X$ and $\xi(\sigma\eta) = \eta(\sigma\xi)$ for every $X, Y \in TM$ and $\xi, \eta \in T^*M$ based on the same point. Reciprocally, each injective endomorphism $\mathcal{K}$ satisfying the conditions mentioned above induces a generalized metric $\mathcal{G}$ defined as in Eq. (4.13) *when the signature of $\mathcal{G}$ remains constant.*

Proof. First, from a generalized metric $\mathcal{G} \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ we can define the three wanted morphisms in the following way: given a $p \in M$, τ transforms each vector $X \in T_pM$ into the element of the cotangent bundle $\tau X \in T_p^*M$ such that $(\tau X)Y = 2\mathcal{G}(X,Y)$ for every $Y \in T_pM$; the morphism σ transforms each $\xi \in T_p^*M$ into the vector $\sigma\xi \in T_pM$ such that $\eta(\sigma\xi) = 2\mathcal{G}(\xi,\eta)$ for every $\eta \in T_p^*M$; and H is the bundle endomorphism of TM such that, given any $X \in T_pM$, we have $\xi(HX) = 2\mathcal{G}(X,\xi)$ for every $\xi \in T_p^*M$. To sum up, we have the following identities:

$$(\tau X)Y = 2\mathcal{G}(X,Y), \quad \eta(\sigma\xi) = 2\mathcal{G}(\xi,\eta), \quad \xi(HX) = 2\mathcal{G}(X,\xi).$$

It is straightforward to see that $(\tau X)Y = (\tau Y)X$ and $\xi(\sigma\eta) = \eta(\sigma\xi)$. Then, the endomorphism $\mathcal{K}$ obtained for such morphisms τ, σ, H fulfills the wanted relation:

$$\begin{aligned}
\mathcal{G}_0(\mathcal{K}(X+\xi), Y+\eta) &= \mathcal{G}_0((HX+\sigma\xi)+(\tau X + H^*\xi), Y+\eta) \\
&= \frac{1}{2}((\tau X + H^*\xi)Y + \eta(HX+\sigma\xi)) \\
&= \frac{1}{2}((\tau X)Y + \xi(HY) + \eta(HX) + \eta(\sigma\xi)) \\
&= \mathcal{G}(X,Y) + \mathcal{G}(Y,\xi) + \mathcal{G}(X,\eta) + \mathcal{G}(\xi,\eta) \\
&= \mathcal{G}(X+\xi, Y+\eta).
\end{aligned}$$

To check the injectivity of the endomorphism $\mathcal{K}$, if we suppose $\mathcal{K}(X+\xi) = 0$ then it must be $\mathcal{G}(X+\xi, Y+\eta) = \mathcal{G}_0(\mathcal{K}(X+\xi), Y+\eta) = 0$ for every $X+\xi \in \mathbb{T}M$. Therefore, as $\mathcal{G}$ is nondegenerate then $X+\xi = 0$ and $\mathcal{K}$ is injective.

To check the converse implication, it is immediate to see that the morphism $\mathcal{G}$ defined as in Eq. (4.13) for a suitable endomorphism $\mathcal{K}$ is bilinear. We check that it is symmetric:

$$\begin{aligned}\mathcal{G}(X,Y) &= \mathcal{G}_0(HX+\tau X,Y) = \frac{1}{2}(\tau X)(Y) = \frac{1}{2}(\tau Y)(X) = \mathcal{G}_0(HY+\tau Y,X)\\ &= \mathcal{G}(Y,X),\end{aligned}$$

$$\begin{aligned}\mathcal{G}(\xi,\eta) &= \mathcal{G}_0(\sigma\xi+H^*\xi,\eta) = \frac{1}{2}\eta(\sigma\xi) = \frac{1}{2}\xi(\sigma\eta) = \mathcal{G}_0(\sigma\eta+H^*\eta,\xi)\\ &= \mathcal{G}(\eta,\xi),\end{aligned}$$

$$\begin{aligned}\mathcal{G}(X,\xi) &= \mathcal{G}_0(HX+\tau X,\xi) = \frac{1}{2}\xi(HX) = \frac{1}{2}(H^*\xi)(X) = \mathcal{G}_0(\sigma\xi+H^*\xi,X)\\ &= \mathcal{G}(\xi,X).\end{aligned}$$

Finally, to see that $\mathcal{G}$ is nondegenerate morphism we assume that there is a $X+\xi \in \mathbb{T}M$ such that $\mathcal{G}(X+\xi,Y+\eta) = \mathcal{G}_0(\mathcal{K}(X+\xi),Y+\eta) = 0$ for all $Y+\eta$. Then, as $\mathcal{G}_0$ is nondegenerate, it must be $\mathcal{K}(X+\xi)=0$. The morphism $\mathcal{K}$ is injective, so $X+\xi=0$ and $\mathcal{G}$ is nondegenerate. □

An analogous definition and proposition can be stated for the case of symplectic structures defined on the generalized tangent bundle. In this case, since we have not found a definition of what could be considered a "strong generalized symplectic structure", the adjective "weak" has been omitted. The proof of the result is parallel to that of Proposition 4.16 and therefore will not be displayed here.

Definition 4.17. A symplectic structure on the generalized tangent bundle (that is to say, a section $\Omega \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ that satisfies the conditions given in Definition 2.2 for $E = \mathbb{T}M$) is called a *generalized symplectic structure*.

Proposition 4.18. *Any generalized symplectic structure $\Omega \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ can be constructed from the natural generalized metric $\mathcal{G}_0$ and an injective endomorphism $\mathcal{K}\colon \mathbb{T}M \to \mathbb{T}M$ such that*

$$\Omega(X+\xi,Y+\eta) = \mathcal{G}_0(\mathcal{K}(X+\xi),Y+\eta), \tag{4.14}$$

for all $X+\xi, Y+\eta \in \mathbb{T}M$ in the same fiber. In matrix notation, $\mathcal{K}$ is written as

$$\mathcal{K} = \begin{pmatrix} H & \sigma \\ \tau & -H^* \end{pmatrix},$$

with $(\tau X)Y = -(\tau Y)X$ and $\xi(\sigma\eta) = -\eta(\sigma\xi)$ for every $X, Y \in TM$ and $\xi,\eta \in T^*M$ based on the same point. Reciprocally, each injective endomorphism $\mathcal{K}$ satisfying the conditions mentioned above induces a generalized symplectic structure Ω defined as in Eq. (4.14).

There are some endomorphisms inducing generalized metrics and generalized symplectic structures that are especially interesting.

Example 4.19. Given an injective endomorphism $H\colon TM \to TM$, we have that

$$\mathcal{K}_{\lambda,H} = \begin{pmatrix} H & 0 \\ 0 & \lambda H^* \end{pmatrix}, \tag{4.15}$$

induces a weak generalized metric for $\lambda = 1$ and a generalized symplectic structure for $\lambda = -1$. Since both almost complex J and almost product structures F are injective, we can infer that the weak generalized almost complex structures $\mathcal{J}_{\lambda,J}$ and the weak generalized almost product structures $\mathcal{F}_{\lambda,F}$, introduced in Eqs. (3.8) and (3.9) respectively, induce interesting examples of weak generalized metrics and generalized symplectic structures. These morphisms are the ones presented in Eqs. (4.7) and (4.8).

Example 4.20. Given a nondegenerate morphism $\varphi \in \Gamma(T^*M \otimes T^*M)$ and $\lambda \in \{1, -1\}$, we have that

$$\mathcal{K}_{\lambda,\varphi} = \begin{pmatrix} 0 & \lambda\sharp_\varphi \\ \flat_\varphi & 0 \end{pmatrix}, \tag{4.16}$$

induces a weak generalized metric when φ is symmetric (i.e., $\varphi(Y, X) = \varphi(X, Y)$), and a generalized symplectic structure when φ is skew-symmetric (i.e., $\varphi(Y, X) = -\varphi(X, Y)$). Since both metrics g and almost symplectic structures ω are nondegenerate, we can infer that the endomorphisms $\mathcal{J}_g$ and $\mathcal{F}_g$ from Eqs. (3.2) and (3.3) induce good examples of weak generalized metrics, which are presented in Eqs. (4.3) and (4.4), while $\mathcal{J}_\omega$ and $\mathcal{F}_\omega$ from Eqs. (3.5) and (3.6) induce generalized symplectic structures, which are presented in Eqs. (4.5) and (4.6).

Example 4.21. In [52, Section 3], Nannicini induces a generalized metric from a Norden manifold (M, J, g). This generalized metric is defined as follows:

$$\mathcal{G}(X + \xi, Y + \eta) = g(X, Y) + \frac{1}{2}g(JX, \sharp_g\eta) + \frac{1}{2}g(\sharp_g\xi, JY) + g(\sharp_g\xi, \sharp_g\eta). \tag{4.17}$$

If we analyze this metric in the sense of Proposition 4.16, it can be checked that it is induced by the endomorphism

$$\mathcal{K} = \mathcal{K}_{1,J} + 2\mathcal{K}_{1,g} = \begin{pmatrix} J & 2\sharp_g \\ 2\flat_g & J^* \end{pmatrix},$$

using the notation from Eqs. (4.15) and (4.16). In fact,

$$\begin{aligned}
\mathcal{G}_0(\mathcal{K}(X + \xi), Y + \eta) &= \mathcal{G}_0(JX + 2\sharp_g\xi + 2\flat_g X + J^*\xi, Y + \eta) \\
&= \frac{1}{2}((2\flat_g X + J^*\xi)(Y) + \eta(JX + 2\sharp_g\xi)) \\
&= (\flat_g X)(Y) + \frac{1}{2}\xi(JY) + \frac{1}{2}\eta(JX) + \eta(\sharp_g\xi) \\
&= g(X, Y) + \frac{1}{2}g(JX, \sharp_g\eta) + \frac{1}{2}g(\sharp_g\xi, JY) + g(\sharp_g\xi, \sharp_g\eta).
\end{aligned}$$

After having defined the concepts of weak generalized metrics and generalized symplectic structures, two additional types of generalized structures can be incorporated into the diagram in Figure 3.2. Taking into account all the morphisms presented in Eqs. (4.3)-(4.12) (and, in particular, the discussions in Examples 4.19 and 4.20), further relationships can be added to this diagram, resulting in Figure 4.1. In this way, a much more comprehensive view of the interconnections between classical geometry on the tangent bundle and generalized is obtained: beyond those already shown in Figure 3.1, many additional and equally significant relationships emerge.

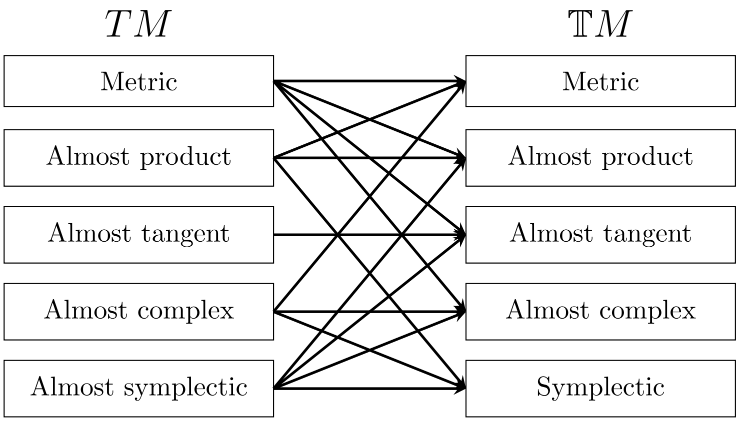


FIGURE 4.1: Scheme of all the studied weak generalized structures that can be induced from geometric structures on the manifold. The arrows represent when a geometric structure on the tangent bundle induces a generalized geometric structure.

Example 4.20 gives the most straightforward way to induce a weak generalized metric using a metric g defined on a manifold. Therefore, we consider the following definition and properties.

Definition 4.22. Let (M, g) be a Riemannian or pseudo-Riemannian manifold. We define the *generalized metric induced by g* as the morphism $\mathcal{G}_g \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ such that

$$\mathcal{G}_g(X + \xi, Y + \eta) = g(X, Y) + g(\sharp_g \xi, \sharp_g \eta). \tag{4.18}$$

We can then see the connection between $\mathcal{G}_0$ and $\mathcal{G}_g$ via Proposition 4.16: the generalized metric induced by g is constructed from $\mathcal{G}_0$ and the endomorphism $2\mathcal{F}_g$ via Eq. (4.13).

This generalized metric agrees with the definition of a metric on a dual vector bundle $E^* \to M$ and on a Whitney sum $E \oplus F \to M$ given in [54, Definitions 3.6, 3.9]. Also, this metric has been used in some previous works, such as [37].

There are some easy properties to check. For example, it is immediate to see that TM and T^*M are orthogonal distributions of $\mathbb{T}M$ with the metric $\mathcal{G}_g$. With respect to the signature of $\mathcal{G}_g$, we make the following affirmation.

Proposition 4.23. *If (M, g) is a pseudo-Riemannian manifold with signature (r, s), then the generalized metric $\mathcal{G}_g$ induced by g is a generalized pseudo-Riemannian metric with signature $(2r, 2s)$. In particular, $\mathcal{G}_g$ is positive-definite if g is positive-definite, and $\mathcal{G}_g$ is neutral if g is neutral.*

Proof. We fix a point $p \in M$ and work within the spaces T_pM, T_p^*M, $\mathbb{T}_pM$. We can take a basis of the tangent space $\{X_1, \ldots, X_{r+s}\} \subset T_pM$ such that $g(X_i, X_i) > 0$ for $i = 1, \ldots, r$ and $g(X_j, X_j) < 0$ for $j = r + 1, \ldots, r + s$. As we can use the isomorphism $\flat_g$ to obtain a basis $\{\flat_g X_1, \ldots, \flat_g X_{r+s}\} \subset T_p^*M$, we compute $\mathcal{G}_g(X_i, X_i)$ and $\mathcal{G}_g(\flat_g X_i, \flat_g X_i)$ for every i: if we take

$i \in \{1, \ldots, r\}$, then

$$\mathcal{G}_g(X_i, X_i) = g(X_i, X_i) > 0,$$
$$\mathcal{G}_g(\flat_g X_i, \flat_g X_i) = g(\sharp_g \flat_g X_i, \sharp_g \flat_g X_i) = g(X_i, X_i) > 0,$$

whilst for $j \in \{r+1, \ldots, r+s\}$ we can see that

$$\mathcal{G}_g(X_j, X_j) = g(X_j, X_j) < 0,$$
$$\mathcal{G}_g(\flat_g X_j, \flat_g X_j) = g(\sharp_g \flat_g X_j, \sharp_g \flat_g X_j) = g(X_j, X_j) < 0.$$

Therefore, as $\{X_1, \flat_g X_1, \ldots, X_{r+s}, \flat_g X_{r+s}\}$ is a basis of $\mathbb{T}_p M$, the signature of the generalized metric $\mathcal{G}_g$ is $(2r, 2s)$. □

As is the case with generalized polynomial structures, the original definition of metrics on the generalized tangent bundle required a somewhat convoluted compatibility condition. This definition is detailed below and it is justified from the perspective of G-structures.

Definition 4.24. A metric on the generalized tangent bundle $\mathcal{G} \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ is called a *strong generalized metric* if $\mathbb{T}M$ can be splitted in two subbundles $\mathbb{T}M = E_+ \oplus E_-$, where $\mathcal{G}_0$ is positive-definite in the subbundle E_+ and E_- is the orthogonal complement of E_+ with respect to the canonical metric $\mathcal{G}_0$, and $\mathcal{G}$ can be described as

$$\mathcal{G}(u, v) = \mathcal{G}_0(u_+, v_+) - \mathcal{G}_0(u_-, v_-),$$

where $u = u_+ + u_-$ and $v = v_+ + v_-$, such that $u_+, v_+ \in E_+$ and $u_-, v_- \in E_-$.

From the perspective of the theory of G-structures (see, for instance, [29, Proposition 6.1]), it can be shown that this definition of strong generalized metrics is equivalent to a reduction of the principal frame bundle associated with the canonical generalized metric $\mathcal{G}_0$ (see Section 2.2), whose structure group is $\mathrm{O}(n, n)$, to a bundle with structure group $\mathrm{O}(n) \times \mathrm{O}(n)$. This is justified by noting that $\mathrm{O}(n) \times \mathrm{O}(n)$ is the maximal compact subgroup of $\mathrm{O}(n, n)$ (i.e., a compact subgroup such that it is not properly contained in another compact subgroup). Exactly as classical Riemannian metrics correspond to a reduction of the frame bundle to one with structure group $\mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$, where $\mathrm{O}(n)$ is the maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$, an analogous reduction is carried out here for the group $\mathrm{O}(n, n)$, which arises due to the presence of $\mathcal{G}_0$.

If this definition is compared with the previous results, it is easy to deduce that, given a strong generalized metric $\mathcal{G}$ associated with a decomposition $\mathbb{T}M = E_+ \oplus E_-$, the endomorphism $\mathcal{K} \colon \mathbb{T}M \to \mathbb{T}M$ associated with $\mathcal{G}$ via Proposition 4.13 must be defined as $\mathcal{K}(u_+ + u_-) = u_+ - u_-$, where $u_+ \in E_+$ and $u_- \in E_-$. In this way, we have $\mathcal{K}^2 = \mathcal{I}d$; in other words, $\mathcal{K}$ is a weak generalized almost paracomplex structure whose ± 1-eigenbundles are $E_\pm$. Moreover, since the map $\mathcal{G}$ must be symmetric, $\mathcal{K}$ must be symmetric with respect to the canonical pairing $\mathcal{G}_0$. This further highlights the relevance of studying generalized polynomial structures that are not necessarily strong.

As a final remark, we would like to highlight that other authors, such as M. Garcia-Fernandez (see [27]) have worked with a definition of strong generalized metrics that allows for metrics that are not necessarily positive-definite. From the perspective of G-structures, a strong generalized metric of signature (r, s) would thus be equivalent to a reduction of the indefinite orthogonal group $\mathrm{O}(n, n)$ to the group $\mathrm{O}(r, s) \times \mathrm{O}(n - r, n - s) \subset \mathrm{O}(n, n)$.

4.3 Generalized (α, ε)-metric structures

Based on the definitions given in Sections 2.1, 3.1 and 4.2, we can also consider generalized polynomial structures and generalized metrics on the generalized tangent bundle of a manifold such that they are compatible between them. In fact, we have already studied this in Section 4.1.3 for the specific case of the canonical generalized metric $\mathcal{G}_0$. We will now consider a broader range of generalized metrics.

Definition 4.25. An (α, ε)-metric structure on the generalized tangent bundle (that is to say, an endomorphism $\mathcal{K} \colon \mathbb{T}M \to \mathbb{T}M$ and a metric $\mathcal{G} \in \Gamma((\mathbb{T}M)^* \otimes (\mathbb{T}M)^*)$ that satisfy the conditions given in Definition 2.6 for $E = \mathbb{T}M$) is called a *generalized* (α, ε)*-metric structure*.

Based on the classification given after Definition 2.6, a generalized (α, ε)-metric structure $(\mathbb{T}M, \mathcal{K}, \mathcal{G})$ will be called in a different way depending on the values of α and ε:

- If $\alpha = 1$ and $\varepsilon = 1$, then the structure $(\mathbb{T}M, \mathcal{K}, \mathcal{G})$ is named *generalized almost product Riemannian* or *pseudo-Riemannian*, depending on whether the generalized metric $\mathcal{G}$ is Riemannian or pseudo-Riemannian. If $\mathcal{G}$ is Riemannian and $\mathcal{K}$ is a weak generalized almost paracomplex structure, $(\mathbb{T}M, \mathcal{K}, \mathcal{G})$ is called a *generalized almost para-Norden structure*.
- If $\alpha = 1$ and $\varepsilon = -1$, then the structure $(\mathbb{T}M, \mathcal{K}, \mathcal{G})$ is named *generalized almost para-Hermitian*. The compatibility condition between $\mathcal{K}$ and $\mathcal{G}$ from Eq. (2.5) forces the generalized metric to have a neutral signature.
- If $\alpha = -1$ and $\varepsilon = 1$, then the structure $(\mathbb{T}M, \mathcal{K}, \mathcal{G})$ is named *generalized almost Hermitian* or *pseudo-Hermitian*, depending on whether g is Riemannian or pseudo-Riemannian.
- If $\alpha = -1$ and $\varepsilon = -1$, then the structure $(\mathbb{T}M, \mathcal{K}, \mathcal{G})$ is named *generalized almost Norden*. As in the case of almost para-Hermitian structures, $\mathcal{G}$ must be a generalized pseudo-Riemannian metric with neutral signature.

The fundamental tensor associated to a generalized (α, ε)-metric structure will be called its *generalized fundamental tensor*. As it is indicated in Eq. (2.6), it is defined as

$$\Phi(X + \xi, Y + \eta) = \mathcal{G}(\mathcal{K}(X + \xi), Y + \eta),$$

for every $X + \xi, Y + \eta \in \mathbb{T}M$ in the same fiber. If $\alpha\varepsilon = 1$, Φ is a generalized metric (the *generalized twin metric*), while if $\alpha\varepsilon = -1$ it is a generalized symplectic structure (the *generalized fundamental symplectic structure*).

Our goal in this section is to study the compatibility of the induced generalized polynomial structures that have already been introduced with the induced generalized metric $\mathcal{G}_g$ from Definition 4.22. Because of this, the following annotation is relevant. We define a *generalized* (α, ε)*-metric* *structure induced by* g as an (α, ε)-metric structure on the fiber bundle $\mathbb{T}M$ with the generalized metric $\mathcal{G}_g$ from Eq. (4.18).

First, we work with a given (pseudo-)Riemannian manifold (M, g). In this case, we have seen in Subsections 3.1.1 and 4.1.1 that g induces a weak generalized almost complex structure, $\mathcal{J}_g$ introduced in Eq. (3.2), and a weak generalized almost paracomplex structure, $\mathcal{F}_g$ introduced in Eq. (3.3). Then, these structures are related to the generalized induced metric $\mathcal{G}_g$ as follows.

Proposition 4.26. *Let (M, g) be a (pseudo-)Riemannian manifold, and $\mathcal{J}_g, \mathcal{F}_g$ the weak generalized almost complex and almost paracomplex structure induced from g. Then, $(\mathbb{T}M, \mathcal{J}_g, \mathcal{G}_g)$ is a generalized almost Hermitian or almost pseudo-Hermitian structure (depending on g). Similarly, $(\mathbb{T}M, \mathcal{F}_g, \mathcal{G}_g)$ is a generalized almost para-Norden structure (or pseudo-Riemannian almost product, depending on g). Their generalized fundamental tensors are, respectively,*

$$\begin{aligned} \Phi_{\mathcal{J}_g}(X+\xi, Y+\eta) &= -2\Omega_0(X+\xi, Y+\eta), \\ \Phi_{\mathcal{F}_g}(X+\xi, Y+\eta) &= 2\mathcal{G}_0(X+\xi, Y+\eta). \end{aligned} \tag{4.19}$$

Proof. We check that $\mathcal{J}_g$ and $\mathcal{F}_g$ fulfill the requirement with respect to $\mathcal{G}_g$:

$$\begin{aligned} \mathcal{G}_g(\mathcal{J}_g(X+\xi), \mathcal{J}_g(Y+\eta)) &= \mathcal{G}_g(-\sharp_g\xi + \flat_g X, -\sharp_g\eta + \flat_g Y) \\ &= g(-\sharp_g\xi, -\sharp_g\eta) + g(\sharp_g\flat_g X, \sharp_g\flat_g Y) \\ &= g(X, Y) + g(\sharp_g\xi, \sharp_g\eta) \\ &= \mathcal{G}_g(X+\xi, Y+\eta), \end{aligned}$$

$$\begin{aligned} \mathcal{G}_g(\mathcal{F}_g(X+\xi), \mathcal{F}_g(Y+\eta)) &= \mathcal{G}_g(\sharp_g\xi + \flat_g X, \sharp_g\eta + \flat_g Y) \\ &= g(\sharp_g\xi, \sharp_g\eta) + g(\sharp_g\flat_g X, \sharp_g\flat_g Y) \\ &= g(X, Y) + g(\sharp_g\xi, \sharp_g\eta) \\ &= \mathcal{G}_g(X+\xi, Y+\eta). \end{aligned}$$

Also, it is easy to check the expressions in Eq. (4.19):

$$\begin{aligned} \Phi_{\mathcal{J}_g}(X+\xi, Y+\eta) &= \mathcal{G}_g(\mathcal{J}_g(X+\xi), Y+\eta) = \mathcal{G}_g(-\sharp_g\xi + \flat_g X, Y+\eta) \\ &= -g(\sharp_g\xi, Y) + g(\sharp_g\flat_g X, \sharp_g\eta) = -\xi(Y) + \eta(X) \\ &= -2\Omega_0(X+\xi, Y+\eta), \end{aligned}$$

$$\begin{aligned} \Phi_{\mathcal{F}_g}(X+\xi, Y+\eta) &= \mathcal{G}_g(\mathcal{F}_g(X+\xi), Y+\eta) = \mathcal{G}_g(\sharp_g\xi + \flat_g X, Y+\eta) \\ &= g(\sharp_g\xi, Y) + g(\sharp_g\flat_g X, \sharp_g\eta) = \xi(Y) + \eta(X) \\ &= 2\mathcal{G}_0(X+\xi, Y+\eta). \end{aligned}$$

Therefore, the result is proved. □

We work now with a given (α, ε)-metric manifold (M, J, g) such that its fundamental tensor is $\phi(\cdot, \cdot) = g(J\cdot, \cdot)$. If $\alpha\varepsilon = 1$, the fundamental tensor is a metric and, as such, induces two generalized polynomial structures: the weak generalized complex structure $\mathcal{J}_\phi$ from Eq. (3.2) and the weak generalized paracomplex structure $\mathcal{F}_\phi$ from Eq. (3.3). On the other hand, if $\alpha\varepsilon = -1$ the fundamental tensor is an almost symplectic structure, and hence induces the strong generalized complex structure $\mathcal{J}_\phi$ from Eq. (3.5) and the strong generalized paracomplex structure $\mathcal{F}_\phi$ from Eq. (3.6). Their behavior with respect to $\mathcal{G}_g$ is shown in the following statement.

Proposition 4.27. *Let (M, J, g) be an (α, ε)-metric manifold with $\phi(\cdot,\cdot) = g(J\cdot,\cdot)$ as its fundamental tensor. Then, the induced generalized metric structure $(\mathbb{T}M, \mathcal{J}_\phi, \mathcal{G}_g)$ is almost Hermitian or almost pseudo-Hermitian (depending on g) when $\varepsilon = 1$, and almost Norden when $\varepsilon = -1$. On the other hand, $(\mathbb{T}M, \mathcal{F}_\phi, \mathcal{G}_g)$ is an induced generalized almost para-Norden structure or generalized pseudo-Riemannian almost product (depending on g) for $\varepsilon = 1$, and a generalized almost para-Hermitian structure when $\varepsilon = -1$. When $\varepsilon = 1$, their respective generalized fundamental tensors are*

$$\begin{aligned} \Phi_{\mathcal{J}_\phi}(X+\xi, Y+\eta) &= -2\Omega_0(JX+\xi, JY+\eta), \\ \Phi_{\mathcal{F}_\phi}(X+\xi, Y+\eta) &= 2\mathcal{G}_0(JX+\xi, JY+\eta), \end{aligned} \tag{4.20}$$

while for $\varepsilon = -1$ we have

$$\begin{aligned} \Phi_{\mathcal{J}_\phi}(X+\xi, Y+\eta) &= 2\mathcal{G}_0(JX+\xi, JY+\eta), \\ \Phi_{\mathcal{F}_\phi}(X+\xi, Y+\eta) &= -2\Omega_0(JX+\xi, JY+\eta). \end{aligned} \tag{4.21}$$

Proof. We first check that $\mathcal{J}_\phi$ and $\mathcal{F}_\phi$ are compatible with $\mathcal{G}_g$. In order to do that, we use Proposition 2.7:

$$\begin{aligned} \mathcal{G}_g(\mathcal{J}_\phi(X+\xi), \mathcal{J}_\phi(Y+\eta)) &= \mathcal{G}_g(-\sharp_\phi\xi + \flat_\phi X, -\sharp_\phi\eta + \flat_\phi Y) \\ &= g(-\sharp_\phi\xi, -\sharp_\phi\eta) + g(\sharp_g\flat_\phi X, \sharp_g\flat_\phi Y) \\ &= g(\alpha J\sharp_g\xi, \alpha J\sharp_g\eta) + g(\sharp_g\flat_g JX, \sharp_g\flat_g JY) \\ &= g(J\sharp_g\xi, J\sharp_g\eta) + g(JX, JY) \\ &= \varepsilon(g(\sharp_g\xi, \sharp_g\eta) + g(X, Y)) \\ &= \varepsilon\mathcal{G}_g(X+\xi, Y+\eta), \end{aligned}$$

$$\begin{aligned} \mathcal{G}_g(\mathcal{F}_\phi(X+\xi), \mathcal{J}_\phi(Y+\eta)) &= \mathcal{G}_g(\sharp_\phi\xi + \flat_\phi X, \sharp_\phi\eta + \flat_\phi Y) \\ &= g(\sharp_\phi\xi, \sharp_\phi\eta) + g(\sharp_g\flat_\phi X, \sharp_g\flat_\phi Y) \\ &= g(\alpha J\sharp_g\xi, \alpha J\sharp_g\eta) + g(\sharp_g\flat_g JX, \sharp_g\flat_g JY) \\ &= g(J\sharp_g\xi, J\sharp_g\eta) + g(JX, JY) \\ &= \varepsilon(g(\sharp_g\xi, \sharp_g\eta) + g(X, Y)) \\ &= \varepsilon\mathcal{G}_g(X+\xi, Y+\eta). \end{aligned}$$

Hence, the generalized induced metric structures are the indicated ones. We compute now their respective fundamental tensors:

$$\begin{aligned}\Phi_{\mathcal{J}_\phi}(X+\xi,Y+\eta) &= \mathcal{G}_g(\mathcal{J}_\phi(X+\xi),Y+\eta) = \mathcal{G}_g(-\sharp_\phi\xi+\flat_\phi X,Y+\eta)\\ &= g(-\sharp_\phi\xi,Y)+g(\sharp_g\flat_\phi X,\sharp_g\eta) = -\varepsilon g(\sharp_g J^*\xi,Y)+g(\sharp_g\flat_g JX,\sharp_g\eta)\\ &= -\varepsilon\xi(JY)+\eta(JX),\end{aligned}$$

$$\begin{aligned}\Phi_{\mathcal{F}_\phi}(X+\xi,Y+\eta) &= \mathcal{G}_g(\mathcal{F}_\phi(X+\xi),Y+\eta) = \mathcal{G}_g(\sharp_\phi\xi+\flat_\phi X,Y+\eta)\\ &= g(\sharp_\phi\xi,Y)+g(\sharp_g\flat_\phi X,\sharp_g\eta) = \varepsilon g(\sharp_g J^*\xi,Y)+g(\sharp_g\flat_g JX,\sharp_g\eta)\\ &= \varepsilon\xi(JY)+\eta(JX).\end{aligned}$$

Therefore, it is immediate to see that if $\varepsilon=1$ then $\Phi_{\mathcal{J}_\phi}(X+\xi,Y+\eta)=-2\Omega_0(JX+\xi,JY+\eta)$ and $\Phi_{\mathcal{F}_\phi}(X+\xi,Y+\eta)=2\mathcal{G}_0(JX+\xi,JY+\eta)$, obtaining Eq. (4.21); on the other hand, if $\varepsilon=-1$ then $\Phi_{\mathcal{J}_\phi}(X+\xi,Y+\eta)=2\mathcal{G}_0(JX+\xi,JY+\eta)$ and $\Phi_{\mathcal{F}_\phi}(X+\xi,Y+\eta)=-2\Omega_0(JX+\xi,JY+\eta)$, resulting in Eq. (4.20). Therefore, the result is proved. □

We now consider that (M,J,g) is an (α,ε)-metric manifold with $\alpha=-1$. As J is an almost complex structure, we can induce the generalized almost complex structures $\mathcal{J}_{\lambda,J}$ given in Eq. (3.8), the generalized almost complex structures $\mathcal{J}_{J,g,\flat},\mathcal{J}_{J,g,\sharp}$ given in Eq. (3.10), and the generalized almost paracomplex structure $\mathcal{F}_{J,g}$ given in Eq. (3.13). It is easy to check that $\mathcal{J}_{J,g,\flat},\mathcal{J}_{J,g,\sharp}$ are not compatible with the induced generalized metric $\mathcal{G}_g$:

$$\begin{aligned}\mathcal{G}_g(\mathcal{J}_{J,g,\flat}(X+\xi),\mathcal{J}_{J,g,\flat}(Y+\eta)) &= \mathcal{G}_g(JX+\flat_g X+\varepsilon J^*\xi,JY+\flat_g Y+\varepsilon J^*\eta)\\ &= g(JX,JY)+g(\sharp_g(\flat_g X+\varepsilon J^*\xi),\sharp_g(\flat_g Y+\varepsilon J^*\eta))\\ &= \varepsilon g(X,Y)+g(X,Y)+\varepsilon(\xi(JY)+\eta(JX))+g(J\sharp_g\xi,J\sharp_g\eta)\\ &= (\varepsilon+1)g(X,Y)+\varepsilon g(\sharp_g\xi,\sharp_g\eta)+\varepsilon\mathcal{G}_0(JX+\xi,JY+\eta),\end{aligned}$$

$$\begin{aligned}\mathcal{G}_g(\mathcal{J}_{J,g,\sharp}(X+\xi),\mathcal{J}_{J,g,\sharp}(Y+\eta)) &= \mathcal{G}_g(JX+\sharp_g\xi+\varepsilon J^*\xi,JY+\sharp_g\eta+\varepsilon J^*\eta)\\ &= g(JX+\sharp_g\xi,JY+\sharp_g\eta)+g(\sharp_g(\varepsilon J^*\xi),\sharp_g(\varepsilon J^*\eta))\\ &= g(JX,JY)+\xi(JY)+\eta(JX)+g(\sharp_g\xi,\sharp_g\eta)+g(J\sharp_g\xi,J\sharp_g\eta)\\ &= \varepsilon g(X,Y)+(\varepsilon+1)g(\sharp_g\xi,\sharp_g\eta)+\mathcal{G}_0(JX+\xi,JY+\eta).\end{aligned}$$

In fact, these two endomorphisms are compatible with the generalized metric given in Eq. (4.17) (see [52, Section 3] for the case $\varepsilon=-1$). Anyway, the other structures are compatible with $\mathcal{G}_g$. The following statements gather these results.

Proposition 4.28. *Let (M,J,g) be an (α,ε)-metric manifold with $\alpha=-1$ and $\phi(\cdot,\cdot)=g(F\cdot,\cdot)$ its fundamental tensor. Then, the weak generalized almost complex structure $\mathcal{J}_{\lambda,J}$ given in Eq.* (3.8) *is compatible with $\mathcal{G}_g$. In particular, if $\varepsilon=1$ then $(\mathbb{T}M,\mathcal{J}_{\lambda,J},\mathcal{G}_g)$ is an induced generalized almost Hermitian or pseudo-Hermitian structure (depending on g); and if $\varepsilon=-1$ then $(\mathbb{T}M,\mathcal{J}_{\lambda,J},\mathcal{G}_g)$ is a generalized almost Norden structure. The generalized fundamental tensor of $(\mathbb{T}M,\mathcal{J}_{\lambda,J},\mathcal{G}_g)$ is*

$$\Phi_{\mathcal{J}_{\lambda,J}}(X+\xi,Y+\eta)=\phi(X,Y)-\lambda\phi(\sharp_\phi\xi,\sharp_\phi\eta). \tag{4.22}$$

Proof. First, we check the compatibility between the generalized almost complex structure $\mathcal{J}_{\lambda,J}$ and the generalized induced metric $\mathcal{G}_g$:

$$\begin{aligned}\mathcal{G}_g(\mathcal{J}_{\lambda,J}(X+\xi),\mathcal{J}_{\lambda,J}(Y+\eta)) &= \mathcal{G}_g(JX+\lambda J^*\xi, JY+\lambda J^*\eta)\\ &= g(JX,JY)+g(\lambda\sharp_g J^*\xi,\lambda\sharp_g J^*\eta)\\ &= g(JX,JY)+g(J\sharp_g\xi,J\sharp_g\eta)\\ &= \varepsilon(g(X,Y)+g(\sharp_g\xi,\sharp_g\eta))\\ &= \varepsilon\mathcal{G}_g(X+\xi,Y+\eta).\end{aligned}$$

We can easily check Eq. (4.22) for the fundamental tensor of the structure:

$$\begin{aligned}\Phi_{\mathcal{J}_{\lambda,J}}(X+\xi,Y+\eta) &= \mathcal{G}_g(\mathcal{J}_{\lambda,J}(X+\xi),Y+\eta) = \mathcal{G}_g(JX+\lambda J^*\xi,Y+\eta)\\ &= g(JX,Y)+\lambda g(\sharp_g J^*\xi,\sharp_g\eta) = \phi(X,Y)+\lambda\varepsilon g(\sharp_\phi\xi,\sharp_g\eta)\\ &= \phi(X,Y)-\lambda\phi(\sharp_\phi\xi,\sharp_\phi\eta).\end{aligned}$$

Hence, we finish the proof. □

Proposition 4.29. *Let (M,J,g) be an (α,ε)-metric manifold with $\alpha=-1$. Then, the weak generalized almost paracomplex structure $\mathcal{F}_{J,g}$ given in Eq.* (3.13) *is compatible with $\mathcal{G}_g$ if and only if $\varepsilon=-1$. In this case, $(\mathbb{T}M,\mathcal{F}_{J,g},\mathcal{G}_g)$ is an induced generalized pseudo-Riemannian almost product structure. If we denote the twin metric of (M,J,g) as ϕ, the generalized twin metric of $(\mathbb{T}M,\mathcal{F}_{J,g},\mathcal{G}_g)$ is given by*

$$\Phi_{\mathcal{F}_{J,g}}(X+\xi,Y+\eta) = 2\sqrt{2}\,\mathcal{G}_0(X+\xi,Y+\eta)+\phi(X,Y)+\phi(\sharp_\phi\xi,\sharp_\phi\eta). \tag{4.23}$$

Proof. We show here the calculations to check the compatibility between the generalized almost paracomplex structure $\mathcal{F}_{J,g}$ and the induced generalized metric $\mathcal{G}_g$, taking into account Proposition 2.7:

$$\begin{aligned}\mathcal{G}_g(\mathcal{F}_{J,g}(X+\xi),\mathcal{F}_{J,g}(Y+\eta)) &= \mathcal{G}_g(JX+\sqrt{2}\,\sharp_g\xi+\sqrt{2}\,\flat_g X+\varepsilon J^*\xi,\\ &\qquad JY+\sqrt{2}\,\sharp_g\eta+\sqrt{2}\,\flat_g Y+\varepsilon J^*\eta)\\ &= g(JX+\sqrt{2}\,\sharp_g\xi, JY+\sqrt{2}\,\sharp_g\eta)\\ &\quad + g(\sharp_g(\sqrt{2}\,\flat_g X+\varepsilon J^*\xi),\sharp_g(\sqrt{2}\,\flat_g Y+\varepsilon J^*\eta))\\ &= g(JX,JY)+\sqrt{2}\,(g(\sharp_g\xi,JY)+g(JX,\sharp_g\eta))\\ &\quad +2g(\sharp_g\xi,\sharp_g\eta)+2g(X,Y)\\ &\quad +\varepsilon\sqrt{2}\,(g(\sharp_g J^*\xi,Y)+g(X,\sharp_g J^*\eta))+g(\sharp_g J^*\xi,\sharp_g J^*\eta)\\ &= \sqrt{2}(1+\varepsilon)(\xi(JY)+\eta(JX))+(2+\varepsilon)(g(X,Y)+g(\sharp_g\xi,\sharp_g\eta))\\ &= \sqrt{2}(1+\varepsilon)(\xi(JY)+\eta(JX))+(2+\varepsilon)\mathcal{G}_g(X+\xi,Y+\eta).\end{aligned}$$

Therefore, the weak generalized almost paracomplex structure $\mathcal{F}_{J,g}$ is compatible with $\mathcal{G}_g$ if and only if $\varepsilon=-1$, that is, if (M,J,g) is an almost Norden manifold. As $\mathcal{G}_g$ must be a generalized neutral metric, $(\mathbb{T}M,\mathcal{F}_{J,g},\mathcal{G}_g)$ is a generalized pseudo-Riemannian almost product structure. We compute its generalized twin metric:

$$\begin{aligned}
\Phi_{\mathcal{F}_{J,g}}(X+\xi, Y+\eta) &= \mathcal{G}_g(\mathcal{F}_{J,g}(X+\xi), Y+\eta) = \mathcal{G}_g(JX + \sqrt{2}\,\sharp_g\xi + \sqrt{2}\,\flat_g X - J^*\xi, Y+\eta) \\
&= g(JX + \sqrt{2}\,\sharp_g\xi, Y) + g(\sharp_g(\sqrt{2}\,\flat_g X - J^*\xi), \sharp_g\eta) \\
&= g(JX, Y) + \sqrt{2}(g(\sharp_g\xi, Y) + g(X, \sharp_g\eta)) - g(\sharp_g J^*\xi, \sharp_g\eta) \\
&= \sqrt{2}(\xi(Y) + \eta(X)) + \phi(X, Y) + \phi(\sharp_\phi\xi, \sharp_\phi\eta) \\
&= 2\sqrt{2}\,\mathcal{G}_0(X+\xi, Y+\eta) + \phi(X, Y) + \phi(\sharp_\phi\xi, \sharp_\phi\eta).
\end{aligned}$$

Therefore, the result is proved. □

Analogously, when (M, F, g) is an (α, ε)-metric manifold with $\alpha = 1$, we have seen that we can induce the weak generalized almost product structure $\mathcal{F}_{1,F}$ and the strong generalized almost paracomplex structure $\mathcal{F}_{-1,F}$, given in Eq. (3.9); the weak generalized almost product structures $\mathcal{F}_{F,g,\flat}, \mathcal{F}_{F,g,\sharp}$ from Eq. (3.11); and the weak generalized almost complex structure $\mathcal{J}_{F,g}$ from Eq. (3.14). As in the previous situation, the structures $\mathcal{F}_{F,g,\flat}$ and $\mathcal{F}_{F,g,\sharp}$ are not compatible with $\mathcal{G}_g$:

$$\begin{aligned}
\mathcal{G}_g(\mathcal{F}_{F,g,\flat}(X+\xi), \mathcal{F}_{F,g,\flat}(Y+\eta)) &= \mathcal{G}_g(FX + \flat_g X - \varepsilon F^*\xi, FY + \flat_g Y - \varepsilon F^*\eta) \\
&= g(FX, FY) + g(\sharp_g(\flat_g X - \varepsilon F^*\xi), \sharp_g(\flat_g Y - \varepsilon F^*\eta)) \\
&= \varepsilon g(X, Y) + g(X, Y) - \varepsilon(\xi(FY) + \eta(FX)) + g(F\sharp_g\xi, F\sharp_g\eta) \\
&= (\varepsilon+1)g(X, Y) + \varepsilon g(\sharp_g\xi, \sharp_g\eta) - \varepsilon\mathcal{G}_0(FX+\xi, FY+\eta),
\end{aligned}$$

$$\begin{aligned}
&= g(FX + \sharp_g\xi, FY + \sharp_g\eta) + g(\sharp_g(\varepsilon F^*\xi), \sharp_g(\varepsilon F^*\eta)) \\
&= \varepsilon g(X, Y) + \xi(FY) + \eta(FX) + g(\sharp_g\xi, \sharp_g\eta) + g(F\sharp_g\xi, F\sharp_g\eta) \\
&= \varepsilon g(X, Y) + (\varepsilon+1)g(\sharp_g\xi, \sharp_g\eta) + \mathcal{G}_0(FX+\xi, FY+\eta).
\end{aligned}$$

With regard to the rest of structures, the following results show the necessary conditions for being compatible with $\mathcal{G}_g$. As their proofs are analogous to those of Propositions 4.28 and 4.29, they are not specified here.

Proposition 4.30. *Let (M, F, g) be an (α, ε)-metric manifold with $\alpha = 1$ and $\phi(\cdot,\cdot) = g(F\cdot,\cdot)$ its fundamental tensor. Then, the weak generalized almost product structure $\mathcal{F}_{\lambda,F}$ from Eq.* (3.9) *is compatible with $\mathcal{G}_g$. In particular, when $\varepsilon = 1$ the induced generalized (α, ε)-metric structure $(\mathbb{T}M, \mathcal{F}_{\lambda,F}, \mathcal{G}_g)$ is a generalized Riemannian or pseudo-Riemannian almost product structure (depending on g); and if $\varepsilon = -1$ then $(\mathbb{T}M, \mathcal{F}_{\lambda,F}, \mathcal{G}_g)$ is a generalized almost para-Hermitian structure. The generalized fundamental tensor of $(\mathbb{T}M, \mathcal{F}_{\lambda,F}, \mathcal{G}_g)$ is*

$$\Phi_{\mathcal{F}_{\lambda,F}}(X+\xi, Y+\eta) = \phi(X, Y) + \lambda\phi(\sharp_\phi\xi, \sharp_\phi\eta). \tag{4.24}$$

Proposition 4.31. *Let (M, F, g) be an (α, ε)-metric manifold with $\alpha = 1$. Then, the weak generalized almost complex structure $\mathcal{J}_{F,g}$ from Eq.* (3.14) *is compatible with $\mathcal{G}_g$ if and only if $\varepsilon = -1$. In this case, $(\mathbb{T}M, \mathcal{J}_{F,g}, \mathcal{G}_g)$ is a generalized almost pseudo-Hermitian structure. If we denote the fundamental tensor of (M, F, g) as ϕ, the generalized fundamental tensor of $(\mathbb{T}M, \mathcal{J}_{F,g}, \mathcal{G}_g)$ is given* *by*

$$\Phi_{\mathcal{J}_{F,g}}(X+\xi, Y+\eta) = -2\sqrt{2}\,\Omega_0(X+\xi, Y+\eta) + \phi(X,Y) + \phi(\sharp_\phi\xi, \sharp_\phi\eta). \tag{4.25}$$

Chapter 5

Integrability of weak generalized structures and the six-dimensional sphere

Both this chapter and Chapter 6 are devoted to the analysis of the integrability of certain generalized polynomial structures. In order to study this, it is necessary to define a bracket product, which will be the Dorfman bracket. This will allow for a proper definition of the integrability of a generalized polynomial structure using their associated eigenbundles.

In this sense, Gualtieri proved that, for an almost complex structure $J\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, the strong generalized almost complex structure $\mathcal{J}_{-1,J}$ from (3.8) is integrable if and only if J is integrable in the classical way. Also, for an almost symplectic structure $\omega \in \Omega^2(M)$, we find that the strong generalized almost complex structure $\mathcal{J}_\omega$ from (3.5) is integrable if and only if $d\omega = 0$. Therefore, any manifold that admits a complex structure or a symplectic structure admits a strong integrable generalized complex structure. However, the converse implication is not true: there are manifolds that admit strong generalized complex structures but do not admit complex or symplectic structures (see, for example, [13]).

This motivates us to focus on the six-dimensional sphere $\mathbb{S}^6$. Specifically, we address a version of the Hopf problem, which concerns the existence of integrable complex structures on $\mathbb{S}^6$, from the point of view of generalized geometry. In particular, our aim is to discuss the existence of examples of weak generalized complex structures on $\mathbb{S}^6$ that are integrable and can be written in a certain way.

Multiple approaches have been taken to study the classical Hopf problem (see, for example, [1] for a historical point of view and [2] for the state-of-the-art). One of these points of view concerns the search for partial results, studying the existence of complex structures on $\mathbb{S}^6$ that satisfy additional requirements. Following this line, an important result by A. Blanchard (see [8]) shows that $\mathbb{S}^6 \subset \mathbb{R}^7$ does not admit any complex structure compatible with the metric inherited by the Euclidean metric of $\mathbb{R}^7$. This result was popularized by C. LeBrun in [42].

The main result we will prove in this chapter is also a partial result. In the two-dimensional case, the sphere $\mathbb{S}^2$ admits a complex and a symplectic structure, and hence strong generalized complex structures. In the case of the six-dimensional sphere $\mathbb{S}^6$, since $H^2(\mathbb{S}^6, \mathbb{R}) = 0$ we know that it does not admit a symplectic structure, and it is not yet known whether it admits integrable complex structures. However, since there are manifolds that admit strong generalized complex structures but do not admit complex or symplectic ones, we can study the problem on the generalized tangent bundle $\mathbb{T}\mathbb{S}^6$ and determine if there are some particular integrable generalized complex structures. The main result proved in this chapter is summarized in the following statement.

Theorem 5.1. *There are no weak generalized complex structures on $\mathbb{S}^6$ that can be written as a spherical combination of the weak generalized almost complex structures $\mathcal{J}_{1,J}, \mathcal{J}_g$ from Eqs.* (3.8) and (3.2) *and the strong generalized almost complex structure $\mathcal{J}_\omega$ from Eq.* (3.2)*, where (J, g) is the nearly Kähler structure on $\mathbb{S}^6$ inherited from the pure octonion product.*

We must note that, since the bracket product that we will use works with sections of the generalized tangent bundle, in this chapter we will understand weak generalized complex structures as endomorphisms of sections $\mathcal{J} : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ and generalized metrics as morphisms $\mathcal{G} : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \mathfrak{F}(M)$.

5.1 Bracket products on the generalized tangent bundle

In order to discuss the integrability of geometric structures on the generalized tangent bundle of a manifold, it is necessary to define a bracket. A first approach could be to propose a bracket that maps pairs of sections of the generalized tangent bundle to sections of the same vector bundle and behaves analogously to the Lie bracket on the tangent bundle. Within the framework of vector bundles, there exists a mathematical structure that formally captures this resemblance between a vector bundle and the tangent bundle equipped with the Lie bracket. These structures, known as Lie algebroids, are described here below.

Definition 5.2 ([55])**.** Let M be a smooth manifold. A *Lie algebroid* over M is defined as a vector bundle $E \to M$ equipped with a bracket product $[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and a vector bundle morphism $\pi : E \to TM$ such that:

- The bracket is skew-symmetric, i.e., $[v, u] = -[u, v]$ for all $u, v \in \Gamma(E)$.
- The bracket satisfies the Jacobi identity, i.e., $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$ for all $u, v, w \in \Gamma(E)$.
- The bracket satisfies the Leibniz rule, i.e., $[u, fv] = \pi(u)(f)v + f[u, v]$ for all $f \in \mathfrak{F}(M)$ and $u, v \in \Gamma(E)$.

From the definition, it follows that π descends to a morphism of sections $\pi : \Gamma(E) \to \mathfrak{X}(M)$ such that $\pi([u, v]) = [\pi(u), \pi(v)]$ for every $u, v \in \Gamma(E)$, where the right-hand side represents the Lie bracket of the vector fields $\pi(u)$ and $\pi(v)$.

With this definition, it is easy to see that the tangent bundle $E = TM$ of a manifold constitutes one of the principal examples of a Lie algebroid, using the classical Lie bracket $[\cdot,\cdot]$ and the identity map $\pi(X) = X$. Therefore, it is not unreasonable to expect that a similar structure can be found in the case of the vector bundle $E = \mathbb{T}M$.

Dorfman introduced a first proposal for a bracket on $\mathbb{T}M$, which is also the most widely used. Before presenting it, we should recall the definitions and notations of the Lie derivative and the interior product. The *Lie derivative* of a 1-form $\xi \in \Omega^1(M)$ with respect to a vector field $X \in \mathfrak{X}(M)$ is the 1-form $\mathcal{L}_X\xi \in \Omega^1(M)$ that satisfies the following property for every $Y \in \mathfrak{X}(M)$:

$$(\mathcal{L}_X\xi)Y = X(\xi(Y)) - \xi([X, Y]).$$

On the other hand, the *interior product* of a differential form with respect to a field $X \in \mathfrak{X}(M)$ is the operation ι_X that maps r-forms $\theta \in \Omega^r(M)$ to $(r-1)$-forms $\iota_X\theta \in \Omega^{r-1}(M)$, defined as

$$(\iota_X\theta)(X_1, \ldots, X_{r-1}) = \theta(X, X_1, \ldots, X_{r-1}),$$

for every $X_1, \ldots, X_{r-1} \in \mathfrak{X}(M)$. We should also recall some properties between the Lie derivative, the interior product and the exterior derivative on any manifold. These relations will be important when proving different properties of the new bracket product.

Lemma 5.3 ([43]). *The following relations between the Lie derivative, the interior product and the exterior derivative are held for any vector fields $X, Y \in \mathfrak{X}(M)$:*

$$\begin{aligned} \mathcal{L}_X &= d\iota_X + \iota_X d, \\ \mathcal{L}_{[X,Y]} &= \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X, \\ \iota_{[X,Y]} &= \mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X, \\ \mathcal{L}_X d &= d\mathcal{L}_X. \end{aligned} \tag{5.1}$$

We are now in position to present the bracket product that we will use throughout the rest of the document.

Definition 5.4 ([18]). Let M be a smooth manifold and $\mathbb{T}M$ its generalized tangent bundle. The *Dorfman bracket* is defined as the morphism $[\![\cdot,\cdot]\!]_D \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ such that

$$[\![X+\xi, Y+\eta]\!]_D = [X, Y] + (\mathcal{L}_X\eta - \iota_Y d\xi), \tag{5.2}$$

for every $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$.

We may well think that the Dorfman bracket behaves as a Lie bracket and thus the generalized tangent bundle of a manifold, according to Definition 5.2, would be a Lie algebroid together with $[\![\cdot,\cdot]\!]_D$. However, this is not true: if we check whether the bracket is skew-symmetric, using Eq. (5.1) we have

$$[\![X+\xi, X+\xi]\!]_D = [X, X] + (\mathcal{L}_X\xi - \iota_X d\xi) = \iota_X d\xi + d\iota_X\xi - \iota_X d\xi = d(\xi(X)) \neq 0.$$

However, this should not be seen as a drawback, since this bracket possesses other equally interesting properties, which are described below.

Proposition 5.5. *The Dorfman bracket satisfies the following relation for every $u, v, w \in \Gamma(\mathbb{T}M)$:*

$$[\![u, [\![v, w]\!]_D]\!]_D = [\![[\![u, v]\!]_D, w]\!]_D + [\![v, [\![u, w]\!]_D]\!]_D. \tag{5.3}$$

Proof. We take three sections $X+\xi, Y+\eta, Z+\theta \in \Gamma(\mathbb{T}M)$ of the generalized tangent bundle, where $X, Y, Z \in \mathfrak{X}(M)$ and $\xi, \eta, \theta \in \Omega^1(M)$. To check this statement, we must use the relations given in Eq. (5.1):

$$\begin{aligned}
[\![X+\xi,[\![Y+\eta,Z+\theta]\!]_D]\!]_D &= [\![X+\xi,[Y,Z]+\mathcal{L}_Y\theta-\iota_Z d\eta]\!]_D \\
&= [X,[Y,Z]]+\mathcal{L}_X(\mathcal{L}_Y\theta-\iota_Z d\eta)-\iota_{[Y,Z]}d\xi \\
&= [X,[Y,Z]]+\mathcal{L}_X\mathcal{L}_Y\theta-\mathcal{L}_X\iota_Z d\eta-\mathcal{L}_Y\iota_Z d\xi+\iota_Z\mathcal{L}_Y d\xi,
\end{aligned}$$

$$\begin{aligned}
[\![Y+\eta,[\![X+\xi,Z+\theta]\!]_D]\!]_D &= [Y,[X,Z]]+\mathcal{L}_Y\mathcal{L}_X\theta-\mathcal{L}_Y\iota_Z d\xi-\iota_{[X,Z]}d\eta \\
&= -[Y,[Z,X]]+\mathcal{L}_Y\mathcal{L}_X\theta-\mathcal{L}_Y\iota_Z d\xi-\mathcal{L}_X\iota_Z d\eta+\iota_Z\mathcal{L}_X d\eta,
\end{aligned}$$

$$\begin{aligned}
[\![[\![X+\xi,Y+\eta]\!]_D,Z+\theta]\!]_D &= [\![[X,Y]+\mathcal{L}_X\eta-\iota_Y d\xi,Z+\theta]\!]_D \\
&= [[X,Y],Z]+\mathcal{L}_{[X,Y]}\theta-\iota_Z d(\mathcal{L}_X\eta-\iota_Y d\xi) \\
&= -[Z,[X,Y]]+\mathcal{L}_X\mathcal{L}_Y\theta-\mathcal{L}_Y\mathcal{L}_X\theta-\iota_Z d\mathcal{L}_X\eta+\iota_Z\mathcal{L}_Y d\xi.
\end{aligned}$$

Therefore, since $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$, Eq. (5.3) holds true. □

Proposition 5.6. *The Dorfman bracket satisfies the relation*

$$[\![X+\xi,f(Y+\eta)]\!]_D = X(f)(Y+\eta)+f[\![X+\xi,Y+\eta]\!]_D, \tag{5.4}$$

for every $X, Y \in \mathfrak{X}(M)$, every $\xi, \eta \in \Omega^1(M)$ and every smooth function $f \in \mathfrak{F}(M)$.

Proof. Taking two vector fields $X, Y \in \mathfrak{X}(M)$, two 1-forms $\xi, \eta \in \Omega^1(M)$ and a function $f \in \mathfrak{F}(M)$, it is immediately seen that

$$\begin{aligned}
[\![X+\xi,f(Y+\eta)]\!]_D &= [X,fY]+\mathcal{L}_X(f\eta)-\iota_{fY}d\xi \\
&= X(f)Y+f[X,Y]+X(f)\eta+f\mathcal{L}_X\eta-f\iota_Y d\xi \\
&= X(f)(Y+\eta)+f[\![X+\xi,Y+\eta]\!]_D,
\end{aligned}$$

and hence the identity from Eq. (5.4) is proved. □

Proposition 5.7. *The Dorfman bracket satisfies the relation*

$$X(\mathcal{G}_0(Y+\eta,Z+\theta)) = \mathcal{G}_0([\![X+\xi,Y+\eta]\!]_D,Z+\theta)+\mathcal{G}_0(Y+\eta,[\![X+\xi,Z+\theta]\!]_D), \tag{5.5}$$

for every vector fields $X, Y, Z \in \mathfrak{X}(M)$ and 1-forms $\xi, \eta, \theta \in \Omega^1(M)$.

Proof. It is immediate to prove:

$$\begin{aligned}
\mathcal{G}_0([\![X+\xi,Y+\eta]\!]_D,Z+\theta) &= \mathcal{G}_0([X,Y]+\mathcal{L}_X\eta-\iota_Y d\xi,Z+\theta) \\
&= \frac{1}{2}((\mathcal{L}_X\eta)(Z)-\iota_Z\iota_Y d\xi+\theta([X,Y])) \\
&= \frac{1}{2}(X(\eta(Z))-\eta([X,Z])-\iota_Z\iota_Y d\xi+\theta([X,Y])),
\end{aligned}$$

$$\mathcal{G}_0(Y+\eta, [\![X+\xi, Z+\theta]\!]_D) = \frac{1}{2}(X(\theta(Y)) - \theta([X,Y]) - \iota_Y\iota_Z d\xi + \eta([X,Z])).$$

Then, due to the definition of $\mathcal{G}_0$, the result is obtained. □

All these various properties make it possible to introduce a type of vector bundle distinct from Lie algebroids, in which the existence of an intrinsic inner product is taken into account.

Definition 5.8 ([58])**.** Let M be a smooth manifold. A *Courant algebroid* over M is defined as a vector bundle $E \to M$ equipped with a bracket product $[\![\cdot,\cdot]\!]\colon \Gamma(E)\times\Gamma(E)\to\Gamma(E)$, a nondegenerate inner product $\langle\cdot,\cdot\rangle\colon \Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M)\to\mathfrak{F}(M)$ and a vector bundle morphism $\pi\colon E\to TM$ such that:

- The bracket satisfies the equality $[\![u,u]\!] = \frac{1}{2}\sharp_{\langle\cdot,\cdot\rangle}\pi^*(d\langle u,u\rangle)$ for all $u\in\Gamma(E)$.
- The bracket satisfies a Jacobi-type identity, i.e., $[\![u,[\![v,w]\!]]\!] = [\![[\![u,v]\!],w]\!] + [\![v,[\![u,w]\!]]\!]$ for all $u,v,w\in\Gamma(E)$.
- The bracket satisfies the Leibniz rule, i.e., $[\![u,fv]\!] = \pi(u)(f)v + f[\![u,v]\!]$ for all $f\in\mathfrak{F}(M)$ and $u,v\in\Gamma(E)$.
- The bracket derives the inner product, i.e., $\pi(u)(\langle v,w\rangle) = \langle[\![u,v]\!],w\rangle + \langle v,[\![u,w]\!]\rangle$ for all $u,v,w\in\Gamma(E)$.

We can observe certain similarities between Definition 5.2 and Definition 5.8: both Lie algebroids and Courant algebroids are required to satisfy a version of the Jacobi identity and a Leibniz-type rule. Also, it can be proved that the morphism π in a Courant algebroid descends to a morphism of sections $\pi\colon\Gamma(E)\to\mathfrak{X}(M)$ such that $\pi([\![u,v]\!]) = [\pi(u),\pi(v)]$ for every $u,v\in\Gamma(E)$, where the right-hand side is the Lie bracket of the fields $\pi(u)$ and $\pi(v)$. However, there are essential differences between the two: Courant algebroids have an intrinsic inner product and, by definition, cannot be antisymmetric.

It is straightforward to verify that the generalized tangent bundle $\mathbb{T}M$ of a smooth manifold M constitutes a Courant algebroid together with the Dorfman bracket $[\![\cdot,\cdot]\!]_D$, the canonical generalized metric $\mathcal{G}_0$ and the projection $\pi(X+\xi)=X$. Nevertheless, there are other interesting examples related to $\mathbb{T}M$. To explore these, we introduce a new definition: a Courant algebroid is said to be *exact* when the short sequence

$$0 \longrightarrow T^*M \xrightarrow{\sharp_{\langle\cdot,\cdot\rangle}\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0\,,$$

is exact, that is, when $\operatorname{im}\sharp_{\langle\cdot,\cdot\rangle}\pi^* = \ker\pi$. We recall that $\pi^*\colon T^*M\to E^*$ is the pullback of the morphism π and $\sharp_{\langle\cdot,\cdot\rangle}\colon E^*\to E$ is the sharp isomorphism associated with the inner product $\langle\cdot,\cdot\rangle$. Then, the following theorem can be proved, which relates each exact Courant algebroid with the generalized tangent bundle via closed 3-forms.

Theorem 5.9 ([58])**.** *Each exact Courant algebroid $E\to M$ is isomorphic to the generalized tangent bundle $\mathbb{T}M$ together with the canonical generalized metric $\mathcal{G}_0$, the projection $\pi(X+\xi)=X$, and* *with a twist of the Dorfman bracket $[\![\cdot,\cdot]\!]_h$ by a closed 3-form $h \in \Omega^3(M)$, such that*

$$[\![X+\xi, Y+\eta]\!]_h = [\![X+\xi, Y+\eta]\!]_D + \iota_Y \iota_X h. \tag{5.6}$$

This theorem provides strong justification for using the Dorfman bracket $[\![\cdot,\cdot]\!]_D$ or its twisted version $[\![\cdot,\cdot]\!]_h$: it allows working with a broad class of vector bundles over the manifold. However, this is not the only possible choice for the bracket on $\mathbb{T}M$. An admissible alternative is to consider the skew-symmetrization of the Dorfman bracket. This new bracket will present slightly different properties from those of the previous bracket, as will be shown below.

Definition 5.10. The *Courant bracket* is the morphism $[\![\cdot,\cdot]\!]_C \colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ such that

$$\begin{aligned}[\![X+\xi, Y+\eta]\!]_C &:= \frac{1}{2}\left([\![X+\xi, Y+\eta]\!]_D - [\![Y+\eta, X+\xi]\!]_D\right) \\ &= [X,Y] + \left(\mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}d\Omega_0(X+\xi, Y+\eta)\right),\end{aligned} \tag{5.7}$$

for every $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$, where Ω_0 is the natural generalized symplectic structure from Eq. (2.12).

Proposition 5.11 ([29, Section 3.2])**.** *The Courant bracket satisfies the following properties:*

- *It is skew-symmetric, i.e.,* $[\![v,u]\!]_C = -[\![u,v]\!]_C$ *for all* $u, v \in \Gamma(\mathbb{T}M)$.
- *It fulfills the rule*

  $$[\![X+\xi, f(Y+\eta)]\!]_C = X(f)(Y+\eta) + f[\![X+\xi, Y+\eta]\!]_C + \mathcal{G}_0(X+\xi, Y+\eta)df,$$

  for all $f \in \mathfrak{F}(M)$, $X, Y \in \mathfrak{X}(M)$ *and* $\xi, \eta \in \Omega^1(M)$.
- *It fulfills the rule*

  $$\sum_{(u,v,w)\text{ cyclic}} [\![u, [\![v,w]\!]_C]\!]_C = \frac{1}{3}d\left(\sum_{(u,v,w)\text{ cyclic}} \mathcal{G}_0([\![u,v]\!]_C, w)\right),$$

  for all $u, v, w \in \Gamma(\mathbb{T}M)$.
- *It fulfills the rule*

  $$\begin{aligned}X(\mathcal{G}_0(Y+\eta, Z+\theta)) &= \mathcal{G}_0([\![X+\xi, Y+\eta]\!]_C, Z+\theta) + \mathcal{G}_0(Y+\eta, [\![X+\xi, Z+\theta]\!]_C) \\ &\quad + \mathcal{G}_0(d\mathcal{G}_0(X+\xi, Y+\eta), Z+\theta) + \mathcal{G}_0(Y+\eta, d\mathcal{G}_0(X+\xi, Z+\theta)),\end{aligned}$$

  for all $X, Y, Z \in \mathfrak{X}(M)$ *and* $\xi, \eta, \theta \in \Omega^1(M)$.

This second bracket sacrifices some of the nice properties of $[\![\cdot,\cdot]\!]_D$ in order to obtain a skew-symmetric product. Therefore, working with this bracket would require a redefinition of the desirable properties for a Courant algebroid.

Another plausible option for the bracket product is to use a connection defined on the manifold $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ to introduce a new morphism. This new bracket is defined as follows.

Definition 5.12. Let ∇ be a connection on the manifold. The ∇-*bracket* is defined as the morphism $[\![\cdot,\cdot]\!]_\nabla\colon \Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M)\to\Gamma(\mathbb{T}M)$ such that, for every $X,Y\in\mathfrak{X}(M)$ and $\xi,\eta\in\Omega^1(M)$,

$$[\![X+\xi,Y+\eta]\!]_\nabla=[X,Y]+(\nabla_X\eta-\nabla_Y\xi), \tag{5.8}$$

where $\nabla_X\eta\in\Omega^1(M)$ such that $(\nabla_X\eta)Y=X(\eta(Y))-\eta(\nabla_XY)$.

Proposition 5.13 ([49, Lemma 6])**.** *The ∇-bracket satisfies the following properties:*

- *It is skew-symmetric, i.e., $[\![v,u]\!]_\nabla=-[\![u,v]\!]_\nabla$ for all $u,v\in\Gamma(\mathbb{T}M)$.*
- *It fulfills the rule*

$$[\![X+\xi,f(Y+\eta)]\!]_\nabla=X(f)(Y+\eta)+f[\![X+\xi,Y+\eta]\!]_\nabla,$$

for all $f\in\mathfrak{F}(M)$, $X,Y\in\mathfrak{X}(M)$ and $\xi,\eta\in\Omega^1(M)$.

- *It fulfills the rule*

$$[\![u,[\![v,w]\!]_\nabla]\!]_\nabla+[\![v,[\![w,u]\!]_\nabla]\!]_\nabla+[\![w,[\![u,v]\!]_\nabla]\!]_\nabla=0,$$

for all $u,v,w\in\Gamma(\mathbb{T}M)$ if and only if the curvature tensor associated to ∇ is null.

It can be seen that the latter bracket presents a main disadvantage: we need to endow the manifold with additional structures, such as a connection ∇ or a metric g (typically working with its Levi-Civita connection ∇^g). Moreover, this bracket does not derive the canonical generalized metric; that is, $X(\mathcal{G}_0(Y+\eta,Z+\theta))\neq\mathcal{G}_0([\![X+\xi,Y+\eta]\!]_\nabla,Z+\theta)+\mathcal{G}_0(Y+\eta,[\![X+\xi,Z+\theta]\!]_\nabla)$. This can be verified through the following computations:

$$\begin{aligned}\mathcal{G}_0([\![X+\xi,Y+\eta]\!]_\nabla,Z+\theta)&=\mathcal{G}_0([X,Y]+\nabla_X\eta-\nabla_Y\xi,Z+\theta)\\&=\frac{1}{2}((\nabla_X\eta)(Z)-(\nabla_Y\xi)(Z)+\theta([X,Y]))\\&=\frac{1}{2}(X(\eta(Z))-\eta(\nabla_XZ)-Y(\xi(Z))+\xi(\nabla_YZ)+\theta([X,Y])),\end{aligned}$$

$$\mathcal{G}_0([\![X+\xi,Z+\theta]\!]_\nabla,Y+\eta)=\frac{1}{2}(X(\theta(Y))-\theta(\nabla_XY)-Z(\xi(Y))+\xi(\nabla_ZY)+\eta([X,Z])).$$

The following table summarizes all the computations and properties associated with each of the proposed brackets from Eqs. (5.2), (5.6), (5.7) and (5.8). Based on the discussion presented in this section, from this point onward we will work with the Dorfman bracket. The only drawback of using a bracket that is not skew-symmetric will be discussed later in Theorem 5.24.

Bracket	Jacobi	Skew-symmetry	Leibniz	Invariance of $\mathcal{G}_0$
(Twisted) Dorfman	✓	✗	✓	✓
Courant	✗	✓	✗	✗
Connection	$\approx$	✓	✓	✗

TABLE 5.1: Comparison of the properties of the brackets $[\![\cdot,\cdot]\!]_D$, $[\![\cdot,\cdot]\!]_C$ and $[\![\cdot,\cdot]\!]_\nabla$.

5.2 Integrability of generalized polynomial structures

Having selected a bracket on the generalized tangent bundle, we are now in a good position to define the integrability conditions that certain generalized geometric structures must satisfy. From the perspective of classical differential geometry, an almost complex structure J on a manifold is said to be integrable if its i-eigenbundle $L_J^{1,0}$, defined in Eq. (2.1), is involutive with respect to the Lie bracket; that is, if $[X - iJX, Y - iJY] \in L_J^{1,0}$ for every $X, Y \in \mathfrak{X}(M)$. This concept can be translated into the setting of generalized geometry by simply replacing vector fields with sections of $\mathbb{T}M$ and the Lie bracket with the Dorfman bracket.

Definition 5.14. A weak (resp. strong) generalized almost complex structure $\mathcal{J}$ is said to be *integrable* when its i-eigenbundle $\mathbb{L}_{\mathcal{J}}^{1,0}$ is involutive with respect to the Dorfman bracket; in other words, if $[\![u - i\mathcal{J}u, v - i\mathcal{J}v]\!]_D \in \Gamma(\mathbb{L}_{\mathcal{J}}^{1,0})$ for every $u, v \in \Gamma(\mathbb{T}M)$. Then $\mathcal{J}$ is called a *weak* (resp. *strong*) *generalized complex structure*.

It should be noted that the Dorfman bracket can be extended to the complexification $\mathbb{T}M_{\mathbb{C}}$ as $[\![iu, v]\!]_D = [\![u, iv]\!]_D = i[\![u, v]\!]_D$, so it is a $\mathbb{C}$-bilinear map.

It can be shown (see, for instance, [40, Chapter IX]) that the definition of integrability for almost complex structures J on manifolds is equivalent to the vanishing of the Nijenhuis tensor associated to J, $N_J\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is defined as

$$N_J(X, Y) = [JX, JY] - J([JX, Y] + [X, JY]) - [X, Y].$$

This notion can also be extended to the generalized tangent bundle, and it turns out that the equivalence between the vanishing of this application and the integrability of a generalized complex structure likewise holds in this context.

Definition 5.15. Let $\mathcal{J}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ a weak generalized almost complex structure. The *generalized Nijenhuis map* associated to $\mathcal{J}$ is the morphism $\mathcal{N}_{\mathcal{J}}\colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined as

$$\mathcal{N}_{\mathcal{J}}(u, v) = [\![\mathcal{J}u, \mathcal{J}v]\!]_D - \mathcal{J}([\![\mathcal{J}u, v]\!]_D + [\![u, \mathcal{J}v]\!]_D) - [\![u, v]\!]_D, \tag{5.9}$$

for $u, v \in \Gamma(\mathbb{T}M)$.

Proposition 5.16. *A weak generalized almost complex structure $\mathcal{J}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ is integrable if and only if $\mathcal{N}_{\mathcal{J}} \equiv 0$.*

Proof. If $[\![u - i\mathcal{J}u, v - i\mathcal{J}v]\!]_D$ is computed for any $u, v \in \Gamma(\mathbb{T}M)$, we have

$$[\![u - i\mathcal{J}u, v - i\mathcal{J}v]\!]_D = [\![u, v]\!]_D - [\![\mathcal{J}u, \mathcal{J}v]\!]_D - i([\![\mathcal{J}u, v]\!]_D + [\![u, \mathcal{J}v]\!]_D).$$

Then, $\mathcal{J}$ will be integrable if and only if $[\![u - i\mathcal{J}u, v - i\mathcal{J}v]\!]_D$ can be written as $w - i\mathcal{J}w$ for some $w \in \Gamma(\mathbb{T}M)$. From the last expression, it is inferred that it should be $w = [\![u, v]\!]_D - [\![\mathcal{J}u, \mathcal{J}v]\!]_D$ and $\mathcal{J}w = [\![\mathcal{J}u, v]\!]_D + [\![u, \mathcal{J}v]\!]_D$. Therefore, the morphism $\mathcal{J}$ will be integrable if and only if we have $[\![\mathcal{J}u, v]\!]_D + [\![u, \mathcal{J}v]\!]_D = \mathcal{J}([\![u, v]\!]_D - [\![\mathcal{J}u, \mathcal{J}v]\!]_D)$ or, equivalently, if

$$[\![\mathcal{J}u, \mathcal{J}v]\!]_D - [\![u, v]\!]_D - \mathcal{J}([\![\mathcal{J}u, v]\!]_D + [\![u, \mathcal{J}v]\!]_D) = \mathcal{N}_{\mathcal{J}}(u, v) = 0,$$

as we wanted to prove. □

The previous result is well known for strong generalized almost complex structures (see, for example, [36, Lemma 2.7]. However, it also holds for weak generalized almost complex structures. Moreover, as shown above, its proof is analogous to that of the strong structures.

Regarding the use of other brackets to study the integrability of generalized geometric structures, one can compare the integrability of a weak generalized almost complex structure with respect to the Dorfman bracket and the Courant bracket. In this sense, by using the generalized Nijenhuis map, we prove the following result.

Proposition 5.17. *Let $\mathcal{J} \colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ be a weak generalized almost complex structure. If $\mathcal{J}$ is integrable with respect to the Dorfman bracket, then it is integrable with respect to the Courant bracket.*

Proof. It is a straightforward calculation: if we denote the generalized Nijenhuis map associated to $\mathcal{J}$ using the Courant bracket as $\mathcal{N}^C_{\mathcal{J}}$, then

$$\begin{aligned}
\mathcal{N}^C_{\mathcal{J}}(u, v) &:= [\![\mathcal{J}u, \mathcal{J}v]\!]_C - \mathcal{J}([\![\mathcal{J}u, v]\!]_C + [\![u, \mathcal{J}v]\!]_C) - [\![u, v]\!]_C \\
&= \frac{1}{2}\left([\![\mathcal{J}u, \mathcal{J}v]\!]_D - [\![\mathcal{J}v, \mathcal{J}u]\!]_D - \mathcal{J}([\![\mathcal{J}u, v]\!]_D - [\![v, \mathcal{J}u]\!]_D + [\![u, \mathcal{J}v]\!]_D - [\![\mathcal{J}v, u]\!]_D)\right. \\
&\quad \left. - [\![u, v]\!]_D + [\![v, u]\!]_D\right) \\
&= \frac{1}{2}(\mathcal{N}^D_{\mathcal{J}}(u, v) - \mathcal{N}^D_{\mathcal{J}}(v, u)),
\end{aligned}$$

where $\mathcal{N}^D_{\mathcal{J}}$ is the generalized Nijenhuis map associated to $\mathcal{J}$ using the Dorfman bracket, as is defined in Eq. (5.9). Therefore, if $\mathcal{N}^D_{\mathcal{J}} \equiv 0$ then $\mathcal{N}^C_{\mathcal{J}} \equiv 0$. □

However, the converse implication does not always hold. As shown in the following result, when the structure under consideration is strong, integrability with respect to the Dorfman bracket and the Courant bracket becomes equivalent.

Proposition 5.18. *Let $\mathcal{J} \colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ be a strong generalized almost complex structure. If $\mathcal{J}$ is integrable with respect to the Courant bracket, then it is integrable with respect to the Dorfman bracket.*

Proof. We begin computing the difference between the Courant bracket and the Dorfman one. If we take two sections $X+\xi, Y+\eta \in \Gamma(\mathbb{T}M)$, then

$$\begin{aligned}
[\![X+\xi, Y+\eta]\!]_C - [\![X+\xi, Y+\eta]\!]_D &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}d\Omega_0(X+\xi, Y+\eta) \\
&\quad - ([X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi) \\
&= -\mathcal{L}_Y\xi + \frac{1}{2}d(\xi(Y) - \eta(X)) + \iota_Y d\xi \\
&= -(\iota_Y d\xi + d\iota_Y\xi) + \frac{1}{2}d(\xi(Y) - \eta(X)) + \iota_Y d\xi \\
&= -\frac{1}{2}d(\xi(Y) + \eta(X)) \\
&= -d\mathcal{G}_0(X+\xi, Y+\eta).
\end{aligned}$$

From this equality, we expand the generalized Nijenhuis map with respect to the Courant bracket, expressing it in terms of the Dorfman bracket and the exterior derivative of several products involving the canonical pairing.

$$\begin{aligned}
\mathcal{N}^C_{\mathcal{J}}(u,v) &= [\![\mathcal{J}u, \mathcal{J}v]\!]_C - \mathcal{J}([\![\mathcal{J}u, v]\!]_C + [\![u, \mathcal{J}v]\!]_C) - [\![u,v]\!]_C \\
&= [\![\mathcal{J}u, \mathcal{J}v]\!]_D - d\mathcal{G}_0(\mathcal{J}u, \mathcal{J}v) - \mathcal{J}\,([\![\mathcal{J}u, v]\!]_D - d\mathcal{G}_0(\mathcal{J}u, v) \\
&\quad + [\![u, \mathcal{J}v]\!]_D - d\mathcal{G}_0(u, \mathcal{J}v)) - [\![u,v]\!]_D + d\mathcal{G}_0(u,v) \\
&= [\![\mathcal{J}u, \mathcal{J}v]\!]_D - d\mathcal{G}_0(u,v) - \mathcal{J}\,([\![\mathcal{J}u, v]\!]_D + d\mathcal{G}_0(u, \mathcal{J}v) \\
&\quad + [\![u, \mathcal{J}v]\!]_D - d\mathcal{G}_0(u, \mathcal{J}v)) - [\![u,v]\!]_D + d\mathcal{G}_0(u,v) \\
&= [\![\mathcal{J}u, \mathcal{J}v]\!]_D - \mathcal{J}([\![\mathcal{J}u, v]\!]_D + [\![u, \mathcal{J}v]\!]_D) - [\![u,v]\!]_D \\
&= \mathcal{N}^D_{\mathcal{J}}(u,v).
\end{aligned}$$

Therefore, if $\mathcal{N}^C_{\mathcal{J}} \equiv 0$ then $\mathcal{N}^D_{\mathcal{J}} \equiv 0$, as we wanted to prove. □

Analogous definitions and results can be formulated for weak generalized almost product and almost paracomplex structures. They are stated below without their proofs.

Definition 5.19. A weak (resp. strong) generalized almost product structure $\mathcal{F}$ is said to be *integrable* when their ± 1-eigenbundles $\mathbb{L}^+_{\mathcal{F}}, \mathbb{L}^-_{\mathcal{F}}$ are involutive with respect to the Dorfman bracket; in other words, if $[\![u+\mathcal{F}u, v+\mathcal{F}v]\!]_D \in \Gamma(\mathbb{L}^+_{\mathcal{F}})$ and $[\![u-\mathcal{F}u, v-\mathcal{F}v]\!]_D \in \Gamma(\mathbb{L}^-_{\mathcal{F}})$ for every $u, v \in \Gamma(\mathbb{T}M)$. Then $\mathcal{F}$ is called a *weak* (resp. *strong*) *generalized product structure*.

Definition 5.20. Let $\mathcal{F}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ be a weak generalized almost product structure. The *generalized Nijenhuis map* associated to $\mathcal{F}$ is the map $\mathcal{N}_{\mathcal{F}}\colon \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined as

$$\mathcal{N}_{\mathcal{F}}(u,v) = [\![\mathcal{F}u, \mathcal{F}v]\!]_D - \mathcal{F}([\![\mathcal{F}u, v]\!]_D + [\![u, \mathcal{F}v]\!]_D) + [\![u,v]\!]_D, \tag{5.10}$$

for $u, v \in \Gamma(\mathbb{T}M)$.

Proposition 5.21. *A weak generalized almost product structure $\mathcal{F}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ is integrable if and only if $\mathcal{N}_{\mathcal{F}} \equiv 0$.*

Example 5.22 ([60, Example 1]). Since the presentation of the canonical generalized paracomplex structure in Definition 2.14, the term "almost" has been deliberately avoided. This is because this structure is, in fact, integrable. This can be verified through straightforward computations:

$$\begin{aligned}\llbracket \mathcal{F}_0(X+\xi), \mathcal{F}_0(Y+\eta)\rrbracket_D &= \llbracket -X+\xi, -Y+\eta\rrbracket_D = [X,Y] - \mathcal{L}_X\eta + \iota_Y d\xi,\\ \llbracket \mathcal{F}_0(X+\xi), Y+\eta\rrbracket_D &= \llbracket -X+\xi, Y+\eta\rrbracket_D = -[X,Y] - \mathcal{L}_X\eta - \iota_Y d\xi,\\ \llbracket X+\xi, \mathcal{F}_0(Y+\eta)\rrbracket_D &= \llbracket X+\xi, -Y+\eta\rrbracket_D = -[X,Y] + \mathcal{L}_X\eta + \iota_Y d\xi\end{aligned}$$

Therefore, its Nijenhuis map is null:

$$\mathcal{N}_{\mathcal{F}_0}(X+\xi, Y+\eta) = 2[X,Y] - \mathcal{F}_0(-2[X,Y]) + [X,Y] = 2[X,Y] - 2[X,Y] = 0.$$

This show that the problem of finding integrable generalized paracomplex structures on any manifold is trivial, since every manifold admits the canonical generalized paracomplex structure $\mathcal{F}_0$.

It is of utmost importance to comment on some properties of the generalized Nijenhuis map and the differences with respect to the usual Nijenhuis tensor of two vector fields. In other works where strong generalized structures are studied, such as [15] by V. Cortés and L. David, it has been checked that the generalized Nijenhuis map behaves as a tensor on a manifold:

Proposition 5.23 ([15, Lemma 9]). *The Nijenhuis generalized map $\mathcal{N}_{\mathcal{J}}(u,v)$ associated with a strong generalized almost complex structure is $\mathfrak{F}(M)$-linear in both u and v.*

Then, the generalized Nijenhuis map of a strong generalized almost complex structure can be called its *generalized Nijenhuis tensor*. However, this is not the case for weak generalized almost complex structures: $\mathcal{N}_{\mathcal{J}}$ is not always $\mathfrak{F}(M)$-linear. To see this, take any almost Hermitian manifold (M,J,g). This geometric structure induces the weak generalized almost complex structure $\mathcal{J}_{J,g,\sharp}(X+\xi) = (JX + \sharp_g\xi) + J^*\xi$, introduced in Eq. (3.10). This structure is symmetric with respect to the canonical generalized metric $\mathcal{G}_0$, and therefore is not strong. In order to see that its generalized Nijenhuis map does not behave as a tensor, we may calculate $\mathcal{N}_{\mathcal{J}_{J,g,\sharp}}(\xi,\eta)$ for any $\xi,\eta\in\Omega^1(M)$:

$$\begin{aligned}\mathcal{N}_{\mathcal{J}_{J,g,\sharp}}(\xi,\eta) &= \llbracket \mathcal{J}\xi, \mathcal{J}\eta\rrbracket_D - \mathcal{J}(\llbracket \mathcal{J}\xi,\eta\rrbracket_D + \llbracket \xi, \mathcal{J}\eta\rrbracket_D) - \llbracket \xi,\eta\rrbracket_D\\ &= \llbracket \sharp_g\xi + J^*\xi, \sharp_g\eta + J^*\eta\rrbracket_D - \mathcal{J}(\llbracket \sharp_g\xi + J^*\xi, \eta\rrbracket_D + \llbracket \xi, \sharp_g\eta + J^*\eta\rrbracket_D)\\ &= [\sharp_g\xi, \sharp_g\eta] + \mathcal{L}_{\sharp_g\xi}(J^*\eta) - \iota_{\sharp_g\eta}d(J^*\xi) - \mathcal{J}(\mathcal{L}_{\sharp_g\xi}\eta - i_{\sharp_g\eta}d\xi)\\ &= [\sharp_g\xi, \sharp_g\eta] - \sharp_g\mathcal{L}_{\sharp_g\xi}\eta + \sharp_g i_{\sharp_g\eta}d\xi + \mathcal{L}_{\sharp_g\xi}(J^*\eta) - i_{\sharp_g\eta}d(J^*\xi) - J^*\mathcal{L}_{\sharp_g\xi}\eta + J^* i_{\sharp_g\eta}d\xi.\end{aligned}$$

If we now multiply ξ by a function $f\in\mathfrak{F}(M)$, it is clear that the map $\mathcal{N}_{\mathcal{J}_{J,g,\sharp}}$ is not $\mathfrak{F}(M)$-linear:

$$\begin{aligned}\mathcal{N}_{\mathcal{J}_{J,g,\sharp}}(\xi,\eta) = {}& [f\sharp_g\xi, \sharp_g\eta] - \sharp_g\mathcal{L}_{f\sharp_g\xi}\eta + \sharp_g i_{\sharp_g\eta}d(f\xi) + \mathcal{L}_{f\sharp_g\xi}(J^*\eta)\\ & - i_{\sharp_g\eta}d(fJ^*\xi) - J^*\mathcal{L}_{f\sharp_g\xi}\eta + J^* i_{\sharp_g\eta}d(f\xi)\\ = {}& f\mathcal{N}_{\mathcal{J}_{J,g,\sharp}}(\xi,\eta) - (\sharp_g\eta)(f)\sharp_g\xi - \sharp_g df\eta(\sharp_g\xi) + (\sharp_g\eta)(f)\sharp_g\xi\\ & - \sharp_g df\xi(\sharp_g\eta) + df(J^*\eta)(\sharp_g\xi) - (\sharp_g\eta)(f)J^*\xi + df(J^*\xi)(\sharp_g\eta)\\ & - J^*df\eta(\sharp_g\xi) + (\sharp_g\eta)(f)J^*\xi - J^*df\xi(\sharp_g\eta)\\ = {}& f\mathcal{N}_{\mathcal{J}_{J,g,\sharp}}(\xi,\eta) - 2g(\sharp_g\xi, \sharp_g\eta)\sharp_g df - 2g(\sharp_g\xi, \sharp_g\eta)J^*df.\end{aligned}$$

The fact that the generalized Nijenhuis map is not $\mathfrak{F}(M)$-linear imposes certain restrictions when using coordinate charts $(U,(x^1,\dots,x^n))$. Nevertheless, it is still possible to derive necessary conditions that must be satisfied for a weak generalized almost complex structure in order to be integrable. These conditions are presented in the following result. It is worth noting that, whenever local coordinates are used, Einstein summation convention will be applied: when an index appears twice in a single term, summation of that term over all the values of the index is implied. For example, $g^{ik}J_k^j$ means

$$g^{ik}J_k^j = \sum_{k=1}^{n} g^{ik}J_k^j = g^{i1}J_1^j + \ldots + g^{in}J_n^j$$

Theorem 5.24. *Let $\mathcal{J}\colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ be a weak generalized almost complex structure that is written in matrix form as*

$$\mathcal{J} = \begin{pmatrix} H & \sigma \\ \tau & K \end{pmatrix}.$$

Let us take local coordinates $(U,(x^1,\dots,x^n))$ such that

$$H\frac{\partial}{\partial x^i} = H_i^j\frac{\partial}{\partial x^j}, \quad \sigma dx^i = \sigma^{ij}\frac{\partial}{\partial x^j}, \quad \tau\frac{\partial}{\partial x^i} = \tau_{ij}dx^j, \quad Kdx^i = K_j^i dx^j.$$

If the generalized structure $\mathcal{J}$ is integrable, then the following conditions are met for the indices $i,j,l = 1,\dots,n$:

$$H_i^k\frac{\partial H_j^l}{\partial x^k} - H_j^k\frac{\partial H_i^l}{\partial x^k} + H_k^l\left(\frac{\partial H_i^k}{\partial x^j} - \frac{\partial H_j^k}{\partial x^i}\right) - \sigma^{kl}\left(\frac{\partial \tau_{jk}}{\partial x^i} - \frac{\partial \tau_{ik}}{\partial x^j} + \frac{\partial \tau_{ij}}{\partial x^k}\right) = 0, \tag{5.11}$$

$$\begin{aligned} H_i^k\frac{\partial \tau_{jl}}{\partial x^k} + \tau_{jk}\frac{\partial H_i^k}{\partial x^l} + H_j^k\left(\frac{\partial \tau_{ik}}{\partial x^l} - \frac{\partial \tau_{il}}{\partial x^k}\right) \\ + \tau_{kl}\left(\frac{\partial H_i^k}{\partial x^j} - \frac{\partial H_j^k}{\partial x^i}\right) - K_l^k\left(\frac{\partial \tau_{jk}}{\partial x^i} - \frac{\partial \tau_{ik}}{\partial x^j} + \frac{\partial \tau_{ij}}{\partial x^k}\right) = 0, \end{aligned} \tag{5.12}$$

$$H_i^k\frac{\partial \sigma^{jl}}{\partial x^k} - \sigma^{jk}\frac{\partial H_i^l}{\partial x^k} - H_k^l\frac{\partial \sigma^{jk}}{\partial x^i} - \sigma^{kl}\left(\frac{\partial H_i^j}{\partial x^k} + \frac{\partial K_k^j}{\partial x^i}\right) = 0, \tag{5.13}$$

$$H_i^k\frac{\partial K_l^j}{\partial x^k} + K_k^j\frac{\partial H_i^k}{\partial x^l} + \sigma^{jk}\left(\frac{\partial \tau_{ik}}{\partial x^l} - \frac{\partial \tau_{il}}{\partial x^k}\right) - \tau_{kl}\frac{\partial \sigma^{jk}}{\partial x^i} - K_l^k\left(\frac{\partial H_i^j}{\partial x^k} + \frac{\partial K_k^j}{\partial x^i}\right) = 0, \tag{5.14}$$

$$\sigma^{ik}\frac{\partial H^l_j}{\partial x^k} - H^k_j\frac{\partial \sigma^{il}}{\partial x^k} + H^l_k\frac{\partial \sigma^{ik}}{\partial x^j} + \sigma^{kl}\left(\frac{\partial K^i_k}{\partial x^j} - \frac{\partial K^i_j}{\partial x^k}\right) = 0, \tag{5.15}$$

$$\sigma^{ik}\frac{\partial \tau_{jl}}{\partial x^k} + \tau_{jk}\frac{\partial \sigma^{ik}}{\partial x^l} + H^k_j\left(\frac{\partial K^i_k}{\partial x^l} - \frac{\partial K^i_l}{\partial x^k}\right) + \tau_{kl}\frac{\partial \sigma^{ik}}{\partial x^j} + K^k_l\left(\frac{\partial K^i_k}{\partial x^j} - \frac{\partial K^i_j}{\partial x^k}\right) = 0, \tag{5.16}$$

$$\sigma^{ik}\frac{\partial \sigma^{jl}}{\partial x^k} - \sigma^{jk}\frac{\partial \sigma^{il}}{\partial x^k} - \sigma^{kl}\frac{\partial \sigma^{ij}}{\partial x^k} = 0, \tag{5.17}$$

$$\sigma^{ik}\frac{\partial K^j_l}{\partial x^k} + K^j_k\frac{\partial \sigma^{ik}}{\partial x^l} + \sigma^{jk}\left(\frac{\partial K^i_k}{\partial x^l} - \frac{\partial K^i_l}{\partial x^k}\right) - K^k_l\frac{\partial \sigma^{ij}}{\partial x^k} = 0. \tag{5.18}$$

Proof. To obtain Eqs. (5.11)–(5.18), one must calculate $\mathcal{N}_{\mathcal{J}}(X,Y), \mathcal{N}_{\mathcal{J}}(X,\eta), \mathcal{N}_{\mathcal{J}}(\xi,Y), \mathcal{N}_{\mathcal{J}}(\xi,\eta)$ for any vector fields and differential forms $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$. We compute $\mathcal{N}_{\mathcal{J}}(X,Y)$ explicitly:

$$\begin{aligned}
\mathcal{N}_{\mathcal{J}}(X,Y) &= [\![\mathcal{J}X,\mathcal{J}Y]\!]_D - \mathcal{J}([\![\mathcal{J}X,Y]\!]_D + [\![X,\mathcal{J}Y]\!]_D) - [\![X,Y]\!]_D \\
&= [\![HX+\tau X, HY+\tau Y]\!]_D - \mathcal{J}([\![HX+\tau X,Y]\!]_D + [\![X,HY+\tau Y]\!]_D) - [\![X,Y]\!]_D \\
&= [HX,HY] + \mathcal{L}_{HX}(\tau Y) - \iota_{HY}d(\tau X) \\
&\quad - \mathcal{J}([HX,Y] - \iota_Y d(\tau X) + [X,HY] + \mathcal{L}_X(\tau Y)) - [X,Y] \\
&= [HX,HY] - H([HX,Y]+[X,HY]) + \sigma(\iota_Y d(\tau X) - \mathcal{L}_X(\tau Y)) - [X,Y] \\
&\quad + \mathcal{L}_{HX}(\tau Y) - \iota_{HY}d(\tau X) - \tau([HX,Y]+[X,HY]) + K(\iota_Y d(\tau X) - \mathcal{L}_X(\tau Y)).
\end{aligned}$$

The vector fields X, Y can now be replaced by the coordinate fields $X = \partial_i$, $Y = \partial_j$. Then, we expand the vector field and differential form components of $\mathcal{N}_{\mathcal{J}}(\partial_i,\partial_j)$ separately, obtaining two different expressions:

$$\begin{aligned}
\mathcal{N}_{\mathcal{J}}(\partial_i,\partial_j)|_{\mathfrak{X}(U)} &= [H\partial_i, H\partial_j] - H([H\partial_i,\partial_j] + [\partial_i, H\partial_j]) + \sigma(i_{\partial_j}d(\tau\partial_i) - \mathcal{L}_{\partial_i}(\tau\partial_j)) - [\partial_i,\partial_j] \\
&= [H^k_i\partial_k, H^l_j\partial_l] - H([H^k_i\partial_k,\partial_j] + [\partial_i, H^k_j\partial_k]) \\
&\quad + \sigma(i_{\partial_j}d(\tau_{ik}dx^k) - \mathcal{L}_{\partial_i}(\tau_{jk}dx^k)) \\
&= \left(H^k_i\frac{\partial H^l_j}{\partial x^k} - H^k_j\frac{\partial H^l_i}{\partial x^k}\right)\partial_l - H\left(\frac{\partial H^k_j}{\partial x^i} - \frac{\partial H^k_i}{\partial x^j}\right)\partial_k \\
&\quad + \sigma\left(\frac{\partial \tau_{ik}}{\partial x^j} - \frac{\partial \tau_{ij}}{\partial x^k} - \frac{\partial \tau_{jk}}{\partial x^i}\right)dx^k \\
&= \left[H^k_i\frac{\partial H^l_j}{\partial x^k} - H^k_j\frac{\partial H^l_i}{\partial x^k} + H^l_k\left(\frac{\partial H^k_i}{\partial x^j} - \frac{\partial H^k_j}{\partial x^i}\right)\right. \\
&\quad \left. -\sigma^{kl}\left(\frac{\partial \tau_{jk}}{\partial x^i} - \frac{\partial \tau_{ik}}{\partial x^j} + \frac{\partial \tau_{ij}}{\partial x^k}\right)\right]\partial_l,
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_{\mathcal{J}}(\partial_i,\partial_j)|_{\Omega^1(U)} &= \mathcal{L}_{H\partial_i}(\tau\partial_j) - i_{H\partial_j}d(\tau\partial_i) - \tau([H\partial_i,\partial_j]+[\partial_i,H\partial_j]) \\
&\quad + K(i_{\partial_j}d(\tau\partial_i) - \mathcal{L}_{\partial_i}(\tau\partial_j)) \\
&= \mathcal{L}_{H_i^k\partial_k}(\tau_{jl}dx^l) - i_{H_j^k\partial_k}d(\tau_{il}dx^l) - \tau([H_i^k\partial_k,\partial_j]+[\partial_i,H_j^k\partial_k]) \\
&\quad + K(i_{\partial_j}d(\tau_{ik}dx^k) - \mathcal{L}_{\partial_i}(\tau_{jk}dx^k)) \\
&= \left(H_i^k\frac{\partial\tau_{jl}}{\partial x^k} + \tau_{jk}\frac{\partial H_i^k}{\partial x^l}\right)dx^l + H_j^k\left(\frac{\partial\tau_{ik}}{\partial x^l} - \frac{\partial\tau_{il}}{\partial x^k}\right)dx^l \\
&\quad - \tau\left(\frac{\partial H_j^k}{\partial x^i} - \frac{\partial H_i^k}{\partial x^j}\right)\partial_k + K\left(\frac{\partial\tau_{ik}}{\partial x^j} - \frac{\partial\tau_{ij}}{\partial x^k} - \frac{\partial\tau_{jk}}{\partial x^i}\right)dx^k \\
&= \left[H_i^k\frac{\partial\tau_{jl}}{\partial x^k} + \tau_{jk}\frac{\partial H_i^k}{\partial x^l} + H_j^k\left(\frac{\partial\tau_{ik}}{\partial x^l} - \frac{\partial\tau_{il}}{\partial x^k}\right) + \tau_{kl}\left(\frac{\partial H_i^k}{\partial x^j} - \frac{\partial H_j^k}{\partial x^i}\right)\right. \\
&\quad \left. - K_l^k\left(\frac{\partial\tau_{ij}}{\partial x^k} - \frac{\partial\tau_{ik}}{\partial x^j} + \frac{\partial\tau_{jk}}{\partial x^i}\right)\right]dx^l.
\end{aligned}$$

Therefore, it is proved that $\mathcal{N}_{\mathcal{J}}(\partial_i,\partial_j)=0$ if and only if Eqs. (5.11) and (5.12) hold true.

Similar calculations can be done for $\mathcal{N}_{\mathcal{J}}(\partial_i,dx^j)$, $\mathcal{N}_{\mathcal{J}}(dx^i,\partial_j)$ and $\mathcal{N}_{\mathcal{J}}(dx^i,dx^j)$, splitting each of them into its vector field and differential form components and obtaining the conditions from the rest of equations. □

Several important observations can be made about the preceding result. First, if the Dorfman bracket were skew-symmetric, one would obtain six equations instead of eight. Furthermore, the theorem provides a necessary but not sufficient condition for a structure to be integrable. This limitation arises from the fact that the Nijenhuis tensor does not behave as a tensor, as noted by Aldi, Da Silva, and Grandini in [3]. However, because of the $\mathfrak{F}(M)$-linearity of the generalized Nijenhuis tensor of a strong generalized almost complex structure, it is immediate to check that the converse implication is also true when the structure is strong. Then, we have the following proposition.

Proposition 5.25. *If the generalized almost complex structure $\mathcal{J}$ in Theorem 5.24 is strong, then $\mathcal{J}$ is integrable if and only if Eqs.* (5.11)–(5.18) *hold true.*

The result in Proposition 5.25 can be compared with some of the generalized structures that have been studied in the previous chapters.

Example 5.26 ([29, Example 4.21])**.** We work first with an almost complex manifold (M,J) and analyze the generalized almost complex structures $\mathcal{J}_{\lambda,J}$ from Eq. (3.8). In this case, we have $H=J$, $\sigma=0$, $\tau=0$ and $K=\lambda J^*$, and hence $H_j^i=J_j^i$ and $K_j^i=\lambda J_j^i$. Therefore, the only expressions of Theorem 5.24 that are not trivial are Eqs. (5.11), (5.14) and (5.16). The first one transforms into

$$J_i^k\frac{\partial J_j^l}{\partial x^k} - J_j^k\frac{\partial J_i^l}{\partial x^k} + J_k^l\left(\frac{\partial J_i^k}{\partial x^j} - \frac{\partial J_j^k}{\partial x^i}\right) = 0,$$

while the Nijenhuis tensor of J, in coordinates, is equal to

$$N_J(\partial_i,\partial_j) = \left[J_i^k\frac{\partial J_j^l}{\partial x^k} - J_j^k\frac{\partial J_i^l}{\partial x^k} + J_k^l\left(\frac{\partial J_i^k}{\partial x^j} - \frac{\partial J_j^k}{\partial x^i}\right)\right]\partial_l.$$

Then, Eq. (5.11) holds true if and only if $N_J \equiv 0$; in other words, if and only if J is integrable. In respect of Eqs. (5.14) and (5.16), one must also use the fact that $J^2 = -Id$; taking local coordinates, $J^i_k J^k_j = -\delta^i_j$ and, consequently,

$$J^i_k \frac{\partial J^k_j}{\partial x^l} = -J^k_j \frac{\partial J^i_k}{\partial x^l}.$$

Thus, it is easy to check that, as expected, these two equations are true if and only if J is integrable and $\lambda = -1$ Therefore, $\mathcal{J}_{1,J}$ is not integrable, while $\mathcal{J}_{-1,J}$ is integrable if and only if J is integrable.

Example 5.27 ([29, Example 4.20])**.** Other example of a generalized complex structure comes from an almost symplectic manifold (M,ω). The almost symplectic structure induces the strong generalized almost complex structure $\mathcal{J}_\omega$, introduced in Eq. (3.5). In this case, we have $H = 0$, $\sigma = -\sharp_\omega$, $\tau = \flat_\omega$ and $K = 0$, and hence $\sigma^{ij} = -\omega^{ij}$ and $\tau_{ij} = \omega_{ij}$, where $\omega^{ik}\omega_{kj} = \delta^i_j$ because $\sharp_\omega = \flat_\omega^{-1}$. Then, Eqs. (5.11), (5.14), (5.16) and (5.17) must be studied because of their non-triviality. The first equation transforms into

$$\omega^{kl}\left(\frac{\partial \omega_{jk}}{\partial x^i} - \frac{\partial \omega_{ik}}{\partial x^j} + \frac{\partial \omega_{ij}}{\partial x^k}\right) = 0,$$

while the local representation of the exterior derivative of ω is given by

$$(d\omega)_{ijk} = \frac{\partial \omega_{jk}}{\partial x^i} - \frac{\partial \omega_{ik}}{\partial x^j} + \frac{\partial \omega_{ij}}{\partial x^k}.$$

Therefore, Eq. (5.11) is true if and only if $d\omega = 0$. Regarding Eqs. (5.14) and (5.16), in a similar way to Example 5.26, knowing that

$$\omega^{ik}\frac{\partial \omega_{kj}}{\partial x^l} = -\omega_{kj}\frac{\partial \omega^{ik}}{\partial x^l},$$

we have that these two equations are true if and only if ω is symplectic. Finally, to check Eq. (5.17) we can modify the previous relation in order to obtain

$$\frac{\partial \omega^{ik}}{\partial x^l} = \omega^{ir}\omega^{ks}\frac{\partial \omega_{rs}}{\partial x^l}.$$

Then, Eq. (5.17) is true if and only if $d\omega = 0$, that is, if ω is symplectic. Therefore, $\mathcal{J}_\omega$ is integrable if and only if $d\omega = 0$.

5.3 Proof of Theorem 5.1

Before presenting the proof of Theorem 5.1, we must introduce the nearly Kähler structure that arises when working with the six-dimensional sphere. To this end, we consider $\mathbb{S}^6$ as a subset of the seven-dimensional Euclidean space $\mathbb{R}^7 = \langle e_1, e_2, \ldots, e_7\rangle$, i.e., $\mathbb{S}^6 \subset \mathbb{R}^7$. This space can be endowed with the Cayley product $(\mathbb{R}^7, \times)$, associated with pure octonions. The explicit multiplication table is given below.

This product in $\mathbb{R}^7$ induces a vector bundle endomorphism $J\colon T\mathbb{S}^6 \to T\mathbb{S}^6$ on the six-dimensional sphere. If we denote this product by $\times$, at any point $p \in \mathbb{S}^6$ the morphism $J_p\colon T_p\mathbb{S}^6 \to T_p\mathbb{S}^6$ is

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

TABLE 5.2: Pure octonions product in $\mathbb{R}^7$. Table taken from [30, Chapter 19].

defined as

$$J_p w = p \times w,$$

for each $w \in T_pM$. This map descends to an endomorphism of vector fields $J\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$. It can be proved that $J^2 = -Id$ and, therefore, J is an almost complex structure on $\mathbb{S}^6$. However, this structure is not integrable, that is, the Nijenhuis tensor associated with J is not null.

In addition, this morphism is compatible with the metric $g\colon \mathfrak{X}(\mathbb{S}^6) \times \mathfrak{X}(\mathbb{S}^6) \to \mathfrak{F}(\mathbb{S}^6)$ which is induced by the Euclidean metric on $\mathbb{R}^7$, such that $g(JX, JY) = g(X, Y)$ for every $X, Y \in \mathfrak{X}(\mathbb{S}^6)$. Therefore, according to Definition 2.6, $(\mathbb{S}^6, J, g)$ is an almost Hermitian structure on $\mathbb{S}^6$, with ω as it fundamental almost symplectic structure. In fact, this structure is usually known as a nearly Kähler structure (see [28]), that is, it satisfies the condition

$$(\nabla^g_X J)X = 0,$$

for every vector field $X \in \mathfrak{X}(\mathbb{S}^6)$, where ∇^g is the Levi-Civita associated with the metric g and $(\nabla^g_X J)Y = X(JY) - J(\nabla^g_X Y)$ for every $Y \in \mathfrak{X}(\mathbb{S}^6)$.

We should also justify why we use certain $\mathfrak{F}(M)$-linear combinations of the structures $\mathcal{J}_{1,J}$, $\mathcal{J}_g$, $\mathcal{J}_\omega$ in order to find a weak generalized complex structure. The argument is grounded on the properties of commutation and anti-commutation of the previously introduced structures. As seen in Proposition 3.18, when the base manifold is endowed with an almost Hermitian structure (M, J, g), then $(\mathcal{J}_g, \mathcal{J}_{1,J}, \mathcal{J}_\omega)$ is a generalized almost hypercomplex structure.

Since this is the only generalized almost hypercomplex structure that can be generated from (M, J, g) using the endomorphisms $\mathcal{J}_g, \mathcal{J}_{\lambda,J}, \mathcal{J}_\omega$ introduced in Eqs. (3.2), (3.5) and (3.8) respectively, the following corollary is easily inferred.

Corollary 5.28. *Let (M, J, g) be an almost Hermitian manifold. Then, the endomorphism obtained as a linear combination $a\mathcal{J}_{1,J} + b\mathcal{J}_g + c\mathcal{J}_\omega$ with $a, b, c \in \mathfrak{F}(M)$ is a weak generalized almost complex structure if and only if $a^2 + b^2 + c^2 = 1$. The structure is strong if and only if $a = b \equiv 0$.*

Proof. As has been commented before, the endomorphisms $\mathcal{J}_{1,J}$, $\mathcal{J}_g$ and $\mathcal{J}_\omega$ conform a generalized almost hypercomplex structure. We compute the square of any endomorphism obtained as a linear combination $\mathcal{J} = a\mathcal{J}_{1,J} + b\mathcal{J}_g + c\mathcal{J}_\omega$ with $a, b, c \in \mathfrak{F}(M)$:

$$\begin{aligned}\mathcal{J}^2 &= (a\mathcal{J}_{1,J} + b\mathcal{J}_g + c\mathcal{J}_\omega)^2 \\ &= a^2\mathcal{J}_{1,J}^2 + b^2\mathcal{J}_g^2 + c^2\mathcal{J}_\omega^2 \\ &\quad + ab(\mathcal{J}_{1,J}\mathcal{J}_g + \mathcal{J}_g\mathcal{J}_{1,J}) + ac(\mathcal{J}_{1,J}\mathcal{J}_\omega + \mathcal{J}_\omega\mathcal{J}_{1,J}) + bc(\mathcal{J}_g\mathcal{J}_\omega + \mathcal{J}_\omega\mathcal{J}_g) \\ &= -(a^2 + b^2 + c^2)\mathcal{I}d.\end{aligned}$$

Therefore, it must be $a^2+b^2+c^2 = 1$. To check whether $\mathcal{J}$ is strong or not, we calculate $\mathcal{G}_0(\mathcal{J}u, \mathcal{J}v)$:

$$\begin{aligned}\mathcal{G}_0(\mathcal{J}u, \mathcal{J}v) &= \mathcal{G}_0(a\mathcal{J}_{1,J}u + b\mathcal{J}_g u + c\mathcal{J}_\omega u, a\mathcal{J}_{1,J}v + b\mathcal{J}_g v + c\mathcal{J}_\omega v) \\ &= a^2\mathcal{G}_0(\mathcal{J}_{1,J}u, \mathcal{J}_{1,J}v) + b^2\mathcal{G}_0(\mathcal{J}_g u, \mathcal{J}_g v) + c^2\mathcal{G}_0(\mathcal{J}_\omega u, \mathcal{J}_\omega v) \\ &\quad + ab(\mathcal{G}_0(\mathcal{J}_{1,J}u, \mathcal{J}_g v) + \mathcal{G}_0(\mathcal{J}_g u, \mathcal{J}_{1,J}v)) + ac(\mathcal{G}_0(\mathcal{J}_{1,J}u, \mathcal{J}_\omega v) \\ &\quad + \mathcal{G}_0(\mathcal{J}_\omega u, \mathcal{J}_{1,J}v)) + bc(\mathcal{G}_0(\mathcal{J}_g u, \mathcal{J}_\omega v) + \mathcal{G}_0(\mathcal{J}_\omega u, \mathcal{J}_g v)) \\ &= -a^2\mathcal{G}_0(u, v) - b^2\mathcal{G}_0(u, v) + c^2\mathcal{G}_0(u, v) + ab(\mathcal{G}_0(\mathcal{J}_\omega u, v) + \mathcal{G}_0(u, \mathcal{J}_\omega v)) \\ &\quad + ac(\mathcal{G}_0(u, \mathcal{J}_g v) + \mathcal{G}_0(\mathcal{J}_g u, v)) - bc(\mathcal{G}_0(\mathcal{J}_{1,J}u, v) + \mathcal{G}_0(u, \mathcal{J}_{1,J}v)) \\ &= (-a^2 - b^2 + c^2)\mathcal{G}_0(u, v) + 2ac\mathcal{G}_0(u, \mathcal{J}_g v) - 2bc\mathcal{G}_0(\mathcal{J}_{1,J}u, v).\end{aligned}$$

Thus, if the structure $\mathcal{J}$ is strong then $-a^2 - b^2 + c^2 = 1$. Combined with $a^2 + b^2 + c^2 = 1$, this condition shows that it must be $a = b \equiv 0$. The converse statement is immediate to check. □

We will call such a $\mathfrak{F}(M)$-linear combination $a\mathcal{J}_{1,J} + b\mathcal{J}_g + c\mathcal{J}_\omega$ a *spherical combination* of the generalized almost hypercomplex structure.

Our main goal is to ask ourselves if, for the nearly Kähler structure $(\mathbb{S}^6, J, g)$ inherited from the pure octonions product, there is any integrable spherical combination of the generalized almost hypercomplex structure conformed by the endomorphisms $\mathcal{J}_{1,J}$, $\mathcal{J}_g$ and $\mathcal{J}_\omega$. We will work in spherical coordinates for $\mathbb{S}^6 \subset \mathbb{R}^7$: if we take angular coordinates $u^1, \ldots, u^6$ such that $u^1, \ldots, u^5 \in (0, \pi)$ and $u^6 \in (0, 2\pi)$, then the coordinates of the six-dimensional sphere are the following:

$$\begin{cases} x^1 = \cos(u^1), \\ x^2 = \sin(u^1)\cos(u^2), \\ x^3 = \sin(u^1)\sin(u^2)\cos(u^3), \\ \quad\vdots \\ x^6 = \sin(u^1)\sin(u^2)\sin(u^3)\sin(u^4)\sin(u^5)\cos(u^6), \\ x^7 = \sin(u^1)\sin(u^2)\sin(u^3)\sin(u^4)\sin(u^5)\sin(u^6). \end{cases} \tag{5.19}$$

The metric g on $\mathbb{S}^6$ is the one induced by the Euclidean metric on $\mathbb{R}^7$, where $\mathbb{S}^6$ is seen as a submanifold of the seven-dimensional Euclidean space. Therefore, in spherical coordinates this metric is given by

$$g_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \\ \sin^2(u^1)\ldots\sin^2(u^{i-1}) & \text{if } i = j \neq 1, \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.20}$$

On the other hand, the local representation of the inverse metric can be easily computed from the data in Eq. (5.20):

$$g^{ij} = \begin{cases} 1 & \text{if } i = j = 1, \\ \dfrac{1}{\sin^2(u^1)\dots\sin^2(u^{i-1})} & \text{if } i = j \neq 1, \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.21}$$

As the explicit expression of J in local coordinates is really long and it would not fit in one page, we will just give some specific values of J^i_j. In particular, we will use that

$$J\partial_1 = \frac{\cos(u^3)}{\sin(u^1)}\partial_2 - \frac{\cos(u^2)\sin(u^3)}{\sin(u^1)\sin(u^2)}\partial_3 + \frac{\cos(u^5)}{\sin(u^1)}\partial_4 - \frac{\cos(u^4)\sin(u^5)}{\sin(u^1)\sin(u^4)}\partial_5 - \frac{1}{\sin(u^1)}\partial_6, \tag{5.22}$$

and

$$\begin{aligned} J^*du^1 = &-\sin(u^1)\cos(u^3)du^2 + \sin(u^1)\cos(u^2)\sin(u^2)\sin(u^3)du^3 \\ &- \sin(u^1)\sin^2(u^2)\sin^2(u^3)\cos(u^5)du^4 \\ &+ \sin(u^1)\sin^2(u^2)\sin^2(u^3)\cos(u^4)\sin(u^4)\sin(u^5)du^5 \\ &+ \sin(u^1)\sin^2(u^2)\sin^2(u^3)\sin^2(u^4)\sin^2(u^5)du^6. \end{aligned} \tag{5.23}$$

Therefore, as $J\partial_1 = J^k_1\partial_k$ and $J^*du^1 = J^1_k du^k$, we have the expressions for each J^k_1 and J^1_k.

Because the proof of Theorem 5.1 is quite extensive, we first check what happens when the contribution of $\mathcal{J}_g$ in the spherical combination is null (i.e. when $b \equiv 0$).

Proposition 5.29. *Let $(\mathbb{S}^6, J, \omega)$ be the six-dimensional sphere endowed with its usual nearly Kähler structure. Then, there is no combination $\mathcal{J} = a\mathcal{J}_{1,J} + c\mathcal{J}_\omega$ with $a, c \in \mathfrak{F}(\mathbb{S}^6)$ and $a^2 + c^2 = 1$ such that the weak generalized almost complex structure $\mathcal{J}$ is integrable.*

Proof. To see that such a weak generalized almost complex structure $\mathcal{J}$ cannot be integrable, we will analyze the necessary conditions shown in Theorem 5.24. In particular, we will use both Eqs. (5.13) and (5.15). Firstly, we interchange the indices i, j in Eq. (5.15), obtaining the following expression:

$$\sigma^{jk}\frac{\partial H^l_i}{\partial x^k} - H^k_i\frac{\partial \sigma^{jl}}{\partial x^k} + H^l_k\frac{\partial \sigma^{jk}}{\partial x^i} + \sigma^{kl}\left(\frac{\partial K^j_k}{\partial x^i} - \frac{\partial K^j_i}{\partial x^k}\right) = 0.$$

This equation is quite similar to Eq. (5.13); in fact, if both expressions are added up some terms will vanish, resulting in the following identity:

$$\sigma^{kl}\left(\frac{\partial H^j_i}{\partial x^k} + \frac{\partial K^j_i}{\partial x^k}\right) = 0.$$

This condition must be fulfilled by any integrable structure $\mathcal{J}$ with $\mathcal{J}^2 = -\mathcal{I}d$. Taking now the spherical combination $\mathcal{J} = a\mathcal{J}_{1,J} + c\mathcal{J}_\omega$ and using its matrix form, it is $H = aJ$, $\sigma = -c\sharp_\omega = cJ\sharp_g$, $\tau = c\flat_\omega = c\flat_g J$ and $K = aJ^*$. The local representations of these morphisms are

$$H^j_i = aJ^j_i, \quad \sigma^{ij} = cg^{ik}J^j_k, \quad \tau_{ij} = cJ^k_i g_{kj}, \quad K^i_j = aJ^i_j,$$

so the previous condition is turned into

$$cg^{ks}J^l_s\frac{\partial}{\partial u^k}\left(aJ^j_i\right)=0.$$

If this equation is developed, it can be written in the following form:

$$ac\left(g^{ks}J^l_s\frac{\partial J^j_i}{\partial u^k}\right)+c\frac{\partial a}{\partial u^k}\left(g^{ks}J^l_sJ^j_i\right)=0. \tag{5.24}$$

In this last equation, there are two clearly separated parts: one related to the product ac, and other related to the product of c with each derivative of a with respect to each u^k. The idea is to compare Eq. (5.24) for different values of the indices i, j, k and, by combining them, to obtain restraints for the functions a and c. We use the combinations of indices $i = l = 1$, $j = 2$; and $i = l = 1$, $j = 3$:

- $i = l = 1, j = 2$:
$$ac\left(g^{ks}J^1_s\frac{\partial J^2_1}{\partial u^k}\right)+c\frac{\partial a}{\partial u^k}\left(g^{ks}J^1_sJ^2_1\right)=0. \tag{5.25}$$

- $i = l = 1, j = 3$:
$$ac\left(g^{ks}J^1_s\frac{\partial J^3_1}{\partial u^k}\right)+c\frac{\partial a}{\partial u^k}\left(g^{ks}J^1_sJ^3_1\right)=0. \tag{5.26}$$

From Eq. (5.22), we know that

$$J^2_1=\frac{\cos(u^3)}{\sin(u^1)},\qquad J^3_1=-\frac{\cos(u^2)\sin(u^3)}{\sin(u^1)\sin(u^2)}.$$

We can multiply Eq. (5.25) by $-J^3_1$ and Eq. (5.26) by J^2_1 and add them up, obtaining

$$ac\left[g^{ks}J^1_s\left(J^2_1\frac{\partial J^3_1}{\partial u^k}-J^3_1\frac{\partial J^2_1}{\partial u^k}\right)\right]=0.$$

If the sum over the indices k, s explicitly, as J^2_1, J^3_1 only depend on u^1, u^2, u^3, and knowing that $g^{ij} = 0$ for $i \neq j$ and $J^1_1 = 0$, the following expression is inferred:

$$\begin{aligned}0&=ac\left[g^{1s}J^1_s\left(J^2_1\frac{\partial J^3_1}{\partial u^1}-J^3_1\frac{\partial J^2_1}{\partial u^1}\right)+g^{2s}J^1_sJ^2_1\frac{\partial J^3_1}{\partial u^2}+g^{3s}J^1_s\left(J^2_1\frac{\partial J^3_1}{\partial u^3}-J^3_1\frac{\partial J^2_1}{\partial u^3}\right)\right]\\&=ac\left[g^{22}J^1_2J^2_1\frac{\partial J^3_1}{\partial u^2}+g^{33}J^1_3\left(J^2_1\frac{\partial J^3_1}{\partial u^3}-J^3_1\frac{\partial J^2_1}{\partial u^3}\right)\right].\end{aligned}$$

We calculate explicitly the function that is multiplying ac, taking from Eq. (5.23) the functions $J^1_2 = -\sin(u^1)\cos(u^3)$ and $J^1_3 = \sin(u^1)\cos(u^2)\sin(u^2)\sin(u^3)$:

$$g^{22}J^1_2 J^2_1 \frac{\partial J^3_1}{\partial u^2} = -\frac{\sin(u^1)\cos(u^3)}{\sin^2(u^1)}\frac{\cos(u^3)}{\sin(u^1)}\frac{\sin(u^3)}{\sin(u^1)\sin^2(u^2)} = -\frac{\cos^2(u^3)\sin(u^3)}{\sin^3(u^1)\sin^2(u^2)},$$

$$\begin{aligned} g^{33}J^1_3 J^2_1 \frac{\partial J^3_1}{\partial u^3} &= -\frac{\sin(u^1)\cos(u^2)\sin(u^2)\sin(u^3)}{\sin^2(u^1)\sin^2(u^2)}\frac{\cos(u^3)}{\sin(u^1)}\frac{\cos(u^2)\cos(u^3)}{\sin(u^1)\sin(u^2)} \\ &= -\frac{\cos^2(u^2)\cos^2(u^3)\sin(u^3)}{\sin^3(u^1)\sin^2(u^2)}, \end{aligned}$$

$$\begin{aligned} g^{33}J^1_3 J^3_1 \frac{\partial J^2_1}{\partial u^3} &= \frac{\sin(u^1)\cos(u^2)\sin(u^2)\sin(u^3)}{\sin^2(u^1)\sin^2(u^2)}\frac{\cos(u^2)\sin(u^3)}{\sin(u^1)\sin(u^2)}\frac{\sin(u^3)}{\sin(u^1)} \\ &= \frac{\cos^2(u^2)\sin^3(u^3)}{\sin^3(u^1)\sin^2(u^2)}. \end{aligned}$$

Joining all together, for all $u^1, u^2, u^3 \in (0,\pi)$ it must be

$$-ac\frac{\sin(u^3)(\cos^2(u^3)+\cos^2(u^2))}{\sin^3(u^1)\sin^2(u^2)} = 0.$$

Therefore, it must be $a \equiv 0$ or $c \equiv 0$. However, that would imply $\mathcal{J} = \pm\mathcal{J}_\omega$, which is not integrable because ω is not a symplectic form; or $\mathcal{J} = \pm\mathcal{J}_{1,J}$, which is also not integrable because J is not integrable. In consequence, there are no functions $a, c \in \mathfrak{F}(\mathbb{S}^6)$ such that $a^2+c^2 = 1$ and $a\mathcal{J}_{1,J}+c\mathcal{J}_\omega$ is a weak generalized complex structure. □

After proving Proposition 5.29, we are in a good position to prove the general case in Theorem 5.1, taking any possible expression for the function b.

Proof of Theorem 5.1. We take a weak generalized almost complex structure that can be written as a spherical combination $\mathcal{J} = a\mathcal{J}_{1,J} + b\mathcal{J}_g + c\mathcal{J}_\omega$ with $a, b, c \in \mathfrak{F}(\mathbb{S}^6)$. We will work mainly with Eq. (5.17); specifically, we symmetrize the expression with respect to the indices i, j. To do this, we interchange the indices i, j in Eq. (5.17):

$$\sigma^{jk}\frac{\partial \sigma^{il}}{\partial x^k} - \sigma^{ik}\frac{\partial \sigma^{jl}}{\partial x^k} - \sigma^{kl}\frac{\partial \sigma^{ji}}{\partial x^k} = 0.$$

Adding this formula to Eq. (5.17), we obtain the following expression in local coordinates:

$$\sigma^{kl}\frac{\partial}{\partial x^k}(\sigma^{ij}+\sigma^{ji}) = 0. \tag{5.27}$$

This condition must be satisfied for any weak generalized complex structure. Using the matrix notation from Eq. (2.8), we have $\sigma = -(b\sharp_g + c\sharp_\omega)$ for the structure $\mathcal{J}$ considered. Taking local coordinates, as $\sharp_\omega = -J\sharp_g = \sharp_g J^*$, the local representation for σ is

$$\sigma^{ij} = cg^{ik}J^j_k - bg^{ij} = -cg^{jk}J^i_k - bg^{ij}.$$

Then, applying Eq. (5.27) for this particular structure, we have that the following condition must be met:

$$bc\left(g^{ks}J^l_s\frac{\partial g^{ij}}{\partial u^k}\right)-b^2\left(g^{kl}\frac{\partial g^{ij}}{\partial u^k}\right)+c\frac{\partial b}{\partial u^k}\left(g^{ks}J^l_sg^{ij}\right)-b\frac{\partial b}{\partial u^k}\left(g^{kl}g^{ij}\right)=0. \tag{5.28}$$

We now want to compare the expression in Eq. (5.28) for different values of i, j, l and hopefully find any condition for the functions b, c. We work in the spherical coordinates given in Eq. (5.19) and use the explicit local expressions for g and J; in particular, we will need to use the following values:

$$g^{11}=1,\quad g^{22}=\frac{1}{\sin^2(u^1)},\quad J^1_1=0.$$

Eq. (5.28) is compared for the indices $i=j=l=1$; and $i=j=2, l=1$:

- $i=j=l=1$:

$$\begin{aligned}0&=bc\left(g^{ks}J^1_s\frac{\partial g^{11}}{\partial u^k}\right)-b^2\left(g^{k1}\frac{\partial g^{11}}{\partial u^k}\right)+c\frac{\partial b}{\partial u^k}\left(g^{ks}J^1_sg^{11}\right)-b\frac{\partial b}{\partial u^k}\left(g^{k1}g^{11}\right)\\&=c\frac{\partial b}{\partial u^k}\left(g^{ks}J^1_s\right)-b\frac{\partial b}{\partial u^1}.\end{aligned}$$

- $i=j=2, l=1$:

$$\begin{aligned}0&=bc\left(g^{ks}J^1_s\frac{\partial g^{22}}{\partial u^k}\right)-b^2\left(g^{k1}\frac{\partial g^{22}}{\partial u^k}\right)+c\frac{\partial b}{\partial u^k}\left(g^{ks}J^1_sg^{22}\right)-b\frac{\partial b}{\partial u^k}\left(g^{k1}g^{22}\right)\\&=bc\left(g^{11}J^1_1\frac{\partial g^{22}}{\partial u^1}\right)-b^2\left(g^{11}\frac{\partial g^{22}}{\partial u^1}\right)+c\frac{\partial b}{\partial u^k}\left(g^{ks}J^1_sg^{22}\right)-b\frac{\partial b}{\partial u^1}\left(g^{11}g^{22}\right)\\&=\frac{2\cos(u^1)}{\sin^3(u^1)}b^2+\frac{1}{\sin^2(u^1)}\left[c\frac{\partial b}{\partial u^k}\left(g^{ks}J^1_s\right)-b\frac{\partial b}{\partial u^1}\right].\end{aligned}$$

Taking both expressions together, it is immediate to see that, for every $u^1\in(0,\pi)$,

$$\frac{2\cos(u^1)}{\sin^3(u^1)}b^2=0,$$

so we have that $b\equiv 0$. In order to be integrable, the structure must be $\mathcal{J}=a\mathcal{J}_{1,J}+c\mathcal{J}_\omega$ with $a^2+c^2=1$. However, Proposition 5.29 states that any such structure cannot be integrable. Therefore, there are no functions $a, b, c\in\mathfrak{F}(\mathbb{S}^6)$ such that $a^2+b^2+c^2=1$ and $a\mathcal{J}_{1,J}+b\mathcal{J}_g+c\mathcal{J}_\omega$ is a weak generalized complex structure, thus proving the theorem. □

Chapter 6

Integrability of strong generalized complex structures on nilmanifolds

In the previous chapter, we examined the integrability of weak generalized polynomial structures, with a particular focus on the six-dimensional sphere. We now turn our attention to the integrability of strong generalized geometric structures. By restricting our study to this kind of morphism, the requirement of compatibility with the canonical metric $\mathcal{G}_0$ not only constrains the class of admissible endomorphisms, but also imposes topological restrictions on the underlying manifolds. In this regard, the following result can be proved.

Proposition 6.1 ([29, Proposition 4.16]). *A smooth manifold M admits a strong generalized almost complex structure if and only if it admits an almost complex structure.*

This implies that all obstructions to the existence of classical almost complex structures carry over to strong generalized almost complex structures. In particular, a manifold that admits the latter type of structure must be even-dimensional. Therefore, unless otherwise stated, we will assume that the underlying manifold M has a dimension equal to $2n$. This assumption will be relevant when discussing the different types of strong generalized complex structures.

In this chapter, we focus on the study of two-step nilmanifolds of dimension eight. To this end, we carry out a detailed analysis of the Lie algebras associated with these manifolds. Given that the classification of such algebras is quite recent (see, for example, [6, 10]), it was previously unknown which of them admit certain types of geometric structures. Among other results, we determine which algebras can be endowed with classical complex and symplectic structures. In addition, we explore other strong generalized complex structures that are not derived from the classical complex and symplectic ones.

6.1 Another perspective for strong generalized complex structures

Throughout this document, we have worked with strong generalized almost complex structures, which can be understood as endomorphisms $\mathcal{J} \colon \mathbb{T}M \to \mathbb{T}M$ satisfying conditions $\mathcal{J}^2 = -\mathcal{I}d$ and $\mathcal{G}_0(\mathcal{J}u, v) = \mathcal{G}_0(u, \mathcal{J}v)$ for every $u, v \in \mathbb{T}M$ in the same fiber. As is shown in Eq. (2.1), each of these endomorphisms is associated with the eigenbundles

$$\mathbb{L}^{1,0}_{\mathcal{J}} = \{u - i\mathcal{J}u \in \mathbb{T}M_{\mathbb{C}} \colon u \in \mathbb{T}M\}, \quad \mathbb{L}^{0,1}_{\mathcal{J}} = \{u + i\mathcal{J}u \in \mathbb{T}M_{\mathbb{C}} \colon u \in \mathbb{T}M\},$$

corresponding to the eigenvalues i and $-i$ of $\mathcal{J}$, respectively. It is clear that $\mathbb{L}^{0,1}_{\mathcal{J}}$ is the conjugate of $\mathbb{L}^{1,0}_{\mathcal{J}}$, that is, $\overline{\mathbb{L}^{1,0}_{\mathcal{J}}} = \mathbb{L}^{0,1}_{\mathcal{J}}$. In addition, the intersection between the two eigenbundles is trivial, that is, $\mathbb{L}^{1,0}_{\mathcal{J}} \cap \mathbb{L}^{0,1}_{\mathcal{J}} = \{0\}$. Furthermore, the eigenbundle $\mathbb{L}^{1,0}_{\mathcal{J}}$ is *isotropic* with respect to the canonical generalized metric $\mathcal{G}_0$, which means that $\mathbb{L}^{1,0}_{\mathcal{J}} \subset (\mathbb{L}^{1,0}_{\mathcal{J}})^{\perp}$. This can be easily checked: for any $u, v \in \mathbb{T}M$, if $\mathcal{J}$ is strong then

$$\begin{aligned} \mathcal{G}_0(u - i\mathcal{J}u, v - i\mathcal{J}v) &= \mathcal{G}_0(u,v) - \mathcal{G}_0(\mathcal{J}u, \mathcal{J}v) - i(\mathcal{G}_0(\mathcal{J}u, v) + \mathcal{G}_0(u, \mathcal{J}v)) \\ &= \mathcal{G}_0(u,v) - \mathcal{G}_0(u,v) - i(\mathcal{G}_0(\mathcal{J}u, v) - \mathcal{G}_0(\mathcal{J}u, \mathcal{J}v)) \\ &= 0. \end{aligned}$$

In fact, it can be proved that working with such an endomorphism $\mathcal{J} \colon \mathbb{T}M \to \mathbb{T}M$ is equivalent to working with a subbundle $\mathbb{L}$ that satisfies the conditions noted above. If we also add the integrability condition given in Definition 5.14, we obtain the following result.

Theorem 6.2 ([29, Proposition 3.27])**.** *Every strong generalized complex structure on a smooth manifold M is equivalent to a maximally isotropic subbundle $\mathbb{L} \subset \mathbb{T}M_{\mathbb{C}}$ such that $\mathbb{L} \cap \overline{\mathbb{L}} = \{0\}$ and is involutive with respect to the Dorfman bracket $[\![\cdot, \cdot]\!]_D$.*

Since in this chapter we work with nilmanifolds, it may be useful to provide another alternative, yet equivalent, interpretation of strong generalized complex structures. A comprehensive exposition of this interpretation can be found, for example, in [56] by R. Rubio.

Given a $2n$-dimensional smooth manifold M, we introduce its *exterior bundle* $\wedge^{\bullet}(T^*M)$. This vector bundle is defined as the following Whitney sum:

$$\wedge^{\bullet}(T^*M) := \bigoplus_{i=0}^{2n} \wedge^i(T^*M) \to M.$$

The space of sections of this vector bundle consists of sums of differential forms of varying degrees. If we denote this space by $\Omega^{\bullet}(M)$, then we have

$$\Omega^{\bullet}(M) = \bigoplus_{i=0}^{2n} \Omega^i(M),$$

and any differential form $\rho \in \Omega^{\bullet}(M)$ shall be uniquely written as a sum $\rho = \rho_0 + \rho_1 + \ldots + \rho_{2n}$, with $\rho_k \in \Omega^k(M)$.

In Chapter 5, we define the interior product with respect to a vector field X as an operation sending r-forms to $(r-1)$-forms. However, it can also be defined pointwise: given a vector $X \in T_pM$, we can consider the map ι_X that takes an element $\theta \in \wedge^r(T^*_pM)$ to $\iota_X\theta \in \wedge^{r-1}(T^*_pM)$, satisfying

$$(\iota_X\theta)(X_1, \ldots, X_{r-1}) = \theta(X, X_1, \ldots, X_{r-1}),$$

for every $X_1, \dots, X_{r-1} \in T_pM$. If we take this into account, we can define an action of the generalized tangent bundle $\mathbb{T}M$ on the exterior bundle $\wedge^\bullet(T^*M)$ using the exterior and interior product of a differential form. This action is defined as follows.

Definition 6.3. For every $X + \xi \in \mathbb{T}M$ and $\rho \in \wedge^\bullet(T^*M)$ based on the same point of the manifold, we define the *action* of $X + \xi$ on ρ as

$$(X + \xi) \cdot \rho := \iota_X \rho + \xi \wedge \rho, \tag{6.1}$$

where $\iota_X \rho$ is the interior product of ρ with respect to X and $\xi \wedge \rho$ is the exterior product of ξ and ρ.

Definition 6.4. The *annihilator* of a differential form $\rho \in \Omega^\bullet(M)$ is defined as

$$\operatorname{Ann}(\rho) = \{X + \xi \in \mathbb{T}M \colon (X + \xi) \cdot \rho = 0\}. \tag{6.2}$$

Note that in the previous definition we must actually understand the action $(X + \xi) \cdot \rho$ as $(X + \xi) \cdot \rho_p$, where the point $p \in M$ is the base of the element $X + \xi \in \mathbb{T}M$, in other words, $X + \xi \in \mathbb{T}_pM$.

It can be proved that the annihilator of any differential form is an isotropic subbundle: if we take a $X + \xi \in \mathbb{T}M$, we can calculate $(X + \xi) \cdot ((X + \xi) \cdot \rho)$ directly from Eq. (6.1):

$$\begin{aligned}(X + \xi) \cdot ((X + \xi) \cdot \rho) &= (X + \xi) \cdot (\iota_X \rho + \xi \wedge \rho) = \iota_X(\iota_X \rho + \xi \wedge \rho) + \xi \wedge (\iota_X \rho + \xi \wedge \rho) \\ &= \iota_X(\xi \wedge \rho) + \xi \wedge \iota_X \rho = \xi(X)\rho - \xi \wedge \iota_X \rho + \xi \wedge \iota_X \rho = \xi(X)\rho \\ &= \mathcal{G}_0(X + \xi, X + \xi)\rho.\end{aligned}$$

Consequently, for an element $X + \xi \in \operatorname{Ann}(\rho)$ it must be $\mathcal{G}_0(X + \xi, X + \xi) = 0$ and, since

$$2\mathcal{G}_0(u, v) = \mathcal{G}_0(u + v, u + v) - \mathcal{G}_0(u, u) - \mathcal{G}_0(v, v),$$

for every $u, v \in \mathbb{T}M$, we have $\mathcal{G}_0(X + \xi, Y + \eta) = 0$ for every $X + \xi, Y + \eta \in \operatorname{Ann}(\rho)$. This implies that $\operatorname{Ann}(\rho)$ is an isotropic subbundle.

However, although the annihilator of any differential form is an isotropic subspace, not all such annihilators are maximally isotropic, that is, it may be strictly contained in other isotropic subspace. In order to make a precise description, we must first introduce the exponential of a form. Given a differential 2-form $\varphi \in \Omega^2(M)$, the *exponential* of φ is defined as the form

$$\exp \varphi := \sum_{i=0}^{\infty} \frac{1}{i!} \varphi^i = 1 + \varphi + \frac{1}{2!}\varphi^2 + \frac{1}{3!}\varphi^3 + \dots, \tag{6.3}$$

where $\varphi^k := \varphi \wedge \overset{k}{\cdots} \wedge \varphi$. Of course, this definition is valid not only for real differential 2-forms, but also for complex 2-forms $\varphi \in \Omega^2(M)_\mathbb{C}$.

Having introduced all these new concepts, we can now provide an equivalent interpretation of the condition for a subbundle of $\mathbb{T}M_\mathbb{C}$ to be maximally isotropic, in terms of the annihilators of differential forms defined in Eq. (6.2).

Proposition 6.5. *Each maximally isotropic subbundle* $\mathbb{L} \subset \mathbb{T}M_{\mathbb{C}}$ *can be locally described as the annihilator of a complex differential form* $\rho \in \Omega^{\bullet}(U)_{\mathbb{C}}$ *that can be written as*

$$\rho = \Theta \wedge \exp(B + i\omega), \tag{6.4}$$

where $U \subset M$ *is open,* $B, \omega \in \Omega^2(U)$ *and* $\Theta = \theta_1 \wedge \ldots \wedge \theta_k$ *with* $\theta_i \in \Omega^1(U)_{\mathbb{C}}$ *for* $i = 1, \ldots, k$.

It is clear that a maximally isotropic subbundle $\mathbb{L}$ of $\mathbb{T}M_{\mathbb{C}}$ does not necessarily correspond to the eigenbundle of a strong generalized almost complex structure: for this to be the case, the intersection with its complex conjugate must be trivial, that is, $\mathbb{L} \cap \overline{\mathbb{L}} = \{0\}$. If we describe such a subbundle in terms of Proposition 6.5 via a differential form ρ, the following result outlines the conditions that this form must satisfy in order to describe an i-eigenbundle.

Proposition 6.6. *A maximally isotropic subbundle* $\mathbb{L} \subset \mathbb{T}M_{\mathbb{C}}$ *that is determined in terms of Proposition 6.5 by the differential form* $\rho = \Theta \wedge \exp(B + i\omega)$ *fulfills the condition* $\mathbb{L} \cap \overline{\mathbb{L}} = \{0\}$ *if and only if*

$$\omega^{n-k} \wedge \Theta \wedge \overline{\Theta} \neq 0, \tag{6.5}$$

where $2n$ *is the dimension of the manifold.*

Before continuing with an alternative interpretation for the integrability condition, there is an important concept that should not be overlooked, as it will be key in classifying the strong generalized complex structures defined on the manifolds of interest. It concerns the degree of the differential forms that induce each structure.

Definition 6.7. Let $\rho = \rho_0 + \ldots + \rho_{2n} \in \wedge^{\bullet}(T^*M)$ be a differential form with $\rho_k \in \wedge^k(T^*M)$. We define the *type* of ρ as the minimum number $k \in \{0, 1, \ldots, 2n\}$ such that $\rho_k \neq 0$.

Definition 6.8. Given a strong generalized almost complex structure that, in $U \subset M$, is described by the differential form $\rho \subset \Omega^{\bullet}(U)_{\mathbb{C}}$, we define the *type of the structure at* $p \in U$ as the type of the differential form ρ_p.

By definition, the type of a strong generalized almost complex structure does not need to be constant along the manifold. Also, it can be inferred from Eq. (6.5) that the type of such a structure cannot be greater than n. This means that the type k of these structures on a $2n$-dimensional nilmanifold must take a value $k \in \{0, 1, \ldots, n\}$.

The interpretations in Propositions 6.5 and 6.6 can be analyzed for some of the strong generalized almost complex structures that have been studied in the previous chapters.

Example 6.9 ([29, Example 4.21]). In Proposition 3.9, we showed that a classical almost complex structure $J \colon TM \to TM$ induces two generalized almost complex structures, $\mathcal{J}_{\lambda,J}$ for $\lambda \in \{1, -1\}$. In Proposition 4.9, we verified that $\mathcal{J}_{-1,J}$ is, in fact, a strong generalized almost complex structure. The i-eigenbundle of the latter structure is given by $\mathbb{L}^{1,0}_{\mathcal{J}_{-1,J}} = L^{1,0}_J \oplus L^{0,1}_{J^*}$.

Then, it can be checked that the subbundle $\mathbb{L}^{1,0}_{\mathcal{J}_{-1,J}}$ is the annihilator of a complex n-form

$$\rho = \theta_1 \wedge \ldots \wedge \theta_n \in \wedge^n(L^{0,1}_{J^*}).$$

Indeed, if we calculate $(X+\xi)\cdot\rho$ for any $X+\xi \in \mathbb{T}M_{\mathbb{C}}$, we have

$$\begin{aligned}(X+\xi)\cdot(\theta_1 \wedge \ldots \wedge \theta_n) &= \iota_X(\theta_1 \wedge \ldots \wedge \theta_n) + \xi \wedge \theta_1 \wedge \ldots \wedge \theta_n \\ &= \sum_{i=1}^n \theta_i(X)\theta_1 \wedge \ldots \wedge \widehat{\theta_i} \wedge \ldots \wedge \theta_n + \xi \wedge \theta_1 \wedge \ldots \wedge \theta_n,\end{aligned}$$

where $\theta_1 \wedge \ldots \wedge \widehat{\theta_i} \wedge \ldots \wedge \theta_n$ is the form obtained by omitting θ_i from the product $\theta_1 \wedge \ldots \wedge \theta_n$. If $X+\xi \in \mathrm{Ann}(\rho)$, then it must be $\sum_{i=1}^n \theta_i(X)\theta_1\wedge\ldots\wedge\widehat{\theta_i}\wedge\ldots\wedge\theta_n = 0$ and $\xi\wedge\theta_1\wedge\ldots\wedge\theta_n = 0$. Since the forms $\theta_1\wedge\ldots\wedge\widehat{\theta_i}\wedge\ldots\wedge\theta_n$ are linearly independent, we have that $\theta_i(X) = 0$ for $i = 1, \ldots, n$. It is easy to prove (see, for example, [39, Section 9.1]) that, since $\theta_i \in L^{0,1}_{J^*}$, this implies that $X \in L^{1,0}_J$. Also, taking into account that the complex dimension of $L^{0,1}_{J^*}$ is n and $\xi \wedge \theta_1 \wedge \ldots \wedge \theta_n = 0$, it must be $\xi \in L^{0,1}_{J^*}$. Therefore, $\mathrm{Ann}(\rho) \subseteq L^{1,0}_J \oplus L^{0,1}_{J^*}$. The converse inclusion is almost identical to check.

It is also clear that, as $\overline{\theta_i} \in L^{1,0}_{J^*}$ for $i = 1, \ldots, n$, then $\theta_1 \wedge \ldots \wedge \theta_n \wedge \overline{\theta_1} \wedge \ldots \wedge \overline{\theta_n} \neq 0$. Therefore, the strong generalized almost complex structure $\mathcal{J}_{-1,J}$ can be described by a globally defined complex n-form $\theta_1 \wedge \ldots \wedge \theta_n \in \Omega^n(M)_{\mathbb{C}}$ that fulfills the conditions given in Propositions 6.5 and 6.6. These calculations also show that, given an almost complex structure J, the type of the strong generalized almost complex structure $\mathcal{J}_{-1,J}$ at any point is equal to n.

Example 6.10 ([29, Example 4.20])**.** In Proposition 3.6, we checked that an almost symplectic structure on a manifold $\omega \in \Gamma(T^*M \otimes T^*M)$ induces, among other structures, the strong generalized almost complex structure $\mathcal{J}_\omega$. The expression for the i-eigenbundle of this endomorphism is $\mathbb{L}^{1,0}_{\mathcal{J}_\omega} = \{X - i\flat_\omega X \in \mathbb{T}M_{\mathbb{C}} \colon X \in TM_{\mathbb{C}}\}$.

Then, the distribution $\mathbb{L}^{1,0}_{\mathcal{J}_\omega}$ is the annihilator of the globally defined differential form

$$\rho = \exp(i\omega).$$

This can be easily be proved. To see that $\mathbb{L}^{1,0}_{\mathcal{J}_\omega} \subseteq \mathrm{Ann}(\rho)$ we take an element $X - i\flat_\omega X \in \mathbb{T}M_{\mathbb{C}}$ and calculate $(X - i\flat_\omega X)\cdot\rho$:

$$\begin{aligned}(X - i\flat_\omega X)\cdot\exp(i\omega) &= (X - i\flat_\omega X)\cdot\sum_{j=0}^\infty \frac{i^j}{j!}\omega^j = \sum_{j=0}^\infty \frac{i^j}{j!}\iota_X(\omega^j) - i\sum_{j=0}^\infty \frac{i^j}{j!}\flat_\omega X \wedge \omega^j \\ &= i\sum_{j=1}^\infty \frac{i^{j-1}j}{j!}\iota_X\omega \wedge \omega^{j-1} - i\sum_{j=0}^\infty \frac{i^j}{j!}\iota_X\omega \wedge \omega^j \\ &= i\sum_{j=1}^\infty \frac{i^{j-1}}{(j-1)!}\iota_X\omega \wedge \omega^{j-1} - i\sum_{j=0}^\infty \frac{i^j}{j!}\iota_X\omega \wedge \omega^j \\ &= 0.\end{aligned}$$

To see the opposite implication, we take an $X+\xi \in \mathrm{Ann}(\rho)$ and compute $(X+\xi)\cdot\rho$:

$$\begin{aligned}(X+\xi)\cdot\exp(i\omega) &= (X+\xi)\cdot\sum_{j=0}^{\infty}\frac{i^j}{j!}\omega^j = \sum_{j=0}^{\infty}\frac{i^j}{j!}\iota_X(\omega^j)+\sum_{j=0}^{\infty}\frac{i^j}{j!}\xi\wedge\omega^j \\ &= i\iota_X\omega+\sum_{j=2}^{\infty}\frac{i^j}{j!}\iota_X(\omega^j)+\xi+\sum_{j=1}^{\infty}\frac{i^j}{j!}\xi\wedge\omega^j \\ &= (\xi+i\iota_X\omega)+\sum_{j=2}^{\infty}\frac{i^j}{j!}\iota_X(\omega^j)+\sum_{j=1}^{\infty}\frac{i^j}{j!}\xi\wedge\omega^j.\end{aligned}$$

The action of $X+\xi$ on ρ must be null, and hence its part in $\Omega^1(M)$, i.e.,

$$[(X+\xi)\cdot\exp(i\omega)]_{\Omega^1(M)} = \xi + i\iota_X\omega = 0.$$

This proves that $\mathrm{Ann}(\rho)\subseteq \mathbb{L}^{1,0}_{\mathcal{J}_\omega}$.

It is also immediate to see that, since ω is a nondegenerate 2-form, then $\omega^n \neq 0$. This means that the strong generalized almost complex structure $\mathcal{J}_{-1,\omega}$ is described by the globally defined complex differential form $\exp(i\omega)\in\Omega^\bullet(M)_\mathbb{C}$ satisfying the conditions in Propositions 6.5 and 6.6. These calculations also show that, given an almost symplectic structure ω, the type of the strong generalized almost complex structure $\mathcal{J}_\omega$ at any point is equal to 0.

Finally, we aim to find a condition equivalent to the involutivity of the subbundle $\mathbb{L}$ with respect to the Dorfman bracket $[\![\cdot,\cdot]\!]_D$, which corresponds to the integrability of the strong generalized complex structure. In order to properly address this condition, it will be necessary to work with the exterior derivative d of a differential form. The following result will be helpful when computing the exterior derivative of an exponential form.

Lemma 6.11. *For any 2-form $\varphi\in\Omega^2(M)_\mathbb{C}$, we have $d\exp(\varphi) = d\varphi\wedge\exp(\varphi)$.*

Proof. Using Eq. (6.3), it is a straightforward calculation:

$$\begin{aligned}d\exp(\varphi) &= d\sum_{i=0}^{\infty}\frac{1}{i!}\varphi^i = \sum_{i=0}^{\infty}\frac{1}{i!}d(\varphi^i) = \sum_{i=1}^{\infty}\frac{i}{i!}d\varphi\wedge(\varphi^{i-1}) \\ &= \sum_{i=1}^{\infty}\frac{1}{(i-1)!}d\varphi\wedge(\varphi^{i-1}) = d\varphi\wedge\sum_{i=0}^{\infty}\frac{1}{i!}\varphi^i \\ &= d\varphi\wedge\exp(\varphi).\end{aligned}$$

□

We now present the condition that a differential form must satisfy in order for the associated subbundle to be involutive with respect to $[\![\cdot,\cdot]\!]_D$.

Proposition 6.12. *A maximally isotropic subbundle $\mathbb{L}\subset\mathbb{T}M_\mathbb{C}$ that is described in terms of Proposition 6.5 by the differential form ρ is involutive with respect to the Dorfman bracket if and only if there is a $X+\xi\in\Gamma(\mathbb{T}M)$ such that*

$$d\rho = (X+\xi)\cdot\rho. \tag{6.6}$$

In particular, it can be seen that the condition given in the previous proposition is satisfied when the exterior derivative of ρ vanishes, that is, when $d\rho = 0$. This condition, which is stronger than the one previously stated, corresponds to a special class of strong generalized complex structures known as *generalized Calabi-Yau structures*. These structures were first studied by Hitchin in [33].

It is now possible to combine Propositions 6.5, 6.6 and 6.12 to obtain a result that provides an alternative interpretation of strong generalized complex structures in terms of differential forms.

Theorem 6.13 ([29, Theorem 4.8])**.** *Every strong generalized complex structure on a $2n$-dimensional smooth manifold M can be locally described by a complex differential form $\rho \in \Omega^\bullet(U)_\mathbb{C}$ that can be written as $\rho = \Theta \wedge \exp(B + i\omega)$, where $U \subset M$ is an open set, $B, \omega \in \Omega^2(U)$ and $\Theta = \theta_1 \wedge \ldots \wedge \theta_k$ with each $\theta_i \in \Omega^1(U)_\mathbb{C}$, and such that $\omega^{n-k} \wedge \Theta \wedge \overline{\Theta} \neq 0$ and $d\rho = (X + \xi) \cdot \rho$ for some section $X + \xi \in \Gamma(\mathbb{T}M)$.*

In this way, we obtain a total of three different interpretations of the three conditions that a strong generalized complex structure $\mathcal{J} \colon \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ must satisfy, which are $\mathcal{J}^2 = -\mathcal{I}d$, $\mathcal{G}_0(\mathcal{J}u, v) = -\mathcal{G}_0(u, \mathcal{J}v)$ and $\mathcal{N}_\mathcal{J} \equiv 0$. In Figure 6.1 we present three diagrams that illustrate the various interpretations of each of these conditions. It is worth noting that, for the remainder of this chapter, we will adhere to the interpretation provided in Theorem 6.13 and use the relations given in Eqs. (6.4), (6.5) and (6.6).

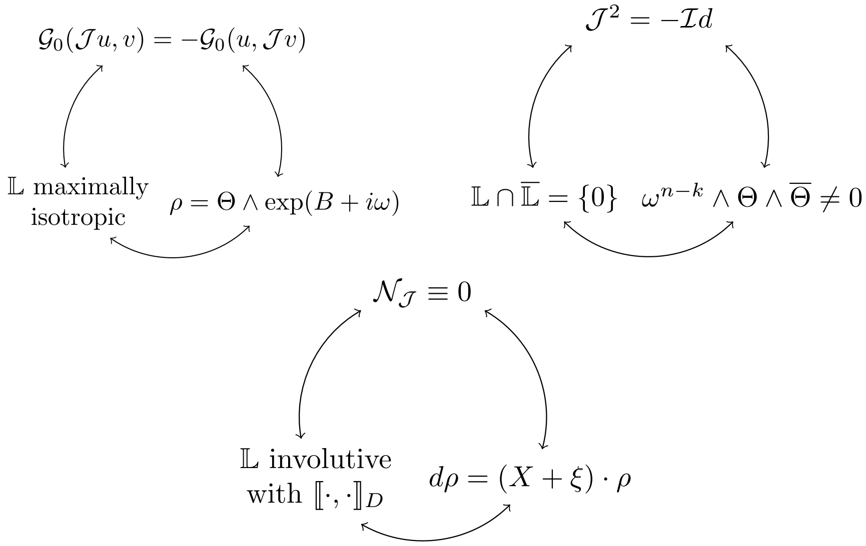


FIGURE 6.1: Equivalent interpretations of the conditions that a strong generalized complex structure must fulfill.

6.2 Fundamentals of nilmanifolds

In the remainder of the chapter, we will work with a specific class of manifolds known as nilmanifolds. The first definition of a nilmanifold was given by K. Malcev in [45], and its study is closely related to the research field of Lie groups and Lie algebras. Therefore, it will be necessary to introduce some basic notions about nilpotent Lie algebras before addressing the search for strong generalized complex structures on nilmanifolds. We will follow the book [31] by B. C. Hall, as well as the article [5] by G. Bazzoni, as our main references to introduce the most relevant concepts.

Let $\mathfrak{g}$ be a real Lie algebra equipped with a Lie bracket $[\cdot,\cdot]$. We define the *lower central series* of $\mathfrak{g}$ as the sequence of algebras $\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \ldots$ such that

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_i = \{[X,Y] \in \mathfrak{g} \colon X \in \mathfrak{g},\ Y \in \mathfrak{g}_{i-1}\} \quad \text{for } i = 1,2,3,\ldots$$

Since $[X,Y] \in \mathfrak{g}$ for every $X, Y \in \mathfrak{g}$, it is clear that $\mathfrak{g}_0 \supseteq \mathfrak{g}_1$. In fact, it can be shown that there is an inclusion relation $\mathfrak{g}_i \supseteq \mathfrak{g}_{i+1}$ for any $i \geq 0$: if we assume $\mathfrak{g}_{i-1} \supseteq \mathfrak{g}_i$ as our induction hypothesis, then

$$\mathfrak{g}_i = \{[X,Y] \in \mathfrak{g} \colon X \in \mathfrak{g},\ Y \in \mathfrak{g}_{i-1}\} \supseteq \{[X,Y] \in \mathfrak{g} \colon X \in \mathfrak{g},\ Y \in \mathfrak{g}_i\} = \mathfrak{g}_{i+1}.$$

This sequence of algebras motivates the following definition.

Definition 6.14. A Lie algebra $\mathfrak{g}$ is called *nilpotent* if $\mathfrak{g}_k = \{0\}$ for some value of k.

Thus, the lower central series of a nilpotent Lie algebra produces the following sequence of inclusions for a certain value of k:

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_k = \{0\}. \tag{6.7}$$

Definition 6.15. Let $\mathfrak{g}$ be a nilpotent Lie algebra. The smallest k such that $\mathfrak{g}_k = \{0\}$ is called the *step* of $\mathfrak{g}$.

We will say that $\mathfrak{g}$ is *k-step*. In this chapter, we focus our attention on two-step nilpotent Lie algebras, that is, in Lie algebras such that $\mathfrak{g}_2 = \{0\}$. Consequently, the sequence in Eq. (6.7) can be simply written as

$$\mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}] \supseteq \{0\},$$

where $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$ unless $\mathfrak{g}$ is the commutative algebra (that is, if $[X,Y] = 0$ for every $X, Y \in \mathfrak{g}$).

We now explain how a Lie algebra can be described using its structure constants. To this end, we will work with a specific type of basis for the underlying vector space. Given any n-dimensional nilpotent Lie algebra $\mathfrak{g}$ with a Lie bracket $[\cdot,\cdot]$, we can always find a basis $\{X_1, \ldots, X_n\}$ such that there are constants $c_{ij}^k \in \mathbb{R}$ with

$$[X_i, X_j] = \sum_{k>i,j} c_{ij}^k X_k,$$

in other words, with $c_{ij}^k = 0$ for $k < i, j$. A basis that meets this requirement is called a *Malcev basis* of the algebra $\mathfrak{g}$.

From a Malcev basis of $\mathfrak{g}$, we can construct its dual basis $\{e_1, \ldots, e_n\}$ of the dual space $\mathfrak{g}^*$, where $e_i(X_j) = \delta_{ij}$. In addition, we can also define a differential map $d \colon \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ that is compatible with the Lie bracket of $\mathfrak{g}$, so that

$$de_k(X_i, X_j) = -e_k([X_i, X_j]), \tag{6.8}$$

for each X_i, X_j of the Malcev basis. This implies that

$$de_k = -\sum_{k>i,j} c_{ij}^k e_i \wedge e_j.$$

We will also call $\{e_1, \ldots, e_n\}$ a Malcev basis of the dual space $\mathfrak{g}^*$ associated to $\{X_1, \ldots, X_n\}$. The map d can be extended to the whole exterior algebra $\wedge^\bullet \mathfrak{g}^*$ taking into account the graded Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta,$$

where $\alpha \in \wedge^r \mathfrak{g}^*$ is a r-form and $\beta \in \wedge^s \mathfrak{g}^*$ is a s-form.

An analogous approach to the one previously presented for the lower central series of a Lie algebra $\mathfrak{g}$ can be followed to study the behavior of its dual space $\mathfrak{g}^*$, making use of the differential map d that has been defined in Eq. (6.8). To this end, we define the following subspaces of $\mathfrak{g}^*$:

$$W_0 = \{0\}, \quad W_i = \{v \in \mathfrak{g}^* \colon dv \in \wedge^2 W_{i-1}\} \quad \text{for } i = 1, 2, 3, \ldots$$

Similarly to what happens with the lower central series of $\mathfrak{g}$, it can be checked that $W_i \subseteq W_{i+1}$ for every $i \geq 0$. First, it is trivial that $W_0 \subseteq W_1$. Assuming that $W_{i-1} \subseteq W_i$, it is immediate to see that W_i is a subspace of W_{i+1}:

$$W_i = \{v \in \mathfrak{g} \colon dv \in \wedge^2 W_{i-1}\} \subseteq \{v \in \mathfrak{g} \colon dv \in \wedge^2 W_i\} = W_{i+1}.$$

It is not complicated to prove (see, for example, [5, Lemma 2]) that a Lie algebra $\mathfrak{g}$ is k-step nilpotent if and only if $W_k = \mathfrak{g}^*$ and $W_{k-1} \neq \mathfrak{g}^*$. Therefore, if we work with a k-step nilpotent Lie algebra $\mathfrak{g}$, the subspaces W_i will produce the following sequence of inclusions:

$$\{0\} = W_0 \subseteq W_1 \subset W_2 \subseteq \cdots \subseteq W_k = \mathfrak{g}^*. \tag{6.9}$$

This sequence is usually called *characteristic filtration* of $\mathfrak{g}^*$. Since we are interested in two-step nilpotent Lie algebras, the characteristic filtration will simply be

$$\{0\} \subseteq \{v \in \mathfrak{g}^* \colon dv = 0\} \subseteq \mathfrak{g}^*,$$

where $W_1 \neq \{0\}$ unless $\mathfrak{g}$ is the commutative algebra.

Taking into account the space $W_1 = \{v \in \mathfrak{g}^* \colon dv = 0\}$, it is immediately seen that the dual space $\mathfrak{g}^*$ of a two-step nilpotent Lie algebra is isomorphic to the sum $W_1 \oplus (\mathfrak{g}^*/W_1)$. This decomposition produces two invariant numbers, $f_0 = \dim W_1$ and $f_1 = \dim(\mathfrak{g}^*/W_1)$. Clearly, its sum is equal to the dimension of $\mathfrak{g}^*$, i.e., $f_0 + f_1 = \dim \mathfrak{g}^*$. We will say that the pair (f_0, f_1) is the *signature* of $\mathfrak{g}^*$.

In summary, if we want to work with a real two-step nilpotent Lie algebra $\mathfrak{g}$ of dimension eight, it suffices to describe the internal structure of the dual algebra $\mathfrak{g}^*$ using the differential map d and taking into account the signature of the algebra. These Lie algebras have already been fully classified: in [10], M. Borovoi, B. A. Dina and W. A. de Graaf make a full description of every real two-step nilpotent Lie algebra of dimension eight that is nondegenerate (i.e., those algebras that cannot be written as a direct sum of a two-step nilpotent Lie algebra of lower dimension and a nonzero abelian Lie algebra). This classification can be combined with the one by Bazzoni in [5], which classifies every two-step nilpotent Lie algebra of dimension seven, not necessarily nondegenerate. If we add a one-dimensional abelian Lie algebra to each one of the algebras in [5], we obtain the two-step Lie algebras of dimension eight that are not considered in [10], obtaining in this way a complete classification of the two-step nilpotent real Lie algebras of dimension eight.

An attentive reader may have noticed that we have not yet mentioned how nilpotent Lie algebras are related to the study of smooth manifolds. The following definition connects these two categories through the concept of nilmanifolds.

Definition 6.16 ([45])**.** A *nilmanifold* is defined as a quotient $M = G/\Gamma$, where G is a connected, simply connected, nilpotent Lie group (i.e., its associated Lie algebra is nilpotent); and Γ is a discrete, cocompact subgroup (i.e., G/Γ is compact).

Using the same terminology as before, we will say that a nilmanifold $M = G/\Gamma$ is *two-step* if the Lie algebra $\mathfrak{g}$ associated with G is two-step. Also, we will define the *signature* of M as the signature (f_0, f_1) of the algebra $\mathfrak{g}^*$.

Let G be a connected, simply connected, nilpotent Lie group, and let $\mathfrak{g}$ be its associated Lie algebra. Then, any tensor defined on $\mathfrak{g}$ corresponds to a *left-invariant tensor* on G. All such tensors descend to tensors on the quotient $M = G/\Gamma$. With a slight abuse of notation, we will refer to these as *left-invariant tensors* on the nilmanifold M. In particular, differential forms on the exterior algebra $\wedge^\bullet \mathfrak{g}^*$ are equivalent to left-invariant differential forms on M. Therefore, we will say that a strong generalized complex structure on M is *left-invariant* if it can be described, in terms of Theorem 6.13, by a left-invariant differential form $\rho \in \Omega^\bullet(M)$.

6.3 Strong generalized complex structures on nilmanifolds

As discussed in the previous section, our goal is to find left-invariant strong generalized complex structures on $2n$-dimensional nilmanifolds $M = G/\Gamma$. To this end, and in light of the interpretation provided in Theorem 6.13, it suffices to identify differential forms ρ that fulfill certain properties. Since the structures studied are required to be left-invariant, these forms will be defined globally, that is, $\rho \in \Omega^\bullet(M)$. Therefore, we will work with differential 2-forms that can be written as $\rho = \Theta \wedge \exp(B + i\omega)$, where $B, \omega \in \Omega^2(M)$ are left-invariant differential forms and $\Theta = \theta_1 \wedge \ldots \wedge \theta_k$ with each $\theta_i \in \Omega^1(M)_\mathbb{C}$ being a left-invariant complex 1-form.

As every considered differential form is left-invariant, we will just need to deal with such structures in the Lie algebra $\mathfrak{g}$ associated to each nilmanifold. Therefore, we have to construct differential forms $\rho \in \wedge^\bullet \mathfrak{g}^*$ such that

$$\rho = \Theta \wedge \exp(B + i\omega),$$

where $B, \omega \in \wedge^2\mathfrak{g}^*$ and Θ can be decomposed as $\Theta = \theta_1 \wedge \ldots \wedge \theta_k$ with each $\theta_i \in \mathfrak{g}^*_{\mathbb{C}}$. Also, these differential forms must satisfy the condition

$$\omega^{n-k} \wedge \Theta \wedge \overline{\Theta} \neq 0.$$

Because of the left-invariant condition, it is easily surmised that the type of the structure is invariant along the nilmanifold and therefore it does not depend on the point $p \in M$.

With respect to the integrability condition, there is a result that proves that, in this context, the condition that these differential forms must satisfy is stronger than the one given in Eq. (6.6).

Theorem 6.17 ([12, Theorem 3.1]). *Each left-invariant strong generalized complex structure on a nilmanifold must be generalized Calabi-Yau; in other words, any left-invariant differential form ρ that determines the integrable structure must be closed, that is, $d\rho = 0$.*

This condition will greatly simplify the computations involved in the search for such structures. Furthermore, it will impose restrictions on the 1-forms $\theta_1, \ldots, \theta_k$. The following result illustrates one such restriction.

Proposition 6.18 ([12, Example 3]). *Given a left-invariant strong generalized complex structure on a nilmanifold defined by a differential form $\rho = \Theta \wedge \exp(B + i\omega)$, it is always possible to choose a decomposition $\Theta = \theta_1 \wedge \ldots \wedge \theta_k$ such that $d\theta_1 = 0$.*

This latter restriction will be particularly relevant when verifying the non-existence of certain types of strong generalized complex structures on some of the Lie algebras under consideration.

6.4 Strong generalized complex structures on two-step nilmanifolds of dimension 8

From this point onward, we will work with two-step nilpotent Lie algebras $\mathfrak{g}$ of dimension eight. To distinguish and handle the dual space $\mathfrak{g}^*$ of each of these algebras, we will assign to each one a Malcev basis $\{e_1, \ldots, e_8\}$ and indicate the value of the exterior derivative of each basis element, that is, the value of each de_i for $i = 1, \ldots, 8$. All the algebras have been taken from [5, 10].

In this case, the only signatures allowed for each algebra $\mathfrak{g}^*$ are $(8, 0)$, $(7, 1)$, $(6, 2)$, $(5, 3)$ and $(4, 4)$. This is the case because, for other values of (f_0, f_1), the induced map $d\colon \mathfrak{g}^*/W_1 \to \wedge^2\mathfrak{g}^*$ would not be injective, which would contradict the fundamental isomorphism theorem.

According to the discussion following Definition 6.8, the type of a left-invariant strong generalized complex structure on an eight-dimensional nilmanifold does not depend on the point of the manifold and can be equal to 0, 1, 2, 3, or 4. Moreover, as discussed in Examples 6.9 and 6.10, every left-invariant strong generalized complex structure of type 4 can be associated with a left-invariant complex structure on the nilmanifold, while each such structure of type 0 corresponds to a left-invariant symplectic structure on the nilmanifold. Structures of other types are not directly related to either left-invariant symplectic or complex structures on M.

Before moving forward, we include a brief remark on the notation used in this section: we write e_{ij} to refer to the exterior product $e_i \wedge e_j$, we write e_{ijk} to denote the product $e_i \wedge e_j \wedge e_k$, and so on, where each e_i refers to the element of a Malcev basis $\{e_1, \ldots, e_8\}$.

6.4.1 Signature $(8, 0)$

We first analyze those nilmanifolds associated with Lie algebras such that $\dim W_1 = 8$, using the notation from Eq. (6.9). The only eight-dimensional Lie algebra that satisfies this condition is the commutative one of dimension eight, that is, the algebra such that $de_1 = \ldots = de_8 = 0$. In the following result, we check that a nilmanifold associated with this Lie algebra admits every type of left-invariant strong generalized complex structure.

Proposition 6.19. *A nilmanifold associated with the eight-dimensional commutative Lie algebra admits left-invariant strong generalized complex structures of each type.*

Proof. In Table 6.1 we provide specific examples of each type of strong generalized complex structure for this Lie algebra. To understand the notation used in the table, it may prove useful to examine Section 6.5, where this notation is detailed. □

6.4.2 Signature $(7, 1)$

We now analyze the nilmanifolds associated with Lie algebras that, according to Eq. (6.9), satisfy $\dim W_1 = 7$; in other words, those for which $de_1 = \ldots = de_7 = 0$ and $de_8 \neq 0$. In this case, the classification in [10] shows that there are no nondegenerate two-step Lie algebras of dimension eight that satisfy this condition. Therefore, we restrict our attention to the three degenerate Lie algebras identified in [5], which are listed in Table 6.2.

With respect to the strong generalized complex structures that each of them admits, there are some types of structure that can be found on any of the studied algebras, as is shown in the following result.

Proposition 6.20. *Every two-step nilmanifold of dimension eight and signature $(7, 1)$ admits a left-invariant strong generalized complex structure of type 4 (complex), type 3 and type 2.*

Proof. In Table 6.2 we provide specific examples of strong generalized complex structures of type 4, type 3 and type 2 for each Lie algebra. □

With regard to the other types of structure, some of the analyzed Lie algebras cannot be endowed with certain structures. This is proved in the following two statements.

Proposition 6.21. *The only two-step nilmanifolds of dimension eight and signature $(7, 1)$ that can be endowed with a left-invariant symplectic structure are those associated with the Lie algebra defined with $de_1 = \ldots = de_7 = 0$ and $de_8 = e_{12}$.*

Proof. In Table 6.2 we provide an example of a symplectic structure for the Lie algebra such that $de_1 = \ldots = de_7 = 0$ and $de_8 = e_{12}$.

If we now take the Lie algebra such that $de_1 = \ldots = de_7 = 0$ and $de_8 = e_{12} + e_{34}$, and consider a 2-form $\omega = \sum_{i<j} \omega_{ij} e_{ij}$, we can calculate $d\omega$:

$$\begin{aligned} d\omega &= \sum_{1\le i<j\le 8} \omega_{ij} de_{ij} = \sum_{i=1}^{7} \omega_{i8} de_{i8} = -\sum_{i=1}^{7} \omega_{i8} e_{i12} - \sum_{i=1}^{7} \omega_{i8} e_{i34} \\ &= -\sum_{i=3}^{7} \omega_{i8} e_{12i} - \sum_{i=1}^{2} \omega_{i8} e_{i34} - \sum_{i=5}^{7} \omega_{i8} e_{34i}. \end{aligned}$$

Using this calculation, it can be easily deduced that if $d\omega = 0$ then, since the forms e_{ijk} are linearly independent, it must be $\omega_{18} = \ldots = \omega_{78} = 0$. This implies that ω cannot be nondegenerate and hence the algebra does not admit a symplectic structure.

The reasoning is analogous when $de_1 = \ldots = de_7 = 0$ and $de_8 = e_{12} + e_{34} + e_{56}$:

$$\begin{aligned} d\omega &= \sum_{1\le i<j\le 8} \omega_{ij} de_{ij} = \sum_{i=1}^{7} \omega_{i8} de_{i8} = -\sum_{i=1}^{7} \omega_{i8} e_{i12} - \sum_{i=1}^{7} \omega_{i8} e_{i34} - \sum_{i=1}^{7} \omega_{i8} e_{i56} \\ &= -\sum_{i=3}^{7} \omega_{i8} e_{12i} - \sum_{i=1}^{2} \omega_{i8} e_{i34} - \sum_{i=5}^{7} \omega_{i8} e_{34i} - \sum_{i=1}^{4} \omega_{i8} e_{i56} - \omega_{78} e_{567}. \end{aligned}$$ □

Proposition 6.22. *The only nilmanifolds that cannot be endowed with a left-invariant strong generalized complex structure of type 1 are those related to the Lie algebra such that* $de_1 = \ldots = de_7 = 0$ *and* $de_8 = e_{12} + e_{34} + e_{56}$.

Proof. In Table 6.2 we provide examples of strong generalized complex structures of type 1 for the other two Lie algebras.

According to Proposition 6.18, we must search for a differential form $\rho = \Theta \wedge \exp(B + i\omega)$ such that $\Theta \in \mathfrak{g}^*_{\mathbb{C}}$ is a complex 1-form and $d\Theta = 0$. Therefore, since $de_8 \neq 0$ we can write

$$\Theta = \sum_{i=1}^{7} z_i e_i, \quad B + i\omega = \sum_{1\le i<j\le 8} h_{ij} e_{ij},$$

where each coefficient $z_1, \ldots, z_7 \in \mathbb{C}$ is a complex number, as well as each h_{ij} for $1 \le i < j \le 8$.

First, we calculate $d(B + i\omega)$:

$$\begin{aligned} d(B+i\omega) &= \sum_{1\le i<j\le 8} h_{ij} de_{ij} = \sum_{j=1}^{7} h_{j8} de_{j8} = -\sum_{j=1}^{7} h_{j8} e_{j12} - \sum_{j=1}^{7} h_{j8} e_{j34} - \sum_{j=1}^{7} h_{j8} e_{j56} \\ &= -\sum_{j=3}^{7} h_{j8} e_{12j} - \sum_{j=1}^{2} h_{j8} e_{j34} - \sum_{j=5}^{7} h_{j8} e_{34j} - \sum_{j=1}^{4} h_{j8} e_{j56} - h_{78} e_{567}. \end{aligned}$$

We now compute $d\rho$, using that $d\Theta = 0$ and the exterior derivative of the exponential map from Lemma 6.11:

$$\begin{aligned} d\rho &= d\Theta \wedge \exp(B + i\omega) - \Theta \wedge d(\exp(B + i\omega)) = -\Theta \wedge d(B + i\omega) \wedge \exp(B + i\omega) \\ &= d(B + i\omega) \wedge \rho. \end{aligned}$$

If we compute the product $d(B+i\omega)\wedge\rho$ explicitly and check whether it is equal to zero, we will obtain multiple equations from the coefficients of the products of elements in the Malcev basis. Joining all these relations together, it can be checked that, for every pair of indices $(i, j) \notin \{(1, 2), (3, 4), (5, 6)\}$, the following equation must be held:

$$h_{i8}z_j - h_{j8}z_i = 0.$$

Also, the following three equations must be true:

$$\begin{cases} h_{18}z_2 - h_{28}z_1 + h_{38}z_4 - h_{48}z_3 = 0, \\ h_{18}z_2 - h_{28}z_1 + h_{58}z_6 - h_{68}z_5 = 0, \\ h_{38}z_4 - h_{48}z_3 + h_{58}z_6 - h_{68}z_5 = 0. \end{cases}$$

Therefore, all these equations can be shortened in only one expression: for every $1 \leq i < j \leq 7$,

$$h_{i8}z_j - h_{j8}z_i = 0.$$

Since Θ cannot be equal to zero, there must be an index i_0 such that $z_{i_0} \neq 0$. Just by rescaling Θ, it can be considered $z_{i_0} = 1$. Then, the equation above transforms into

$$h_{j8} = h_{i_0 8}z_j,$$

for all $j = 1, \dots, 7$, and therefore the expressions of Θ and $B + i\omega$ transform into

$$\Theta = \sum_{i=1}^{7} z_i e_i, \quad B + i\omega = \sum_{1 \leq i < j \leq 7} h_{ij}e_{ij} + h_{i_0 8} \sum_{j=1}^{7} z_j e_{j8}.$$

However, it can be checked computationally that, in this situation, the product $\omega^3 \wedge \Theta \wedge \overline{\Theta}$ would be equal to zero, and thus ρ does not fulfill the required conditions. The computations that are necessary to see this are extensive (they can be done using specific mathematical software, such as Sage or Maxima, [46, 57]). This shows that there is no left-invariant strong generalized complex structure of type 1 on a nilmanifold whose associated Lie algebra is the one considered. □

6.4.3 Signature $(6, 2)$

Now it is time to study the nilmanifolds related to Lie algebras such that, using the notation of Eq. (6.9), satisfy the condition $\dim W_1 = 6$. This implies that $de_1 = \ldots = de_6 = 0$ and $de_7, de_8 \neq 0$. From the classifications [5, 10] we can see that there are thirteen Lie algebras that have signature

$(6, 2)$, of which there are seven nondegenerate algebras and six degenerate ones. All these algebras are listed in Tables 6.3 and 6.4.

In this case, most of the types of left-invariant strong generalized complex structures can be found on any nilmanifold associated to a two-step Lie algebra of dimension eight and signature $(6, 2)$. However, only five of them admit a left-invariant symplectic structure, as proved in the results below.

Proposition 6.23. *Every two-step nilmanifold of dimension eight and signature* $(6, 2)$ *admits a left-invariant strong generalized complex structure of type 4 (complex), type 3, type 2 and type 1.*

Proof. In Tables 6.3 and 6.4 we provide specific examples of strong generalized complex structures of type 4, type 3, type 2 and type 1 for each Lie algebra. □

Proposition 6.24. *A two-step nilmanifold of dimension eight and signature* $(6, 2)$ *admits a left-invariant symplectic structure if and only if it is degenerate and not associated with the Lie algebra characterized by* $de_1 = \ldots = de_6 = 0$, $de_7 = e_{12}$ *and* $de_8 = e_{13} + e_{45}$.

Proof. In Table 6.4 we provide examples of symplectic structures for each of the $(6, 2)$ Lie algebras that admit a symplectic structure.

If we now consider the Lie algebra with $de_1 = \ldots = de_6 = 0$, $de_7 = e_{12}$ and $de_8 = e_{13} + e_{45}$, and take a 2-form $\omega = \sum_{i<j} \omega_{ij} e_{ij}$, we calculate $d\omega$:

$$\begin{aligned} d\omega &= \sum_{1\leq i<j\leq 8} \omega_{ij} de_{ij} = \sum_{i=1}^{6} \omega_{i7} de_{i7} + \sum_{i=1}^{7} \omega_{i8} de_{i8} \\ &= -\sum_{i=1}^{6} \omega_{i7} e_{i12} - \sum_{i=1}^{6} \omega_{i8} e_{i13} - \sum_{i=1}^{6} \omega_{i8} e_{i45} + \omega_{78}(e_{128} - e_{713} - e_{745}) \\ &= -\sum_{i=3}^{6} \omega_{i7} e_{12i} + \omega_{28} e_{123} - \sum_{i=4}^{6} \omega_{i8} e_{13i} - \sum_{i=1}^{3} \omega_{i8} e_{i45} - \omega_{68} e_{456} + \omega_{78}(e_{128} - e_{137} - e_{457}). \end{aligned}$$

Just by grouping the terms and asking the 2-form to be closed, we can see from the coefficients of the elements e_{i45} $(i = 1, 2, 3)$ and e_{13j} $(j = 4, 5, 6, 7)$ that if $d\omega = 0$ then $\omega_{18} = \ldots = \omega_{78} = 0$. Therefore, ω is degenerate and this Lie algebra does not admit a symplectic structure.

The reasoning is analogous for each nondegenerate two-step Lie algebra of dimension eight and signature $(6, 2)$. We can take, for example, the one such that $de_1 = \ldots = de_6 = 0$, $de_7 = e_{34} + e_{56}$ and $de_8 = e_{12}$. In this case, the expression for $d\omega$ is

$$\begin{aligned} d\omega &= \sum_{1\leq i<j\leq 8} \omega_{ij} de_{ij} = \sum_{i=1}^{6} \omega_{i7} de_{i7} + \sum_{i=1}^{7} \omega_{i8} de_{i8} \\ &= -\sum_{i=1}^{6} \omega_{i7} e_{i34} - \sum_{i=1}^{6} \omega_{i7} e_{i56} - \sum_{i=1}^{6} \omega_{i8} e_{i12} + \omega_{78}(e_{348} + e_{568} - e_{712}) \\ &= -\sum_{i=1}^{2} \omega_{i7} e_{i34} - \sum_{i=5}^{6} \omega_{i7} e_{34i} - \sum_{i=1}^{4} \omega_{i7} e_{i56} - \sum_{i=3}^{6} \omega_{i8} e_{12i} + \omega_{78}(e_{348} + e_{568} - e_{127}). \end{aligned}$$

If we take into account the condition $d\omega = 0$, from the coefficients of the elements e_{134}, e_{234}, e_{356}, e_{456}, e_{345}, e_{346} and e_{348} we can deduce that if $d\omega = 0$ then $\omega_{17} = \ldots = \omega_{67} = \omega_{78} = 0$. Therefore, ω is degenerate and the Lie algebra does not admit a symplectic structure.

The calculations for the other Lie algebras are analogous and, therefore, are not displayed here. □

6.4.4 Signature $(5, 3)$

We face now the study of the nilmanifolds whose associated Lie algebra, using the notation from Eq. (6.9), satisfy the condition $\dim W_1 = 5$. This means that $de_1 = \ldots = de_5 = 0$ and $de_6, de_7, de_8 \neq 0$. Using the classifications [5, 10] of two-step nilpotent Lie algebras, we obtain twenty-three non-isomorphic Lie algebras with signature $(5, 3)$. Of these twenty-three algebras, sixteen are nondegenerate, while the remaining ones are degenerate. All these algebras are listed in Tables 6.5 and 6.6.

In this case, while all of these Lie algebras admit strong generalized complex structures of types 3, 2 and 1, one of them does not admit a symplectic structure. As for complex structures, we have shown that all but three of the algebras admit such a structure. For the remaining three, we have not been able to determine whether or not they admit a complex structure. This is proved in the following three results.

Proposition 6.25. *Every two-step nilmanifold of dimension eight and signature* $(5, 3)$ *admits a left-invariant strong generalized complex structure of type 3, type 2 and type 1.*

Proof. In Tables 6.5 and 6.6 we provide specific examples of strong generalized complex structures of type 3, type 2 and type 1 for each Lie algebra. □

Proposition 6.26. *The only two-step nilmanifolds of dimension eight and signature* $(5, 3)$ *that cannot be endowed with a left-invariant symplectic structure are those related to the Lie algebra such that* $de_1 = \ldots = de_5 = 0$, $de_6 = e_{25} + e_{34}$, $de_7 = e_{12}$ *and* $de_8 = e_{15}$.

Proof. In Table 6.6 we provide examples of symplectic structures for every two-step nilpotent Lie algebra of signature $(5, 3)$ that is not the one mentioned above.

If we now take the Lie algebra with $de_1 = \ldots = de_5 = 0$, $de_6 = e_{25} + e_{34}$, $de_7 = e_{12}$ and $de_8 = e_{15}$, and consider a 2-form $\omega = \sum_{i<j} \omega_{ij} e_{ij}$, we can compute $d\omega$ and obtain the following

3-form:

$$
\begin{aligned}
d\omega &= \sum_{1\le i<j\le 8} \omega_{ij} de_{ij} = \sum_{i=1}^{5} \omega_{i6} de_{i6} + \sum_{i=1}^{6} \omega_{i7} de_{i7} + \sum_{i=1}^{7} \omega_{i8} de_{i8} \\
&= -\sum_{i=1}^{5} \omega_{i6} e_{i25} - \sum_{i=1}^{5} \omega_{i6} e_{i34} - \sum_{i=1}^{5} \omega_{i7} e_{i12} - \sum_{i=1}^{5} \omega_{i8} e_{i15} \\
&\quad + \omega_{67}(e_{257} + e_{347} - e_{612}) + \omega_{68}(e_{258} + e_{348} - e_{615}) + \omega_{78}(e_{128} - e_{715}) \\
&= -\omega_{16} e_{125} + \sum_{i=3}^{4} \omega_{i6} e_{2i5} - \sum_{i=1}^{2} \omega_{i6} e_{i34} - \omega_{56} e_{345} - \sum_{i=3}^{5} \omega_{i7} e_{12i} - \sum_{i=2}^{4} \omega_{i8} e_{1i5} \\
&\quad + \omega_{67}(e_{257} + e_{347} - e_{126}) + \omega_{68}(e_{258} + e_{348} - e_{156}) + \omega_{78}(e_{128} - e_{157}).
\end{aligned}
$$

We now group the terms and ask ω to be closed. Then, from the coefficients of the elements e_{134}, e_{234}, e_{235}, e_{245}, e_{345}, e_{257} and e_{258}, we deduce that if $d\omega = 0$ then $\omega_{16} = \ldots = \omega_{56} = \omega_{67} = \omega_{68} = 0$. Therefore, ω is degenerate and the nilmanifold cannot be endowed with a left-invariant symplectic structure. □

Proposition 6.27. *Let M be a two-step nilmanifold of dimension eight and signature $(5,3)$ such that it is not associated to one of the following Lie algebras:*

- $de_1 = \ldots = de_5 = 0$, $de_6 = e_{13}$, $de_7 = e_{24}$ *and* $de_8 = e_{15} + e_{25}$.
- $de_1 = \ldots = de_5 = 0$, $de_6 = e_{24}$, $de_7 = e_{15} + e_{23}$ *and* $de_8 = e_{13}$.
- $de_1 = \ldots = de_5 = 0$, $de_6 = e_{15} + e_{24}$, $de_7 = e_{14} + e_{23}$ *and* $de_8 = e_{13}$.

Then, it admits a left-invariant complex structure.

Proof. In Table 6.5 we provide specific examples of complex structures for each Lie algebra not mentioned above. □

6.4.5 Signature $(4,4)$

Lastly, we analyze the nilmanifolds whose associated two-step Lie algebra has dimension eight and signature $(4,4)$. From Eq. (6.9), we deduce that this means that $de_1 = \ldots = de_4 = 0$ and $de_5, \ldots, de_8 \neq 0$. In this case, there exist only four nondegenerate such Lie algebras, which are presented in [10] and collected in Tables 6.7 and 6.8.

In the result below, we show that all these Lie algebras admit strong generalized complex structures of every possible type.

Proposition 6.28. *Every two-step nilmanifold of dimension eight and signature $(4,4)$ admits left-invariant strong generalized complex structures of each possible type.*

Proof. In Tables 6.7 and 6.8 we provide specific examples of each type of strong generalized complex structure for each Lie algebra. □

All the results from Propositions 6.19-6.28 can be summarized in the following theorem, where we see that, for an eight-dimensional nilmanifold, being two-step is a sufficient condition for admitting a left-invariant strong generalized complex structure.

Theorem 6.29. *Every two-step nilmanifold of dimension eight admits a left-invariant strong generalized complex structure. In particular, every two-step nilmanifold of dimension eight admits a left-invariant complex structure or a left-invariant symplectic structure. Also, every such nilmanifold admits a left-invariant strong generalized complex structure that is not induced by either a left-invariant complex or a symplectic structure on the manifold.*

Naturally, being two-step is not a necessary condition for eight-dimensional nilmanifolds in order to admit a left-invariant strong generalized complex structure. We can study, for example, a nilmanifold whose associated nilpotent Lie algebra $\mathfrak{g}$ is characterized by the following description:

$$de_1 = \cdots = de_4 = 0, \quad de_5 = e_{12}, \quad de_6 = e_{13}, \quad de_7 = e_{14}, \quad de_8 = e_{15}.$$

Using the notation from Eq. (6.9), we have that W_1 is generated by $\{e_1, \ldots, e_4\}$ for this Lie algebra; W_2 is generated by $\{e_1, \ldots, e_7\}$; and $W_3 = \mathfrak{g}^*$. Therefore, it is a three-step nilpotent Lie algebra of dimension eight. We can induce a left-invariant symplectic structure using the following 2-form:

$$\omega = e_{18} + e_{27} + e_{36} + e_{45}.$$

It is immediate to see that ω is not degenerate. In addition, it is easy to compute $d\omega$:

$$d\omega = -e_{115} - e_{214} - e_{313} - e_{412} = e_{124} - e_{124} = 0.$$

Therefore, ω is a symplectic structure and thus induces a left-invariant strong generalized complex structure of type 0.

6.5 Tables of strong generalized complex structures on nilmanifolds

The following six pages present, in landscape format, the tables listing the various strong generalized complex structures found for each of the two-step nilpotent Lie algebras $\mathfrak{g}$ of dimension eight.

Some clarifications on the notation used in these tables may be helpful before proceeding. To ensure maximum clarity and readability, all elements of the Malcev basis $\{e_1, \ldots, e_8\}$ of the dual space $\mathfrak{g}^*$ are denoted by bold numbers: for example, $\mathbf{1} + i\mathbf{2}$ denotes the 1-form $e_1 + ie_2$, and $d\mathbf{8}$ stands for de_8. Other numbers or symbols that are not in bold, such as $2i$, correspond to coefficients in $\mathbb{C}$. Furthermore, in order to further ease the notation, all exterior product symbols $\wedge$ have been omitted; for example, $\mathbf{12}$ stands for $e_{12} = e_1 \wedge e_2$, and $(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})$ denotes the 2-form $(e_1+ie_2)\wedge(e_3+ie_4)$.

Bearing this in mind, if in one table there is a strong generalized complex structure of type 2 that is written as

$$(\mathbf{2} + i\mathbf{3})(\mathbf{1} + i\mathbf{7}) \exp(\mathbf{46} + \mathbf{58} + i(\mathbf{48} + \mathbf{56})),$$

it actually represents the differential form

$$(e_2 + ie_3) \wedge (e_1 + ie_7) \wedge \exp(e_{46} + e_{58} + i(e_{48} + e_{56})).$$

Comparing this differential form with the expression in Eq. (6.4), we have that the previous form is $\rho = \Theta \wedge \exp(B + i\omega)$, with $B = e_{46} + e_{58}$, $\omega = e_{48} + e_{56}$ and $\Theta = \theta_1 \wedge \theta_2$ such that $\theta_1 = e_2 + ie_3$ and $\theta_2 = e_1 + ie_7$.

Finally, note that a dash $(-)$ in a box of the tables indicates that we have proved the non-existence of a strong generalized complex structure of the specified type for the corresponding Lie algebra. In contrast, question marks (??) indicate that no such structure has been found, but neither has its non-existence been proven.

Algebra	Structures				
$d\mathbf{i}$	Type 4 (complex)	Type 3	Type 2	Type 1	Type 0 (symplectic)
0	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})(\mathbf{7}+i\mathbf{8})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})\exp(i\mathbf{78})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})\exp(i(\mathbf{56}+\mathbf{78}))$	$(\mathbf{1}+i\mathbf{2})\exp(i(\mathbf{34}+\mathbf{56}+\mathbf{78}))$	$\mathbf{12}+\mathbf{34}+\mathbf{56}+\mathbf{78}$

TABLE 6.1: Examples of strong generalized complex structures of type 4 (complex), type 3, type 2, type 1 and type 0 (symplectic) for the two-step nilpotent Lie algebra of dimension eight and signature $(8,0)$.

Algebra	Structures				
$d\mathbf{8}$	Type 4 (complex)	Type 3	Type 2	Type 1	Type 0 (symplectic)
$\mathbf{12}$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})(\mathbf{7}+i\mathbf{8})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})\exp(i\mathbf{78})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})\exp(i(\mathbf{56}+\mathbf{78}))$	$(\mathbf{1}+i\mathbf{2})\exp(i(\mathbf{34}+\mathbf{56}+\mathbf{78}))$	$\mathbf{18}+\mathbf{23}+\mathbf{45}+\mathbf{67}$
$\mathbf{12}+\mathbf{34}$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})(\mathbf{7}+i\mathbf{8})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})\exp(i\mathbf{78})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})\exp(i(\mathbf{56}+\mathbf{78}))$	$(\mathbf{1}+i\mathbf{2})\exp(i(\mathbf{38}+\mathbf{45}+\mathbf{67}))$	–
$\mathbf{12}+\mathbf{34}+\mathbf{56}$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})(\mathbf{7}+i\mathbf{8})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})(\mathbf{5}+i\mathbf{6})\exp(i\mathbf{78})$	$(\mathbf{1}+i\mathbf{2})(\mathbf{3}+i\mathbf{4})\exp(i(\mathbf{58}+\mathbf{67}))$	–	–

TABLE 6.2: Examples of strong generalized complex structures of type 4 (complex), type 3, type 2, type 1 and type 0 (symplectic) for two-step nilpotent Lie algebras of dimension eight and signature $(7,1)$.

Algebra		Structures	
$d7$	$d8$	Type 4 (complex)	Type 3
12	13	$(1+i4)(2+i3)(5+i6)(7+i8)$	$(1+i4)(2+i3)(7+i8)\exp(i56)$
12	34	$(1+i2)(3+i4)(5+i6)(7+i8)$	$(1+i2)(3+i4)(7+i8)\exp(i56)$
12	$13+24$	$(1+i2)(3+i4)(5+i6)(7+i8)$	$(1+i2)(3+i4)(7+i8)\exp(i56)$
12	$13+45$	$(1+i6)(2+i3)(4+i5)(7+i8)$	$(2+i3)(4+i5)(7+i8)\exp(i16)$
$12+34$	$13+25$	$(1+i6)(2+3+i3)(4-2i5)(7+8+i8)$	$(1+i3)(2+i4)(6+i7)\exp(i58)$
$13-24$	$14+23$	$(1+i2)(3-i4)(5+i6)(7+i8)$	$(1+i2)(3-i4)(7+i8)\exp(i56)$
$12+56$	$34+56$	$(1+i2)(3+i4)(5+i6)(7+i8)$	$(3+i4)(5+i6)(1+i8)\exp(i27)$
$12-36+45$	$34-56$	$(1+i2)(3+i4)(5+i6)(7+i8)$	$(3+i5)(4-i6)(1+i8)\exp(i27)$
$15+46$	$14+23$	$(1+i4)(2+i3)(5+i6)(7+i8)$	$(1+i4)(2+i3)(5+i6)\exp(i78)$
$15+26+34$	$14+23$	$(1+i2)(3-i4)(5+i6)(7+i8)$	$(1+i2)(3-i4)(5+i6)\exp(i78)$
$13+25$	$16+24$	$(1+i2)(3+i5)(4-i6)(7+i8)$	$(1+i2)(3+i5)(4-i6)\exp(i78)$
$34+56$	12	$(1+i2)(3+i4)(5+i6)(7+i8)$	$(3+i4)(5+i6)(1+i7)\exp(i28)$
$16+25+34$	12	$(1+i2)(3+i4)(5-i6)(7+i8)$	$(1+i2)(5-i6)(3+i8)\exp(i47)$

TABLE 6.3: Examples of strong generalized complex structures of type 4 (complex) and type 3 for two-step nilpotent Lie algebras of dimension eight and signature $(6,2)$.

Algebra		Structures		
$d7$	$d8$	Type 2	Type 1	Type 0 (symplectic)
12	13	$(1+i4)(5+i6)\exp(i(27+38))$	$(1+i4)\exp(i(27+38+56))$	$14+27+38+56$
12	34	$(1+i4)(5+i6)\exp(i(27+38))$	$(1+i4)\exp(i(27+38+56))$	$14+27+38+56$
12	$13+24$	$(1+i3)(5+i6)\exp(i(27+48))$	$(1+i3)\exp(i(27+48+56))$	$14+28+37+56$
12	$13+45$	$(1+i3)(5+i6)\exp(i(27+48))$	$(1+i3)\exp(i(27+48+56))$	$-$
$12+34$	$13+25$	$(1+i4)(5+i6)\exp(i(37+28))$	$(1+i4)\exp(i(28+37+56))$	$14+28+37+56$
$13-24$	$14+23$	$(3+i4)(5+i6)\exp(i(18+27))$	$(3+i4)\exp(i(18+27+56))$	$18+27+34+56$
$12+56$	$34+56$	$(2+i4)(5+i6)\exp(i(17+38))$	$(5+i6)\exp(i(17+24+38))$	$-$
$12-36+45$	$34-56$	$(1+i2)(3+i5)\exp(i(47-68))$	$(1+i2)\exp(i(35+47-68))$	$-$
$15+46$	$14+23$	$(1+i5)(2+i3)\exp(i(48+67))$	$(2+i3)\exp(i(16+47+58))$	$-$
$15+26+34$	$14+23$	$(1+i4)(3+i5)\exp(i(28+67))$	$(1+i5)\exp(i(27+36-48))$	$-$
$13+25$	$16+24$	$(1+i3)(2+i4)\exp(i(57+68))$	$(1+i2)\exp(37+68+i(36+48+57))$	$-$
$34+56$	12	$(3+i4)(5+i6)\exp(i(17+28))$	$(3+i4)\exp(i(18+26+57))$	$-$
$16+25+34$	12	$(2+i5)(3+i4)\exp(i(18+67))$	$(3+i4)\exp(i(15+27+68))$	$-$

TABLE 6.4: Examples of strong generalized complex structures of type 2, type 1 and type 0 (symplectic) for two-step nilpotent Lie algebras of dimension eight and signature $(6,2)$.

Algebra			Structures	
$d6$	$d7$	$d8$	Type 4 (complex)	Type 3
12	13	14	$(1+i4)(2+i3)(5+i8)(6+i7)$	$(1+i5)(2+i3)(6+i7)\exp(i48)$
12	13	23	$(1+i4)(2+i3)(5+i8)(6+i7)$	$(1+i5)(2+i3)(6+i7)\exp(i48)$
12	13	$14+23$	$(1+i4)(2+i3)(5+i8)(6+i7)$	$(1+i5)(2+i3)(6+i7)\exp(i48)$
12	34	13	$(1+i4)(2+i3)(5-8+i7)(6+i8)$	$(1+i5)(2+i3)(6+i8)\exp(i47)$
12	34	$14+23$	$(1+i2)(3-i4)(5+i8)(6+i7)$	$(1+i2)(3-i4)(6+i7)\exp(i58)$
$14+23$	$-13+24$	12	$(1+i2)(3+i4)(5+i8)(6+i7)$	$(1+i5)(3+i4)(6+i7)\exp(i28)$
$14+23$	$-13+24$	$12+34$	$(1+i2)(3+i4)(5+i8)(6+i7)$	$(1+i2)(3+i4)(6+i7)\exp(i58)$
$12+45$	$25+34$	$15+23$	$(2+5-i5)(2\cdot 3+i4)(1+i7)(6+8-i8)$	$(2+5-i5)(2\cdot 3+i4)(6+8-i8)\exp(i17)$
$12+2\cdot(23+45)$	$25-2\cdot 34$	$-15+2\cdot(24-35)$	$(2-i5)(3+i4)(1+i7)(6+i8)$	$(2-i5)(3+i4)(6+i8)\exp(i17)$
$14+35$	$15+24$	23	$(2\cdot 2+i3)(4+5+i5)(1+i8)(6+7+i7)$	$(1+i4)(2+i5)(6+i8)\exp(i37)$
$14+24-35$	$-15+25+34$	-23	$(2+i3)(4-i5)(1+i8)(6+i7)$	$(2+i3)(4-i5)(6+i7)\exp(i18)$
$15+34$	23	$14+24$	$(1+2+i4)(3+i5)(2+i8)(6+i7)$	$(1+2+i4)(3+i5)(6+i7)\exp(i28)$
$15+34$	$14-23$	$13+24$	$(1+i5)(3-i4)(2+i6)(7+i8)$	$(1+i5)(3-i4)(7+i8)\exp(i26)$
$25+34$	$15+23$	12	$(1+i2)(2+3-i4)(5+i8)(6-i7)$	$(1+i2)(2+3-i4)(6-i7)\exp(i58)$
13	24	$15+25$	??	$(1+i3)(2+i4)(6+i7)\exp(i58)$
13	$15-24$	$14+25$	$(1+i3)(4-i5)(2+i6)(7+i8)$	$(1+i3)(4-i5)(7+i8)\exp(i26)$
24	$15+23$	13	??	$(1+i3)(2+i4)(6+i8)\exp(27+i57)$
$15+24$	$14+23$	13	??	$(1+i3)(2+i4)(7+i8)\exp(i56)$
$25+34$	12	15	$(1+i5)(3+i4)(2+i8)(6+i7)$	$(1+i5)(3+i4)(6+i7)\exp(i28)$
$13+25$	$15+24$	12	$(2\cdot 1+2+i2)(3+2i4)(5+i8)(6+7-i7)$	$(1+i5)(2-i3)(6+2i8)\exp(i47)$
$14+35$	23	12	$(1+i3)(4+i5)(2+i6)(7+i8)$	$(1+i3)(4+i5)(7+i8)\exp(i26)$
23	14	15	$(2+i3)(4+i5)(1+i6)(7+i8)$	$(2+i3)(4+i5)(7+i8)\exp(i16)$
$15+23$	14	12	$(1+i2)(3+4-i5)(4+i8)(6+i7)$	$(1+i5)(2-i4)(7+i8)\exp(i36)$

TABLE 6.5: Examples of strong generalized complex structures of type 4 (complex) and type 3 for two-step nilpotent Lie algebras of dimension eight and signature $(5,3)$.

Algebra			Structures		
$d6$	$d7$	$d8$	Type 2	Type 1	Type 0 (symplectic)
12	13	14	$(1+i4)(5+i8)\exp(i(26+37))$	$(1+i5)\exp(i(26+37+48))$	$15+26+37+48$
12	13	23	$(2+i3)(6+i7)\exp(i(15+48))$	$(4+i5)\exp(i(16+28+37))$	$16+28+37+45$
12	13	$14+23$	$(2+i3)(6+i7)\exp(i(15+48))$	$(1+i5)\exp(i(28+37+46))$	$15+28+37+46$
12	34	13	$(2+i3)(6+i8)\exp(i(15+47))$	$(1+i5)\exp(i(26+38+47))$	$15+26+38+47$
12	34	$14+23$	$(1+i2)(3-i4)\exp(i(58+67))$	$(1+i5)\exp(i(28+37+46))$	$15+28+37+46$
$14+23$	$-13+24$	12	$(1+i2)(3-i4)\exp(i(58+67))$	$(1+i5)\exp(i(28+36-47))$	$15+28+37+46$
$14+23$	$-13+24$	$12+34$	$(1+i2)(3-i4)\exp(i(58+67))$	$(1+i5)\exp(i(2\cdot 26+35+48))$	$17+2\cdot 26+35+48$
$12+45$	$25+34$	$15+23$	$(2+5-i5)(2\cdot 3+i4)\exp(2\cdot 38+67-78+i(-16+78))$	$(3+i5)\exp(i(16-27+48))$	$16-27+35+48$
$12+2\cdot(23+45)$	$25-2\cdot 34$	$-15+2\cdot(24-35)$	$(2-i5)(3+i4)\exp(78+i(18+67))$	$(1+i3)\exp(i(26+4\cdot 47+58))$	$13+26+4\cdot 47+58$
$14+35$	$15+24$	23	$(2+i3)(1+i8)\exp(i(47+56))$	$(1+i2)\exp(i(38+47+56))$	$12+38+47+56$
$14+24-35$	$-15+25+34$	-23	$(2+i3)(1+i8)\exp(i(46-57))$	$(1+i2)\exp(i(38+46-57))$	$12+38+46-57$
$15+34$	23	$14+24$	$(2+i3)(1+i7)\exp(46+58+i(48+56))$	$(2+i5)\exp(i(16+38+47))$	$16+25+38+47$
$15+34$	$14-23$	$13+24$	$(1+i2)(3+i4)\exp(i(56+78))$	$(2+i5)\exp(i(16+2\cdot 37+48))$	$16+25+2\cdot 37+48$
$25+34$	$15+23$	12	$(1+i2)(4+i8)\exp(i(36+57))$	$(1+i4)\exp(i(28+36+57))$	$14+28+36+57$
13	24	$15+25$	$(1+i3)(2+i4)\exp(i(58+67))$	$(1+i2)\exp(i(36+47+58))$	$12+36+47+58$
13	$15-24$	$14+25$	$(1+i2)(4+i5)\exp(i(36+78))$	$(1+i2)\exp(i(36+47+58))$	$12+36+47+58$
24	$15+23$	13	$(1+i3)(4+i8)\exp(27+i(26+57))$	$(3+i5)\exp(i(17+28+46))$	$17+28+35+46$
$15+24$	$14+23$	13	$(1+i3)(2+i4)\exp(i(56+78))$	$(4+i5)\exp(i(16+27+38))$	$16+27+38+45$
$25+34$	12	15	$(2+i5)(7+i8)\exp(i(13+46))$	$(3+i4)\exp(i(16+2\cdot 28+57))$	$-$
$13+25$	$15+24$	12	$(1+i2)(3+i8)\exp(37+56+i(46+57))$	$(3+i4)\exp(i(16+2\cdot 27+58))$	$16+2\cdot 27+34+58$
$14+35$	23	12	$(1+i3)(7+i8)\exp(46+i(24+56))$	$(1+i3)\exp(i(26+48-57))$	$13+26+48-57$
23	14	15	$(2+i3)(1+i6)\exp(i(47+58))$	$(1+i2)\exp(i(36+47+58))$	$12+36+47+58$
$15+23$	14	12	$(1+i2)(3-i5)\exp(i(47+68))$	$(2+i5)\exp(i(16-38+47))$	$16+25-38+47$

TABLE 6.6: Examples of strong generalized complex structures of type 2, type 1 and type 0 (symplectic) for two-step nilpotent Lie algebras of dimension eight and signature $(5,3)$.

Algebra				Structures	
$d5$	$d6$	$d7$	$d8$	Type 4 (complex)	Type 3
12	34	13	24	$(1+i4)(2+i3)(5-i8)(6+i7)$	$(1+i4)(5-i8)(6+i7)\exp(i23)$
12	34	$13-24$	$-14-23$	$(1+i2)(3+i4)(5+i6)(7-i8)$	$(1+i2)(3+i4)(7-i8)\exp(i56)$
13	24	$14+23$	12	$(1+i3)(2-i4)(5+i6)(7+2i8)$	$(1+i3)(2-i4)(7+2i8)\exp(i56)$
14	23	12	13	$(1+i4)(2+i3)(5+i6)(7+i8)$	$(1+i4)(2+i3)(7+i8)\exp(i56)$

TABLE 6.7: Examples of strong generalized complex structures of type 4 (complex) and type 3 for two-step nilpotent Lie algebras of dimension eight and signature $(4,4)$.

Algebra				Structures		
$d5$	$d6$	$d7$	$d8$	Type 2	Type 1	Type 0 (symplectic)
12	34	13	24	$(1+i4)(6+i7)\exp(38+i(28+35))$	$(1+i4)\exp(57+68+i(23-56+78))$	$15+28+37+46$
12	34	$13-24$	$-14-23$	$(1+i2)(7-i8)\exp(i(36+45))$	$(1+i2)\exp(48-57+i(36+47+58))$	$17-28+36+2\cdot 45$
13	24	$14+23$	12	$(1+i4)(6+i8)\exp(i(27+35))$	$(1+i2)\exp((5-6)8+i(35+46+78))$	$15+28+36+47$
14	23	12	13	$(2+i3)(7+i8)\exp(i(15+46))$	$(1+i2)\exp(78+i(38+45-67))$	$18+27+36+45$

TABLE 6.8: Examples of strong generalized complex structures of type 2, type 1 and type 0 (symplectic) for two-step nilpotent Lie algebras of dimension eight and signature $(4,4)$.

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