First and second order optimality conditions for the control of infinite horizon Navier-Stokes equations

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ABSTRACT
First and second order optimality conditions for optimal control problems over the infinite time horizon subject to the Navier Stokes equations are derived. The cost functional enhances temporal sparsity of the controls, which implies that the optimal controls shut down in finite time. The problem formulation also includes explicit constraints on the control which may be non-smooth and non-affine.

KEYWORDS
Infinite horizon optimal control, Navier Stokes equations, second order optimality conditions, sparsity promoting cost functional, nonsmooth optimization, Euclidean norm constraints.

AMS CLASSIFICATION
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1. Introduction

This paper concerns the following optimal control problem

$$\text{(P)} \quad \min_{\mathbf{u}(t)\in\mathbf{K} \text{ for a.a } t\in I} J(\mathbf{u}),$$

where

$$J(\mathbf{u}) := \frac{1}{2}\int_I \|\mathbf{y}_\mathbf{u}(t) - \mathbf{y}_d(t)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t + \frac{\alpha}{2}\int_I \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t + \beta\int_I \|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t,$$

I denotes the infinite horizon $(0,\infty)$, $\alpha,\beta \geq 0$ with $\alpha+\beta>0$, $\mathbf{y_u}$ is the solution of the Navier-Stokes system

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla\mathfrak{p} = \mathbf{f}_0 + \chi_\omega \mathbf{u} \ \text{ in } Q = \Omega\times I, \\ \operatorname{div}\mathbf{y} = 0 \ \text{ in } Q, \ \mathbf{y} = 0 \ \text{ on } \Sigma = \Gamma\times I, \ \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega, \end{cases} \tag{1}$$

and $\mathbf{K}$ is a closed, bounded, and convex subset of $\mathbf{L}^\sigma(\omega)$ with $\sigma\in[1,\infty]$. Further Ω denotes a bounded domain in $\mathbb{R}^2$ with a C^3 boundary Γ, and ω is a subset of Ω with positive Lebesgue measure. Assumptions on $\mathbf{f}_0$ and $\mathbf{y}_d$ will be made below.

The specificities of this problem are the following: it is posed over the infinite time horizon $[0,\infty)$, it may contain a non-differential term for the control cost, and it allows quite general explicit constraints on the control. Concerning the infinite time horizon, observe that due to the energy conserving property of the Navier-Stokes nonlinearity, the existence of feasible controls does not create a difficulty and the control problem (P) is really on optimal control problem and not a stabilization problem in disguise. But still, standard results from the analysis of optimality conditions for optimal control over finite horizons cannot directly be utilized here, since typically the dependence of constants on the time horizon within a priori estimates, for the adjoint equation, for instance, is not analyzed. The L^1 term with respect to time in the cost of the control is sparsity promoting. As a consequence the control will shut off and be identically zero, for all t sufficiently large, if $\beta\neq 0$. To the best of our knowledge such a term has not been considered for optimal control of Navier-Stokes equations before. Concerning explicit constraints on vector valued controls, the existing literature almost exclusively treats affine coordinate-wise constraints, whereas we allow integral constraints which involve the Euclidean norm of the vector-valued controls. A second order analysis for this kind of constraints has apparently not been carried out in the PDE-constrained optimal control literature before. It can also be of use for vector-valued controls independently of the Navier Stokes context.

On a technical level, the difficulties which need to be overcome include the following: On the infinite time horizon the Aubin-Lions lemma does not hold. The resulting lack of compactness necessitates to treat separately the long time behavior of the solutions to the primal as well as the adjoint equations. Differently from the finite horizon case, the transversality condition in the first order optimality system is more complicated to specify. It amounts to characterizing the behavior of the adjoint state as time tends to infinity. Within the derivation of second order necessary conditions the construction of a sufficiently rich set of feasible directions approximating the optimal control, is quite delicate. As we shall see it requires to take into consideration the geometry of the set of admissible controls. Let us point out here, that our constraints are not of affine or polygonal nature, rather we can think of them as to allow curved boundaries.

Let us mention some of the literature which is related to the contributions of this paper. Optimal control for problems over an infinite time horizon has been investigated by the authors of this paper for semilinear parabolic equations in [14,16], and in [5] for bilinear control problems. The sparsifying effect for stabilization problems was first pointed out [11]. Differential from the PDE-context, infinite horizon optimal control for ordinary differential equation has received much attention. Likely its analysis started with Halkin's work [22]. Much of the earlier work is described in [7]. More recent contributions can be found for instance in [3] and [4].

Concerning optimal control of the Navier Stokes equations there is a vast literature; see, for instance, the monographs [20] and [26] and the references there in. Thus, we restrict ourselves to those publications which are directly concerned with the research of our paper. Regarding second order analysis for Navier-Stokes control problems we mention [8], [29], and [30]. In these references, the control constraints are of pointwise type. In [21], first order conditions were derived for constraints of type $\|u(t)\|_{L^2(\Omega)} \leq \gamma$. We are not aware of any result on infinite horizon open loop control for the Navier-Stokes equations and second order conditions for constraints which are not of pointwise type. New estimates involving the asymptotic behavior as $t \to \infty$ were necessary to deal with the state and adjoint state equations; see Lemma 2.5 and Theorem 3.2.

We turn to a brief description of the results of this paper. Section 2 contains the state space analysis of the 2-D Navier Stokes equation on the temporal interval $[0, \infty)$ in the setting which is relevant for the remainder of the paper. In particular this means that the forcing function is admitted to be of low regularity only, so that the case $\sigma = 1$ is included. The analysis of the optimal control problem and first formulations of first-order necessary conditions are given in Section 3. Here we distinguish the cases whether α, respectively β are zero or not. In Section 4 we choose $\mathbf{K}$ the closed ball in $\mathbf{L}^\sigma(\omega)$ centered at zero with radius $\gamma > 0$, and consider separately the cases $\sigma = 1$, $\sigma = 2$, and $\sigma = \infty$. In these cases the vector norm on the control vectors is chosen as the Euclidean norm in $\mathbb{R}^2$. More detailed first order conditions than in Section 3, as well as necessary and sufficient second order conditions are given for these cases. The Appendix contains the proofs for two technical lemmas from Section 2.

We mention that the second order analysis carried out in section 4 for the cases $\sigma = 1$ and $\sigma = 2$ is new and its treatment is very different from the frequently studied case $\sigma = \infty$. Even, the case $\sigma = \infty$ cannot be treated in the usual way due to the fact that the control takes vector values and the constraints are not imposed separately on each component of $\mathbf{u}(t)$, but on its Euclidean norm; see Theorem 4.15. In the case of semilinear parabolic equations on finite horizon and a scalar control, the case $\sigma = 1$ was studied in [15]. The proof of second order necessary conditions for $\sigma = 1$ given in section 4.2 is inspired by arguments used in [10] and [15] for finite horizon problems. In [10] we consider measure value controls. An extended cone was required to establish the sufficient second order conditions. In [15], the term promoting sparsity was not present. The technique of proof for $\sigma = 2$ is new and completely different.

Notation

We denote $\mathbf{W}_0^{1,s}(\Omega) = W_0^{1,s}(\Omega) \times W_0^{1,s}(\Omega)$ for $s \in (1, \infty)$, endowed with the norm

$$\|\mathbf{y}\|_{\mathbf{W}_0^{1,s}(\Omega)} = \|\nabla \mathbf{y}\|_{\mathbf{L}^s(\Omega)} = \left(\int_\Omega |\nabla \mathbf{y}|^s \, dx\right)^{\frac{1}{s}} = \left(\int_\Omega [|\nabla y_1|^2 + |\nabla y_2|^2]^{\frac{s}{2}} \, dx\right)^{\frac{1}{s}}.$$

As usual, for $s = 2$ we set $\mathbf{H}_0^1(\Omega) = \mathbf{W}_0^{1,2}(\Omega)$. We also consider the spaces

$$\mathbf{H} = \text{closure of } \{\boldsymbol{\phi} \in \mathbf{C}_0^\infty(\Omega) : \ \operatorname{div} \boldsymbol{\phi} = 0\} \text{ in } \mathbf{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega),$$
$$\mathbf{W}_s(\Omega) = \{\mathbf{y} \in \mathbf{W}_0^{1,s}(\Omega) : \operatorname{div} \mathbf{y} = 0\}, \quad \mathbf{V} = \mathbf{W}_2(\Omega).$$

For $r, s \in (1, \infty)$ and $0 < T \leq \infty$ we define the reflexive Banach spaces

$$\mathbf{W}_{r,s}(0,T) = \{\mathbf{y} \in L^r(I; \mathbf{W}_s(\Omega)) : \frac{\partial \mathbf{y}}{\partial t} \in L^r(I; \mathbf{W}_{s'}(\Omega)')\},$$
$$\mathbf{W}_r^{2,1}(0,T) = \{\mathbf{y} \in L^r(I; \mathbf{H}^2(\Omega) \cap \mathbf{V}) : \frac{\partial \mathbf{y}}{\partial t} \in L^r(I; \mathbf{H})\},$$

with the norms

$$\|\mathbf{y}\|_{\mathbf{W}_{r,s}(0,T)} = \|\mathbf{y}\|_{L^r(I;\mathbf{W}_0^{1,s}(\Omega))} + \|\frac{\partial \mathbf{y}}{\partial t}\|_{L^r(I;\mathbf{W}_{s'}(\Omega)')},$$
$$\|\mathbf{y}\|_{\mathbf{W}_r^{2,1}(0,T)} = \|\mathbf{y}\|_{L^r(I;\mathbf{H}^2(\Omega))} + \|\frac{\partial \mathbf{y}}{\partial t}\|_{L^r(I;\mathbf{H})}.$$

Above s' stands for the conjugate of s: $s' = \frac{s}{s-1}$. If $r = s = 2$ we denote $\mathbf{W}(0,T) = \mathbf{W}_{2,2}(0,T)$ and $\mathbf{V}^{2,1}(0,T) = \mathbf{W}_r^{2,1}(0,T)$. In the case $T = \infty$ the notation $(0,T)$ will be replaced by I in the above spaces.

Now we consider the interpolation spaces $\mathbf{B}_{s,r}(\Omega) = (\mathbf{W}_{s'}(\Omega)', \mathbf{W}_s(\Omega))_{1-1/r,r}$. From [1, Chap. III/4.10.2] we know that $\mathbf{W}_{r,s}(0,T) \subset C([0,T]; \mathbf{B}_{s,r}(\Omega))$ and the trace mapping $\mathbf{y} \in \mathbf{W}_{r,s}(0,T) \to \mathbf{y}(0) \in \mathbf{B}_{s,r}(\Omega)$ is surjective. If $r = s = 2$, then it is known that $\mathbf{B}_{2,2}(\Omega) = (\mathbf{V}', \mathbf{V})_{\frac{1}{2},2} = \mathbf{H}$. Hence, the embedding $\mathbf{W}(0,T) \subset C([0,T]; \mathbf{H})$ holds; see [23, Page 22, Proposition I-2.1] and [28, Page 143, Remark 3].

2. Analysis of the state equation

The aim of this section is to study the well-posedness of the following problem

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial t} - \nu \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla \mathfrak{p} = \mathbf{f} \ \text{ in } Q, \\ \operatorname{div} \mathbf{y} = 0 \ \text{ in } Q, \ \mathbf{y} = 0 \ \text{ on } \Sigma, \ \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega, \end{cases} \tag{2}$$

where $\nu > 0$, $\mathbf{f} \in L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega))$, and $\mathbf{y}_0 \in \mathbf{B}_{2,4}(\Omega) + \mathbf{B}_{p,q}(\Omega)$. The parameters p and q are fixed throughout this manuscript, and it is assumed that

$$\frac{4}{3} \leq p < 2 \text{ and } q \geq 8 \tag{3}$$

holds. All the results of this paper remain valid if we assume that $p \in [2, \infty)$ because of the embedding $\mathbf{W}_0^{1,s}(\Omega) \subset \mathbf{W}_0^{1,p}(\Omega)$ for every $s > p$. For $p < 2$ we have that $\mathbf{L}^1(\Omega) \subset \mathbf{W}^{-1,p}(\Omega)$, which is necessary to deal with the case where $\mathbf{f} = \mathbf{f}_0 + \chi_\omega \mathbf{u}$ with controls $\mathbf{u} \in L^\infty(I; \mathbf{L}^1(\omega))$. We give the results for $p < 2$ to simplify the presentation avoiding different cases depending on the value of p. Moreover, the analysis for $p \geq 2$ is simpler.

Now we introduce the following spaces:

$$\mathbf{Y} = [L^2(I; \mathbf{V}) \cap L^\infty(I; \mathbf{H})] + [L^q(I; \mathbf{W}_p(\Omega)) \cap L^4(I; \mathbf{W}_p(\Omega))], \ \ \mathbf{Y}_0 = \mathbf{H} + \mathbf{B}_{p,q}(\Omega),$$
$$\mathcal{Y} = \mathbf{W}_{4,2}(I) + \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I), \qquad \mathcal{Y}_0 = \mathbf{B}_{2,4}(\Omega) + \mathbf{B}_{p,q}(\Omega),$$

and observe that $\mathcal{Y} \subset \mathbf{Y}$ and $\mathcal{Y}_0 \subset \mathbf{Y}_0$. They are Banach spaces for the canonical norms, for instance, for $\mathbf{Y}$ and $\mathcal{Y}$ we have

$$\|\mathbf{y}\|_{\mathbf{Y}} = \inf_{\mathbf{y}=\mathbf{y}_1+\mathbf{y}_2} \|\mathbf{y}_1\|_{L^2(I;\mathbf{V})} + \|\mathbf{y}_1\|_{L^\infty(I;\mathbf{H})} + \|\mathbf{y}_2\|_{L^q(I;\mathbf{W}_p(\Omega))} + \|\mathbf{y}_2\|_{L^4(I;\mathbf{W}_p(\Omega))},$$
$$\|\mathbf{y}\|_{\mathcal{Y}} = \inf_{\mathbf{y}=\mathbf{y}_1+\mathbf{y}_2} \|\mathbf{y}_1\|_{\mathbf{W}_{4,2}(I)} + \|\mathbf{y}_2\|_{\mathbf{W}_{q,p}(I)} + \|\mathbf{y}_2\|_{\mathbf{W}_{4,p}(I)}.$$

The choice of these spaces is inspired by those chosen for measure-valued controls in [12,13], adapted to the infinite horizon case which leads to the power 4 for the Sobolev index in time in the above definitions of $\mathbf{Y}$ and $\mathcal{Y}$. They are sufficiently large such that controls in $L^q(I;\mathbf{L^1}(\omega)) \cap \mathbf{L^4}(\mathbf{I};\mathbf{L^1}(\omega))$ and quite general initial conditions are admitted. Moreover they allow first and second order derivatives for the control to state mapping for the choice $\mathcal{Y}$.

We note that for the finite horizon the continuous embedding $\mathbf{W}_{q,p}(0,T) \subset \mathbf{W}_{4,p}(0,T)$ is fulfilled. Hence, we have the embedding of the trace spaces $\mathbf{B}_{p,q}(\Omega) \subset \mathbf{B}_{p,4}(\Omega)$. Therefore, taking into account that the image space for the trace mappings $\mathbf{y} \in \mathbf{W}_{q,p}(0,T) \to \mathbf{y}(0) \in \mathbf{B}_{p,q}(\Omega)$ is the same for any finite interval $(0,T)$ as for I, the continuity and surjectivity of the mapping $\mathbf{y} \in \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I) \to \mathbf{y}(0) \in \mathbf{B}_{p,q}(\Omega)$ follows.

In order to define the notion of solution of (2), we need the following technical lemma, whose proof is given in the Appendix.

Lemma 2.1. *If* (3) *holds, the bilinear operators* $B : \mathbf{Y} \times \mathbf{Y} \longrightarrow L^2(I;\mathbf{H}^{-1}(\Omega))$ *and* $\mathcal{B} : \mathcal{Y} \times \mathcal{Y} \longrightarrow L^4(I;\mathbf{H}^{-1}(\Omega))$ *defined by* $B(\mathbf{y}_1,\mathbf{y}_2) = \mathcal{B}(\mathbf{y}_1,\mathbf{y}_2) = (\mathbf{y}_1 \cdot \nabla)\mathbf{y}_2$ *are continuous.*

As usual, we can remove the pressure from the equation (2) by using divergence free test functions.

Definition 2.2. We say that $\mathbf{y} \in \mathbf{W}(I) + \mathbf{W}_{q,p}(I)$ is a variational solution of (2) if for almost every $t \in I$

$$\begin{cases} \langle \frac{d}{dt}\mathbf{y}(t), \boldsymbol{\psi}\rangle_{\mathbf{W}_{p'}(\Omega)',\mathbf{W}_{p'}(\Omega)} + a(\mathbf{y}(t),\boldsymbol{\psi}) + b(\mathbf{y}(t),\mathbf{y}(t),\boldsymbol{\psi}) \\ = \langle \mathbf{f}(t),\boldsymbol{\psi}\rangle_{\mathbf{W}^{-1,p}(\Omega),\mathbf{W}_0^{1,p'}(\Omega)} \quad \forall \boldsymbol{\psi} \in \mathbf{W}_{p'}(\Omega), \\ \mathbf{y}(0) = \mathbf{y}_0, \end{cases} \tag{4}$$

where

$$a(\mathbf{y}(t),\boldsymbol{\psi}) = \nu \int_\Omega \nabla\mathbf{y}(x,t) : \nabla\boldsymbol{\psi}(x)\,dx = \nu \sum_{i=1}^{2} \int_\Omega \nabla\mathbf{y}_i(x,t)\nabla\boldsymbol{\psi}_i(x)\,dx,$$
$$b(\mathbf{y}(t),\mathbf{y}(t),\boldsymbol{\psi}) = \langle B(\mathbf{y}(t),\mathbf{y}(t)),\boldsymbol{\psi}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = \int_\Omega [\mathbf{y}(t)\cdot\nabla]\mathbf{y}(t)\cdot\nabla\boldsymbol{\psi}\,dx.$$

A distribution $\mathfrak{p}$ in Q is called an associated pressure if the equation

$$\frac{\partial\mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla\mathfrak{p} = \mathbf{f} \ \text{ in } Q$$

is satisfied in the distribution sense. Then, $(\mathbf{y},\mathfrak{p})$ is called a solution of (2).

Given $\mathbf{y}$ satisfying (4), the pressure $\mathfrak{p}$ is obtained by using De Rham's theorem; see [25, Lemma IV-1.4.1]. The next theorem is the main result of this section.

Theorem 2.3. *Suppose that* (3) *holds. Then, system* (2) *has a unique solution* $(\mathbf{y}, \mathfrak{p}) \in [\mathbf{W}(I) + \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)] \times [W^{-1,q}(I; L^p(\Omega)/\mathbb{R}) \cap W^{-1,4}(I; L^p(\Omega)/\mathbb{R})]$ *for every* $\mathbf{y}_0 \in \mathbf{Y}_0$ *and* $\mathbf{f} \in L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega))$*, and there exists a nondecreasing continuous function* $\eta_{p,q} : [0, \infty) \longrightarrow [0, \infty)$ *with* $\eta_{p,q}(0) = 0$ *such that*

$$\|\mathbf{y}\| \leq \eta_{p,q}\Big(\|\mathbf{f}\|_{L^q(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{f}\|_{L^4(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{y}_0\|_{\mathbf{Y}_0}\Big), \tag{5}$$

where $\|\cdot\|$ *denotes the norm in* $\mathbf{W}(I) + \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)$*. Furthermore, if* $\mathbf{y}_0 \in \mathcal{Y}_0$*, then the regularity* $\mathbf{y} \in \mathcal{Y}$ *holds.*

We establish two lemmas to carry out the proof of this theorem.

Lemma 2.4. *Given* $\mathbf{g} \in L^r(I; \mathbf{W}^{-1,s}(\Omega))$ *and* $\mathbf{y}_{S0} \in \mathbf{B}_{s,r}(\Omega)$ *with* $1 < r, s < \infty$*, there exists a unique solution* $(\mathbf{y}_S, \mathfrak{p}_S) \in \mathbf{W}_{r,s}(I) \times W^{-1,r}(I; L^s(\Omega)/\mathbb{R})$ *of the following equation*

$$\begin{cases} \dfrac{\partial \mathbf{y}_S}{\partial t} - \nu\Delta\mathbf{y}_S + \nabla\mathfrak{p}_S = \mathbf{g} \quad \text{in } Q, \\ \operatorname{div}\mathbf{y}_S = 0 \quad \text{in } Q,\ \mathbf{y}_S = 0 \quad \text{on } \Sigma,\ \mathbf{y}_S(0) = \mathbf{y}_{S0} \ \text{in } \Omega. \end{cases} \tag{6}$$

Moreover, there exists a constant $C_{r,s}$ *independent of* $(\mathbf{g}, \mathbf{y}_{S0})$ *such that*

$$\|\mathbf{y}_S\|_{\mathbf{W}_{r,s}(I)} \leq C_{r,s}\Big(\|\mathbf{g}\|_{L^r(I;\mathbf{W}_{s'}(\Omega)')} + \|\mathbf{y}_{S0}\|_{\mathbf{B}_{s,r}(\Omega)}\Big). \tag{7}$$

The reader is referred to [13, Therem 2.5] for the proof of this result, where the C^3 regularity of Γ is needed to use the maximal parabolic regularity for the Stokes system. There the proof was made for finite horizon intervals $(0, T)$, but the same is valid without changes for I. The only issue to take into account is that the maximal parabolic regularity results used there are also valid for infinite horizon intervals; see [18] or [19].

Lemma 2.5. *Given* $(\mathbf{g}, \mathbf{y}_{N0}) \in L^2(I; \mathbf{H}^{-1}(\Omega)) \times \mathbf{H}$*,* $\mathbf{e}_1, \mathbf{e}_2 \in \mathbf{Y}$*, and* $\nu_0 \geq 0$*, the system*

$$\begin{cases} \dfrac{\partial \mathbf{y}_N}{\partial t} - \nu\Delta\mathbf{y}_N + \nu_0(\mathbf{y}_N \cdot \nabla)\mathbf{y}_N + (\mathbf{e}_1 \cdot \nabla)\mathbf{y}_N + (\mathbf{y}_N \cdot \nabla)\mathbf{e}_2 + \nabla\mathfrak{p}_N = \mathbf{g} \quad \text{in } Q, \\ \operatorname{div}\mathbf{y}_N = 0 \quad \text{in } Q,\ \mathbf{y}_N = 0 \quad \text{on } \Sigma,\ \mathbf{y}_N(0) = \mathbf{y}_{N0} \ \text{in } \Omega \end{cases} \tag{8}$$

has a unique solution $(\mathbf{y}_N, \mathfrak{p}_N) \in \mathbf{W}(I) \times W^{-1,\infty}(I; L^2(\Omega)/\mathbb{R})$*. Furthermore, there* *exists a nondecreasing function* $\eta_N : [0,\infty) \longrightarrow (0,\infty)$ *such that*

$$\begin{aligned}
&\|\mathbf{y}_N\|_{L^2(I;\mathbf{H}_0^1(\Omega))} + \|\mathbf{y}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} \le \eta_N\Big(\|\mathbf{e}_2\|_{\mathbf{Y}}\Big)\Big(\|\mathbf{g}\|_{L^2(I;\mathbf{V}')} + \|\mathbf{y}_{N0}\|_{\mathbf{L}^2(\Omega)}\Big),\\
&\|\mathbf{y}_N\|_{\mathbf{W}(I)} \le \nu_0\eta_N^2(\|\mathbf{e}_2\|_{\mathbf{Y}})\Big(\|\mathbf{g}\|_{L^2(I;\mathbf{V}')} + \|\mathbf{y}_{N0}\|_{\mathbf{L}^2(\Omega)}\Big)^2\\
&\qquad +[(1+\nu+\|\mathbf{e}_1\|_{\mathbf{Y}}+\|\mathbf{e}_2\|_{\mathbf{Y}})\eta_N(\|\mathbf{e}_2\|_{\mathbf{Y}})+1]\Big(\|\mathbf{g}\|_{L^2(I;\mathbf{V}')} + \|\mathbf{y}_{N0}\|_{\mathbf{L}^2(\Omega)}\Big).
\end{aligned} \tag{9}$$

In addition, if $\mathbf{g} \in L^4(I;\mathbf{H}^{-1}(\Omega))$, $\mathbf{e}_1, \mathbf{e}_2 \in L^8(I;\mathbf{L}^4(\Omega))$, *and* $\mathbf{y}_{N0} \in \mathbf{B}_{2,4}(\Omega)$, *then the regularity* $\mathbf{y}_N \in \mathbf{W}_{4,2}(I)$ *holds.*

The proof is carried out in the Appendix.

Proof of Theorem 2.3. Using [13, Theorem 2.4] and arguing as in the proof of Lemma 2.5, we infer the existence and uniqueness of a solution $(\mathbf{y},\mathfrak{p}) \in [\mathbf{W}(I) + \mathbf{W}_{q,p}(I)]\times W^{-1,q}(I;L^p(\Omega)/\mathbb{R})$ of (2) as well as the estimate (5). To prove the additional regularity under the assumption $\mathbf{y}_0 \in \mathcal{Y}_0$ we decompose (2) into the following two systems

$$\begin{cases} \dfrac{\partial \mathbf{y}_S}{\partial t} - \nu\Delta\mathbf{y}_S + \nabla\mathfrak{p}_S = \mathbf{f} \ \text{ in } Q,\\ \operatorname{div}\mathbf{y}_S = 0 \ \text{ in } Q,\ \mathbf{y}_S = 0 \ \text{ on } \Sigma,\ \mathbf{y}_S(0) = \mathbf{y}_{S0} \text{ in } \Omega, \end{cases} \tag{10}$$

$$\begin{cases} \dfrac{\partial \mathbf{y}_N}{\partial t} - \nu\Delta\mathbf{y}_N + (\mathbf{y}_N\cdot\nabla)\mathbf{y}_N + (\mathbf{y}_S\cdot\nabla)\mathbf{y}_N + (\mathbf{y}_N\cdot\nabla)\mathbf{y}_S + \nabla\mathfrak{p}_N\\ \qquad\qquad = -(\mathbf{y}_S\cdot\nabla)\mathbf{y}_S \ \text{ in } Q,\\ \operatorname{div}\mathbf{y}_N = 0 \ \text{ in } Q,\ \mathbf{y}_N = 0 \ \text{ on } \Sigma,\ \mathbf{y}_N(0) = \mathbf{y}_{N0} \text{ in } \Omega, \end{cases} \tag{11}$$

where $\mathbf{y}_0 = \mathbf{y}_{N0}+\mathbf{y}_{S0}$. From Lemma 2.4 we infer the existence and uniqueness of a solution $(\mathbf{y}_S,\mathfrak{p}_S) \in [\mathbf{W}_{q,p}(I)\times W^{-1,q}(I;L^p(\Omega)/\mathbb{R})]\cap[\mathbf{W}_{4,p}(I)\times W^{-1,4}(I;L^p(\Omega)/\mathbb{R})]$ and the estimate (6) holds with $s = p$, $r = q$ and also $r = 4$. For equation (11) we observe that the right hand side $(\mathbf{y}_S\cdot\nabla)\mathbf{y}_S$ is an element of $L^2(I;\mathbf{H}^{-1}(\Omega))$, which follows from Lemma 2.1. Then, applying Lemma 2.5 with $\nu_0 = 1$ and $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{y}_S$ we get the existence and uniqueness of a solution $(\mathbf{y}_N,\mathfrak{p}_N) \in \mathbf{W}(I)\times W^{-1,\infty}(I;L^2(\Omega)/\mathbb{R})$ satisfying the estimate (9). Looking at (11), we also deduce that $\mathfrak{p}_N \in W^{-1,2}(I;L^2(\Omega)/\mathbb{R})$; see [25, Lemma IV-1.4.1]. Then, by interpolation we get that $\mathfrak{p}_N \in W^{-1,q}(I;L^2(\Omega)/\mathbb{R}) \cap W^{-1,4}(I;L^p(\Omega)/\mathbb{R})$. Finally, using the embedding $L^2(\Omega) \subset L^p(\Omega)$ we conclude that $\mathfrak{p}_N \in W^{-1,q}(I;L^p(\Omega)/\mathbb{R})$. Now, it is immediate to check that $(\mathbf{y},\mathfrak{p}) = (\mathbf{y}_N+\mathbf{y}_S,\mathfrak{p}_N+\mathfrak{p}_S) \in [\mathbf{W}(I)+\mathbf{W}_{q,p}(I)\cap\mathbf{W}_{4,p}(I)]\times[W^{-1,q}(I;L^p(\Omega)/\mathbb{R})\cap W^{-1,4}(I;L^p(\Omega)/\mathbb{R})]$ solves the equation (2). The uniqueness was established in [13, Theorem 2.4].

Now we assume that $\mathbf{y}_0 \in \mathcal{Y}_0$ and prove that $\mathbf{y} \in \mathcal{Y}$. By Lemma 2.1 we get that $(\mathbf{y}_S\cdot\nabla)\mathbf{y}_S \in L^4(I;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{y}_S \in L^8(I;\mathbf{L}^4(\Omega))$. Then, we infer from Lemma 2.5 that $\mathbf{y}_N \in \mathbf{W}_{4,2}(I)$. Hence, the regularity $\mathbf{y} = \mathbf{y}_N+\mathbf{y}_S \in \mathbf{W}_{4,2}(I)+[\mathbf{W}_{q,p}(I)\cap\mathbf{W}_{4,p}(I)] = \mathcal{Y}$ holds. □

We finish this section analyzing the differentiability of the mapping

$$\mathcal{G} : L^q(I;\mathbf{W}^{-1,p}(\Omega)) \cap L^4(I;\mathbf{W}^{-1,p}(\Omega)) \longrightarrow \mathcal{Y}$$

associating to each element $\mathbf{f} \in L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega))$ the solution $\mathbf{y_f} \in \mathcal{Y}$ of (4).

Theorem 2.6. *The mapping $\mathcal{G}$ is of class C^∞. Further, given $\mathbf{f}, \mathbf{g}, \mathbf{g}_1, \mathbf{g}_2 \in L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega))$ we have that $\mathbf{z_g} = \mathcal{G}'(\mathbf{f})\mathbf{g}$ and $\mathbf{z}_{\mathbf{g}_1,\mathbf{g}_2} = \mathcal{G}''(\mathbf{f})(\mathbf{g}_1, \mathbf{g}_2)$ are the unique solutions of the systems*

$$\left\{ \begin{array}{l} \dfrac{\partial \mathbf{z}}{\partial t} - \nu \Delta \mathbf{z} + (\mathbf{y_f} \cdot \nabla)\mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{y_f} + \nabla \mathfrak{q} = \mathbf{g} \quad \text{in } Q, \\ \operatorname{div} \mathbf{z} = 0 \quad \text{in } Q, \ \mathbf{z} = 0 \quad \text{on } \Sigma, \ \mathbf{z}(0) = 0 \ \text{in } \Omega, \end{array} \right. \tag{12}$$

and

$$\left\{ \begin{array}{l} \dfrac{\partial \mathbf{z}}{\partial t} - \nu \Delta \mathbf{z} + (\mathbf{y_f} \cdot \nabla)\mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{y_f} + \nabla \mathfrak{q} = -(\mathbf{z}_{\mathbf{g}_2} \cdot \nabla)\mathbf{z}_{\mathbf{g}_1} - (\mathbf{z}_{\mathbf{g}_1} \cdot \nabla)\mathbf{z}_{\mathbf{g}_2} \quad \text{in } Q, \\ \operatorname{div} \mathbf{z} = 0 \quad \text{in } Q, \ \mathbf{z} = 0 \quad \text{on } \Sigma, \ \mathbf{z}(0) = 0 \ \text{in } \Omega, \end{array} \right. \tag{13}$$

respectively, where $\mathbf{y_f} = \mathcal{G}(\mathbf{f})$ and $\mathbf{z}_{\mathbf{g}_i} = \mathcal{G}'(\mathbf{f})\mathbf{g}_i$ for $i = 1, 2$.

Proof. Let us define the space $\mathcal{W} = L^4(I; \mathbf{V}') + L^q(I; \mathbf{W}_{p'}(\Omega)') \cap L^4(I; \mathbf{W}_{p'}(\Omega)')$ endowed with the norm

$$\|\mathbf{h}\|_{\mathcal{W}} = \inf\{\|\mathbf{h}_1\|_{L^4(I;\mathbf{V}')} + \|\mathbf{h}_2\|_{L^q(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{h}_2\|_{L^4(I;\mathbf{W}_{p'}(\Omega)')} : \mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2\}.$$

Thus, $\mathcal{W}$ is a Banach space. We also consider the operators $A_{\mathbf{V}} : \mathbf{V} \longrightarrow \mathbf{V}'$ and $A_{\mathbf{W}} : \mathbf{W}_p(\Omega) \longrightarrow \mathbf{W}_{p'}(\Omega)'$ given by

$$\langle A_{\mathbf{V}}\mathbf{y}, \mathbf{z}\rangle_{\mathbf{V}',\mathbf{V}} = \nu \int_\Omega \nabla \mathbf{y} : \nabla \mathbf{z}\, dx, \quad \forall \mathbf{y}, \mathbf{z} \in \mathbf{V},$$
$$\langle A_{\mathbf{W}}\mathbf{y}, \mathbf{z}\rangle_{\mathbf{W}_{p'}(\Omega)',\mathbf{W}_p(\Omega)} = \nu \int_\Omega \nabla \mathbf{y} : \nabla \mathbf{z}\, dx, \quad \forall (\mathbf{y}, \mathbf{z}) \in \mathbf{W}_p(\Omega) \times \mathbf{W}_{p'}(\Omega).$$

Associated with these two continuous operators we define

$$A : \mathcal{Y} \longrightarrow \mathcal{W}, \quad A\mathbf{y} = A_{\mathbf{V}}\mathbf{y}_1 + A_{\mathbf{W}}\mathbf{y}_2,$$

where $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ with $\mathbf{y}_1 \in \mathbf{W}_{4,2}(I)$ and $\mathbf{y}_2 \in \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)$. It is immediate to check that $A\mathbf{y}$ is independent of the chosen representation $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$, and it is continuous. Now, we introduce the mapping

$$\mathcal{F} : \mathcal{Y} \times L^q(I; \mathbf{W}_{p'}(\Omega)') \cap L^4(I; \mathbf{W}_{p'}(\Omega)') \longrightarrow \mathcal{W} \times \mathcal{Y}_0,$$
$$\mathcal{F}(\mathbf{y}, \mathbf{f}) = \Big(\frac{\partial \mathbf{y}}{\partial t} + A\mathbf{y} + \mathcal{B}(\mathbf{y}, \mathbf{y}) - \mathbf{f}, \mathbf{y}(0) - \mathbf{y}_0\Big),$$

where $\mathbf{y}_0 \in \mathcal{Y}_0$ is the initial condition in (4). Recall that $\mathcal{Y} \subset C(I; \mathcal{Y}_0)$ holds. Hence, $\mathbf{y} \in \mathcal{Y} \to \mathbf{y}(0) \in \mathcal{Y}_0$ is a linear and continuous mapping. Moreover, Lemma 2.1 implies that $\mathbf{y} \in \mathcal{Y} \to \mathcal{B}(\mathbf{y}, \mathbf{y}) \in L^4(I; \mathbf{H}^{-1}(\Omega)) \subset L^4(I; \mathbf{V}') \subset \mathcal{W}$ is bilinear and continuous. By definition of $\mathbf{W}_{4,2}(I)$ and $\mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)$ we also have that $\frac{\partial}{\partial t} : \mathcal{Y} \to \mathcal{W}$ is a linear and continuous operator. All together this implies that $\mathcal{F}$ is a C^∞ mapping.

Given $\mathbf{f} \in L^q(I; \mathbf{W}_{p'}(\Omega)') \cap L^4(I; \mathbf{W}_{p'}(\Omega)')$, we denote by $\mathbf{y_f} \in \mathcal{Y}$ the solution of (4). Then, we have that

$$\begin{aligned} &\frac{\partial \mathcal{F}}{\partial \mathbf{y}}(\mathbf{y}_f, \mathbf{f}) : \mathcal{Y} \longrightarrow \mathcal{W} \times \mathcal{Y}_0, \\ &\frac{\partial \mathcal{F}}{\partial \mathbf{y}}(\mathbf{y}_f, \mathbf{f})\mathbf{z} = \Big(\frac{\partial \mathbf{z}}{\partial t} + A\mathbf{z} + \mathcal{B}(\mathbf{y}_f, \mathbf{z}) + B(\mathbf{z}, \mathbf{y}_f), \mathbf{z}(0)\Big) \quad \forall \mathbf{z} \in \mathcal{Y} \end{aligned} \tag{14}$$

is a linear and continuous mapping. Actually, it is an isomorphism. Let us prove this. Given an arbitrary element $(\mathbf{g}, \mathbf{z}_0) \in \mathcal{W} \times \mathcal{Y}_0$, we set $\mathbf{g} = \mathbf{g}_N + \mathbf{g}_S$ and $\mathbf{z}_0 = \mathbf{z}_{N0} + \mathbf{z}_{S0}$ with $\mathbf{g}_N \in L^4(I; \mathbf{V}')$, $\mathbf{g}_S \in L^q(I; \mathbf{W}_{p'}(\Omega)') \cap L^4(I; \mathbf{W}_{p'}(\Omega)')$, $\mathbf{z}_{N0} \in \mathbf{W}_{2,4}(\Omega)$, and $\mathbf{z}_{S0} \in \mathbf{B}_{p,q}(\Omega)$. Now, we show the existence and uniqueness of a solution $\mathbf{z} \in \mathcal{Y}$ of the equation

$$\left\{ \begin{array}{l} \dfrac{\partial \mathbf{z}}{\partial t} + A\mathbf{z} + B(\mathbf{y}_f, \mathbf{z}) + \mathcal{B}(\mathbf{z}, \mathbf{y}_f) = \mathbf{g} \text{ in } I, \\ \mathbf{z}(0) = \mathbf{z}_0. \end{array} \right. \tag{15}$$

We decompose the system in two parts

$$\left\{ \begin{array}{l} \dfrac{\partial \mathbf{z}_S}{\partial t} + A_{\mathbf{W}}\mathbf{z}_S = \mathbf{g}_S \text{ in } I, \\ \mathbf{z}_S(0) = \mathbf{z}_{S0}, \end{array} \right. \tag{16}$$

and

$$\left\{ \begin{array}{l} \dfrac{\partial \mathbf{z}_N}{\partial t} + A_{\mathbf{V}}\mathbf{z}_N + B(\mathbf{y_f}, \mathbf{z}_N) + B(\mathbf{z}_N, \mathbf{y_f}) = \mathbf{g}_N - \mathcal{B}(\mathbf{y_f}, \mathbf{z}_S) - \mathcal{B}(\mathbf{z}_S, \mathbf{y_f}) \text{ in } I, \\ \mathbf{z}_N(0) = \mathbf{z}_{N0} \text{ in } \Omega. \end{array} \right. \tag{17}$$

The existence and uniqueness of a solution $\mathbf{z}_S \in \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)$ of (16) follows from Lemma 2.4. In equation (17), we have that $\mathbf{z}_{N0} \in \mathbf{B}_{2,4}(\Omega)$, $\mathbf{y_f} \in \mathcal{Y}$, and from Lemma 2.1 we get that the right hand side of the partial differential equation belongs to $L^4(I; \mathbf{H}^{-1}(\Omega)) \cap L^2(I; \mathbf{H}^{-1}(\Omega))$. Hence, applying Lemma 2.5 with $\nu_0 = 0$ and $\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{z}_S$ we infer the existence and uniqueness of a solution $\mathbf{z}_N \in \mathbf{W}_{4,2}(I)$. Now, setting $\mathbf{y} = \mathbf{y}_N + \mathbf{y}_S \in \mathcal{Y}$, we deduce that $\mathbf{y}$ is a solution of (15). The uniqueness follows from Gronwall's inequality.

Then, we apply the implicit function theorem to deduce the existence of a C^∞ mapping $\tilde{\mathcal{G}} : \mathbf{W}_{p'}(\Omega)' \longrightarrow \mathcal{Y}$ such that $\mathcal{F}(\tilde{\mathcal{G}}(\mathbf{f}), \mathbf{f}) = 0$ for every function $\mathbf{f} \in L^q(I; \mathbf{W}_{p'}(\Omega)') \cap L^4(I; \mathbf{W}_{p'}(\Omega)')$. Hence, $\tilde{\mathcal{G}}(\mathbf{f}) = \mathbf{y_f}$ is the solution of (4). Moreover, by differentiation of the identity $\mathcal{F}(\tilde{\mathcal{G}}(\mathbf{f}), \mathbf{f}) = 0$ with respect to $\mathbf{f}$, setting $\mathbf{z_g} = D\tilde{\mathcal{G}}(\mathbf{f})\mathbf{g}$ for $\mathbf{g} \in L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega))$, and using (14) and De Rham's theorem equation (12) follows. Differentiating twice the identity $\mathcal{F}(\tilde{\mathcal{G}}(\mathbf{f}), \mathbf{f}) = 0$ with respect to $\mathbf{f}$ and setting $\mathbf{z} = D^2\tilde{\mathcal{G}}(\mathbf{f})(\mathbf{g}_1, \mathbf{g}_2)$, equation (13) follows easily from the identity

$$\frac{\partial^2 \mathcal{F}}{\partial \mathbf{y}^2}(\mathbf{y_f}, \mathbf{f})(\mathbf{g}_1, \mathbf{g}_2) = \Big(\frac{\partial \mathbf{z}}{\partial t} + A\mathbf{z} + \mathcal{B}(\mathbf{y}_f, \mathbf{z}) + \mathcal{B}(\mathbf{z}, \mathbf{y_f}) + \mathcal{B}(\mathbf{z}_{\mathbf{g}_1}, \mathbf{z}_{\mathbf{g}_2}) + \mathcal{B}(\mathbf{z}_{\mathbf{g}_2}, \mathbf{z}_{\mathbf{g}_1}), \mathbf{z}(0)\Big).$$

Observing that $\mathcal{G} : L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega)) \longrightarrow \mathcal{Y}$ is given by $\mathcal{G} = \tilde{G} \circ R_\sigma$, where $R_\sigma : L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega)) \longrightarrow L^q(I; \mathbf{W}_{p'}(\Omega)') \cap L^4(I; \mathbf{W}_{p'}(\Omega)')$ is the restriction operator, that is linear and continuous, the theorem follows. □

3. Analysis of the optimal control problem (P)

In this section we study the control problem (P) associated with the state equation (1). We recall that $\mathbf{K}$ is a convex, closed, and bounded subset of $L^\sigma(\omega)$. We keep the assumptions made in section 2 and impose the following additional requirements: $0 \in \mathbf{K} \subset \mathbf{L}^\sigma(\omega)$ with $\sigma \in [1, \infty]$ and

$$(\alpha + \beta > 0) \text{ and } (\alpha > 0 \text{ if } \sigma < 2), \tag{18}$$

$$\mathbf{K} \text{ is weakly}^* \text{ closed in } \mathbf{L}^\infty(\omega) \text{ if } \sigma = \infty, \tag{19}$$

$$\mathbf{y}_d \in L^q(I; \mathbf{L}^2(\Omega)) \cap L^2(I; \mathbf{L}^2(\Omega)) \text{ and } \mathbf{f}_0 \in L^q(I; \mathbf{W}^{-1,p}(\Omega)) \cap L^4(I; \mathbf{W}^{-1,p}(\Omega)), \tag{20}$$

where q and p satisfy (3). The space of controls will be specified as

$$\mathbf{U} = L^\infty(I; \mathbf{L}^\sigma(\omega)) \cap L^2(I; \mathbf{L}^2(\omega)) \quad \text{if} \quad \beta = 0, \tag{21}$$

$$\mathbf{U} = L^\infty(I; \mathbf{L}^\sigma(\omega)) \cap L^1(I; \mathbf{L}^2(\omega)) \quad \text{if} \quad \alpha = 0, \tag{22}$$

$$\mathbf{U} = L^\infty(I; \mathbf{L}^\sigma(\omega)) \cap L^2(I; \mathbf{L}^2(\omega)) \cap L^1(I; \mathbf{L}^2(\omega)) \quad \text{if} \quad \alpha > 0 \text{ and } \beta > 0. \tag{23}$$

Let us observe that $\mathbf{U} \subset L^2(I; \mathbf{L}^2(\omega))$. Indeed, for the cases (21) and (23) the embedding is obvious. Concerning (22), using (18) and interpolation we get that $L^\infty(I; \mathbf{L}^\sigma(\omega)) \cap L^1(I; \mathbf{L}^2(\omega)) \subset L^2(I; \mathbf{L}^2(\omega))$.

The set of feasible controls is defined by $\mathbf{U}_{\mathrm{ad}} = \{\mathbf{u} \in \mathbf{U} : \mathbf{u}(t) \in \mathbf{K} \text{ for a.a. } t \in I\}$.

By using interpolation between Lebesgue spaces we get that $\mathbf{U} \subset L^q(I; \mathbf{L}^1(\omega)) \subset L^q(I; \mathbf{W}^{-1,p}(\Omega))$. Therefore, Theorem 2.3 implies that the existence and uniqueness of $\mathbf{y}_\mathbf{u} \in \mathcal{Y}$ and, hence, $J(\mathbf{u}) < \infty$ for every $\mathbf{u} \in \mathbf{U}$.

We point out that the assumption $0 \in \mathbf{K}$ is natural, in the sense that $J(\mathbf{u}) = \infty$ for every $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$ if this assumption does not hold. Indeed, if $J(\mathbf{u}) < \infty$ and $\mathbf{u}(t) \in \mathbf{K}$ for almost all $t \in I$, then there exists a sequence of points $\{t_k\}_{k=1}^\infty$ converging to ∞ such that $\|\mathbf{u}(t_k)\|_{\mathbf{L}^2(\omega)} \to 0$ and $\{\mathbf{u}(t_k)\}_{k=1}^\infty \subset \mathbf{K}$. Since $\mathbf{K}$ is bounded and weakly* closed in $\mathbf{L}^\sigma(\omega)$ there exists a subsequence, denoted in the same way, such that $\mathbf{u}(t_k) \stackrel{*}{\rightharpoonup} \mathbf{v}$ in $\mathbf{L}^\sigma(\omega)$ and $\mathbf{v} \in \mathbf{K}$. This together with the strong convergence of $\{\mathbf{u}(t_k)\}_{k=1}^\infty$ to 0 in $\mathbf{L}^2(\omega)$ implies that $\mathbf{v} = 0$ and, consequently, $0 \in \mathbf{K}$.

As a first step we address the existence of an optimal control for (P).

Theorem 3.1. *Problem* (P) *has at least one solution.*

Proof. Firstly, we observe that $0 \in \mathbf{U}_{\mathrm{ad}}$. Hence, the existence of a minimizing sequence $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathbf{U}_{\mathrm{ad}}$ follows. From assumptions (18) and (21)–(23) and the fact that $J(\mathbf{u}_k) \leq J(0)$ we deduce that $\{\mathbf{u}_k\}_{k=1}^\infty$ is bounded in $L^2(I; \mathbf{L}^2(\omega))$. Then, by taking a subsequence, we get that $\bar{\mathbf{u}}_k \rightharpoonup \bar{\mathbf{u}}$ in $L^2(I; \mathbf{L}^2(\omega))$. Let us prove that $\bar{\mathbf{u}}$ is a solution of (P).

Step 1. $\bar{\mathbf{u}} \in \mathbf{U}_{\rm ad}$. From Mazur's Theorem we infer the existence of a convex combination $\{\mathbf{v}_k\}_{k=1}^\infty$ of the sequence $\{\mathbf{u}_k\}_{k=1}^\infty$ converging to $\bar{\mathbf{u}}$ strongly in $L^2(I;\mathbf{L}^2(\omega))$. Taking a subsequence we can assume that $\mathbf{v}_k(t) \to \bar{\mathbf{u}}(t)$ strongly in $\mathbf{L}^2(\omega)$ for almost every $t \in I$. The convexity of $\mathbf{K}$ implies that $\mathbf{v}_k(t) \in \mathbf{K}$ for every k and almost every $t \in I$. If $\sigma \le 2$, then we have that $\mathbf{v}_k(t) \to \bar{\mathbf{u}}(t)$ strongly in $\mathbf{L}^\sigma(\omega)$ and, hence, the closedness of $\mathbf{K}$ in $\mathbf{L}^\sigma(\omega)$ implies that $\bar{\mathbf{u}}(t) \in \mathbf{K}$ for almost every $t \in I$. If $\sigma > 2$, then for almost every $t \in I$ there exists a subsequence of $\{\mathbf{v}_k\}_{k=1}^\infty$, denoted in the same way, such that $\mathbf{v}_k(t) \rightharpoonup \mathbf{v}$ (or $\stackrel{*}{\rightharpoonup}$ if $\sigma = \infty$) in $\mathbf{L}^\sigma(\omega)$. Since $\mathbf{K}$ is convex and closed (or weakly* closed if $\sigma = \infty$) we get that $\mathbf{v} \in \mathbf{K}$. But $\sigma > 2$, therefore $\mathbf{v}_k(t) \rightharpoonup \mathbf{v}$ in $\mathbf{L}^2(\omega)$ as well. Combining this with the strong convergence $\mathbf{v}_k(t) \to \bar{\mathbf{u}}(t)$ in $\mathbf{L}^\sigma(\omega)$, we obtain that $\bar{\mathbf{u}}(t) = \mathbf{v} \in \mathbf{K}$. Since $\bar{\mathbf{u}}$ satisfies the control constraint, we infer that $\bar{\mathbf{u}} \in L^\infty(I;\mathbf{L}^\sigma(\omega))$. It remains to prove that $\bar{\mathbf{u}} \in \mathbf{U}$. Actually the only thing that remains to be proved is that $\bar{\mathbf{u}} \in L^1(I;\mathbf{L}^2(\omega))$ if $\beta > 0$. For every $T < \infty$, the continuous embedding $L^2(0,T;\mathbf{L}^2(\omega)) \subset L^1(0,T;\mathbf{L}^2(\omega))$ implies that $\mathbf{u}_k \rightharpoonup \bar{\mathbf{u}}$ in $L^1(0,T;\mathbf{L}^2(\omega))$. This yields

$$\begin{aligned}\|\bar{\mathbf{u}}\|_{L^1(0,T;\mathbf{L}^2(\omega))} &\le \liminf_{k\to\infty} \|\mathbf{u}_k\|_{L^1(0,T;\mathbf{L}^2(\omega))}\\ &\le \liminf_{k\to\infty} \|\mathbf{u}_k\|_{L^1(I;\mathbf{L}^2(\omega))} \le \frac{1}{\beta}\liminf_{k\to\infty} J(\mathbf{u}_k) \le \frac{1}{\beta}J(0).\end{aligned}$$

This implies that $\|\bar{\mathbf{u}}\|_{L^1(I;\mathbf{L}^2(\omega))} \le \frac{1}{\beta}J(0) < \infty$ and, consequently, $\bar{\mathbf{u}} \in \mathbf{U}$.

Step 2. $J(\bar{\mathbf{u}}) \le \liminf_{k\to\infty} J(\mathbf{u}_k)$. Since $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathbf{U}_{\rm ad}$ we have that $\{\mathbf{u}_k\}_{k=1}^\infty$ is bounded in $L^\infty(I;\mathbf{L}^1(\omega))$. We also have the boundedness of $\{\mathbf{u}_k\}_{k=1}^\infty$ in $L^2(I;\mathbf{L}^2(\omega)) \subset L^2(I;\mathbf{L}^1(\omega))$. Hence, by interpolation we infer that $\{\chi_\omega\mathbf{u}_k\}_{k=1}^\infty$ is bounded in $L^q(I,\mathbf{L}^1(\Omega)) \cap L^4(I;\mathbf{L}^1(\Omega)) \subset L^q(I;\mathbf{W}^{-1,p}(\Omega)) \cap L^4(I;\mathbf{W}^{-1,p}(\Omega))$. Applying Theorem 2.3 we deduce the boundedness of $\{\mathbf{y}_{\mathbf{u}_k}\}_{k=1}^\infty$ in $\mathbf{W}(I) + \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)$. Hence, taking a subsequence, denoted in the same way, we get $\mathbf{y}_{\mathbf{u}_k} \rightharpoonup \bar{\mathbf{y}}$ in for some $\bar{\mathbf{y}} \in \mathbf{W}(I) + \mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)$. Let us fix $T < \infty$ arbitrarily. It is well known that $\mathbf{W}(0,T)$ is compactly embedded in $L^2(0,T;\mathbf{L}^2(\Omega))$. Moreover, applying [27, Theorem III-2.1] to the spaces $\mathbf{W}_0^{1,p}(\Omega) \subset \mathbf{L}^2(\Omega) \subset \mathbf{W}^{-1,p}(\Omega)$ we infer the compactness of the embedding $\mathbf{W}_{q,p}(0,T) \cap \mathbf{W}_{4,p}(0,T) \subset L^2(0,T;\mathbf{L}^2(\Omega))$. Then, the convergence $\mathbf{y}_{\mathbf{u}_k} \rightharpoonup \bar{\mathbf{y}}$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ holds. Using these convergences, it is easy to pass to the limit in the equation satisfied by $(\mathbf{y}_{\mathbf{u}_k},\mathbf{u}_k)$ to get that $\bar{\mathbf{y}}$ is the state associated with $\bar{\mathbf{u}}$. Moreover, $\mathbf{y}_{\mathbf{u}_k} \rightharpoonup \bar{\mathbf{y}}$ in $L^2(I;\mathbf{L}^2(\Omega))$.

From the established convergences we infer

$$\begin{aligned}&\frac{1}{2}\int_I \|\bar{\mathbf{y}}(t)-\mathbf{y}_d(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t + \frac{\alpha}{2}\int_I \|\bar{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t + \beta\int_I \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t\\ &\le \liminf_{k\to\infty}\Big(\frac{1}{2}\int_I \|\mathbf{y}_k(t)-\mathbf{y}_d(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t + \frac{\alpha}{2}\int_I \|\mathbf{u}_k(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t + \beta\int_I \|\mathbf{u}_k(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t\Big)\\ &= \liminf_{k\to\infty} J(\mathbf{u}_k) = \inf\,(\mathrm{P}).\end{aligned}$$

Thus we get that $J(\bar{\mathbf{u}}) \le \inf\,(\mathrm{P})$, hence $\bar{\mathbf{u}}$ is a solution of (P). □

Before analyzing the optimality conditions satisfied by a local minimizer of (P), we address the issue of differentiability of J. To this end, we first introduce the mapping $G : \mathbf{U} \longrightarrow \mathcal{Y}$ by $\mathbf{y}_\mathbf{u} = G(\mathbf{u}) = \mathcal{G}(\mathbf{f}_0 + \chi_\omega\mathbf{u})$, where $\mathcal{G} : L^q(I;\mathbf{W}^{-1,p}(\Omega)) \cap$

$L^4(I;\mathbf{W}^{-1,p}(\Omega)) \longrightarrow \mathcal{Y}$ is the mapping introduced in section 2. It was shown in the proof of Theorem 3.1 that $\chi_\omega \mathbf{u} \in L^q(I;\mathbf{W}^{-1,p}(\Omega)) \cap L^4(I;\mathbf{W}^{-1,p}(\Omega))$ for every $\mathbf{u} \in \mathbf{U}$. Hence, recalling the assumption on $\mathbf{f}_0$ in (20), we have that G is well defined and in view of Theorem 2.6 it is of class C^∞. Now, we decompose J into two functions: $J(\mathbf{u}) = F(\mathbf{u}) + \beta j(\mathbf{u})$, where

$$j(\mathbf{u}) = \|\mathbf{u}\|_{L^1(I;\mathbf{L}^2(\omega))} = \int_I \sqrt{\|u_1(t)\|^2_{L^2(\omega)} + \|u_2(t)\|^2_{L^2(\omega)}}\,\mathrm{d}t.$$

Obviously, j is not differentiable, but it is Lipschitz, convex, and continuous in $\mathbf{U}$. Now, we analyze the differentiability of F. To this end, let us introduce the space with $\mathbf{X} = L^q(I;\mathbf{W}^{2,4}(\Omega)) \cap \mathbf{W}^{1,q}(I;\mathbf{L}^4(\Omega))$.

Theorem 3.2. *The functional $F : \mathbf{U} \longrightarrow \mathbb{R}$ is of class C^∞ and its first and second derivatives are given by the following expressions:*

$$F'(\mathbf{u})\mathbf{v} = \int_I \int_\omega (\boldsymbol{\varphi}_\mathbf{u} + \alpha \mathbf{u})\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t, \tag{24}$$

$$F''(\mathbf{u})\mathbf{v}^2 = \int_I \int_\Omega \left\{|\mathbf{z}_\mathbf{v}|^2 + 2(\mathbf{z}_\mathbf{v}\cdot\nabla)\boldsymbol{\varphi}_\mathbf{u}\mathbf{z}_\mathbf{v}\right\}\mathrm{d}x\,\mathrm{d}t + \alpha\int_I\int_\omega |\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{25}$$

where $\mathbf{z}_\mathbf{v} = G'(\mathbf{u})\mathbf{v}$ is the solution of (12) with $\mathbf{g}$ replaced by $\chi_\omega\mathbf{v}$, and the function $\boldsymbol{\varphi}_\mathbf{u} \in \mathbf{V}^{2,1}(I) \cap \mathbf{X}$ is the adjoint state, the unique solution along with the pressure $\pi_\mathbf{u} \in \Pi = L^q(I;W^{1,4}(\Omega))/\mathbb{R}$ of

$$\begin{cases} -\dfrac{\partial\boldsymbol{\varphi}}{\partial t} - \nu\Delta\boldsymbol{\varphi} - (\mathbf{y}_\mathbf{u}\cdot\nabla)\boldsymbol{\varphi} - (\nabla\boldsymbol{\varphi})^T\mathbf{y}_\mathbf{u} + \nabla\pi = \mathbf{y}_\mathbf{u} - \mathbf{y}_d \ \text{ in } Q, \\ \operatorname{div}\boldsymbol{\varphi} = 0 \ \text{ in } Q,\ \boldsymbol{\varphi} = 0 \ \text{ on } \Sigma,\ \lim_{T\to\infty}\|\boldsymbol{\varphi}(T)\|_{\mathbf{H}^1_0(\Omega)} = 0 \ \text{ in } \Omega. \end{cases} \tag{26}$$

Moreover, there exists a constant C depending of $\mathbf{y}_\mathbf{u}$ and a nondecreasing monotone real value function $\tilde\eta$ such that

$$\begin{aligned} \|(\boldsymbol{\varphi},\pi)\|_{\mathbf{X}\times\Pi} &\le C\|\mathbf{y}_\mathbf{u} - \mathbf{y}_d\|_{L^q(I;\mathbf{L}^4(\Omega))} + \tilde\eta\Big(\|\mathbf{f}_0\|_{L^q(I;\mathbf{W}_{p'}(\Omega)')} \\ &+ \|\mathbf{f}\|_{L^4(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{u}\|_{L^q(I;\mathbf{L}^1(\omega))} + \|\mathbf{y}_0\|_{\mathbf{Y}_0}\Big)\|\mathbf{y}_u - \mathbf{y}_d\|_{L^2(I;\mathbf{L}^2(\Omega))}. \end{aligned} \tag{27}$$

Proof. The C^∞ differentiability of F follows from Theorem 2.6. The formulas (24) and (25) are consequences of (12), (13), and (26). We only need to prove that (26) has a unique solution in $\mathbf{X}\times\Pi$ and (27) holds. First, we fixed $T < \infty$ and consider the problems

$$\begin{cases} -\dfrac{\partial\boldsymbol{\varphi}}{\partial t} - \nu\Delta\boldsymbol{\varphi} - (\mathbf{y}_\mathbf{u}\cdot\nabla)\boldsymbol{\varphi} - (\nabla\boldsymbol{\varphi})^T\mathbf{y}_\mathbf{u} + \nabla\pi = \mathbf{y}_\mathbf{u} - \mathbf{y}_d \ \text{ in } Q_T, \\ \operatorname{div}\boldsymbol{\varphi} = 0 \ \text{ in } Q_T,\ \boldsymbol{\varphi} = 0 \ \text{ on } \Sigma_T,\ \boldsymbol{\varphi}(T) = 0 \ \text{ in } \Omega. \end{cases} \tag{28}$$

In [12, Theorem 3.2] it was proved that (28) has a unique solution $\boldsymbol{\varphi}_T \in \mathbf{V}^{2,1}(0,T)$ satisfying $\|\boldsymbol{\varphi}_T\|_{\mathbf{V}^{2,1}(0,T)} \le \eta(\|\mathbf{y}_\mathbf{u}\|_{L^4(0,T;\mathbf{L}^4(\Omega))})\|\mathbf{y}_u - \mathbf{y}_d\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$, where η is a nondecreasing monotone function. Then, using (5) we get

$$\|\boldsymbol{\varphi}_T\|_{\mathbf{V}^{2,1}(0,T)} \leq \hat{\eta}(\|\mathbf{f}_0\|_{L^q(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{f}\|_{L^4(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{u}\|_{L^q(I;\mathbf{L}^1(\omega))} + \|\mathbf{y}_0\|_{\mathbf{Y}_0}\Big)\|\mathbf{y}_u - \mathbf{y}_d\|_{L^2(I;\mathbf{L}^2(\Omega))} \tag{29}$$

for a nondecreasing monotone function $\hat{\eta}$. Moreover, in [12, Lemma 4.9] it was proved that $\boldsymbol{\varphi}_T$ belongs to the space $\mathbf{X}_T = L^q(0,T;\mathbf{W}^{2,4}(\Omega)) \cap \mathbf{W}^{1,q}(0,T;\mathbf{L}^4(\Omega))$ and the associated pressure π_T belongs to $\Pi_T = L^q(0,T;W^{1,4}(\Omega))/\mathbb{R}$. Extending $(\boldsymbol{\varphi}_T,\pi_T)$ by zero to Q we get that $(\boldsymbol{\varphi}_T,\pi_T) \in \mathbf{X}\times\Pi$. Applying [1, Theorem III-4.10.2] with $E_0 = \mathbf{L}^4(\Omega)$, $E_1 = \mathbf{W}^{2,4}(\Omega)$, and $p = q$, we get that $\mathbf{X}$ is continuously embedded in $C(I;(\mathbf{L}^4(\Omega),\mathbf{W}^{2,4}(\Omega))_{1-\frac{1}{q},q})$. We also have that

$$(\mathbf{L}^4(\Omega),\mathbf{W}^{2,4}(\Omega))_{1-\frac{1}{q},q} \subset (\mathbf{L}^4(\Omega),\mathbf{W}^{2,4}(\Omega))_{1-\frac{1}{q},4} = \mathbf{W}^{2(1-\frac{1}{q}),4}(\Omega) \subset \mathbf{C}^1(\bar{\Omega})$$

Therefore, we have that $\mathbf{X} \subset C(I;\mathbf{C}^1(\bar{\Omega}))$ and there exists a constant C_1 such that $\|\boldsymbol{\varphi}\|_{C(I;\mathbf{C}^1(\bar{\Omega}))} \leq C_1\|\boldsymbol{\varphi}\|_{\mathbf{X}}$ for every $\boldsymbol{\varphi} \in X$. Moreover, the embedding $\mathbf{X}_T \subset C([0,T];\mathbf{C}^1(\bar{\Omega}))$ is compact for every $T < \infty$; see [2, Theorem 3].

Since $\mathbf{y_u} \in L^q(I;\mathbf{L}^4(\Omega))$, for every $\varepsilon > 0$ there exists $T_\varepsilon \in (0,\infty)$ such that $\|\mathbf{y_u}\|_{L^q(T_\varepsilon,\infty;\mathbf{L}^4(\Omega))} < \varepsilon$. Moreover, from the maximal parabolic regularity of the Stokes system we infer for every $T > T_\varepsilon$

$$\begin{aligned}
&\|(\boldsymbol{\varphi}_T,\pi_T)\|_{\mathbf{X}\times\Pi} = \|(\boldsymbol{\varphi}_T,\pi_T)\|_{\mathbf{X}_T\times\Pi_T}\\
&\leq C_2\Big(\|\mathbf{y_u}-\mathbf{y}_d\|_{L^q(0,T;\mathbf{L}^4(\Omega))} + \|(\mathbf{y_u}\cdot\nabla)\boldsymbol{\varphi}_T\|_{L^q(0,T;\mathbf{L}^4(\Omega))} + \|(\nabla\boldsymbol{\varphi}_T)^T\mathbf{y_u}\|_{L^q(0,T;\mathbf{L}^4(\Omega))}\Big)\\
&\leq C_2\Big(\|\mathbf{y_u}-\mathbf{y}_d\|_{L^q(I;\mathbf{L}^4(\Omega))} + \|(\mathbf{y_u}\cdot\nabla)\boldsymbol{\varphi}_T\|_{L^q(0,T_\varepsilon;\mathbf{L}^4(\Omega))} + \|(\nabla\boldsymbol{\varphi}_T)^T\mathbf{y_u}\|_{L^q(0,T_\varepsilon;\mathbf{L}^4(\Omega))}\\
&+ \|(\mathbf{y_u}\cdot\nabla)\boldsymbol{\varphi}_T\|_{L^q(T_\varepsilon,\infty;\mathbf{L}^4(\Omega))} + \|(\nabla\boldsymbol{\varphi}_T)^T\mathbf{y_u}\|_{L^q(T_\varepsilon\infty;\mathbf{L}^4(\Omega))}\Big)\\
&\leq C_2\Big(\|\mathbf{y_u}-\mathbf{y}_d\|_{L^q(I;\mathbf{L}^4(\Omega))} + 2\|\boldsymbol{\varphi}_T\|_{C([0,T_\varepsilon];\mathbf{C}^1(\bar{\Omega}))}\|\mathbf{y_u}\|_{L^q(0,T_\varepsilon;\mathbf{L}^4(\Omega))}\\
&+ 2\|\boldsymbol{\varphi}_T\|_{C(I;\mathbf{C}^1(\bar{\Omega}))}\|\mathbf{y_u}\|_{L^q(T_\varepsilon,\infty;\mathbf{L}^4(\Omega))}\Big)\\
&\leq C_2\Big(\|\mathbf{y_u}-\mathbf{y}_d\|_{L^q(I;\mathbf{L}^4(\Omega))} + 2\|\boldsymbol{\varphi}_T\|_{C([0,T_\varepsilon];\mathbf{C}^1(\bar{\Omega}))}\|\mathbf{y_u}\|_{L^q(0,T_\varepsilon;\mathbf{L}^4(\Omega))}\Big)\\
&+ 2C_2C_1\varepsilon\|(\boldsymbol{\varphi}_T,\pi_T)\|_{\mathbf{X}\times\Pi},
\end{aligned}$$

where C_2 is independent of T. Hence, it is enough to take $\varepsilon \leq \frac{1}{4}C_1C_2$ to deduce that

$$\begin{aligned}
&\|(\boldsymbol{\varphi}_T,\pi_T)\|_{\mathbf{X}\times\Pi}\\
&\leq 2C_2\Big(\|\mathbf{y_u}-\mathbf{y}_d\|_{L^q(I;\mathbf{L}^4(\Omega))} + 2\|\boldsymbol{\varphi}_T\|_{C([0,T_\varepsilon];\mathbf{C}^1(\bar{\Omega}))}\|\mathbf{y_u}\|_{L^q(0,T_\varepsilon;\mathbf{L}^4(\Omega))}\Big).
\end{aligned} \tag{30}$$

Now, applying Lions Lemma [24, Lemma III-1.1] to the spaces $X_{T_\varepsilon} \subset C([0,T_\varepsilon];\mathbf{C}^1(\bar{\Omega})) \subset L^2(0;T_\varepsilon;\mathbf{L}^2(\Omega))$ and using (29), we deduce the existence of a constant C_3 such that

$$
\begin{aligned}
&\|\boldsymbol{\varphi}_T\|_{C([0,T_\varepsilon];\mathbf{C}^1(\bar\Omega))} \le \frac{1}{8C_2\|\mathbf{y_u}\|_{L^q(I;\mathbf{L}^4(\Omega))}}\|\boldsymbol{\varphi}_T\|_{\mathbf{X}_{T_\varepsilon}} + C_3\|\boldsymbol{\varphi}_T\|_{L^2(I;\mathbf{L}^2(\Omega))}\\
&\le \frac{1}{8C_2\|\mathbf{y_u}\|_{L^q(I;\mathbf{L}^4(\Omega))}}\|(\boldsymbol{\varphi}_T,\pi_T)\|_{\mathbf{X}\times\Pi} + C_3\hat\eta\Big(\|\mathbf{f}_0\|_{L^q(I;\mathbf{W}_{p'}(\Omega)')}\\
&+\|\mathbf{f}\|_{L^4(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{u}\|_{L^q(I;\mathbf{L}^1(\omega))} + \|\mathbf{y}_0\|_{\mathbf{Y}_0}\Big)\|\mathbf{y}_u-\mathbf{y}_d\|_{L^2(I;\mathbf{L}^2(\Omega))}.
\end{aligned}
$$

Inserting this inequality in (30) we obtain

$$
\begin{aligned}
&\|(\boldsymbol{\varphi}_T,\pi_T)\|_{\mathbf{X}\times\Pi} \le 4C_2\|\mathbf{y_u}-\mathbf{y}_d\|_{L^q(I;\mathbf{L}^4(\Omega))}+\tilde\eta\Big(\|\mathbf{f}_0\|_{L^q(I;\mathbf{W}_{p'}(\Omega)')}\\
&+\|\mathbf{f}\|_{L^4(I;\mathbf{W}_{p'}(\Omega)')} + \|\mathbf{u}\|_{L^q(I;\mathbf{L}^1(\omega))} + \|\mathbf{y}_0\|_{\mathbf{Y}_0}\Big)\|\mathbf{y}_u-\mathbf{y}_d\|_{L^2(I;\mathbf{L}^2(\Omega))}.
\end{aligned}
$$

Then, we deduce the existence of a sequence $\{T_k\}_{k=1}^\infty$ and a pair $(\boldsymbol{\varphi},\pi)\in\mathbf{X}\times\Pi$ such that $(\boldsymbol{\varphi}_{T_k},\pi_{T_k})\rightharpoonup(\boldsymbol{\varphi},\pi)$ in $\mathbf{X}\times\Pi$. Moreover, $(\boldsymbol{\varphi},\pi)$ satisfies inequality (27). It is easy to pass to the limit in (28) and to deduce that $(\boldsymbol{\varphi},\pi)$ is a solution of (26), except for the identity $\lim_{T\to\infty}\|\boldsymbol{\varphi}(T)\|_{\mathbf{H}_0^1(\Omega)}=0$. To establish this equality we observe that the regularity $\boldsymbol{\varphi}\in\mathbf{V}^{2,1}(I)$ follows from (29). Hence, we have that $\boldsymbol{\varphi}\in L^2(I;\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega))$ and $\frac{\partial\boldsymbol{\varphi}}{\partial t}\in L^2(I;\mathbf{L}^2(\Omega))$. The fact that $\boldsymbol{\varphi}\in L^2(I;\mathbf{H}_0^1(\Omega))$ implies the existence of a sequence $\{t_k\}_{k=1}^\infty$ converging to ∞ such that $\lim_{k\to\infty}\|\boldsymbol{\varphi}(t_k)\|_{\mathbf{H}_0^1(\Omega)}=0$. For every $T<t_k$ the following relation holds

$$
\begin{aligned}
&\|\boldsymbol{\varphi}(T)\|^2_{\mathbf{H}_0^1(\Omega)} = \|\boldsymbol{\varphi}(t_k)\|^2_{\mathbf{H}_0^1(\Omega)} - 2\int_T^{t_k}\int_\Omega\Delta\boldsymbol{\varphi}(x,t)\frac{\partial\boldsymbol{\varphi}}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\le\|\boldsymbol{\varphi}(t_k)\|_{\mathbf{H}_0^1(\Omega)} + 2\Big(\int_T^{t_k}\|\Delta\boldsymbol{\varphi}(t)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\Big)^{\frac12}\Big(\int_T^{t_k}\|\frac{\partial\boldsymbol{\varphi}}{\partial t}(t)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\Big)^{\frac12}.
\end{aligned}
$$

Taking limit as $k\to\infty$ we get

$$
\|\boldsymbol{\varphi}(T)\|^2_{\mathbf{H}_0^1(\Omega)} \le 2\Big(\int_T^{\infty}\|\Delta\boldsymbol{\varphi}(t)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\Big)^{\frac12}\Big(\int_T^{\infty}\|\frac{\partial\boldsymbol{\varphi}}{\partial t}(t)\|^2_{\mathbf{L}^2(\Omega)}\,\mathrm{d}t\Big)^{\frac12}.
$$

This yields the desired identity: $\lim_{T\to\infty}\|\boldsymbol{\varphi}(T)\|_{\mathbf{H}_0^1(\Omega)}=0$. □

Remark 3.3. *Using Lemma 2.5 we infer that the linear and quadratic forms* $F'(\mathbf{u})$ *and* $F''(\mathbf{u})$ *can be extended to continuous forms on* $L^2(I;\mathbf{L}^2(\omega))$ *by the same expressions* (24) *and* (25).

The next lemma establishes some properties of the function $j: L^1(I;\mathbf{L}^2(\omega))\longrightarrow\mathbb{R}$.

Lemma 3.4. (i) *For the subdifferential* $\partial j(\mathbf{u})$ *we have the following characterization:* $\boldsymbol{\lambda}\in\partial j(\mathbf{u})$ *if and only if* $\boldsymbol{\lambda}\in L^\infty(I;\mathbf{L}^2(\omega))$ *and*

$$
\left\{\begin{array}{ll}
\|\boldsymbol{\lambda}(t)\|_{\mathbf{L}^2(\omega)}\le 1 & \text{for a.a. } t\in I^0_{\mathbf{u}},\\
\boldsymbol{\lambda}(x,t)=\dfrac{\mathbf{u}(x,t)}{\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}} & \text{for a.a. } t\in I^+_{\mathbf{u}},
\end{array}\right. \tag{31}
$$

where $I^0_{\mathbf{u}}=\{t\in I:\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}=0\}$ *and* $I^+_{\mathbf{u}}=I\setminus I^0_{\mathbf{u}}$.

(ii) *For every* $\mathbf{u}, \mathbf{v} \in L^1(I; \mathbf{L}^2(\omega))$ *the directional derivative of* j *is given by*

$$j'(\mathbf{u}; \mathbf{v}) = \int_{I^0_{\mathbf{u}}} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \,\mathrm{d}t + \int_{I^+_{\mathbf{u}}} \frac{1}{\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}} \int_\omega \mathbf{u}(t)\mathbf{v}(t) \,\mathrm{d}x \,\mathrm{d}t. \tag{32}$$

For the proof the reader is referred to [9]. Next we establish the first order optimality conditions for a local minimizer of (P).

Theorem 3.5. *Let* $\bar{\mathbf{u}}$ *be a local minimizer of* (P) *in* $\mathbf{U}$*, then there exists* $\bar{\boldsymbol{\lambda}} \in \partial j(\bar{\mathbf{u}})$ *such that for almost every* $t \in I$ *the inequality*

$$\int_\omega (\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))(\mathbf{v}(x) - \bar{\mathbf{u}}(x,t)) \,\mathrm{d}x \geq 0 \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{L}^2(\omega) \tag{33}$$

holds, where $\bar{\boldsymbol{\varphi}}$ *is the adjoint state associated to* $\bar{\mathbf{u}}$*.*

Proof. Let us consider the case $\beta > 0$. Using (24) and the convexity of j we get for all $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$

$$0 \leq \lim_{\rho \searrow 0} \frac{J(\bar{\mathbf{u}} + \rho(\mathbf{u} - \bar{\mathbf{u}})) - J(\bar{\mathbf{u}})}{\rho} \leq \int_I \int_\omega (\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) \,\mathrm{d}x \,\mathrm{d}t + \beta(j(\mathbf{u}) - j(\bar{\mathbf{u}})).$$

Setting $\mathcal{J} : \mathbf{U} \longrightarrow \mathbb{R} \cup \{+\infty\}$ with

$$\mathcal{J}(\mathbf{u}) = \int_I \int_\omega (\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}}) \mathbf{u} \,\mathrm{d}x \,\mathrm{d}t + \beta j(\mathbf{u}) + I_{\mathbf{U}_{\mathrm{ad}}}(\mathbf{u}),$$

where $I_{\mathbf{U}_{\mathrm{ad}}}$ is the indicator function of $\mathbf{U}_{\mathrm{ad}}$, we infer from the above inequality that $\bar{\mathbf{u}}$ is a minimizer of $\mathcal{J}$ in $\mathbf{U}$. We observe that due to the assumption (18) and the definition of $\mathbf{U}$ given by (21)–(23) the functional $\mathcal{J}$ is well defined and convex. Moreover, the only term in the definition of $\mathcal{J}$ that is not continuous is $I_{\mathbf{U}_{\mathrm{ad}}}$. Therefore, from the calculus with convex functions we obtain that $0 \in \partial \mathcal{J}(\bar{\mathbf{u}}) = \bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}} + \beta \partial j(\bar{\mathbf{u}}) + \partial I_{\mathbf{U}_{\mathrm{ad}}}(\bar{\mathbf{u}})$. Hence, the existence of $\bar{\boldsymbol{\lambda}} \in \partial j(\bar{\mathbf{u}})$ follows such that $-(\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}} + \beta \bar{\boldsymbol{\lambda}}) \in \partial I_{\mathbf{U}_{\mathrm{ad}}}$, which is equivalent to

$$\int_I \int_\omega (\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))(\mathbf{u}(x,t) - \bar{\mathbf{u}}(x,t)) \,\mathrm{d}x \,\mathrm{d}t \geq 0 \quad \forall \mathbf{u} \in \mathbf{U}_{\mathrm{ad}}. \tag{34}$$

The proof of (34) for $\beta = 0$ is standard. Let us deduce (33) from (34). Given $\mathbf{v} \in \mathbf{K} \cap \mathbf{L}^2(\omega)$, we introduce the set

$$E_{\mathbf{v}} = \Big\{ t \in I : \int_\omega (\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))(\mathbf{v}(x) - \bar{\mathbf{u}}(x,t)) \,\mathrm{d}x < 0 \Big\}.$$

Let us prove that $|E_{\mathbf{v}}| = 0$. For every integer $k \geq 1$ we set $E^k_{\mathbf{v}} = E_{\mathbf{v}} \cap (0, k)$ and consider the function $\mathbf{w}_k(x,t) = \chi_{E^k_{\mathbf{v}}}(t)\mathbf{v}(x) + (1 - \chi_{E^k_{\mathbf{v}}}(t))\bar{\mathbf{u}}(x,t)$. Obviously we have that $\mathbf{w}_k \in \mathbf{U}_{\text{ad}}$. Then, we get with (34)

$$\begin{aligned} 0 &\leq \int_I \int_\omega (\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))(\mathbf{w}_k(x,t) - \bar{\mathbf{u}}(x,t))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{E_{\mathbf{v}}^k} \int_\omega (\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))(\mathbf{v}(x) - \bar{\mathbf{u}}(x,t))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

By definition of $E_{\mathbf{v}}$ this is holds only if $|E_{\mathbf{v}}^k| = 0$, hence $|E_{\mathbf{v}}| = \lim_{k\to\infty} |E_{\mathbf{v}}^k| = 0$. □

The next theorem analyzes the sparsity properties for any local minimizer $\bar{\mathbf{u}}$ of (P).

Theorem 3.6. *Let $(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\lambda}})$ satisfy the first order optimality condition (33) and assume that $\beta > 0$. Then, the following expression for $\bar{\boldsymbol{\lambda}}$ holds*

$$\bar{\boldsymbol{\lambda}}(x,t) = \begin{cases} \frac{\bar{\mathbf{u}}(x,t)}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}} & \text{if } \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \neq 0, \\ -\frac{1}{\beta}\bar{\boldsymbol{\varphi}}(x,t) & \text{if } \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = 0. \end{cases} \tag{35}$$

Moreover, the following sparsity property is fulfilled

$$\text{if } \alpha > 0 \text{ then } \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = 0 \Leftrightarrow \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \leq \beta, \tag{36}$$

$$\text{if } \alpha = 0 \text{ then } \begin{cases} \text{if } \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} < \beta & \Rightarrow \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = 0, \\ \text{if } \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} > \beta & \Rightarrow \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \neq 0. \end{cases} \tag{37}$$

Proof. The first equality in (35) follows from the fact that $\bar{\boldsymbol{\lambda}} \in \partial j(\bar{\mathbf{u}})$ and (31). To prove the second identity we observe that if $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = 0$, then (33) implies

$$\int_\omega (\bar{\boldsymbol{\varphi}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(t,x))\mathbf{v}(x)\,\mathrm{d}x \geq 0 \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{L}^2(\omega),$$

which leads to $\bar{\boldsymbol{\varphi}}(t) + \beta \bar{\boldsymbol{\lambda}}(t) = 0$. We now prove (36) and (37). For $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = 0$ we combine (31) and (35) to infer

$$\frac{1}{\beta}\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} = \|\bar{\boldsymbol{\lambda}}(t)\|_{\mathbf{L}^2(\omega)} \leq 1,$$

which proves the left to right implication of (36) and the second implication of (37). To prove the remaining implications we proceed by contradiction. Assume that $\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \leq \beta$ ($\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} < \beta$ if $\alpha = 0$) and $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \neq 0$. Then from (33) and (35) we infer

$$\int_\omega \Big(\bar{\boldsymbol{\varphi}}(x,t) + [\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}]\bar{\mathbf{u}}(x,t)\Big)(\mathbf{v}(x) - \bar{\mathbf{u}}(x,t))\,\mathrm{d}x \geq 0 \quad \forall \mathbf{v} \in \mathbf{K} \cap \mathbf{L}^2(\omega).$$

This implies

$$\bar{\mathbf{u}}(x,t) = \mathrm{Proj}_{\mathbf{K}}\Big(-[\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}]^{-1}\bar{\boldsymbol{\varphi}}(x,t)\Big), \tag{38}$$

where $\mathrm{Proj}_{\mathbf{K}} : \mathbf{L}^2(\omega) \longrightarrow \mathbf{K} \cap \mathbf{L}^2(\omega)$ denotes the $\mathbf{L}^2(\omega)$-projection onto the convex and closed subset $\mathbf{K} \cap \mathbf{L}^2(\omega)$. Since $0 \in \mathbf{K}$, we have

$$\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \leq [\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}]^{-1} \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)}$$

This is equivalent to $\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \geq \alpha \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\Omega)} + \beta$, which contradicts our assumption. □

4. Detailed Analysis for some special choices of K.

In this section we consider three different selections for $\mathbf{K}$ corresponding to $\sigma = 1, 2,$ and ∞. In each case we first deduce some properties from the first order optimality analysis carried out in Section 3 and then we perform the second order analysis.

Second order sufficient optimality conditions are useful for several purposes, including the proof of stability of optimal controls with respect to small perturbations in the data of the control problem, error estimates for the numerical approximation, analysis of the convergence rate of the numerical algorithms, estimates of the difference between finite and infinite time solutions.

4.1. Case $\sigma = 2$.

Here we assume that $\mathbf{K} = \mathbf{B}_\gamma$ the closed $\mathbf{L}^2(\omega)$-ball centered at zero with radius $\gamma > 0$. As a consequence of the first order optimality conditions (33) we get the following result.

Theorem 4.1. *Let $(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\lambda}})$ satisfy the first order optimality condition* (33) *and assume that $\alpha > 0$. Then the following representation formula for $\bar{\mathbf{u}}$ holds*

$$\bar{\mathbf{u}}(x,t) = -\min\big\{\gamma, \frac{1}{\alpha}(\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} - \beta)^+\big\} \frac{1}{\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)}} \bar{\boldsymbol{\varphi}}(x,t). \tag{39}$$

Consequently, if ω is an open subset of Ω the regularity property $\bar{\mathbf{u}} \in L^q(I; \mathbf{W}^{2,4}(\omega)) \cap \mathbf{W}^{1,q}(I; \mathbf{L}^4(\omega))$ holds.

Proof. According to (36), (39) holds if $\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \leq \beta$. Let us study the case $\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} > \beta$. Once again applying (36) we deduce that $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} > 0$ in this case. First, let us assume that $\big\|\big[\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}\big]^{-1} \bar{\boldsymbol{\varphi}}(t)\big\|_{\mathbf{L}^2(\omega)} \leq \gamma$. Then, from (38) we infer that

$$\bar{\mathbf{u}}(x,t) = -\big[\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}\big]^{-1} \bar{\boldsymbol{\varphi}}(x,t).$$

From here we deduce that $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = \frac{1}{\alpha}\big(\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} - \beta\big)$. Inserting this in the above identity we obtain

$$\bar{\mathbf{u}}(x,t) = -\frac{\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} - \beta}{\alpha\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)}}\bar{\boldsymbol{\varphi}}(x,t) \quad \text{and} \quad \frac{\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} - \beta}{\alpha} = \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \leq \gamma.$$

Hence, the identity (39) holds. On the other hand, if $\big\|\big[\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}\big]^{-1}\bar{\boldsymbol{\varphi}}(t)\big\|_{\mathbf{L}^2(\omega)} > \gamma$, then (38) implies that $\bar{\mathbf{u}}(x,t) = \gamma\frac{\bar{\boldsymbol{\varphi}}(x,t)}{\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)}}$. This yields

$$\frac{\gamma}{\alpha\gamma + \beta}\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} = \Big\|\Big[\alpha + \frac{\beta}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}\Big]^{-1}\bar{\boldsymbol{\varphi}}(t)\Big\|_{\mathbf{L}^2(\omega)} > \gamma$$

and, consequently, $\frac{\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} - \beta}{\alpha} > \gamma$. Then, once again (39) holds. The regularity of $\bar{\mathbf{u}}$ follows from the regularity of $\bar{\boldsymbol{\varphi}}$ established in Theorem 3.2 and the representation formula (39). □

Now, we address the second order analysis. We consider local minimizers of (P) in the $L^2(I;\mathbf{L}^2(\omega))$-sense. More precisely, we say that $\bar{\mathbf{u}}$ is an $L^2(I;\mathbf{L}^2(\omega))$ local minimizer if there exists $\varepsilon > 0$ such that

$$J(\bar{\mathbf{u}}) \leq J(\mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{U}_{\text{ad}} \text{ such that } \|\mathbf{u} - \bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))} \leq \varepsilon.$$

We observe that (21)–(23) with $\sigma = 2$ imply that $\mathbf{U} \subset L^p(I;\mathbf{L}^2(\omega))$ for every $p \in [2,\infty]$. Obviously we have that if $\bar{\mathbf{u}}$ is an $L^2(I;\mathbf{L}^2(\omega))$ local minimizer, then it is a local minimizer in the $\mathbf{U}$ sense because $\|\cdot\|_{L^2(I;\mathbf{L}^2(\omega))} \leq \|\cdot\|_{\mathbf{U}}$. Of course, any global minimizer is an $L^2(I;\mathbf{L}^2(\omega))$ local minimizer. Moreover, we have the following result.

Lemma 4.2. *The control $\bar{\mathbf{u}}$ is an $L^2(I;\mathbf{L}^2(\omega))$ local minimizer if and only it is an $L^p(I;\mathbf{L}^2(\omega))$ local minimizer for every $p \in (2,\infty)$.*

Proof. From the inequality

$$\|\mathbf{u} - \bar{\mathbf{u}}\|_{L^p(I;\mathbf{L}^2(\omega))} \leq \|\mathbf{u} - \bar{\mathbf{u}}\|^{\frac{p-2}{p}}_{L^\infty(I;\mathbf{L}^2(\omega))}\|\mathbf{u} - \bar{\mathbf{u}}\|^{\frac{2}{p}}_{L^2(I;\mathbf{L}^2(\omega))} \leq (2\gamma)^{\frac{p-2}{p}}\|\mathbf{u} - \bar{\mathbf{u}}\|^{\frac{2}{p}}_{L^2(I;\mathbf{L}^2(\omega))}$$

satisfied by every $\mathbf{u} \in \mathbf{U}_{\text{ad}}$, we infer that $\bar{\mathbf{u}}$ is an $L^2(I;\mathbf{L}^2(\omega))$ local minimizer if it is an $L^p(I;\mathbf{L}^2(\omega))$ local minimizer. We prove the converse by contradiction. Assume that $\bar{\mathbf{u}}$ is an $L^2(I;\mathbf{L}^2(\omega))$ local minimizer, but it is not an $L^p(I;\mathbf{L}^2(\omega))$ local minimizer. Then, there exists a sequence $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathbf{U}_{\text{ad}}$ converging to $\bar{\mathbf{u}}$ in $L^p(I;\mathbf{L}^2(\omega))$ and such that $J(\mathbf{u}_k) < J(\bar{\mathbf{u}})$ holds for every k. If $\alpha > 0$, the boundedness of $\{\mathbf{u}_k\}_{k=1}^\infty$ in $L^2(I;\mathbf{L}^2(\omega))$ follows from the inequality $J(\mathbf{u}_k) < J(\bar{\mathbf{u}})$. If $\alpha = 0$, then this inequality implies the boundedness of $\{\mathbf{u}_k\}_{k=1}^\infty$ in $L^1(I;\mathbf{L}^2(\omega))$. Combining this with the convergence $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $L^p(I;\mathbf{L}^2(\omega))$, we deduce by interpolation that $\{\mathbf{u}_k\}_{k=1}^\infty$ is bounded in $L^2(I;\mathbf{L}^2(\omega))$ and, consequently, $\mathbf{u}_k \rightharpoonup \bar{\mathbf{u}}$ in $L^2(I;\mathbf{L}^2(\omega))$ holds in both cases. We prove that this convergence is strong. By taking a subsequence, we deduce from the strong convergence $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $L^p(I;\mathbf{L}^2(\omega))$ that $\|\mathbf{u}_k(t)\|_{\mathbf{L}^2(\omega)} \to \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}$ for almost all $t \in I$. Using these convergences - and Fatou's Lemma if $\beta > 0$ - we infer

$$J(\bar{\mathbf{u}}) \leq \liminf_{k\to\infty} J(\mathbf{u}_k) \leq \limsup_{k\to\infty} J(\mathbf{u}_k) \leq J(\bar{\mathbf{u}}).$$

From [17, Lemma 5.2] we infer that $\|\mathbf{u}_k\|_{L^2(I;\mathbf{L}^2(\omega))} \to \|\bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))}$ if $\alpha > 0$ and $\|\mathbf{u}_k\|_{L^1(I;\mathbf{L}^2(\omega))} \to \|\bar{\mathbf{u}}\|_{L^1(I;\mathbf{L}^2(\omega))}$ if $\beta > 0$. Therefore, we deduce that $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $L^2(I;\mathbf{L}^2(\omega))$ if $\alpha > 0$. If $\alpha = 0$, then (18) implies that $\beta > 0$ and, consequently, we have that $\bar{\mathbf{u}}(t) \equiv 0$ for $t \geq T^*$. From the convergences $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $L^p(0,T^*;\mathbf{L}^2(\omega))$ and $\|\mathbf{u}_k\|_{L^1(I;\mathbf{L}^2(\omega))} \to \|\bar{\mathbf{u}}\|_{L^1(I;\mathbf{L}^2(\omega))}$ we get

$$\begin{aligned}\|\bar{\mathbf{u}}\|_{L^1(0,T^*;\mathbf{L}^2(\omega))} &= \|\bar{\mathbf{u}}\|_{L^1(I;\mathbf{L}^2(\omega))} = \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^1(I;\mathbf{L}^2(\omega))}\\ &= \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^1(0,T^*;\mathbf{L}^2(\omega))} + \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^1(T^*,\infty;\mathbf{L}^2(\omega))}\\ &= \|\bar{\mathbf{u}}\|_{L^1(0,T^*;\mathbf{L}^2(\omega))} + \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^1(T^*,\infty;\mathbf{L}^2(\omega))}.\end{aligned}$$

This implies that $\lim_{k\to\infty} \|\mathbf{u}_k\|_{L^1(T^*,\infty;\mathbf{L}^2(\omega))} = 0$. Moreover, we have that

$$\lim_{k\to\infty} \|\mathbf{u}_k - \bar{\mathbf{u}}\|_{L^p(0,T^*;\mathbf{L}^2(\omega))} + \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^p(T^*,\infty;\mathbf{L}^2(\omega))} = \lim_{k\to\infty} \|\mathbf{u}_k - \bar{\mathbf{u}}\|_{L^p(I;\mathbf{L}^2(\omega))} = 0$$

holds. Then we obtain by interpolation that $\lim_{k\to\infty} \|\mathbf{u}_k\|_{L^2(T^*,\infty;\mathbf{L}^2(\omega))} = 0$. We conclude that

$$\lim_{k\to\infty} \|\mathbf{u}_k - \bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))} = \lim_{k\to\infty} \|\mathbf{u}_k - \bar{\mathbf{u}}\|_{L^2(0,T^*;\mathbf{L}^2(\omega))} + \lim_{k\to\infty} \|\mathbf{u}_k\|_{L^2(T^*,\infty;\mathbf{L}^2(\omega))} = 0.$$

Hence, independently of whether $\alpha > 0$ or 0, we obtain that $\mathbf{u}_k \to \bar{\mathbf{u}}$ in $L^2(I;\mathbf{L}^2(\omega))$. This contradicts the $L^2(I;\mathbf{L}^2(\omega))$ local optimality of $\bar{\mathbf{u}}$ and the fact that $J(\mathbf{u}_k) < J(\bar{\mathbf{u}})$ for every k. ☐

We define the Lagrange function:

$$\mathcal{L} : \mathbf{U} \times L^\infty(I) \longrightarrow \mathbb{R}, \quad \mathcal{L}(\mathbf{u},\mu) = J(\mathbf{u}) + \frac{1}{2\gamma}\int_I \mu(t)\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t.$$

According to Theorem 3.2 and Lemma 3.4 $\mathcal{L}$ has a partial directional derivative at any point of $\mathbf{U}$ and in any direction $\mathbf{v} \in \mathbf{U}$ given by

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial u}(\mathbf{u},\mu;\mathbf{v}) &= \int_I\int_\omega (\boldsymbol{\varphi}_{\mathbf{u}} + \alpha\mathbf{u})\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \beta\int_{I^0_{\mathbf{u}}} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t\\ &+ \beta\int_{I^+_{\mathbf{u}}} \frac{1}{\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}}\int_\omega \mathbf{u}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\gamma}\int_I \mu(t)\int_\omega \mathbf{u}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{I^+_{\mathbf{u}}}\int_\omega (\boldsymbol{\varphi}_{\mathbf{u}} + \alpha\mathbf{u} + \beta\boldsymbol{\lambda})\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_{I^0_{\mathbf{u}}}\Big[\int_\omega \boldsymbol{\varphi}_{\mathbf{u}}\mathbf{v}\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big]\,\mathrm{d}t\\ &+ \frac{1}{\gamma}\int_I \mu(t)\int_\omega \mathbf{u}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{40}$$

If $(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\lambda}})$ satisfies the first order optimality condition (33), we define the associated Lagrange multiplier by $\bar{\mu}(t) = \|\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t)\|_{\mathbf{L}^2(\omega)}$. We introduce the following active constraint sets

$$I_\gamma = \{t \in I : \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = \gamma\} \quad\text{and}\quad I^+_\gamma = \{t \in I_\gamma : \bar{\mu}(t) > 0\}.$$

Let us observe that $\bar{\mu}(t) = 0$ if $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} < \gamma$. Indeed, given $\varepsilon \in (0, \gamma)$ we define the set $I_\varepsilon = \{t \in I : \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \leq \gamma - \varepsilon\}$. For every $\mathbf{v} \in B_\varepsilon$ we have that $\mathbf{v} + \bar{\mathbf{u}}(t) \in B_\gamma$ for $t \in I_\varepsilon$. Hence, from (33) we infer that

$$\int_\omega (\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}(t))\mathbf{v}\,\mathrm{d}x \geq 0 \quad \forall \mathbf{v} \in B_\varepsilon \text{ and a.a. } t \in I_\varepsilon,$$

which implies that $\bar{\mu}(t) = 0$ for almost all $t \in I_\varepsilon$. Since $\varepsilon > 0$ is arbitrary this proves our claim. As a consequence, the expression (40) is reduced to

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}) = \int_{I_\gamma^+} \int_\omega (\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}})\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}\mathbf{v}\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big]\,\mathrm{d}t$$
$$+ \frac{1}{\gamma}\int_{I_\gamma^+} \bar{\mu}(t) \int_\omega \bar{\mathbf{u}}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \quad \forall \mathbf{v} \in \mathbf{U}. \tag{41}$$

Lemma 4.3. *With the above notation, the following properties hold:*

$$\frac{1}{\gamma}\bar{\mu}(t)\bar{\mathbf{u}}(t) = -(\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t)) \quad \textit{for} \quad t \in I, \tag{42}$$

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}) = \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}\mathbf{v}\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big]\,\mathrm{d}t \geq 0 \quad \forall \mathbf{v} \in \mathbf{U}. \tag{43}$$

If $\beta > 0$, then (42) *implies that $\bar{\boldsymbol{\lambda}} \in L^2(I; \mathbf{L}^2(\omega))$.*

Proof. From the above comments and the definition of $\bar{\mu}$ we get that both sides of (42) are zero if $t \notin I_\gamma^+$. Let us prove the identity for $t \in I_\gamma^+$. Using (33) we obtain

$$\bar{\mu}(t) = \|\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}(t)\|_{\mathbf{L}^2(\omega)} = \frac{1}{\gamma}\sup_{\mathbf{v} \in \mathbf{B}_\gamma} \int_\omega -(\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}(t))\mathbf{v}\,\mathrm{d}x$$
$$\leq \frac{1}{\gamma}\int_\omega -(\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}(t))\bar{\mathbf{u}}(t)\,\mathrm{d}x \leq \frac{1}{\gamma}\bar{\mu}(t)\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} = \bar{\mu}(t).$$

Since Schwartz's inequality is satisfied as an equality we deduce the existence of a constant $c(t)$ such that $\bar{\mathbf{u}}(t) = c(t)(\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t))$. Inserting this identity in the above inequalities we infer

$$-\frac{c(t)}{\gamma}\int_\omega |\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}(t)|^2\,\mathrm{d}x = \bar{\mu}(t).$$

This implies that $-\frac{c(t)}{\gamma}\bar{\mu}(t) = 1$ and, hence, (42) holds. The equality in (43) is an immediate consequence of (41) and (42). The inequality in (43) follows from (36) and (37) and Schwarz's inequality. □

As a consequence of the above lemma we further obtain the following complementarity condition.

Corollary 4.4. *Let* $\bar{\mathbf{u}} \in \mathbf{U}_{\rm ad}$ *satisfy* (33). *Then* $(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\lambda}}, \bar{\mu})$ *satisfy*

$$\bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t) + \frac{1}{\gamma}\bar{\mu}(t)\bar{\mathbf{u}}(x,t) = 0, \ \textit{a.e. in } Q,$$
$$\bar{\mu}(t) \geq 0, \ \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \leq \gamma, \ \bar{\mu}(t)(\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} - \gamma) = 0 \ \textit{a.e. in } I.$$

Remark 4.5. *From Remark 3.3 and Lemma 4.3 the expressions given by* (41) *and* (43) *can be extended to continuous mappings*

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{\mathbf{u}}, \bar{\mu}) : L^2(I;\mathbf{L}^2(\omega)) \cap L^1(I;\mathbf{L}^2(\omega)) \longrightarrow \mathbb{R}.$$

In the case where $\beta = 0$*, then they can be extended to* $L^2(I;\mathbf{L}^2(\omega))$*.*

In order to formulate the second order conditions for optimality we introduce the cone of critical directions as follows:

$$C_{\bar{\mathbf{u}}} = \{\mathbf{v} \in \mathbf{S} : J'(\bar{\mathbf{u}};\mathbf{v}) = 0 \ \text{ and } \ \int_\omega \bar{\mathbf{u}}(t)\mathbf{v}(t)\,\mathrm{d}x \leq 0 \text{ if } t \in I_\gamma\}$$

where $\mathbf{S} = L^2(I;\mathbf{L}^2(\omega)) \cap L^1(I;\mathbf{L}^2(\omega))$ if $\beta > 0$ and $\mathbf{S} = L^2(I;\mathbf{L}^2(\omega))$ otherwise. Let us observe that $\int_\omega \bar{\mathbf{u}}(t)\mathbf{v}(t)\,\mathrm{d}x = 0$ if $t \in I_\gamma^+$ and $\mathbf{v} \in C_{\bar{\mathbf{u}}}$. Indeed, from the condition $\int_\omega \bar{\mathbf{u}}(t)\mathbf{v}(t)\,\mathrm{d}x \leq 0$ if $t \in I_\gamma$ and (42) we deduce that $\int_\omega [\bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t)]\mathbf{v}(t)\,\mathrm{d}x \geq 0$ for $t \in I$. Now, using that

$$0 = J'(\bar{\mathbf{u}};\mathbf{v}) = \int_{I^+_{\bar{\mathbf{u}}}} \int_\omega [\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}]\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_{I^0_{\bar{\mathbf{u}}}} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big]\,\mathrm{d}t$$

and (43) we infer that $\int_\omega [\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}}]\mathbf{v}\,\mathrm{d}x = 0$ in I. Taking into account again (42), the desired identity follows.

We also provide the following formal definition for every $\mathbf{v} \in L^2(I;\mathbf{L}^2(\omega))$

$$j''(\mathbf{u};\mathbf{v}^2) = \begin{cases} \displaystyle\int_{I^+_{\mathbf{u}}} \frac{1}{\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}}\Big[\int_\omega |\mathbf{v}|^2\,\mathrm{d}x - \Big(\int_\omega \frac{\mathbf{u}\mathbf{v}}{\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}}\,\mathrm{d}x\Big)^2\Big]\,\mathrm{d}t & \text{if } \mathbf{u} \not\equiv 0, \\ 0 & \text{if } \mathbf{u} \equiv 0. \end{cases}$$

This does not mean that j has second order directional derivatives in any direction $\mathbf{v}$. For some directions $\mathbf{v}$ the second derivative exists and it is given by the above formula, but for some others the above integral is infinity. In any case, since the integrand for variable t is non negative, the integral is always defined. Now, we set $\forall \mathbf{v} \in L^2(I;\mathbf{L}^2(\omega))$

$$\frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}^2}(\mathbf{u}, \mu; \mathbf{v}^2) = F''(\mathbf{u})\mathbf{v}^2 + \beta j''(\mathbf{u};\mathbf{v}^2) + \frac{1}{\gamma}\int_I \mu(t)\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t, \tag{44}$$

where $F''(\mathbf{u})\mathbf{v}^2$ was given in (25).

Theorem 4.6. *If* $\bar{\mathbf{u}}$ *is an* $L^2(I;\mathbf{L}^2(\omega))$ *local minimizer of* (P)*, then the following second order condition holds:* $\frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2) \geq 0$ *for all* $\mathbf{v} \in C_{\bar{\mathbf{u}}}$*.*

Proof. The proof follows the lines of the [16, Theorem 3.2]. There are only some differences in the case $\beta > 0$, that we analyze here. Let $\varepsilon > 0$ be such that J achieves its minimum value in the set $\mathbf{U}_{\rm ad} \cap B_\varepsilon(\bar{\mathbf{u}})$ at $\bar{\mathbf{u}}$. First we take $\mathbf{v} \in C_{\bar{\mathbf{u}}} \cap L^\infty(I;\mathbf{L}^2(\omega)) \subset \mathbf{U}$, the assumption $L^\infty(I;\mathbf{L}^2(\omega))$ will be removed later. We define for every integer $k \geq 1$

$$\mathbf{v}_k(x,t) = \begin{cases} 0 & \text{if } \gamma^2 - \frac{1}{k} < \|\bar{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\omega)} < \gamma^2 \text{ or } 0 < \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} < \frac{1}{k}, \\ \mathbf{v}(x,t) & \text{otherwise.} \end{cases}$$

Though the definition of $\mathbf{v}_k$ is slightly different from the one given in [16, Theorem 3.2] to deal with the case $\beta = 0$. Arguing as in [16] we get that for k big enough there exists $\alpha_k > 0$ sufficiently small such that the function

$$\phi_k : (-\alpha_k, +\alpha_k) \longrightarrow \mathbf{U}, \quad \phi_k(\rho) = \sqrt{1 - \frac{\rho^2}{\gamma^2}\|\mathbf{v}_k\|^2_{\mathbf{L}^2(\omega)}}\,\bar{\mathbf{u}} + \rho \mathbf{v}_k$$

enjoys the following properties: $\phi_k(\rho) \in \mathbf{U}_{\rm ad}$ and $\|\phi_k(\rho) - \bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))} \leq \varepsilon \; \forall \rho \in [0, \alpha_k)$. Even more, it is immediate to check that $\|\phi_k(\rho) - \bar{\mathbf{u}}\|_{L^\infty(I;\mathbf{L}^2(\omega))} \leq \varepsilon$ for every $\rho > 0$ small enough. Now, we define $\psi_k : (-\alpha_k, +\alpha_k) \longrightarrow \mathbb{R}$ by $\psi_k(\rho) = J(\phi_k(\rho))$. For every $\rho > 0$ small enough it is easy to check that $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} < \frac{1}{k}$ if $\|\phi_k(\rho)\|_{\mathbf{L}^2(\omega)} < \frac{1}{4k}$ and consequently $\mathbf{v}_k(t) = 0$ there if in addition $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} > 0$. Hence, one can verify that ψ_k is twice continuously differentiable in an interval $[0, \rho_0]$ for $\rho_0 > 0$ small enough.

We have that $\psi_k(0) = J(\bar{\mathbf{u}}) \leq \psi_k(\rho)$ for every $\rho \in [0, \alpha_k)$. Let us compute $\psi_k'(0)$

$$\psi_k'(0) = J'(\bar{\mathbf{u}};\mathbf{v}_k) = \int_{I_\gamma^+}\int_\omega (\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}})\mathbf{v}_k \,\mathrm{d}x\,\mathrm{d}t + \int_{I^0_{\bar{\mathbf{u}}}} [\bar{\varphi}\mathbf{v}_k + \beta\|\mathbf{v}_k\|_{\mathbf{L}^2(\omega)}]\,\mathrm{d}t.$$

By definition $\mathbf{v}_k(x,t) = \mathbf{v}(x,t)$ holds for $t \in I_\gamma$ and for $t \in I^0_{\bar{\mathbf{u}}}$. Further, the fact that $\mathbf{v} \in C_{\bar{\mathbf{u}}}$ implies that $J'(\bar{\mathbf{u}};\mathbf{v}_k) = J'(\bar{\mathbf{u}};\mathbf{v}) = 0$, hence $\psi_k'(0) = 0$. Together with the fact that ψ_k achieves the minimum in $[0, \alpha_k)$ at 0 this implies that $\psi_k''(0) \geq 0$. Then, we get with (42) and (44)

$$\begin{aligned} 0 \leq \psi_k''(0) &= J''(\phi_k(0))\phi_k'(0)^2 + J'(\phi_k(0))\phi_k''(0) = F''(\bar{\mathbf{u}})\mathbf{v}_k^2 + \beta j''(\bar{\mathbf{u}};\mathbf{v}_k^2) \\ &- \frac{1}{\gamma^2}\int_{I_\gamma^+}\int_\omega (\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}})\bar{\mathbf{u}}\|\mathbf{v}_k(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}x\,\mathrm{d}t \\ &= F''(\bar{\mathbf{u}})\mathbf{v}_k^2 + \beta j''(\bar{\mathbf{u}};\mathbf{v}_k^2) + \frac{1}{\gamma}\int_{I_\gamma^+} \bar{\mu}(t)\|\mathbf{v}_k(t)\|^2_{\mathbf{L}^2(\omega)}\,\mathrm{d}t = \frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}_k^2). \end{aligned}$$

We know pass to the limit in the above inequality as $k \to \infty$. To this end, we observe that applying Lebesgue's dominated convergence theorem the convergence $\mathbf{v}_k \to \mathbf{v}$ in $L^p(I;\mathbf{L}^2(\omega))$ holds for every $p \in [2, \infty)$. Then Theorems 2.6 and 3.2 implies that $F''(\bar{\mathbf{u}})\mathbf{v}_k^2 \to F''(\bar{\mathbf{u}})\mathbf{v}^2$. Moreover, the convergence of the third term of $\frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}_k^2)$ is immediate. Finally, it is obvious that $j''(\bar{\mathbf{u}};\mathbf{v}^2) \geq j''(\bar{\mathbf{u}};\mathbf{v}_k^2)$. All together this yields that $\frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2) \geq 0$.

To remove the assumption $\mathbf{v} \in L^\infty(I;\mathbf{L}^2(\omega))$ one can proceed as at the end of the proof of [16, Theorem 3.2]. □

For the property $\phi_k(\rho) \in \mathbf{U}_{\rm ad}$ the definition of ϕ_k as non-affine function is essential.

It reflects the fact that the constraint $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \le \gamma$ is not of affine structure as well.

Theorem 4.7. *Assume that* $\alpha > 0$ *and* $(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\lambda}})$ *satisfies the first order optimality condition* (33). *We also suppose that the second order condition* $\frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2) > 0$ *for all* $\mathbf{v} \in C_{\bar{\mathbf{u}}} \setminus \{0\}$ *holds. Then, there exist* $\varepsilon > 0$ *and* $\kappa > 0$ *such that*

$$J(\bar{\mathbf{u}}) + \frac{\kappa}{2}\|\mathbf{u} - \bar{\mathbf{u}}\|^2_{L^2(I;\mathbf{L}^2(\omega))} \le J(\mathbf{u}) \ \forall \mathbf{u} \in \mathbf{U}_{\rm ad} \text{ with } \|\mathbf{u} - \bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))} \le \varepsilon. \tag{45}$$

Proof. The proof of the theorem when $\beta = 0$ is the same as the proof of [16, Theorem 3.3]. Here we concentrate again on the case $\beta > 0$ and provide the necessary changes. We argue by contradiction and assume that (45) does not hold. Then, for every integer $k \ge 1$ there exists a control $u_k \in \mathbf{U}_{\rm ad}$ such that

$$\rho_k = \|\mathbf{u}_k - \bar{\mathbf{u}}\|_{L^2(Q_\omega)} < \frac{1}{k} \quad \text{and} \quad J(\mathbf{u}_k) < J(\bar{\mathbf{u}}) + \frac{1}{2k}\|\mathbf{u}_k - \bar{\mathbf{u}}\|^2_{L^2(I;\mathbf{L}^2(\omega))}. \tag{46}$$

We define $\mathbf{v}_k = \frac{1}{\rho_k}(\mathbf{u}_k - \bar{\mathbf{u}})$. Since $\|\mathbf{v}_k\|_{L^2(I;\mathbf{L}^2(\omega))} = 1$ for every k, taking a subsequence, we can assume that $\mathbf{v}_k \rightharpoonup \mathbf{v}$ in $L^2(I; \mathbf{L}^2(\omega))$.

According to (26) there exists $T^* < \infty$ such that $\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \le \beta$ for all $t \ge T^*$. Then, (36) implies that $\bar{\mathbf{u}}(t) \equiv 0$ for $t \ge T^*$.

We split the proof into four steps.

Step I - If $\{\mathbf{w}_k\}_{k=1}^\infty \subset \mathbf{U}_{\rm ad}$ *converges to* $\bar{\mathbf{u}}$ *in* $L^2(I; \mathbf{L}^2(\omega))$*, then* $\mathbf{y}_{\mathbf{w}_k} \to \bar{\mathbf{y}}$ *in* $\mathcal{Y}$ *and* $\boldsymbol{\varphi}_{\mathbf{w}_k} \to \bar{\boldsymbol{\varphi}}$ *in* $\mathbf{X}$. Indeed, since $\{\mathbf{w}_k\}_{k=1}^\infty \subset \mathbf{U}_{\rm ad}$, then it is bounded in $L^\infty(I; \mathbf{L}^2(\omega))$. Hence, we have that $\mathbf{w}_k \to \bar{\mathbf{u}}$ in $L^p(I; \mathbf{L}^2(\omega))$ for every $p \in [2, \infty)$. Therefore, applying Theorem 2.6 we get that $\mathbf{y}_{\mathbf{w}_k} = G(\mathbf{w}_k) = \mathcal{G}(\mathbf{f}_0 + \chi_\omega \mathbf{w}_k) \to \mathcal{G}(\mathbf{f}_0 + \chi_\omega \bar{\mathbf{u}}) = G(\bar{\mathbf{u}}) = \bar{\mathbf{y}}$ in $\mathcal{Y}$. Since $\mathcal{Y}$ is continuously embedded in $L^q(I; \mathbf{L}^4(\Omega)) \cap L^2(I; \mathbf{L}^2(\Omega))$ we deduce from Theorem 3.2 that $\boldsymbol{\varphi}_{\mathbf{w}_k} \to \bar{\boldsymbol{\varphi}}$ in $\mathbf{X}$.

Step II - $\mathbf{v} \in C_{\bar{\mathbf{u}}}$. For every $0 < T < \infty$ the continuous embedding $L^2(0; T; \mathbf{L}^2(\omega)) \subset L^1(0, T; \mathbf{L}^2(\omega))$ implies that $\mathbf{v}_k \rightharpoonup \mathbf{v}$ in $L^1(0, T; \mathbf{L}^2(\omega))$ and, hence, $\mathbf{v} \in L^1(0, T; \mathbf{L}^2(\omega))$ for all T finite. On the other hand, from (46), the convexity of j, and the mean value theorem we get

$$\begin{aligned} j'(\bar{\mathbf{u}}; \mathbf{v}_k) &\le \frac{j(\mathbf{u}_k) - j(\bar{\mathbf{u}})}{\rho_k} = \frac{J(\mathbf{u}_k) - J(\bar{\mathbf{u}})}{\beta \rho_k} - \frac{F(\mathbf{u}_k) - F(\bar{\mathbf{u}})}{\beta \rho_k} \\ &\le \frac{\rho_k}{2\beta k} - \frac{1}{\beta} F'(\mathbf{u}_{\theta_k})\mathbf{v}_k \to -\frac{1}{\beta} F'(\bar{\mathbf{u}})\mathbf{v}, \end{aligned} \tag{47}$$

where $\mathbf{u}_{\theta_k} = \bar{\mathbf{u}} + \theta_k(\mathbf{u}_k - \bar{\mathbf{u}})$ for some $\theta_k \in (0, 1)$. The convergence $F'(\mathbf{u}_{\theta_k})\mathbf{v}_k \to F'(\bar{\mathbf{u}})\mathbf{v}$ follows from Step I and the expression for F' in (24). As a consequence we deduce the existence of a constant $C > 0$ such that $j'(\bar{\mathbf{u}}; \mathbf{v}_k) \le C$ for every k. Using that

$I_{\bar{\mathbf{u}}}^+ \subset [0,T^*]$ we deduce from (32) that for $T^* < T < \infty$

$$\int_{T^*}^{T} \|\mathbf{v}_k(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t \le \int_{I_{\bar{\mathbf{u}}}^0} \|\mathbf{v}_k(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t = j'(\bar{\mathbf{u}};\mathbf{v}_k) - \int_{I_{\bar{u}}^+}\int_\omega \frac{\bar{\mathbf{u}}\mathbf{v}}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C + \int_0^{T^*} \|\mathbf{v}_k(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t \le C + \sqrt{T^*}.$$

This leads

$$\int_{T^*}^{T} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t \le \liminf_{k\to\infty} \int_{T^*}^{T} \|\mathbf{v}_k(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t \le C + \sqrt{T^*} \quad \forall T > T^*.$$

Thus, the regularity $\mathbf{v} \in L^1(I;\mathbf{L}^2(\omega))$ holds. Using (34), the fact that $\bar{\mu}(t) = 0$ if $t \notin I_\gamma^+$, that $\bar{\boldsymbol{\lambda}} \in L^2(I;\mathbf{L}^2(\omega))$ (Lemma 4.3), and that $\int_I\int_\omega \bar{\boldsymbol{\lambda}}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \le j'(\bar{\mathbf{u}};\mathbf{v})$ we deduce

$$0 \le \int_{I_\gamma^+}\int_\omega (\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}})\mathbf{v}_k\,\mathrm{d}x\,\mathrm{d}t \to \int_{I_\gamma^+}\int_\omega (\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}})\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \le J'(\bar{\mathbf{u}};\mathbf{v}).$$

Next we prove the contrary inequality. Since $\{\mathbf{v}_k\}_{k=1}^\infty$ converges weakly to $\mathbf{v}$ in $L^2(I;\mathbf{L}^2(\omega)) \cap L^1(0;T;\mathbf{L}^2(\omega))$ for every $T < \infty$ we have

$$\int_{I_{\bar{\mathbf{u}}}^+}\int_\omega \bar{\boldsymbol{\lambda}}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_{I_{\bar{\mathbf{u}}}^0\cap[0,T]} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t$$
$$\le \liminf_{k\to\infty}\Big\{\int_{I_{\bar{\mathbf{u}}}^+}\int_\omega \bar{\boldsymbol{\lambda}}\mathbf{v}_k\,\mathrm{d}x\,\mathrm{d}t + \int_{I_{\bar{\mathbf{u}}}^0\cap[0,T]} \|\mathbf{v}_k(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t\Big\} \le \liminf_{k\to\infty} j(\bar{\mathbf{u}};\mathbf{v}_k).$$

Taking the supremum on T we deduce that $j'(\bar{\mathbf{u}};\mathbf{v}) \le \liminf_{k\to\infty} j'(\bar{\mathbf{u}};\mathbf{v}_k)$. Using this fact and (47) we infer $j'(\bar{\mathbf{u}};\mathbf{v}) \le \liminf_{k\to\infty} j'(\bar{\mathbf{u}};\mathbf{v}_k) \le -\frac{1}{\beta}F'(\bar{\mathbf{u}})\mathbf{v}$, which is equivalent to $J'(\bar{\mathbf{u}};\mathbf{v}) = F'(\bar{\mathbf{u}})\mathbf{v} + \beta j'(\bar{\mathbf{u}};\mathbf{v}) \le 0$. Thus, we have $J'(\bar{\mathbf{u}};\mathbf{v}) = 0$. To conclude that $\mathbf{v} \in C_{\bar{\mathbf{u}}}$ we need to check the inequality $\int_\omega \bar{\mathbf{u}}(t)\mathbf{v}(t)\,\mathrm{d}t \le 0$ for $t \in I_\gamma$. This proof is the same as in [16, Theorem 3.3].

Step III - $\frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}},\bar{\boldsymbol{\lambda}};\mathbf{v}^2) \le 0$. Applying Egoroff's Theorem we deduce that for every $\delta \in (0,T^*)$ there exists a measurable set $I_\delta \subset (0,T^*)$ such that $\|\mathbf{u}_k - \bar{\mathbf{u}}\|_{L^\infty(I_\delta;\mathbf{L}^2(\omega))} \to 0$ as $k \to \infty$ and $|I_\delta| > T^* - \delta$. Now, for every integer $l \ge 1$ we define

$$I_{\delta,l} = \{t \in I_\delta : \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \ge \frac{1}{l}\},\ j_\delta(\mathbf{u}) = \int_{I_\delta} \|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t,\ j_{\delta,l}(\mathbf{u}) = \int_{I_{\delta,l}} \|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t.$$

We also set $g_{\delta,l}(\mathbf{u}) = j(\mathbf{u}) - j_{\delta,l}(\mathbf{u})$,

$$\mathcal{L}_\delta(\mathbf{u},\bar{\mu}) = F(\mathbf{u}) + \beta j_\delta(\mathbf{u}) + \frac{1}{2\gamma}\int_{I_\gamma^+} \bar{\mu}(t)\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)}^2\,\mathrm{d}t,$$

and define $\mathcal{L}_{\delta,l}(\mathbf{u},\bar{\mu})$ as above replacing j_δ by $j_{\delta,l}$. Denoting by $\mathbf{B}_{\delta,l}(\bar{\mathbf{u}})$ the ball in

$L^\infty(I_{\delta,l};\mathbf{L}^2(\omega))$ centered at $\bar{\mathbf{u}}$ and radius $\frac{1}{2l}$, we have for $t \in I_{\delta,l}$

$$\|\mathbf{u}(t)\|_{\mathbf{L}^2(\omega)} \geq \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} - \|\mathbf{u}(t) - \bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \geq \frac{1}{l} - \frac{1}{2l} = \frac{1}{2l} \quad \forall \mathbf{u} \in \mathbf{B}_{\delta,l}(\bar{\mathbf{u}}).$$

We denote by $k_{\delta,l}$ an integer such that $\mathbf{u}_k \in \mathbf{B}_{\delta,l}(\bar{\mathbf{u}})$ for all $k \geq k_{\delta,l}$.

It is immediate that $\mathcal{L}_{\delta,l}(\mathbf{u},\bar{\mu})$ is of class C^2 with respect to $\mathbf{u}$ in $\mathbf{B}_{\delta,l}(\bar{\mathbf{u}})$. Since $\|\mathbf{u}_k(t)\|_{\mathbf{L}^2(\omega)} \leq \gamma = \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}$ for $t \in I^+_\gamma$, we get $\mathcal{L}(\mathbf{u}_k,\bar{\mu}) - \mathcal{L}(\bar{\mathbf{u}},\bar{\mu}) \leq J(\mathbf{u}_k) - J(\bar{\mathbf{u}})$. Now, using (46), the fact that $\mathcal{L} = \mathcal{L}_{\delta,l} + g_{\delta,l}$, the inequality $g'_{\delta,l}(\bar{\mathbf{u}};\mathbf{u}_k - \bar{\mathbf{u}}) \leq g_{\delta,l}(\mathbf{u}_k) - g_{\delta,l}(\bar{\mathbf{u}})$, a second order Taylor expansion, and (43) we obtain for every $k \geq k_{\delta,l}$

$$\begin{aligned}
&\frac{\rho_k^2}{2k} \geq \mathcal{L}(\mathbf{u}_k,\bar{\mu}) - \mathcal{L}(\bar{\mathbf{u}},\bar{\mu}) = \beta g_{\delta,l}(\mathbf{u}_k) - \beta g_{\delta,l}(\bar{\mathbf{u}}) + \frac{\partial \mathcal{L}_{\delta,l}}{\partial \mathbf{u}}(\bar{\mathbf{u}},\bar{\mu})(\mathbf{u}_k - \bar{\mathbf{u}}) \\
&+ \frac{1}{2}\frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\mathbf{u}_{\theta_k},\bar{\mu})(\mathbf{u}_k - \bar{\mathbf{u}})^2 \geq \frac{\partial \mathcal{L}}{\partial \mathbf{u}}(\bar{\mathbf{u}},\bar{\mu})(\mathbf{u}_k - \bar{\mathbf{u}}) + \frac{1}{2}\frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\mathbf{u}_{\theta_k},\bar{\mu})(\mathbf{u}_k - \bar{\mathbf{u}})^2 \\
&\geq \frac{1}{2}\frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\mathbf{u}_{\theta_k},\bar{\mu})(\mathbf{u}_k - \bar{\mathbf{u}})^2.
\end{aligned}$$

Dividing the above inequality by $\frac{\rho_k^2}{2}$ we infer $\frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\mathbf{u}_{\theta_k},\bar{\mu})\mathbf{v}_k^2 \leq \frac{1}{k}$. Below we will prove that

$$\frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}},\bar{\mu})\mathbf{v}^2 \leq \liminf_{k\to\infty} \frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\mathbf{u}_{\theta_k},\bar{\mu})\mathbf{v}_k^2 \leq 0. \tag{48}$$

Using this fact and observing that $\frac{\partial^2 \mathcal{L}_{\delta,l}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}},\bar{\mu})\mathbf{v}^2$ increases when $l \to \infty$, we infer that $\frac{\partial^2 \mathcal{L}_\delta}{\partial \mathbf{u}^2}(\bar{\mathbf{u}},\bar{\mu};\mathbf{v}^2) \leq 0$. The same argument applies as $\delta \to 0$ monotonically, hence $\frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}},\bar{\mu};\mathbf{v}^2) \leq 0$ as desired. It remains to prove the first inequality of (48). In view of the weak convergence $\mathbf{v}_k \rightharpoonup \mathbf{v}$ in $L^2(I;\mathbf{L}^2(\omega))$, the strong convergence $\mathbf{u}_{\theta_k} \to \bar{\mathbf{u}}$ in $L^p(I_\delta;\mathbf{L}^2(\omega))$ for every $p \in [2,\infty]$, and the expression for the second derivative of the Lagrange function, it is obvious that the only delicate point is to prove that

$$\int_I \int_\omega (\mathbf{z}\cdot\nabla)\bar{\boldsymbol{\varphi}}\mathbf{z}\,\mathrm{d}x\,\mathrm{d}t \leq \liminf_{k\to\infty} \int_I \int_\omega (\mathbf{z}_k\cdot\nabla)\boldsymbol{\varphi}_k\mathbf{z}_k\,\mathrm{d}x\,\mathrm{d}t$$

with $\mathbf{z}_k = G'(\mathbf{u}_{\theta_k})\mathbf{v}_k$, $\mathbf{z} = G'(\bar{\mathbf{u}})\mathbf{v}$, and $\boldsymbol{\varphi}_{\theta_k}$ the adjoint state associated to $\mathbf{u}_{\theta_k}$. The boundedness of $\{\mathbf{v}_k\}_{k=1}^\infty$ in $L^2(I;\mathbf{L}^2(\omega))$ implies that $\{\mathbf{z}_k\}_{k=1}^\infty$ is bounded in $\mathbf{W}(I)$ by a constant C. The same constant applies to $\mathbf{z}$. From Step I we know that $\mathbf{y}_{\mathbf{u}_{\theta_k}} \to \bar{\mathbf{y}}$ in $\mathcal{Y}$ and $\boldsymbol{\varphi}_{\theta_k} \to \bar{\boldsymbol{\varphi}}$ in $\mathbf{X} \subset C(I;\mathbf{C}^1(\bar{\Omega}))$. Then, from Lemma 2.5 it is easy to infer that $\mathbf{z}_k \rightharpoonup \mathbf{z}$ in $\mathbf{W}(I)$. As a consequence we have that $\mathbf{z}_k \to \mathbf{z}$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ for every $T < \infty$. According to (26), for every $\eta > 0$ there exists $T_\eta < \infty$ such that

$\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{H}_0^1(\Omega)} \leq \eta$ for all $t \geq T_\eta$. Using these facts we obtain for all $T \geq T_\eta$

$$
\begin{aligned}
&\int_I \int_\Omega (\mathbf{z} \cdot \nabla)\bar{\boldsymbol{\varphi}}\mathbf{z}\,\mathrm{d}x\,\mathrm{d}t = \lim_{k\to\infty} \int_0^T \int_\Omega (\mathbf{z}_k \cdot \nabla)\boldsymbol{\varphi}_{\theta_k}\mathbf{z}_k\,\mathrm{d}x\,\mathrm{d}t + \int_T^\infty \int_\Omega (\mathbf{z} \cdot \nabla)\bar{\boldsymbol{\varphi}}\mathbf{z}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \liminf_{k\to\infty} \int_I \int_\Omega (\mathbf{z}_k \cdot \nabla)\boldsymbol{\varphi}_{\theta_k}\mathbf{z}_k\,\mathrm{d}x\,\mathrm{d}t + \limsup_{k\to\infty} \int_T^\infty \|\mathbf{z}_k(t)\|^2_{\mathbf{L}^4(\Omega)} \|\boldsymbol{\varphi}_{\theta_k}(t)\|_{\mathbf{H}_0^1(\Omega)}\,\mathrm{d}t \\
&+ \int_T^\infty \|\mathbf{z}(t)\|^2_{\mathbf{L}^4(\Omega)} \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{H}_0^1(\Omega)}\,\mathrm{d}t \leq \liminf_{k\to\infty} \int_I \int_\Omega (\mathbf{z}_k \cdot \nabla)\boldsymbol{\varphi}_{\theta_k}\mathbf{z}_k\,\mathrm{d}x\,\mathrm{d}t \\
&+ \sqrt{|\Omega|}\limsup_{k\to\infty} \int_T^\infty \|\mathbf{z}_k(t)\|^2_{\mathbf{L}^4(\Omega)}\,\mathrm{d}t\,\|\boldsymbol{\varphi}_{\theta_k} - \bar{\boldsymbol{\varphi}}\|_{C(I;\mathbf{C}^1(\bar\Omega))} \\
&+ \Big(\limsup_{k\to\infty} \int_T^\infty \|\mathbf{z}_k(t)\|^2_{\mathbf{L}^4(\Omega)}\,\mathrm{d}t + \int_T^\infty \|\mathbf{z}(t)\|^2_{\mathbf{L}^4(\Omega)}\,\mathrm{d}t\Big) \sup_{t\geq T} \|\bar{\varphi}(t)\|_{\mathbf{H}_0^1(\Omega)} \\
&\leq \liminf_{k\to\infty} \int_I \int_\Omega (\mathbf{z}_k \cdot \nabla)\boldsymbol{\varphi}_{\theta_k}\mathbf{z}_k\,\mathrm{d}x\,\mathrm{d}t + C'\eta.
\end{aligned}
$$

Since η can be selected arbitrarily small, the desired inequality follows.

Step IV - Final contradiction. Arguing as in the proof of [16, Theorem 3.3] we infer that $\mathbf{v} = 0$ and then $\frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2) \geq \alpha > 0$, which contradicts Step III. □

4.2. Case $\boldsymbol{\sigma = 1}$.

Here we assume that $\mathbf{K}$ is the $\mathbf{L}^1(\omega)$-ball centered at zero and radius γ, i.e. we choose $\mathbf{K}$ as $\mathbf{B}_\gamma = \{\mathbf{v} \in \mathbf{L}^2(\omega) : \|\mathbf{v}\|_{\mathbf{L}^1(\omega)} \leq \gamma\}$. We recall that $\mathbf{U} \subset L^2(I; \mathbf{L}^2(\omega))$; see the comments after (21)–(23). Here, we also consider local minimizers in the $L^2(I; \mathbf{L}^2(\omega))$ sense.

First, in the analysis of this case we introduce the Lagrange multiplier associated with the control constraint. If $\bar{\mathbf{u}} \in \mathbf{U}$ is a local minimizer of (P), then we infer from (33) that, for almost all $t \in I$, $\bar{\mathbf{u}}(t)$ is a global minimizer of the optimization problem

$$
\min_{\mathbf{v}\in\mathbf{U}} \mathcal{J}(\mathbf{v}) := \int_\omega (\bar{\varphi}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t))\mathbf{v}\,\mathrm{d}x + I_{\mathbf{B}_\gamma}(\mathbf{v}).
$$

Then, we have that $0 \in \partial\mathcal{J}(\bar{\mathbf{u}}(t)) = \bar{\varphi}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t) + \partial I_{\mathbf{B}_\gamma}(\bar{\mathbf{u}}(t))$. Hence, the existence of a Lagrange multiplier follows

$$
\bar{\boldsymbol{\mu}}(t) \in \partial I_{\mathbf{B}_\gamma}(\bar{\mathbf{u}}(t)) \ \text{ such that } \ \bar{\boldsymbol{\varphi}}(t) + \alpha\bar{\mathbf{u}}(t) + \beta\bar{\boldsymbol{\lambda}}(t) + \bar{\boldsymbol{\mu}}(t) = 0. \tag{49}
$$

In the rest of this section, except if some other thing is indicated, $(\bar{\boldsymbol{\varphi}}, \bar{\mathbf{u}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ will denote functions satisfying (49), where $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$, $\bar{\boldsymbol{\lambda}} \in \partial j(\bar{\mathbf{u}})$ and $\bar{\boldsymbol{\varphi}} \in \mathbf{V}^{2,1}(I) \cap \mathbf{X}$ is the adjoint state associated with $\bar{\mathbf{u}}$; see Theorem 3.2.

Arguing as in [15, Corollary 3.1] and replacing $\mathrm{sign}(\bar{\mathbf{u}}(x,t))$ and $\mathrm{sign}(\bar{\boldsymbol{\mu}}(x,t))$ by

$\frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}$ and $\frac{\bar{\boldsymbol{\mu}}(x,t)}{|\bar{\boldsymbol{\mu}}(x,t)|}$, respectively, we deduce the following properties:

$$\begin{cases} \bar{\mathbf{u}}(x,t)\bar{\boldsymbol{\mu}}(x,t) = |\bar{\mathbf{u}}(x,t)||\bar{\boldsymbol{\mu}}(x,t)| \text{ for a.a. } (x,t) \in \omega \times I, \\ \text{if } \|\bar{\mathbf{u}}(t)\|_{L^1(\Omega)} < \gamma \text{ then } \bar{\boldsymbol{\mu}}(t) \equiv 0 \text{ in } \omega \text{ a.e. in } I, \\ \text{if } \|\bar{\mathbf{u}}(t)\|_{L^1(\Omega)} = \gamma \text{ and } \bar{\boldsymbol{\mu}}(t) \not\equiv 0 \text{ in } \omega, \\ \qquad\qquad \text{then } \operatorname{supp}(\bar{\mathbf{u}}(t)) \subset \{x \in \omega : |\bar{\boldsymbol{\mu}}(x,t)| = \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}\}. \end{cases} \tag{50}$$

Lemma 4.8. *Under the above notations, we have that $\bar{\mathbf{u}}, \bar{\boldsymbol{\mu}} \in L^\infty(I; \mathbf{L}^\infty(\omega))$ holds. Furthermore, if $\beta > 0$ then $\bar{\boldsymbol{\lambda}}$ also enjoys the $L^\infty(I; \mathbf{L}^\infty(\omega))$ regularity.*

Proof. First we assume that $\beta = 0$. Observe that the first relation of (50) implies that $|\alpha\bar{\mathbf{u}}(x,t) + \bar{\boldsymbol{\mu}}(x,t)| = \alpha|\bar{\mathbf{u}}(x,t)| + |\bar{\boldsymbol{\mu}}(x,t)|$. We deduce from (49) that $\alpha|\bar{\mathbf{u}}(x,t)| + |\bar{\boldsymbol{\mu}}(x,t)| \le |\bar{\boldsymbol{\varphi}}(x,t)|$. Since $\bar{\boldsymbol{\varphi}} \in L^\infty(I; \mathbf{L}^\infty(\omega))$, this inequality proves that $\bar{\mathbf{u}}, \bar{\boldsymbol{\mu}} \in L^\infty(I; \mathbf{L}^\infty(\omega))$ as well.

If $\beta > 0$, then we know the existence of $T^* < \infty$ such that $\bar{\mathbf{u}}(t) \equiv 0$ for $t \ge T^*$. This along with the second relation of (50) implies that $\bar{\boldsymbol{\mu}}(t) \equiv 0$ for $t \ge T^*$ too. We prove that $\|\bar{\boldsymbol{\mu}}\|_{L^\infty(I;\mathbf{L}^\infty(\omega))} < M$ for every $M > \frac{1}{\gamma}\int_I\int_\omega \bar{\boldsymbol{\mu}}\bar{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t$. If this is not the case for one of these constants M we define

$$E_M = \{(x,t) \in \omega \times (0,T^*) : |\bar{\boldsymbol{\mu}}(x,t)| \ge M\} \;\text{ and }\; \mathbf{v}(x,t) = \frac{\gamma}{|E_M|}\frac{\bar{\boldsymbol{\mu}}(x,t)}{|\bar{\boldsymbol{\mu}}(x,t)|}\chi_{E_M}(x,t).$$

Then, we have that $\mathbf{v} \in \mathbf{U}_{\rm ad}$ and

$$\int_I\int_\omega \bar{\boldsymbol{\mu}}\bar{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t < \gamma M \le \frac{\gamma}{|E_M|}\int_{E_M} |\bar{\boldsymbol{\mu}}|\,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\omega \bar{\boldsymbol{\mu}}\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t,$$

which contradicts (34). Using again (49) we infer that $\bar{\boldsymbol{\lambda}} \in L^\infty(I; \mathbf{L}^\infty(\omega))$ if $\beta > 0$. □

From Lemma 4.8 we get the following representation for $\bar{\boldsymbol{\mu}}$.

Lemma 4.9. *If $\bar{\mathbf{u}}(x,t) \neq 0$, then $\bar{\boldsymbol{\mu}}(x,t) = \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}\frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}$ holds.*

Proof. It $\|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)} = 0$, then the equality obviously holds. Let us consider the case $\|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)} \neq 0$. Then $\|\bar{\mathbf{u}}(t)\|_{L^1(\Omega)} = \gamma$, by the second equation (50). From the first relation of (50) and the assumption that $\bar{\mathbf{u}}(x,t) \neq 0$ we deduce the existence of a constant $c(x,t)$ such that $\bar{\boldsymbol{\mu}}(x,t) = c(x,t)\bar{\mathbf{u}}(x,t)$. Inserting this identity in the first equality of (50) and using the third statement of (50) we obtain $c(x,t)|\bar{\mathbf{u}}(x,t)|^2 = |\bar{\boldsymbol{\mu}}(x,t)||\bar{\mathbf{u}}(x,t)| = \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}|\bar{\mathbf{u}}(x,t)|$. Hence, $c(x,t) = \frac{\|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}}{|\bar{\mathbf{u}}(x,t)|}$ and the statement of the lemma follows. □

Next we address the second order analysis of the control problem (P). To this end we introduce the function

$$g : \mathbf{L}^1(\omega) \longrightarrow \mathbb{R}, \quad g(\mathbf{v}) = \int_\omega |\mathbf{v}(x)|\,\mathrm{d}x = \int_\omega \sqrt{v_1^2(x) + v_2^2(x)}\,\mathrm{d}x,$$

and the critical cone

$$C_{\bar{\mathbf{u}}} = \{\mathbf{v} \in \mathbf{S} : J'(\bar{\mathbf{u}}; \mathbf{v}) = 0 \text{ and } g'(\mathbf{u}(t); \mathbf{v}(t)) \le 0 \text{ for a.a. } t \in I_\gamma\}$$

where $I_\gamma = \{t \in I : g(\bar{\mathbf{u}}(t)) = \gamma\}$ and $\mathbf{S} = L^2(I; \mathbf{L}^2(\omega)) \cap L^1(I; \mathbf{L}^2(\omega))$ if $\beta > 0$ or $\mathbf{S} = L^2(I; \mathbf{L}^2(\omega))$ if $\beta = 0$. We also denote $I_\gamma^+ = \{t \in I_\gamma : \bar{\boldsymbol{\mu}}(t) \not\equiv 0\}$.

Let us observe that

$$g'(\mathbf{u}; \mathbf{v}) = \int_{\omega_{\mathbf{u}}^+} \frac{\mathbf{u}(x)}{|\mathbf{u}(x)|} \mathbf{v}(x)\,\mathrm{d}x + \int_{\omega_{\mathbf{u}}^0} |\mathbf{v}(x)|\,\mathrm{d}x \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{L}^1(\omega), \tag{51}$$

where $\omega_{\mathbf{u}}^+ = \{x \in \omega : \mathbf{u}(x) \neq 0\}$ and $\omega_{\mathbf{u}}^0 = \omega \setminus \omega_{\mathbf{u}}^+$.

Lemma 4.10. *The following properties hold:*

1 - If $\mathbf{v} \in C_{\bar{\mathbf{u}}}$ then $g'(\bar{\mathbf{u}}(t), \mathbf{v}(t)) = 0$ holds for almost all $t \in I_\gamma^+$.

2 - Let $\mathbf{v} \in \mathbf{S}$ satisfy $g'(\bar{\mathbf{u}}(t); \mathbf{v}(t)) = 0$ for almost all $t \in I_\gamma^+$. Then, $J'(\bar{\mathbf{u}}; \mathbf{v}) = 0$ if and only if the following two conditions are fulfilled:

$$\|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)} |\mathbf{v}(x,t)| = \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t) \quad \text{for a.a. } (x,t) \in \omega_{\bar{u}(t)}^0 \times I_\gamma^+,$$

$$\mathbf{v}(x,t) = \begin{cases} 0 & \text{if } \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} < \beta, \\ -\dfrac{\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}}{\beta} \bar{\boldsymbol{\varphi}}(x,t) & \text{if } \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} = \beta, \end{cases} \quad \text{for a.a. } (x,t) \in \omega \times I_{\bar{u}}^0.$$

Proof. Let us prove the first statement. From the identity

$$0 = J'(\bar{\mathbf{u}}; \mathbf{v}) = \int_{I_{\bar{\mathbf{u}}}^+} \int_\omega [\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}} + \beta \bar{\boldsymbol{\lambda}}] \mathbf{v}\,\mathrm{d}x\,\mathrm{d}t + \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \mathrm{d}t$$

and the fact that

$$\int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \mathrm{d}t \ge \int_{I_{\bar{\mathbf{u}}}^0} [\beta - \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)}] \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\,\mathrm{d}t \ge 0$$

due to (36), we deduce that the first integral in $J'(\bar{\mathbf{u}}; \mathbf{v})$ is ≤ 0. Then, using (49) and (50) we get

$$\int_{I_\gamma^+} \int_\omega \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{I_{\bar{\mathbf{u}}}^+} \int_\omega \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t \ge 0.$$

This inequality and Lemma 4.9 yield

$$
\begin{aligned}
0 &\le \int_{I_\gamma^+}\int_\omega \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&= \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}\int_{I_\gamma^+}\int_{\omega^+_{\bar{\mathbf{u}}(t)}} \frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{I_\gamma^+}\int_{\omega^0_{\bar{\mathbf{u}}(t)}} \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\le \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}\Big[\int_{I_\gamma^+}\int_{\omega^+_{\bar{\mathbf{u}}(t)}} \frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{I_\gamma^+}\int_{\omega^0_{\bar{\mathbf{u}}(t)}} |\mathbf{v}(x,t)|\,\mathrm{d}x\,\mathrm{d}t\Big]\\
&= \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}\int_{I_\gamma^+} g'(\bar{\mathbf{u}}(t);\mathbf{v}(t))\,\mathrm{d}x\,\mathrm{d}t \le 0.
\end{aligned}
$$

Since $g'(\bar{\mathbf{u}}(t);\mathbf{v}(t))\,\mathrm{d}x \le 0$ for almost all $t \in I_\gamma^+$ and its integral in I_γ^+ is zero, the first statement of the lemma follows.

To prove the second statement we use (49), (50), Lemma 4.9, and the assumption $g'(\bar{\mathbf{u}}(t);\mathbf{v}(t)) = 0$ to infer

$$
\begin{aligned}
&J'(\bar{\mathbf{u}};\mathbf{v}) = -\int_{I_\gamma^+}\int_\omega \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{I^0_{\bar{\mathbf{u}}}}\Big(\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big)\,\mathrm{d}t\\
&= -\int_{I_\gamma^+}\int_{\omega^+_{\bar{u}(t)}} \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_{I_\gamma^+}\int_{\omega^0_{\bar{u}(t)}} \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_{I^0_{\bar{\mathbf{u}}}}\Big(\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big)\,\mathrm{d}t\\
&= -\int_{I_\gamma^+}\int_{\omega^+_{\bar{u}(t)}} \|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}\frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_{I_\gamma^+}\int_{\omega^0_{\bar{u}(t)}} \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_{I^0_{\bar{\mathbf{u}}}}\Big(\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big)\,\mathrm{d}t\\
&= \int_{I_\gamma^+}\int_{\omega^0_{\bar{u}(t)}} \Big(\|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}|\mathbf{v}(x,t)| - \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_{I^0_{\bar{\mathbf{u}}}}\Big(\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}\Big)\,\mathrm{d}t
\end{aligned}
$$

Since the integrands in the last two integrals are nonnegative, the identity $J'(\bar{\mathbf{u}};\mathbf{v}) = 0$ holds if and only if the following equalities are fulfilled

$$
\begin{aligned}
&\|\bar{\boldsymbol{\mu}}(t)\|_{\mathbf{L}^\infty(\omega)}|\mathbf{v}(x,t)| = \bar{\boldsymbol{\mu}}(x,t)\mathbf{v}(x,t) \text{ for a.a. } (x,t) \in \omega^0_{\bar{\mathbf{u}}(t)} \times I_\gamma^+,\\
&\int_\omega \bar{\boldsymbol{\varphi}}(t)\mathbf{v}(t)\,\mathrm{d}x + \beta\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} = 0 \text{ for a.a. } (x,t) \in \omega \times I^0_{\bar{u}}.
\end{aligned}
$$

Since $\|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \le \beta$ in $I^0_{\bar{\mathbf{u}}}$, the second identity is equivalent to the equality for $\mathbf{v}$ written in the second statement of the lemma. □

The next theorem states the results concerning the second order analysis.

Theorem 4.11. *Let* $\bar{\mathbf{u}}$ *be a local solution of* (P) *in the* $L^2(I;\mathbf{L}^2(\omega))$ *sense. Then, the inequality* $J''(\bar{\mathbf{u}};\mathbf{v}^2) \ge 0$ *holds for all* $\mathbf{v} \in C_{\bar{\mathbf{u}}}$. *Conversely, if* $\bar{\mathbf{u}} \in \mathbf{U}_{\mathrm{ad}}$ *satisfies* *the first order optimality conditions and the second order condition* $J''(\bar{\mathbf{u}};\mathbf{v}^2) > 0$ $\forall\mathbf{v} \in C_{\bar{\mathbf{u}}} \setminus \{0\}$, *then there exist* $\kappa > 0$ *and* $\varepsilon > 0$ *such that*

$$J(\bar{\mathbf{u}}) + \frac{\kappa}{2}\|\mathbf{u} - \bar{\mathbf{u}}\|^2_{L^2(I;\mathbf{L}^2(\omega))} \leq J(\mathbf{u}) \quad \forall \mathbf{u} \in \mathbf{U}_{\mathrm{ad}} \cap B_\varepsilon(\bar{\mathbf{u}}), \tag{52}$$

where $B_\varepsilon(\bar{\mathbf{u}}) = \{\mathbf{u} \in L^2(I;\mathbf{L}^2(\omega)) : \|\mathbf{u} - \bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))} \leq \varepsilon\}$.

Proof. The proof of the sufficient second order conditions is almost the same as the one of Theorem 4.7 replacing when necessary $\mathcal{L}$ and $\mathcal{L}_{\delta,l}$ by J and $J_{\delta,l}$. For the proof of the necessary conditions we follow [15, Theorem 5.1]. However, changes are necessary to deal with the infinite horizon, the non differentiable term in the cost functional, and the fact that the controls are vector rather than scalar functions.

Let $\mathbf{v}$ be an element of $C_{\bar{u}} \cap L^\infty(I;\mathbf{L}^1(\omega)) \subset \mathbf{U}$. We will prove that $J''(\bar{\mathbf{u}})\mathbf{v}^2 \geq 0$. Later, we will remove the assumption $\mathbf{v} \in L^\infty(I;\mathbf{L}^1(\omega))$. Set

$$\mathbf{h}(x,t) = \begin{cases} \dfrac{\mathbf{v}(x,t)}{|\bar{\mathbf{u}}(x,t)|} & \text{if } x \notin \omega^0_{\bar{\mathbf{u}}(t)}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad a(t) = \int_\omega \mathbf{h}(x,t)\bar{\mathbf{u}}(x,t)\,\mathrm{d}x.$$

Thus, we have $g'(\bar{\mathbf{u}}(t);\mathbf{v}(t)) = a(t) + \int_{\omega^0_{\bar{\mathbf{u}}(t)}} |\mathbf{v}(x,t)|\,\mathrm{d}x$; see (51).

For every integer $k \geq 1$ we put

$$\begin{aligned}
P_k(\mathbf{h}(x,t)) &= \begin{cases} \mathbf{h}(x,t) & \text{if } |\mathbf{h}(x,t)| \leq k, \\ k\dfrac{\mathbf{h}(x,t)}{|\mathbf{h}(x,t)|} & \text{otherwise,} \end{cases} \\
a_k(t) &= \int_\omega P_k(\mathbf{h}(x,t))\bar{\mathbf{u}}(x,t)\,\mathrm{d}x, \\
\mathbf{h}_k(x,t) &= P_k(\mathbf{h}(x,t))|\bar{\mathbf{u}}(x,t)| + \frac{a(t) - a_k(t)}{\gamma}\bar{\mathbf{u}}(x,t), \\
\mathbf{v}_k(x,t) &= \begin{cases} 0 & \text{if } \gamma - \frac{1}{k} < \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^1(\omega)} < \gamma, \\ 0 & \text{if } 0 < \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} < \frac{1}{k}, \\ \mathbf{h}_k(x,t) + \mathbf{v}(x,t)\chi_{\omega^0_{\bar{\mathbf{u}}(t)}}(x) & \text{if } \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^1(\omega)} = \gamma, \\ \mathbf{v}(x,t) & \text{otherwise,} \end{cases}
\end{aligned}$$

where $\chi_{\omega^0_{\bar{\mathbf{u}}(t)}}(x)$ takes the value 1 if $x \in \omega^0_{\bar{\mathbf{u}}(t)}$ and 0 otherwise. We observe that P_k denotes the radial projection in $\mathbb{R}^2$ onto the ball $\{r \in \mathbb{R}^2 : |r| \leq k\}$.

Using the pointwise convergence $P_k(\mathbf{h}(x,t))\bar{\mathbf{u}}(x,t) \to \mathbf{h}(x,t)\bar{\mathbf{u}}(x,t)$ almost everywhere in $\omega \times I$ and that $|P_k(\mathbf{h}(x,t))\bar{\mathbf{u}}(x,t)| \leq |\mathbf{v}(x,t)|$, we deduce with Lebesgue's Theorem that $\lim_{k\to\infty} a_k(t) = a(t)$ for almost all $t \in I$. Therefore, we have that $\mathbf{v}_k(x,t) \to \mathbf{v}(x,t)$ for almost all $(x,t) \in \omega \times I$. Moreover, we have

$$|\mathbf{h}_k(x,t)| \leq |\mathbf{v}(x,t)| + \frac{2}{\gamma}\|\mathbf{v}\|_{L^\infty(I;\mathbf{L}^1(\omega))}|\bar{\mathbf{u}}(x,t)|$$

and consequently

$$|\mathbf{v}_k(x,t)| \le |\mathbf{v}(x,t)| + \frac{2}{\gamma}\|\mathbf{v}\|_{L^\infty(I;\mathbf{L}^1(\omega))}|\bar{\mathbf{u}}(x,t)| \text{ for a.a. } (x,t)\in\omega\times I.$$

Once again, since $\mathbf{v},\bar{\mathbf{u}}\in\mathbf{U}$ we obtain with Lebesgue's Theorem that $\mathbf{v}_k\to\mathbf{v}$ in $\mathbf{S}$.

Let us prove that $J'(\bar{\mathbf{u}};\mathbf{v}_k)=0$. To this end, we apply Lemma 4.10. Given $t\in I_\gamma$, taking into account that $g(\bar{\mathbf{u}}(t))=\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^1(\omega)}=\gamma$ we get with (51)

$$\begin{aligned}
&g'(\bar{\mathbf{u}}(t);\mathbf{v}_k(t))\\
&=\int_{\omega^+_{\bar{\mathbf{u}}(t)}} P_k(\mathbf{h}(x,t))\bar{\mathbf{u}}(x,t)\,\mathrm{d}x + \frac{a(t)-a_k(t)}{\gamma}\int_{\omega^+_{\bar{\mathbf{u}}(t)}}|\bar{\mathbf{u}}(x,t)|\,\mathrm{d}x + \int_{\omega^0_{\bar{\mathbf{u}}(t)}}|\mathbf{v}(x,t)|\,\mathrm{d}x\\
&= a(t) + \int_{\omega^0_{\bar{\mathbf{u}}(t)}}|\mathbf{v}(x,t)|\,\mathrm{d}x = g'(\bar{\mathbf{u}}(t);\mathbf{v}(t)) \begin{cases} =0 & \text{if } t\in I^+_\gamma,\\ \le 0 & \text{if } t\in I_\gamma\setminus I^+_\gamma,\end{cases}
\end{aligned}$$

where we used that $\mathbf{v}\in C_{\bar{\mathbf{u}}}$ in the last step.

We observe that $\mathbf{v}_k(x,t)=\mathbf{v}(x,t)$ for $(x,t)\in(\omega^0_{\bar{\mathbf{u}}(t)}\times I^+_\gamma)\cup(\omega\times I^0_{\bar{\mathbf{u}}})$. Since $\mathbf{v}\in C_{\bar{\mathbf{u}}}$, it satisfies the conditions of Lemma 4.10-2 and $\mathbf{v}_k$ does it as well, and thus $J'(\bar{\mathbf{u}};\mathbf{v}_k)=0$.

Take $\rho_k>0$ satisfying $\rho_k\big(k+\frac{2}{\gamma}\|\mathbf{v}\|_{L^\infty(I;\mathbf{L}^1(\omega))}\big)<\frac{1}{k\max\{1,\gamma\}}$. Then, we have for each fixed k and $\forall\rho\in(0,\rho_k)$

$$\rho\big(|P_k(\mathbf{h}(x,t))| + \big|\frac{|a(t)-a_k(t)|}{\gamma}\big|\big) \le \rho\big(k+\frac{2}{\gamma}\|\mathbf{v}\|_{L^\infty(I;\mathbf{L}^1(\omega))}\big) < \frac{1}{k}.$$

Using this estimate we get that $\|\bar{\mathbf{u}}(t)+\rho\mathbf{v}_k(t)\|_{L^1(\omega)}\le\gamma$ if $g(\bar{\mathbf{u}}(t))=\gamma$ and $0<\rho<\rho_k$:

$$\begin{aligned}
&\|\bar{\mathbf{u}}(t)+\rho\mathbf{v}_k(t)\|_{\mathbf{L}^1(\omega)}\\
&=\int_{\omega^+_{\bar{\mathbf{u}}(t)}}\Big|\bar{\mathbf{u}}(t)[1+\rho\big[P_k(\mathbf{h}(x,t))\frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}+\frac{a(t)-a_k(t)}{\gamma}\big]\Big|\,\mathrm{d}x + \rho\int_{\omega^0_{\bar{\mathbf{u}}(t)}}|\mathbf{v}(x,t)|\,\mathrm{d}x\\
&=\int_{\omega^+_{\bar{\mathbf{u}}(t)}}|\bar{\mathbf{u}}(t)|[1+\rho\big[P_k(\mathbf{h}(x,t))\frac{\bar{\mathbf{u}}(x,t)}{|\bar{\mathbf{u}}(x,t)|}+\frac{a(t)-a_k(t)}{\gamma}\big]\,\mathrm{d}x + \rho\int_{\omega^0_{\bar{\mathbf{u}}(t)}}|\mathbf{v}(x,t)|\,\mathrm{d}x\\
&=\gamma+\rho\Big\{\int_\omega\big[P_k(\mathbf{h}(x,t))\bar{\mathbf{u}}(x,t)+\frac{a(t)-a_k(t)}{\gamma}|\bar{\mathbf{u}}(x,t)|\big]\,\mathrm{d}x + \int_{\Omega^0_{\bar{\mathbf{u}}(t)}}|\mathbf{v}(x,t)|\,\mathrm{d}x\Big\}\\
&=\gamma+\rho\Big\{a(t)+\int_{\omega^0_{\bar{\mathbf{u}}(t)}}|\mathbf{v}(x,t)|\,\mathrm{d}x\Big\} = \gamma+\rho g'(\bar{\mathbf{u}}(t);\mathbf{v}(t))\le\gamma.
\end{aligned}$$

In the case $\gamma-\frac{1}{k}<\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^1(\omega)}<\gamma$, we have that $\mathbf{v}_k(t)=0$ and, consequently, $\|\bar{\mathbf{u}}(t)+\rho\mathbf{v}_k(t)\|_{\mathbf{L}^1(\omega)}=\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^1(\omega)}<\gamma$. If $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^1(\omega)}<\gamma-\frac{1}{k}$, then we get

$$\|\bar{\mathbf{u}}(t)+\rho\mathbf{v}_k(t)\|_{\mathbf{L}^1(\omega)}\le\gamma-\frac{1}{k}+\rho\|\mathbf{v}\|_{L^\infty(I;\mathbf{L}^1(\omega))}<\gamma.$$

Using the local optimality of $\bar{\mathbf{u}}$, the fact that $\bar{\mathbf{u}}+\rho\mathbf{v}_k\in\mathbf{U}_{\rm ad}$, that $\mathbf{v}_k$ vanishes as $\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}\in(0,\frac{1}{k})$, and $J'(\bar{\mathbf{u}};\mathbf{v}_k)=0$, making a Taylor expansion we get for every

$\rho < \rho_k$ small enough

$$0 \le J(\bar{\mathbf{u}} + \rho \mathbf{v}_k) - J(\bar{\mathbf{u}}) = \rho J'(\bar{\mathbf{u}}; \mathbf{v}_k) + \frac{\rho^2}{2} J''(\bar{\mathbf{u}} + \theta\rho\mathbf{v}_k; \mathbf{v}_k^2) = \frac{\rho^2}{2} J''(\bar{\mathbf{u}} + \theta\rho\mathbf{v}_k)\mathbf{v}_k^2.$$

Dividing the above inequality by $\rho^2/2$ and taking $\rho \to 0$ we obtain that $J''(\bar{\mathbf{u}}; \mathbf{v}_k^2) \ge 0$. Since $\mathbf{v}_k \to \mathbf{v}$ in $\mathbf{S}$, we pass to the limit as $k \to \infty$ and conclude that $J''(\bar{\mathbf{u}})\mathbf{v}^2 \ge 0$.

Finally, we take $\mathbf{v} \in C_{\bar{\mathbf{u}}}$ arbitrary and set $\mathbf{v}_k(x,t) = \frac{\mathbf{v}(x,t)}{1 + \frac{1}{k}\|\mathbf{v}(t)\|_{\mathbf{L}^1(\omega)}}$ for every $k \ge 1$. Then, we have

$$\begin{aligned} J'(\bar{\mathbf{u}}; \mathbf{v}_k) &= \frac{1}{1 + \frac{1}{k}\|\mathbf{v}(t)\|_{\mathbf{L}^1(\omega)}} J'(\bar{\mathbf{u}}; \mathbf{v}) = 0 \quad \text{and} \\ g'(\bar{\mathbf{u}}(t); \mathbf{v}_k(t)) &= \frac{1}{1 + \frac{1}{k}\|\mathbf{v}(t)\|_{\mathbf{L}^1(\omega)}} g'(\bar{\mathbf{u}}(t); \mathbf{v}(t)) \left\{ \begin{array}{ll} = 0 & \text{if } t \in I_\gamma^+, \\ \le 0 & \text{if } t \in I_\gamma \setminus I_\gamma^+. \end{array} \right. \end{aligned}$$

Therefore, $\mathbf{v}_k \in C_{\bar{\mathbf{u}}} \cap L^\infty(I; \mathbf{L}^1(\omega))$ and $\mathbf{v}_k \to \mathbf{v}$ in $\mathbf{S}$ is satisfied. Hence, we get $J''(\bar{\mathbf{u}})\mathbf{v}^2 = \lim_{k\to\infty} J''(\bar{\mathbf{u}})\mathbf{v}_k^2 \ge 0$, which concludes the proof. □

4.3. Case $\sigma = \infty$.

In this case, we take $\mathbf{K} = \mathbf{B}_\gamma = \{\mathbf{v} \in \mathbf{L}^\infty(\omega) : \|\mathbf{v}\|_{\mathbf{L}^\infty(\omega)} \le \gamma\}$. We observe that $\mathbf{U} \subset L^2(I; \mathbf{L}^2(\omega))$. Indeed, if $\alpha = 0$, then (22) implies that $\mathbf{U} = L^\infty(I; \mathbf{L}^\infty(\omega)) \cap L^1(I; \mathbf{L}^2(\omega))$. Since $L^\infty(I; \mathbf{L}^\infty(\omega)) \subset L^\infty(I; \mathbf{L}^2(\omega))$, we deduce by interpolation that $\mathbf{U} \subset L^2(I; \mathbf{L}^2(\omega))$.

From the optimality conditions (33) we infer the following properties of any local minimizer $\bar{\mathbf{u}}$.

Lemma 4.12. *Let $\bar{\mathbf{u}} \in \mathbf{U}_{\rm ad}$ satisfy* (33)*. Then, the following properties hold*

$$\left\{ \begin{array}{l} \text{if } |\bar{\mathbf{u}}(x,t)| < \gamma \Rightarrow \bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t) = 0, \\ \text{if } \bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t) \ne 0 \Rightarrow \bar{\mathbf{u}}(x,t) = -\gamma \frac{\bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t)}{|\bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t)|}. \end{array} \right. \tag{53}$$

Proof. Let us observe that (33) is equivalent to

$$[\bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t)] \cdot [\xi - \bar{\mathbf{u}}(x,t)] \ge 0 \quad \forall \xi \in \mathbb{R}^2 \text{ with } |\xi| \le \gamma.$$

We recall that $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^2$. This inequality implies (53). □

We define $\bar{\mu}(x,t) = |\bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t)|$. We consider the following sets where the constraint is active:

$$A_\gamma = \{(x,t) \in \omega \times I : |\bar{\mathbf{u}}(x,t)| = \gamma\} \quad \text{and} \quad A_\gamma^+ = \{(x,t) \in A_\gamma : \bar{\mu}(x,t) \ne 0\}.$$

Corollary 4.13. *Let $\bar{\mathbf{u}} \in \mathbf{U}_{\rm ad}$ satisfy* (33). *Then $(\bar{\mathbf{u}}, \bar{\boldsymbol{\varphi}}, \bar{\boldsymbol{\lambda}}, \bar{\mu})$ satisfy a.e. in $\omega \times I$*

$$\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}} + \frac{1}{\gamma}\bar{\mu}\bar{\mathbf{u}} = 0, \quad \bar{\mu} \geq 0,\ |\bar{\mathbf{u}}| \leq \gamma,\ \bar{\mu}(|\bar{\mathbf{u}}| - \gamma) = 0. \tag{54}$$

Moreover, we have that $\|\bar{\mu}\|_{L^\infty(I;\mathbf{L}^\infty(\omega))} \leq \|\bar{\boldsymbol{\varphi}}\|_{L^\infty(I;\mathbf{L}^\infty(\omega))}$ and, if $\beta > 0$, then $\bar{\lambda} \in L^\infty(I;\mathbf{L}^\infty(\omega))$ holds as well.

Proof. The first part of the corollary is a straightforward consequence of Lemma 4.12. From (54) we get for $(x,t) \in A^+_\gamma$

$$\begin{aligned} 0 &= (\bar{\boldsymbol{\varphi}}(x,t) + \alpha\bar{\mathbf{u}}(x,t) + \beta\bar{\boldsymbol{\lambda}}(x,t) + \frac{1}{\gamma}\bar{\mu}\bar{\mathbf{u}}(x,t))\bar{\mathbf{u}}(x,t) \\ &= \bar{\boldsymbol{\varphi}}(x,t)\bar{\mathbf{u}}(x,t) + \alpha\gamma^2 + \beta\bar{\boldsymbol{\lambda}}(x,t)\bar{\mathbf{u}}(x,t) + \gamma\bar{\mu}(x,t). \end{aligned}$$

We observe that $\bar{\boldsymbol{\lambda}}(x,t)\bar{\mathbf{u}}(x,t) = 0$ if $t \in I^0_{\bar{\mathbf{u}}(t)}$ and $\bar{\boldsymbol{\lambda}}(x,t)\bar{\mathbf{u}}(x,t) = \frac{|\bar{\mathbf{u}}(x,t)|^2}{\|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)}}$ otherwise. In any case, we have that $\bar{\boldsymbol{\lambda}}(x,t)\bar{\mathbf{u}}(x,t) \geq 0$ for almost all $(x,t) \in \omega \times I$. Using this property in the above identity we obtain for $(x,t) \in A^+_\gamma$

$$0 < \gamma\bar{\mu}(x,t) \leq \alpha\gamma^2 + \beta\bar{\boldsymbol{\lambda}}(x,t)\bar{\mathbf{u}}(x,t) + \gamma\bar{\mu}(x,t) = -\bar{\boldsymbol{\varphi}}(x,t)\bar{\mathbf{u}}(x,t) \leq \|\bar{\varphi}\|_{\mathbf{L}^\infty(I;\mathbf{L}^\infty(\omega))}\gamma,$$

which proves the estimate for $\bar{\mu}$. Now, the boundedness of $\bar{\boldsymbol{\lambda}}$ follows from (54). □

In order to carry out the second order analysis, we define the cone of critical directions for $\bar{\mathbf{u}} \in \mathbf{U}_{\rm ad}$ satisfying (33) as follows

$$C_{\bar{\mathbf{u}}} = \left\{\mathbf{v} \in \mathbf{S} : J'(\bar{\mathbf{u}};\mathbf{v}) = 0 \ \text{ and } \ \bar{\mathbf{u}}(x,t)\cdot\mathbf{v}(x,t) \leq 0 \text{ if } |\bar{\mathbf{u}}(x,t)| = \gamma\right\},$$

where $\mathbf{S} = L^2(I;\mathbf{L}^2(\omega)) \cap L^1(I;\mathbf{L}^2(\omega))$ if $\beta > 0$ and $\mathbf{S} = L^2(I;\mathbf{L}^2(\omega))$ if $\beta = 0$. We also consider the Lagrange function $\mathcal{L} : \mathbf{U} \times L^\infty(I;\mathbf{L}^\infty(\omega)) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}(\mathbf{u},\mu) = J(\mathbf{u}) + \frac{1}{2\gamma}\int_I\int_\omega \mu(x,t)|\mathbf{u}(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

The Lagrangian $\mathcal{L}$ enjoys the following properties.

Lemma 4.14. *Let $\bar{\mathbf{u}}$ and $\bar{\mu}$ be as above. Then, we have*

$$\frac{\partial\mathcal{L}}{\partial\mathbf{u}}(\bar{\mathbf{u}},\bar{\mu};\mathbf{v})\begin{cases} = 0 & \text{if } \beta = 0, \\ \geq 0 & \text{if } \beta > 0, \end{cases} \quad \forall\mathbf{v} \in \mathbf{S}, \tag{55}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{u}}(\bar{\mathbf{u}},\bar{\mu};\mathbf{v}) = 0 \quad \forall\mathbf{v} \in C_{\bar{\mathbf{u}}}, \tag{56}$$

$$\bar{\mathbf{u}}(x,t)\mathbf{v}(x,t) = 0 \quad \text{for a.a. } (x,t) \in A^+_\gamma \text{ and } \forall\mathbf{v} \in C_{\bar{\mathbf{u}}}. \tag{57}$$

Proof. From (24), (31), (32), and (53) we get $\forall \mathbf{v} \in \mathbf{U}$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}) = \int_{A_\gamma^+} (\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}} + \beta \bar{\boldsymbol{\lambda}}) \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t + \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t) \mathbf{v}(t) \, \mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \, \mathrm{d}t$$
$$+ \frac{1}{\gamma} \int_{A_\gamma^+} \bar{\mu} \bar{\mathbf{u}} \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t = \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t) \mathbf{v}(t) \, \mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \, \mathrm{d}t. \tag{58}$$

Now, we observe that the mapping $\mathbf{v} \in \mathbf{U} \longrightarrow \frac{\partial \mathcal{L}}{\partial \mathbf{u}}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}) \in \mathbb{R}$ can be extended to a continuous functional on $\mathbf{S}$.

If $\beta = 0$, then (53) implies that $\bar{\boldsymbol{\varphi}} \equiv 0$ in $\omega \times I_{\bar{\mathbf{u}}}^0$ and the fist identity of (55) follows. If $\beta > 0$ then (36) and (37) along with Schwarz's inequality implies the inequality of (55). Now, we prove (56) ad (57). Given $\mathbf{v} \in C_{\bar{\mathbf{u}}}$, since $\bar{\boldsymbol{\lambda}} \in \partial j(\bar{\mathbf{u}})$ and $\mathbf{v} \in C_{\bar{\mathbf{u}}}$ we get

$$0 = J'(\bar{\mathbf{u}}; \mathbf{v}) \geq \int_I \int_\omega (\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}} + \beta \bar{\boldsymbol{\lambda}}) \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t.$$

But (53) and the fact that $\bar{\mathbf{u}}(x,t)\mathbf{v}(x,t) \leq 0$ if $|\bar{\mathbf{u}}(x,t)| = \gamma$ imply that $(\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))\mathbf{v}(x,t) \geq 0$ for almost all $(x,t) \in \omega \times I$. Combining this with the above inequality and Corollary 4.13 we infer that $\frac{1}{\gamma}\bar{\mu}(x,t)\bar{\mathbf{u}}(x,t)\mathbf{v}(x,t) = -(\bar{\boldsymbol{\varphi}}(x,t) + \alpha \bar{\mathbf{u}}(x,t) + \beta \bar{\boldsymbol{\lambda}}(x,t))\mathbf{v}(x,t) = 0$ a.e. in $\omega \times I$. This implies that $\bar{\mathbf{u}}(x,t)\mathbf{v}(x,t) = 0$ if $\bar{\mu}(x,t) \neq 0$. Using once again (24), (31), and (32) we deduce

$$0 = J'(\bar{\mathbf{u}}; \mathbf{v}) = \int_{I_{\bar{\mathbf{u}}}^+} \int_\omega (\bar{\boldsymbol{\varphi}} + \alpha \bar{\mathbf{u}} + \beta \bar{\boldsymbol{\lambda}}) \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t + \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t) \mathbf{v}(t) \, \mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \, \mathrm{d}t$$
$$= \int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t) \mathbf{v}(t) \, \mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \, \mathrm{d}t.$$

This identity and (58) yield (56). □

Now, we establish the second order necessary optimality conditions.

Theorem 4.15. *Let* $\bar{\mathbf{u}}$ *be a local solution of* (P) *in the* $L^2(I; \mathbf{L}^2(\omega))$ *sense. Then, the inequality* $\frac{\partial^2 \mathcal{L}}{\partial \mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2) \geq 0$ *holds for all* $\mathbf{v} \in C_{\bar{\mathbf{u}}}$.

Proof. Let us take $\mathbf{v} \in C_{\bar{\mathbf{u}}} \cap L^\infty(I; \mathbf{L}^2(\omega))$. It was established in the proof of Lemma 4.14 that

$$\int_{I_{\bar{\mathbf{u}}}^0} \Big[\int_\omega \bar{\boldsymbol{\varphi}}(t) \mathbf{v}(t) \, \mathrm{d}x + \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \Big] \, \mathrm{d}t = 0.$$

Using this inequality, (36), and (37) we get for almost all $t \in I_{\bar{\mathbf{u}}}^0$

$$\beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} = -\int_\omega \bar{\boldsymbol{\varphi}}(t) \mathbf{v}(t) \, \mathrm{d}x \leq \|\bar{\boldsymbol{\varphi}}(t)\|_{\mathbf{L}^2(\omega)} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)} \leq \beta \|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}.$$

This yields $\beta \mathbf{v}(x,t) = -\bar{\boldsymbol{\varphi}}(x,t)\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}$ for almost every $t \in I_{\bar{\mathbf{u}}}^0$. Since $\mathbf{v} \in L^\infty(I; \mathbf{L}^2(\omega))$ and $\bar{\boldsymbol{\varphi}} \in L^\infty(I; \mathbf{L}^\infty(\Omega))$, we infer that $\mathbf{v} \in L^\infty(I_{\bar{\mathbf{u}}}^0; \mathbf{L}^\infty(\omega))$.

For every integer $k > \frac{1}{\gamma}$ and $0 < \rho_k < \min\{\frac{1}{k^2}, \frac{\gamma}{1+\|\mathbf{v}\|_{L^\infty(I^0_{\bar{\mathbf{u}}};\mathbf{L}^\infty(\omega))}}\}$ we define the function $\phi_k : [0, \rho_k] \longrightarrow \mathbf{S}$ by

$$\phi_k(\rho) = \begin{cases} \bar{\mathbf{u}}(x,t) & \text{if } \gamma - \dfrac{1}{k} < |\bar{\mathbf{u}}(x,t)| < \gamma, \\ \bar{\mathbf{u}}(x,t) & \text{if } 0 < \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} \text{ and } |\mathbf{v}(x,t)| > k, \\ \bar{\mathbf{u}}(x,t) & \text{if } \beta > 0 \text{ and } 0 < \|\bar{\mathbf{u}}(t)\|_{\mathbf{L}^2(\omega)} < \dfrac{1}{k}, \\ \sqrt{1 - \dfrac{\rho^2|\mathbf{v}(x,t)|^2}{\gamma^2}}\,\bar{\mathbf{u}}(x,t) + \rho\mathbf{v}(x,t) & \text{if } |\bar{\mathbf{u}}(x,t)| = \gamma \text{ and } |\mathbf{v}(x,t)| \le k, \\ \bar{\mathbf{u}}(x,t) + \rho\mathbf{v}(x,t) & \text{otherwise.} \end{cases}$$

It is immediate that $|\phi_k(\rho)| \le |\mathbf{v}(x,t)| + |\bar{\mathbf{u}}(x,t)|$, hence $\phi_k(\rho) \in \mathbf{S}$ and ϕ_k is well defined. Moreover, $\phi_k(0) = \bar{\mathbf{u}}$ and, using that $\bar{\mathbf{u}}(x,t)\mathbf{v}(x,t) \le 0$ if $|\bar{\mathbf{u}}(x,t)| = \gamma$, we get that $\phi_k(\rho) \in \mathbf{U}_{\rm ad}$ for every $[0, \rho_k]$. Therefore, the function $\psi_k : [0, \rho_k] \longrightarrow \mathbb{R}$, defined by $\psi_k(\rho) = J(\phi_k(\rho))$, has a local minimum at 0 and $\psi_k'(0) = J'(\bar{\mathbf{u}}; \phi_k'(0)) = 0$ by definition of ϕ_k and the fact that $J'(\bar{\mathbf{u}}; \mathbf{v}) = 0$. Consequently, we have that $0 \le \psi_k''(0) = J''(\bar{\mathbf{u}}; \phi_k'(0)^2) + J'(\bar{\mathbf{u}}; \phi_k''(0))$. We observe that $\phi_k'(0) \to \mathbf{v}$ in $\mathbf{S}$ and $\phi_k''(0) \to -\frac{1}{\gamma^2}|\mathbf{v}|^2\bar{\mathbf{u}}\chi_{A_\gamma}$ as $k \to \infty$. Then, we obtain with (53)

$$\begin{aligned} \lim_{k\to\infty} J'(\bar{\mathbf{u}}; \phi_k''(0)) &= J'(\bar{\mathbf{u}}; \frac{1}{\gamma^2}|\mathbf{v}|^2\bar{\mathbf{u}}\chi_{A_\gamma}) \\ &= -\frac{1}{\gamma^2}\int_{A_\gamma^+} (\bar{\boldsymbol{\varphi}} + \alpha\bar{\mathbf{u}} + \beta\bar{\boldsymbol{\lambda}})\bar{\mathbf{u}}|\mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\gamma}\int_{A_\gamma^+} \bar{\mu}|\mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Therefore, we have

$$0 \le \lim_{k\to\infty} \psi_k''(0) = J''(\bar{\mathbf{u}}; \mathbf{v}^2) + \frac{1}{\gamma}\int_{A_\gamma^+} \bar{\mu}|\mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}t = \frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2).$$

Finally we remove the assumption $\mathbf{v} \in C_{\bar{\mathbf{u}}} \cap L^\infty(I; \mathbf{L}^2(\omega))$. Given $\mathbf{v} \in C_{\bar{\mathbf{u}}}$ we define

$$\mathbf{v}_k(x,t) = \frac{\mathbf{v}(x,t)}{1 + \frac{1}{k}\|\mathbf{v}(t)\|_{\mathbf{L}^2(\omega)}} \quad \text{for } \; k \ge 1.$$

Then we have that $\{\mathbf{v}_k\}_{k=1}^\infty \subset C_{\bar{\mathbf{u}}} \cap L^\infty(I; \mathbf{L}^2(\omega))$ and $\mathbf{v}_k \to \mathbf{v}$ in $\mathbf{S}$. Then, it is easy to pass to the limit in the inequality $\frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}_k^2) \ge 0$ and to get the desired result. □

The proof of the next theorem is analogous to the one of Theorem 4.7 with obvious changes.

Theorem 4.16. *If $\alpha > 0$ and $\bar{\mathbf{u}} \in \mathbf{U}_{\rm ad}$ satisfies the first order optimality conditions and the second order condition $\frac{\partial^2\mathcal{L}}{\partial\mathbf{u}^2}(\bar{\mathbf{u}}, \bar{\mu}; \mathbf{v}^2) > 0$ $\forall\mathbf{v} \in C_{\bar{\mathbf{u}}} \setminus \{0\}$, then there exist $\kappa > 0$ and $\varepsilon > 0$ such that*

$$J(\bar{\mathbf{u}}) + \frac{\kappa}{2}\|\mathbf{u} - \bar{\mathbf{u}}\|^2_{L^2(I;\mathbf{L}^2(\omega))} \le J(\mathbf{u}) \quad \forall\mathbf{u} \in \mathbf{U}_{\rm ad} \cap B_\varepsilon(\bar{\mathbf{u}}), \tag{59}$$

where $B_\varepsilon(\bar{\mathbf{u}}) = \{\mathbf{u} \in L^2(I; \mathbf{L}^2(\omega)) : \|\mathbf{u} - \bar{\mathbf{u}}\|_{L^2(I;\mathbf{L}^2(\omega))} \le \varepsilon\}$.

5. Appendix

Proof of Lemma 2.1. First, we analyze the bilinear form B. Let $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{Y}$ and $(\mathbf{y}_{i,H}, \mathbf{y}_{i,W}) \in [L^2(I;\mathbf{V}) \cap L^\infty(I;\mathbf{H})] \times [L^q(I;\mathbf{W}_p(\Omega)) \cap L^4(I;\mathbf{W}_p(\Omega))]$ be elements such that $\mathbf{y}_i = \mathbf{y}_{i,H} + \mathbf{y}_{i,W}$ for $i = 1, 2$. Then, we are going to prove estimates for the terms $B(\mathbf{y}_{1,H}, \mathbf{y}_{2,H})$, $B(\mathbf{y}_{1,H}, \mathbf{y}_{2,W})$, $B(\mathbf{y}_{1,W}, \mathbf{y}_{2,H})$, and $B(\mathbf{y}_{1,W}, \mathbf{y}_{2,W})$. Given $\boldsymbol{\psi} \in \mathbf{H}_0^1(\Omega)$, we observe that

$$\langle B(\mathbf{y}_1, \mathbf{y}_2), \boldsymbol{\psi}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} = \sum_{i,j=1}^{2} \int_\Omega \mathbf{y}_{1,i}(x,t)\partial_{x_i}\mathbf{y}_{2,j}(x,t)\boldsymbol{\psi}_j(x)\,dx.$$

To deduce the estimates we will use the Gagliardo inequality

$$\|\mathbf{y}\|_{\mathbf{L}^r(\Omega)} \le C_r \|\mathbf{y}\|_{\mathbf{L}^2(\Omega)}^{\frac{2}{r}} \|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)}^{\frac{r-2}{r}} \quad \forall r \in (2,\infty) \text{ and } \forall \mathbf{y} \in \mathbf{H}_0^1(\Omega); \tag{60}$$

see [6, page 313]. Now, we proceed in four steps.

Step 1.- Using that $\operatorname{div} \mathbf{y}_{1,H} = 0$, we know that

$$\int_\Omega [(\mathbf{y}_{1,H} \cdot \nabla)\mathbf{y}_{2,H}]\boldsymbol{\psi}\,dx = -\int_\Omega [(\mathbf{y}_{1,H} \cdot \nabla)\boldsymbol{\psi}]\mathbf{y}_{2,H}\,dx. \tag{61}$$

Then, from Schwarz's inequality and (60) with $r = 4$ it follows

$$\begin{aligned}
&\left(\int_I |\langle B(\mathbf{y}_{1,H}(t), \mathbf{y}_{2,H}(t)), \boldsymbol{\psi}\rangle|^2\,dt\right)^{\frac{1}{2}} = \left(\int_I |\langle B(\mathbf{y}_{1,H}(t), \boldsymbol{\psi}), \mathbf{y}_{2,H}(t)\rangle|^2\,dt\right)^{\frac{1}{2}} \\
&\le \left(\int_I \|\mathbf{y}_{1,H}(t)\|_{\mathbf{L}^4(\Omega)}^2 \|\mathbf{y}_{2,H}(t)\|_{\mathbf{L}^4(\Omega)}^2\,dt\right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C_4^2 \left(\int_I \|\mathbf{y}_{1,H}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{y}_{1,H}(t)\|_{\mathbf{H}_0^1(\Omega)} \|\mathbf{y}_{2,H}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{y}_{2,H}(t)\|_{\mathbf{H}_0^1(\Omega)}\,dt\right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C_4^2 \|\mathbf{y}_{1,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))}^{\frac{1}{2}} \|\mathbf{y}_{1,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))}^{\frac{1}{2}} \|\mathbf{y}_{2,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))}^{\frac{1}{2}} \|\mathbf{y}_{2,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))}^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le \frac{C_4^2}{4} \Big(\|\mathbf{y}_{1,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} + \|\mathbf{y}_{1,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))}\Big) \\
&\quad \times \Big(\|\mathbf{y}_{2,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} + \|\mathbf{y}_{2,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))}\Big) \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}.
\end{aligned} \tag{62}$$

Step 2.- Using Hölder's inequality and (60) with $r = 2p' = \frac{2p}{p-1}$ we get

$$\begin{aligned}
&\left(\int_I |\langle B(\mathbf{y}_{1,H}(t), \mathbf{y}_{2,W}(t)), \boldsymbol{\psi}\rangle|^2 \, dt\right)^{\frac{1}{2}} \\
&\le \left(\int_I \|\mathbf{y}_{1,H}(t)\|^2_{\mathbf{L}^{2p'}(\Omega)} \|\boldsymbol{\psi}\|^2_{\mathbf{L}^{2p'}(\Omega)} \|\mathbf{y}_{2,W}(t)\|^2_{\mathbf{W}_0^{1,p}(\Omega)} \, dt\right)^{\frac{1}{2}} \\
&\le C_{2p'} \left(\int_I \|\mathbf{y}_{1,H}(t)\|^{\frac{2}{p'}}_{\mathbf{L}^2(\Omega)} \|\mathbf{y}_{1,H}(t)\|^{\frac{2}{p}}_{\mathbf{H}_0^1(\Omega)} \|\mathbf{y}_{2,W}(t)\|^2_{\mathbf{W}_0^{1,p}(\Omega)} \, dt\right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C_{2p'} \|\mathbf{y}_{1,H}\|^{\frac{1}{p'}}_{L^\infty(I;\mathbf{L}^2(\Omega))} \|\mathbf{y}_{1,H}\|^{\frac{1}{p}}_{L^2(I;\mathbf{H}_0^1(\Omega))} \|\mathbf{y}_{2,W}\|_{L^{2p'}(I;\mathbf{W}_0^{1,p}(\Omega))} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}.
\end{aligned}$$

Taking into account that $4 < 2p' \le 8 \le q$, we get from above by interpolation between L^4 and L^q and Young's inequality that

$$\begin{aligned}
&\left(\int_I |\langle B(\mathbf{y}_{1,H}(t), \mathbf{y}_{2,W}(t)), \boldsymbol{\psi}\rangle|^2 \, dt\right)^{\frac{1}{2}} \le C\Big(\|\mathbf{y}_{1,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} + \|\mathbf{y}_{1,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))}\Big) \\
&\times \Big(\|\mathbf{y}_{2,W}\|_{L^4(I;\mathbf{W}_0^{1,p}(\Omega))} + \|\mathbf{y}_{2,W}\|_{L^q(I;\mathbf{W}_0^{1,p}(\Omega))}\Big) \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}. \qquad (63)
\end{aligned}$$

Step 3.- Using again Hölder's inequality and (60) with $r = 4$ we obtain

$$\begin{aligned}
&\left(\int_I |\langle B(\mathbf{y}_{1,W}(t), \mathbf{y}_{2,H}(t)), \boldsymbol{\psi}\rangle|^2 \, dt\right)^{\frac{1}{2}} \\
&\le \left(\int_I \|\mathbf{y}_{1,W}(t)\|^2_{\mathbf{L}^4(\Omega)} \|\mathbf{y}_{2,H}(t)\|^2_{\mathbf{L}^4(\Omega)} \, dt\right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C_4^2 \left(\int_I \|\mathbf{y}_{1,W}(t)\|^2_{\mathbf{W}_0^{1,p}(\Omega)} \|\mathbf{y}_{2,H}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{y}_{2,H}(t)\|_{\mathbf{H}_0^1(\Omega)} \, dt\right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C_4^2 \|\mathbf{y}_{1,W}\|_{L^4(I;\mathbf{W}_0^{1,p}(\Omega))} \|\mathbf{y}_{2,H}\|^{\frac{1}{2}}_{L^\infty(I;\mathbf{L}^2(\Omega))} \|\mathbf{y}_{2,H}\|^{\frac{1}{2}}_{\mathbf{H}_0^1(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C_4^2 \|\mathbf{y}_{1,W}\|_{L^4(I;\mathbf{W}_0^{1,p}(\Omega))} \Big(\|\mathbf{y}_{2,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} + \|\mathbf{y}_{2,H}\|_{\mathbf{H}_0^1(\Omega)}\Big) \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}. \qquad (64)
\end{aligned}$$

Step 4.- Using again the property (61), Hölder's inequality, the embedding $\mathbf{W}_0^{1,p}(\Omega) \subset \mathbf{L}^4(\Omega)$, and the fact that $p \ge \frac{4}{3}$ we obtain

$$\begin{aligned}
&\left(\int_I |\langle B(\mathbf{y}_{1,W}(t), \mathbf{y}_{2,W}(t)), \boldsymbol{\psi}\rangle|^2 \, dt\right)^{\frac{1}{2}} \\
&\le \left(\int_I \|\mathbf{y}_{1,W}\|^2_{\mathbf{L}^4(\Omega)} \|\mathbf{y}_{2,W}\|^2_{\mathbf{L}^4(\Omega)} dt\right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \\
&\le C\Big(\|\mathbf{y}_{1,W}\|_{L^4(I;\mathbf{W}_0^{1,p}(\Omega))} + \|\mathbf{y}_{2,W}\|_{L^4(I;\mathbf{W}_0^{1,p}(\Omega))}\Big) \|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}. \qquad (65)
\end{aligned}$$

Finally, adding the estimates (62)-(65) we obtain

$$\|B(\mathbf{y}_1,\mathbf{y}_2)\|_{L^2(I;\mathbf{H}^{-1}(\Omega))} \leq C\Big(\|\mathbf{y}_{1,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} + \|\mathbf{y}_{1,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))} + \|\mathbf{y}_{1,W}\|_{L^q(I;\mathbf{W}_0^{1,p}(\Omega))}\Big)$$
$$\times \Big(\|\mathbf{y}_{2,H}\|_{L^\infty(I;\mathbf{L}^2(\Omega))} + \|\mathbf{y}_{2,H}\|_{L^2(I;\mathbf{H}_0^1(\Omega))} + \|\mathbf{y}_{2,W}\|_{L^q(I;\mathbf{W}_0^{1,p}(\Omega))}\Big). \tag{66}$$

Taking the infimum on the right hand side of the above inequality among all functions $(\mathbf{y}_{i,H},\mathbf{y}_{i,W}) \in [L^2(I;\mathbf{H}_0^1(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))] \times L^q(0,T;\mathbf{W}_0^{1,p}(\Omega))$ satisfying that $\mathbf{y}_i = \mathbf{y}_{i,H} + \mathbf{y}_{i,W}$, $i = 1,2$, we conclude

$$\|B(\mathbf{y}_1,\mathbf{y}_2)\|_{L^2(I;\mathbf{H}^{-1}(\Omega))} \leq C'\|\mathbf{y}_1\|_{\mathbf{Y}}\|\mathbf{y}_2\|_{\mathbf{Y}}.$$

Now we turn to the estimates for $\mathcal{B}$. First, we point out that $\mathcal{Y}$ is continuously embedding in $L^8(I;\mathbf{L}^4(\Omega))$. Indeed, since $\mathbf{W}_0^{1,p}(\Omega) \subset \mathbf{L}^4(\Omega)$ for every $p \geq \frac{4}{3}$, we deduce that $\mathbf{W}_{q,p}(I) \subset L^q(I;\mathbf{L}^4(\Omega))$ and $\mathbf{W}_{4,p}(I) \subset L^4(I;\mathbf{L}^4(\Omega))$. Then, recalling that $4 < 8 \leq q$, we infer by interpolation that $\mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I) \subset L^8(I;\mathbf{L}^4(\Omega))$, the embedding being continuous.

Let us prove that $\mathbf{W}_{4,2}(I) \subset L^8(I;\mathbf{L}^4(\Omega))$ holds as well. For this purpose we point out that $\mathbf{W}_{4,2}(I) \subset C(I;(\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega))_{\frac{3}{4},4})$ and there exists a constant C_1 such that $\|\mathbf{y}\|_{C(I;(\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega))_{\frac{3}{4},4})} \leq C_1\|\mathbf{y}\|_{\mathbf{W}_{4,2}(I)}$; see [1, Th. III-4.10.2]. Using that

$$(\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega))_{\frac{3}{4},4} \subset (\mathbf{W}^{-1,4}(\Omega),\mathbf{W}^{\frac{1}{2},4}(\Omega))_{\frac{3}{4},4} = \mathbf{W}^{\frac{1}{8},4}(\Omega) \subset \mathbf{L}^4(\Omega),$$

we infer that $\mathbf{W}_{4,2}(I) \subset C(I;\mathbf{L}^4(\Omega))$ and $\|\mathbf{y}\|_{C(I;\mathbf{L}^4(\Omega))} \leq C_2\|\mathbf{y}\|_{\mathbf{W}_{4,2}(I)}$ for some constant C_2. Now, using Gagliardo's inequality (60) with $r = 4$, we get for every $\mathbf{y} \in \mathbf{W}_{4,2}(I)$

$$\|\mathbf{y}\|_{L^8(I;\mathbf{L}^4(\Omega))} \leq \|\mathbf{y}\|^{\frac{1}{2}}_{C(I;\mathbf{L}^4(\Omega))}\|\mathbf{y}\|^{\frac{1}{2}}_{\mathbf{L}^4(Q)}$$
$$\leq C_4\|\mathbf{y}\|^{\frac{1}{2}}_{C(I;\mathbf{L}^4(\Omega))}\|\mathbf{y}\|^{\frac{1}{4}}_{L^\infty(I;\mathbf{L}^2(\Omega))}\|\mathbf{y}\|^{\frac{1}{4}}_{L^2;\mathbf{H}_0^1(\Omega))} \leq C_5\|\mathbf{y}\|_{\mathbf{W}_{4,2}(I)}.$$

All together this implies that $\mathcal{Y} = \mathbf{W}_{4,2}(I) + [\mathbf{W}_{q,p}(I) \cap \mathbf{W}_{4,p}(I)] \subset L^8(I;\mathbf{L}^4(\Omega))$ with continuous embedding. Then, we have for all $\mathbf{y}_1,\mathbf{y}_2 \in \mathcal{Y}$

$$\Big(\int_I |\langle\mathcal{B}(\mathbf{y}_1(t),\mathbf{y}_2(t)),\boldsymbol{\psi}\rangle|^4\,dt\Big)^{\frac{1}{4}} = \Big(\int_I |\langle\mathcal{B}(\mathbf{y}_1(t),\boldsymbol{\psi}),\mathbf{y}_2(t)\rangle|^4\,dt\Big)^{\frac{1}{4}}$$
$$\leq \Big(\int_I \|\mathbf{y}_1(t)\|^4_{\mathbf{L}^4(\Omega)}\|\mathbf{y}_2(t)\|^4_{\mathbf{L}^4(\Omega)}\,dt\Big)^{\frac{1}{4}}\|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}$$
$$\leq \|\mathbf{y}_1\|_{L^8(I;\mathbf{L}^4(\Omega))}\|\mathbf{y}_2\|_{L^8(I;\mathbf{L}^4(\Omega))}\|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)} \leq C\|\mathbf{y}_1\|_{\mathcal{Y}}\|\mathbf{y}_2\|_{\mathcal{Y}}\|\boldsymbol{\psi}\|_{\mathbf{H}_0^1(\Omega)}, \tag{67}$$

which proves the continuity of $\mathcal{B}$. □

Proof of Lemma 2.5. For every $T < \infty$ the existence and uniqueness of a solution $(\mathbf{y}_N,\mathfrak{p}_N) \in \mathbf{W}(0,T) \times W^{-1,\infty}(0,T;L^2(\Omega)/\mathbb{R})$ of (8) in $Q_T = \Omega \times (0,T)$ was proved in

[13, Proposition 2.7]. Additionally the estimates

$$\|\mathbf{y}_N\|_{L^2(0,T;\mathbf{H}_0^1(\Omega))} + \|\mathbf{y}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \le \eta_N\Big(\|\mathbf{e}_2\|_{\mathbf{Y}}\Big)\Big(\|\mathbf{g}\|_{L^2(I;\mathbf{V}')} + \|\mathbf{y}_{N0}\|_{\mathbf{L}^2(\Omega)}\Big),$$
$$\|\mathbf{y}_N\|_{\mathbf{W}(0,T)} \le \nu_0\eta_N^2(\|\mathbf{e}_2\|_{\mathbf{Y}})\Big(\|\mathbf{g}\|_{L^2(I;\mathbf{V}')} + \|\mathbf{y}_{N0}\|_{\mathbf{L}^2(\Omega)}\Big)^2$$
$$+[(1+\nu+\|\mathbf{e}_1\|_{\mathbf{Y}} + \|\mathbf{e}_2\|_{\mathbf{Y}})\eta_N(\|\mathbf{e}_2\|_{\mathbf{Y}})+1]\Big(\|\mathbf{g}\|_{L^2(I;\mathbf{V}')} + \|\mathbf{y}_{N0}\|_{\mathbf{L}^2(\Omega)}\Big).$$

are deduced from the same result, where a explicit definition of η_N was given. Hence, taking the supremum on T (9) follows and the first part of the lemma is proved.

It remains to prove that, under the additional regularity of the data of the equation (9), the solution belongs to $\mathbf{W}_{4,2}(I)$. First, we show the existence of $T \in (0,\infty)$ such that $\mathbf{y}_N \in \mathbf{W}_{4,2}(0,T)$. To this end we apply the fixed point Schauder's theorem as follows. Fixed $T < \infty$ we define the mapping $F : L^8(I;\mathbf{L}^4(\Omega)) \longrightarrow L^8(I;\mathbf{L}^4(\Omega))$ such that $F(\mathbf{z}) = \mathbf{y}_{\mathbf{z}}\chi_{[0,T]}$, where $\mathbf{y}_{\mathbf{z}}$ is the solution of the equation

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial t} - \nu\Delta\mathbf{y} + \nabla\mathfrak{p} = \mathbf{g}_{\mathbf{z}} \ \text{ in } Q, \\ \operatorname{div}\mathbf{y} = 0 \ \text{ in } Q, \ \mathbf{y} = 0 \ \text{ on } \Sigma, \ \mathbf{y}(0) = \mathbf{y}_{N0} \text{ in } \Omega, \end{cases}$$

with $\mathbf{g}_{\mathbf{z}} = \mathbf{g} - \chi_{(0,T)}[(\mathbf{z}\cdot\nabla)\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{e}_2 + (\mathbf{e}_1\cdot\nabla)\mathbf{z}]$. Let us prove that F is well defined. From (67) and using the regularity of $\mathbf{g}$, $\mathbf{e}_1$, and $\mathbf{e}_2$ we deduce that

$$\|\mathbf{g}_{\mathbf{z}}\|_{L^4(I;\mathbf{H}^{-1}(\Omega))} \le \Big(\|\mathbf{g}\|_{L^4(I;\mathbf{H}^{-1}(\Omega))}$$
$$+ \|\mathbf{z}\|_{L^8(0,T;\mathbf{L}^4(\Omega))}[\|\mathbf{z}\|_{L^8(0,T;\mathbf{L}^4(\Omega))} + \|\mathbf{e}_1\|_{L^8(I;\mathbf{L}^4(\Omega))} + \|\mathbf{e}_2\|_{L^8(I;\mathbf{L}^4(\Omega))}]\Big).$$

Using this regularity for $\mathbf{g}_{\mathbf{z}}$ and the fact that $\mathbf{y}_{N0} \in \mathbf{B}_{2,4}(\Omega)$, we infer from Lemma 2.4 that $\mathbf{y}_{\mathbf{z}} \in \mathbf{W}_{4,2}(I)$ and the following estimate holds

$$\|\mathbf{y}_{\mathbf{z}}\|_{\mathbf{W}_{4,2}(I)} \le C_{4,2}(\|\mathbf{g}_{\mathbf{z}}\|_{L^4(I;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{y}_{N0}\|_{\mathbf{B}_{2,4}(\Omega)}). \tag{68}$$

Taking $s = \frac{1}{8}$ and $\theta = \frac{3}{4}$ in [2, Theorem 3], we get that $\mathbf{W}_{4,2}(0,T)$ is compactly embedded in $L^8(0,T;(\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega))_{\frac{3}{4},1}) \subset L^8(0,T;\mathbf{H}^{\frac{1}{2}}(\Omega)) \subset L^8(0,T;\mathbf{L}^4(\Omega))$. Therefore, the mapping F is well defined and compact for every $T < \infty$. It was established in the proof of Lemma 2.1 that $\mathbf{W}_{4,2}(I) \subset C(I;\mathbf{L}^4(\Omega))$ with continuous embedding. If $\|\mathbf{z}\|_{L^8(I;\mathbf{L}^4(\Omega))} \le 1$, then we have with (68)

$$\|\mathbf{y}_{\mathbf{z}}\|_{L^8(0,T;\mathbf{L}^4(\Omega))} \le T^{\frac{1}{8}}\|\mathbf{y}_{\mathbf{z}}\|_{C(I;\mathbf{L}^4(\Omega))} \le CT^{\frac{1}{8}}\|\mathbf{y}_{\mathbf{z}}\|_{\mathbf{W}_{4,2}(I)}$$
$$\le CC_{4,2}T^{\frac{1}{8}}(\|\mathbf{g}_{\mathbf{z}}\|_{L^4(I;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{y}_{N0}\|_{\mathbf{B}_{2,4}(\Omega)})$$
$$\le CC_{4,2}T^{\frac{1}{8}}\Big(\|\mathbf{g}\|_{L^4(I;\mathbf{H}^{-1}(\Omega))} + 1 + \|\mathbf{e}_1\|_{L^8(I;\mathbf{L}^4(\Omega))} + \|\mathbf{e}_2\|_{L^8(I;\mathbf{L}^4(\Omega))} + \|\mathbf{y}_{N0}\|_{\mathbf{B}_{2,4}(\Omega)}\Big).$$

Now, selecting T sufficiently small we infer that F applies the unit ball of $L^8(0,T;\mathbf{L}^4(\Omega))$ into itself. Hence, Schauder's Theorem implies the existence of a fixed point for F. Of course this fixed point belongs to $\mathbf{W}_{4,2}(0,T)$ and solves the equation (9) in the interval $(0,T)$. Since $\mathbf{y}_N$ is the unique solution of this equation in any interval $(0,T)$ we conclude that this fixed point is precisely the restriction of $\mathbf{y}_N$ to

$(0,T)$. There are two possibilities: either $\mathbf{y}_N \in \mathbf{W}_{4,2}(I)$ or there exists a maximal time $T^* \leq \infty$ such that $\mathbf{y}_N \in \mathbf{W}_{4,2}(0,T)$ for every $T < T^*$ and $\lim_{T\to T^*} \|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T)} = \infty$. Let us prove that the second option can not occur. We know that there exists a constant C independent of T such that $\|\mathbf{y}\|_{C([0,T];\mathbf{L}^4(\Omega))} \leq C\|\mathbf{y}\|_{\mathbf{W}_{4,2}(0,T)}$; see [1, Theorem III-4.10.2].

From the first part of the proof we have that $\mathbf{y}_N \in \mathbf{Y}$. By using Gagliardo's inequality (60) with $r = 4$ we obtain that $\mathbf{Y}$ is continuously embedded in $\mathbf{L}^4(Q)$. Hence, given $\epsilon > 0$ we can select $T_\varepsilon < T^*$ such that

$$\int_{T_\varepsilon}^{\infty} \|\mathbf{y}_N(t)\|^4_{\mathbf{L}^4(\Omega)}\,\mathrm{d}t < \varepsilon.$$

Let us denote $C_0 = \|\mathbf{g}\|_{L^4(I;\mathbf{H}^{-1}(\Omega))} + \|\mathbf{e}_1\|^2_{L^8(I;\mathbf{L}^4(\Omega))} + \|\mathbf{e}_2\|^2_{L^8(I;\mathbf{L}^4(\Omega))}$ and set $\varepsilon = [2CC_{4,2}]^{-1}$. Then, for every $T \in (T_\varepsilon, T^*)$ we infer from Lemma 2.4 that

$$\begin{aligned}
&\|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T)} \leq \|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T_\varepsilon)} + \|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(T_\varepsilon,T)}\\
&\leq \|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T_\varepsilon)} + C_{4,2}(C_0 + \|\mathbf{y}_N(T_\varepsilon)\|_{\mathbf{B}_{2,4}(\Omega)} + \|\mathbf{y}_N\|^2_{L^8(T_\varepsilon,T;\mathbf{L}^4(\Omega))})\\
&\leq \|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T_\varepsilon)} + C_{4,2}(C_0 + \|\mathbf{y}_N(T_\varepsilon)\|_{\mathbf{B}_{2,4}(\Omega)} + \|\mathbf{y}_N\|_{C([0,T];\mathbf{L}^4(\Omega))}\|\mathbf{y}_N\|_{L^4(T_\varepsilon,T;\mathbf{L}^4(\Omega))})\\
&\leq \|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T_\varepsilon)} + C_{4,2}(C_0 + \|\mathbf{y}_N(T_\varepsilon)\|_{\mathbf{B}_{2,4}(\Omega)}) + \frac{1}{2}\|\mathbf{y}_N\|_{\mathbf{W}_{4,2}(0,T)},
\end{aligned}$$

which proves that $T^* = \infty$ and $\mathbf{y}_N \in \mathbf{W}_{4,2}(I)$. □

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