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ORIGINAL PAPER

Surfaces with Central Configuration and Dulac's Problem for a Three Dimensional Isolated Hopf Singularity

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Abstract

Let ξ be a real analytic vector field with an elementary isolated singularity at $0 \in \mathbb{R}^3$ and eigenvalues $\pm bi, c$ with $b, c \in \mathbb{R}$ and $b \neq 0$. We prove that all cycles of ξ in a sufficiently small neighborhood of 0, if they exist, are contained in the union of finitely many subanalytic invariant surfaces, each one entirely composed of a continuum of cycles. In particular, we solve Dulac's problem for such vector fields, i.e., finiteness of limit cycles.

Keywords Hopf-zero singularity · Dulac problem in $\mathbb{R}^3$ · Local finiteness of limit cycles · Invariant surfaces · Reduction of singularities

Mathematics Subject Classification 34C05 (Primary) · 34C08 · 34C07 · 34C20 · 37C10 · 37C25 (Secondary)

1 Introduction and Statements

Dulac's problem is a central topic in the study of the dynamics of real analytic vector fields. In general terms, it consists in proving that there are no infinitely many limit cycles accumulating and collapsing to a singular point. Recall that in general, a *cycle* (or a *closed orbit*) of a vector field in a given manifold M is the image of a non-trivial periodic solution $\gamma : \mathbb{R} \to M$ (also denoted by γ), and a *limit cycle* is a cycle possessing a neighborhood free of other cycles.

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In dimension two, the problem was answered by Dulac in 1923 [9], but his proof had an important gap. It was solved nearly 70 years after by Ilyashenko [15] and Écalle [11], with two independent and different proofs, both very intricate. Recently, alternative proofs in some particular cases have been published, using *o-minimal geometry* [8, 12, 17, 30]. Dulac's result can be used to prove the finiteness statement of (the second part of) Hilbert's 16th problem. Namely, any polynomial vector field in $\mathbb{R}^2$ has finitely many limit cycles (see Ilyashenko's survey [16] for more information).

Non-accumulation of limit cycles for planar analytic vector fields implies a stronger property: either there are none in a neighborhood, or there is a continuum family of nested cycles filling a whole punctured neighborhood (a *central configuration*). In fact, given a cycle γ, we can define the *Poincaré first return* map in a transversal segment at some point $p \in \gamma$. It is an analytic local diffeomorphism whose fixed points correspond exactly to cycles in a neighborhood of γ. Thus, there are necessarily finitely many of them or they form a continuum annulus around γ. This is also the argument for proving Dulac's result in the easiest case in dimension two (apart, of course, from the trivial hyperbolic or semi-hyperbolic situations, when no local cycles exist). Namely, the case where the linear part of the vector field has purely imaginary non-zero eigenvalues (a so-called *Hopf singularity*) since after a blowing-up centered at zero, the exceptional divisor is a cycle and the Poincaré first return map is an analytic map. Hence, the set of fixed points is an analytic set and it can only be either a finite set or a continuum.

In this paper, we solve Dulac's problem for analytic three-dimensional vector fields with isolated singularity with a pair of conjugated imaginary non-zero eigenvalues (a *three-dimensional Hopf singularity*). In fact, we determine a finite number of invariant surfaces where local cycles may be placed and theses surfaces present a central configuration. Let us provide precise statements.

Denote by $\mathfrak{X}^{\omega}(\mathbb{R}^3, 0)$ the family of germs of analytic vector fields ξ at $0 \in \mathbb{R}^3$, singular at the origin, that is, $\xi(0) = 0$. If $\xi \in \mathfrak{X}^{\omega}(\mathbb{R}^3, 0)$ and U is an open neighborhood of 0 where (a representative of) ξ is defined, we denote by $\mathcal{C}_U = \mathcal{C}_U(\xi)$ the union of all cycles of $\xi|_U$ (that is, entirely contained in U). It is called the *cycle-locus* of ξ in U. Notice that this cycle-locus depends strongly on the neighborhood U and that it does not behave as a germ of a set that we can associate to the germ ξ (i.e., if $U' \subset U$ we can only assert that $\mathcal{C}_{U'} \subset \mathcal{C}_U$, but not $\mathcal{C}_{U'} = U' \cap \mathcal{C}_U$).

Consider the following family:

$$\mathcal{H}^3 := \{\xi \in \mathfrak{X}^{\omega}(\mathbb{R}^3, 0) : \ \mathrm{Spec}(D\xi(0)) = \{\pm bi, c\}, \ \text{where } b, c \in \mathbb{R} \text{ and } b \neq 0\}.$$

Observe that any $\xi \in \mathcal{H}^3$ has a unique formal invariant curve $\widehat{\Omega} = \widehat{\Omega}_{\xi}$ at 0, which is non-singular and tangent to the eigenspace corresponding to the eigenvalue c. It is called the *(formal) rotational axis* of ξ. When $c \neq 0$ (the *semi-hyperbolic case*), the rotational axis is convergent and provides an analytic invariant curve, since in this case $\widehat{\Omega}$ coincides with the stable or unstable manifold of ξ (see for instance [7]). On the contrary, when $c = 0$ (the *completely hyperbolic case* or *zero-Hopf singularity*), the rotational axis $\widehat{\Omega}$ may be convergent or not, although there is always an invariant $\mathcal{C}^{\infty}$-curve whose Taylor expansion at 0 coincides with $\widehat{\Omega}$. This is a result by Bonckaert and Dumortier in [3] in the case where ξ has an isolated singularity since ξ satisfies the required Łojasiewicz inequality condition in this case). It is trivially true if the singularity is not isolated since, in this case, $\widehat{\Omega}$ coincides with the singular locus $\mathrm{Sing}(\xi)$, an analytic curve. Notice that in the semi-hyperbolic case, ξ has an isolated singularity.

The main result in this paper can be stated as follows.

Theorem 1.1 (Structure of cycle-locus) *Let $\xi \in \mathcal{H}^3$ with isolated singularity. Then there is some neighborhood U of $0 \in \mathbb{R}^3$, where a representative of ξ is defined, for which exactly one of the following possibilities holds:*

(i) $\mathcal{C}_U(\xi) = \emptyset$.

(ii) *There is a finite non-empty family $\mathcal{S} = \{S_1, \ldots, S_r\}$ of connected regular analytic two-dimensional submanifolds of $U \setminus \{0\}$, mutually disjoint, invariant for ξ, subanalytic sets in U and satisfying $\overline{S_j} \cap U = S_j \cup \{0\}$ for any j, such that, for any element $V \subset U$ in some neighborhood basis at 0, we have*

$$\mathcal{C}_V(\xi) = (S_1 \cup S_2 \cup \cdots \cup S_r) \cap V.$$

As a consequence, Dulac's property is true for these vector fields:

Corollary 1.2 *Let $\xi \in \mathcal{H}^3$ with an isolated singularity. Then there are not infinitely many limit cycles of ξ accumulating and collapsing to $0 \in \mathbb{R}^3$.*

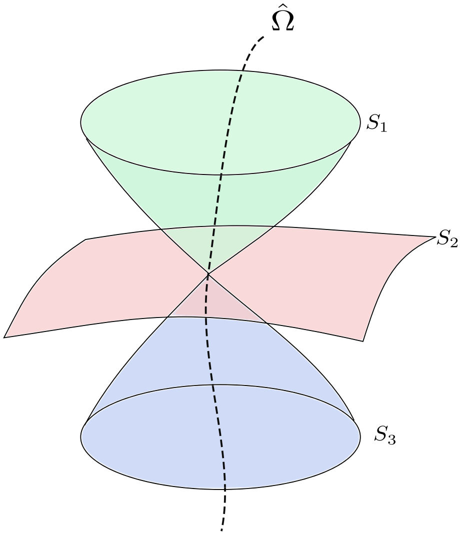

Fig. 1 Illustration of case (ii). Each surface has a center configuration

In the second possibility (ii), see Fig. 1, the germs of the surfaces $S_j \in \mathcal{S}$ are uniquely determined. Each of them, in a sufficiently small neighborhood, is composed of a continuum of nested cycles around the singularity, i.e., each S_j is a surface with a central configuration as in the planar case (although S_j could be singular at the origin). Let us call each $S_j \in \mathcal{S}$ a *limit central surface*, by analogy with the concept of limit cycle. The following example defines two limit central surfaces, both being singular at the origin.

Example 1.3 Consider the following vector field in $\mathcal{H}^3$.

$$\xi = (-y - xz^2 + x(x^2 + y^2))\frac{\partial}{\partial x} + (x - yz^2 + y(x^2 + y^2))\frac{\partial}{\partial y} + (z^3 - z(x^2 + y^2))\frac{\partial}{\partial z}.$$

It has isolated singularity. The two half-cones $S_1 = \{(x, y, z) : x^2 + y^2 - z^2 = 0, z > 0\}$ and $S_2 = \{(x, y, z) : x^2 + y^2 - z^2 = 0, z < 0\}$ are invariant. The restriction of ξ to any of the surfaces S_i is $\xi|_{S_i} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, which proves that ξ defines a central configuration in S_i, for $i = 1, 2$.

As an application of Theorem 1.1 to this example, one can see that there are not cycles outside $S_1 \cup S_2$ in a neighborhood of 0.

The result stated in Theorem 1.1 for the semi-hyperbolic case ($c \neq 0$) has been already proved by Aulbach [1], in a more general situation of n-dimensional analytic vector fields with a pair of purely imaginary non-zero eigenvalues and $n - 2$ eigenvalues with non-zero real part. Before Aulbach, the same situation has been considered in the literature by other authors [18, 19, 27, 28], under the assumption that the vector field has a first integral (as in the classical Lyapunov's result [20]). Using that any center manifold contains every local cycle (see [6]), one obtains that the possibility (ii) can only occur for a unique limit central surface ($r = 1$), which coincides with the center manifold W^c of ξ (hence unique, non-singular and analytic in this case).

Vector fields with a Hopf singularity in the completely non-hyperbolic case ($c = 0$) have been studied in the literature. For instance, Dumortier in [10] considered such vector fields of class $\mathcal{C}^\infty$ at $0 \in \mathbb{R}^3$, satisfying two *Łojasiewicz-type* inequalities: one for the vector field itself, which implies that 0 is an isolated singularity; and a second one for the *infinitesimal generator* of the Poincaré first-return map associated to the cycle that appears after the blowing-up of an invariant $\mathcal{C}^\infty$ realization of $\widehat{\Omega}$. He obtains a complete description of the asymptotic behavior of all trajectories in a neighborhood of the origin, as well as a *weak topological* classification of the vector field. However, those assumptions prevent the existence of any local cycle (that is, one has only the possibility (i) of Theorem 1.1). In our situation where ξ is analytic, Łojasiewicz's inequality for ξ is equivalent to the property of isolated singularity, but we do not require the second assumption, thus permitting the existence of cycles and hence the possibility (ii).

We should mention that vector fields in $\mathcal{H}^3$ have also been considered in families for different purposes. We can mention Guckenheimer and Holmes [14], where there is a complete description of the bifurcation diagrams for small codimension singularities; Baldoma, Ibáñez and Martínez-Seara [2] where the appearance of certain chaotic behavior, associated to a *Shilnikov configuration*, is studied; García [13], where, for each $k \in \mathbb{N}$, it is shown the existence of a bound for the number of limit cycles, appearing in certain generic families inside $\mathcal{H}^3$, which make at most k turns around the rotational axis.

Let us summarize the ideas for the proof of Theorem 1.1 and the plan of the article.

In Sect. 2, we propose a simple proof in the semi-hyperbolic case, in spite of the existing references already mentioned for this situation. Our aim, apart for the sake of completeness, is to introduce some of the arguments involved in the proof of the general case, absent in Aulbach's proof [1] but appearing in Dumortier's work [10]. Namely, blowing-up techniques and Poincaré first return map along the cycles emerging from the blowing-up.

The rest of the article is devoted to the proof in the completely non-hyperbolic case. We fix a formal normal form $\hat{\xi}$ (for instance in the sense of Takens [31]) and a sequence of analytic vector fields $\{\xi_\ell\}_\ell$ that approximate $\hat{\xi}$. The approximation must be understood in terms of jet equalities, that is, $j_\ell(\xi_\ell) = j_\ell(\hat{\xi})$. Neither the formal normal form $\hat{\xi}$ nor the sequence $\{\xi_\ell\}_\ell$ are univocally determined. However, once $\hat{\xi}$ is fixed, we choose the vector fields ξ_ℓ to be analytically conjugated to ξ. Thus, it is enough to prove Theorem 1.1 for some ξ_ℓ, with ℓ large enough.

In Sect. 3, we use blowing-ups to study $\hat{\xi}$ and its jets approximations ξ_ℓ. The rotational axis $\widehat{\Omega}$ is not necessarily convergent and we cannot blow it up (or any realization of it) if we want to preserve analyticity. Starting from the blowing-up of the origin, we define recursively *sequences of admissible blowing-ups*: a composition of blowing-ups centered at either the infinitely near points of $\widehat{\Omega}$ (*characteristic singularities*) or invariant closed circles of the corresponding strict transforms of $\hat{\xi}$ (*characteristic cycles*). The main result of this section is a *reduction of singularities* of the normal form $\hat{\xi}$ *adapted* to our problem. This process may be understood as a refinement, for this situation, of Panazzolo's result on reduction of singularities of general three-dimensional analytic vector fields [26] (notice that a Hopf singularity is already in the final elementary situation in the sense of Panazzolo). Essentially, the formal normal form $\hat{\xi}$ can be viewed as a vector field of revolution by rotating a planar vector field $\hat{\eta}$; the adapted reduction of singularities corresponds to the reduction of singularities of $\hat{\eta}$. In a second part of Sect. 3, we discuss how to apply sequences of admissible blowing-ups to the jet approximations ξ_ℓ. We find lower bounds for ℓ so that certain dynamical properties of $\hat{\xi}$, that depend on a finite jet, are inherited by ξ_ℓ. In particular, the characteristic cycles are actual cycles of the strict transform of ξ_ℓ.

In Sect. 4, we prove that, after any sequence of admissible blowing-ups, the characteristic cycles and the characteristic singularities are the only possible limit sets of families of cycles of the transform of ξ_ℓ, provided that ℓ is large enough. Thus, in order to prove Theorem 1.1, we only search for cycles near the characteristic cycles and characteristic singularities.

In Sect. 5, we study the different local situations appearing after an adapted reduction of singularities $\pi : (M, E) \longrightarrow (\mathbb{R}^3, 0)$ of $\hat{\xi}$. We have specific monotonic functions along the trajectories of the transformed vector field $\widetilde{\xi}_\ell = \pi^*\xi_\ell$ in neighborhoods of characteristic singularities or corner-characteristic cycles, preventing the existence of cycles of $\widetilde{\xi}_\ell$ in sufficiently small neighborhoods of them. Around a non-corner characteristic cycle γ, we work with the associated Poincaré first return map P_γ of $\widetilde{\xi}_\ell$. First, we find a formal invariant nonsingular surface S_γ of $\widetilde{\xi}_\ell$ supported by γ and transversal to the divisor, using that this is the case for the transform $\pi^*\hat{\xi}$ of $\hat{\xi}$. This surface S_γ provides a formal invariant curve Γ_γ for P_γ and, around Γ_γ, we can describe the periodic orbits of P_γ. Namely, there is a conic neighborhood Σ_γ around Γ_γ such that: if $\Gamma_\gamma \not\subseteq \mathrm{Fix}(P_\gamma)$, there are not periodic points of P_γ inside Σ_γ; if, otherwise, $\Gamma_\gamma \subseteq \mathrm{Fix}(P_\gamma)$, then Γ_γ is exactly the set of periodic points (thus fixed) inside Σ_γ.

Finally in Sect. 6, we give the proof of Theorem 1.1 gathering the results of the previous sections. First, we fix a vector field ξ_ℓ to which the reduction of singularities π can be applied. By means of the results in Sects. 4 and 5, cycles of $\widetilde{\xi}_\ell$ sufficiently near to (but not contained in) the divisor E can only be located in neighborhoods of the non-corner characteristic cycles γ. Moreover, the conic neighborhoods Σ_γ above provide solid conic neighborhoods $\widetilde{\Sigma}_\gamma$ of S_γ in such a way that if a cycle of $\widetilde{\xi}_\ell$ is contained in $\widetilde{\Sigma}_\gamma$, then the curve Γ_γ is contained in $\mathrm{Fix}(P_\gamma)$ and supports a continuum of cycles inside the saturation of Γ_γ by the flow, an analytic surface around γ. The projection of this surface under π provides a limit central surface. This would finish the proof of Theorem 1.1 if we could guarantee that all cycles of $\widetilde{\xi}_\ell$ in a neighborhood of γ are contained in the cone $\widetilde{\Sigma}_\gamma$. This is achieved by "opening" the cones Σ_γ to actual neighborhoods of γ by means of further blowing-ups. In this way, it is possible that we need a larger jet approximation $\xi_{\ell'}$ with $\ell' \geq \ell$, for which the order of its cones could change, a priori. We overcome this last difficulty showing that the order of a cone around γ where the cycles have the desired properties may be uniformly bounded for $\ell' \geq \ell$.

Notation and Conventions About the Power Series

If A is a $\mathbb{R}-$algebra and $\mathbf{x} = (x_1, \ldots, x_n)$ are variables, $A[[\mathbf{x}]]$ denotes the $\mathbb{R}-$algebra of formal power series in $\mathbf{x}$ with coefficients in A. Elements $f \in A[[\mathbf{x}]]$ are written as

$$f = \sum_{\alpha \in \mathbb{N}^n_{\geq 0}} f_\alpha \mathbf{x}^\alpha, \text{ where } \mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_n).$$

For any $k \geq 0$, the *$k-$jet of f* is defined as

$$j_k(f) = j_k^{\mathbf{x}}(f) := \sum_{\alpha : |\alpha| \leq k} f_\alpha \mathbf{x}^\alpha$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_n$. The *order of f*, denoted by $\nu(f)$ is the first $k \geq 0$ (or $+\infty$ if it does not exist) such that $j_k(f) \neq 0$. If we separate the variables into two groups $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ where $\mathbf{y} = (y_1, \ldots, y_r)$ and $\mathbf{z} = (z_1, \ldots, z_s)$, the *$k-$jet $j_k^{\mathbf{z}}(f)$ of f with respect to the variable* $\mathbf{z}$ is the $k-$jet of f as an element of $A[[\mathbf{y}]][[\mathbf{z}]]$ under the natural identification $A[[\mathbf{x}]] \xrightarrow{\sim} A[[\mathbf{y}]][[\mathbf{z}]]$, that is, the jet $j_k^{\mathbf{z}}(f)$ is given by

$$j_k^{\mathbf{z}}(f) := \sum_{|\beta| \leq k} \left(\sum_{\gamma \in \mathbb{N}^r_{\geq 0}} f_{(\gamma,\beta)} \mathbf{y}^\gamma \right) \mathbf{z}^\beta.$$

We will use freely the following basic properties of jets:

- $j_k(f \cdot g) = j_k(j_k(f) \cdot j_k(g))$, for $f, g \in A[[\mathbf{x}]]$. In fact, this property can be refined: if $k \geq \max\{\nu(f), \nu(g)\}$, then $j_k(f \cdot g) = j_k(j_{k-\nu(g)}(f) \cdot j_{k-\nu(f)}(g))$.
- $j_k(f^{-1}) = j_k((j_k(f))^{-1})$ if f is a unit in $A[[\mathbf{x}]]$.
- $j_k^{\mathbf{z}}(f(x_1, \ldots, x_i + a, \ldots, x_n)) = j_k^{\mathbf{z}}(f)(x_1, \ldots, x_i + a, \ldots, x_n)$ for $i \leq r$ and $a \in A$.
- $j_k(f) = j_k(j_k^{\mathbf{z}}(f))$.

We extend the use of $k-$jets (respectively with respect to $\mathbf{z}$) for formal vector fields $\hat{\eta} = \eta_1 \frac{\partial}{\partial x_1} + \cdots + \eta_n \frac{\partial}{\partial x_n}$ with $\eta_j \in A[[\mathbf{x}]]$ or tuples $F = (f_1, \ldots, f_m) \in A[[\mathbf{x}]]^m$ in the obvious way

$$j_k^u(\hat{\eta}) := j_k^u(\eta_1) \frac{\partial}{\partial x_1} + \cdots + j_k^u(\eta_n) \frac{\partial}{\partial x_n}, \quad j_k^u(F) := (j_k^u(f_1), \ldots, j_k^u(f_m)),$$

with $u = \mathbf{x}$ (respectively $u = \mathbf{z}$).

When A is a normed space, the subalgebra of convergent series with coefficients on A is the subalgebra of $A[[\mathbf{x}]]$ defined by

$$A\{\mathbf{x}\} := \bigcup_{\delta > 0} A\{\mathbf{x}\}_\delta$$

where, by definition, a series $f = \sum_{\alpha \in \mathbb{N}^n_{\geq 0}} f_\alpha \mathbf{x}^\alpha \in A[[\mathbf{x}]]$ belongs to $A\{\mathbf{x}\}_\delta$ if there exists $C > 0$ such that $||f_\alpha|| < C\delta^{|\alpha|}$ for any α. The main examples for the algebra of the coefficients used along the article are the following:

- $A = \mathbb{R}$ with the standard norm of the absolute value.
- $A = \mathbb{R}[\cos\theta, \sin\theta]$, the algebra of trigonometric polynomials, whose elements are considered indistinctively as a function on $\mathbb{R}$ or on $\mathbb{S}^1$, via the covering $\tau : \theta \to (\cos\theta, \sin\theta)$. It will be endowed with the supremum norm $||f|| := \sup_{\theta \in \mathbb{R}} f(\theta)$. Notice that given a convergent series $F \in \mathbb{R}[\cos\theta, \sin\theta]\{\mathbf{x}\}_\delta$, its partial sums converge absolutely and uniformly in the compact sets of the neighborhood $V = \mathbb{S}^1 \times (-\delta, \delta)^n$ of $\mathbb{S}^1 \times \{0\}$ (or the neighborhood $V = \mathbb{R} \times (-\delta, \delta)^n$ of $\mathbb{R} \times \{0\}$), thus providing an analytic function that we denote again f.

- In the case of $A = \mathbb{R}[\mathbf{z}]$ (respectively $\mathbb{R}[\cos\theta, \sin\theta, \mathbf{z}]$), where $\mathbf{z} = (z_1, \ldots, z_r)$, there is no unique natural norm on A. We will consider a norm for each compact set K of $\mathbb{R}^r$ (resp. $\mathbb{S}^1 \times \mathbb{R}^r$) with non-empty interior, defined by

$$||f||_K := \sup_{a \in K}\{|f(a)|\}.$$

Denoting $A_K = (A, ||\cdot||_K)$ such a normed space, we have the corresponding algebra of convergent series $A_K\{\mathbf{x}\}$. We define the *algebra of convergent series with coefficients in A* as the intersection of algebras $A_K\{\mathbf{x}\}$ where K runs all compact sets of such form. With an abuse of notation, we name this algebra $A\{\mathbf{x}\}$ for convenience. Each element $f \in A\{\mathbf{x}\}$ defines an analytic function on a neighborhood of $\mathbb{R}^r \times \{0\}$ (resp. $\mathbb{S}^1 \times \mathbb{R}^r \times \{0\}$) in $\mathbb{R}^r \times \mathbb{R}^n$ (resp. in $\mathbb{S}^1 \times \mathbb{R}^r \times \mathbb{R}^n$).

Moreover, for a formal vector field $\hat{\xi}$, we will use the expression $\hat{\xi}(z)$ to denote the formal series obtained by the application of the formal derivation $\hat{\xi}$ to the function z, and it coincides with the coefficient of $\frac{\partial}{\partial z}$ in $\hat{\xi}$.

2 The Semi-hyperbolic Case

In this section, we provide a proof of Theorem 1.1 in the semi-hyperbolic case, i.e., the linear part $D\xi(0)$ has eigenvalues $\{\pm bi, c\}$ with both b, c different from zero.

Assume for instance that $c < 0$. Then, the stable manifold W^s of ξ at 0 is one-dimensional and, as we have said, it coincides with the rotational axis, which is therefore convergent. Fix some center manifold W^c of ξ at 0 of class $\mathcal{C}^k$, with $k \geq 2$. In general, it is not analytic, nor unique. But it contains any cycle of ξ that is contained in a sufficiently small neighborhood U of the origin, i.e., $\mathcal{C}_U(\xi) \subset W^c$ (see [6]).

Take a neighborhood U_0 inside which, both the stable manifold W^s and the chosen center manifold W^c are regular embedded submanifolds, and such that $\mathcal{C}_{U_0}(\xi) \subset W^c$. Let $\pi : M \to U_0$ be the polar blowing-up with center W^s. It is a proper analytic map. The divisor $E = \pi^{-1}(W^s)$ is a cylinder and the fiber $\gamma = \pi^{-1}(0)$ over the origin is a cycle of the transformed vector field $\widetilde{\xi} := \pi^*\xi$. The strict transform $\widetilde{W^c} = \overline{\pi^{-1}(W^c \backslash \{0\})}$ is a surface of class $\mathcal{C}^{k-1}$, invariant for $\widetilde{\xi}$ and transversal to E. Moreover, $\gamma = E \cap \widetilde{W^c}$.

Now, consider a point $a \in \gamma$, and two nested analytic discs $\Delta' \subset \Delta$ transverse to $\widetilde{\xi}$ close to a so that the Poincaré first-return map $P_\gamma : \Delta' \to \Delta$ of $\widetilde{\xi}$ associated to γ is well defined and analytic. Notice that if ζ is any cycle of $\widetilde{\xi}$ such that $\zeta \cap \Delta = \zeta \cap \Delta'$, then the intersection $\zeta \cap \Delta$ is a periodic orbit of P_γ (see Fig. 2). In particular, if ζ is the inverse image by π of a cycle inside $\mathcal{C}_{U_0}(\widetilde{\xi})$, then, ζ is contained in $\widetilde{W^c}$. Taking into account that W^c is two-dimensional and using classical arguments based on the Jordan Curve Theorem (see for instance [25]), we conclude in this case that ζ cuts Δ' in a single point, necessarily a fixed point of P_γ. Hence, the family of cycles of ξ in a given neighborhood of the origin are in bijection with the set of fixed points $\mathrm{Fix}(P_\gamma)$ of P_γ not in E, and hence, $\mathrm{Fix}(P_\gamma)$ is contained in the intersection $H = \widetilde{W^c} \cap \Delta'$. Let us prove now Theorem 1.1.

Suppose that item (i) does not hold, i.e., $\mathcal{C}_U(\xi) \neq \emptyset$ for any open neighborhood U of 0. Then we have infinitely many cycles of ξ that accumulate and collapse to 0. By the above, there are infinitely many fixed points of P_γ in H accumulating to the point a. Being P_γ an analytic map, its set $\mathrm{Fix}(P_\gamma)$ of fixed points is an analytic set of positive dimension. Since $\mathrm{Fix}(P_\gamma) \subset H$ and H is a curve of class $\mathcal{C}^{k-1}$ (transversal intersection of $\widetilde{W^c}$ and Δ'), we conclude that $H = \mathrm{Fix}(P_\gamma)$.

Let $\widetilde{U}$ be a neighborhood of γ in M satisfying:

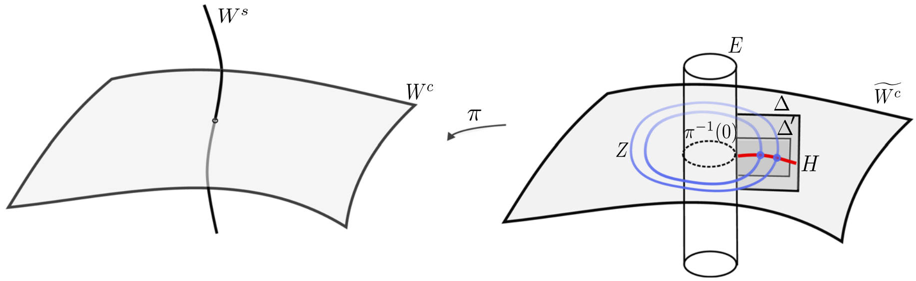

Fig. 2 Definition of the Poincaré map P_γ

- $\widetilde{U} \cap \Delta = \Delta'$.
- $\widetilde{U} \cap \widetilde{W}^c$ is the saturation of $H \cap \widetilde{U}$ by the flow of $\widetilde{\xi}$.
- $U = \pi(\widetilde{U})$ is contained in U_0.

We get that U is a neighborhood of 0 and $\mathcal{C}_U(\xi) = W^c \cap U \setminus \{0\}$. Notice also that $\widetilde{W^c} \cap \widetilde{U}$ is an analytic set since H is an analytic curve. Since π is proper, we conclude that $W^c \cap U$ is a subanalytic set and Theorem 1.1 is proved.

Remark 2.1 The proof above shows that, in the semi-hyperbolic case, there is at most one limit central surface S_1. Moreover, if S_1 exists, then $\overline{S_1} = W^c$ is a center manifold which is unique and analytic (using Tamm's Theorem [32], because W^c is of class $\mathcal{C}^k$ and subanalytic in this case).

3 Admissible Blowing-Ups and Adapted Reduction of Singularities

Consider a vector field ξ in the family $\mathcal{H}^3$ with completely non-hyperbolic linear part, that is, $\text{Spec}(\xi) = \{\pm bi, 0\}$. Without loss of generality for the study of the foliation generated by ξ, we will assume $b = 1$. In some coordinates, the vector field is written as

$$\xi = (-y + A_1(x, y, z))\frac{\partial}{\partial x} + (x + A_2(x, y, z))\frac{\partial}{\partial y} + (A_3(x, y, z))\frac{\partial}{\partial z}, \qquad (1)$$

with $A_1, A_2, A_3 \in \mathbb{R}\{x, y, z\}$ of order at least two.

3.1 Formal Normal Form and Truncated Normal Forms

Using Takens' theorem on normal forms (see [31]), there exists a formal automorphism at 0, expressed in terms of the chosen coordinates as

$$\hat{\varphi}(x, y, z) = (x + \hat{\varphi}_1(x, y, z), y + \hat{\varphi}_2(x, y, z), z + \hat{\varphi}_3(x, y, z)) \in \mathbb{R}[[x, y, z]]^3,$$

with $j_1(\hat{\varphi}_j) = 0$ for $j = 1, 2, 3$, such that the formal vector field $\hat{\xi} = \hat{\varphi}^*(\xi)$ is written in the form

$$\hat{\xi} = T(x^2 + y^2, z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + R(x^2 + y^2, z)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$
$$+ Z(x^2 + y^2, z)\frac{\partial}{\partial z}, \qquad (2)$$

where $R, T, Z \in \mathbb{R}[[u, v]]$ and $T(0, 0) = 1$. Note that $R(u, v), Z(u, v) \in (u, v)$ and $Z(0, v) \in (v^2)$. Remark also that neither the automorphism $\hat{\varphi}$ need to be convergent, nor the components of $\hat{\xi}$ need to belong to $\mathbb{R}\{x, y, z\}$. Any formal vector field $\hat{\xi}$ as in (2) obtained as above is called a formal normal form of ξ. We remark that $\hat{\xi}$ is not uniquely determined by ξ.

Remark 3.1 The z-axis is sent to the formal rotational axis $\widehat{\Omega}$ of ξ by $\hat{\varphi}$, that is, $\widehat{\Omega} = \hat{\varphi}(0, 0, z)$. On the other hand, since $\hat{\varphi}$ must preserve the (formal) singular locus, the hypothesis that ξ has isolated singularity implies that $Z(0, v) \neq 0$.

Once we fix a formal normal form $\hat{\xi}$ of ξ given by $\hat{\xi} = \hat{\varphi}^*\xi$, we can consider truncated normal forms of ξ in the following way. For any $\ell \in \mathbb{N}_{\geq 2}$, let φ_ℓ be the polynomial tangent to the identity diffeomorphism of $(\mathbb{R}^3, 0)$ given by

$$\varphi_\ell(x, y, z) = (j_{\ell+1}\hat{\varphi})(x, y, z) = (j_{\ell+1}(x \circ \hat{\varphi}), j_{\ell+1}(y \circ \hat{\varphi}), j_{\ell+1}(z \circ \hat{\varphi})).$$

The vector field $\xi_\ell = (\varphi_\ell)^*(\xi)$ has the same $\ell-$jet as the formal one $\hat{\xi}$ in coordinates (x, y, z), that is, $j_\ell(\xi_\ell) = j_\ell(\hat{\xi})$. Notice that the vector field ξ_ℓ is analytically conjugated to ξ and formally conjugated to $\hat{\xi}$ for any ℓ. More precisely, we have the following formal equation:

$$\hat{\xi} = \psi_\ell^*\xi_\ell, \text{ where } \psi_\ell := \varphi_\ell^{-1} \circ \hat{\varphi}. \tag{3}$$

It is sufficient to prove Theorem 1.1 for ξ_ℓ for any ℓ. The strategy is the following: we use $\hat{\xi}$ as a guiding vector field so that, after a sequence of blowing-ups, we get a transform of $\hat{\xi}$ with a specific good expression. Both the choice of the sequence of blowing-ups and the expression of the transform will depend only on a finite jet of $\hat{\xi}$, allowing us to choose ℓ sufficiently large so that all the construction is applied to ξ_ℓ.

The blowing-ups will be real (oriented) ones, thus generating boundary and corners, either with center at a point or at an analytic curve isomorphic to the circle $\mathbb{S}^1$. See for instance the work [24] for intrinsic and general definitions of real blowing-ups.

3.2 The First Blowing-Up

The first blowing-up to be done is the real blowing-up $\sigma_0 : (M_0, E_0) \longrightarrow (\mathbb{R}^3, 0)$ with center at the origin. The blown-up space M_0 is a manifold having the divisor $E_0 = \sigma_0^{-1}(0)$ as its boundary. This divisor is homeomorphic to a sphere and represents the space of all the half-lines through 0. The morphism σ_0 defines an analytic isomorphism from $M_0 \backslash E_0$ to $\mathbb{R}^3 \backslash \{0\}$. We consider M_0 covered by three charts $(C_0, (\theta, z^{(0)}, \rho^{(0)}))$, $(C_\infty, (x^{(\infty)}, y^{(\infty)}, z^{(\infty)}))$ and $(C_{-\infty}, (x^{(-\infty)}, y^{(-\infty)}, z^{(-\infty)}))$ where $C_0 \simeq \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ and $C_{\pm\infty} \simeq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. In these charts, the expression of σ_0 is given by:

$$\text{In } C_0 : \begin{cases} x = \rho^{(0)} \cos\theta \\ y = \rho^{(0)} \sin\theta \\ z = \rho^{(0)} z^{(0)} \end{cases} \quad (\cos\theta, \sin\theta) \in \mathbb{S}^1, z^{(0)} \in \mathbb{R}, \rho^{(0)} \geq 0 \tag{4}$$

$$\text{In } C_\infty : \begin{cases} x = x^{(\infty)} z^{(\infty)} \\ y = y^{(\infty)} z^{(\infty)} \\ z = z^{(\infty)} \end{cases} \quad x^{(\infty)}, y^{(\infty)} \in \mathbb{R}, z^{(\infty)} \geq 0 \tag{5}$$

$$\text{In } C_{-\infty} : \begin{cases} x = x^{(-\infty)} z^{(-\infty)} \\ y = y^{(-\infty)} z^{(-\infty)} \\ z = -z^{(-\infty)} \end{cases} \quad x^{(-\infty)}, y^{(-\infty)} \in \mathbb{R}, z^{(-\infty)} \geq 0. \tag{6}$$

Remark 3.2 Strictly speaking, C_0 is not the domain of a usual chart of M_0, since it is not homeomorphic to an open set of $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. Considering the usual covering $\tilde{C}_0 = \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ with $\tau : \tilde{C}_0 \longrightarrow C_0$ given by $(\theta, z, \rho) \mapsto (\sin\theta, \cos\theta, z, \rho)$, we can treat θ as a true coordinate (and we will tacitly do), so that $\sigma_0 \circ \tau$ has the expression in (4). This convention justifies our abuse of terminology in expressions like "a chart $(C_0, (\theta, z^{(0)}, \rho^{(0)}))$".

The origins of the charts C_∞ and $C_{-\infty}$ will be denoted by γ_∞ and $\gamma_{-\infty}$, respectively. They are the points of the divisor E_0 corresponding to the half-lines contained in the $z-$axis and they are the only points of E_0 not covered by C_0. More explicitly, $\sigma_0(C_0) = \mathbb{R}^3 \backslash \{x = y = 0\}$.

We define the *(total) transform of* $\hat{\xi}$ *by* σ_0 *in the chart* C_0 as the pull-back

$$\hat{\xi}^{(0)} := (\sigma_0|_{C_0})^* \hat{\xi}.$$

Using simplified notation $(z, \rho) := (z^{(0)}, \rho^{(0)})$ and Eqs. (2) and (4), the vector field $\hat{\xi}^{(0)}$ is given by

$$\hat{\xi}^{(0)} = B_\theta(z, \rho)\frac{\partial}{\partial\theta} + B_z(z, \rho)\frac{\partial}{\partial z} + B_\rho(z, \rho)\frac{\partial}{\partial\rho}, \tag{7}$$

where $B_\theta(z, \rho) = T(\rho^2, \rho z)$, $B_z(z, \rho) = \frac{1}{\rho} Z(\rho^2, \rho z) - zR(\rho^2, \rho z)$ and $B_\rho(z, \rho) = \rho R(\rho^2, \rho z)$. Notice that, by the definition of the blowing-up, we have that $B_\theta, B_z, B_\rho \in \mathbb{R}[z][[\rho]]$ since z is replaced by $z\rho$. Moreover, $(B_z, B_\rho) \neq (0, 0)$ since $Z(u, v) \neq 0$ by Remark 3.1 and ρ divides B_z, B_ρ.

The coefficient $B_\theta(z, \rho)$ is a unit in $\mathbb{R}[z][[\rho]]$ since $B_\theta(z, 0) = 1$. This allows us to consider θ as the "time variable" and, consequently, $\hat{\xi}^{(0)}$ is completely described by the *associated two dimensional* formal vector field $\hat{\eta}_0$ given by the system of formal ODEs

$$\hat{\eta}_0 : \begin{cases} \frac{dz}{d\theta} = B_\theta(z, \rho)^{-1} B_z(z, \rho) = \rho^{n^{(0)}} A_z(z, \rho) \\ \frac{d\rho}{d\theta} = B_\theta(z, \rho)^{-1} B_\rho(z, \rho) = \rho^{n^{(0)}} A_\rho(z, \rho). \end{cases} \tag{8}$$

In this expression, $A_i \in \mathbb{R}[z][[\rho]]$ for $i = z, \rho$ and $n^{(0)}$ is the maximum exponent n such that ρ^n divides both B_ρ and B_z. The *associated reduced vector field* is by definition $\hat{\eta}_0' := \rho^{-n^{(0)}} \hat{\eta}_0$.

There are two possible scenarios determined in the following definition.

Definition 3.3 The blowing-up σ_0 is called *non-dicritical* if $A_\rho(z, 0) \equiv 0$ and *dicritical* if $A_\rho(z, 0) \neq 0$. Alternatively, we say that E_0 is *non-dicritical* or that E_0 is *dicritical*, respectively.

Despite of the fact that $\hat{\eta}_0$ is just formal, the restriction $\hat{\eta}_0'|_{F_0}$ to the curve $F_0 := E_0 \cap \{\theta = 0\}$ is a well defined vector field (under the natural identification $\{\theta = 0\} \cong \mathbb{R}^2$, $(0, z, \rho) = (z, \rho)$). This restriction has polynomial coefficients in the coordinate z. Therefore, its singular locus:

$$\mathrm{Sing}(\hat{\eta}_0'|_{F_0}) := \{a \in F_0 : \hat{\eta}_0'|_{F_0}(a) = 0\} = \{(z, 0) : A_\rho(z, 0) = A_z(z, 0) = 0\}$$

is finite. Singular points are points where we have to focus in order to define successive blowing-ups. But, in the dicritical case, we have to add those non-singular points where the vector field is tangent to the divisor. To be used for later, we recall the definition of such non-transversal points in the general situation of a normal crossing divisor (see Cano, Cerveau and Deserti's book [5] in the complex holomorphic context).

Let χ be a formal vector field defined at 0 and F a non-empty normal crossing divisor. Consider a chart $(U, (x, y))$ centered at 0 where $F = \{xy^\epsilon = 0\}$ and the coefficients of the vector field in these coordinates belong to $\mathbb{R}[y][[x]]$ if $\epsilon = 0$ or to $\mathbb{R}[y][[x]] \cap \mathbb{R}[x][[y]]$ if $\epsilon = 1$. Take any point $a = (a_1, a_2) \in F \cap U$, the vector field $\chi_a := \chi(\tilde{x} + a_1, \tilde{y} + a_2)$ is well defined as a formal vector field in coordinates $(\tilde{x}, \tilde{y})$.

Definition 3.4 Let F be a non-empty normal crossing divisor and let χ be a formal vector field defined at F. The *adapted singular locus* $\widetilde{\mathrm{Sing}}(\chi, F)$ *of* χ *relatively to* F, is the set of points $p \in F$ in which either $\chi(p) = 0$ or $C \cup F$ has no normal crossings at p, where C is the formal invariant curve of χ through p.

Applied to our reduced vector field $\hat{\eta}'_0$ and to F_0, we have

(a) If E_0 is non-dicritical, then $\widetilde{\mathrm{Sing}}(\hat{\eta}'_0, F_0) = \mathrm{Sing}(\hat{\eta}'_0|_{F_0})$.
(b) If E_0 is dicritical, then $\widetilde{\mathrm{Sing}}(\hat{\eta}_0, F_0) = \mathrm{Sing}(\hat{\eta}'_0|_{F_0}) \cup \{(z, 0) : A_\rho(z, 0) = 0\}$

In both cases, the adapted singular locus $\widetilde{\mathrm{Sing}}(\hat{\eta}_0, F_0)$ is finite.

We define also the transforms $\hat{\xi}^{(\infty)} := (\sigma_0|_{C_\infty})^* \hat{\xi}$ and $\hat{\xi}^{(-\infty)} := (\sigma_0|_{C_{-\infty}})^* \hat{\xi}$ of $\hat{\xi}$ in the charts $C_\infty, C_{-\infty}$, respectively. The expressions for $\hat{\xi}^{(\infty)}$, using simplified notation $(x, y, z) := (x^{(\infty)}, y^{(\infty)}, z^{(\infty)})$ is the following:

$$\hat{\xi}^{(\infty)} = R^{(\infty)}(x^2 + y^2, z)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + T^{(\infty)}(x^2 + y^2, z)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + Z^{(\infty)}(x^2 + y^2, z)\frac{\partial}{\partial z}, \tag{9}$$

where $R^{(\infty)}, T^{(\infty)}, Z^{(\infty)} \in \mathbb{R}[x^2 + y^2][[z]]$ are given by:

$$\begin{aligned} R^{(\infty)}(x^2 + y^2, z) &= R((x^2 + y^2)z^2, z) - \frac{1}{z} Z((x^2 + y^2)z^2, z), \\ T^{(\infty)}(x^2 + y^2, z) &= T((x^2 + y^2)z^2, z) \text{ and} \\ Z^{(\infty)}(x^2 + y^2, z) &= Z((x^2 + y^2)z^2, z). \end{aligned}$$

In a similar way, we obtain expressions for $\hat{\xi}^{(-\infty)}$.

3.3 Characteristic Cycles and Successive Blowing-Ups

Recall that the adapted singular locus $\widetilde{\mathrm{Sing}}(\hat{\eta}'_0, F_0)$ of $\hat{\eta}'_0$ relative to F_0 is finite. Its elements, belonging to $F_0 = \{\theta = \rho^{(0)} = 0\}$ are determined by the $z^{(0)}-$coordinate in the chart C_0. Denote them by

$$\widetilde{\mathrm{Sing}}(\hat{\eta}'_0, F_0) = \{(\omega_i^{(0)}, 0) : i = 1, \dots, m_0\}, \quad \text{with } \omega_i^{(0)} < \omega_j^{(0)} \text{ if } i < j.$$

Definition 3.5 The *characteristic cycles* of $\hat{\xi}$ in M_0 are the connected components of the set $\mathbb{S}^1 \times \widetilde{\mathrm{Sing}}(\eta'_0|F_0) \subset C_0$, that is, the circles in the divisor E_0 given by $\gamma_i := \{z^{(0)} = \omega_i^{(0)}, \rho^{(0)} = 0\}$ for $i = 1, 2, \dots, m_0$. The origins $\gamma_\infty, \gamma_{-\infty}$ of the charts C_∞ and $C_{-\infty}$ (cf. Eqs. (5) and (6)) are called the *characteristic singularities of* $\hat{\xi}$ *in* M_0. We use the term *characteristic elements* to refer to either the characteristic cycles or characteristic singularities.

In the rest of this section, we inductively define certain sequences of blowing-ups attached to $\hat{\xi}$ starting from the data defined above for the first blowing-up σ_0. More precisely, consider the tuple $\mathcal{M}_0 := (M_0, \sigma_0, \mathcal{A}_0, \mathcal{D}_0)$, where:

- $\mathcal{A}_0$ is the atlas of M_0 composed by the charts $C_{-\infty}, C_0, C_\infty$,
- $\mathcal{D}_0$ is the family of *characteristic elements* of $\hat{\xi}$ in M_0, that is, $\mathcal{D}_0 := \{\gamma_{-\infty}, \gamma_1, \dots, \gamma_{m_0}, \gamma_\infty\}$.

By definition, we say that $\mathcal{M}_0$ is a *sequence of admissible blowing-ups of length* $l = 0$ *for* $\hat{\xi}$. Suppose that we have already defined sequences of admissible blowing-ups for $\hat{\xi}$ of length $l-1$, consisting on tuples $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ satisfying the following hypothesis:

(H1) $\pi : (M, E) \longrightarrow (\mathbb{R}^3, 0)$ is a sequence of (real) blowing-ups with smooth analytic closed centers and factorizing through σ_0 (i.e., $\pi = \sigma_0 \circ \bar{\pi}$, where $\bar{\pi} : M \longrightarrow M_0$ is either the identity or a sequence of blowing-ups with smooth analytic closed centers).

(H2) $\mathcal{D} = \{\gamma_I\}_{I\in\mathcal{I}}$ is a finite family of disjoint closed subsets of the divisor $E = \pi^{-1}(0)$, such that:

- There are two elements in $\mathcal{D}$ with indices $I_\infty^{\mathcal{M}} = (\infty, \overset{s}{\dots}, \infty)$ and $I_{-\infty}^{\mathcal{M}} = (-\infty, \overset{t}{\dots}, -\infty)$ for some $s, t \in \mathbb{N}_{\geq 1}$, that are the two points where E intersects the strict transform $\pi^*(\{x = y = 0\})$ of the $z-$axis. They are called the *characteristic singularities of* $\hat{\xi}$ *in* M.
- The rest of the elements γ_I, with $I \neq I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}$, are analytic embedded circles called *characteristic cycles (of* $\hat{\xi}$ *in* M).
- The intersection of any pair of components of E is an element of $\mathcal{D}$. Each of them is called a *corner characteristic cycle*. The corner characteristic cycles are those indexed by tuples $I = (i_1, \dots, i_r) \neq I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}$ for which $i_r = \pm\infty$.

(H3) $\mathcal{A} = \{C_J\}_{J\in\mathcal{J}}$ is an atlas of M with the following properties:

(1) There are charts $(C_J, (x^{(J)}, y^{(J)}, z^{(J)}))$ centered at the characteristic singularities γ_J, with $J \in \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$, satisfying $E \cap C_J = \{z^{(J)} = 0\}$. Moreover, the expression of π in the chart C_J with $J = I_{\epsilon\infty}^{\mathcal{M}}$, for $\epsilon = \pm 1$, is

$$\pi(x^{(J)}, y^{(J)}, z^{(J)}) = ((z^{(J)})^r x^{(J)}, (z^{(J)})^r y^{(J)}, \epsilon z^{(J)})$$

with $r \in \mathbb{N}_{\geq 1}$ (r and ϵ depend on J). Furthermore, the coefficients of $\hat{\xi}^{(J)} := (\pi|_{C_J})^*\hat{\xi}$ belong to $\mathbb{R}[x^{(J)}, y^{(J)}][[z^{(J)}]]$.

(2) If $J \notin \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$, the chart $(C_J, (\theta, z^{(J)}, \rho^{(J)}))$ is defined for $\theta \in \mathbb{R}$, $z^{(J)}$ in $\mathbb{R}$ or $\mathbb{R}_{\geq 0}$ and $\rho^{(J)} \in \mathbb{R}_{\geq 0}$ (with the same convention as in Remark 3.2), and satisfies $E \cap C_J = \{\rho^{(J)}(z^{(J)})^\epsilon = 0\}$ with $\epsilon = 0$ or 1 according to $z^{(J)}$ being defined either in $\mathbb{R}$ or $\mathbb{R}_{\geq 0}$, respectively. In the case $\epsilon = 0$, the chart C_J is a *non-corner chart* and the characteristic cycles contained in $E \cap C_J$ are given by equations $\{z^{(J)} = a_i, \rho^{(J)} = 0\}$, where $\{a_i\}_i$ is a finite collection of real numbers. In the case $\epsilon = 1$, the chart C_J is a *corner chart* and the family of characteristic cycles contained in $E \cap C_J$ consists of a unique corner characteristic cycle given by $\{z^{(J)} = 0, \rho^{(J)} = 0\}$ and a collection of non-corner characteristic cycles given either by $\{z^{(J)} = b_j, \rho^{(J)} = 0\}_j$ for a family $\{b_j\}_j$ of positive numbers or by $\{z^{(J)} = 0, \rho^{(J)} = c_k\}_k$ for a family $\{c_k\}_k$ of positive numbers.

(3) For any $J \notin \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$ the expression of $\pi|_{C_J}$ is polynomial in $(\cos\theta, \sin\theta, z^{(J)}, \rho^{(J)})$ and the transformed vector field $\hat{\xi}^{(J)} := (\pi|_{C_J})^*\hat{\xi}$ written, with simplified notation $(\theta, z, \rho) = (\theta, z^{(J)}, \rho^{(J)})$, as

$$\hat{\xi}^{(J)} = B_\theta^{(J)}(z, \rho)\frac{\partial}{\partial\theta} + B_z^{(J)}(z, \rho)\frac{\partial}{\partial z} + B_\rho^{(J)}(z, \rho)\frac{\partial}{\partial\rho}, \tag{10}$$

satisfies that, for $i = \theta, z, \rho$, the coefficient $B_i^{(J)}$ belongs to $\mathbb{R}[z][[\rho]]$ if C_J is a non-corner chart, or to both algebras $\mathbb{R}[z][[\rho]]$ and $\mathbb{R}[\rho][[z]]$, if C_J is a corner characteristic chart. In any case, $B_\theta^{(J)}(z,0) = 1$ and hence it is a unit of the corresponding algebra.

When $J = I_{-\infty}^{\mathcal{M}}$ or $J = I_{\infty}^{\mathcal{M}}$, we define $n^{(J)}$ to be the maximum n such that $\hat{\xi}^{(J)}(z^{(J)}) = (z^{(J)})^n \cdot \tilde{B}(x^{(J)}, y^{(J)}, z^{(J)})$ with $\tilde{B}$ an element in $\mathbb{R}[x^{(J)}, y^{(J)}][[z^{(J)}]]$.

Observing (H3)-(2), for $J \notin \{I_{\infty}^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$, we define the *vector field associated to the transform* $\hat{\xi}^{(J)}$, as the formal two dimensional vector field $\hat{\eta}_J$ given by the following system of ODEs (using (10) and simplifying $(z,\rho) = (z^{(J)}, \rho^{(J)})$):

$$\hat{\eta}_J : \begin{cases} \frac{dz}{d\theta} = B_\theta^{(J)}(z,\rho)^{-1} B_z^{(J)}(z,\rho) = \rho^{n_1^{(J)}} z^{n_2^{(J)}} A_z^{(J)}(z,\rho) \\ \frac{d\rho}{d\theta} = B_\theta^{(J)}(z,\rho)^{-1} B_\rho^{(J)}(z,\rho) = \rho^{n_1^{(J)}} z^{n_2^{(J)}} A_\rho^{(J)}(z,\rho). \end{cases} \tag{11}$$

Here, $n_1^{(J)}$ is the maximum n such that ρ^n divides both $B_\rho^{(J)}$ and $B_z^{(J)}$ (and thus, $A_\rho^{(J)}$ and $A_z^{(J)}$ are formal series not both together divisible by ρ). On the other hand, if C_J is a non-corner chart, we take $n_2^{(J)} = 0$, and, if C_J is a corner chart, we take $n_2^{(J)}$ to be the maximum m such that z^m divides both $B_\rho^{(J)}$ and $B_z^{(J)}$. We define $n^{(J)} := \max\{n_1^{(J)}, n_2^{(J)}\}$ in all cases.

It is clear that $\mathcal{M}_0$ fulfills (H1-H3). Now, a *sequence of admissible blowing-ups for* $\hat{\xi}$ *of length* l is a tuple $\mathcal{M}' := (M', \pi', \mathcal{A}', \mathcal{D}')$ built from a sequence of admissible blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ of length $l-1$ in such a way that $\pi' = \pi \circ \sigma_{\gamma_I}$, where

$$\sigma_{\gamma_I} : M' \longrightarrow M$$

is the blowing-up centered at some $\gamma_I \in \mathcal{D}$. The expression of σ_{γ_I} in charts and the description of the families $\mathcal{D}'$, $\mathcal{A}'$ are exposed in what follows (see Fig. 3 for an illustration of the different situations). We consider two cases: the blowing-up σ_{γ_I} is centered at a characteristic singularity or at a characteristic cycle.

(i) *First case*. The blowing-up σ_{γ_I} is centered at the singular point γ_I with $I = I_{\infty}^{\mathcal{M}}$ (or analogously for $I = I_{-\infty}^{\mathcal{M}}$). Put $J = I$, $J_\infty = (J, \infty)$ and $J_0 = (J, 0)$. The point γ_I is the origin of a chart $(C_J, (x^{(J)}, y^{(J)}, z^{(J)})) \in \mathcal{A}$ where the divisor $E = \pi^{-1}(0)$ is given by $\{z^{(J)} = 0\}$. Then, the exceptional divisor $\sigma_{\gamma_I}^{-1}(\gamma_I)$ is covered by two charts of M', say $(C_{J_\infty}, (x^{(J_\infty)}, y^{(J_\infty)}, z^{(J_\infty)}))$ and $(C_{J_0}, (\theta, z^{(J_0)}, \rho^{(J_0)}))$, so that σ_{γ_I} is written as:

$$\text{In } C_{J_0} : \begin{cases} x^{(J)} = \rho^{(J_0)} \cos\theta \\ y^{(J)} = \rho^{(J_0)} \sin\theta \\ z^{(J)} = \rho^{(J_0)} z^{(J_0)} \end{cases} \qquad \theta \in \mathbb{R}, z^{(J_0)}, \rho^{(J_0)} \geq 0 \tag{12}$$

$$\text{In } C_{J_\infty} : \begin{cases} x^{(J)} = x^{(J_\infty)} z^{(J_\infty)} \\ y^{(J)} = y^{(J_\infty)} z^{(J_\infty)} \\ z^{(J)} = z^{(J_\infty)} \end{cases} \qquad x^{(J_\infty)}, y^{(J_\infty)} \in \mathbb{R}, z^{(J_\infty)} \geq 0. \tag{13}$$

We set the atlas of $\mathcal{M}'$ to be $\mathcal{A}' = (\mathcal{A} \backslash \{C_J\}) \cup \{C_{J_0}, C_{J_\infty}\}$ (under the identification $\sigma_{\gamma_J} : M' \backslash \sigma_{\gamma_J}^{-1}(\gamma_J) \to M \backslash \gamma_J$). The chart C_{J_0} is a corner chart in this case.

Let us define $\mathcal{D}$. We consider first the vector field $\hat{\xi}^{(J_\infty)} := (\sigma_{\gamma_I}|_{C_{J_\infty}})^* \hat{\xi}^{(I)}$. Define $n^{(J_\infty)}$ as the maximum $n \in \mathbb{N}$ such that $(z^{(J_\infty)})^n$ divides $\hat{\xi}^{(J_\infty)}(z^{(J_\infty)})$. In the chart C_{J_∞}, the expression of the vector field $\hat{\xi}^{(J_\infty)}$ is similar to (9). The origin of C_{J_∞} is named γ_{J_∞}. Secondly, consider the formal vector field $\hat{\xi}^{(J_0)} := (\sigma_{\gamma_I}|_{C_{J_0}})^*(\hat{\xi}^{(I)})$. Use the expression in (10) for $\hat{\xi}^{(I)}$, and

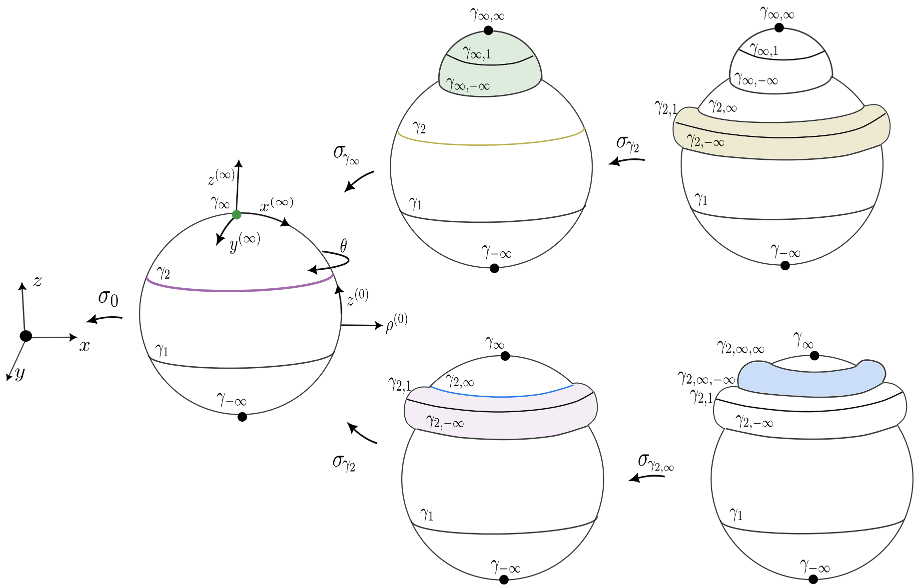

Fig. 3 Several sequences of admissible blowing-ups

rename $(z, \rho) = (z^{(J_0)}, \rho^{(J_0)})$. Then, $\hat{\xi}^{(J_0)}$ is given by

$$\hat{\xi}^{(J_0)} = B_\theta^{(J_0)}(z, \rho)\frac{\partial}{\partial\theta} + B_z^{(J_0)}(z, \rho)\frac{\partial}{\partial z} + B_\rho^{(J_0)}(z, \rho)\frac{\partial}{\partial\rho}, \tag{14}$$

where $B_\theta^{(J_0)} \in \mathbb{R}[z][[\rho]]$ is a unit in the algebra $\mathbb{R}[z][[\rho]]$ (because $B_\theta^{(J_0)}(z, 0) = 1$) and

$$B_z^{(J_0)} = \rho^{n_1^{(J_0)}} z^{n_2^{(J_0)}} \tilde{B}_z^{(J_0)}, \quad B_\rho^{(J_0)} = \rho^{n_1^{(J_0)}} z^{n_2^{(J_0)}} \tilde{B}_\rho^{(J_0)}, \qquad \tilde{B}_\rho^{(J_0)}, \tilde{B}_z^{(J_0)} \in \mathbb{R}[z][[\rho]], \tag{15}$$

where $n_1^{(J_0)}, n_2^{(J_0)}$ are defined similarly as we have defined $n_1^{(J)}$ and $n_2^{(J)}$. The *vector field associated to* $\hat{\xi}^{(J_0)}$ is the two dimensional vector field $\hat{\eta}_{J_0}$ with coefficients in $\mathbb{R}[z][[\rho]]$, defined in a similar manner as $\hat{\eta}_0$ in (8). That is,

$$\hat{\eta}_{J_0} = \rho^{n_1^{(J_0)}} z^{n_2^{(J_0)}} \left(A_z^{(J_0)}(z, \rho)\frac{\partial}{\partial z} + A_\rho^{(J_0)}(z, \rho)\frac{\partial}{\partial\rho} \right), \tag{16}$$

where $A_k^{(J_0)} = \tilde{B}_k^{(J_0)} \cdot (B_\theta^{(J_0)})^{-1}$ for $k = z, \rho$. The vector field $\hat{\eta}'_{J_0} := \frac{1}{\rho^{n_1^{(J_0)}} z^{n_2^{(J_0)}}}\hat{\eta}_{J_0}$ is called the *reduced vector field associated to* $\hat{\xi}^{(J_0)}$. We distinguish two cases:

- The blowing-up σ_{γ_I} is *non-dicritical* if $A_\rho^{(J_0)}(z, 0) \equiv 0$. In this case, we say that the divisor $\sigma_{\gamma_I}^{-1}(\gamma_I)$ is a *non-dicritical component* of the total divisor $E' := (\pi')^{-1}(0)$.
- The blowing-up σ_{γ_I} is *dicritical* if $A_\rho^{(J_0)}(z, 0) \neq 0$ and $\sigma_{\gamma_I}^{-1}(\gamma_I)$ a *dicritical component* of the total divisor E'.

Put $E_{J_0} := \sigma_{\gamma_I}^{-1}(\gamma_I) \cap C_{J_0}$ and $F_{J_0} := E_{J_0} \cap \{\theta = 0\}$ and consider $\widetilde{\text{Sing}}(\hat{\eta}'_{J_0}, F_{J_0})$ the adapted singular locus of $\hat{\eta}'_{J_0}$ relatively to F_{J_0}, it is a finite set, taking into account that the coefficients of $\hat{\eta}'_{J_0}$ belong to $\mathbb{R}[z][[\rho]]$. Denote those elements contained in the regular part

$\dot{F}_{J_0} := F_{J_0} \cap \{z > 0\}$ of F_{J_0} as (in coordinates $(z^{(J_0)}, \rho^{(J_0)})$)

$$\widetilde{\mathrm{Sing}}(\hat{\eta}'_{J_0}, F_{J_0}) \cap \dot{F}_{J_0} = \{(\omega_i^{(J_0)}, 0) : i = 1, \ldots, m_{J_0}\}, \text{ with } 0 < \omega_i^{(J_0)} < \omega_j^{(J_0)} \text{ if } i < j.$$

The circles $\gamma_{I,i} := \{z = \omega_i^{(J_0)}, \rho = 0\} \subset E_{J_0}$ for each $i = 1, \ldots, m_{J_0}$ are by definition the *non-corner characteristic cycles in* C_{J_0}. The circle $\gamma_{I,-\infty} := \{z = 0, \rho = 0\}$ is by definition *a corner characteristic cycle*.

Gathering all the above objects, we define the family $\mathcal{D}' := \{\gamma_I\}_{I \in \mathcal{I}'}$ of *characteristic elements of* $\mathcal{M}'$, where

$$\mathcal{I}' = (\mathcal{I} \backslash \{I\}) \cup \left(\bigcup_{i=1}^{m_{J_0}} \{(I, i)\} \right) \cup \{(I, -\infty), (I, \infty)\}.$$

The elements of $\mathcal{D}'$ are subsets of $E' = (\pi')^{-1}(0)$, once we identify $\gamma_L = \sigma_{\gamma_I}^{-1}(\gamma_L)$ for $L \in \mathcal{I} \backslash \{I\}$. They are either the two points $\gamma_{\infty,\ldots,\infty}$ and $\gamma_{-\infty,\ldots,-\infty}$ (whose indices are denoted also by $I_{\infty}^{\mathcal{M}'}$ and $I_{-\infty}^{\mathcal{M}'}$, respectively) called the *characteristic singularities of* $\hat{\xi}$ *in* $\mathcal{M}'$ or circles (the *characteristic cycles of* $\hat{\xi}$ *in* $\mathcal{M}'$).

(ii) *Second case.* σ_{γ_I} is centered at one of the characteristic cycles $\gamma_I \in \mathcal{D}$ with $I = (i_1, \ldots, i_r)$. It can be a corner characteristic cycle (in which case $i_r = \pm\infty$) or not. The charts after the blowing-up σ_{γ_I} are defined in a different manner in each case. In order to simplify the notation, name $I' = (i_1, \ldots, i_{r-1})$.

(a) When γ_I is a corner characteristic cycle, it can be seen as $\{\rho^{(J)} = 0, z^{(J)} = 0\}$ in a chart C_J by (H3). Put $J_0 = (I, 0)$ and $J_\infty = (I, \infty)$. The set $\sigma_{\gamma_I}^{-1}(\gamma_I)$ is covered by two new charts $(C_{J_\infty}, (\theta, z^{(J_\infty)}, \rho^{(J_\infty)}))$ and $(C_{J_0}, (\theta, z^{(J_0)}, \rho^{(J_0)}))$, where the blowing-up σ_{γ_I} is written as:

$$\text{In} \quad C_{J_\infty} : \begin{cases} \theta &= \theta \\ z^{(J)} &= z^{(J_\infty)} \\ \rho^{(J)} &= \rho^{(J_\infty)} z^{(J_\infty)} \end{cases}, \quad \theta \in \mathbb{R},\ z^{(J_\infty)}, \rho^{(J_\infty)} \geq 0, \tag{17}$$

$$\text{In} \quad C_{J_0} : \begin{cases} \theta &= \theta \\ z^{(J)} &= \rho^{(J_0)} z^{(J_0)} \\ \rho^{(J)} &= \rho^{(J_0)} \end{cases}, \quad \theta \in \mathbb{R},\ z^{(J_0)}, \rho^{(J_0)}, \geq 0. \tag{18}$$

The new atlas is defined by $\mathcal{A}' := (\mathcal{A} \backslash \{C_J\}) \cup \{C_{J_0}, C_{J_\infty}\}$, where we have identified $M' \backslash \sigma_{\gamma_I}^{-1}(\gamma_I)$ and $M \backslash \gamma_I$ via σ_{γ_I}.

To determine the new family $\mathcal{D}'$ of characteristic elements in this case, we write the transformed formal vector fields $\hat{\xi}^{(J_0)} := (\sigma_{\gamma_I}|_{C_{J_0}})^* \hat{\xi}^{(J)}, \hat{\xi}^{(J_\infty)} := (\sigma_{\gamma_I}|_{C_{J_\infty}})^* \hat{\xi}^{(J)}$ in the two charts. Both are similar and, in fact, to determine $\mathcal{D}'$ only one of the expressions is sufficient. Considering for instance the chart C_{J_0}, and with similar computations and notations as in the precedent paragraphs, we write (simplifying $(z, \rho) = (z^{(J_0)}, \rho^{(J_0)})$)

$$\hat{\xi}^{(J_0)} = B_\theta^{(J_0)}(z, \rho) \frac{\partial}{\partial \theta} + B_z^{(J_0)}(z, \rho) \frac{\partial}{\partial z} + B_\rho^{(J_0)}(z, \rho) \frac{\partial}{\partial \rho}, \tag{19}$$

where $B_\theta^{(J_0)}, B_z^{(J_0)}, B_\rho^{(J_0)} \in \mathbb{R}[z][[\rho]]$ and $B_\theta^{(J_0)}$ is a unit. The *vector field associated to* $\hat{\xi}^{(J_0)}$ is $\hat{\eta}_{J_0} := (B_\theta^{J_0})^{-1} B_z^{(J_0)} \frac{\partial}{\partial z} + (B_\theta^{J_0})^{-1} B_\rho^{(J_0)} \frac{\partial}{\partial \rho}$. We put

$$\hat{\eta}_{J_0} = \rho^{n_1^{(J_0)}} z^{n_2^{(J_0)}} \hat{\eta}'_{J_0} = \rho^{n_1^{(J_0)}} z^{n_2^{(J_0)}} \left(A_z^{(J_0)}(z, \rho) \frac{\partial}{\partial z} + A_\rho^{(J_0)}(z, \rho) \frac{\partial}{\partial \rho} \right),$$

where the natural numbers $n_k^{(J_0)}$ for $k = 1, 2$ are defined as in case (i). The vector field $\hat{\eta}'_{J_0}$ is the *reduced associated vector field*. We distinguish the cases when σ_{γ_I}, or the component $E_{J_0} := \sigma_{\gamma_I}^{-1}(\gamma_I)$, is dicritical ($A_\rho^{(J_0)}(z, 0) \neq 0$) or non-dicritical ($A_\rho^{(J_0)}(z, 0) \equiv 0$). Put $F_{J_0} := E_{J_0} \cap \{\theta = 0\}$, $\dot{F}_{J_0} := F_{J_0} \cap \{z > 0\}$ and denote

$$\widetilde{\mathrm{Sing}}(\hat{\eta}'_{J_0}, F_{J_0}) \cap \dot{F}_{J_0} = \{(\omega_i^{(J_0)}, 0) : i = 1, \ldots, m_{J_0}\}, \text{ with } 0 < \omega_i^{(J_0)} < \omega_j^{(J_0)} \text{ if } i < j.$$

With these data, we set:

$$\begin{aligned} \gamma_{I,i} &:= \mathbb{S}^1 \times \{(z^{(J_0)}, \rho^{(J_0)}) = (\omega_i^{(J_0)}, 0)\}, i = 1, \ldots m_{J_0} \\ \gamma_{I,-\infty} &:= \mathbb{S}^1 \times \{(z^{(J_0)}, \rho^{(J_0)}) = (0, 0)\} \subset C_{J_0} \\ \gamma_{I,\infty} &:= \mathbb{S}^1 \times \{(z^{(J_\infty)}, \rho^{(J_\infty)}) = (0, 0)\} \subset C_{J_\infty} \end{aligned}$$

and we define the family of *characteristic elements* of $\mathcal{M}'$ as $\mathcal{D}' := \{\gamma_I\}_{I \in \mathcal{I}}$, where

$$\mathcal{I}' = (\mathcal{I} \backslash \{I\}) \cup \{(I, i)\}_{i=1}^{m_{J_0}} \cup \{(I, -\infty)\} \cup \{(I, \infty)\},$$

again identifying γ_L with $\sigma_{\gamma_I}^{-1}(\gamma_L)$ for $L \in \mathcal{I} \backslash \{I\}$. Notice that, among the new characteristic cycles, $\gamma_{I,\infty}, \gamma_{I,-\infty}$ are corner cycles and the other ones are non-corner characteristic cycles.

(b) When γ_I is a non-corner characteristic cycle (that is, by (H2), when $I = (i_1, \ldots, i_r)$ with $i_r \neq \pm\infty$), it can be seen as the set $\gamma_I = \{z^{(J)} = \omega_k^{(J)}, \rho^{(J)} = 0\}$ for some $\omega_k^{(J)}$ in the domain of $z^{(J)}$ of a chart C_J, by (H3). Set $J_{-\infty} := (I, -\infty)$, $J_\infty := (I, \infty)$ and $J_0 := (I, 0)$. The blowing-up $\sigma_{\gamma_I} : (M', E') \longrightarrow (M, \gamma_I)$ of γ_I is given in three new charts $(C_u, (\theta, z^{(u)}, \rho^{(u)}))$, for $u \in \{J_\infty, J_0, J_{-\infty}\}$, by

$$\text{In } C_{J_\infty} : \begin{cases} \theta = \theta \\ z^{(J)} = z^{(J_\infty)} - \omega_k^{(J)} \\ \rho^{(J)} = \rho^{(J_\infty)} z^{(J_\infty)} \end{cases} \quad \theta \in \mathbb{R}, \rho^{(J_\infty)}, z^{(J_\infty)} \geq 0. \tag{20}$$

$$\text{In } C_{J_0} : \begin{cases} \theta = \theta \\ z^{(J)} = \rho^{(J_0)}(z^{(J_0)} - \omega_k^{(J)}) \\ \rho^{(J)} = \rho^{(J_0)} \end{cases} \quad \theta \in \mathbb{R}, z^{(J_0)} \in \mathbb{R}, \rho^{(J_0)} \geq 0. \tag{21}$$

$$\text{In } C_{J_{-\infty}} : \begin{cases} \theta = \theta \\ z^{(J)} = -z^{(J_{-\infty})} + \omega_k^{(J)} \\ \rho^{(J)} = \rho^{(J_{-\infty})} z^{(J_{-\infty})} \end{cases} \quad \theta \in \mathbb{R}, \rho^{(J_{-\infty})}, z^{(J_{-\infty})} \geq 0. \tag{22}$$

The new atlas is $\mathcal{A}' := (\mathcal{A} \backslash \{C_J\}) \cup \{C_{J_0}, C_{J_\infty}, C_{J_{-\infty}}\}$. The family $\mathcal{D}'$ of *characteristic elements* of $\hat{\xi}$ in $\mathcal{M}'$ is defined analogously as in case (a), studying the corresponding transformed vector field $\hat{\xi}^{(J_0)} = (\sigma_{\gamma_I}|_{C_{J_0}})^* \hat{\xi}^{(J)}$, its associated two-dimensional vector field $\hat{\eta}_{J_0}$ and the adapted singular locus of the reduced associated vector field $\hat{\eta}'_{J_0}$ relatively to $F_{J_0} = C_{J_0} \cap \sigma_{\gamma_I}^{-1}(\gamma_I) \cap \{\theta = 0\}$. We just observe the following:

- The charts C_{J_∞} and $C_{J_{-\infty}}$ are corner charts and the curves $\gamma_{J_\infty} = \{z^{(J_\infty)} = 0, \rho^{(J_\infty)} = 0\}$ and $\gamma_{J_{-\infty}} = \{z^{(J_{-\infty})} = 0, \rho^{(J_{-\infty})} = 0\}$ are corner characteristic cycles.
- The chart C_{J_0} is a non-corner chart and contains the new non-corner characteristic cycles $\gamma_{I,i}$ where $\gamma_{I,i} = \mathbb{S}^1 \times \{(\omega_i^{(J_0)}, 0)\}$, $i = 1, \ldots, m_{J_0}$ being $\widetilde{\mathrm{Sing}}(\hat{\eta}'_{J_0}, F_{J_0}) = \{(\omega_i^{(J_0)}, 0),\ i = 1, \ldots, m_{J_0}\}$.

From the construction, we can check that the hypothesis (H1-H3) are fulfilled for $\mathcal{M}'$. Thus, we have defined admissible sequences of blowing-ups of $\hat{\xi}$ of any length.

Remark 3.6 By construction, a non-corner characteristic cycle $\gamma_I \in \mathcal{D}$ is defined in some chart C_J by equations $\gamma_I = \{z^{(J)} = \omega_k^{(J)}, \rho^{(J)} = 0\}$ for some $k \in \mathbb{N}_{\geq 1}$. It may happen that the same characteristic cycle γ_I is defined in a corner chart $C_{\tilde{J}}$ by $\{z^{(\tilde{J})} = 0, \rho^{(\tilde{J})} = c\}$, for some $c \in \mathbb{R}$.

Remark 3.7 Notice that for any given sequence of admissible blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ and for any chart C_J of $\mathcal{A}$, the associated vector field $\hat{\eta}_J$ is not identically zero. This can be seen from the construction of $\mathcal{M}$ and using Remark 3.1.

3.4 Adapted Reduction of Singularities

Recall (see the book [5]) that a formal vector field $\chi = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ at $(\mathbb{R}^2, 0)$ has a *(real) simple singularity* if $\mathrm{Sing}(\chi) = \{0\}$, the eigenvalues λ_1, λ_2 of the linear part $D\chi(0)$ are real and at least one of them is different from zero, for instance $\lambda_2 \neq 0$, and $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}_{>0}$. In this case, χ has exactly two formal invariant curves, also called *separatrices,* which are tangent to the corresponding eigenspaces, non-singular and mutually transverse. We need an extended notion of simple singularity, also taken from that reference, that takes into account the existence of a divisor and the possibility that the singularity is not isolated.

Definition 3.8 Let $F = \{xy^\epsilon = 0\}$, where $\epsilon \in \{0, 1\}$, be a normal crossing divisor at $0 \in \mathbb{R}^2$. A formal vector field χ at $(\mathbb{R}^2, 0)$ has an *adapted simple singularity relatively to* F if one of the two following situations occurs:

(1) $\mathrm{Sing}(\chi) = \{0\}$, the singularity is simple and each component of F is invariant for χ (thus, if $\epsilon = 1$, the two components of F are the two separatrices).

(2) $\epsilon = 0$, there is a formal non-singular curve Γ transversal to $F = \{x = 0\}$ given by an equation $\Gamma = \{y - \hat{g}(x) = 0\}$ contained in $\mathrm{Sing}(\chi)$, and $\chi = (y - \hat{g}(x))^r \bar{\chi}$, with $r \geq 1$, such that either $\bar{\chi}$ is non-singular at 0 and F is the only invariant curve of $\bar{\chi}$ through 0, or $\bar{\chi}$ has a simple singularity at 0 and the set of separatrices of $\bar{\chi}$ at 0 is $\{F, \Gamma\}$.

To distinguish the two cases of this definition, in the situation of *(2)*, we say that χ has a *non-saturated adapted simple singularity*. Usually in this situation, one divides χ by an equation of $\mathrm{Sing}(\chi)$ to get the situation in (1) or a non-singular point. However for us, the vector field χ will come from some three dimensional vector field, hence it will be important to keep unaltered the singular locus placed outside the divisor.

Before introducing the reduction of singularities of $\hat{\xi}$ adapted to our problem, we recall Seidenberg's Theorem ([29]) of reduction of singularities of a two dimensional analytic (or formal) vector field ξ, following the lines of the book [5]. In this reference, it is assumed that the vector field is saturated, i.e. that χ has an isolated singularity at the origin. For us, it is important to consider the non-saturated case: that χ writes as $\chi = f\bar{\chi}$, where f is non-zero, non-unit and a generator of $\mathrm{Sing}(\chi)$. Moreover, we cannot "saturate" χ just by dividing by f since we treat the formal case and we want to preserve the analytic nature of the given coordinates. Instead, we adapt the result in [5] to the non-saturated case, which only involves a slightly modification and encompasses both a reduction of singularities of the singular locus of χ and a reduction of singularities of $\bar{\chi}$. For the sake of completeness, we provide here a precise statement and we sketch the modifications to be made for its proof.

Theorem 3.9 *Let χ be a formal vector field at $(\mathbb{R}^2, 0)$ not identically zero, saturated or not, $F^{(0)}$ be a normal crossings divisor and $0 \in \widetilde{Sing}(\chi, F^{(0)})$. Then there is a composition of a finite number of punctual blowing-ups $\pi : (\tilde{N}, \tilde{F}) \to (\mathbb{R}^2, F^{(0)})$ fulfilling the following conditions:*

(a) *For any point* $q \in \tilde{F} = \pi^{-1}(F^{(0)})$, *if* χ'_q *is the strict transform of* χ *by* π *at* q *(that is,* $\chi'_q = \frac{1}{u^k v^l}\pi^*(\chi)$, *where* uv^ϵ *is a local reduced equation of* $\tilde{F}$ *at* q *(*$\epsilon = 0$ *or 1) and* k, l *are maximal so that* χ'_q *has no pole), then* $q \in \widetilde{Sing}(\chi'_q, \tilde{F})$ *if and only if* $q \in Sing(\chi'_q)$.

(b) *If* $q \in \tilde{F}$ *is a singular point of* χ'_q, *then* q *is an adapted simple singularity relatively to* $\tilde{F}$ *(cf. Definition 3.8).*

(c) *Any dicritical component is isolated as a dicritical component (i.e. any other component that intersects it is non dicritical).*

Proof In the case where χ has as an isolated singularity at 0, the existence of the reduction of singularities τ is given by the result in [5], where one eliminates points in the adapted singular locus relatively to the divisor that are not singular points (to get (a)). In the case where the singular locus $S := \mathrm{Sing}(\chi)$ of χ at 0 is not reduced to $\{0\}$ (thus S is a finite union of formal curves), we first consider a reduction of singularities $\psi : (N^{(1)}, F^{(1)}) \to (\mathbb{R}^2, F^{(0)})$ of S. Then, let χ'_1 be the strict transform of χ by ψ, we blow up any $q \in \widetilde{\mathrm{Sing}}(\chi'_1, F^{(1)})$ that is not a singular point to get (a). After that, we may assume that such strict transform, named χ'_q, either has an isolated singularity at q (and hence we apply again [5]) or q is a point in the strict transform $S^{(1)}_q$ of the curve S by ψ. In this last case, there are coordinates (x, y) at q such that $F^{(1)} = \{x = 0\}$, and χ'_q is written as $\chi'_q = (y - \hat{g}(x))^r \bar{\chi}'_q$, where $\{y - \hat{g}(x) = 0\}$ is an equation of $S^{(1)}_q$, $r \geq 1$ and $\bar{\chi}'_q$ has at most an isolated singularity at q.

If $\bar{\chi}'_q(q) = 0$, after a reduction of singularities of $\bar{\chi}'_q$, we may assume that q is a simple singularity of $\bar{\chi}'_q$. By further blowing-ups, we separate $\tilde{S}^{(1)}_q$ from the two separatrices of $\bar{\chi}'_q$ unless one of them coincides with $\tilde{S}^{(1)}_q$. We will get in this way adapted simple singularities of (the transform of) $\tilde{\chi}'_q$ either saturated (cf. Definition 3.8-(1)) or non saturated (cf. Definition 3.8-(2)).

When $\bar{\chi}'_q(q) \neq 0$, if Γ is the formal solution of $\bar{\chi}'_q$ through q, a new blowing-up at q produces an adapted simple singularity for the transform of $\bar{\chi}'_q$ at the point corresponding to the tangent line of Γ. If Γ coincides with $\tilde{S}^{(1)}_q$, we get an adapted simple singular point for $\tilde{\chi}'_q$. Otherwise, by further blowing-ups, we separate Γ from $\tilde{S}^{(1)}_q$ and we get either adapted simple singularities or points in the situation already treated.

Note that condition *(c)* is obtained as a consequence of the result in [5] since only normal crossings are allowed. ◻

Now, we can state the result which gives the reduction of singularities of a formal normal form $\hat{\xi}$ of a vector field $\xi \in \mathcal{H}^3$ with isolated singularity.

Proposition 3.10 (Adapted resolution of singularities) *Let* $\hat{\xi}$ *be a formal vector field written as in Eq. (2) with isolated singularity at* $0 \in \mathbb{R}^3$. *Then there exists a sequence of admissible blowing-ups* $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ *for* $\hat{\xi}$ *with* $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$, $\mathcal{D} = \{\gamma_I\}_{I \in \mathcal{I}}$ *and total divisor* $E = \pi^{-1}(0)$ *such that*

(1) *For* $J \in \{I^{\mathcal{M}}_{\infty}, I^{\mathcal{M}}_{-\infty}\}$, *the transformed vector field* $\hat{\xi}^{(J)} = (\pi|_{C_J})^*\hat{\xi}$ *satisfies* $\hat{\xi}^{(J)}(z^{(J)}) = (z^{(J)})^t \cdot G$ *where* $t \geq 1$ *and* G *is a unit in* $\mathbb{R}[[x^{(J)}, y^{(J)}, z^{(J)}]]$.

(2) *For any* $J \in \mathcal{J} \backslash \{I^{\mathcal{M}}_{\infty}, I^{\mathcal{M}}_{-\infty}\}$, *the singularities of the reduced associated vector field* $\hat{\eta}'_J$ *are adapted simple singularities relatively to the divisor* $E \cap C_J \cap \{\theta = 0\}$.

(3) *If* E_0 *is a dicritical component of* E, *then* E_0 *is isolated as a dicritical component (i.e. any other component that intersects* E_0 *is non dicritical). Moreover, for any* $J \in \mathcal{J} \backslash \{I^{\mathcal{M}}_{\infty}, I^{\mathcal{M}}_{-\infty}\}$, *one has* $\widetilde{Sing}(\hat{\eta}'_J, F_{0,J}) = \emptyset$ *where* $F_{0,J} = E_0 \cap C_J \cap \{\theta = 0\}$, *in particular,* $\hat{\eta}'_J$ *is everywhere transversal to* $F_{0,J}$.

Proof From Remark 3.1, there exists a term $c_j z^j$ in the coefficient $\hat{\xi}(z)$ with $c_j \neq 0$. Assume, without loss of generality, that j is the minimum exponent with this condition. Notice that $j > 0$. Write $\hat{\xi}(z) \in \mathbb{R}[[x, y, z]]$ as

$$\hat{\xi}(z) = Z(x^2 + y^2, z) = z^{t_0} G(x, y, z) = z^{t_0} \sum_{k=\nu(G)}^{\infty} G_k(x, y, z),$$

where G_k is an homogeneous polynomial of degree k for each k, $t_0 \geq 0$ is defined as the maximum integer such that z^{t_0} divides $\hat{\xi}(z)$ and $\nu(G)$ is the order of G as a series as defined in the Introduction. Then $G_{j-t_0}(x, y, z)$ contains the monomial $c_j z^{j-t_0}$ (notice that $j \geq t_0$ and the equality holds if and only if $\nu(G) = 0$). Consider the first blowing-up σ_0 and study $\hat{\xi}^{(\infty)}(z^{(\infty)})$, where $\hat{\xi}^{(\infty)} = (\sigma_0|_{C_\infty})^* \hat{\xi}$. Omitting super-indices for the coordinates $(x^{(\infty)}, y^{(\infty)}, z^{(\infty)})$, we have:

$$\hat{\xi}^{(\infty)}(z) = z^{t_0} \sum_{k=\nu(G)}^{\infty} G_k(x, y, 1) z^k = z^{t_1} \sum_{k=\nu(G)}^{\infty} G_k(x, y, 1) z^{k-\nu(G)},$$

where $t_1 = t_0 + \nu(G) \geq t_0$. Rewrite the series $G^{(1)} := \sum_{k=\nu(G)}^{\infty} G_k(x, y, 1) z^{k-\nu(G)}$ in homogeneous components:

$$\hat{\xi}^{(\infty)}(z) = z^{t_1} G^{(1)}(x, y, z) = z^{t_1} \sum_{k=\nu(G^{(1)})}^{\infty} G_k^{(1)}(x, y, z).$$

If $j = t_1$, we see that $G_0^{(1)} = c_j$ and thus $G^{(1)}$ is a unit, which gives statement *(1)* of the proposition for $t = t_1$. Otherwise, if $t_1 < j$, we see that $G_{j-t_0}^{(1)}(x, y, z)$ contains the term $c_j z^{j-t_1}$. Notice that, in this case, we have $t_1 \geq t_0$ since, otherwise, if $t_1 = t_0$ then $\nu(G) = 0$ and $j = t_0 = t_1$. Thus, $j - t_0 > j - t_1 \geq 0$. By recurrence over $j - t_0$, there exists an admissible sequence of blowing-ups $\widetilde{\mathcal{M}} = (\widetilde{M}, \widetilde{\pi}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{D}})$ with $\widetilde{\pi}$ a composition of s blowing-ups at the corresponding characteristic singularities $\gamma_{I_\infty^{\mathcal{M}_i}}$ such that, defining $t_0, t_1, \ldots, t_s$ as above, we have $j = t_s$. We conclude *(1)* for $\widetilde{\pi}^* \hat{\xi}$ at the characteristic singularity $\gamma_{I_\infty^{\widetilde{\mathcal{M}}}}$ with $t = t_s$. Analogously, up to blowing-up repeatedly the characteristic singularity $\gamma_{I_{-\infty}^{\widetilde{\mathcal{M}}}}$, we may assume that *(1)* holds at $\gamma_{I_{-\infty}^{\widetilde{\mathcal{M}}}}$.

According to the construction of sequences of admissible blowing-ups in the Sect. 3.3, $\widetilde{\mathcal{A}}$ is composed by the two charts $C_{I_{-\infty}^{\widetilde{\mathcal{M}}}}$ and $C_{I_\infty^{\widetilde{\mathcal{M}}}}$ and a finite number of charts named:

$$\{C_J\}_{J \in \widetilde{\mathcal{J}}_0}, \text{ where } \widetilde{\mathcal{J}}_0 = \{0, (\infty, 0), \ldots, (\infty, \ldots, \infty, 0), (-\infty, 0), \ldots, (-\infty, \ldots, -\infty, 0)\}$$

with coordinates of the form $(\theta, z, \rho) \in \mathbb{R} \times (\mathbb{R}_{\geq 0})^2$, except for the first one with z taking values in $\mathbb{R}$. For any $J \in \widetilde{\mathcal{J}}_0$, consider the transformed vector field $\hat{\xi}^{(J)} = (\widetilde{\pi}|_{C_J})^* \hat{\xi}$ and the corresponding reduced associated vector field $\hat{\eta}'_J$. Denote by $\tilde{E} := \tilde{\pi}^{-1}(0)$ the total divisor of $\widetilde{\mathcal{M}}$. Notice that the coordinate θ is well defined in the union $U = \bigcup_{J \in \widetilde{\mathcal{J}}_0} C_J$ so that $\tilde{F} := \tilde{E} \cap \{\theta = 0\} \cap U$ has a perfect sense. In fact, $\tilde{F} = \tilde{E} \cap \overline{\pi^{-1}(\{y = 0, x > 0\})}$. Now, given $J \in \widetilde{\mathcal{J}}_0$, we apply Theorem 3.9 at each point $a \in \widetilde{\mathrm{Sing}}(\hat{\eta}'_J, \tilde{F})$ in order to obtain a reduction of singularities τ_a of the two dimensional vector field $\hat{\eta}'_J$ at a adapted to $\tilde{F}$.

Notice that in the sequence of blowing-ups that Theorem 3.9 provides, we start blowing up with center at points $a \in \widetilde{\mathrm{Sing}}(\hat{\eta}'_J, \tilde{F})$ for the different $J \in \widetilde{\mathcal{J}}_0$. The points in $\widetilde{\mathrm{Sing}}(\hat{\eta}'_J, \tilde{F})$ correspond exactly to the family of characteristic cycles of $\widetilde{\mathcal{M}}$ (elements of $\widetilde{\mathcal{D}}$). Considering admissible blowing-ups σ_{γ_I} centered at those $\gamma_I \in \widetilde{\mathcal{D}}$, the restriction $\sigma_{\gamma_I}|_{\{\theta=0\}}$ is exactly the blowing-up centered at the corresponding point $\gamma_I \cap \{\theta = 0\}$ of the two-dimensional vector field $\hat{\eta}'_J$. Moreover, this property repeats for the subsequent points to be blown up to achieve τ_a and the corresponding strict transform of $\hat{\eta}'_J$. In other words, having defined the sequence of blowing-ups τ_a as above, satisfying (a), (b) and (c) for any $a \in \widetilde{\mathrm{Sing}(\hat{\eta}'_J, \tilde{F})}$ and for any $J \in \widetilde{J}_0$, the composition of these sequences of two dimensional blowing-ups τ_a provides a sequence of admissible blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ factorizing through $\widetilde{\pi}$ (i.e. $\pi = \widetilde{\pi} \circ \pi'$) such that $\mathcal{M}$ satisfies (2) and (3) of the statement. Since π' does not modify the characteristic singularities $\gamma_{I^{\widetilde{\mathcal{M}}}_{-\infty}}, \gamma_{I^{\widetilde{\mathcal{M}}}_{-\infty}}$, we have also (1), and we are done. □

Remark 3.11 Notice that after an adapted reduction of singularities, the non-corner characteristic cycles that we obtain are contained in non-dicritical components of the total divisor.

3.5 Behavior of Jet Approximations of Normal Forms Under Blowing-Ups

In this section, we study the effect of sequences of admissible blowing-ups to the jet approximations ξ_ℓ of the formal normal form $\hat{\xi}$, for convenient values of ℓ. First, we establish the jet dependence of the transform of $\hat{\xi}$ by such blowing-ups in the different charts.

Proposition 3.12 *Let $\hat{\xi}$ be a formal normal form of $\xi \in \mathcal{H}_3$. Consider an admissible sequence of blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ for $\hat{\xi}$ of length $l > 0$, with $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$. For every $J \in \mathcal{J}$ and for every $k \geq 1$, if u is a coordinate of the chart C_J such that $\{u = 0\} \subset E = \pi^{-1}(0)$, then we have*

$$j^u_k(\hat{\xi}^{(J)}) = j^u_k((\pi|_{C_J})^* j_{k+l+1}(\hat{\xi})). \tag{23}$$

Proof The proof uses the following standard fact.

Fact Let η be a vector field with coefficients in $A[[x_1, \ldots, x_n]]$ and let τ be a quadratic morphism of the form $\tau(x_1, \ldots, x_n) = (x_1x_i, \ldots, x_{i-1}x_i, x_i, x_{i+1}x_i, \ldots, x_nx_i)$. Then,

$$\begin{aligned} j^{x_i}_k(\tau^*\eta) &= j^{x_i}_k(\tau^* j_{k+1}(\eta)) \\ j^{x_j}_k(\tau^*\eta) &= j^{x_j}_k(\tau^* j^{x_j}_k(\eta)) = j^{x_j}_k(\tau^* j^{x_j}_{k+1}(\eta)),\ \ j \neq i. \end{aligned} \tag{24}$$

We proceed by induction on the length l of $\mathcal{M}$. If $l = 0$, that is, $\pi = \sigma_0$ is the blowing-up of the origin $0 \in \mathbb{R}^3$ described in Sect. 3.2. We have (with simplified notation $\rho := \rho^{(0)}$, $z := z^{(0)}$)

$$\begin{aligned} j^\rho_k((\sigma_0|_{C_0})^*\hat{\xi}) &= j^\rho_k((\sigma_0|_{C_0})^* j_{k+1}(\hat{\xi})), \\ j^z_k((\sigma_0|_{C_\infty})^*\hat{\xi}) &= j^z_k((\sigma_0|_{C_\infty})^* j_{k+1}(\hat{\xi})), \\ j^z_k((\sigma_0|_{C_{-\infty}})^*\hat{\xi}) &= j^z_k((\sigma_0|_{C_{-\infty}})^* j_{k+1}(\hat{\xi})), \end{aligned} \tag{25}$$

which proves the result.

Suppose $l > 0$ and that $\pi = \widetilde{\pi} \circ \sigma_{\gamma_I}$, where σ_{γ_I} is the blowing-up centered at some characteristic element γ_I of a sequence of admissible blowing-ups $\widetilde{\mathcal{M}} = (\widetilde{M}, \widetilde{\pi}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{D}})$ of length $l - 1$. It is enough to study the transform $\hat{\xi}^{(J)}$ in the charts C_J when $\sigma^{-1}_{\gamma_I}(\gamma_I) \cap C_J \neq \emptyset$, since the map σ_{γ_I} is an isomorphism out of $\sigma^{-1}_{\gamma_I}(\gamma_I)$. According to the construction of $\mathcal{M}$ from $\widetilde{\mathcal{M}}$ and using the same notations as in Sect. 3.3, we have several cases:

(1) The point γ_I is the origin of a chart $(C_{J_I}, (x^{(J_I)}, y^{(J_I)}, z^{(J_I)}))$ of $\widetilde{\mathcal{A}}$ (for instance $I = I_\infty^{\widetilde{\mathcal{M}}}$) where $z^{(J_I)} = 0$ is the equation of the divisor $\widetilde{E} \cap C_{J_I}$, and $J = I_\infty^{\mathcal{M}} = (\infty, \overset{s}{\ldots}, \infty)$. In this case, $u = z^{(J)}$ is the only coordinate of the chart C_J in the conditions of the statement. Using the induction hypothesis $j_{k'}^{z^{(J_I)}}(\hat{\xi}^{(J_I)}) = j_{k'}^{z^{(J_I)}}((\pi|_{C_{J_I}})^* j_{k'+(l-1)+1}(\hat{\xi}))$ for $k' = k+1$, we have that

$$\begin{aligned} j_k^u(\hat{\xi}^{(J)}) &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* \hat{\xi}^{(I)}) = j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}^{z^{(J_I)}}(\hat{\xi}^{(I)})) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}^{z^{(J_I)}}((\tilde{\pi}|_{C_I})^*(j_{(k+1)+(l-1)+1}(\hat{\xi}))) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^*((\tilde{\pi}|_{C_I})^*(j_{k+l+1}(\hat{\xi}))) \\ &= j_k^u((\pi|_{C_J})^*(j_{k+l+1}(\hat{\xi})) \end{aligned}$$

(2) The point γ_I is the origin of a chart $(C_{J_I}, (x^{(J_I)}, y^{(J_I)}, z^{(J_I)}))$ of $\widetilde{\mathcal{A}}$ where $z^{(J_I)} = 0$ is the equation of the divisor $\widetilde{E} \cap C_{J_I}$ and $\sigma_{\gamma_I}|_{C_J}: C_J \to C_{J_I}$ has the same expression as (4) for σ_0, considering coordinates $(\theta, z^{(J)}, \rho^{(J)})$ for C_J and with the obvious change of notation. Notice that in C_J the two coordinates $u = \rho^{(J)}$ and $u = z^{(J)}$ are in the conditions of the statement. By the induction hypothesis, renaming $z = z^{(J_I)}$ for simplicity, we have, for any $k \geq 1$, that $j_k^z(\hat{\xi}^{(J_I)}) = j_k^z((\tilde{\pi}|_{C_{J_I}})^* j_{k+l}(\hat{\xi}))$. By the fact that $j_k(\chi) = j_k(j_k^z(\chi))$ for any vector field χ, we also have $j_k(\hat{\xi}^{(J_I)}) = j_k((\tilde{\pi}|_{C_{J_I}})^* j_{k+l}(\hat{\xi}))$. From this last equality, the result follows for $u = \rho^{(J)}$ similarly to the case of the first blowing-up σ_0. For $u = z^{(J)}$, it is a consequence of the second equation of (24).

(3) γ_I is a characteristic cycle of $\widetilde{\mathcal{M}}$. Taking into account Remark 3.6, we may assume $\gamma_I \subset \{\rho^{(J_I)} = 0\}$ for some chart $(C_{J_I}, (\theta, z^{(J_I)}, \rho^{(J_I)})) \in \tilde{\mathcal{A}}$. Let us put for simplicity $(z, \rho) = (z^{(J_I)}, \rho^{(J_I)})$. We distinguish two cases:

   (a) γ_I is a corner characteristic cycle. In this case, $\sigma_{\gamma_I}^{-1}(\gamma_I)$ is covered by two charts $(C_J, (\theta, z^{(J)}, \rho^{(J)}))$ with $J = J_\infty, J_0$, for which the expression of σ_{γ_I} is given by (17) and (18), respectively. By symmetry, both are treated similarly, and we assume the case $J = J_\infty$. Notice that the coordinates $u = z^{(J)}$ and $u = \rho^{(J)}$ are in the condition of the statement. For $u = z^{(J)}$, we have, for any $k \geq 1$:

$$\begin{aligned} j_k^u(\hat{\xi}^{(J)}) &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}(\hat{\xi}^{(J_I)})) = j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}(j_{k+1}^z(\hat{\xi}^{(J_I)}))) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}(j_{k+1}^z((\widetilde{\pi}|_{C_{J_I}})^*(j_{k+l+1}(\hat{\xi}))))) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}((\widetilde{\pi}|_{C_{J_I}})^*(j_{k+l+1}(\hat{\xi})))) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^*(\widetilde{\pi}|_{C_{J_I}})^*(j_{k+l+1}(\hat{\xi}))) = j_k^u((\pi|_{C_J})^*(j_{k+l+1}(\hat{\xi}))). \end{aligned} \tag{26}$$

   Here, we have used the first formula of Eq. (24) for the quadratic map σ_{γ_I} in the first and fifth equalities, general properties of jets (cf. Sect. 1) in the second and fourth equalities and the induction hypothesis in the third equality. This proves (23) for $u = z^{(J)}$. For $u = \rho^{(J)}$, we have, for any $k \geq 1$:

$$\begin{aligned} j_k^u(\hat{\xi}^{(J)}) &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}^\rho(\hat{\xi}^{(J_I)})) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^* j_{k+1}^\rho((\widetilde{\pi}|_{C_{J_I}})^*(j_{k+l+1}(\hat{\xi})))) \\ &= j_k^u((\sigma_{\gamma_I}|_{C_J})^*(\widetilde{\pi}|_{C_{J_I}})^*(j_{k+l+1}(\hat{\xi}))) = j_k^u((\pi|_{C_J})^*(j_{k+l+1}(\hat{\xi}))), \end{aligned} \tag{27}$$

   where we have used the second formula of (24) for the quadratic map σ_{γ_I} in the first and third equality and the induction hypothesis in the second equality. This proves (23) for $u = \rho^{(J)}$.

(b) γ_I is a non-corner characteristic cycle. In this case $\sigma_{\gamma_I}^{-1}(\gamma_I)$ is covered by three charts C_{J_∞}, C_{J_0} and $C_{J_{-\infty}}$, for which the expression of σ_{γ_I} is given by Eqs. (20), (21) and (22), respectively. In the chart $(C_{J_\infty}, (\theta, z^{(J_\infty)}, \rho^{(J_\infty)}))$, the two coordinates $u = z^{(J_\infty)}$ and $u = \rho^{(J_\infty)}$ are in the hypothesis of the statement. The proof of the result is analogous to the one in case (a), namely Eqs. (26) and (27). The chart $C_{J_{-\infty}}$ is similar to C_{J_∞}. Finally, in the chart $(C_{J_0}, (\theta, z^{(J_0)}, \rho^{(J_0)}))$ only $u = \rho^{(J_0)}$ is in the hypothesis of the statement. The proof for this coordinate is just the same sequence of equalities as in (26) with the interchange of the role of the coordinates z and ρ in C_{J_I}.

□

Now, let us discuss the validity of Proposition 3.12 for the jets approximations of the normal form ξ_ℓ.

Consider the first blowing-up σ_0 at $0 \in \mathbb{R}^3$, a singular point of ξ_ℓ for any ℓ. Being ξ_ℓ analytic, the total transform $\sigma_0^*\xi_\ell$ exists and is analytic in a neighborhood of the divisor $E_0 = \sigma_0^{-1}(0)$. Moreover, in terms of coordinates of the charts $C_{-\infty}$, C_0, C_∞ (cf. Sect. 3.2), we can prove (see for instance the computations in [1, sec. 3])

(a) For $(C_\infty, (x^{(\infty)}, y^{(\infty)}, z^{(\infty)}))$ (and analogously for $C_{-\infty}$) the coefficients of $\xi_\ell^{(\infty)} := (\sigma_0|_{C_\infty})^*\xi_\ell$ belong to $\mathbb{R}[x^{(\infty)}, y^{(\infty)}][[z^{(\infty)}]] \cap \mathbb{R}\{x^{(\infty)}, y^{(\infty)}, z^{(\infty)}\}$. In fact, they belong to the algebra $\mathbb{R}[x^{(\infty)}, y^{(\infty)}]\{z^{(\infty)}\}$ of convergent series with polynomial coefficients (cf. notations at the end of Sect. 1).
(b) For $(C_0, (\theta, z, \rho))$, the coefficients of $\xi_\ell^{(0)} := (\sigma_0|_{C_0})^*\xi_\ell$ belong to $\mathbb{R}[\cos\theta, \sin\theta, z][[\rho]] \cap \mathbb{R}[\cos\theta, \sin\theta]\{z, \rho\}$. In fact, they belong to $\mathbb{R}[\cos\theta, \sin\theta, z]\{\rho\}$.

Finally, taking into account that $\ell \geq 1$ (i.e. ξ_ℓ has the same linear part as ξ or $\hat{\xi}$), we may observe that $\xi_\ell^{(J)}|_{E_0 \cap C_J} = \hat{\xi}^{(J)}|_{E_0 \cap C_J}$ for any $J \in \{-\infty, 0, \infty\}$. In particular, the characteristic elements of $\hat{\xi}$ in E_0 are invariant for the transform $\sigma_0^*\xi_\ell$, then $\sigma_0^*\xi_\ell$ admits a transform which is analytic if we blow up again one of those characteristic elements. Using recursively the same kind of arguments, and with a similar proof, we obtain the following version of Proposition 3.12 for the jets approximations of the normal form.

Proposition 3.13 *Let $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ be an admissible sequence of blowing-ups of length l with $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$. Then, for $\ell \geq l + 1$ and $J \in \mathcal{J}$, the transform $\xi_\ell^{(J)} := (\pi|_{C_J})^*\xi_\ell$ is analytic. Moreover, for any $k \in \mathbb{N}$, if u is a coordinate of C_J such that $\{u = 0\} \subset E = \pi^{-1}(0)$ and $\ell \geq k + l + 1$, then, we have*

$$j_k^u(\xi_\ell^{(J)}) = j_k^u(\hat{\xi}^{(J)}).$$

Remark 3.14 As a part of the proof, we can see that the restriction of $\xi_\ell^{(J)}$ and $\hat{\xi}^{(J)}$ to the divisor coincide. Hence, the characteristic elements $\gamma_I \in \mathcal{D}$ are invariant for the total transform $\pi^*\xi_\ell$. They are called *characteristic singularities or characteristic cycles*, accordingly, of $\xi_\ell^{(J)}$. Moreover, we observe that the coefficients of $\xi_\ell^{(J)}$ are convergent series in the coordinates of the chart C_J, i.e., they satisfy the corresponding property (a), respectively (b), above when $J \in \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ (resp. $J \notin \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$). In the last case, we can also interchange the roles of the coordinates z and ρ if C_J is a corner chart.

Recall, from Sect. 3.3, the definition of the associated two dimensional vector fields $\hat{\eta}'_J$ to $\hat{\xi}^{(J)}$ for $J \in \mathcal{J} \setminus \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$ and the corresponding reduced vector fields $\hat{\eta}'_J = (\rho^{n_1^{(J)}} z^{n_2^{(J)}})^{-1}\hat{\eta}_J$, where (θ, z, ρ) are the coordinates in C_J. Write the transform $\xi_\ell^{(J)}$ as

$$\xi_\ell^{(J)} = B_{\ell,\theta}^{(J)}(\theta, z, \rho)\frac{\partial}{\partial\theta} + B_{\ell,z}^{(J)}(\theta, z, \rho)\frac{\partial}{\partial z} + B_{\ell,\rho}^{(J)}(\theta, z, \rho)\frac{\partial}{\partial\rho}. \tag{28}$$

Then, the *associated (to* $\xi_\ell^{(J)}$*) system of ODEs* $\eta_{\ell,J}$ is defined as:

$$\begin{cases} \frac{dz}{d\theta} = B_{\ell,\rho}^{(J)}(\theta, z, \rho) \cdot (B_{\ell,\theta}^{(J)}(\theta, z, \rho))^{-1} \\ \frac{d\rho}{d\theta} = B_{\ell,z}^{(J)}(\theta, z, \rho) \cdot (B_{\ell,\theta}^{(J)}(\theta, z, \rho))^{-1} \end{cases} \tag{29}$$

Recall also that, if $J \in \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$ and we use simplified notation $(x, y, z) := (x^{(J)}, y^{(J)}, z^{(J)})$, we have defined $n^{(J)}$ as the maximum $n \in \mathbb{N}$ such that $\hat{\xi}^{(J)}(z)$ is divisible by z^n. As well, if $J \in \mathcal{J} \setminus \{I_\infty^{\mathcal{M}}, I_{-\infty}^{\mathcal{M}}\}$, we have defined $n^{(J)} := \max\{n_1^{(J)}, n_2^{(J)}\}$. With those notations, we have the following Corollary of Proposition 3.13.

Corollary 3.15 *Let* $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ *be an admissible sequence of blowing-ups of length* $l > 0$ *with* $\mathcal{A} = \{C_J\}_{J \in \mathcal{J}}$. *Define* $\ell_{\mathcal{M}} := \max\{n^{(J)} : J \in \mathcal{J}\} + l + 1$. *Fix* $k \in \mathbb{N}_{\geq 0}$.

(1) *Let* $J \in \mathcal{J} \setminus \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ *and put* $(z, \rho) := (z^{(J)}, \rho^{(J)})$. *For every* $\ell \geq \ell_{\mathcal{M}} + k$, *the monomial* $(\rho)^{n_1^{(J)}}(z)^{n_2^{(J)}}$ *divides the system* $\eta_{\ell,J}$. *Moreover, putting* $\eta'_{\ell,J} := (\rho^{n_1^{(J)}} z^{n_2^{(J)}})^{-1} \eta_{\ell,J}$, *if* u *is a coordinate with* $\{u = 0\} \subset E \cap C_J$, *then*

$$j_k^u(\eta'_{\ell,J}) = j_k^u(\hat{\eta}'_J).$$

(2) *Let* $J \in \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ *and put* $(x, y, z) := (x^{(J)}, y^{(J)}, z^{(J)})$. *For every* $\ell \geq \ell_{\mathcal{M}} + k$, *the series* $\xi_\ell^{(J)}(z)$ *is divisible by* $z^{n^{(J)}}$, *and*

$$j_k^z\left(z^{-n^{(J)}} \xi_\ell^{(J)}(z)\right) = j_k^z\left(z^{-n^{(J)}} \hat{\xi}^{(J)}(z)\right).$$

Proof Both statements are direct consequence of the jet equality stated in Proposition 3.13. Since $k + \ell_{\mathcal{M}} \geq n_i^{(J)} + l + 1$ for $i = 1, 2$ and for very $J \in \mathcal{J} \setminus \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$ and $k + \ell_{\mathcal{M}} \geq n^{(J)} + l + 1$ when $J \in \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$, we have that the monomials of type $(\rho^{(J)})^{n_1^{(J)}}(z^{(J)})^{n_2^{(J)}}$ divide the system $\eta_{\ell,J}$ when $J \in \mathcal{J} \setminus \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$, or $(z^{(J)})^{n^{(J)}}$ divides $\xi_\ell^{(J)}(z^{(J)})$ when $J \in \{I_{-\infty}^{\mathcal{M}}, I_\infty^{\mathcal{M}}\}$. □

3.6 Lifting of Automorphisms by Admissible Blowing-Ups

Recall from Eq. (3) that there is a formal automorphism ψ_ℓ at $0 \in \mathbb{R}^3$ that conjugates $\hat{\xi}$ and ξ_ℓ, that is, $\hat{\xi} = \psi_\ell^*(\xi_\ell)$, and that ψ_ℓ is tangent to the identity up to order ℓ, i.e., $j_\ell(\psi_\ell - Id) = 0$. In what follows, we will need to lift such a conjugation to the charts of a sequence of admissible blowing-ups. As well, we will need to lift the (analytic) conjugation between different jet approximations ξ_ℓ and $\xi_{\ell'}$. We provide a proper statement covering all those situations.

Proposition 3.16 *Let* $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ *be a sequence of admissible blowing-ups of length* l. *Take* $\ell > l + 1$ *and let* $\psi \in \mathbb{R}[[x, y, z]]$ *be a formal automorphism satisfying* $j_\ell(\psi - (x, y, z)) = 0$. *Then, for any* $C_J \in \mathcal{A}$, *there exists a formal automorphism* $\psi^{(J)}$ *in the coordinates of* C_J *satisfying*

$$\pi|_{C_J} \circ \psi^{(J)} = \psi \circ \pi|_{C_J}. \tag{30}$$

More precisely, if $\ell = k + l + 1$, *we have:*

(1) *Suppose* $J = I_{\pm\infty}^{\mathcal{M}}$ *and denote* $(u, v, w) := (x^{(J)}, y^{(J)}, z^{(J)})$, *then there is* $\psi^{(J)} \in \mathbb{R}[u, v][[w]]^3$ *that satisfies* $j_k^w(\psi^{(J)} - (u, v, w)) = 0$.

(2) *Suppose* $J \neq I^{\mathcal{M}}_{\pm\infty}$ *and denote* $(\theta, z, \rho) := (\theta, z^{(J)}, \rho^{(J)})$ *(notice that* $\{\rho = 0\} \subset E \cap C_J$*). Then,* $\psi^{(J)} = \psi^{(J)}(\theta, z, \rho) = (\theta + F_\theta, z + F_z, \rho + F_\rho)$ *where each* $F_i \in \mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]$ *and* $\psi^{(J)}$ *satisfies* $j^\rho_k(\psi^{(J)} - (\theta, z, \rho)) = 0$*. Moreover, if* $\{z = 0\} \subset E \cap C_J$ *(*C_J *is a corner chart), then we have also* $F_i \in \mathbb{R}[\cos\theta, \sin\theta, \rho][[z]]$*.*

(3) *In the same conditions, assume moreover that* $\psi \in \mathbb{R}\{x, y, z\}^3$ *is convergent. Then,* $\psi^{(J)} \in \mathbb{R}[u, v]\{w\}$ *in case (1) and* $\psi^{(J)} \in \mathbb{R}[\cos\theta, \sin\theta, z]\{\rho\}$ *in case (2).*

Proof We can write $\pi = \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{r'} \circ \cdots \circ \sigma_r$ with $\sigma_i = \sigma_{\gamma_{I_i}}$ and $0 \le r' \le r$, where γ_{I_i} are characteristic singularities of the form $\gamma_{(\varepsilon\infty, \overset{m_i}{\ldots}, \varepsilon\infty)}$ for $0 \le i \le r'$ and $\varepsilon = \pm 1$, and γ_{I_j} are characteristic cycles for $1 + r' \le j \le r$.

Suppose that $r' \ge 1$. First, the automorphism ψ can be lifted to $\psi^{(\infty)}$ at the point γ_∞ (or, correspondingly, to $\psi^{(-\infty)}$ at the point $\gamma_{-\infty}$): using the chart $(C_\infty, (x^{(\infty)}, y^{(\infty)}, z^{(\infty)}))$ and the quadratic expression of $\sigma_0|_{C_\infty}$ given in (5), the formal automorphism defined by

$$\psi^{(\infty)}(x^{(\infty)}, y^{(\infty)}, z^{(\infty)}) = \left(\frac{x \circ \psi}{z \circ \psi}, \frac{y \circ \psi}{z \circ \psi}, z \circ \psi\right) \circ \sigma_0|_{C_\infty}(x^{(\infty)}, y^{(\infty)}, z^{(\infty)}) \quad (31)$$

satisfies that $\sigma_0|_{C_\infty} \circ \psi^{(\infty)} = \psi \circ \sigma_0|_{C_\infty}$. Moreover, using that $j_\ell(\psi) = Id$ and the explicit expression of $\psi^{(\infty)}$, we get that $j_{\ell-1}(\psi^{(\infty)}) = Id$. In addition, it is standard to prove from (31) that $\psi^{(\infty)}$ belongs to $\mathbb{R}[x^{(\infty)}, y^{(\infty)}][[z^{(\infty)}]]$, and more precisely to $\mathbb{R}[x^{(\infty)}, y^{(\infty)}]\{z^{(\infty)}\}$ when ψ is convergent. Thus, $\psi^{(J)}$ satisfies the required properties *(1)* and *(3)* for $J \in \{-\infty, \infty\}$ when $r = 1$. Moreover, $\psi^{(\infty)}$ satisfies at γ_∞ the same properties as ψ does at 0, renaming $\ell := \ell - 1$. Repeating the same arguments for each blowing-up in the composition $\pi_1 = \sigma_0 \circ \cdots \sigma_{r'-1}$ when $r' \ge 1$, we obtain that there is a formal automorphism $\psi^{I_{r'}}$ at $\gamma_{I_{r'}}$ such that $\pi|_{C_{I_{r'}}} \circ \psi^{I_{r'}} = \psi \circ \pi|_{C_{I_{r'}}}$ and satisfying the same hypothesis as ψ at 0, but putting $\ell - r'$ instead of ℓ. Thus, for $J = I^{\mathcal{M}}_{-\infty}, I^{\mathcal{M}}_{\infty}$, we have shown items *(1)* and *(3)*. Renaming the point $\gamma_{I_{r'}}$ as the origin when $r' \ge 1$, we can assume that $r' = 0$. Let us analyze the chart $(C_0, (\theta, z, \rho))$ of the blowing-up σ_0. Write $\psi(x, y, z) = (x + G_1, y + G_2, z + G_3)$ where each $G_i \in \mathbb{R}[[x, y, z]]$ has order at least $\ell + 1$. We have

$$\begin{aligned}\psi \circ \sigma_0|_{C_0}(\theta, z, \rho) = (&\rho\cos\theta + G_1(\rho\cos\theta, \rho\sin\theta, \rho z), \rho\sin\theta + G_2(\rho\sin\theta, \rho\sin\theta, \rho z),\\ &\rho z + G_3(\rho\cos\theta, \rho\sin\theta, \rho z)).\end{aligned} \quad (32)$$

Hence, $G_i \circ \sigma_0|_{C_0} \in \mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]$ and $\rho^{\ell+1}$ divides each $G_i \circ \sigma_0|_{C_0}$. Moreover, these series $G_i \circ \sigma_0|_{C_0}$ belong to $\mathbb{R}[\cos\theta, \sin\theta, z]\{\rho\}$ if ψ is convergent. We introduce $\tilde{G}_i = \frac{1}{\rho} G_i \circ \sigma_0|_{C_0}$. We look for a formal automorphism of the form

$$\psi^{(0)}(\theta, z, \rho) = (\theta + \rho F_1(\theta, z, \rho), z + \rho F_2(\theta, z, \rho), \rho + \rho F_3(\theta, z, \rho))$$

such that $\sigma_0|_{C_0} \circ \psi^{(0)} = \psi \circ \sigma_0|_{C_0}$. The tuple (F_1, F_2, F_3) must fulfill

$$\begin{aligned}\rho\cos\theta + \rho\tilde{G}_1 &= (\rho + \rho F_3)\cos(\theta + \rho F_1),\\ \rho\sin\theta + \rho\tilde{G}_2 &= (\rho + \rho F_3)\sin(\theta + \rho F_1),\\ \rho z + \rho\tilde{G}_3 &= (z + \rho F_2)(\rho + \rho F_3).\end{aligned}$$

Put $\tilde{F}_1 := \rho F_1$ and $\tilde{F}_2 = \rho F_2$. Using classical formulas for trigonometric functions, and dividing the above expressions by ρ, we obtain the following system of formal equations with coefficients in the ring $\mathbb{R}[\cos\theta, \sin\theta, z, \rho]$ and in the variables $(\tilde{G}, \tilde{F}) = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{F}_1, \tilde{F}_2, F_3)$.

$$
\begin{aligned}
\tilde{G}_1 &= \cos\theta F_3 - \sin\theta \tilde{F}_1 - \sin\theta F_3 \tilde{F}_1 + O(\tilde{F}_1^2),\\
\tilde{G}_2 &= \sin\theta F_3 + \cos\theta \tilde{F}_1 + \cos\theta F_3 \tilde{F}_1 + O(\tilde{F}_1^2),\\
\tilde{G}_3 &= zF_3 + \tilde{F}_2 + F_3\tilde{F}_2.
\end{aligned} \tag{33}
$$

The differential of the system with respect to the unknown variables $\tilde{F}$ at $\tilde{F} = 0$ is invertible, as a matrix with entries in $\mathbb{R}[\cos\theta, \sin\theta, z, \rho]$. We apply the implicit function theorem to find a solution $\tilde{F} \in \mathbb{R}[\cos\theta, \sin\theta, z, \rho][[\tilde{G}]]^3$, see for example [4, A.IV.37]. Notice also that $\tilde{G} \in \mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]$, and hence $\tilde{F} \in \mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]^3$. Since $j^{\rho}_{\ell-1}(\tilde{G}_i) = 0$ for $i = 1, 2, 3$, we find that ρ^{ℓ} divides $\rho F_1, \rho F_2, F_3$ and $j_{\ell-1}(\psi^{(0)}) = Id$. Once more, if ψ is convergent, then we get that $\psi^{(0)} \in \mathbb{R}[\cos\theta, \sin\theta, z]\{\rho\}^3$ since $G_i \in \mathbb{R}[\cos\theta, \sin\theta, z]\{\rho\}$. (Notice that the system (33) is composed by equations that belong to $\mathbb{R}[\cos\theta, \sin\theta, z, \rho]\{\tilde{G}, \tilde{F}\}$, in this case). We get the desired lifting $\psi^{(0)}$ of ψ in the chart C_0.

We proceed studying the rest of the blowing-ups σ_i, $i \geq 1$, by recurrence. Let us simply discuss the first step of the recurrence, the rest is done similarly. Up to a translation $z \mapsto z+\omega$ in $\{\rho = 0\}$, the expression of σ_1 (omitting super-indices) is either $\sigma_1(\theta, z, \rho) = (\theta, \rho z, \rho)$ (in the non-corner chart C_{J_0} as in Eq. (21)) or $\sigma_1(\theta, z, \rho) = (\theta, \pm z, z\rho)$ (in a corner chart, say $C_{J_{\pm\infty}}$, as in Eqs. (20) or (22)). We obtain the desired expression of $\psi^{(J)}$ proceeding as in (31). For example, in the situation of the corner chart $J = J_\infty$, we define

$$
\psi^{(J)}(\theta, z, \rho) := \left(\theta \circ \psi^{(0)}, z \circ \psi^{(0)}, \frac{\rho \circ \psi^{(0)}}{z \circ \psi^{(0)}}\right) \circ \sigma_1|_{C_J}.
$$

Considering the expression of $\sigma_1|_{C_J}$, the coefficients of $\psi^{(J)}$ belong to $\mathbb{R}[\cos\theta, \sin\theta, \rho][[z]]$ and also to $\mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]$ since the coefficients of $\psi^{(0)}$ belong to $\mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]$. By the fact that $j^{\rho}_{\ell-1}(\psi^{(0)} - (\theta, z, \rho)) = 0$ and the above expression, we deduce $j^{u}_{\ell-2}(\psi^{(J)} - (\theta, z, \rho)) = 0$ for $u = \rho, z$. This shows *(2)* for $J = J_\infty$. To prove *(3)* for this same index, we observe that through all the operations made above (including the translation in z), the convergent nature of $\psi^{(J_\infty)}$ is inherited from that of $\psi^{(0)}$. □

4 Characteristic Cycles as Limit Sets

In this section, we use the analytic approximations ξ_ℓ to the formal normal form $\hat{\xi}$ with an objective: proving that the characteristic elements of ξ_ℓ after a sequence of admissible blowing-ups $\mathcal{M}$ are the only possible limit sets of the family of local cycles of ξ_ℓ for ℓ large enough.

Along this section, we fix a sequence of admissible blowing-ups $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ for $\hat{\xi}$, with $\mathcal{A} = \{C_J\}_{J\in\mathcal{J}}$ and $\mathcal{D} = \{\gamma_I\}_{I\in\mathcal{I}}$. Denote by $E = \pi^{-1}(0)$ the total divisor of π. We define also the *support of* $\mathcal{D}$ as $\mathrm{Supp}\mathcal{D} = \bigcup_{I\in\mathcal{I}} \gamma_I$. Recall the definition of $\ell_{\mathcal{M}}$ in Corollary 3.15.

Proposition 4.1 *Let $\ell \geq \ell_{\mathcal{M}} + 1$ and W be a neighborhood of $\mathrm{Supp}\mathcal{D} = \bigcup_{I\in\mathcal{I}} \gamma_I$. There is some neighborhood $U = U(W)$ of $0 \in \mathbb{R}^3$ such that $\pi^{-1}(\mathcal{C}_U(\xi_\ell)) \subseteq W$.*

To prove this result, we need to introduce new notation and a technical lemma. Consider the set

$$\dot{E} := E \setminus \left(\left(\bigcup_{I:\gamma_I \in \mathcal{D} \text{ corner}} \{\gamma_I\} \right) \cup \{\gamma_{I_\infty^{\mathcal{M}}}\} \cup \{\gamma_{I_{-\infty}^{\mathcal{M}}}\} \right).$$

The set $\dot{E}$ has a finite family of connected components denoted by $\mathcal{E}_{\mathcal{M}} = \{L_0, L_1, \ldots, L_{k_{\mathcal{M}}}\}$. Each $L_i \in \mathcal{E}_{\mathcal{M}}$ is open in E and contained in a chart C_{J_i} for $i = 0, 1, \ldots, k_{\mathcal{M}}$. Therefore, we will call them simply *open components (of* E). In addition, in case L_i is contained in two different charts, we choose C_{J_i} such that $L_i \subseteq \{\rho^{(J_i)} = 0\}$, which is always possible by the hypothesis (H3) of sequences of admissible blowing-ups. Then, each open component $L_i = \mathbb{S}^1 \times (\lambda_i^-, \lambda_i^+) \times \{0\}$ in the coordinates of C_{J_i} where $\lambda_i^- \in \mathbb{R} \cup \{-\infty\}$ and $\lambda_i^+ \in \mathbb{R} \cup \{\infty\}$. An element $L_i \in \mathcal{E}_{\mathcal{M}}$ is said to be dicritical (respectively, non-dicritical) if the component of E that contains L_i is dicritical (respectively, non-dicritical).

Fix $L = L_i \in \mathcal{E}_{\mathcal{M}}$, with $L = \mathbb{S}^1 \times (\lambda^-, \lambda^+) \times \{0\}$, and the corresponding chart C_J with $J = J_i$. Consider the formal vector field $\hat{\eta}_J$ associated to $\hat{\xi}^{(J)} = (\pi|_{C_J})^* \hat{\xi}$ as in Eq. (11). For the purpose of this section, we write, removing super-indices in (z, ρ):

$$\hat{\eta}_J = \rho^{n_1^{(J)}} (A_z^{(J)}(z, \rho) \frac{\partial}{\partial z} + A_\rho^{(J)}(z, \rho) \frac{\partial}{\partial \rho}). \tag{34}$$

Notice that there is a small modification here with respect to Eq. (11): we include the factor $z^{n_2^{(J)}}$ in the coefficients $A_j^{(J)}$ for $j = z, \rho$ (which may be non-trivial if C_J is a corner chart) in Eq. (34). The reduced vector field $\hat{\eta}_J'' := \rho^{-n_1^{(J)}} \hat{\eta}_J$ considered here (not necessarily equal to $\hat{\eta}_J'$) has a finite number of points in the adapted singular locus of $\hat{\eta}_J'$ relatively to $F = E \cap \{\theta = 0\}$ along $\{\rho = 0\}$, which determine the characteristic cycles contained in L. The z−coordinates of the characteristic cycles in L are denoted by $\omega_1^L, \ldots, \omega_{m_L}^L$ and the associated characteristic cycles by $\gamma_1^L, \ldots, \gamma_{m_L}^L$.

Define the collection of sets $\mathcal{V}(L, \varepsilon, \delta) := \{V_0, V_1, \ldots, V_{m_L-1}, V_{m_L}\}$ depending on two parameters $\varepsilon, \delta > 0$ by:

$$\begin{aligned} V_0 &= \mathbb{S}^1 \times \Omega_0(\varepsilon) \times (0, \delta], \quad \Omega_0(\varepsilon) = [\mu_-, \omega_1^L - \varepsilon], \\ V_j &= \mathbb{S}^1 \times \Omega_j(\varepsilon) \times (0, \delta], \quad \Omega_j(\varepsilon) = [\omega_j^L + \varepsilon, \omega_{j+1}^L - \varepsilon], \quad j = 1, \ldots, m_L - 1, \\ V_{m_L} &= \mathbb{S}^1 \times \Omega_{m_L}(\varepsilon) \times (0, \delta], \quad \Omega_{m_L}(\varepsilon) = [\omega_{m_L}^L + \varepsilon, \mu_+], \end{aligned} \tag{35}$$

where $\mu_\pm = \lambda^\pm \mp \varepsilon$ when $|\lambda^\pm| < \infty$, $\mu_- = \omega_1^L - \frac{1}{\varepsilon}$ when $\lambda^- = -\infty$, and $\mu_+ = \omega_{m_L}^L + \frac{1}{\varepsilon}$ when $\lambda^+ = \infty$. Define the surfaces $\partial_{min} V_j$ and $\partial_{max} V_j$ as follows:

- $\partial_{min} V_0 = \mathbb{S}^1 \times \{\mu_-\} \times (0, \delta]$ and $\partial_{min} V_j = \mathbb{S}^1 \times \{\omega_j^L + \varepsilon\} \times (0, \delta]$ for $j = 1, 2, \ldots, m_L$.
- $\partial_{max} V_j = \mathbb{S}^1 \times \{\omega_{j+1}^L - \varepsilon\} \times (0, \delta]$ for $j = 0, 1, \ldots, m_L - 1$ and $\partial_{max} V_{m_L} = \mathbb{S}^1 \times \{\mu_+\} \times (0, \delta]$.

Notice that elements of the family $\mathcal{V}(L, \varepsilon, \delta)$ are subsets of the corresponding elements of $\mathcal{V}(L, \varepsilon', \delta)$ for any $\varepsilon' < \varepsilon$.

Lemma 4.2 *Assume* $\ell \geq \ell_{\mathcal{M}} + 1$ *and denote by* $\xi_\ell^{(J)} = (\pi|_{C_J})^* \xi_\ell$. *There exists* $\varepsilon_0 > 0$ *such that for every small* ε *with* $\varepsilon_0 > \varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that the collection* $\mathcal{V}(L, \varepsilon, \delta) = \{V_j\}_{j=0}^{m_L}$ *satisfies:*

(1) *In case L is non-dicritical, the function z is monotonic along the trajectories of* $\xi_\ell^{(J)}$ *in each* V_j *for any j. Otherwise, if L is dicritical, the function* ρ *is monotonic along the trajectories of* $\xi_\ell^{(J)}$ *in each* V_j, *for any j.*

(2) *If L is dicritical and* $\rho^{n_1^{(J)}+1}$ *does not divide* $\xi_\ell^{(J)}(z)$*, then* $\xi_\ell^{(J)}(z)$ *has constant sign along the surfaces* $\partial_{min}V_j$ *and* $\partial_{max}V_j$*, for any* j*.*

(3) *Suppose that L is dicritical and* $\rho^{n_1^{(J)}+1}$ *divides* $\xi_\ell^{(J)}(z)$*. Denote* $\mathcal{V}(L, \frac{\varepsilon}{2}, \delta) = \{V'_0, V'_1, \ldots, V'_{m_L}\}$*. Then, each element* $V'_j \in \mathcal{V}(L, \frac{\varepsilon}{2}, \delta)$ *fulfills* (1) *and, moreover, any trajectory of* $\pi^*\xi_\ell$ *containing a point in* V_j *remains inside* V'_j *either for any positive time* $t \geq 0$ *or for any negative time* $t \leq 0$*.*

Proof Taking into account Corollary 3.15 and since $\ell \geq \ell_{\mathcal{M}}$, the vector field $\xi_\ell^{(J)}$ is described by a non-autonomous two dimensional system of ODEs (see Eq. (29))

$$\begin{cases} \frac{dz}{d\theta} = \rho^{n_1^{(J)}} A_z^{\ell,(J)}(\theta, z, \rho), \\ \frac{d\rho}{d\theta} = \rho^{n_1^{(J)}} A_\rho^{\ell,(J)}(\theta, z, \rho) \end{cases} \tag{36}$$

where $A_u^{\ell,(J)}(\theta, z, 0) = A_u^{(J)}(z, 0)$ for $u = \rho, z$. (As for the formal system of ODEs (34), we include the factor $z^{n_2^{(J)}}$ in $A_u^{\ell,(I)}$).

We choose ε_0 satisfying the following conditions:

- In any case, we require $\varepsilon_0 < \frac{1}{2}\min_{i\neq j}\{|\omega_i^L - \omega_j^L|\}$.
- When L is dicritical, if we have that $A_z^{\ell,(I)}(\theta, z, 0) \not\equiv 0$ and $\{t_1, \ldots, t_s\}$ is its set of zeroes, in order to prove property *(2)*, we require also

$$\varepsilon_0 < \frac{1}{2}\min\{|\omega_j^L - t_k| \ \ 1 \leq j \leq m_L, \ 1 \leq k \leq s, \ \omega_j^L \neq t_k\}.$$

In the non-dicritical case, the function $A_z^{\ell,(J)}(\theta, z, 0) \equiv A_z^{(J)}(z, 0)$ is not identically zero and only depends on z. Being its zeroes $\omega_1^L, \ldots, \omega_{m_L}^L$ by definition, it has constant sign when z belongs to the interval of $\Omega_j(\varepsilon)$ for $j \in \{0, \ldots, m_L\}$ for any $0 < \varepsilon < \varepsilon_0$. By continuity and periodicity in θ, $A_z^{\ell,(J)}(\theta, z, \rho)$ has constant sign for (θ, z, ρ) in a set of the form $\mathbb{S}^1 \times \Omega_j(\varepsilon) \times (0, \delta_j]$ for some $\delta_j = \delta_j(\varepsilon)$. Take δ fulfilling $\delta \leq \min_{i=0,\ldots,m_{L_i}}\{\delta_i\}$ and $B_{\ell,\theta}^{(J)} = \xi_\ell^{(J)}(\theta)$ has positive sign in $\mathbb{S}^1 \times \Omega_j(\varepsilon) \times (0, \delta]$ for every $j = 0, \ldots, m_L$. This is possible since $B_{\ell,\theta}^{(J)}(\theta, 0, 0) = 1$. Then, we define $V_j := \mathbb{S}^1 \times \Omega_j(\varepsilon) \times (0, \delta]$. Taking into account that $\xi_\ell^{(J)}(z) = \rho^{n_1^{(J)}} A_z^{\ell,(J)}(\theta, z, \rho) \cdot B_{\ell,\theta}^{(J)}(\theta, z, \rho)$, we obtain the property *(1)* for the non-dicritical case.

In the dicritical case we proceed in the same way. Notice that $A_\rho^{\ell,(J)}(\theta, z, 0) = A_\rho^{(J)}(z, 0)$ only depends on z and its set of zeros is by definition $\omega_1^L, \ldots, \omega_{m_L}^L$. We get that $\xi_\ell^{(J)}(\rho)$ has constant sign in each V_j and statement *(1)* holds.

Let us show *(2)*, assuming that $\rho^{n_1^{(J)}+1}$ does not divide $\xi_\ell^{(J)}(z)$. By the choice of ε_0, we have that $A_z^{\ell,(J)}(\theta, z, 0) = A_z^{(J)}(z, 0)$ does not vanish at any of the extreme values of $\Omega_j(\varepsilon)$. Since $\xi_\ell^{(J)}(z) = \rho^{n_1^{(J)}} A_z^{(J)}(\theta, z, \rho) \cdot B_{\ell,\theta}^{(J)}(\theta, z, \rho)$, we obtain *(2)*, up to taking a smaller δ.

Finally, we show *(3)*. Assume that L is dicritical and that $A_z^{\ell,(J)}(\theta, z, 0) \equiv 0$. Then, the system (36) associated to $\xi_\ell^{(J)}$ can be written as

$$\begin{cases} \frac{dz}{d\theta} = \rho^{n_1^{(J)}+1} \widetilde{A}_z^{\ell,(J)}(\theta, z, \rho) \\ \frac{d\rho}{d\theta} = \rho^{n_1^{(J)}} A_\rho^{\ell,(J)}(\theta, z, \rho) \end{cases}, \tag{37}$$

where $A_\rho^{\ell,(J)}(\theta, z, 0)$ does not depend on θ (by Corollary 3.15), vanishes exactly for $z \in \{\omega_1^L, \ldots, \omega_{m_L}^L\}$, and $\widetilde{A}_z^{\ell,(J)}(\theta, z, 0) \in \mathbb{R}[\cos\theta, \sin\theta, z]$. Proceeding as in the beginning of the proof, we take a constant $\delta > 0$ such that the collection $\mathcal{V}(L, \frac{\varepsilon}{2}, \delta) = \{V_0', V_1', \dots, V_{m_L}'\}$ fulfills *(1)*, so that ρ is monotonic in every V_j'. Being $\overline{V}_j$ compact, there are constants $a, K > 0$ such that for any $V_j' \in \mathcal{V}(L, \frac{\varepsilon}{2}, \delta)$, we have

$$\inf_{p \in V_j'} \{|A_\rho^{\ell,(J)}(p)|\} \geq a, \quad \sup_{p \in V_j'} \{|\widetilde{A}_z^{\ell,(J)}(p)|\} \leq K. \tag{38}$$

Fix V_j' and suppose, for instance, that $A_\rho^{\ell,(J)}|_{V_j'} < 0$. Then, if $\sigma : \mathbb{R} \longrightarrow M$ is a trajectory of $\xi_\ell^{(J)}$ parameterized as a solution $\sigma(\theta) = (\theta, z(\theta), \rho(\theta))$ of system (37), as long as it remains in $V_j' \backslash L$, the function $\rho(\theta)$ is strictly decreasing. Hence, σ can be parameterized by ρ instead of θ and we obtain from (37) and (38) that

$$\left|\frac{dz}{d\rho}\right| \leq C\rho, \text{ where } C = \frac{K}{a}.$$

Now, consider the collection $\mathcal{V}(L, \varepsilon, \delta) = \{V_0, V_1, \dots, V_{m_L}\}$ whose elements fulfill $V_j \subset V_j'$ for $j = 0, 1, \dots, m_L$. If the trajectory σ starts at a point $p_0 = (\theta_0, z_0, \rho_0) \in V_j \subset V_j'$ with $\rho_0 > 0$, it satisfies, for $\theta > \theta_0$:

$$|z(\theta) - z_0| \leq \frac{C}{2}|\rho(\theta)^2 - \rho_0^2| \leq \frac{C}{2}\rho_0^2 \leq \frac{C}{2}\delta^2$$

as long as $\mathrm{Im}(\sigma|_{[\theta_0,\theta]}) \subset V_j'$. We obtain similar bounds for $|z(\theta) - z_0|$ when $A_\rho^{\ell,(J)}|_{V_j'} > 0$. Imposing $\delta < \sqrt{\frac{\varepsilon}{C}}$, we can conclude that $|z_0 - z(\theta)| < \frac{\varepsilon}{2}$ and guarantee, for any j and for any $p_0 \in V_j \in \mathcal{V}(L, \varepsilon, \delta)$, that the trajectory σ starting at p_0 satisfies $\mathrm{Im}(\sigma|_{[\theta_0,\infty)}) \subset V_j'$ (or $\mathrm{Im}(\sigma|_{(-\infty,\theta_0]}) \subset V_j'$ in case $A_\rho^{\ell,(J)}|_{V_j'} > 0$). □

From the proof above, we may observe that $\mathcal{V}(L, \varepsilon, \delta')$ also fulfills (1–3) of the lemma for any $\delta' < \delta$.

Notation 4.3 Given an open component $L \in \mathcal{E}_\mathcal{M}$ as above, with the notations of Sect. 3.3 for $j \in \{0, \dots, m_L\}$, let $I_j \in \mathcal{I}$ be the index of the corresponding characteristic cycle $\gamma_{I_j} = \{z = \omega_j, \rho = 0\}$. Let I_0, I_{m_L+1} be also the indices of, either the corner characteristic cycles or characteristic singularities in the component $\bar{L}$. We say that the box $V_j \in \mathcal{V}(L, \varepsilon, \delta)$ with $j = 1, \dots, m_L - 1$ is *adjacent* to γ_{I_j} and to $\gamma_{I_{j+1}}$ and we denote $\partial_{I_j} V_j = \partial_{min} V_j$ and $\partial_{I_{j+1}} V_j = \partial_{max} V_j$.

Proof of Proposition 4.1. Let W be a neighborhood of $\mathrm{Supp}\mathcal{D}$. For every $I \in \mathcal{I}$, we consider an open neighborhood $W_I \subset W$ of γ_I such that $W_I \cap W_{I'} = \emptyset$ if $I \neq I'$. Consider the collection $\mathcal{E}_\mathcal{M}$, and apply Lemma 4.2 to each $L_i \in \mathcal{E}_\mathcal{M}$, taking ε and δ small enough so that each family $\mathcal{V}(L_i, \varepsilon, \delta)$ also satisfies:

- For any $V \in \mathcal{V}(L_i, \varepsilon, \delta)$, we impose $V \cap W_I \neq \emptyset$ if and only if γ_I is adjacent to V.
- For any $V \in \mathcal{V}(L_i, \varepsilon, \delta)$, the boundaries $\partial_{min} V$ and $\partial_{max} V$ are contained in the corresponding neighborhoods W_I and $W_{I'}$, where γ_I and $\gamma_{I'}$ are adjacent to V.
- The set

$$\bigcup_{I \in \mathcal{I}} W_I \cup \bigcup_{L \in \mathcal{E}_\mathcal{M}} \bigcup_{V \in \mathcal{V}(L,\varepsilon,\delta)} V$$

is a neighborhood of the divisor $E = \pi^{-1}(0)$ in M.

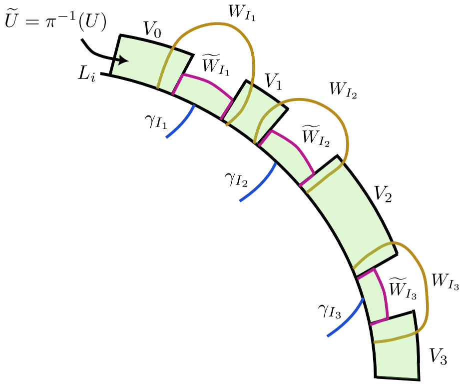

Fig. 4 Cross-section of the neighborhoods $\widetilde{W}_{I_j} \subset W_{I_j}$ and of $\tilde{U}$

Now, we define a closed neighborhood $\widetilde{W}_I \subset W_I$ of γ_I for each $I \in \mathcal{I}$ in such a way that (see Fig. 4):

(i) The set

$$\widetilde{U} = \operatorname{int}\left(\bigcup_{I\in\mathcal{I}} \widetilde{W}_I \cup \bigcup_{L\in\mathcal{E}_{\mathcal{M}}} \bigcup_{V\in\mathcal{V}(L,\varepsilon,\delta)} V\right)$$

is a neighborhood of the divisor E in M.

(ii) For any $I \in \mathcal{I}$, $L \in \mathcal{E}_{\mathcal{M}}$ and $V \in \mathcal{V}(L, \varepsilon, \delta)$, $\widetilde{W}_I \cap \bar{V}$ is empty, in case V is not adjacent to γ_I, or, otherwise, it is of the form $\widetilde{W}_I \cap V = \mathbb{S}^1 \times \{c\} \times (0, \mu]$, where $0 < \mu \leq \delta$ and $c = c(V, I)$ satisfies $\partial_I V = \mathbb{S}^1 \times \{c\} \times (0, \delta]$.

Now, the set $U := \pi(\tilde{U})$ is an open neighborhood of 0 satisfying the requirements of the proposition. More precisely, we claim that $\pi^{-1}(\mathcal{C}_U(\xi_\ell)) \subset \bigcup_{I\in\mathcal{I}} \widetilde{W}_I$.

To prove this, suppose that there is a cycle Z of ξ_ℓ contained in U and such that $\widetilde{Z} := \pi^{-1}(Z)$ intersects some $V \in \mathcal{V}(L, \varepsilon, \delta)$ for some L. Consider a parametrization $\sigma : \mathbb{R} \longrightarrow \tilde{U}$ of $\widetilde{Z}$ as a trajectory of $\pi^*\xi_\ell$ such that $\sigma(0) \in V$. By the property *(1)* of Lemma 4.2, one of the coordinates z or ρ is monotonic along σ inside V, so it cannot be completely contained in V. As a consequence, σ leaves V so that for some $t_0 \geq 0$ we have $\sigma(t_0) \in \operatorname{Fr}(V) \cap \widetilde{W}_I$, where $I \in \mathcal{I}$ and γ_I is adjacent to V. By construction (cf. item (ii) above), $\sigma(t_0)$ belongs to the boundary $\partial_I V$. We have two cases to consider (we take notations as in Lemma 4.2).

- $A_z^{\ell,(J)}(\theta, z, 0) \neq 0$. By statement (2) of Lemma 4.2, the vector field $\pi^*\xi_\ell$ is transverse to $\partial_I V$, so that, for instance, we have $\sigma((t_0 - c, t_0)) \subset \operatorname{int}(V)$ and $\sigma((t_0, t_0 + c)) \subset \operatorname{ext}(V)$ for some $c > 0$. Since σ is periodic, we must have that σ crosses $\operatorname{Fr}(V)$ at a first time $t_1 > t_0$ necessarily along one of the boundaries $\partial_{min} V$, $\partial_{max} V$ where $\pi^*\xi_\ell$ points towards $\operatorname{int}(V)$. If we denote $\{\partial_I V, \partial_{I'} V\} = \{\partial_{min} V, \partial_{max} V\}$, we must have $\sigma(t_0) \in \partial_I V$, $\sigma(t_1) \in \partial_{I'} V$ and $\sigma((t_0, t_1)) \subset \operatorname{ext}(V)$. Now, by construction, $\tilde{U} \setminus V = \tilde{U}_1 \cup \tilde{U}_2$, where $\tilde{U}_1$, $\tilde{U}_2$ are non-empty open sets such that $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ and the closure of each $\tilde{U}_i$ cuts V only along exactly one of the sets $\{\partial_{I'} V, \partial_I V\}$. We get the desired contradiction: $\sigma((t_0, t_1))$, being connected, is contained either in $\tilde{U}_1$ or in $\tilde{U}_2$ and the extremities $\sigma(t_0)$, $\sigma(t_1)$ should belong to the same set among $\partial_I V$, $\partial_{I'} V$.

- $A_z^{\ell,(J)}(\theta, z, 0) \equiv 0$. Using statement (3) of Lemma 4.2, we know that either $\sigma((t_0, \infty))$ or $\sigma((-\infty, t_0))$ is contained in the corresponding element V' of the collection $\mathcal{V}(L, \frac{\varepsilon}{2}, \delta)$ and $\rho \circ \sigma$ is monotonic along that interval. This is also a contradiction with σ being periodic.

Consequently, we have proved that $\tilde{Z} \subset \bigcup_{I \in \mathcal{I}} \widetilde{W}_I$ (in fact, included in a single $\widetilde{W}_I$ by connectedness). Therefore, we have that:

$$\pi^{-1}(\mathcal{C}_U(\xi_\ell)) \subset \bigcup_{I \in \mathcal{I}} \widetilde{W}_I \subseteq \bigcup_{I \in \mathcal{I}} W_I \subseteq W,$$

as we wanted to prove. □

5 Analysis of Final Adapted Simple Singularities

Along this section, we consider some $\xi \in \mathcal{H}^3$ with fixed singularity. We fix a formal normal form $\hat{\xi}$ of ξ and an adapted resolution of singularities $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ of $\hat{\xi}$ according to Proposition 3.10. Denote by $E = \pi^{-1}(0)$ the exceptional divisor of π.

5.1 Infinitely Near Points of the Rotational Axis

We see first that we can find a neighborhood of the two characteristic singular points that does not contain cycles of a jet approximation ξ_ℓ of $\hat{\xi}$.

Proposition 5.1 *Given $\ell \geq \ell_{\mathcal{M}} + 1$, there exist neighborhoods W_∞ of $\gamma_{I_\infty^{\mathcal{M}}}$ and $W_{-\infty}$ of $\gamma_{I_{-\infty}^{\mathcal{M}}}$ in M such that neither $W_\infty \backslash E$ nor $W_{-\infty} \backslash E$ contains cycles of $\pi^*\xi_\ell$.*

Proof According to the construction in Sect. 3.3, the point $\gamma_{I_\infty^{\mathcal{M}}}$ is the origin of the chart $(C_J, (x^{(J)}, y^{(J)}, z^{(J)}))$ with $J = I_\infty^{\mathcal{M}}$ and $E \cap C_J = \{z^{(J)} = 0\}$. Being $\mathcal{M}$ an adapted resolution of singularities of $\hat{\xi}$ and by means of Corollary 3.15, we have in a neighborhood of $\gamma_{I_\infty^{\mathcal{M}}}$ that $\xi_\ell^{(J)}(z^{(J)}) = (z^{(J)})^{n^{(J)}} \cdot F(x^{(J)}, y^{(J)}, z^{(J)})$ where $\xi_\ell^{(J)} = (\pi|_{C_J})^*\xi_\ell$, $n^{(J)} \in \mathbb{N}_{\geq 1}$ and $F(x^{(J)}, y^{(J)}, z^{(J)}) \in \mathbb{R}\{x^{(J)}, y^{(J)}, z^{(J)}\}$ converges and satisfies $F(0, 0, 0) \neq 0$. Take a neighborhood W_∞ of $I_\infty^{\mathcal{M}}$ in M where F has a constant sign, positive or negative. We have that the trajectories of $\pi^*\xi_\ell$ in $W_\infty \backslash E$ can be parameterized by $z^{(J)}$, which avoids the existence of cycles of $\pi^*\xi_\ell$ in $W_\infty \backslash E$. The proof for $\gamma_{I_{-\infty}^{\mathcal{M}}}$ is analogous. □

5.2 Simple Corner Characteristic Cycles

We prove that cycles of $\pi^*\xi_\ell$ cannot accumulate along corner characteristic cycles. Once again, the argument is to find a function, around such corner characteristic cycle, which is monotonic along the trajectories of $\pi^*\xi_\ell$, if ℓ is sufficiently large.

Proposition 5.2 *Let $\ell \geq \ell_{\mathcal{M}} + 1$. Consider a corner characteristic cycle γ_I of $\hat{\xi}$ in $\mathcal{M}$. Then, there exists a neighborhood W_I of γ_I in M such that $\pi^*(\xi_\ell)$ does not contain cycles in $W_I \backslash E$.*

Proof By construction, the corner characteristic cycle γ_I is given by $\{z^{(J)} = \rho^{(J)} = 0\}$ for some chart $(C_J, (\theta, z^{(J)}, \rho^{(J)})) \in \mathcal{A}$ for which $E \cap C_J = \{\rho^{(J)} z^{(J)} = 0\}$. For simplicity, from now on, we remove the super-indices of the coordinates. By definition of π being an adapted resolution of singularities, at least one of the two components of the divisor $\{\rho = 0\}$ and $\{z = 0\}$ is non-dicritical. More precisely, let $\hat{\eta}_J$ be the two dimensional vector field associated to $\hat{\xi}^{(J)} = (\pi|_{C_J})^*(\hat{\xi})$ and consider $\hat{\eta}'_J = \frac{1}{\rho^a z^b}\hat{\eta}_J$ the reduced associated vector field, i.e., $a = n_1^{(J)}$, $b = n_2^{(J)}$. We have two cases:

(a) The origin is not a singular point of $\hat{\eta}'_J$ and one of the components, say $\{z = 0\}$, is the solution of $\hat{\eta}'_J$.
(b) The origin is a simple singularity of $\hat{\eta}'_J$ and both components are invariant for $\hat{\eta}'_J$.

In the case (a) write

$$\hat{\eta}'_J = zF_z(z,\rho)\frac{\partial}{\partial z} + (\lambda_2 + F_\rho(z,\rho))\frac{\partial}{\partial \rho}$$

where $\lambda_2 \neq 0$ and $F_z, F_\rho \in \mathbb{R}[\rho][[z]] \cap \mathbb{R}[z][[\rho]]$ with $F_\rho(0,0) = 0$. We have that

$$\hat{\xi}^{(J)}(\rho) = \rho^a z^b \cdot (\lambda_2 + F_\rho(z,\rho))\hat{\xi}^{(J)}(\theta). \tag{39}$$

Since $\ell \geq \ell_{\mathcal{M}} + 1$, Corollary 3.15 implies that:

$$\xi_\ell^{(J)}(\rho) = \rho^a z^b \cdot (\lambda_2 + F_\rho^\ell(\theta, z, \rho))\xi_\ell^{(J)}(\theta),$$

where F_ρ^ℓ is analytic and $F_\rho^\ell(\theta, 0, 0) = 0$. Considering that the monomial $\rho^a z^b > 0$ for $(z,\rho) \in \mathbb{R}^2_{>0}$, and taking into account that $\lambda_2 \neq 0$ and $\xi_\ell^{(J)}(\theta) > 0$ along γ_I, there is a neighborhood W_I of γ_I such that $\xi_\ell^{(J)}(\rho)$ has constant sign in $W_I \backslash E$. Hence, the trajectories of $\xi_\ell^{(J)}$ can be parameterized by ρ in $W_I \backslash E$ and thus $\xi_\ell^{(J)}$ cannot have cycles in $W_I \backslash E$.

In the case (b) being both components of the divisor invariant, we can write:

$$\hat{\eta}'_J = (\lambda_1 z + zF_z(z,\rho))\frac{\partial}{\partial z} + (\lambda_2 \rho + \rho F_\rho(z,\rho))\frac{\partial}{\partial \rho},$$

where $\lambda_1^2 + \lambda_2^2 \neq 0$ and $F_z, F_\rho \in \mathbb{R}[\rho][[z]] \cap \mathbb{R}[z][[\rho]]$ satisfy $F_\rho(0,0) = F_z(0,0) = 0$. Suppose without loss of generality that $\lambda_1 \neq 0$. Then, we write:

$$\hat{\xi}^{(J)}(z) = \rho^a z^{b+1} \cdot (\lambda_1 + F_z(z,\rho))\hat{\xi}^{(J)}(\theta).$$

Since $\ell \geq \ell_{\mathcal{M}} + 1$, Corollary 3.15 implies that:

$$\xi_\ell^{(J)}(z) = \rho^a z^{b+1} \cdot (\lambda_1 + F_z^\ell(\theta, z, \rho))\xi_\ell^{(J)}(\theta),$$

where F_z^ℓ is analytic and $F_z^\ell(\theta, 0, 0) = 0$. As in the first case, we find that the trajectories of $\xi_\ell^{(J)}$ can be parameterized by z in $W_I \backslash E$ and $\xi_\ell^{(J)}$ cannot have cycles in $W_I \backslash E$. □

5.3 Simple Non-corner Characteristic Cycles

All along this subsection, we suppose that γ_I is a non-corner characteristic cycle of $\mathcal{M}$ contained in a chart C_J for which $\{\rho^{(J)} = 0\}$ is the equation of $E \cap C_J$ and $\gamma_I = \{\rho^{(J)} = 0, z^{(J)} = w_I\}$ for some $\omega_I \in \mathbb{R}$. Consider the transform $\hat{\xi}^{(J)} = (\pi|_{C_J})^*\hat{\xi}$ in the translated coordinates $(z := z^{(J)} - w_I, \rho := \rho^{(J)})$. Its associated two-dimensional vector field is

$$\hat{\eta}_J := \frac{\hat{\xi}^{(J)}(\rho)}{\hat{\xi}^{(J)}(\theta)}\frac{\partial}{\partial \rho} + \frac{\hat{\xi}^{(J)}(z)}{\hat{\xi}^{(J)}(\theta)}\frac{\partial}{\partial z}.$$

More precisely, we write $\hat{\eta}_J = \rho^a \hat{\eta}'_J$ where $a \geq 0$ and $\hat{\eta}'_J$ is a formal vector field in coordinates (z, ρ) with a simple singularity at the origin, One of the separatrices of $\hat{\eta}'_J$ is the divisor $\{\rho = 0\}$ and the other one, denoted by $\hat{\Gamma}_I$, is smooth and transverse to the divisor.

5.3.1 Invariant Formal Surface Along γ_I

Being $\hat{\Gamma}_I$ a formal non-singular curve transverse to $\{\rho = 0\}$, it can be expressed as a formal graph $z = \hat{h}_I(\rho)$, where $\hat{h}_I(\rho) \in \mathbb{R}[[\rho]]$. Since $L_{\frac{\partial}{\partial\theta}} \hat{\xi}^{(J)} = 0$, we have that $\hat{S}_I := \mathbb{S}^1 \times \hat{\Gamma}_I$ is a formal invariant non-singular surface of $\hat{\xi}^{(J)}$ supported along the cycle γ_I. Its vanishing ideal $\mathrm{id}(\hat{S}_I)$ in the ring $\mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$ is generated by $H_I(z, \rho) := z - \hat{h}_I(\rho)$. Using this surface, we can also construct a formal invariant surface for the transformed vector field $\xi_\ell^{(J)} = (\pi|_{C_J})^* \xi_\ell$ along the characteristic cycle γ_I, when ℓ is sufficiently large. More precisely,

Proposition 5.3 *Suppose that $\ell \geq \ell_{\mathcal{M}} + 1$. Then, there is a formal invariant surface $\hat{S}_{\ell,I}$ of $\xi_\ell^{(J)}$ along γ_I expressed in coordinates (θ, z, ρ) as the ideal in $\mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$ generated by some series of the form $H_{\ell,I}(\theta, z, \rho) := z - h_{\ell,I}(\theta, \rho)$, where $h_{\ell,I} \in \mathbb{R}[\cos\theta, \sin\theta][[\rho]]$ with $h_{\ell,I}(\theta, 0) = 0$.*

Proof Consider the formal conjugation $\psi_\ell^* \xi_\ell = \hat{\xi}$ (cf. Eq. (3)). From Proposition 3.16, there is a formal automorphism $\psi_\ell^{(J)}$ defined by:

$$(\theta, z, \rho) \circ \psi_\ell^{(J)} = (\psi_\ell^\theta, \psi_\ell^z, \psi_\ell^\rho) = (\theta + O(\rho^2), z + O(\rho^2), \rho + O(\rho^2)),$$

conjugating $\hat{\xi}^{(J)}$ to $\xi_\ell^{(J)}$ and such that $\psi_\ell^{(J)} - (\theta, z, \rho) \in \mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]^3$. We consider the formal surface $\hat{S}_{\ell,I}$ whose defining ideal is $\mathrm{id}(\hat{S}_{\ell,I}) = (\widetilde{H}_{\ell,I}(\theta, z, \rho))$ where

$$\widetilde{H}_{\ell,I}(\theta, z, \rho) := (\psi_\ell^{(J)})^*(H_I) = H_I \circ \psi_\ell^{(J)} = \psi_\ell^z - \hat{h}_I(\psi_\ell^\rho) \in \mathbb{R}[\cos\theta, \sin\theta, z][[\rho]].$$

Using that $\frac{\partial \widetilde{H}_{\ell,I}}{\partial z}(0, 0, 0) \neq 0$ and applying the implicit function theorem to $\widetilde{H}_{\ell,I}$, we find an expression of the form $H_{\ell,I} = z - h_{\ell,I}(\theta, \rho)$ for a generator of $id(\hat{S}_{\ell,I})$, with $h_{\ell,I} \in \mathbb{R}[\cos\theta, \sin\theta][[\rho]]$. □

5.3.2 Poincaré First-Return Map Associated to γ_I

By Remark 3.14, γ_I is a trajectory of the vector field $\xi_\ell^{(J)} = (\pi|_{C_J})^* \xi_\ell$ for $\ell \geq \ell_{\mathcal{M}} + 1$. Let $P = P_{\ell,I} : \Delta \to \{\theta = 0\}$ be the Poincaré first-return map of $\xi_\ell^{(J)}$ relatively to γ_I, where Δ is a sufficiently small neighborhood of $(z, \rho) = (0, 0)$ in $\{\theta = 0\}$ in which P is analytic.

Notice that the Poincaré map does not depend on the parametrization of the trajectories of the vector field, and hence, we can define it using any equivalent vector field. In particular, we are going to consider the vector field $\tilde{\xi}_\ell^{(J)}$ equivalent to $\xi_\ell^{(J)}$ obtained by the multiplication by the inverse of $\xi_\ell^{(J)}(\theta)$. That is, we put

$$\tilde{\xi}_\ell^{(J)} = \frac{\partial}{\partial\theta} + \chi, \quad \text{where} \;\; \chi = \frac{\xi_\ell^{(J)}(z)}{\xi_\ell^{(J)}(\theta)} \frac{\partial}{\partial z} + \frac{\xi_\ell^{(J)}(\rho)}{\xi_\ell^{(J)}(\theta)} \frac{\partial}{\partial\rho}. \tag{40}$$

Notice that the components of χ are the right members of the system of ODEs $\eta_{\ell,J}$ introduced in Sect. 3.5. They belong to the $\mathbb{R}-$algebra $\mathbb{R}[\cos\theta, \sin\theta]\{z, \rho\}$ (by Remark 3.14). Thus, we consider $\widetilde{\xi}_\ell^{(J)}$ as an analytic vector field on the domain $(\theta, z, \rho) \in \mathbb{R} \times (-\delta, \delta)^2$, for some $\delta > 0$, $2\pi-$periodic in the variable θ. Moreover, from Corollary 3.15, we have that ρ divides χ and hence $\widetilde{\xi}_\ell^{(J)}|_E = \frac{\partial}{\partial\theta}$.

Denote by $\Phi^t := \Phi^t_{\tilde{\xi}_\ell^{(J)}}$ the flow map of $\tilde{\xi}_\ell^{(J)}$. It is defined and analytic for $(t, (\theta, z, \rho)) \in (-\varepsilon, 2\pi + \varepsilon) \times ((-\varepsilon, 2\pi + \varepsilon) \times V)$ where V is a neighborhood of $0 \in \mathbb{R}^2$. Using that $\widetilde{\xi}_\ell^{(J)}(\theta) = 1$, we obtain

$$\Phi^t(\theta, z, \rho) = (\theta + t, \Psi_z^t(\theta, z, \rho), \Psi_\rho^t(\theta, z, \rho)), \tag{41}$$

that is, the angle θ is the natural time for $\tilde{\xi}_\ell^{(J)}$. By definition, the Poincaré map is given by

$$P(z, \rho) = (\Psi_z^{2\pi}(0, z, \rho), \Psi_\rho^{2\pi}(0, z, \rho)). \tag{42}$$

We are going to express the flow via the exponential map. To be precise, given any $G \in \mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$, we define:

$$\mathrm{Exp}(t\tilde{\xi}_\ell^{(J)})(G) := \sum_{i=0}^{\infty} \frac{t^i}{i!}(\tilde{\xi}_\ell^{(J)})^{(i)}(G),$$

where, for any vector field ζ, $\zeta^{(0)}(G) = G$ and $\zeta^{(i)}(G) = \zeta(\zeta^{(i-1)}(G))$, if $i \geq 1$. Taking into account the above properties of the components of $\tilde{\xi}_\ell^{(J)}$, it is immediate to check that $\mathrm{Exp}(t\tilde{\xi}_\ell^{(J)})(G) \in \mathbb{R}[\cos\theta, \sin\theta][[t, z, \rho]]$. In the following result, we get some useful properties of this exponential map and its relation with the flow map. Notice first that, if $G \in \mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$, then the composition $G \circ \Phi^t$, due to the analyticity of Φ^t, has a formal Taylor expansion at $t = 0$, denoted by $T_0(G \circ \Phi^t)$, a formal power series in variables (t, z, ρ), with analytic functions of $\theta \in (-\varepsilon, 2\pi + \varepsilon)$ as coefficients.

Proposition 5.4 *Let $G \in \mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$. We have:*

(1) $T_0(G \circ \Phi^t) = \mathrm{Exp}(t\tilde{\xi}_\ell^{(J)})(G) \in \mathbb{R}[\cos\theta, \sin\theta][[t, z, \rho]]$

(2) *For any $t_0 \in [0, 2\pi]$, the expression* $\mathrm{Exp}(t_0\tilde{\xi}_\ell^{(J)})(G) = \sum_{i=0}^{\infty} \frac{t_0^i}{i!}(\tilde{\xi}_\ell^{(J)})^{(i)}(G)$ *has a sense as a series in $\mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$ and we have*

$$G \circ \Phi^{t_0} = \mathrm{Exp}(t_0\tilde{\xi}_\ell^{(J)})(G) \tag{43}$$

Proof We prove (1) with the same arguments as the case in Loray's text for holomorphic vector fields [23, p. 15]: expand $G \circ \Phi^t$ as a Taylor series in t at $t = 0$, so that we get

$$T_0(G \circ \Phi^t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \left.\frac{\partial^i (G \circ \Phi^t)}{\partial t^i}\right|_{t=0},$$

and check that $\frac{\partial^i (G \circ \Phi^t)}{\partial t^i} = (\tilde{\xi}_\ell^{(J)})^{(i)}(G) \circ \Phi^t$ for any $i \geq 1$.

Let us prove item (2). First, we show that there exists $\alpha > 0$ such that (2) is true for any $t_0 \in [0, \alpha]$. For that, consider the particular case where G is either the coordinate z or ρ (with the notations of (41), $z \circ \Phi^{t_0} = \Psi_z^{t_0}$ and $\rho \circ \Phi^{t_0} = \Psi_\rho^{t_0}$). By analyticity of these functions and by item (1), we get that $\mathrm{Exp}(t\tilde{\xi}_\ell^{(J)})(z)$, $\mathrm{Exp}(t\tilde{\xi}_\ell^{(J)})(\rho) \in \mathbb{R}[\cos\theta, \sin\theta]\{t, z, \rho\}$. More precisely, they belong to $\mathbb{R}[\cos\theta, \sin\theta]\{t\}_\beta[[z, \rho]]$ for some $\beta > 0$ (recall the notations stated in Sect. 1, that is, all coefficients in $\mathbb{R}[\cos\theta, \sin\theta]\{t\}$ have a common radius of convergence).

We conclude that $\Psi_z^{t_0} = z \circ \Phi^{t_0} = \mathrm{Exp}(t_0 \tilde{\xi}_\ell^{(J)})(z)$ and $\Psi_\rho^{t_0} = \rho \circ \Phi^{t_0} = \mathrm{Exp}(t_0 \tilde{\xi}_\ell^{(J)})(\rho)$ for any $t_0 \in [0, \alpha]$ with $0 < \alpha < \beta$.

Let $G \in \mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$ be any formal series and write

$$G = \sum_{u,v} G_{uv}(\theta) z^u \rho^v, \text{ with } G_{uv}(\theta) \in \mathbb{R}[\cos\theta, \sin\theta].$$

Consider the series

$$\bar{G} = \sum_{u,v} G_{uv}(\theta + t) z^u \rho^v$$

which belongs to $\mathbb{R}[\cos\theta, \sin\theta]\{t\}_\beta[[z, \rho]]$ since each $G_{uv}(\theta)$ is a trigonometric polynomial. Taking into account the expression of the flow Φ^t, we have that $G \circ \Phi^t$ is the result of substituting in the series $\bar{G}$ the variables z, ρ by Ψ_z^t, Ψ_ρ^t, respectively. Since the series Ψ_z^t, Ψ_ρ^t belong to $\mathbb{R}[\cos\theta, \sin\theta]\{t\}_\beta[[z, \rho]]$ and have positive order with respect the variables z, ρ, substitution has perfect sense and provides an element in $\mathbb{R}[\cos\theta, \sin\theta]\{t\}_\beta[[z, \rho]]$. Since, by item (1), $T_0(G \circ \Phi^t)$ coincides with $\mathrm{Exp}(t\widetilde{\xi}_\ell^{(J)})(G)$ as a series in $\mathbb{R}[\cos\theta, \sin\theta][[t, z, p]]$, we conclude item (2) and expression (43) for $t_0 \in [0, \alpha]$. Notice that we can choose $\alpha > 0$ which does not depend on G. Let us show that we can extend the property(43) to any $t_0 \in [0, 2\alpha]$ (and hence similar extensions will prove *(2)*). Let $t_0 \in [\alpha, 2\alpha]$ and write $t_0 = s_0 + \alpha$, where $s_0 \in [0, \alpha]$. We have $G \circ \Phi^{t_0} = (G \circ \Phi^{s_0}) \circ \Phi^\alpha$. Applying (43) for the values s_0 and α, and for G and $G \circ \Phi^{s_0}$, respectively, we get

$$\begin{aligned} G \circ \Phi^{t_0} &= \sum_i \frac{\alpha^i}{i!} (\tilde{\xi}_\ell^{(J)})^{(i)} (G \circ \Phi^{s_0}) = \sum_i \frac{\alpha^i}{i!} (\tilde{\xi}_\ell^{(J)})^{(i)} \left(\sum_j \frac{s_0^j}{j!} (\tilde{\xi}_\ell^{(J)})^{(j)}(G) \right) \\ &= \sum_k \left(\sum_{i+j=k} \frac{\alpha^i}{i!} \frac{s_0^j}{j!} (\tilde{\xi}_\ell^{(J)})^{(k)}(G) \right) \\ &= \sum_k \frac{(\alpha + s_0)^k}{k!} (\tilde{\xi}_\ell^{(J)})^{(k)}(G) = \mathrm{Exp}(t_0 \tilde{\xi}_\ell^{(J)})(G), \end{aligned}$$

as it was to be proved. □

We can now prove two important features of the Poincaré map.

Lemma 5.5 *There exists ℓ_I such that, for any $\ell \geq \ell_I$, the Poincaré map $P = P_{\ell,I}$ satisfies:*

(a) P is tangent to the identity but $P \neq Id$ as a germ of diffeomorphisms at $(0, 0) \in \Delta$.
(b) The formal curve $\Gamma = \Gamma_{\ell,I} := \hat{S}_{\ell,I} \cap \Delta$ is invariant for P.

Proof Recall that the two-dimensional formal vector field $\hat{\eta}_J$ associated to the formal vector field $\hat{\xi}^{(J)}$ has an adapted simple singularity corresponding to the characteristic cycle γ_I. As mentioned, the defining ideal of the formal curve $\hat{\Gamma}_I$ is generated by $z - \hat{h}_I(\rho)$, where $\hat{h}_I(\rho) = \sum_{i \geq 1} a_i \rho^i$.

Therefore, we can write $\hat{\eta}_J = \rho^n \hat{\eta}'_J$ with

$$\hat{\eta}'_J = (z - \hat{h}_I(\rho))^r \left((\lambda_1(z - a_1\rho) + B_1(z, \rho)) \frac{\partial}{\partial z} + (\lambda_2 \rho + B_2(z, \rho)) \frac{\partial}{\partial \rho} \right),$$

where $n = n_1^{(J)}$, $r \in \mathbb{N}_{\geq 0}$, $(\lambda_1, \lambda_2) \neq (0, 0)$ and $B_i \in \mathbb{R}[z][[\rho]]$ has order greater or equal than 2 for $i = 1, 2$. Up to making a new admissible blowing-up with center γ_I, we may assume that ρ divides B_1 and that ρ^2 divides B_2.

Define $\ell_I = \ell_{\mathcal{M}} + r + 1$. Applying Corollary 3.15 to $k = r + 1$, we get, for any $\ell \geq \ell_{\mathcal{M}} + k = \ell_I$:

$$j^{\rho}_{r+1}(\eta'_{\ell,J}) = j^{\rho}_{r+1}(\hat{\eta}'_J). \tag{44}$$

Assume first that $\lambda_1 \neq 0$. From (44), and taking into account that χ coincides with the two dimensional system $\eta_{\ell,J}$ (cf. Eq. (40)), we obtain

$$j^{\rho}_n(\tilde{\xi}^{(J)}_\ell) = \frac{\partial}{\partial\theta} + \lambda_1\rho^n z^{r+1}\frac{\partial}{\partial z} \tag{45}$$

Using this in the computation of the exponential $\mathrm{Exp}(t\tilde{\xi}^{(J)}_\ell)(z)$, written as a series as

$$\mathrm{Exp}(t\tilde{\xi}^{(J)}_\ell)(z) = z + tQ_1 + t^2Q_2 + \cdots, \quad \text{with } Q_j \in \mathbb{R}[\cos\theta, \sin\theta][[z,\rho]],$$

one can show by recurrence that

$$Q_1 = \rho^n(\lambda_1 z^{r+1} + O(\rho)),$$
$$Q_j = O(\rho^{jn}),\ j \geq 2.$$

Using Proposition 5.4 and Eq. (42), we deduce, since $\lambda_1 \neq 0$ and $n \geq 1$,

$$z \circ P(z,\rho) = \Psi^{2\pi}_z(0,z,\rho) = \mathrm{Exp}(2\pi\tilde{\xi}^{(J)}_\ell)(z) = z + \rho^n(2\pi\lambda_1 z^{r+1} + O(\rho)) \neq z.$$

This proves *(a)* if $\lambda_1 \neq 0$.

On the contrary, if $\lambda_1 = 0$ and $\lambda_2 \neq 0$, we obtain

$$j^{\rho}_{n+1}(\tilde{\xi}^{(J)}_\ell) = \frac{\partial}{\partial\theta} + \rho^{n+1}g(z)\frac{\partial}{\partial z} + \lambda_2\rho^{n+1}z^r\frac{\partial}{\partial\rho},$$

where $g(z) \in \mathbb{R}\{z\}$. We deduce, similarly, that, if we write again

$$\mathrm{Exp}(t\tilde{\xi}^{(J)}_\ell)(\rho) = \rho + tQ_1 + t^2Q_2 + \cdots, \quad \text{with } Q_j \in \mathbb{R}[\cos\theta, \sin\theta][[z,\rho]],$$

then $Q_1 = \rho^{n+1}(\lambda_2 z^r + O(\rho))$ and $Q_j = O(\rho^{nj+1})$ if $j \geq 2$. Hence,

$$\rho \circ P(z,\rho) = \Psi^{2\pi}_\rho(0,z,\rho) = \mathrm{Exp}(2\pi\tilde{\xi}^{(J)}_\ell)(\rho) = \rho + \rho^{n+1}(2\pi\lambda_2 z^r + O(\rho)) \neq \rho,$$

and *(a)* equally holds.

Let us show (b). Let $H = H_{\ell,I} \in \mathbb{R}[\cos\theta, \sin\theta][[z,\rho]]$ be a generator of the ideal of the invariant surface $\hat{S}_{\ell,I}$ obtained in Proposition 5.3. The series $g(z,\rho) := H(0,z,\rho)$ is a generator of the formal plane curve $\Gamma = \hat{S}_{\ell,I} \cap \{\theta = 0\}$. We need to check that the composition $g \circ P$ is divisible by g. Using Proposition 5.4, we have

$$H \circ \Phi^{2\pi} = \sum_{i\geq 0}\frac{(2\pi)^i}{i!}(\tilde{\xi}^{(J)}_\ell)^{(i)}(H). \tag{46}$$

Since $\hat{S}_{\ell,I}$ is invariant for $\tilde{\xi}^{(J)}_\ell$, we have $\tilde{\xi}^{(J)}_\ell(H) \in \mathrm{id}(\hat{S}_{\ell,I})$, that is, H divides $\tilde{\xi}^{(J)}_\ell(H)$. By recurrence, H divides $(\tilde{\xi}^{(J)}_\ell)^{(i)}(H)$ for any $i \geq 0$. Thus, from Eq. (46), we get

$$H \circ \Phi^{2\pi} = H \cdot K, \text{ where } K \in \mathbb{R}[\cos\theta, \sin\theta][[z,\rho]].$$

We conclude, using that $P(z,\rho) = \Phi^{2\pi}(0,z,\rho)$,

$$g \circ P = (H \circ \Phi^{2\pi})|_{\{\theta=0\}} = H(0,z,\rho)K(0,z,\rho) = g \cdot \tilde{K},\ \tilde{K} \in \mathbb{R}[[z,\rho]],$$

as we wanted to prove. □

5.3.3 Periodic Orbits of the Poincaré Map Around the Invariant Curve

We are interested in the periodic orbits of P near $(0, 0)$ since, as we know, they correspond to cycles of $\xi_\ell^{(J)}$ near γ_I. As it was the case for defining cycles, periodic orbits depend on the domain of (a representative of) P. To be more precise, if P is defined in some neighborhood W of $(0, 0)$, a *periodic orbit in* W is a finite set $\{p_i\}_{i=0}^{n-1}$ contained in W such that $p_i = P(p_{i-1})$ for $i = 1, \ldots, n-1$ and $p_0 = P(p_{n-1})$.

Denote by $\mathrm{Fix}(P)$ the (germ of the) locus of fixed points of P. It is an analytic set with empty interior (since $P \neq id$), thus either reduced to the origin or an analytic curve with finitely many branches. We can distinguish two different situations:

(a) The invariant curve Γ is not contained in $\mathrm{Fix}(P)$.
(b) The invariant curve Γ is contained in $\mathrm{Fix}(P)$.

In particular, in case (b), Γ is a real branch of $\mathrm{Fix}(P)$, and thus it converges. In both cases, we investigate periodic orbits of P in some "neighborhood" of Γ. To be precise, consider a parametrization of Γ of the form $z = h(\rho)$, with $h(\rho) \in \mathbb{R}[[\rho]]$ and $h(0) = 0$. A *conic neighborhood* of Γ is a set of the form

$$\Sigma_{N,\delta}(\Gamma) = \Sigma_{N,\delta}^{(z,\rho)}(\Gamma) := \{(z, \rho) : |z - j_N(h(\rho))| < \rho^N, 0 < \rho < \delta\}.$$

where $N \in \mathbb{N}_{\geq 1}$ and any $\delta > 0$ sufficiently small.

Remark 5.6 By its definition, these conic neighborhoods depend on the chosen coordinates. However, after a simple change of variables consisting in a transformation of type $\bar{z} = z + \alpha(\rho)$ with $\alpha(\rho) \in \mathbb{R}[[\rho]]$, the sets $\Sigma_{N,\delta}^{(z,\rho)}$ and $\Sigma_{N,\delta}^{(\bar{z},\bar{\rho})}$ coincide exactly under this change.

Case (a) In this case, we prove that there are no periodic orbits in a conic neighborhood of Γ. The arguments are inspired by the papers [21, 22], devoted to treat this case for holomorphic diffeomorphisms.

Lemma 5.7 *Suppose that* Γ *is not contained in* $\mathrm{Fix}(P)$. *Then, there is some* $N \in \mathbb{N}_{\geq 1}$ *and some* $\delta > 0$ *such that (a representative of)* P *does not have periodic orbits in* $\Sigma_{N,\delta}(\Gamma)$.

Proof First, since the divisor is contained in $\mathrm{Fix}(P)$, we have that $\rho \circ P - \rho$ can be divided by ρ. On the other hand, being Γ invariant for P, there is a formal diffeomorphism $\Theta(\rho) = \rho + O(\rho^2) \in \mathbb{R}[[\rho]]$ satisfying:

$$P(h(\rho), \rho) = (h(\Theta(\rho)), \Theta(\rho)).$$

The formal diffeomorphism Θ is called the *restriction of* P *to* Γ, denoted by $P|_\Gamma := \Theta$ (see [21]). The *order* of $P|_\Gamma$, defined as $\mathrm{ord}_\rho(\Theta(\rho) - \rho) - 1$, does not depend on the coordinates nor the parametrization $(h(\rho), \rho)$ of Γ. In this case, it is a natural number $m < \infty$, because otherwise, $\Theta(\rho) = \rho$ and Γ would be contained in $\mathrm{Fix}(P)$. We deduce that there is a maximal $k \geq 1$ such that ρ^k divides $\rho \circ P - \rho$, so that we can write

$$\rho \circ P - \rho = \rho^k (A(\rho) + zB(z, \rho)), \tag{47}$$

where $A \in \mathbb{R}\{\rho\}$, $B \in \mathbb{R}\{z, \rho\}$ and $\rho^k(A(\rho) + zB(z, \rho)) \neq 0$. Up to taking new coordinates $(\bar{z}, \rho)$ with $\bar{z} = z - j_{m+1}(h(\rho))$ (which we rename (z, ρ) for simplicity) and using Remark 5.6, we may assume from the beginning that $\mathrm{ord}_\rho(h(\rho)) \geq m + 2$. From equation (47), the definition of $\Theta = P|_\Gamma$ and its order m, we have

$$\alpha\rho^{m+1} + \cdots = \Theta(\rho) - \rho = \rho^k (A(\rho) + h(\rho)B(h(\rho), \rho)),\ \alpha \neq 0,$$

which implies that $A(\rho) = \rho^s \widetilde{A}(\rho)$ with $k+s = m+1$ and $\widetilde{A}(0) = \alpha$ (in particular s is finite). Put $N = m+1$ and let us prove the required property for a cone $\Sigma_{N,\delta} = \Sigma_{N,\delta}^{(z,\rho)}(\Gamma)$ with some $\delta > 0$. Notice first that, for the chosen coordinates (z, ρ), we have $j_N(h(\rho)) = 0$, so $\Sigma_{N,\delta}$ is given simply by equations $|z| < \rho^N$ and $0 < \rho < \delta$. On the other hand, $N = k + s > s$, since $k > 0$. Assume for instance that $\alpha < 0$ (analogous arguments apply if $\alpha > 0$). Take a preliminary $\delta_1 > 0$ such that $\widetilde{A}(\rho) < \frac{\alpha}{2}$ if $\rho < \delta_1$ and let $K > 0$ be a bound for $|B|$ in a neighborhood of $(0, 0)$ that contains Σ_{N,δ_1}. We have in Σ_{N,δ_1}

$$\rho^s \widetilde{A}(\rho) + zB < \rho^s \widetilde{A}(\rho) + \rho^N K < \rho^s (\frac{\alpha}{2} + \rho^{N-s} K). \tag{48}$$

Now take $\delta < \delta_1$ such that $\delta^{N-s} < \frac{|\alpha|}{4K}$, and hence we obtain from (48)

$$\rho \circ P - \rho = \rho^k (\rho^s \widetilde{A}(\rho) + zB) < \rho^k (\rho^s \frac{\alpha}{4}) < 0 \text{ in } \Sigma_{N,\delta}. \tag{49}$$

We conclude that, if $\{p_0, p_1 = P(p_0), p_2 = P(p_1), \ldots\}$ is an orbit (finite or not) of P completely contained in $\Sigma_{N,\delta}$, then the sequence of $\rho-$coordinates decreases strictly:

$$\rho(p_0) > \rho(p_1) > \cdots$$

This proves the result. □

Case (b) Suppose that Γ is convergent and contained in $\mathrm{Fix}(P)$. Convergence means that Γ is an analytic curve given by a graph $\Gamma = \{(h(\rho), \rho) : \rho \in [0, \varepsilon)\}$, where $h(\rho) \in \mathbb{R}\{\rho\}$. Up to taking new analytic coordinates $(z - h(\rho), \rho)$, we will assume that $h(\rho) \equiv 0$. Thus, the conic neighborhoods $\Sigma_{N,\delta} = \Sigma_{N,\delta}^{(z,\rho)}(\Gamma)$ will be simply defined by the equations $|z| < \rho^N$ and $0 < \rho < \delta$. With these assumptions, we prove the following result.

Lemma 5.8 *Suppose that* $\Gamma = \{z = 0\} \subset \mathrm{Fix}(P)$. *Then, there is* $N \in \mathbb{N}_{\geq 1}$ *and* $\delta > 0$ *such that the fixed points of the set* $\Gamma \cap \Sigma_{N,\delta}$ *are the only periodic orbits of* P *in* $\Sigma_{N,\delta}$.

Proof The two coordinate axis $\{\rho = 0\}$ and $\{z = 0\}$ are contained in $\mathrm{Fix}(P)$. Thus, both components $(z \circ P - z, \rho \circ P - \rho)$ of the map $P - Id$ are divisible by a positive power of z and by a positive power of ρ. In particular, we can write $z \circ P = z(1 + \psi(z, \rho))$ where ρ divides ψ. From this, we prove the following observation.

Claim There is a neighborhood V of $(0, 0)$ such that if $\{p_0, p_1, \ldots\}$ is an orbit of P contained in V, then the sign of the z coordinate of its elements is constant.

Proof of the claim: Since $\psi(0, 0) = 0$, we can consider a neighborhood V where we have $(1 + \psi(z, \rho)) > \frac{1}{2}$. Hence, $\mathrm{Sign}(z \circ P) = \mathrm{Sign}(z(1 + \psi(z, \rho))) = \mathrm{Sign}(z)$ and the claim follows.

On the other hand, since $P \neq Id$ (cf. Lemma 5.5), the two components $z \circ P - z$, $\rho \circ P - \rho$ cannot be identically zero simultaneously. Suppose that $\rho \circ P - \rho \neq 0$. Then we can write

$$\rho \circ P - \rho = \rho^{k_1} z^{k_2} (A(\rho) + zB(z, \rho)), \tag{50}$$

where $k_1, k_2 \in \mathbb{N}_{\geq 1}$ and $A(\rho)$ is a convergent non-zero series. We write $A(\rho) = \rho^s(\alpha + \cdots)$, where $s \geq 0$ and $\alpha \neq 0$. Analogously as in the proof of Lemma 5.7, if N is any given natural number with $N > s$, then there exists $\delta > 0$ such that the function $A(\rho) + zB(z, \rho)$ has constant non-zero sign on $\Sigma_{N,\delta}$. Taking into account the claim above, if δ is sufficiently small so that $\Sigma_{N,\delta} \subset V$, we conclude form Eq. (50) that if $\{p_0, p_1, \ldots\}$ is an orbit of P contained in $\Sigma_{N,\delta} \backslash \Gamma = \left(\Sigma_{N,\delta} \cap \{z > 0\}\right) \cup \left(\Sigma_{N,\delta} \cap \{z < 0\}\right)$, then the sequence $\{\rho(p_0), \rho(p_1), \ldots\}$

is strictly increasing or strictly decreasing. This proves the lemma in case $\rho \circ P - \rho \neq 0$. On the contrary, if $\rho \circ P - \rho = 0$ but $z \circ P - z \neq 0$, we obtain analogously that the z−coordinate of elements of an orbit in $\Sigma_{N,\delta} \setminus \Gamma$ is strictly increasing or strictly decreasing, which proves the lemma equally. □

6 Proof of the Main Theorem

In this section we provide a proof of Theorem 1.1. It is enough to prove the result for some jet approximation ξ_ℓ of a formal normal form $\hat{\xi}$ of ξ, since all those vector fields are locally analytically conjugated to ξ at $0 \in \mathbb{R}^3$.

Fix an adapted reduction of singularities $\mathcal{M} = (M, \pi, \mathcal{A}, \mathcal{D})$ of $\hat{\xi}$ given by Proposition 3.10 and denote by $\mathcal{D}^{nc}$ the subset of $\mathcal{D}$ consisting on non-corner characteristic cycles. For any $\gamma_I \in \mathcal{D}^{nc}$, we consider a chart C_J and coordinates $(\theta, z^{(I)} := z^{(J)} - \omega_I, \rho^{(I)} = \rho^{(J)})$ as in Sect. 5.3, such that $\gamma_I \subset C_J$ and given by $\gamma_I = \{z^{(I)} = 0, \rho^{(I)} = 0\}$. Let $\hat{\Gamma}_I$ be the formal plane curve at the origin $(z^{(I)}, \rho^{(I)}) = (0, 0)$ of $\{\theta = 0\}$, invariant for the associated vector field $\hat{\eta}_J$ and transversal to the divisor $\{\rho^{(I)} = 0\}$. Consider the parametrization of $\hat{\Gamma}_I$ given in these coordinates as $z = \hat{h}_I(\rho^{(I)})$, $\hat{h}_I \in \mathbb{R}[[\rho^{(I)}]]$, $\hat{h}_I(0) = 0$. Consider also the formal surface $\hat{S}_I = \mathbb{S}^1 \times \hat{\Gamma}_I$ invariant for $\hat{\xi}^{(J)}$ and given by the same equation $z = \hat{h}_I(\rho^{(I)})$, but considering it in $\mathbb{R}[\cos\theta, \sin\theta, z][[\rho]]$. Let ℓ be the first natural number such that $\ell \geq \ell_I$ for any $\gamma_I \in \mathcal{D}^{nc}$, where ℓ_I is given in Lemma 5.5 (notice that $\ell \geq \ell_{\mathcal{M}} + 1$). Denote by $\hat{S}_{\ell,I}$ the formal invariant surface of $\xi_\ell^{(J)}$ given in Proposition 5.3 with defining ideal $id(\hat{S}_{\ell,I}) = (H_{\ell,I})$, where $H_{\ell,I} = z^{(I)} - h_{\ell,I}(\theta, \rho^{(I)})$ with $h_{\ell,I} \in \mathbb{R}[\cos\theta, \sin\theta][[\rho^{(I)}]]$. Denote by $P_{\ell,I}$ the Poincaré map of $\xi_\ell^{(J)}$ along γ_I defined in some neighborhood of the origin of $\{\theta = 0\}$ and $\Gamma_{\ell,I} = \hat{S}_{\ell,I} \cap \{\theta = 0\}$ its corresponding formal invariant curve (by Lemma 5.5). Denote furthermore $\mathcal{D}^{nc} = \mathcal{D}^{nc,nfix} \cup \mathcal{D}^{nc,fix}$ as a disjoint union, where $\gamma_I \in \mathcal{D}^{nc,fix}$ if and only if $\Gamma_{\ell,I} \subset \mathrm{Fix}(P_{\ell,I})$.

Apply Lemmas 5.7 or 5.8 according to whether $\gamma_I \in \mathcal{D}^{nc,nfix}$ or $\gamma_I \in \mathcal{D}^{nc,fix}$, and get conic neighborhoods $\Sigma_{N_{\ell,I},\delta} = \Sigma_{N_{\ell,I},\delta}(\Gamma_{\ell,I})$ (in coordinates $(z^{(I)}, \rho^{(I)})$ and where we have unified $\delta > 0$) where $P_{\ell,I}$ has no periodic orbits except for the fixed points $\Gamma_{\ell,I} \cap \Sigma_{N_{\ell,I},\delta}$ when $\gamma_I \in \mathcal{D}^{nc,fix}$.

This information is of course relevant in order to describe the union of cycles of the transform $\xi_\ell^{(J)}$ near γ_I, but it is not enough a priori, since $\Sigma_{N_{\ell,I},\delta}$ is not a full neighborhood of the origin at $\{\theta = 0\}$. By further blowing-ups along the characteristic cycles γ_I, one can eventually open these conic neighborhoods to full neighborhoods of γ_I but, if the order of tangency $N_{\ell,I}$ is too large, the sequence of blowing-ups to be done may not follow the formal curve $\hat{\Gamma}_I$, and thus we could skip the context of sequences of admissible blowing-ups. We can take a bigger ℓ' so that the formal curve $\Gamma_{\ell',I}$ approximates $\hat{\Gamma}_I$ better than the curve $\Gamma_{\ell,I}$ does. But the order $N_{\ell',I}$ may increase with ℓ' a priori.

In the strategy that follows we overcome these difficulties. We have to consider first the same kind of conic three-dimensional neighborhoods of the surfaces $\hat{S}_{\ell,I}$. With more generality, consider coordinates (θ, z, ρ) in $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_{\geq 0}$, put $\gamma = \mathbb{S}^1 \times \{(0, 0)\}$ and let S be a formal non-singular surface along γ given by an equation of the form

$$z - h(\theta, \rho) = 0, \text{ where } h(\theta, \rho) \in \mathbb{R}[\cos\theta, \sin\theta][[\rho]].$$

For $N \in \mathbb{N}_{\geq 0}$ and constants $C, \delta > 0$, we define

$$\widetilde{\Sigma}_{N,\delta,C}(S) = \widetilde{\Sigma}^{(\theta,z,\rho)}_{N,\delta,C}(S) = \{(\theta, z, \rho) \;\; |z - j^\rho_N(h(\theta, \rho))| < C\rho^N, 0 < \rho < \delta\}.$$

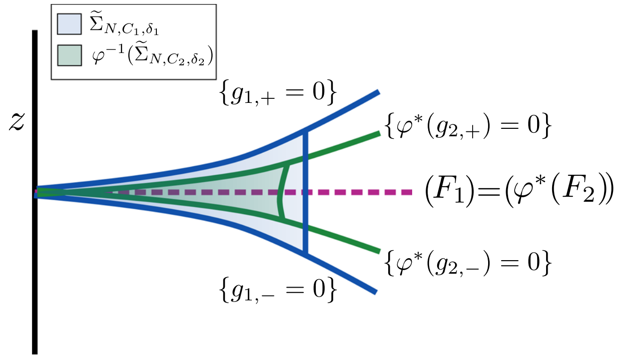

Fig. 5 $\widetilde{\Sigma}_{N,C_1,\delta_1}(S_1)$ and $\varphi^{-1}(\widetilde{\Sigma}_{N,C_2,\delta_2}(S_2))$

In particular, notice that $\Sigma_{N,\delta}(\Gamma_{\ell,I}) = \widetilde{\Sigma}_{N,\delta,1}(\hat{S}_{\ell,I}) \cap \{\theta = 0\}$. We need two results, the first one is just a remark that follows from the construction of sequences of admissible blowing-ups in Sect. 3 and the definition of adapted simple singularities.

Remark 6.1 If $\gamma_I \in \mathcal{D}^{nc}$ and $\widetilde{\mathcal{M}} = (\widetilde{M}, \widetilde{\pi}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{D}})$ is the sequence of admissible blowing-ups obtained from $\mathcal{M}$ by blowing-up along γ_I (that is, $\widetilde{\pi} = \pi \circ \sigma_{\gamma_I}$), then $\widetilde{\mathcal{M}}$ is again an adapted reduction of singularities of $\hat{\xi}$ with $\widetilde{\mathcal{D}} = (\mathcal{D}\backslash\{\gamma_I\}) \cup \{\gamma_{I,-\infty}, \gamma_{I,\infty}, \gamma_{I,1}\}$ and $\widetilde{\mathcal{D}}^{nc} = (\mathcal{D}^{nc}\backslash\{\gamma_I\}) \cup \{\gamma_{I,1}\}$. We will say that $\gamma_{I,1}$ *emerges from* γ_I. Moreover, the new non-corner characteristic cycle $\gamma_{I,1}$ is given by equations $\{z^{(I,0)} = \rho^{(I,0)} = 0\}$ in coordinates for which σ_{γ_I} is written as

$$z^{(I)} = \rho^{(I,0)}(z^{(I,0)} + a_{I,1}),\ \rho^{(I)} = \rho^{(I,0)},$$

where $\hat{h}'_I(0) = a_{I,1}$. We deduce that if $N \in \mathbb{N}$ and $j_N^{\rho^{(I)}}(\hat{h}_I) = j_N^{\rho^{(I)}}(h_{\ell,I})$, then the strict transform $\sigma^*_{\gamma_I}\hat{S}_{\ell,I}$ by σ_{γ_I} is a formal surface along $\gamma_{I,1}$ and

$$\sigma_{\gamma_I}^{-1}(\widetilde{\Sigma}_{N,\delta,1}^{(\theta,z^{(I)},\rho^{(I)})}(\hat{S}_{\ell,I})) = \widetilde{\Sigma}_{N-1,\delta,1}^{(\theta,z^{(I,0)},\rho^{(I,0)})}(\sigma^*_{\gamma_I}\hat{S}_{\ell,I}).$$

Lemma 6.2 *Let $\varphi(\theta,z,\rho) = (\theta+\varphi_\theta, z+\varphi_z, \rho+\varphi_\rho)$ be a diffeomorphism along $\gamma = \{z=0, \rho=0\}$, where $\varphi_\theta, \varphi_z, \varphi_\rho \in \mathbb{R}[\cos\theta, \sin\theta]\{z,\rho\}$ are of order at least two in (z,ρ) and divisible by ρ. Let S_i, $i = 1, 2$, be formal surfaces with defining ideals generated by $F_i = z - f_i(\theta,\rho)$ where $f_i(\theta,\rho) \in \mathbb{R}[\cos\theta, \sin\theta][[\rho]]$ and such that $S_1 = \varphi^*(S_2)$, i.e. in terms of ideals $(F_1) = (F_2 \circ \varphi)$. Then, for every cone $\widetilde{\Sigma}_{N,C_1,\delta_1}(S_1)$ with δ_1 sufficiently small, there exist some constants $C_2, \delta_2 > 0$ such that:*

$$\varphi^{-1}(\widetilde{\Sigma}_{N,C_2,\delta_2}(S_2)) \subseteq \widetilde{\Sigma}_{N,C_1,\delta_1}(S_1).$$

See Fig. 5 for an illustration of this lemma.

Proof Consider the functions $g_{i,\epsilon}(\theta,z,\rho) = z - j_N^\rho(f_i(\theta,\rho)) - \epsilon C_i \rho^N$ for $i = 1, 2$ and $\epsilon \in \{-1,+1\}$. The boundary of the cone $\widetilde{\Sigma}_{N,C_i,\delta_i}(S_i)$ is given by three surfaces with equations $g_{i,+} = 0$, $g_{i,-} = 0$ and $\rho = \delta_i$. It is enough to prove that there exists $C_2, \delta_2 > 0$ such that the following holds.

(i) The function $\rho \circ \varphi - \delta_2$ is positive in the points $K := \{\rho = \delta_1\} \cap \overline{\widetilde{\Sigma}_{N,C_1,\delta_1}(S_1)}$.
(ii) The function $g_{2,+} \circ \varphi$ is positive in the set $\{g_{1,+} = 0, 0 < \rho \circ \varphi < \delta_2\}$.

(iii) The function $g_{2,-} \circ \varphi$ is negative in the set $\{g_{1,-} = 0, 0 < \rho \circ \varphi < \delta_2\}$.

To get (i), we can take any $\delta_2 < \inf((\rho \circ \varphi)|_K)$, taking into account that K is compact and that $\rho \circ \varphi = \rho + \varphi_\rho$ only vanishes along the divisor $\rho = 0$ in a neighborhood of $\mathbb{S}^1 \times \{0\}$.

Notice that we have $z - j_N^\rho(f_i) = F_i + O(\rho^{N+1}),\ i = 1, 2$, and $F_2 \circ \varphi = F_1 \cdot U$ where $U = 1 + T$ and $T \in \mathbb{R}[\cos\theta, \sin\theta][[z, \rho]]$ such that ρ divides T. Then, we deduce

$$g_{2,+} \circ \varphi = (F_2 - C_2\rho^N + O(\rho^{N+1})) \circ \varphi = F_1 \cdot U + (-C_2\rho^N + O(\rho^{N+1})) \circ \varphi.$$

Using that $\rho \circ \varphi = \rho + \varphi_\rho = \rho(1 + \tilde{\varphi}_\rho)$ with $\tilde{\varphi}_\rho \in \mathbb{R}[\cos\theta, \sin\theta]\{z, \rho\}$ of positive order, we have:

$$g_{2,+} \circ \varphi = F_1 \cdot U - C_2\rho^N + O(\rho^{N+1}).$$

Now, we evaluate $g_{2,+} \circ \varphi$ on the sets $\{g_{1,+} = 0\}$, in other words, when $z = j_N^\rho(f_1) + C_1\rho^N$. Considering that $F_1(\theta, j_N^\rho(f_1) + C_1\rho^N, \rho) = f_1 + O(\rho^{N+1}) + C_1\rho^N - f_1 = C_1\rho^N + O(\rho^{N+1})$, we get:

$$\begin{aligned} &g_{2,+} \circ \varphi(\theta, j_N^\rho(f_1) + C_1\rho^N, \rho) \\ &\quad = (C_1\rho^N + O(\rho^{N+1})) \cdot (1 + T(\theta, j_N^\rho(f_1) + C_1\rho^N, \rho)) - C_2\rho^N + O(\rho^{N+1}). \end{aligned}$$

Since ρ divides T, we obtain finally

$$g_{2,+} \circ \varphi(\theta, j_N^\rho(f_1) + C_1\rho^N, \rho) = (C_1 - C_2)\rho^N + O(\rho^{N+1}).$$

Thus, taking $C_2 < C_1$, we have that $g_{2,+} \circ \varphi(\theta, j_N^\rho(f_1) + C_1\rho^N, \rho) > 0$ for ρ small enough. We get (ii), up to taking a smaller δ_2. The arguments to get (iii) are similar. □

Denote $\mathcal{D}^{nc,fix} = \{\gamma_1, \ldots, \gamma_r\}$ and $\mathcal{D}^{nc,nfix} = \{\gamma_{r+1}, \ldots, \gamma_s\}$ and, for any j, let $I_j \in \mathcal{I}$ be defined by $\gamma_j = \gamma_{I_j}$. Let $J_j \in \mathcal{J}$ be the index of the chart $(C_{J_j}, (\theta, z^{(J_j)}, \rho^{(J_j)}))$ as presented in the beginning of Sect. 5.3, where the cycle γ_{I_j} is given by $z^{(J_j)} = \rho^{(J_j)} = 0$. Denote also $N_j = N_{\ell, I_j}$.

Consider the sequence of admissible blowing-ups $\mathcal{M}' = (M', \pi', \mathcal{A}', \mathcal{D}')$ over $\mathcal{M}$ constructed as follows. For each $j = 1, \ldots, s$, let τ_j be the composition

$$\tau_j = \sigma_{j,1} \circ \cdots \circ \sigma_{j,N_j},$$

where $\sigma_{j,1}$ is the admissible blowing-up whose center is the characteristic cycle $\gamma_j = \gamma_{I_j}$, $\sigma_{j,2}$ is the admissible blowing-up whose center is the non-corner characteristic cycle $\gamma_{I_j,1}$ emerging form γ_{I_j} (cf. Remark 6.1), and so on. Then, $\mathcal{M}'$ is the resulting sequence of admissible blowing-ups by setting $\pi' = \pi \circ \tau_1 \circ \cdots \circ \tau_s$.

Notice that $\mathcal{M}'$ is an adapted reduction of singularities of $\hat{\xi}$ with the same number of non-corner characteristic cycles as $\mathcal{M}$. We put $\mathcal{D}'^{nc} = \{\gamma'_1, \cdots, \gamma'_s\}$ where γ'_j emerges from γ_j by the composition of τ_j. Now, we take $\ell' \in \mathbb{N}$ satisfying $\ell' \geq \max\{\ell, \ell_{\mathcal{M}'} + 1\}$.

Proposition 6.3 *The vector field $\xi_{\ell'}$ satisfies Theorem 1.1.*

Proof Choose $0 < \delta' \leq \delta$ sufficiently small and an open set V_j with $\widetilde{\Sigma}_{N_j,\delta',1}(\hat{S}_{\ell,I_j}) \subset V_j \subset \widetilde{\Sigma}_{N_j,\delta,1}(\hat{S}_{\ell,I_j})$ for any $j = 1, \ldots, s$ so that they satisfy

(a) The Poincaré map P_{ℓ,I_j} is defined in $\widetilde{\Sigma}_{N_j,\delta,1}(\hat{S}_{\ell,I_j}) \cap \{\theta = 0\} = \Sigma_{N_j,\delta}(\Gamma_{\ell,I_j})$ and satisfies there the conclusions of Lemmas 5.7 or 5.8, correspondingly.

(b) If Z is a cycle of the transform $\xi_\ell^{(J_j)}$ contained in V_j, it intersects $\{\theta = 0\}$.

(c) If $j \in \{1, \ldots, r\}$ then $\Gamma_{\ell,I_j} \subset \mathrm{Fix}(P_{\ell,I_j})$ admits a representative in $\widetilde{\Sigma}_{N_j,\delta,1}(\hat{S}_{\ell,I_j})$ denoted again Γ_{ℓ,I_j} whose intersection with $V_j \cap \{\theta = 0\}$ is a connected analytic regular curve.
(d) For any $a \in \Gamma_{\ell,I_j} \cap V_j$ when $j \in \{1, \ldots, r\}$, the cycle of $\xi_\ell^{(J_j)}$ through a is contained in V_j.

The existence of these objects with such properties is guaranteed by standard arguments using the continuity of the flow of $\xi_\ell^{(J_j)}$ and the fact that each γ_j is a cycle of $\xi_\ell^{(J_j)}$. Notice that if $j \in \{r+1, \ldots, s\}$ we can take $\delta' = \delta$ and V_j to be equal to the solid cone $\widetilde{\Sigma}_{N_j,\delta,1}(\hat{S}_{\ell,I_j})$

Define, for $j = 1, \ldots, r$, the set $\widetilde{S}_j$ given by the saturation of $\Gamma_{\ell,I_j} \cap V_j$ by the flow of $\xi_\ell^{(J_j)}$. By the above properties, $\widetilde{S}_j$ is an analytic submanifold of $V_j \subset M$, intersecting the divisor $\pi^{-1}(0)$ along γ_j and completely filled up with cycles of $\xi_\ell^{(J_j)}$. We have, furthermore from (a):

$$\mathcal{C}_{\bigcup_{j=1}^s V_j}(\pi^*\xi_\ell) = \widetilde{S}_1 \cup \cdots \cup \widetilde{S}_r. \tag{51}$$

With the notations of Eq. (3), the diffeomorphism $\psi_{\ell,\ell'} := \psi_{\ell'} \circ \psi_\ell^{-1}$ is analytic and conjugates ξ_ℓ and $\xi_{\ell'}$, namely $\xi_\ell = \psi_{\ell,\ell'}^* \xi_{\ell'}$. Moreover, since $\ell' \geq \ell \geq \ell_{\mathcal{M}} + 1$, we have that $j_\ell(\psi_{\ell,\ell'}) = Id$ and we can apply Proposition 3.16 to $\psi_{\ell,\ell'}$. We obtain an analytic conjugation $\psi_{\ell,\ell'}^{(J_j)}$ between $\xi_\ell^{(J_j)}$ and $\xi_{\ell'}^{(J_j)}$ in a neighborhood of γ_j. Up to shrinking δ and δ', we may assume that $\psi_{\ell,\ell'}^{(J_j)}$ is well defined and one-to-one in V_j for any j. Moreover, $\psi_{\ell,\ell'}^{(J_j)}$ is in the conditions of Lemma 6.2 with respect to the coordinates $(\theta, z^{(J_j)}, \rho^{(J_j)})$ and satisfies $\hat{S}_{\ell,I_j} = (\psi_{\ell,\ell'}^{(J_j)})^*(\hat{S}_{\ell',I_j})$. Let $W_j := \psi_{\ell,\ell'}^{(J_j)}(V_j)$ for $j = 1, \ldots, r$. Using the conclusion of Lemma 6.2 and the fact that $\widetilde{\Sigma}_{N_j,\delta',1}(\hat{S}_{\ell,I_j}) \subset V_j$ for any j, we have that each W_j has a solid cone around $\hat{S}_{\ell',I_j}$ of the form $\widetilde{\Sigma}_{N_j,\delta',1}(\hat{S}_{\ell',I_j})$ (same order N_j).

We deduce from (51) and by conjugation the following:

$$\mathcal{C}_{\bigcup_{j=1}^s W_j}(\pi^*\xi_{\ell'}) = \widetilde{S}'_1 \cup \cdots \cup \widetilde{S}'_r, \tag{52}$$

where $\widetilde{S}'_j := (\psi_{\ell,\ell'}^{(J_j)})(\widetilde{S}_j) \subset W_j$, for $j = 1, \ldots, r$. By Remark 6.1 and taking into account that $\ell \geq l + N_j + 1$ for any j, we have that $W'_j := \tau_j^{-1}(W_j)$, together with the intersection of its closure with the divisor $\tau_j^{-1}(\gamma_j)$, is a neighborhood of γ'_j in M'. We may assume that $W'_j \cap W'_k = \emptyset$ if $j \neq k$. Complete the union $\bigcup_{j=1}^s W'_j$ to a neighborhood W' of $\mathrm{Supp}(\mathcal{D}')$ in M' adding two by two disjoint neighborhoods of the elements $\gamma \in \mathcal{D}' \setminus \mathcal{D}'^{nc}$ where, correspondingly, Propositions 5.1 or 5.2 holds. We apply finally Proposition 4.1 to W' (recall $\ell' \geq \ell_{\mathcal{M}'} + 1$): there is a neighborhood U of $0 \in \mathbb{R}^3$ such that $(\pi')^{-1}(\mathcal{C}_U(\xi_{\ell'})) \subset W'$. By Propositions 5.1 or 5.2 we get moreover that

$$(\pi')^{-1}(\mathcal{C}_U(\xi_{\ell'})) \subset \bigcup_{j=1}^s W'_j.$$

This equation, together with (52) shows that, if we put $S_j := \pi(\widetilde{S}'_j) \cap U$ for $j = 1, \ldots, r$, then

$$\mathcal{C}_U(\xi_{\ell'}) \subset S_1 \cup \cdots \cup S_r. \tag{53}$$

Notice that each S_j is an analytic smooth surface in $U\setminus\{0\}$ and that $0 \in \bar{S}_j$. Up to taking a smaller U, we may assume that $\bar{S}_j \cap U = S_j \cup \{0\}$. Moreover, S_j is a subanalytic set, since π is proper and $\widetilde{S}'_j$ is semi-analytic. Finally, by construction, we have that $S_j \subset \pi(\widetilde{S}'_j)$ for $j = 1, \ldots, r$, the two sets have the same germ at $0 \in \mathbb{R}^3$ and the later is entirely composed of cycles for every $j = 1, \ldots, r$. This implies, together with (53) that, if $V \subset U$ is any open neighborhood of $0 \in \mathbb{R}^3$ such that $\mathrm{Fr}(V) \cap S_j$ coincides with one of such cycles for every $j = 1, \ldots, r$, then we have

$$\mathcal{C}_V(\xi_{\ell'}) = (S_1 \cup \cdots \cup S_r) \cap V.$$

This ends the proof. □

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