Fractional radial transport in cylindrical geometry

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A transport equation is derived from microscopic considerations, aimed at modeling fractional radial transport in cylindrical-like geometries. The procedure generalizes existing work on one-dimensional Cartesian systems. The transport equation emerges as the fluid limit of an underlying continuous-time random walk (CTRW) that preserves the required symmetries and conservation laws. In the process, appropriate radial fractional operators are identified and defined through their Hankel transforms, providing a smooth interpolation between standard radial differential operators. Finally, propagators for the radial fractional transport equation are obtained in terms of Fox H functions.

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I. INTRODUCTION

Over the past few decades, one-dimensional (1D) fractional transport equations of the form

$$\frac{\partial n}{\partial t} = {}_0D_t^{1-\beta}\left[\kappa \frac{\partial^\alpha n}{\partial |x|^\alpha}\right] + S(x,t) \tag{1}$$

have been proposed as more suitable than classical diffusion for systems exhibiting nonlocality and memory [1,2]. Here, $n(x,t)$ represents a particle or mass density (or any other conserved field). On the other hand, ${}_0D_t^s$ is the Riemann–Liouville fractional derivative of order s, and $\partial^b/\partial|x|^b$ is the Riesz derivative of order b [3,4]. The parameter ranges (0,1] for β and (0,2] for α ensure positivity of $n(x,t)$ at all times $t>0$ if integration starts from a positive initial condition $n(x,0)>0,\ \forall x$. $\kappa>0$ is a constant diffusivity and $S(x,t)$ is an external source. Such models have found application to transport across fractal landscapes [5], crack propagation in ice [6], tracer transport by turbulence [7], and—of particular relevance here—energy, momentum, and particle transport driven by near-marginal turbulence along the minor radius of a tokamak [8–10]. In these systems, the absence of characteristic spatial and temporal scales over broad ranges produces self-similarity and memory effects that classical diffusion cannot capture [1,2]. Fractional derivatives, being nonlocal operators that account for the system's full spatial and temporal extent, can incorporate these features.

Tokamaks are toroidal devices where a plasma is confined by the magnetic field generated by surrounding magnetic coils with the intent of producing fusion [11]. Radial transport in tokamaks is often modeled in a simplified cylindrical geometry, accurate for large aspect ratio $\epsilon = R/a$, where R and a are the major and minor radii of the tokamak. A straight periodic cylinder thus represents the plasma column, and transport is described by the advection-diffusion equation,

$$\frac{\partial n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma_r) + S(r,t), \qquad \Gamma_r = Vn - D\frac{\partial n}{\partial r}, \tag{2}$$

where V is the pinch velocity, D the diffusion coefficient, and S an external source. The aim of this paper is to construct a fractional counterpart to Eq. (2), focusing on its diffusive term, for transport regimes where nonlocality and memory are significant.

The classical diffusion equation arises as the fluid limit of a one-dimensional, homogeneous, temporally invariant continuous-time random walk (CTRW) [12]. Its fractional form, Eq. (1), follows when scale invariance is incorporated [13,14]. For that reason, our derivation starts from a two-dimensional (2D), isotropic, and homogeneous CTRW whose fluid limit will provide the desired fractional radial transport equation, together with a suitable fractional generalization of the radial part of the cylindrical Laplacian,

$$\Delta_r^2 := \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{3}$$

The resulting equation is applicable beyond tokamak physics, to any cylindrical-like system with negligible azimuthal variation.

The paper is thus organized as follows. The solution of the general d-dimensional CTRW model is reviewed, in terms of its two defining probability density functions (pdfs), in Sec. II. The pdfs that are more appropriate to build an isotropic, homogeneous 2D CTRW are discussed in Sec. III. The fluid limit of this 2D CTRW is calculated in Sec. IV, resulting in the proposed fractional radial transport equation. Expressions amenable of numerical computation for the propagators of the

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fractional radial transport equation are obtained in Sec. V. Finally, some conclusions are drawn in Sec. VI.

II. d-DIMENSIONAL CTRW MODEL: GENERAL SOLUTION

In an unbounded, temporally invariant, homogeneous and d-dimensional CTRW, each "walker" changes its position $\mathbf{r} \to \mathbf{r} + \Delta\mathbf{r}$, with both $\mathbf{r}$ and $\Delta\mathbf{r}$ vectors in $\mathbb{R}^d$, after having remained at rest for a waiting-time $\tau > 0$. The pdfs chosen for the waiting time τ, $\psi(\tau)$, and for the step size $\Delta\mathbf{r}$, $p(\Delta\mathbf{r})$, fully define the CTRW. Its solution requires the calculation of the probability of finding a walker at any position $\mathbf{r}$ at an arbitrary time $t > 0$, $G(\mathbf{r}, t|\mathbf{r}_0, 0)$, given its initial location $\mathbf{r}_0$. $G(\mathbf{r}, t|\mathbf{r}_0, 0)$ is often referred to as the propagator. Once it is known, the density of walkers at any time is given by the convolution

$$n(\mathbf{r}, t) = \int_0^t dt' \int d\mathbf{r}' G(\mathbf{r}, t|\mathbf{r}', t') S_{\text{aug}}(\mathbf{r}', t'), \quad (4)$$

with $S_{\text{aug}}(\mathbf{r}, t) = S(\mathbf{r}, t) + n_0(\mathbf{r})\delta(t)$, $n_0(\mathbf{r}) = n(\mathbf{r}, 0)$ being the initial walker distribution and with $S(r, t)$ including all external sources of walkers. The propagator for the d-dimensional CTRW can be evaluated in the following way [12]. First, by rewriting it as,

$$G(\mathbf{r}, t) = \int_0^\infty d\tau \eta(t-\tau) Q(\mathbf{r}, \tau), \quad (5)$$

where $Q(\mathbf{r}, \tau)$ is the total probability of the walker arriving at $\mathbf{r}$ at time τ and $\eta(t-\tau)$ gives the probability of staying there until time t. Next, Fourier transforms in space and (unilateral) Laplace transforms in time are used to take advantage of the unboundedness of the domain. Transforms will be distinguished from space-time functions by means of an "overbar" $[\bar{f}(k)]$ (Fourier transform in space, only), a "tilde" $[\tilde{f}(s)]$ (Laplace transform in time, only), or a "hat" $[\hat{f}(k, s)]$ (Fourier and Laplace) as well as by their dependencies on k and/or s. Thus, Laplace-Fourier transforming Eq. (5), one gets to

$$\hat{G}(\mathbf{k}, s) = \tilde{\eta}(s)\hat{Q}(\mathbf{k}, s), \quad (6)$$

with $\mathbf{k} \in \mathbb{R}^d$ and $s \in \mathbb{C}$. Next, $\eta(t)$ is found by considering that

$$\eta(t) = 1 - \int_0^t \psi(t')dt' = \int_t^\infty \psi(t')dt', \quad (7)$$

since $\psi(t)$ is the probability of the next waiting time being t. By Laplace transforming Eq. (7), it is obtained that

$$\tilde{\eta}(s) = \frac{1 - \tilde{\psi}(s)}{s}. \quad (8)$$

Regarding the probability $Q(\mathbf{r}, \tau)$, it is rewritten as

$$Q(\mathbf{r}, \tau) = \sum_{j=0}^\infty Q^j(\mathbf{r}, \tau) = \sum_{j=0}^\infty \psi_j(\tau) G_j(\mathbf{r}), \quad (9)$$

where Q^j is the total probability of arriving at the location $\mathbf{r}$ in j jumps at time τ, ψ_j is the probability of the particle executing its jth jump at time τ, and $G_j(\mathbf{r})$ is the probability of reaching $\mathbf{r}$ in j steps. ψ_j is obtained from the recursive relation:

$$\psi_j(\tau) = \int_0^\tau \psi(\tau - \tau')\psi_{j-1}(\tau')d\tau', \;\; \psi_1(\tau) = \psi(\tau). \quad (10)$$

Laplace transforming Eq. (10), it follows that $\hat{\psi}_j(s) = [\hat{\psi}(s)]^j$. $G_j(\mathbf{r})$ satisfies the recursive relation,

$$G_j(\mathbf{r}) = \int p(\mathbf{r} - \mathbf{r}')G_{j-1}(\mathbf{r}')d\mathbf{r}', \quad (11)$$

that, moving to Fourier space, becomes $\bar{G}_j(\mathbf{k}) = \bar{p}(\mathbf{k})\bar{G}_{j-1}(\mathbf{k})$. Exploiting recursiveness it is found that $\bar{G}_j(\mathbf{k}) = [\bar{p}(\mathbf{k})]^j$. Collecting all these results together, the CTRW propagator comes to be [12]

$$\hat{G}(\mathbf{k}, s) = \frac{1 - \tilde{\psi}(s)}{s} \sum_{j=0}^\infty [\tilde{\psi}(s)\bar{p}(\mathbf{k})]^j = \frac{[1 - \tilde{\psi}(s)]/s}{1 - \tilde{\psi}(s)\bar{p}(\mathbf{k})}. \quad (12)$$

The space-time expression of the CTRW propagator can be obtained by inverting Eq. (12) via Laplace-Fourier transforms, once the waiting time and step pdfs are specified.

III. SUITABLE PDFS FOR ISOTROPIC 2D CTRWS

The specific choices made for the waiting time and the step-size pdfs define the CTRW and set its properties. It is worthwhile to start this section by reviewing the choices that lead to the classical diffusion equation and the 1D fractional diffusion equation [Eq. (1)].

A. One-dimensional case

One-dimensional diffusive transport requires the existence of well-defined temporal and spatial characteristic transport scales that can be introduced through the average waiting time, τ_0, and the step-size standard deviation, σ. These are, respectively, the first and second moment of $\psi(\tau)$ and $p(\Delta x)$. Under these conditions, the fluid limit of the CTRW naturally connects the CTRW with the usual diffusive equation [12]. The traditional diffusive choices are an exponential, $\psi(\tau) = \tau_0^{-1}\exp(-\tau/\tau_0)$, for waiting times, and a Gaussian distribution, $p(\Delta x) = (2\pi\sigma^2)^{-1/2}\exp(-(\Delta x)^2/2\sigma^2)$, for the step sizes. Physically, the Gaussian choice reflects the fact that mesoscopic displacements are the result of an underlying random additive process (for instance, of interparticle collisions in a gas or plasma) [15]. Similarly, memoryless processes are often Poisson processes, which leads to an exponential waiting-time pdf [16].

Alternative choices must be made if transport is self-similar or exhibits memory. The central limit theorem advises to pick $\psi(\tau)$ and $p(\Delta x)$ from the family of Lévy strictly stable distributions [2], since they are attractors of additive processes if characteristic scales are absent [17]. This is reflected in the first moment of $\psi(\tau)$ and the second moment of $p(\Delta x)$ which diverge. Most Lévy distributions lack an analytic form, but are defined via the characteristic function (the Fourier transform of the pdf)

$$\bar{L}_{\alpha|\lambda,\sigma}(k) = \exp\left\{-\sigma^\alpha |k|^\alpha \left[1 - \imath\lambda \,\text{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right\} \quad (13)$$

for $0 < \alpha \leqslant 2,\ \alpha \neq 1$ (here, $\imath = \sqrt{-1}$), and

$$\bar{L}_{1|0,\sigma}(k) = \exp\left\{-\sigma^\alpha |k|^\alpha \left[1 + \imath\left(\frac{2}{\pi}\right)\mathrm{sgn}(k)\ln|k|\right]\right\} \quad (14)$$

if $\alpha = 1$. The most important parameter is α due to the asymptotic scaling [17]:

$$L_{\alpha|\beta,\sigma}(x) \sim (1 \pm \lambda)\sigma^\alpha |x|^{-(1+\alpha)}, \quad x \to \pm\infty. \quad (15)$$

As a result, their second moment diverges if $\alpha < 2$ and also their first moment if $0 < \alpha \leqslant 1$. With regard to the other two parameters, λ varies between -1 and 1 and determines the degree of symmetry (around $x = 0$) of the distribution. Symmetry ensues only for $\lambda = 0$ [i.e., $L_{\alpha|0,\sigma}(x) = L_{\alpha|0,\sigma}(-x)$]. A particularly interesting pdf is one in which $0 < \alpha < 1$ and $\lambda = \pm 1$. They are known as extremal Lévy pdfs and are one-sided in the sense that they are only defined for positive or negative values of x (depending on the sign of λ), maintaining the tail power-law scaling [Eq. (15)] on that side. Finally, σ is a scaling parameter in the sense that

$$L_{\alpha|\lambda,\sigma}(ax) = L_{\alpha|\mathrm{sgn}(a)\lambda,|a|\sigma}(x). \quad (16)$$

A CTRW leading to transport with self-similar features and memory can thus be built with the choice [13,14]

$$p(\Delta x) = L_{\alpha|0,\sigma}(\Delta x), \quad (0 < \alpha < 2; \sigma > 0) \quad (17)$$

and

$$\psi(\tau) = L_{\beta|1,\tau_0}(\tau), \quad (0 < \beta < 1, \tau_0 > 0) \quad (18)$$

that lead directly to the fractional transport equation, Eq. (1), as we will review later.

B. d-dimensional α-stable distributions

The central limit theorem also provides d-dimensional versions of α-stable distributions [17], which may be used to construct CTRWs in higher dimensions. Introducing the d-dimensional random vector $\mathbf{r} = (x_1, x_2, \cdots, x_d)$, the characteristic function of the α-stable, d-dimensional pdf is

$$\bar{L}^d_{\alpha|\Gamma}(\mathbf{k}) = \exp\left\{-\int_{S_d} |\mathbf{k}\cdot\mathbf{s}|^\alpha \left(1 - \imath\,\mathrm{sgn}(\mathbf{k}\cdot\mathbf{s})\tan\left(\frac{\pi\alpha}{2}\right)\right)\Gamma(d\mathbf{s})\right\} \quad (19)$$

if $0 < \alpha < 2$ and $\alpha \neq 1$, and

$$\bar{L}^d_{\alpha|\Gamma}(\mathbf{k}) = \exp\left\{-\int_{S_d} |\mathbf{k}\cdot\mathbf{s}|^\alpha \left(1 + \imath\left(\frac{2}{\pi}\right)\mathrm{sgn}(\mathbf{k}\cdot\mathbf{s})\ln(\mathbf{k}\cdot\mathbf{s})\right)\Gamma(d\mathbf{s})\right\} \quad (20)$$

for $\alpha = 1$. Here, $\mathbf{k} \in \mathbb{R}^d$ and S_d is the $(d-1)$-dimensional sphere of radius 1 centered at $\mathbf{k} = 0$. For example, for $d = 1$, the sphere is just the set containing the two points -1 and 1. For $d = 2$, S_2 is just a circle with unit radius, S_3 is the sphere of unit radius, and so forth. The vector $\mathbf{s}$ is a unit vector connecting $\mathbf{k} = 0$ with any point on S_d, and $d\mathbf{s}$ represents a surface element on S_d. Finally, $\Gamma(d\mathbf{s}) \geqslant 0$ is a "finite measure" whose integral over S_d must be positive.

Now, the stable distribution is defined by the choices made for the parameter α and the measure Γ. Equations (19) and (20) differ from their 1D counterparts [i.e., Eqs. (13) and (14)] in the integration over S_d. The integral can be interpreted by saying that, since the orientation of the random vector $\mathbf{k}$ is arbitrary, each of its possible directions (determined by $\mathbf{s}$) is treated as an α-stable random variable that contributes to the integral according to the "weight" assigned to it by $\Gamma(d\mathbf{s})$. The measure Γ is not a probability measure [17]. This is clear if we examine the case $d = 1$. The integral then reduces to a sum over the two points that form the 0d sphere, $+1$ and -1:

$$\begin{aligned}\hat{L}^1_{\alpha|\Gamma}(k) &= \exp\left\{-\left[|k|^\alpha\left(1 - \imath\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\Gamma(+1)\right.\right.\\ &\quad \left.\left.+|-k|^\alpha\left(1 - \imath\,\mathrm{sgn}(-k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\Gamma(-1)\right]\right\}\\ &= \exp\left\{-|k|^\alpha\left([\Gamma(1)+\Gamma(-1)] - \imath\,\mathrm{sgn}(k)\right.\right.\\ &\quad \left.\left.\times\tan\left(\frac{\pi\alpha}{2}\right)[\Gamma(1)-\Gamma(-1)]\right)\right\},\end{aligned} \quad (21)$$

that becomes the 1D Lévy α-stable distribution, Eq. (13), after defining $\sigma = [\Gamma(1)+\Gamma(-1)]^{1/\alpha}$ and $\lambda = (\Gamma(1) - \Gamma(-1))/(\Gamma(1)+\Gamma(-1))$. Symmetry (i.e., $\lambda = 0$) requires a uniform measure [i.e., $\Gamma(1) = \Gamma(-1)$]. Since $\Gamma(1)+\Gamma(-1)$ is in general not 1, the "measure" is not a "probability measure." In fact, this sum gives the scaling parameter σ.

C. Isotropic, two-dimensional α-stable distributions

Moving on to the 2D isotropic case, which is relevant to the discussion of radial transport in cylindrical coordinates, its characteristic function can easily be recast into the expressions [17]

$$\bar{L}^2_{\alpha|\Gamma}(\mathbf{k}) = \exp\left\{-k^\alpha\left(\phi_1(\theta_k) - \imath\phi_2(\theta_k)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\},\ \alpha \neq 1, \quad (22)$$

and

$$\bar{L}^2_{1|\Gamma}(\mathbf{k}) = \exp\left\{-k^\alpha\left(\phi_1(\theta_k) + \imath\left(\frac{2}{\pi}\right)\phi_2(\theta_k)\ln(k)\right)\right\},\ \alpha = 1. \quad (23)$$

where, for convenience, the following quantities

$$\phi_1(\theta_k) := \int_0^{2\pi} |\cos(\theta - \theta_k)|^\alpha \Gamma(\theta)d\theta, \quad (24)$$

$$\phi_2(\theta_k) := \int_0^{2\pi} \mathrm{sgn}[\cos(\theta - \theta_k)]|\cos(\theta - \theta_k)|^\alpha \Gamma(\theta)d\theta \quad (25)$$

have been introduced. Here, $k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$ and $\theta_k = \tan^{-1}(k_y/k_x)$. To get to Eqs. (22) and (23), start with Eqs. (19) and (20), with C_2 being the unit circle in 2D Fourier space. Then, use that $\mathbf{s} = \mathbf{u}_r(\theta)$, the polar radial unit vector with θ the azimuthal angle in $\mathbf{k}$ space. The wave vector is expressed as $\mathbf{k} = k\mathbf{u}_r(\theta_k)$. The measure is written as $\Gamma(d\mathbf{s}) = \Gamma(\theta)d\theta$. Clearly, ϕ_1 is a scaling coefficient, whilst ϕ_2 quantifies the degree of azimuthal symmetry of the measure $\Gamma(\theta)$. The isotropic case corresponds to $\Gamma(\theta) = \Gamma_0/2\pi$,which leads to $\phi_2 = 0$, implying that all radial directions are equivalent, and $\phi_1 = \Gamma_0 \int_0^1 |\cos(2\pi\theta)|^\alpha d\theta := \sigma^\alpha$, the scaling parameter. The characteristic function of the isotropic, 2D α-stable pdf is then

$$\bar{L}^2_{\alpha|\Gamma}(\mathbf{k}) = L^2_{\alpha|\sigma}(k) = \exp\{-\sigma^\alpha k^\alpha\},\ k \geqslant 0. \tag{26}$$

It should be emphasized that, in contrast to what happens in the 2D Gaussian case ($\alpha = 2$), the characteristic function of the isotropic, 2D α-stable pdf is not separable. That is, it is not equivalent to the product of two pdfs associated to displacements along perpendicular directions (say, x and y), each following the same α-stable pdf:

$$\exp\{-\sigma^\alpha |k_x|^\alpha\} \cdot \exp\{-\sigma^\alpha |k_y|^\alpha\} \neq \exp\{-\sigma^\alpha k^\alpha\}. \tag{27}$$

As a result, fractional radial diffusion cannot be obtained as the combination of two independent, perpendicular fractional transport processes.

IV. FLUID LIMIT OF THE 2D ISOTROPIC CTRW

Effective models capable of capturing the long-term, large-distance transport dynamics of CTRWs can be obtained by calculating their fluid limit [13,14]. This corresponds to considering the asymptotic behavior for $k \to 0$ and $s \to 0$ of the CTRW solution [Eq. (12)],

$$\hat{n}(\mathbf{k}, s) = \hat{G}(k, s)\bar{n}_0(\mathbf{k}) = \frac{[1 - \tilde{\psi}(s)]/s}{1 - \tilde{\psi}(s)\bar{p}(\mathbf{k})}\bar{n}_0(\mathbf{k}), \tag{28}$$

where $\bar{n}_0(\mathbf{k})$ is the Fourier transform of the initial condition. It is however convenient to recast Eq. (28) into the form

$$s\hat{n}(\mathbf{k}, s)[1 - \tilde{\psi}(s)\bar{p}(\mathbf{k})] = [1 - \tilde{\psi}(s)]\bar{n}_0(\mathbf{k}) \tag{29}$$

before taking the fluid limit. As before, we discuss first the one-dimensional cases.

A. One-dimensional case

1. Classical diffusion

In the diffusive case, waiting times follow an exponential pdf, $\psi(\tau) = \tau_0^{-1}\exp(-\tau/\tau_0)$, and step sizes a Gaussian pdf, $p(\Delta x) = (2\pi\sigma)^{-1}\exp(-|\Delta x|^2/2\sigma^2)$. Their respective Laplace and Fourier transform behave asymptotically as

$$\bar{p}(k) \sim 1 - \sigma^2 k^2,\ k \to 0, \tag{30}$$

$$\tilde{\psi}(s) \sim 1 - \tau_0 s,\ s \to 0. \tag{31}$$

Inserting these scalings into Eq. (29), and keeping only the lowest order, one finds

$$s\hat{n}(k, s)[\tau_0 s + \sigma^2 k^2] = \tau_0 s\bar{n}_0(k), \tag{32}$$

which can be reordered as

$$s\hat{n}(k, s) - \bar{n}_0(k_x) = -\frac{\sigma^2}{\tau_0}k_x^2, \tag{33}$$

whose Laplace-Fourier inverse leads to the (unbounded) classical diffusive equation:

$$\frac{\partial n}{\partial t} = \kappa_{2,1}\frac{\partial^2 n}{\partial x^2}. \tag{34}$$

The diffusivity, $\kappa_{2,1} = \sigma^2/\tau_0$, is determined by the characteristic lengthscale σ, the step-size standard deviation, and timescale τ_0, the mean waiting time. One only needs to add suitable boundary conditions plus an adequate choice for $n(x, 0)$ to use it.

2. Fractional diffusion

Characteristic scales are removed by choosing waiting time and step-size pdfs within the α-stable Lévy family [13,14],

$$p(\Delta x) = L_{\alpha|0,\sigma}(\Delta x), \quad \psi(\Delta t) = L_{\beta|1,\tau_0}(\tau), \tag{35}$$

whose asymptotic behavior is ($k, s \to 0$),

$$\bar{p}(k) \sim 1 - \sigma^\alpha |k|^\alpha, \quad \tilde{\psi}(s) \sim 1 - \tau_0^\beta s^\beta, \tag{36}$$

leading to, when inserted into Eq. (29),

$$s\hat{n}(k_x) - \bar{n}_0(k) = -s^{1-\beta}\left(\frac{\sigma^\alpha}{\tau_0^\beta}|k_x|^\alpha\right), \tag{37}$$

whose Laplace-Fourier inverse yields the 1D fractional diffusion equation [Eq. (1)]:

$$\frac{\partial n}{\partial t} = {}_0D_t^{1-\beta}\left[\kappa_{\alpha,\beta}\frac{\partial^\alpha n}{\partial |x|^\alpha}\right]. \tag{38}$$

Nonlocal and memory effects are now kept in the transport equation. The fractional diffusivity is $\kappa_{\alpha,\beta} = \sigma^\alpha/\tau_0^\beta$. For $\alpha \to 2$ and $\beta \to 1$, Eq. (26) is recovered. Again, suitable initial and boundary conditions must be given for the problem to be well defined. Imposing the latter may be a subtle process, though, since it may require the truncation of the fractional operators [18].

B. Two-dimensional isotropic case

The isotropic CTRW requires a step-size pdf that satisfies $p(\Delta\mathbf{r}) = p(|\Delta\mathbf{r}|) = p(\Delta r)$. For the 2D, isotropic CTRW of interest to us, appropriate choices are

$$p(\Delta\mathbf{r}) = L^2_{\alpha|0,\Gamma}(\Delta r), \quad \psi(\Delta t) = L_{\beta|1,\tau}(\tau), \tag{39}$$

with $\Gamma(\theta) = \Gamma_0/2\pi$. The asymptotic behaviors of their transforms are ($k, s \to 0$):

$$\bar{p}(k) \sim 1 - \sigma^\alpha k^\alpha, \quad \tilde{\psi}(s) \sim 1 - \tau^\beta s^\beta, \tag{40}$$

where $k = (k_x^2 + k_y^2)^{1/2} > 0$ is the radial wave number. Inserting them into Eq. (29),

$$s\hat{n}(k, s) - \bar{n}_0(k) = -s^{1-\beta}\left(\frac{\sigma^\alpha}{\tau^\beta}k^\alpha\right), \tag{41}$$

whose Laplace-Fourier inverse yields the sought fractional radial transport equation. It should be emphasized that it has been implicitly assumed here that the initial walker density $n_0(\mathbf{r}) = n_0(r)$. That is, it exhibits rotational symmetry in space with respect to a certain origin that defines $\mathbf{r} = 0$. Otherwise, $\bar{n}_0(\mathbf{k}) \neq \bar{n}_0(k)$. The isotropic CRTW ensures that this symmetry is maintained at all later times. An external source can also be considered as long as it has the same symmetry. The Laplace inversion of Eq. (41) is

$$\frac{\partial \bar{n}(k,t)}{\partial t} = {}_0D_t^{1-\beta}[-\kappa_{\alpha,\beta}k^\alpha \bar{n}(k,t)]. \tag{42}$$

The Fourier inverse of this equation can be done in terms of the zeroth-order Hankel transform, H_0, that is reviewed in Appendix A. The result can be expressed formally as

$$\frac{\partial n(r,t)}{\partial t} = {}_0D_t^{1-\beta}\left[\kappa_{\alpha,\beta}\Delta_r^\alpha \cdot n(r,t)\right], \tag{43}$$

where a class of fractional radial differential operator has been introduced,

$$\begin{aligned}\Delta_r^\alpha \cdot n(r,t) &\equiv H_0^{-1}[-k^\alpha \bar{n}(k,t)] \\ &= -\int_0^\infty k^{\alpha+1}J_0(rk)\bar{n}(k,t)dk,\end{aligned} \tag{44}$$

with $J_0(x)$ the usual Bessel function. Equation (43), the sought fractional 2D radial transport equation, is the main result of the paper. Once suitable initial and boundary conditions are added, the fractional transport problem becomes well poised and ready for numerical solution, since the action of the radial operator Δ_r^α must be evaluated numerically. This often requires, as in the cartesian case previously discussed, the truncation of the spatial integrals to the system of interest [18]. If $\alpha = 2$, Eq. (43) reduces to the isotropic member of the family of equations known as the time-fractional 2D radial diffusive equation [19,20]:

$$\frac{\partial n(r,t)}{\partial t} = {}_0D_t^{1-\beta}\left[D_{2,\beta}\Delta_r^2 n\right] = {}_0D_t^{1-\beta}\left[D_{2,\beta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right)\right]. \tag{45}$$

In addition, if one also sets $\beta = 1$, the classical radial diffusion equation is obtained.

V. PROPAGATORS OF THE FRACTIONAL RADIAL TRANSPORT EQUATION IN TERMS OF GENERALIZED FOX *H* FUNCTIONS

The general solution of Eq. (43) can be written in terms of its propagator, $G_{\beta;\alpha}(r,t|r_0,t_0)$, that gives the time evolution of the initial condition $\delta(t-t_0)\delta(r-r_0)/(2\pi r)$ [21]. Once the propagator is known, the evolution of the arbitrary initial condition $n(r,0)$ is

$$n(r,t) = \int_0^t dt' \int_0^\infty dr' G_{\beta;\alpha}(r,t|r',t')S_{\rm aug}(r',t'), \tag{46}$$

where the augmented source $S_{\rm aug}(r,t) = n(r,0)\delta(t) + S(r,t)$ includes the possibility of an external source. In this section we provide expressions for the propagator of Eq. (43) in terms of Fox generalized H functions (reviewed in Appendix C), which can be evaluated using well-known numerical techniques [22].

The procedure starts by noting that the Hankel transform of the propagator evolves according to the equation [see Eq. (42)]

$$\begin{aligned}\frac{\partial \bar{G}_{\beta;\alpha}(k,t)}{\partial t} &= {}_0D_t^{1-\beta}[-Dk^\alpha \bar{G}_{\beta;\alpha}(k,t)], \\ \bar{G}_{\beta,\alpha}(k,t_0) &= J_0(kr_0).\end{aligned} \tag{47}$$

Applying now the unilateral Laplace transform in time,

$$s\bar{G}_{\beta;\alpha}(k,s) - J_0(kr_0) = s^{1-\beta}[-Dk^\alpha \bar{G}_{\beta;\alpha}(k,s)], \tag{48}$$

the Hankel-Laplace transform of the propagator is found to be

$$\bar{G}_{\beta;\alpha}(k,s) = \frac{s^{\beta-1}J_0(kr_0)}{s^\beta + Dk^\alpha}. \tag{49}$$

To calculate its space-time form, its temporal Laplace inverse is first evaluated in terms of the Mittag-Leffler function $E_\beta(z)$ (see Appendix B),

$$\bar{G}_{\beta;\alpha}(k,t) = J_0(kr_0)E_\beta(-Dk^\alpha(t-t_0)^\beta). \tag{50}$$

Second, the Hankel inversion in space requires to evaluate the integral,

$$G_{\beta;\alpha}(r,t|r_0,t_0) = \int_0^\infty kE_\beta(-Dk^\alpha(t-t_0)^\beta)J_0(kr_0)J_0(kr)dk. \tag{51}$$

To advance, one must note that both the Bessel and Mittag-Leffler functions can be expressed in terms of Fox H functions (see Appendix C). In particular, using Eq. (C9) combined with the scaling property Eq. (C2), one can express the Mittag-Leffler function as

$$\begin{aligned}&E_\beta(-Dk^\alpha(t-t_0)^\beta) \\ &= H_{1,2}^{1,1}\left[Dk^\alpha(t-t_0)^\beta \middle| \begin{matrix}(0,1) & -- \\ (0,1) & (0,\beta)\end{matrix}\right] \\ &= \left(\frac{\alpha}{2}\right)H_{1,2}^{1,1}\left[D^{2/\alpha}(t-t_0)^{2\beta/\alpha}k^2 \middle| \begin{matrix}(0,2/\alpha) & -- \\ (0,2/\alpha) & (0,2\beta/\alpha)\end{matrix}\right]\end{aligned} \tag{52}$$

and, with the help of Eq. (C10), the Bessel function becomes

$$J_0(kr_0) = H_{0,2}^{1,0}\left[\frac{(kr_0)^2}{4}\middle| \begin{matrix}-- & -- \\ (0,1) & (0,1)\end{matrix}\right]. \tag{53}$$

Using Eqs. (52) and (53), the space-time expression for the propagator becomes the Hankel transform of the product of two different Fox functions, that can itself be expressed as another generalized H function thanks to the identity, Eq. (C24),

$$\begin{aligned} G_{\beta;\alpha}(r,t|r_0,t_0) &= \frac{\alpha}{2}\int_0^\infty H_{1,2}^{1,1}\left[D^{\frac{2}{\alpha}}(t-t_0)^{\frac{2\beta}{\alpha}}k^2\left|\begin{matrix}(0,\frac{2}{\alpha})(--)\\(0,\frac{2}{\alpha})(0,\frac{2\beta}{\alpha})\end{matrix}\right.\right]\cdot H_{0,2}^{1,0}\left[\frac{(kr_0)^2}{4}\left|\begin{matrix}(--)(--)\\(0,1)(0,1)\end{matrix}\right.\right]kJ_0(kr)dk \\ &= \frac{\alpha}{r^2}H^{1,1,0,1,1}_{2,(1,0),0,(2;2)}\left[\begin{matrix}\left(\frac{(2D^{1/\alpha}(t-t_0)^{\beta/\alpha})}{r}\right)^2\\ \\ \left(\frac{r_0}{r}\right)^2\end{matrix}\left|\begin{matrix}(1,1)(1,1)\\(0,1)(--);(--)(--)\\ ---\\(0,\frac{2}{\alpha})\,(0,\frac{2\beta}{\alpha});(0,1)(0,1)\end{matrix}\right.\right]. \end{aligned} \tag{54}$$

This is the final expression for the propagator. Although it looks very complicated, it is amenable for numerical calculation [23]. In the special case $\beta = 1$, Eq. (54) can be somewhat simplified. Indeed, by using both Eq. (C5) and the scaling property Eq. (C19), one finds

$$\begin{aligned} G_{1;\alpha}(r,t|r_0,t_0) &= \frac{\alpha}{2}\int_0^\infty H_{0,1}^{1,0}\left[D^{\frac{2}{\alpha}}(t-t_0)^{\frac{2}{\alpha}}k^2\left|\begin{matrix}--\\(0,\frac{2}{\alpha})\end{matrix}\right.\right]\cdot H_{0,2}^{1,0}\left[\frac{(kr_0)^2}{4}\left|\begin{matrix}-- & --\\(0,1) & (0,1)\end{matrix}\right.\right]kJ_0(kr)dk = \\ &= \frac{\alpha^3}{4r^2}H^{1,0,0,1,1}_{2,(0,0),0,(2;1)}\left[\begin{matrix}\left(\frac{r_0}{r}\right)^\alpha\\ \\ \frac{2^\alpha D(t-t_0)}{r^\alpha}\end{matrix}\left|\begin{matrix}(1,\frac{\alpha}{2})(1,\frac{\alpha}{2})\\(--)(--);(--)\\ ---\\(0,\frac{\alpha}{2})(0,\frac{\alpha}{2});\;(0,1)\end{matrix}\right.\right]. \end{aligned} \tag{55}$$

VI. CONCLUSIONS

We have derived a fractional equation [Eq. (43)] for radial transport in two dimensions by calculating the fluid limit of a 2D isotropic CTRW constructed to lack characteristic scales in both space and time. This equation captures the long-term, large-distance transport dynamics of the unbounded, isotropic, and homogeneous 2D CTRW along the radial direction. The equation contains two fractional exponents, a spatial one ($0 < \alpha \leqslant 2$) and a temporal one ($0 < \beta \leqslant 1$). Several methods to estimate these exponents from experimental data are already available in the literature (see, for instance, Refs. [24], Sec. 6.1.4, and references therein). Some of these methods are based on reconstructing the propagator of the fractional transport equation from experimental data. To support this type of analysis, we have computed explicit expressions, in terms of generalized H functions, for the propagators [Eqs. (54) and (55)] of Eq. (43) for admissible values of α and β. With the help of available techniques for the numerical evaluation of generalized H functions (see, for instance, Ref. [22]), these expressions make it possible to evolve in time any initial condition that is independent of the azimuthal angle, including any external sources with the same symmetry. We expect these results to be useful for constructing effective transport models for any physical system in which radial transport exhibits strong nondiffusive features, as appears to be the case in near-marginal tokamak plasmas. We also believe that our results represent a significant improvement over the Cartesian 1D fractional transport equations that have often been used for this purpose [7,9,10,25,26].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THE HANKEL TRANSFORM

2D Fourier transforms can be reduced to a one-dimensional transform for functions with azimuthal symmetry. The 2D Fourier transform is

$$\begin{aligned} \bar{f}(k_x,k_y) := F[f(x,y)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \\ \times \exp(\imath(xk_x + yk_y))dxdy. \end{aligned} \tag{A1}$$

By transforming it to polar coordinates in real space,

$$x = r\cos\theta; \quad y = r\sin\theta, \tag{A2}$$

and Fourier space,

$$k_x = k\cos\theta_k; \quad k_y = k\sin\theta_k, \tag{A3}$$

and using,

$$k_x x + k_y y = kr(\cos\theta_k\cos\theta + \sin\theta_k\sin\theta) = kr\cos(\theta_k - \theta), \tag{A4}$$

the 2D Fourier transform becomes

$$\bar{f}(k,\theta_k) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty f(r,\theta)\exp[\imath kr\cos(\theta_k-\theta)]rdrd\theta. \tag{A5}$$

When f exhibits azimuthal symmetry [i.e., $f(r,\theta) = f(r)$], Eq. (A5) becomes

$$\begin{aligned} \bar{f}(k,\theta_k) &= \frac{1}{2\pi}\int_0^\infty rdrf(r)\left[\int_0^{2\pi}\exp[\imath kr\cos(\theta_k-\theta)]d\theta\right] \\ &= \int_0^\infty rf(r)J_0(rk)dr = \bar{f}(k) := H_0[f(r)], \end{aligned} \tag{A6}$$

where we have used the integral representation of the zeroth-order Bessel function of the first kind [27]:

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\imath a\cos(\theta-\phi))d\theta,\ \forall\phi, \tag{A7}$$

which is obtained from integrating in θ the Jacobi-Anger expansion of the exponential,

$$e^{\imath kr\cos\theta} = \sum_n \imath^n J_n(kr)e^{-\imath n\theta}. \tag{A8}$$

Equation (A6) defines the one-dimensional integral transform known as the Hankel transform (of order zero). These are a few interesting facts about Hankel transforms:

(i) The inverse Hankel transform is the same as the forward transform:

$$H_0^{-1}(\bar{f}(k)) = \int_0^\infty k\bar{f}(k)J_0(rk)dk = f(r). \tag{A9}$$

This is a consequence of the orthogonality property of Bessel functions [27],

$$\int_0^\infty J_0(rk)J_0(rk')rdr = \delta(k-k')/k,\ k,k'\geqslant 0. \tag{A10}$$

(ii) The Hankel transform, when applied to the radial part of the cylindrical Laplacian operator, verifies that

$$H_0\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = -k^2H_0[f], \tag{A11}$$

which is a consequence of the following property of Bessel functions [27]:

$$\frac{d}{dx}[xJ_n(sx)] = \frac{sx}{2n}[(n+1)J_{n+1}(sx)-(n-1)J_{n-1}(sx)]. \tag{A12}$$

APPENDIX B: MITTAG-LEFLLER FUNCTION

The Mittag-Leffler function is a complex function defined as [28]

$$E_p(z) = \sum_{m=0}^\infty \frac{z^m}{\Gamma(pm+1)}, \tag{B1}$$

and provides a generalization of the exponential [note that $\exp(\pm x) = E_1(\pm x)$]. When restricted to the positive real domain, its Laplace transform satisfies

$$L[E_p(at^p)] = \frac{s^{p-1}}{s^p - a}. \tag{B2}$$

APPENDIX C: FOX *H* FUNCTIONS AND THEIR GENERALIZATION

1. Fox *H* functions

Fox H functions are defined [23] as [assuming $1\leqslant m\leqslant q$, $0\leqslant n\leqslant p, A_j, B_j > 0$ and a_j, b_j complex numbers such that no pole of $\Gamma(b_j - B_js)$ coincides with any pole of $\Gamma(1-a_j+A_js))$ for any j]:

$$\begin{aligned} H_{p,q}^{m,n}&\left(x\middle|\begin{matrix}(a_1,A_1) & (a_2,A_2) & \cdots & (a_p,A_p)\\ (b_1,B_1) & (b_2,B_2) & \cdots & (b_q,B_q)\end{matrix}\right)\\ &:= H_{p,q}^{m,n}\left(x\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right)\\ &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Pi_{j=1}^m\Gamma(b_j-B_js)}{\Pi_{j=m+1}^q\Gamma(1-b_j+B_js)}\\ &\quad\times\frac{\Pi_{j=1}^n\Gamma(1-a_j+A_js)}{\Pi_{j=n+1}^p\Gamma(a_j-A_js)}z^sds. \end{aligned} \tag{C1}$$

The contour must go from $-i\infty$ to $+i\infty$ while leaving all poles of $\Gamma(b_j - B_js)$ to the right, and all poles of $\Gamma(1-a_j+A_js))$ to the left, although other equivalent choices are possible. The resulting function is analytic in $0<|z|<\infty$ if $\mu = \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0$.

Some interesting properties of H-functions are

$$H_{p,q}^{m,n}\left(x\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right) = \frac{1}{k}H_{p,q}^{m,n}\left(x^k\middle|\begin{matrix}(a_p,kA_p)\\(b_q,kB_q)\end{matrix}\right),\ k\in\mathbb{R}^+. \tag{C2}$$

Also that

$$x^\alpha H_{p,q}^{m,n}\left(x\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right) = H_{p,q}^{m,n}\left(x\middle|\begin{matrix}(a_p+\alpha A_p,A_p)\\(b_q+\alpha B_q,B_q)\end{matrix}\right),\ k\in\mathbb{C} \tag{C3}$$

together with

$$H_{p,q}^{m,n}\left(x\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right) = H_{p,q}^{m,n}\left(\frac{1}{x}\middle|\begin{matrix}(1-b_q,B_q)\\(1-a_p,A_p)\end{matrix}\right). \tag{C4}$$

The H function is symmetric in the pairs $(a_1,A_1),\cdots(a_n,A_n)$, likewise in the pairs $(a_{n+1}, A_{n+1}),\cdots(a_p,A_p)$. Same applies for $(b_1,B_1)\cdots(b_m,B_m)$ and in $(b_{m+1},B_{m+1}),\cdots(b_q,B_q)$. And, finally, the order can be reduced if one of the (a_j,A_j), $j=1,\cdots,n$ is equal to one of the $(b_k,B_k), k=m+1,\cdots,q$ [or one of the (b_j,B_j), $j=1,\cdots m$ is equal to one of the $(a_k,A_k), k=n+1,\cdots,p$] then the H function reduces in one order in p and q as well as in n (or in m):

$$H_{p,q}^{m,n}\left(x\middle|\begin{matrix}(a_1,A_1) & (a_2,A_2) & \cdots & (a_p,A_p) & \\ (b_1,B_1) & (b_2,B_2) & \cdots & (b_{q-1},B_{q-1}) & (a_1,A_1)\end{matrix}\right) = H_{p-1,q-1}^{m,n-1}\left(x\middle|\begin{matrix}(a_2,A_2) & \cdots & (a_p,A_p) & \\ (b_1,B_1) & (b_2,B_2) & \cdots & (b_{q-1},B_{q-1})\end{matrix}\right). \tag{C5}$$

Finally, it is interesting to note the asymptotic limits,

$$\lim_{|x|\to\infty} H_{p,q}^{m,n}(x)\sim|x|^d,\ d=\max\left[\mathrm{Re}\left(\frac{a_j-1}{A_j}\right), \jmath=1,\cdots n\right] \tag{C6}$$

and

$$\lim_{|x|\to 0} H_{p,q}^{m,n}(x)\sim|x|^c,\ d=\min\left[\mathrm{Re}\left(\frac{b_j}{B_j}\right),\ j=1,\cdots m\right]. \tag{C7}$$

Many functions can be expressed as H Functions. For instance, the exponential becomes

$$e^z = H^{1,0}_{0,1}\left[-z\middle|\begin{matrix} - \\ (0,1)\end{matrix}\right] \tag{C8}$$

and the usual Mittag-Leffler function,

$$E_\beta(z) = H^{1,1}_{1,2}\left[-z\middle|\begin{matrix}(0,1) & - \\ (0,1) & (0,\beta)\end{matrix}\right], \quad \text{with } E_1(z) = \exp(z). \tag{C9}$$

Also, the νth order Bessel function can be expressed as

$$\left(\frac{z}{2}\right)^a J_\nu(z) = H^{1,0}_{0,2}\left[\frac{z^2}{4}\middle|\begin{matrix} - & - \\ ((a+\nu)/2,1) & ((a-\nu)/2,1)\end{matrix}\right]. \tag{C10}$$

Of particular interest is the Hankel transform of the H function, given by [22]

$$\int_0^\infty kJ_0(kx)H^{m,n}_{p,q}\left[bk^\sigma\middle|\begin{matrix}(a_1,A_1) & \cdots & (a_p,A_p) \\ (b_1,B_1) & \cdots & (b_q,B_q)\end{matrix}\right] = \frac{2}{x^2}H^{m,n+1}_{p+2,q}\left[b\left(\frac{2}{x}\right)^\sigma\middle|\begin{matrix}\left(0,\frac{\sigma}{2}\right) & (a_1,A_1) & \cdots & (a_p,A_p) & \left(0,\frac{\sigma}{2}\right) \\ (b_1,B_1) & \cdots & (b_q,B_q) & - & -\end{matrix}\right], \tag{C11}$$

where it is assumed that $a, b > 0$.

2. Generalized H functions

The generalized H functions are introduced through the double integral [23]

$$H^{u,n,n',m,m'}_{v,(p;p'),w,(q;q')}\left[\begin{matrix}x \\ y\end{matrix}\middle|\begin{matrix}(e_v,E_v) \\ (a_p,A_p);(c_{p'},C_{p'}) \\ (f_w,F_w) \\ (b_q,B_q);(d_{q'},D_{q'})\end{matrix}\right] = -\frac{1}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\chi_1(\xi)\chi_2(\eta)\chi_3(\xi+\eta)x^\xi y^\eta d\xi d\eta \tag{C12}$$

with

$$\chi_1(\xi) = \frac{\Pi_{j=1}^m\Gamma(b_j - B_j\xi)\Pi_{j=1}^n\Gamma(1-a_j+A_j\xi)}{\Pi_{j=1}^q\Gamma(1-b_j+B_j\xi)\Pi_{j=1}^p\Gamma(a_j-A_j\xi)} \tag{C13}$$

$$\chi_2(\eta) = \frac{\Pi_{j=1}^{m'}\Gamma(d_j - D_j\eta)\Pi_{j=1}^{n'}\Gamma(1-c_j+C_j\eta)}{\Pi_{j=1}^{q'}\Gamma(1-d_j+D_j\eta)\Pi_{j=1}^{p'}\Gamma(c_j-C_j\eta)} \tag{C14}$$

and

$$\chi_3(\rho) = \frac{\Pi_{j=1}^u\Gamma(e_j+E_j\rho)}{\Pi_{j=u+1}^v\Gamma(1-e_j+E_j\rho)\Pi_{j=1}^w\Gamma(f_j+F_j\rho)}, \tag{C15}$$

where it is assumed that all indices are non-negative integers with $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q$, $0 \leqslant n' \leqslant p'$, $1 \leqslant m' \leqslant p'$, $0 \leqslant u \leqslant v$. Also, the collection of coefficients A_j, $(j = 1, \cdots n)$, B_j, $(j = 1, \cdots m)$ D_j, $(j = 1, \cdots m')$, C_j, $(j = 1, \cdots n')$, E_j, $(j = 1, \cdots u)$, and F_j, $(j = 1, \cdots v)$ are all positive, real numbers. The paths of integration are indented, if necessary so that the poles of $\Gamma(b_j - B_j\eta)$, $(j = 1, \cdots m)$ and $\Gamma(d_j - D_j\eta)$, $(j = 1, \cdots m')$ lie to the right and those of $\Gamma(a_j - A_j\eta)$, $(j = 1, \cdots n)$, $\Gamma(c_j - C_j\eta)$, $(j = 1, \cdots n')$, and $\Gamma(e_j - E_j\eta)$, $(j = 1, \cdots u)$ to the left of the imaginary axis.

This integral converges absolutely if

$$\rho_1 := \sum_{j=1}^m B_j - \sum_{j=m+1}^p B_j + \sum_{i=1}^n A_i - \sum_{i=n+1}^q A_j + \sum_{k=1}^v E_j - \sum_{k=v+1}^u E_j - \sum_{l=1}^w F_j > 0; \tag{C16}$$

$$\rho_2 := \sum_{j=1}^{m'} D_j - \sum_{j=m'+1}^{p'} D_j + \sum_{i=1}^{n'} C_i - \sum_{i=n'+1}^{q'} C_j + \sum_{k=1}^v E_j - \sum_{k=v+1}^u E_j - \sum_{l=1}^w F_j > 0. \tag{C17}$$

These functions satisfy the following property (for $k > 0$):

$$H^{u,n,n',m,m'}_{v,(p;p'),w,(q;q')}\left[\begin{array}{c|c} x^k & (e_v, E_v) \\ y^k & (a_p, A_p); (c_{p'}, C_{p'}) \\ & (f_w, F_w) \\ & (b_q, B_q); (d_{q'}, D_{q'}) \end{array}\right] \tag{C18}$$

$$= \frac{1}{k^2} H^{u,n,n',m,m'}_{v,(p;p'),w,(q;q')}\left[\begin{array}{c|c} x & (e_v, E_v/k) \\ y & (a_p, A_p/k); (c_{p'}, C_{p'}/k) \\ & (f_w, F_w/k) \\ & (b_q, B_q/k); (d_{q'}, D_{q'}/k) \end{array}\right], \tag{C19}$$

and reduce to the product of two H functions if $u = v = w = 0$,

$$H^{0,n,n',m,m'}_{0,(p;p'),0,(q;q')}\left[\begin{array}{c|c} x & - \\ y & (a_p, A_p); (c_{p'}, C_{p'}) \\ & - \\ & (b_q, B_q); (d_{q'}, D_{q'}) \end{array}\right] = H^{m,n}_{p,q}\left(x\middle|\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\right) \cdot H^{m',n'}_{p',q'}\left(x\middle|\begin{array}{c}(c'_p, C'_p)\\(d'_q, D'_q)\end{array}\right). \tag{C20}$$

Some interesting limits involving the generalized H functions are

$$H^{m,n}_{p,q}\left(x\middle|\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\right) = \lim_{y\to 0} H^{0,n,0,m,1}_{0,(p;0),0,(q;1)}\left[\begin{array}{c|c} x & - \\ y & (a_p, A_p); --- \\ & - \\ & (b_q, B_q); (0, 1) \end{array}\right] \tag{C21}$$

together with

$$H^{m,u}_{v,q}\left(x\middle|\begin{array}{c}(1-e_v, E_v)\\(b_q, B_q)\end{array}\right) = \lim_{y\to 0} H^{u,0,0,m,1}_{v,(0;0),0,(q;1)}\left[\begin{array}{c|c} x & (e_v, E_v) \\ y & -; - \\ & - \\ & (b_q, B_q); (0, 1) \end{array}\right]. \tag{C22}$$

Interestingly, the generalized H funtion can also be recast in terms of the integral of the product of three H functions,

$$\frac{1}{c} H^{u,n,n',m,m'}_{v,(p;p'),w,(q;q')}\left[\begin{array}{c|c} & (e_v, E_v) \\ a/c & (a_p, A_p); (c_{p'}, C_{p'}) \\ b/c & (f_w - 1 + F_w, F_w) \\ & (b_q, B_q); (d_{q'}, D_{q'}) \end{array}\right] = \int_0^\infty dx H^{m,n}_{p,q}\left(ax\middle|\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\right) H^{m',n'}_{p',q'}\left(bx\middle|\begin{array}{c}(c_{p'}, C_{p'})\\(d_{q'}, D_{q'})\end{array}\right) \cdot H^{u,0}_{v,w}\left(cx\middle|\begin{array}{c}(f_w - 1, F_w)\\(-e_v, E_v)\end{array}\right), \tag{C23}$$

which can be used to obtain the ν-order Hankel transform of the product of two H functions ($a > 0$),

$$\int_0^\infty dx x^{\rho-1} J_\nu(ax) H^{m,n}_{p,q}\left(bx^2\middle|\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}\right) H^{m',n'}_{p',q'}\left(cx^2\middle|\begin{array}{c}(c_{p'}, C_{p'})\\(d_{q'}, D_{q'})\end{array}\right) = \frac{2^{\rho-1}}{a^\rho} H^{1,n,n',m,m'}_{2,(p;p'),0,(q;q')}\left[\begin{array}{c|c} & \left(\frac{\rho+\nu}{2}, 1\right); \left(\frac{\rho-\nu}{2}, 1\right) \\ (4b)/a^2 & (a_p, A_p); (c_{p'}, C_{p'}) \\ (4c)/a^2 & - \\ & (b_q, B_q); (d_{q'}, D_{q'}) \end{array}\right], \tag{C24}$$

under the following conditions:

$$\mathrm{Re}\left(\frac{\nu+\rho}{2} + \min\left(\frac{b_j}{B_j}\right) + \min\left(\frac{d_k}{D_k}\right)\right) > 0, \tag{C25}$$

$$\mathrm{Re}\left(\frac{\rho}{2} + \max\left(\frac{a_j - 1}{A_j}\right) + \max\left(\frac{c_k - 1}{C_k}\right) - \frac{3}{4}\right) < 0, \tag{C26}$$

$$\sum_{j=1}^{n} A_j - \sum_{k=1}^{m} B_j \leqslant 0, \ \sum_{j=1}^{n'} C_j - \sum_{k=1}^{m'} D_j \leqslant 0. \tag{C27}$$

Using Eq. (C23), one can also obtain give the Laplace transform of the product of two H functions [Re$(a) > 0$],

$$\int_0^\infty dx x^\rho \exp(-ax) H^{m,n}_{p,q}\left(bx\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right) H^{m',n'}_{p',q'}\left(cx\left|\begin{matrix}(c_{p'}, C_{p'})\\(d_{q'}, D_{q'})\end{matrix}\right.\right) = \frac{1}{a^{\rho+1}} H^{1,n,n',m,m'}_{1,(p;p'),0,(q;q')}\left[\begin{matrix}b/a\\c/a\end{matrix}\left|\begin{matrix}(1+\rho, 1)\\(a_p, A_p); (c_{p'}, C_{p'})\\-\\(b_q, B_q); (d_{q'}, D_{q'})\end{matrix}\right.\right], \quad \text{(C28)}$$

under the conditions

$$\mathrm{Re}\left(\rho + \min\left(\frac{b_j}{B_j}\right) + \min\left(\frac{d_k}{D_k}\right)\right) > 0, \quad \text{(C29)}$$

$$\sum_{j=1}^{n} A_j - \sum_{k=1}^{m} B_j \leqslant 0, \ \sum_{j=1}^{n'} C_j - \sum_{k=1}^{m'} D_j \leqslant 0. \quad \text{(C30)}$$

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