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OPERATORS ON THE KALTON–PECK SPACE Z_2

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ABSTRACT

We study operators on the Kalton–Peck Banach space Z_2 from various points of view: matrix representations, examples, spectral properties and operator ideals. For example, we prove that there are non-compact, strictly singular operators acting on Z_2, but the product of two of them is a compact operator. Among applications, we show that every copy of Z_2 in Z_2 is complemented, and each semi-Fredholm operator on Z_2 has complemented kernel and range, the space Z_2 is Z_2-automorphic and we give a partial solution to a problem of Johnson, Lindenstrauss and Schechtman about strictly singular perturbations of operators on Z_2.

1. Introduction

In spite of being a by now classical Banach space, many aspects of the celebrated Kalton–Peck space Z_2 introduced in [21] remain unknown. For instance, it is not known whether Z_2 is a counterexample to the hyperplane problem: is Z_2 isomorphic to its closed subspaces of codimension 1? The space is clearly isomorphic to its closed subspaces of codimension 2. For a short exposition of the few known facts about the structure of Z_2 and its subspaces we refer to [6, Section 10.8] (The Kalton–Peck spaces) and [6, Section 10.9] (Properties of Z_2 explained by itself) and, for a condensed exposition, to the remainder of this section.

In this paper we study operators on the space Z_2 from several points of view. In Section 3 we introduce a matrix representation for these operators, and the results obtained are refined for operators that admit an upper or lower triangular representation in Section 4. In Section 5 we consider an involution $T \longrightarrow T^+$ on the algebra $\mathfrak{L}(Z_2)$ of bounded operators on Z_2 introduced in [19] and obtain several new results that we apply to show that every copy of Z_2 in Z_2 is complemented, that the space Z_2 is Z_2-automorphic and that each semi-Fredholm operator on Z_2 has complemented kernel and range. Section 6 contains a list of natural examples of operators on Z_2. In Section 7 we follow ideas of Kalton to show that the operator ideals of strictly singular, strictly cosingular and inessential operators coincide and contain each proper operator ideal of $\mathfrak{L}(Z_2)$, we prove that the product of two strictly singular operators on Z_2 is compact, although there are non-compact, strictly singular operators, and we show that the perturbation classes problem has a positive solution for operators on Z_2. The final Section 8 outlines a few open directions of research.

Next we summarize some facts about the space Z_2 and operators on it. We refer the reader to the Preliminaries (Section 2) for any unexplained notation.

1.1. Facts about the space Z_2. Let us briefly describe the space Z_2. We refer to the Preliminaries section for all unexplained notation. Consider the homogeneous map $\mathsf{KP} : \ell_2 \to \ell_\infty$ given by $\mathsf{KP}(x) = 2x \log |x|$ for normalized $x \in \ell_2$. Then

$$Z_2 = \{(\omega, x) \in \ell_\infty \times \ell_2 : \omega - \mathsf{KP}x \in \ell_2\}$$

endowed with the quasinorm $\|(\omega, x)\|_{Z_2} = \|\omega - \mathsf{KP}x\|_2 + \|x\|_2$. The space Z_2 is a nontrivial **twisted Hilbert space** in the sense that it contains an uncomplemented subspace M isomorphic to ℓ_2 such that Z_2/M is again isomorphic to ℓ_2. Indeed, there is a nontrivial exact sequence

$$(P_1) \qquad 0 \longrightarrow \ell_2 \xrightarrow{\ i\ } Z_2 \xrightarrow{\ p\ } \ell_2 \longrightarrow 0,$$

with inclusion $iy = (y, 0)$ and quotient map $p(\omega, x) = x$. The quasinorm above is equivalent to a norm [21, Theorem 4.7] and thus Z_2 is a reflexive and ℓ_2-saturated Banach space (see [12]) isomorphic (actually isometric, see below) to its dual [21, Theorem 5.1], something we will write as $Z_2 \simeq Z_2^*$. We will also consider the following quasi-Banach spaces:

- Dom$\mathsf{KP} = \{x \in \ell_2 : \mathsf{KP}x \in \ell_2\}$ endowed with the quasinorm

$$\|x\| = \|\mathsf{KP}x\|_2 + \|x\|_2,$$

  which is equivalent to the norm of the Orlicz space ℓ_f generated by the Orlicz function $f(t) = t^2 \log^2 t$ [21]; so we can identify Dom$\mathsf{KP} = \ell_f$.
- Ran $\mathsf{KP} = \{\omega \in \ell_\infty : \exists x \in \ell_2, \omega - \mathsf{KP}x \in \ell_2\}$ endowed with the quasinorm

$$\|\omega\| = \inf_{x \in \ell_2} \|\omega - \mathsf{KP}x\|_2 + \|x\|_2,$$

  which is equivalent to the norm of ℓ_f^*; so we can identify Ran $\mathsf{KP} = \ell_f^*$ [5].

There is another natural nontrivial exact sequence

$$(P_2) \qquad 0 \longrightarrow \ell_f \xrightarrow{\ j\ } Z_2 \xrightarrow{\ q\ } \ell_f^* \longrightarrow 0,$$

with inclusion $jx = (0, x)$ and quotient map $q(\omega, x) = \omega$, which has associated a quasilinear map $\mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_\infty$. This map KP^{-1} provides an alternative description of Z_2 as a twisted sum of ℓ_f and ℓ_f^*. Indeed, the space

$$\hat{Z}_2 = \{(x, \omega) \in \ell_\infty \times \ell_f^* : x - \mathsf{KP}^{-1}\omega \in \ell_f\}$$

can be endowed with the quasinorm $\|(x,\omega)\| = \|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\omega\|_{\ell_f^*}$, so that the map $(\omega, x) \longrightarrow (x, \omega)$ defines an isomorphism from Z_2 onto $\hat{Z}_2$. Thus there exists constants $m, M > 0$ so that

$$m\|(\omega,x)\|_{Z_2} \leq \|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\omega\|_{\ell_f^*} \leq M\|(\omega,x)\|_{Z_2}. \tag{1}$$

Moreover, the map $\omega \to (\omega, \mathsf{KP}^{-1}\omega)$ is a bounded homogeneous lifting for q (see [9]) and

- $\mathrm{Dom}\mathsf{KP}^{-1} = \{\omega \in \ell_f^* : \mathsf{KP}^{-1}\omega \in \ell_f\}$ is endowed with the quasinorm
$$\|\omega\| = \|\mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\omega\|_{\ell_f^*},$$
which is equivalent to the norm of ℓ_2. So we can identify $\mathrm{Dom}\mathsf{KP}^{-1} = \ell_2$.
- $\mathrm{Ran}\,\mathsf{KP}^{-1} = \{x \in \ell_\infty : \exists \omega \in \ell_f^*, x - \mathsf{KP}^{-1}\omega \in \ell_f\}$ is endowed with the quasinorm
$$\|x\| = \inf_{\omega \in \ell_f^*} \|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\omega\|_{\ell_f^*},$$
which is equivalent to the norm of ℓ_2. So we can identify $\mathrm{Ran}\,\mathsf{KP}^{-1} = \ell_2$.

There is no known explicit formula for KP^{-1} (see [8]). Both $\mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_f^*$ and $\mathsf{KP}^{-1} \circ \mathsf{KP} : \ell_2 \to \ell_2$ are bounded maps, and condition $(\star)$ in Theorem 3.2 for $T = I_{Z_2}$ implies that the map $I_{Z_2} - \mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ is bounded.

The space Z_2 also appears as the derived space obtained by complex interpolation of the scale (ℓ_∞, ℓ_1) at $1/2$. This means that if $\mathcal{C}$ is the corresponding Calderón space, $\delta : \mathcal{C} \to \ell_\infty$ denotes the evaluation map at $1/2$ and $\delta' : \mathcal{C} \to \ell_\infty$ denotes the evaluation of the derivative map at $1/2$, then Z_2 is the quotient space $\mathcal{C}/(\ker\delta \cap \ker\delta')$, which means that (isomorphically)

$$Z_2 = \{(\omega, x) \in \ell_\infty \times \ell_2 : \exists f \in \mathcal{C} : f(1/2) = x \text{ and } f'(1/2) = \omega\} \tag{2}$$

endowed with the natural quotient norm. We will use this approach in Section 6.4.

The space Z_2 is not only isometric to its dual, but also KP is "self-dual", in the sense that the quasilinear map KP^* associated to the exact sequence dual to (P_1) satisfies $\mathsf{KP}^* = -\mathsf{KP}$ [21, Theorem 5.1], which is reflected in

$$|\langle \mathsf{KP}x, y\rangle - \langle x, \mathsf{KP}y\rangle| \leq 2\|x\|_2\|y\|_2.$$

This implies that the dual of Z_2 is $Z_2^* = \{(\omega^*, x^*) \in \ell_\infty \times \ell_2 : \omega^* - \mathsf{KP}^*x^* \in \ell_2\}$ with duality formula

$$\langle(\omega^*, x^*), (\omega, x)\rangle = \langle \omega^*, x\rangle + \langle \omega, x^*\rangle, \tag{3}$$

and $D : Z_2 \to Z_2^*$ given by $D(\omega, x) = (-\omega, x)$ is a bijective isometry.

1.2. Facts about operators on Z_2. The knowledge about operators on Z_2 is even scarcer than that about Z_2 itself and can be summarized in two results:

Theorem 1.1 ([19, Theorems 7 and 8]): *Let $S \in \mathfrak{L}(Z_2)$ and $T \in \mathfrak{L}(Z_2, Y)$.*

1. *If S is not strictly singular, then there exists a subspace E of Z_2 isomorphic to Z_2 such that $S|_E$ is an isomorphism and $S[E]$ is complemented in Z_2 (hence E also is complemented).*
2. *If T is not strictly singular, then there exists a complemented subspace F of Z_2 isomorphic to Z_2 such that $T|_F$ is an isomorphism.*

Corollary 1.2: *In (P_1) and (P_2), the quotient maps p, q are strictly singular and the inclusions i, j are strictly cosingular.*

Proof. The result for i and p is proved in [21]. Since ℓ_f has cotype 2 [22, Corollary 13], ℓ_f^* has type 2; hence ℓ_f^* contains no copies of Z_2 since $Z_2 \simeq Z_2^*$, and q is strictly singular. The result for j can be derived by duality. ∎

Proposition 1.3 ([2, Lemma 16.15]): *A scalar 2×2 matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right)$ defines an operator in $\mathfrak{L}(Z_2)$ in the obvious way $A(e_n, e_m) = (\alpha e_n + \beta e_m, \delta e_n + \gamma e_m)$ if and only if $\alpha = \gamma$ and $\delta = 0$.*

2. Preliminaries

2.1. General operator theory. An **operator ideal** [29] is a subclass $\mathcal{A}$ of the class $\mathfrak{L}$ of bounded operators between Banach spaces such that finite range operators belong to $\mathcal{A}$, $\mathcal{A} + \mathcal{A} \subset \mathcal{A}$ and $\mathfrak{L}\,\mathcal{A}\,\mathfrak{L} \subset \mathcal{A}$.

Let X and Y be Banach spaces, and let $T \in \mathfrak{L}(X, Y)$. We denote by $N(T)$ the kernel of T and by $R(T)$ the range of T. The operator T is **strictly singular** if no restriction of T to an infinite-dimensional subspace of X is an isomorphism; T is **strictly cosingular** if $q_N T$ is never surjective when q_N is the quotient map onto an infinite-dimensional quotient Y/N. The classes $\mathfrak{S}$ of strictly singular operators and $\mathfrak{C}$ of strictly cosingular operators are operator ideals [29, 1.9 and 1.10]. The operator T is **upper semi-Fredholm,** $T \in \Phi_+$, if $R(T)$ is closed and $N(T)$ is finite-dimensional; it is **lower semi-Fredholm,** $T \in \Phi_-$, if $R(T)$ is finite codimensional (hence closed), $\Phi_\pm = \Phi_+ \cup \Phi_-$ is the class of semi-Fredholm operators, and $\Phi = \Phi_+ \cap \Phi_-$ is the class of Fredholm operators.

For $T \in \Phi_\pm(X,Y)$, the **index** of T is defined by

$$i(T) = \dim N(T) - \dim Y/R(T) \in \mathbb{Z} \cup \{\pm\infty\}.$$

Also T is **inessential**, denoted $T \in \mathfrak{In}$, if $I_X - AT$ is a Fredholm operator for all $A \in \mathfrak{L}(Y,X)$ or, equivalently, $I_Y - TA$ is Fredholm for all $A \in \mathfrak{L}(Y,X)$. The operator ideal $\mathfrak{In}$ was introduced by Kleinecke [23], and contains both $\mathfrak{S}$ and $\mathfrak{C}$.

2.2. Exact sequences and quasilinear maps. Let X and Y be quasi-Banach spaces with quasi-norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. We suppose that Y is a subspace of some vector space Σ. A map $\Omega : X \to \Sigma$ is called **quasilinear from X to Y** with **ambient space** Σ and denoted $\Omega : X \curvearrowright Y$ if it is homogeneous and there exists a constant C so that for each $x, z \in X$,

$$\Omega(x+z) - \Omega x - \Omega z \in Y \quad \text{and} \quad \|\Omega(x+z) - \Omega x - \Omega z\|_Y \le C(\|x\|_X + \|z\|_X).$$

A quasilinear map $\Omega : X \curvearrowright Y$ is said to be **bounded** if there exists a constant D so that $\Omega x \in Y$ and $\|\Omega x\|_Y \le D\|x\|_X$ for each $x \in X$. It is said to be **trivial** if there exists a linear map $L : X \longrightarrow \Sigma$ so that $\Omega - L : X \longrightarrow Y$ is bounded.

A quasilinear map $\Omega : X \curvearrowright Y$ generates an exact sequence $0 \to Y \to Z \to X \to 0$ (namely, a diagram formed by quasi-Banach spaces and continuous operators so that the kernel of each operator coincides with the image of the previous one), as follows:

$$Z = \{(\omega, x) \in \Sigma \times X : \omega - \Omega x \in Y\}$$

endowed with the quasi-norm

$$\|(\omega, x)\|_\Omega = \|\omega - \Omega x\|_Y + \|x\|_X,$$

with inclusion $y \longrightarrow (y, 0)$ and quotient map $(\omega, x) \longrightarrow x$. The quasilinear map (equivalently, the exact sequence) is **trivial** if the image of Y in Z is complemented.

The space Z is called a **twisted sum** of Y and X and denoted $Y \oplus_\Omega X$. If Ω is bounded, then $Y \oplus_\Omega X = Y \times X$ and $\|y - \Omega x\|_Y + \|x\|_X$ and $\|y\|_Y + \|x\|_X$ are equivalent quasi-norms on this space. If Ω is trivial then $Y \oplus_\Omega X$ is isomorphic to $Y \times X$.

The general theory of twisted sums developed in [21] establishes a correspondence between exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ of quasi Banach spaces and quasilinear maps $\Omega : X \curvearrowright Y$. The quasilinear map generating $0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \longrightarrow 0$ is KP.

3. Matrix representation of operators on Z_2

The space Z_2 admits two presentations as a twisted sum space: (P_1) and (P_2). Operators on Z_2 can be represented by a matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \delta & \gamma\end{smallmatrix}\right)$, whose entries α, β, δ and γ are linear maps between sequence spaces and depend on whether one is using (P_1) or (P_2). Unless specified otherwise, we will always refer to (P_1); in which case the matrix above acts as

$$\begin{pmatrix}\alpha & \beta\\ \delta & \gamma\end{pmatrix}(\omega, x) = (\alpha\omega + \beta x, \delta\omega + \gamma x). \tag{4}$$

This same operator acting on $\hat{Z}_2$ has representing matrix $\left(\begin{smallmatrix}\gamma & \delta\\ \beta & \alpha\end{smallmatrix}\right)$ and takes (x, ω) into $(\gamma x + \delta\omega, \beta x + \alpha\omega)$.

If $T = \left(\begin{smallmatrix}\alpha & \beta\\ \delta & \gamma\end{smallmatrix}\right)$ is a bounded operator on Z_2, then $\alpha = qTi : \ell_2 \longrightarrow \ell_f^*$, $\beta = qTj : \ell_f \longrightarrow \ell_f^*$, $\delta = pTi : \ell_2 \longrightarrow \ell_2$ and $\gamma = pTj : \ell_f \longrightarrow \ell_2$ are bounded operators. Note that the operator T is determined by its restriction to $\ell_2 \oplus \ell_f$, which is a dense subspace of Z_2, and on this dense subspace the entries of the matrix are bounded operators. However, we assume that Equation (4) is valid for all $(\omega, x) \in Z_2$.

Lemma 3.1 (Necessary conditions): *Let $T = \left(\begin{smallmatrix}\alpha & \beta\\ \delta & \gamma\end{smallmatrix}\right)$ be a bounded operator on Z_2. The following conditions are satisfied:*

(d) $\delta : \ell_2 \longrightarrow \ell_2$ *is bounded.*

(g) $\gamma : \ell_f \longrightarrow \ell_2$ *and* $(\beta - \mathsf{KP} \circ \gamma) : \ell_f \longrightarrow \ell_2$ *are bounded.*

(d+gK') $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$ *is a bounded map.*

(g+dK) $\gamma + \delta \circ \mathsf{KP} : \ell_2 \longrightarrow \ell_2$ *is a bounded map.*

Proof. (d) and the first part of (g) were shown before. For the second part of (g), let $x \in \ell_f$. Then

$$\|Tjx\|_{Z_2} = \|T(0,x)\|_{Z_2} = \|(\beta x, \gamma x)\|_{Z_2} = \|(\beta - \mathsf{KP}\circ\gamma)x\|_2 + \|\gamma x\|_2 \leq \|Tj\| \cdot \|x\|_{\ell_f}.$$

Thus $\beta - \mathsf{KP} \circ \gamma : \ell_f \longrightarrow \ell_2$ is bounded.

(d+gK') A bounded lifting L_q for q is given by $L_q\omega = (\omega, \mathsf{KP}^{-1}\omega)$. Then for every $\omega \in \ell_f^*$,

$$\|pTL_q\omega\|_2 = \|(\delta + \gamma \circ \mathsf{KP}^{-1})\omega\| \leq \|pT\| \cdot \|L_q\| \cdot \|\omega\|_{\ell_f^*}.$$

Hence $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$ is bounded.

(g+dK) A bounded lifting L_p for p is given by $L_p y = (\mathsf{KP}y, y)$. Then for each $y \in \ell_2$, we have

$$\|pTL_p y\|_2 = \|(\gamma + \delta \circ \mathsf{KP})y\| \leq \|pT\| \cdot \|L_p\| \cdot \|y\|_2.$$

Hence $\gamma + \delta \circ \mathsf{KP} : \ell_2 \longrightarrow \ell_2$ is bounded. ∎

Now we characterize the bounded operators on Z_2 .

Theorem 3.2: *The operator $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right)$ is bounded on Z_2 if and only if the four necessary conditions in Lemma 3.1 hold as well as*

$(\star)$ $\quad \alpha + \beta \circ \mathsf{KP}^{-1} - \mathsf{KP}(\delta + \gamma \circ \mathsf{KP}^{-1}) : \ell_f^* \longrightarrow \ell_2$ *is a bounded map.*

Proof. Condition $(\star)$ is necessary: if T is bounded then

$$\|\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x)\|_2 \leq \|T\| \|(\omega, x)\|_{Z_2}$$

and the choice $(\omega, \mathsf{KP}^{-1}\omega) = L_q \omega$ yields

$$\|\alpha\omega + \beta \circ \mathsf{KP}^{-1}\omega - \mathsf{KP}(\delta\omega + \gamma \circ \mathsf{KP}^{-1}\omega)\|_2 \leq \|L_q\| \|T\| \|\omega\|_{\ell_f^*}.$$

Conversely, we will show that there exists a constant $C > 0$ so that for each $(\omega, x) \in Z_2$ we have $(\alpha\omega + \beta x, \delta\omega + \gamma x) \in Z_2$ and

$$\|(\alpha\omega + \beta x, \delta\omega + \gamma x)\|_{Z_2} \leq C \|(\omega, x)\|_{Z_2}.$$

We need to show that

(1) $\delta\omega + \gamma x \in \ell_2$,
(2) $\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x) \in \ell_2$ (hence $\alpha\omega + \beta x \in \ell_\infty$), and
(3) $\|\delta\omega + \gamma x\|_2 + \|\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x)\|_2 \leq C \|(\omega, x)\|_{Z_2}$.

Recall that there exists $M > 0$ such that $\|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\omega\|_{\ell_f^*} \leq M \|(\omega, x)\|_{Z_2}$ for each $(\omega, x) \in Z_2$.

(1) Observe that $\omega - \mathsf{KP}x \in \ell_2$ and $x - \mathsf{KP}^{-1}\omega \in \ell_f$, and by assumption, the maps $\gamma + \delta \circ \mathsf{KP} : \ell_2 \longrightarrow \ell_2$ and $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$ are bounded. Thus

$$\delta\omega + \gamma x = \frac{1}{2}(\delta(\omega - \mathsf{KP}x) + \gamma(x - \mathsf{KP}^{-1}\omega) + (\gamma + \delta \circ \mathsf{KP})x + (\delta + \gamma \circ \mathsf{KP}^{-1})\omega) \in \ell_2$$

and

$$\begin{aligned}\|\delta\omega + \gamma x\|_2 \leq{}& \frac{1}{2}(\|\delta\| \|\omega - \mathsf{KP}x\|_2 + \|\gamma\| \|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\gamma + \delta \circ \mathsf{KP}\| \|x\|_2 \\ &+ \|\delta + \gamma \circ \mathsf{KP}^{-1}\| \|\omega\|_{\ell_f^*}) \\ \leq{}& \frac{1}{2}(\|\delta\| + \|\gamma\| M + \|\gamma + \delta \circ \mathsf{KP}\| + \|\delta + \gamma \circ \mathsf{KP}^{-1}\| M) \|(\omega, x)\|_{Z_2}.\end{aligned}$$

To prove (2) we decompose $\alpha\omega + \beta x - \mathsf{KP}(\delta\omega + \gamma x)$ in three pieces:

$$\begin{aligned}&\alpha\omega + \beta \circ \mathsf{KP}^{-1}\omega - \mathsf{KP}(\delta + \gamma \circ \mathsf{KP}^{-1})\omega\\ &\quad + \beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega)\\ &\quad + \mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega) + \mathsf{KP}(\delta + \gamma \circ \mathsf{KP}^{-1})\omega - \mathsf{KP}(\delta\omega + \gamma x).\end{aligned}$$

The first piece is bounded by $(\star)$; and for the third piece, note that $\mathsf{KP} : \ell_2 \curvearrowright \ell_2$ is quasilinear, hence $\mathsf{KP}(x+y) - \mathsf{KP}x - \mathsf{KP}y \in \ell_2$ and

$$\|\mathsf{KP}(x+y) - \mathsf{KP}x - \mathsf{KP}y\|_2 \leq \|\mathsf{KP}\|(\|x\|_2 + \|y\|_2)$$

for $x, y \in \ell_2$. Moreover, $\gamma : \ell_f \longrightarrow \ell_2$ and $\delta + \gamma \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$ are bounded. Thus

$$\begin{aligned}\|\mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega) &+ \mathsf{KP}(\delta\omega + \gamma \circ \mathsf{KP}^{-1}\omega) - \mathsf{KP}(\delta\omega + \gamma x)\|_2\\ &\leq \|\mathsf{KP}\|(\|\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega\|_2 + \|\delta\omega + \gamma \circ \mathsf{KP}^{-1}\omega\|_2)\\ &\leq \|\mathsf{KP}\|(\|\gamma\|\|x - \mathsf{KP}^{-1}\omega\|_{\ell_f} + \|\delta + \gamma \circ \mathsf{KP}^{-1}\|\|\omega\|_{\ell_f^*})\\ &\leq \|\mathsf{KP}\|(\|\gamma\| + \|\delta + \gamma \circ \mathsf{KP}^{-1}\|)M\|(\omega, x)\|_{Z_2}.\end{aligned}$$

For the second piece, since $x - \mathsf{KP}^{-1}\omega \in \ell_f$, $(\beta - \mathsf{KP} \circ \gamma) : \ell_f \longrightarrow \ell_2$ is bounded and $\beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP}(\gamma x - \gamma \circ \mathsf{KP}^{-1}\omega)$ is equal to

$$\beta(x - \mathsf{KP}^{-1}\omega) - \mathsf{KP} \circ \gamma(x - \mathsf{KP}^{-1}\omega) = (\beta - \mathsf{KP} \circ \gamma)(x - \mathsf{KP}^{-1}\omega),$$

one has

$$\begin{aligned}\|(\beta - \mathsf{KP} \circ \gamma)(x - \mathsf{KP}^{-1}\omega)\|_2 &\leq \|(\beta - \mathsf{KP} \circ \gamma)\|\|x - \mathsf{KP}^{-1}\omega\|_{\ell_f}\\ &\leq M\|(\beta - \mathsf{KP} \circ \gamma)\|\|(\omega, x)\|_{Z_2}.\end{aligned}$$

(3) clearly follows from the arguments in the proof of (1) and (2). ∎

Condition $(\star)$ for $T = I_{Z_2}$ implies that $I - \mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$ is bounded, from which we can derive the Benyamini–Lindenstrauss characterization of Proposition 1.3: if $\left(\begin{smallmatrix}\alpha & \beta\\ \delta & \gamma\end{smallmatrix}\right)$ is a bounded operator on Z_2 with $\alpha, \beta, \gamma, \delta$ scalars then the boundedness of $\gamma + \delta\mathsf{KP} : \ell_2 \to \ell_2$ implies that $\delta = 0$ while the boundedness of $\alpha + \beta\mathsf{KP}^{-1} - \gamma\mathsf{KP} \circ \mathsf{KP}^{-1} : \ell_f^* \to \ell_2$ yields that, since $\beta\mathsf{KP}^{-1}$ is also bounded for any scalar β and $\alpha - \gamma + \gamma(I - \mathsf{KP} \circ \mathsf{KP}^{-1})$ is bounded, then also $\alpha - \gamma : \ell_f^* \to \ell_2$ is bounded, and thus $\alpha = \gamma$.

Apart from $(\star)$ and the necessary conditions in Lemma 3.1, we have some additional ones that were not needed in the proof of Theorem 3.2:

Lemma 3.3: *If* $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *then the following conditions are satisfied:*

(a) $\alpha : \ell_2 \longrightarrow \ell_f^*$ *is a bounded operator.*

(b) $\beta : \ell_f \longrightarrow \ell_f^*$ *is a bounded operator.*

(c_0) $\alpha - \mathsf{KP} \circ \delta : \ell_2 \longrightarrow \ell_2$ *is bounded.*

(c_1) $\alpha \circ \mathsf{KP} + \beta : \ell_2 \longrightarrow \ell_f^*$ *is bounded.*

(c_2) $\alpha + \beta \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_f^*$ *is bounded.*

(c_3) $\alpha \circ \mathsf{KP} + \beta - \mathsf{KP}(\delta \circ \mathsf{KP} + \gamma) : \ell_2 \longrightarrow \ell_2$ *is bounded.*

(c_4) $\gamma - \mathsf{KP}^{-1} \circ \beta : \ell_f \to \ell_f$ *is bounded.*

Proof. (a) and (b) follow from $\alpha = qTi$ and $\beta = qTj$.

(c_0) For $y \in \ell_2$,

$$\|Tiy\|_{Z_2} = \|T(y,0)\|_{Z_2} = \|(\alpha y, \delta y)\|_{Z_2} = \|(\alpha - \mathsf{KP} \circ \delta)y\|_2 + \|\delta y\|_2 \leq \|Ti\| \cdot \|y\|_2.$$

Therefore $(\alpha - \mathsf{KP} \circ \delta) : \ell_2 \longrightarrow \ell_2$ is bounded.

(c_1) For each $y \in \ell_2$,

$$\|qTL_p y\| = \|qT(\mathsf{KP}y, y)\|_{Z_2} = \|(\alpha \circ \mathsf{KP} + \beta)y\|_{\ell_f^*} \leq \|T\| \|L_p\| \|y\|_2.$$

(c_2) and (c_3) are proved in a similar way, using that $qTL_q\omega = (\alpha + \beta \circ \mathsf{KP}^{-1})\omega$ for each $\omega \in \ell_f^*$ and

$$\|TL_p y\|_{Z_2} = \|((\alpha \circ \mathsf{KP} + \beta)y, (\delta \circ \mathsf{KP} + \gamma)y)\|_{Z_2} \leq \|T\| \|L_p\| \|y\|_2.$$

and (c_4) follows from the continuity of Tj. ∎

Let us see some applications.

Proposition 3.4: *Let* $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right)$ *be an operator on* Z_2 .

(d*) *If* $\gamma : \ell_2 \to \ell_2$ *is bounded then* $\delta : \ell_f^* \longrightarrow \ell_2$ *and* $\delta \circ \mathsf{KP} : \ell_2 \longrightarrow \ell_2$ *are also bounded.*

(d') *If* $\alpha : \ell_2 \to \ell_2$ *is bounded then* $\delta : \ell_2 \longrightarrow \ell_f$ *is bounded. Hence* $\mathsf{KP} \circ \delta : \ell_2 \longrightarrow \ell_2$ *is bounded.*

(d**) *If* $\gamma : \ell_f \to \ell_f$ *is bounded then* $\mathsf{KP}^{-1} \circ \beta : \ell_f \longrightarrow \ell_f$ *is also bounded.*

(d") *If* $\alpha : \ell_f^* \to \ell_f^*$ *is bounded then* $\beta \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_f^*$ *is bounded.*

Proof. (d*) Since $\mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$ and $\gamma : \ell_2 \longrightarrow \ell_2$ are bounded, so is $\gamma \circ \mathsf{KP}^{-1} : \ell_f^* \longrightarrow \ell_2$. By (d+gK') in Lemma 3.1, $\delta : \ell_f^* \longrightarrow \ell_2$ is bounded, hence $\delta \circ \mathsf{KP} : \ell_2 \longrightarrow \ell_2$ is also bounded.

(d') By (c_0) in Lemma 3.3, $\alpha - \mathsf{KP} \circ \delta : \ell_2 \longrightarrow \ell_2$ is bounded. Then $\alpha : \ell_2 \longrightarrow \ell_2$ bounded implies $\mathsf{KP} \circ \delta : \ell_2 \longrightarrow \ell_2$ bounded; hence $\mathsf{KP}^{-1} \circ \mathsf{KP} \circ \delta : \ell_2 \longrightarrow \ell_f$

bounded; equivalently, $\delta(\ell_2) \subset \ell_f$. Hence $\delta : \ell_2 \longrightarrow \ell_f$ is bounded by the closed graph theorem. For the last equivalence observe that, by the definition of domain of KP and KP^{-1}, for $x \in \ell_2$,

$$\mathsf{KP}^{-1} \circ \mathsf{KP}x \in \ell_f \Rightarrow \mathsf{KP}x \in \ell_2 \Rightarrow x \in \ell_f.$$

The assertions (d**) and (d”) can be obtained analogously. ∎

Proposition 3.4 yields the boundedness of $\mathsf{KP} \circ \delta, \delta \circ \mathsf{KP}, \beta \circ \mathsf{KP}^{-1}$ or $\mathsf{KP}^{-1} \circ \beta$ but only under additional conditions that, in general, are not guaranteed. It is for that reason surprising that one has

Proposition 3.5: *Let* $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$. *Then:*

1. $\mathsf{KP} \circ \delta : \ell_2 \curvearrowright \ell_2$ *and* $\delta \circ \mathsf{KP} : \ell_2 \curvearrowright \ell_2$ *are trivial quasilinear maps.*
2. $\mathsf{KP}^{-1} \circ \beta : \ell_f \curvearrowright \ell_f$ *and* $\beta \circ \mathsf{KP}^{-1} : \ell_f^* \curvearrowright \ell_f^*$ *are trivial quasilinear maps.*

Proof. We prove assertion (1). Consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \stackrel{i}{\longrightarrow} & Z_2 & \stackrel{p}{\longrightarrow} & \ell_2 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & \mathrm{PB} & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

Then $\mathsf{KP}\circ\delta$ is a quasilinear map generating the lower exact sequence and $\mathsf{KP} \circ \delta$ is trivial if and only if δ admits a bounded linear lifting $\ell_2 \longrightarrow Z_2$ [6, Lemma 2.8.3]. Since $\alpha - \mathsf{KP} \circ \delta : \ell_2 \to \ell_2$ and $\delta : \ell_2 \longrightarrow \ell_2$ are bounded, $\hat{\delta}x = (\alpha x, \delta x)$ provides the lifting. Indeed, $\|(\alpha x, \delta x)\|_{Z_2} = \|(\alpha - \mathsf{KP} \circ \delta)x\|_2 + \|\delta x\|_2$.

Analogously, if we consider the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \stackrel{i}{\longrightarrow} & Z_2 & \stackrel{p}{\longrightarrow} & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle \delta}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & \mathrm{PO} & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

then $\delta \circ \mathsf{KP}$ is a quasilinear map generating the lower exact sequence which is trivial if and only if δ admits a bounded linear extension $\overline{\delta} : Z_2 \longrightarrow \ell_2$ [6, Lemma 2.6.3]. In the proof of Theorem 3.2 it is shown that $\overline{\delta}(\omega, x) = \delta\omega + \gamma x$ provides such an extension.

The proof for (2) follows in a similar way: to show that $\mathsf{KP}^{-1} \circ \beta : \ell_f \curvearrowright \ell_f$ is trivial, just consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_f & \xrightarrow{j} & Z_2 & \xrightarrow{q} & \ell_f^* & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \ell_f & \longrightarrow & \mathrm{PB} & \longrightarrow & \ell_f & \longrightarrow & 0 \end{array}$$

(where β is continuous by Proposition 3.3 (b)) and observe that $\omega \to (\beta\omega, \gamma\omega)$ is a bounded lifting for β by (c_4). To show that $\beta \circ \mathsf{KP}^{-1} : \ell_f^* \curvearrowright \ell_f^*$ just consider the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_f & \xrightarrow{j} & Z_2 & \xrightarrow{q} & \ell_f^* & \longrightarrow & 0 \\ & & {\scriptstyle \beta}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \ell_f^* & \longrightarrow & \mathrm{PO} & \longrightarrow & \ell_f^* & \longrightarrow & 0 \end{array}$$

and use (c_2). ∎

4. Triangular operators

An operator $T \in \mathfrak{L}(Z_2)$ is said to be **compatible** with the presentation (P_1) if it satisfies $T[i[\ell_2]] \subset i[\ell_2]$. This occurs if and only if its corresponding matrix $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right)$ is **upper triangular**, namely, $\delta = 0$. In that case, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2 & \xrightarrow{i} & Z_2 & \xrightarrow{p} & \ell_2 & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \ell_2 & \xrightarrow{i} & Z_2 & \xrightarrow{p} & \ell_2 & \longrightarrow & 0 \end{array}$$

is commutative. The operator $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right)$ above is said to be **compatible** with the presentation (P_2) if it satisfies $T[j[\ell_f]] \subset j[\ell_f]$, and this occurs if and only if its matrix is **lower triangular**, namely, $\beta = 0$. In that case, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_f & \xrightarrow{j} & Z_2 & \xrightarrow{q} & \ell_f^* & \longrightarrow & 0 \\ & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \ell_f & \xrightarrow{j} & Z_2 & \xrightarrow{q} & \ell_f^* & \longrightarrow & 0. \end{array}$$

is commutative.

Lemma 4.1 (Necessary conditions): *If* $T = \left(\begin{smallmatrix}\alpha & \beta\\ 0 & \gamma\end{smallmatrix}\right) : Z_2 \longrightarrow Z_2$ *is a bounded operator then:*

(a'+g') $\alpha : \ell_2 \longrightarrow \ell_2$ *and* $\gamma : \ell_2 \longrightarrow \ell_2$ *are bounded operators.*

(k) $\alpha - \gamma$ *is compact.*

Proof. Since $\delta = 0$, $\alpha : \ell_2 \longrightarrow \ell_2$ and $\gamma : \ell_2 \longrightarrow \ell_2$ are bounded by (c_0) in Lemma 3.3 and (g+dK) in Lemma 3.1.

(k) was proved in [10, Corollary 5.9]. ∎

Next we give a characterization of upper triangular operators.

Theorem 4.2: *An operator* $T = \left(\begin{smallmatrix}\alpha & \beta\\ 0 & \gamma\end{smallmatrix}\right)$ *is bounded on* Z_2 *if and only if* α, γ *and* $\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma + \beta$ *are bounded maps from* ℓ_2 *into* ℓ_2.

Proof. Suppose $T \in \mathfrak{L}(Z_2)$. If $(\omega, x) \in Z_2$ then $T(\omega, x) = (\alpha\omega + \beta x, \gamma x)$, and α and γ are bounded on ℓ_2 by Lemma 4.1. Moreover, for every $x \in \ell_2$,

$$\|(\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma + \beta)x\|_2 \leq \|T(\mathsf{KP}x, x)\|_{Z_2} \leq \|T\|\|x\|_2.$$

Conversely, if α, γ and $\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma + \beta$ are bounded maps on ℓ_2 then

$$\begin{aligned}\|(\alpha\omega + \beta x, \gamma x)\|_{Z_2} &= \|\alpha\omega + \beta x - \mathsf{KP} \circ \gamma x\|_2 + \|\gamma x\|_2\\ &\leq \|\alpha(\omega - \mathsf{KP}x)\|_2 + \|\alpha \circ \mathsf{KP}x + \beta x - \mathsf{KP} \circ \gamma x\|_2 + \|\gamma x\|_2\\ &\leq (\|\alpha\| + \|\alpha \circ \mathsf{KP} - \mathsf{KP} \circ \gamma + \beta\| + \|\gamma\|)\|(\omega, x)\|_{Z_2}.\end{aligned}$$

Thus $T \in \mathfrak{L}(Z_2)$. ∎

A few variations of the previous result are possible:

Proposition 4.3:

(a) $S = \left(\begin{smallmatrix}\alpha & 0\\ \delta & 0\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\mathsf{KP} \circ \delta - \alpha : \ell_f^* \longrightarrow \ell_2$ *and* $\delta : \ell_f^* \longrightarrow \ell_2$ *are bounded. If so,* $S \in \mathfrak{S}$. *In particular,* $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 0\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\alpha \in \mathfrak{L}(\ell_f^*, \ell_2)$; *and* $\left(\begin{smallmatrix}0 & 0\\ \delta & 0\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\delta \in \mathfrak{L}(\ell_f^*, \ell_f)$.

(b) $R = \left(\begin{smallmatrix}0 & \beta\\ 0 & \gamma\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\mathsf{KP} \circ \gamma - \beta : \ell_2 \longrightarrow \ell_2$ *and* $\gamma : \ell_2 \longrightarrow \ell_2$ *are bounded. If so,* $R \in \mathfrak{S}$. *In particular,* $\left(\begin{smallmatrix}0 & \beta\\ 0 & 0\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\beta \in \mathfrak{L}(\ell_2, \ell_2)$; *and* $\left(\begin{smallmatrix}0 & 0\\ 0 & \gamma\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\gamma \in \mathfrak{L}(\ell_2, \ell_f)$.

(c) $T = \left(\begin{smallmatrix}\alpha & \beta\\ 0 & 0\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\alpha \circ \mathsf{KP} + \beta : \ell_2 \longrightarrow \ell_2$ *and* $\alpha : \ell_2 \longrightarrow \ell_2$ *are bounded. If so,* $T \in \mathfrak{S}$.

(d) $M = \left(\begin{smallmatrix}0 & 0\\ \delta & \gamma\end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ *if and only if* $\delta \circ \mathsf{KP} + \gamma : \ell_2 \longrightarrow \ell_f$ *and* $\delta : \ell_2 \longrightarrow \ell_f$ *are bounded. If so,* $M \in \mathfrak{S}$.

Proof. (a) Suppose $S \in \mathfrak{L}(Z_2)$. If $(\omega, x) \in Z_2$ then $S(\omega, x) = (\alpha\omega, \delta\omega)$ does not depend on x. Moreover,

$$\|S(\omega, \mathsf{KP}^{-1}\omega)\|_{Z_2} = \|(\alpha - \mathsf{KP} \circ \delta)\omega\|_2 + \|\delta\omega\|_2 \leq \|S\|\|(\omega, \mathsf{KP}^{-1}\omega)\|_{Z_2}.$$

By Equation (1),

$$m\|(\omega, \mathsf{KP}^{-1}\omega)\| \leq \|\omega\|_{\ell_f^*}.$$

Hence $\alpha - \mathsf{KP} \circ \delta : \ell_f^* \longrightarrow \ell_2$ and $\delta : \ell_f^* \longrightarrow \ell_2$ are bounded, and so is the operator $B : \ell_f^* \to Z_2$ defined by $B\omega = (\alpha\omega, \delta\omega)$. Since $S = Bq$, we get $S \in \mathfrak{S}$ by Corollary 1.2.

Conversely, if $\mathsf{KP} \circ \delta - \alpha : \ell_f^* \longrightarrow \ell_2$ and $\delta : \ell_f^* \longrightarrow \ell_2$ are bounded, then

$$\|S(\omega, x)\|_{Z_2} = \|(\alpha - \mathsf{KP} \circ \delta)\omega\|_2 + \|\delta\omega\|_2 \leq C_1\|\omega\|_{\ell_f^*}.$$

By Equation (1),

$$\|\omega\|_{\ell_f^*} \leq M\|(\omega, x)\|_{Z_2};$$

hence $S \in \mathfrak{L}(Z_2)$.

The equivalences in (b) and (c) follow directly from Theorem 4.2. Moreover, $R \in \mathfrak{S}$ because $Cx = (\beta x, \gamma x)$ defines $C \in \mathfrak{L}(\ell_2, Z_2)$,

$$\|(\beta x, \gamma x)\|_{Z_2} = \|R(\mathsf{KP}x, x)\|_{Z_2} \leq \|R\|\|x\|_2,$$

and $R = Cp$, and $T \in \mathfrak{S}$ because $R(T) \subset i[\ell_2]$.

(d) Suppose $M \in \mathfrak{L}(Z_2)$. If $(\omega, x) \in Z_2$ then $M(\omega, x) = (0, \delta\omega + \gamma x)$. Since $R(M) \subset j[\ell_2]$, we get $M \in \mathfrak{S}$. Moreover, if $\omega \in \ell_2$, then $M(\omega, 0) = (0, \delta\omega)$. Thus $\delta : \ell_2 \longrightarrow \ell_f$ is bounded. Also, if $x \in \ell_2$, then

$$M(\mathsf{KP}x, x) = (0, (\delta \circ \mathsf{KP} + \gamma)x).$$

Hence $\delta \circ \mathsf{KP} + \gamma : \ell_2 \longrightarrow \ell_f$ is bounded.

Conversely, if $\delta \circ \mathsf{KP} + \gamma : \ell_2 \longrightarrow \ell_f$ and $\delta : \ell_2 \longrightarrow \ell_f$ are bounded, since $\delta\omega + \gamma x = \delta(\omega - \mathsf{KP}x) + (\delta \circ \mathsf{KP} + \gamma)x$, we obtain

$$\begin{aligned}\|T(\omega, x)\|_{Z_2} &\leq C_3\|\delta\omega + \gamma x\|_{\ell_f} \leq C_3(\|\delta(\omega - \mathsf{KP}x)\|_{\ell_f} + \|(\delta \circ \mathsf{KP} + \gamma)x\|_{\ell_f}) \\ &\leq C_4\|(\omega, x)\|_{Z_2}. \qquad \blacksquare\end{aligned}$$

Johnson, Lindenstrauss and Schechtman [18] asked whether every operator $T : Z_2 \to Z_2$ is a strictly singular perturbation of an operator sending $i[\ell_2]$ into itself. We could also consider operators sending $j[\ell_f]$ into itself and formulate the corresponding conjecture. One has the following partial result:

PROPOSITION 4.4: *Let* $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$.

(1) *If* $\delta \in \mathfrak{L}(\ell_f^*, \ell_f)$ *then* T *is a strictly singular perturbation of an upper triangular operator.*
(2) *If* $\beta \in \mathfrak{L}(\ell_2, \ell_2)$ *then* T *is a strictly singular perturbation of a lower triangular operator.*

Proof. (1) follows from $\left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)$ and Proposition 4.3(a), and (2) follows from $\left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & 0 \\ \delta & \gamma \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)$ and Proposition 4.3(b). ∎

5. Complemented copies of Z_2 in Z_2

Kalton [19] considered the continuous alternating bilinear form $\Omega : Z_2 \times Z_2 \to \mathbb{R}$ given by

$$\Omega\langle(\omega_1, x_1), (\omega_2, x_2)\rangle = \langle\omega_1, x_2\rangle - \langle\omega_2, x_1\rangle,$$

and for every $T \in \mathfrak{L}(Z_2)$ he defined an operator $T^+ : Z_2 \longrightarrow Z_2$ by

$$\Omega(T^+y, z) = \Omega(y, Tz) \quad \text{for } y, z \in Z_2.$$

Note that $(Dx)y = -\Omega(x, y)$ for $x, y \in Z_2$. Hence $T^+ = D^{-1}T^*D \in \mathfrak{L}(Z_2)$, where $T^* \in \mathfrak{L}(Z_2^*)$ is the conjugate operator of T. Indeed, for $x, y \in Z_2$ we have

$$(DT^+x)y = -\Omega(T^+x, y) = -\Omega(x, Ty) = Dx(Ty) = (T^*Dx)y;$$

hence $DT^+ = T^*D$. Moreover, the map $T \longrightarrow T^+$ is an involution on $\mathfrak{L}(Z_2)$ [31, Definition 11.14].

Since D is a bijective isometry, most of the properties of T^+ coincide with those of T^*. Namely, $\|T\| = \|T^+\|$, $R(T)$ is closed if and only if $R(T^+)$ is so, T is an isomorphism into if and only if T^+ is surjective, and $T \in \Phi_-$ if and only if $T^+ \in \Phi_+$. In particular, $T^+ = T$ and $T \in \Phi_+$ imply $T \in \Phi$.

It is easy to check that if $T = \left(\begin{smallmatrix} \alpha & \beta \\ \delta & \gamma \end{smallmatrix}\right)$ then $T^* = \left(\begin{smallmatrix} \gamma^* & \beta^* \\ \delta^* & \alpha^* \end{smallmatrix}\right)$ and $T^+ = \left(\begin{smallmatrix} \gamma^* & -\beta^* \\ -\delta^* & \alpha^* \end{smallmatrix}\right)$.

Definition 5.1 ([19]): A subspace E of Z_2 is said to be **isotropic** when

$$\Omega(u, v) = 0 \quad \text{for every } u, v \in E.$$

An operator $T \in \mathfrak{L}(Z_2)$ is **isotropic** when its range is isotropic, equivalently, when $T^+T = 0$.

Clearly $i[\ell_2]$, $j[\ell_f]$ and $\{(x, x) \in Z_2 : x \in \ell_f\}$ are isotropic subspaces of Z_2. Moreover, the operator $i\,p = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{L}(Z_2)$ is isotropic, non-compact, strictly singular and strictly cosingular, with $R(i\,p) = N(i\,p)$ and $(i\,p)^+ = -i\,p$.

Lemma 5.2: *For each $T \in \mathfrak{L}(Z_2)$, $R(T) \cap N(T^+)$ is an isotropic subspace.*

Proof. Let $(\omega_1, x_1), (\omega_2, x_2) \in R(T) \cap N(T^+)$. Then $(\omega_2, x_2) = T(\omega, x)$ for some $(\omega, x) \in Z_2$ and

$$\Omega\langle(\omega_1, x_1), (\omega_2, x_2)\rangle = \Omega\langle(\omega_1, x_1), T(\omega, x)\rangle = \Omega\langle T^+(\omega_1, x_1), (\omega, x)\rangle = 0,$$

concluding the proof. ∎

Proposition 5.3: *Let $T \in \mathfrak{L}(Z_2)$.*

(a) *If T^+T is strictly singular then so is T.*
(b) *If Ti is strictly singular then so is T.*
(c) *If $Ti \in \Phi_+$ then $T \in \Phi_+$.*

Proof. (a) and (b) are [19, Theorem 9 and Lemma 5].

(c) Suppose that $T \notin \Phi_+$. Then there exists an infinite-dimensional subspace $M \subset Z_2$ such that $T|_M$ is compact. Since p is strictly singular, we can find a normalized basic sequence (x_n) in M and sequences (x_n^*) in Z_2^* biorthogonal to (x_n) with $C = \sup_n \|x_n^*\| < \infty$ and (y_n) in $i[\ell_2] = N(p)$ with $\|x_n - y_n\| < 2^{-n}/C$. Then

$$Kx = \sum_{i=1}^{\infty} x_i^*(x)(x_i - y_i)$$

defines a compact operator $K \in \mathfrak{L}(Z_2)$ with $\|K\| < 1$ such that, denoting by N the closed subspace generated by (x_n), we have $(I - K)[N] \subset i[\ell_2]$. We claim that $T|_{(I-K)[N]}$ is compact, hence $Ti \notin \Phi_+$.

Indeed, if (w_n) is a bounded sequence in $(I - K)[N]$ then $w_n = (I - K)z_n$ with (z_n) a bounded sequence in N, and $(Tw_n) = (Tz_n - TKz_n)$ has a convergent subsequence because both (Tz_n) and (TKz_n) are relatively compact sequences. ∎

Lemma 5.4: *Let $T \in \mathfrak{L}(Z_2)$. If $T \in \Phi_+$ then $T^+T \in \Phi$.*

Proof. Since $(T^+T)^+ = T^+T$, it is enough to show that $T^+T \in \Phi_+$. Suppose that $T^+T \notin \Phi_+$. Then $T^+Ti \notin \Phi_+$; hence there exists a normalized block basis sequence $(w^{(n)})$ in ℓ_2 such that, if we denote by $i_w : \ell_2 \to \ell_2$ the isometric embedding defined by $i_w e_n = w^{(n)}$, then T^+Tii_w is compact. Let W be the block operator associated to the sequence $(w^{(n)})$ in [19, Section 4]. Since $T^+TWie_n = T^+Tii_w e_n$ for each $n \in \mathbb{N}$, T^+TWi is compact; hence $T^+TW \in \mathfrak{S}$ by (b) in Proposition 5.3. Thus $W^+T^+TW \in \mathfrak{S}$, hence $TW \in \mathfrak{S}$ by (a) in Proposition 5.3, implying $T \notin \Phi_+$. ∎

It is known [2, Theorem 16.16] that every infinite-dimensional complemented subspace of Z_2 contains a further complemented subspace isomorphic to Z_2. It is not known whether every infinite-dimensional complemented subspace of Z_2 is isomorphic to Z_2 (which in particular would imply that Z_2 is isomorphic to its hyperplanes). We add now a new piece of knowledge:

Theorem 5.5: *Every subspace of Z_2 isomorphic to Z_2 is complemented.*

Proof. Let $T \in \mathfrak{L}(Z_2)$ be an isomorphism into with range $R(T)$. Then $T^+T \in \Phi$, hence there exists a finite codimensional subspace N of $R(T)$ such that $T^+|_N$ is an isomorphism and $T^+[N]$ is finite codimensional, hence complemented; thus N is complemented and so is $R(T)$. ∎

An extension of Theorem 5.5 in operator terms is available now:

Theorem 5.6: *Every semi-Fredholm operator on Z_2 has complemented kernel and range.*

Proof. Let T be an operator on Z_2. If $T \in \Phi_+$ then the kernel is finite-dimensional, and we can prove that $R(T)$ is complemented with the proof of Theorem 5.5. If $T \in \Phi_-$ then $R(T)$ is closed finite codimensional and $T^* \in \Phi_+$ (in Z_2^*). Since $Z_2^* \simeq Z_2$, $R(T^*)$ is complemented by the first part, hence $N(T) = ^\perp R(T^*)$ is also complemented. ∎

Recall from [1] that a Banach space X is said to be Y**-automorphic** if every isomorphism between two infinite codimensional subspaces of X isomorphic to Y can be extended to an automorphism of X. It is clear that ℓ_2 is ℓ_2-automorphic and that Z_2 is not ℓ_2-automorphic: indeed, $Z_2 \simeq Z_2 \oplus Z_2$, and an isomorphism between the subspaces $\ell_2 \oplus 0$ and $\ell_2 \oplus \ell_2$ cannot be extended to an automorphism of $Z_2 \oplus Z_2$, because the corresponding quotients $\ell_2 \oplus Z_2$ and $\ell_2 \oplus \ell_2$ are not isomorphic.

Surprisingly, one has:

Proposition 5.7: *Z_2 is Z_2-automorphic.*

Proof. As Kalton remarks in [19, p. 110], Pełczyński's decomposition argument shows that if E is a complemented subspace of Z_2 and $E \oplus E \simeq E$ then E is isomorphic to Z_2. Suppose that $Z_2 \simeq Z_2 \oplus F$. By Theorem 1.1(2) one has $F \simeq Z_2 \oplus N$, and thus $F \oplus F \simeq F \oplus Z_2 \oplus N \simeq Z_2 \oplus N \simeq F$. Hence $F \simeq Z_2$. ∎

Now if E is an infinite-dimensional complemented subspace of Z_2, then

$$E \simeq Z_2 \oplus E' \simeq Z_2 \oplus Z_2 \oplus E' \simeq Z_2 \oplus E$$

and thus E is isomorphic to its 2-codimensional subspaces since

$$E \oplus \mathbb{K}^2 \simeq Z_2 \oplus E \oplus \mathbb{K}^2 \simeq Z_2 \oplus E \simeq E.$$

We conjecture that $E \oplus E \simeq Z_2$.

Since $Z_2^* \simeq Z_2$ and i is strictly cosingular [21] (hence i^* strictly singular), one can easily derive the following results, by duality, from the previous ones:

Proposition 5.8: *Let $T \in \mathfrak{L}(Z_2)$.*

(a) *If T^+T is strictly cosingular then so is T^+.*
(b) *If pT is strictly cosingular then so is T.*
(c) *If $pT \in \Phi_-$ then $T \in \Phi_-$.*
(d) *If $T^+ \in \Phi_-$ then $T^+T \in \Phi$.*

6. Examples of operators on Z_2

6.1. Rank-one operators. Given $(x^*, \omega^*) \in Z_2^*$ and $(u, v) \in Z_2$, the rank-one operator $(x^*, \omega^*) \otimes (u, v)$ acts on Z_2 as follows. For each $(\omega, x) \in Z_2$,

$$[(x^*, \omega^*) \otimes (u, v)](\omega, x) = \langle (x^*, \omega^*), (\omega, x) \rangle \cdot (u, v) = (\omega\omega^* + x^*x) \cdot (u, v).$$

Thus the matrix associated to $(x^*, \omega^*) \otimes (u, v)$ is

$$\begin{pmatrix} \omega^* \otimes u & x^* \otimes u \\ \omega^* \otimes v & x^* \otimes v \end{pmatrix}.$$

6.2. Nuclear operators. Fix $u^* \in Z_2^*$ and $v \in Z_2$ and obtain the matrix representation for the one-dimensional map $u^* \otimes v$. If $u^* = \sum a_n u_n^*$ and $v = \sum b_n v_n$, set

$$\mathbf{u}(2n-1)^* = \sum a_{2n-1} u_{2n-1}$$

the "odd" part of u^* and

$$\mathbf{u}(2n)^* = \sum a_{2n} u_{2n}$$

the "even" part of u^*. Define in the same manner the odd and even parts $\mathbf{v}(2n-1)$ and $\mathbf{v}(2n)$ of v to obtain

$$u^* \otimes v = \begin{pmatrix} \mathbf{u}(2n-1)^* \otimes \mathbf{v}(2n-1) & \mathbf{u}(2n)^* \otimes \mathbf{v}(2n-1) \\ \mathbf{u}(2n-1)^* \otimes \mathbf{v}(2n) & \mathbf{u}(2n)^* \otimes \mathbf{v}(2n) \end{pmatrix}.$$

Consequently, if $T = \sum u_k^* \otimes v_k$ is a nuclear operator on Z_2 one gets

$$T = \begin{pmatrix} \sum_k \mathbf{u}_k(2n-1)^* \otimes \mathbf{v}_k(2n-1) & \sum_k \mathbf{u}_k(2n)^* \otimes \mathbf{v}_k(2n-1) \\ \sum_k \mathbf{u}_k(2n-1)^* \otimes \mathbf{v}_k(2n) & \sum_k \mathbf{u}_k(2n)^* \otimes \mathbf{v}_k(2n) \end{pmatrix}.$$

6.3. Operators acting on the scale of ℓ_p spaces. The fact that Z_2 is the derived space at ℓ_2 for the scale of ℓ_p spaces provides some elements of $\mathfrak{L}(Z_2)$. We say that an operator $\alpha : \ell_2 \longrightarrow \ell_2$ **acts on the scale** when there are $1 \leq p < 2 < q \leq \infty$ such that both $\alpha : \ell_p \to \ell_p$ and $\alpha : \ell_q \to \ell_q$ are bounded. In this case $\tau_\alpha = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ is a continuous operator on Z_2 (see [9, Proposition 3.6]).

Against the naïve intuition, τ_α can be a bounded operator on Z_2 and still α is not necessarily an operator acting on the scale, as the following simple example shows: Let $z \in \ell_f$ such that $z \notin \ell_p$ for $p < 2$. Then $\alpha_0(x) = e_1^*(x)z$ defines a bounded operator on ℓ_2 but $\alpha_0(\ell_p) \not\subset \ell_p$ for $p < 2$; thus α_0 does not act in the scale. However, τ_{α_0} is a bounded operator on Z_2 because $(0, e_1^*), (e_1^*, 0) \in Z_2^*$, $(z, 0), (0, z) \in Z_2$ and $\tau_{\alpha_0} = (0, e_1^*) \otimes (z, 0) + (e_1^*, 0) \otimes (0, z)$.

We next present several natural examples of operators α acting on the scale:

(1) α a diagonal operator D_σ with $\sigma \in \ell_\infty$, or α a right (or left) shift operator.

(2) α a surjective isometry on ℓ_p for some $p \neq 2$. It was proved in [25, Proposition 2.f.14] that these operators have the form $\alpha((x_n)_n) = (\varepsilon_n x_{\sigma(n)})_n$ for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and some sequence of signs $(\varepsilon_n)_n$. The induced operator τ_α is then an isometry [20].

(3) α the **Cesàro operator** C defined by $C((x_n)_n) = (\frac{1}{n} \sum_{k=1}^n x_k)_n$.

It is bounded on ℓ_p for $p > 1$ with $\|C\|_p = \frac{p}{p-1}$ [30]. It is not bounded on ℓ_1 since $C(e_1)$ is the harmonic series. See [4] for the properties of C as an operator on ℓ_2.

(4) α a **Hilbert matrix operator** H_λ defined by a Hilbert matrix

$$\left(\frac{1}{n+m+\lambda}\right)_{n,m=0}^{\infty}, \quad \lambda \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$

The operator H_λ is bounded on ℓ_p for $1 < p < \infty$. See [32].

(5) α a **Hausdorff operator** A associated to a sequence of complex scalars $\{\mu_n : n = 0, 1, 2, \ldots\}$ (see [3, Section 3.4]). It has the form

$$A((x_k)_k) = \left(\sum_{k=0}^{n} a_{nk} x_k \right)_n$$

with

$$a_{nk} = \begin{cases} \binom{n}{k} \Delta^{n-k} \mu_k, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

where $\Delta(\mu_n) = \mu_n - \mu_{n+1}$. Many Hausdorff operators are bounded on some spaces ℓ_p [3, 30]:

(i) The **generalized Cesàro operator** C_a^α arising from

$$\mu_n = \frac{\Gamma(a+\alpha)\,\Gamma(n+a)}{\Gamma(a)\,\Gamma(n+a+\alpha)},$$

where Γ is the Gamma function. For $\alpha > 0$, $a > 1/p$ the operator C_a^α is bounded on ℓ_p $(p > 1)$ with norm equal to

$$\frac{\Gamma(a+\alpha)\Gamma(a-1/p)}{\Gamma(a+\alpha-1/p)}.$$

Note that $C_1^1 = C$, the Cesàro operator.

(ii) The **Hölder operator** H_α arising from $\mu_n = (n+1)^{-\alpha}$. For $\alpha > 0$, H_α is bounded on ℓ_p with norm $(\frac{p}{p-1})^\alpha$.

(iii) The **Euler operator** (E, r) arising from $\mu_n = a^n$, where $0 < a < 1$, $r = \frac{1-a}{a}$ and its norm on ℓ_p is $(1+r)^{1/p}$.

(iv) The **Gamma operator** Γ_a^α arising from $\mu_n = (\frac{a}{n+a})^\alpha$, where $\alpha > 0$, and it is bounded on ℓ_p with norm $(\frac{a}{a-1/p})^\alpha$ whenever $a > 1/p$.

We do not know whether every Hausdorff operator which is bounded on ℓ_2 acts on the scale.

The following result belongs to Sneiberg [33]: Let (X_0, X_1) and (Y_0, Y_1) be two interpolation pairs such that $T : X_i \to Y_i$ is bounded for $i = 0, 1$. If $T^{-1} : X_\theta \to Y_\theta$ exists and is bounded for some $0 < \theta < 1$, then there is $\varepsilon > 0$ such that $T^{-1} : X_s \to Y_s$ exists and is bounded for $|s - \theta| < \varepsilon$. We can infer from that

PROPOSITION 6.1: *For an operator $\alpha : \ell_2 \to \ell_2$ acting on the scale, the following statements are equivalent:*

(i) τ_α *is an isomorphism.*
(ii) $\alpha : \ell_2 \to \ell_2$ *is an isomorphism.*
(iii) *There exists* $\varepsilon > 0$ *such that* $\alpha : \ell_p \to \ell_p$ *is an isomorphism for all* $|2 - p| < \varepsilon$.

Consequently $\sigma(\tau_\alpha) = \sigma(\alpha)$.

Proof. If α acts on the couple (ℓ_p, ℓ_{p^*}) then the operator $\alpha - \lambda I$ also acts on that same couple. Moreover, if τ_α is an isomorphism then $\alpha : \ell_2 \to \ell_2$ is an isomorphism. And that if both α and α^{-1} act on some scale (ℓ_p, ℓ_{p^*}) then τ_α is an isomorphism on Z_2. ∎

If α is an operator acting on the scale, its spectrum on ℓ_p may be independent of p, as is the case of diagonal operators or the left and right shift operators, but it also may vary with p: Leibowitz [24] proved that, for $1 < p < \infty$ the Cesàro operator C on ℓ_p has no eigenvalues and its spectrum is

$$\sigma(C) = \{\lambda \in \mathbb{C} \colon |\lambda - p^*/2| \leq p^*/2\}.$$

Moreover, $\lambda I - C$ is a Fredholm operator with index -1 for $|\lambda - p^*/2| < p^*/2$, and has dense proper range for $|\lambda - p^*/2| = p^*/2$ [15].

The Hilbert matrix operator H_1 on ℓ_2 has no eigenvalues and $\sigma(H_1)$ is the interval $[0, \pi]$, see [26], while the spectrum on ℓ_p varies with p and has eigenvalues for $p > 2$ and residual points for $p < 2$ [32]. Since the matrix representing H_1 is symmetric, the conjugate operator of $H_1 : \ell_p \to \ell_p$ is $H_1 : \ell_{p^*} \to \ell_{p^*}$. Thus the spectra of H_1 on ℓ_p and ℓ_{p^*} coincide.

6.4. OPERATORS ON THE CALDERÓN SPACE. We obtain operators on Z_2 by picking operators on the Calderón space $T : \mathcal{C} \to \mathcal{C}$ such that

$$T[\ker \delta \cap \ker \delta'] \subset \ker \delta \cap \ker \delta'.$$

The simplest way to do that is to pick an operator τ on the scale and then set $T(f)(z) = \tau(f(z))$. If $\varphi : \mathbb{S} \to \mathbb{D}$ is a conformal map, then the operator $S(f)(z) = \tau(\varphi(z)f(z))$ induces $\left(\begin{smallmatrix} 0 & \tau \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, given two operators α, ϕ on the scale, $T(f)(z) = \alpha(f(z)) + \beta(\varphi(z)f(z))$ induces the upper triangular operator $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right)$ on Z_2.

6.5. Diagonal operators. The continuity of diagonal operators D_σ on Z_2 is related to the unconditional structure of the space. Recall that the sequence $(u_n)_{n\in\mathbb{N}}$ given by $u_{2n-1} = (e_n, 0)$ and $u_{2n} = (0, e_n)$, where $(e_n)_n$ is the canonical basis of ℓ_2, is a basis on Z_2 which is not unconditional. Therefore not all $\sigma \in \ell_\infty$ define a diagonal operator on Z_2. Let us denote $a = (\sigma_{2n-1})$ and $b = (b_n = \sigma_{2n})$. If $D_\sigma = \left(\begin{smallmatrix} D_a & 0 \\ 0 & D_b \end{smallmatrix}\right)$ is an operator on Z_2 then $D_a - D_b = D_{a-b}$ is compact by Lemma 4.1; thus $a - b \in c_0$. It is startling that an additional condition is required:

Proposition 6.2: *A diagonal operator $D_\sigma : Z_2 \to Z_2$ defined by a monotone decreasing sequence σ is bounded if and only if* $(|\sigma_{2k-1} - \sigma_{2k}|)_{k\in\mathbb{N}} = \mathrm{O}(\frac{1}{\log n})$.

Proof. Cabello and García showed in [7] that a diagonal operator $D_a : \ell_2 \to \ell_2$ can be lifted to Z_2 if and only if the decreasing rearrangement sequence $(a_n^*)_n$ is $\mathrm{O}(\frac{1}{\log n})$. The self duality of the Kalton–Peck space yields the result. ∎

6.6. Block operators. Let $U = (u_n)$ be a bounded sequence of disjointly supported blocks in ℓ_2. We define a bounded operator $u : \ell_2 \to \ell_2$ by

$$ux = \sum x_n u_n.$$

Kalton [19] defined the operator T_U on Z_2 by

$$T_U(e_n, 0) = u_n \quad \text{and} \quad T_U(0, e_n) = (\mathsf{KP}u_n, u_n),$$

and proved that it is an into isometry. Let us call T_U a **block operator.** As we said in Section 6.3, if α is an operator acting on the scale then τ_α is an operator on Z_2. In general, a perturbation of τ_α is required to make u an upper triangular operator. In particular, the operator u defined by a sequence of disjointly supported normalized blocks in ℓ_1 is not an operator on the scale. In [11, Theorem 4.6] it is explained how to obtain the required perturbation and how this perturbation yields the Kalton block operator $T_U = \left(\begin{smallmatrix} u & \mathsf{KP}u \\ 0 & u \end{smallmatrix}\right)$ mentioned above.

7. Operator ideals on Z_2

The classes $\mathfrak{S}$ and $\mathfrak{C}$ are not dual to each other. In general $T^* \in \mathfrak{S} \Longrightarrow T \in \mathfrak{C}$ and $T^* \in \mathfrak{C} \Longrightarrow T \in \mathfrak{S}$. But since Z_2 is reflexive, it turns out that an operator $T : Z_2 \to Z_2$ is strictly singular (resp. cosingular) if and only if $T^* : Z_2^* \to Z_2^*$ is strictly cosingular (resp. singular). One moreover has

Theorem 7.1: *$\mathfrak{S}(Z_2) = \mathfrak{C}(Z_2) = \mathfrak{In}(Z_2)$. Moreover, that set contains every proper ideal of $\mathfrak{L}(Z_2)$.*

Proof. By the first part of Theorem 1.1, if $S \in \mathfrak{L}(Z_2) \setminus \mathfrak{S}(Z_2)$ then there exist $A, B \in \mathfrak{L}(Z_2)$ so that $ASB = I_{Z_2}$; hence S does not belong to any proper operator ideal, and $\mathfrak{In}(Z_2) \subset \mathfrak{S}(Z_2)$. Since $\mathfrak{S}(X)$ and $\mathfrak{C}(X)$ are contained in $\mathfrak{In}(X)$ for each X and $Z_2 \simeq Z_2^*$, the equalities follow. ∎

Observe that $\mathfrak{S}(Z_2, \ell_\infty)) \neq \mathfrak{L}(Z_2, \ell_\infty) = \mathfrak{In}(Z_2, \ell_\infty)$: let $T \in \mathfrak{L}(Z_2, \ell_\infty)$. For every $A \in \mathfrak{L}(\ell_\infty, Z_2) = \mathfrak{S}(\ell_\infty, Z_2)$, $I - AT$ is Fredholm. Similarly,

$$\mathfrak{C}(\ell_1, Z_2)) \neq \mathfrak{L}(\ell_1, Z_2) = \mathfrak{In}(\ell_1, Z_2).$$

The next result is a dual version of the second part of Theorem 1.1.

Proposition 7.2: *Let $T \in \mathfrak{L}(X, Z_2)$. If $T \notin \mathfrak{C}$, then there exists a complemented subspace N of Z_2 with Z_2/N isomorphic to Z_2 such that $Q_N T$ is surjective, where $Q_N : Z_2 \to Z_2/N$ is the quotient map.*

Proof. If $T \in \mathfrak{L}(X, Z_2)$ is not in $\mathfrak{C}$ then $T^* \in \mathfrak{L}(Z_2^*, X^*)$ is not in $\mathfrak{S}$. Since $Z_2^* \simeq Z_2$, by Theorem 1.1 there exists a complemented subspace M of Z_2^* isomorphic to Z_2^* such that $T^*|_M$ is an isomorphism. Then $N = {}^{\perp}M$ is a subspace of Z_2 satisfying the required conditions. ∎

Let $\mathfrak{K}$ be the class of compact operators and let $L_p \equiv L_p(0,1)$ for $1 \leq p \leq \infty$. Then $\mathfrak{S}(L_p) \neq \mathfrak{K}(L_p)$ for $p \neq 2$ [27], but $T \in \mathfrak{S}(L_p)$ implies $T^2 \in \mathfrak{K}(L_p)$ [14].

Theorem 7.3: *$\mathfrak{S}(Z_2) \neq \mathfrak{K}(Z_2)$, and $S, T \in \mathfrak{S}(Z_2)$ implies $ST \in \mathfrak{K}$.*

Proof. As we mentioned before, $i\,p \in \mathfrak{L}(Z_2)$ is strictly singular but not compact.

For the remaining part, recall that an operator S acting on a reflexive space X is compact if and only if for every normalized weakly null sequence (x_n) in X, (Sx_n) has a norm null subsequence; and it was proved in [21, Theorem 5.4] that every normalized weakly null sequence (x_n) in Z_2 has a subsequence equivalent either to the (usual) basis of ℓ_2 or to the (usual) basis on ℓ_f.

Let $S, T \in \mathfrak{S}(Z_2)$ and let (x_n) be a normalized weakly null sequence Z_2. If (x_{n_k}) is a subsequence equivalent to the basis of ℓ_f, then (Tx_{n_k}) has no subsequence equivalent to the basis of ℓ_f because T is strictly singular; hence (Tx_{n_k}) has a subsequence equivalent to the basis of ℓ_2 or it is norm null. Also, if (x_{n_k}) is a subsequence equivalent to the basis of ℓ_2 then (Tx_{n_k}) has no subsequence equivalent to the basis of ℓ_2, and has no subsequence equivalent to the basis of ℓ_f because $\ell_f \subsetneq \ell_2$: a bounded operator cannot take the unit basis of ℓ_2 to the unit basis of ℓ_f; hence it is norm null. In each case, (STx_n) has a norm null subsequence. ∎

The **perturbation class** of a class of operators $\mathcal{A} \subset \mathfrak{L}$ is defined by its components as follows when $\mathcal{A}(X,Y) \neq \emptyset$:

$$P\mathcal{A}(X,Y) = \{L \in \mathfrak{L}(X,Y)\colon T + L \in \mathcal{A}(X,Y) \text{ for all } T \in \mathcal{A}(X,Y)\}.$$

Kato and Vladimirskii (see [29, Section 26.6]) proved that $\mathfrak{S} \subset P\Phi_+$ and $\mathfrak{C} \subset P\Phi_-$, and it is known that $\mathfrak{In} = P\Phi$. The perturbation classes problem asks whether $\mathfrak{S} = P\Phi_+$ and $\mathfrak{C} = P\Phi_-$. This problem has a positive answer under certain conditions but not in general (see [16]), and also for Z_2 although this space does not verify those conditions.

Proposition 7.4: *We have* $P\Phi(Z_2) = P\Phi_+(Z_2) = P\Phi_-(Z_2) = \mathfrak{S}(Z_2)$.

Proof. In general,

$$\mathfrak{S}(X) \subset P\Phi_+(X) \subset P\Phi(X) = \mathfrak{In}(X) \quad \text{and} \quad \mathfrak{C}(X) \subset P\Phi_-(X) \subset \mathfrak{In}(X),$$

where the inclusions of $P\Phi_+(X)$ and $P\Phi_-(X)$ in $P\Phi(X)$ are a consequence of the continuity of the index $i : \Phi_\pm(X,Y) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ (see [25, Proposition 2.c.9]). Hence, if $T \in \Phi_+(X)$ and $A \in P\Phi_+(X)$ then $i(T+tA) = i(T)$ for each $t \in [0,1]$. But Theorem 7.1 implies $\mathfrak{S}(Z_2) = \mathfrak{C}(Z_2) = \mathfrak{In}(Z_2)$; so all these inclusions are equalities for $X = Z_2$. ∎

8. Further directions of research

The overall tone of these suggestions is to determine which properties of operators on ℓ_2 are valid for operators on Z_2 and which are not.

8.1. The convolution on $\mathfrak{L}(Z_2)$. Here we consider the relation between T and T^+ as operators in $\mathfrak{L}(\ell_2)$.

Question 1: Suppose that $T \in \mathfrak{L}(Z_2)$ is an isomorphism into. Is T^+T bijective? What does $T^+T = I$ or $TT^+T = T$ for $T \in \mathfrak{L}(Z_2)$ mean?

Question 2: Is $\mathfrak{L}(Z_2)/\mathfrak{S}(Z_2)$ isomorphic to a C*-algebra?

Clearly $\mathfrak{L}(Z_2)$ is not isomorphic to a C*-algebra since there exists $T \neq 0$ such that $T^+T = 0$. However, $T^+T \in \mathfrak{S}(Z_2)$ implies $T \in \mathfrak{S}(Z_2)$, and $T \in \mathfrak{S}(Z_2)$ if and only if $T^+ \in \mathfrak{S}(Z_2)$.

8.2. Polynomially bounded operators. A **contraction** is an operator T with $\|T\| \leq 1$, and for a polynomial p we denote

$$\|p\|_\infty = \sup_{|z|<1} |p(z)|.$$

Question 3: Is every contraction in $\mathfrak{L}(Z_2)$ polynomially bounded? Equivalently,

(5) $\exists\, C > 0$ so that $\|T\| \leq 1 \Rightarrow \|p(T)\| \leq C\|p\|_\infty$ for every polynomial p?

Note that (5) with $C = 1$ isometrically characterizes Hilbert spaces. Moreover, if X is isomorphic to a Hilbert space, clearly (5) holds for some $C \geq 1$; however, the converse implication fails (see [34]).

8.3. The group of invertible operators. We denote by $\mathfrak{GL}(X)$ the group of invertible operators on a Banach space X. It is known that $\mathfrak{GL}(\ell_2)$ is connected in the complex case, while $\mathfrak{GL}(\ell_p \times \ell_q)$ is not connected for $1 \leq p < q < \infty$ [13, 28].

Question 4: In the case $\mathbb{K} = \mathbb{C}$, is $\mathfrak{GL}(Z_2)$ connected?

Is the subgroup $\{T \in \mathfrak{GL}(Z_2) : T \text{ is upper triangular}\}$ connected?

The latter question could be tackled by obtaining a characterization of the invertible operators $T \in \mathfrak{L}(Z_2)$ in terms of the components $\alpha, \beta, \delta, \gamma$ of the matrix representation of T.

8.4. Representations of Z_2. A basic question whose meaning is not even clear is whether there are other "natural" presentations of Z_2 beyond the ℓ_2 and the ℓ_f presentations considered in this paper. In homological terms, since $Z_2 \simeq Z_2 \oplus Z_2$ one could obtain other nontrivial representations such as

$$0 \longrightarrow \ell_2 \longrightarrow Z_2 \longrightarrow \ell_2 \oplus Z_2 \longrightarrow 0\ ,$$

$$0 \longrightarrow \ell_f \longrightarrow Z_2 \longrightarrow \ell_f^* \oplus Z_2 \longrightarrow 0\ ,$$

etc., or even the trivial one

$$0 \longrightarrow Z_2 \longrightarrow Z_2 \longrightarrow Z_2 \longrightarrow 0\ .$$

None of these presentations are even isomorphic to either (P_1) or (P_2).

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