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INSTITUTO DE FÍSICA DE CANTABRIA (UC-CSIC)

Unravelling the dark nature of the Universe with gravitational lensing

Desentrañando la naturaleza oscura del Universo con
lentes gravitacionales

MEMORIA PRESENTADA PARA OPTAR AL TÍTULO DE DOCTOR POR LA
UNIVERSIDAD DE CANTABRIA. DOCTORADO EN CIENCIA Y TECNOLOGÍA.

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October, 2025

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En Santander, a FECHA,

Fdo.: Dr. Jose María Diego

Fdo.: Dr. Bradley James Kavanagh

Acknowledgements

This thesis gathers the last four years, during which I've learned a lot, and represents the first stage of what I hope will be a long research career. I don't really want to get sentimental, but I will nonetheless express a few words of gratitude for all the people involved.

First, I would like to thank the reviewers, for whom I feel a bit sorry as they will have to read this long and boring manuscript.

I couldn't avoid saying something about my supervisors, Chema and Bradley. I'm truly grateful to have had you by my side during these past years; teaching me, helping me through all the bottlenecks this thesis has endured, and encouraging me. I truly consider you both kind and generous people, and I hope to continue collaborating with you in the years to come. Even with Chema's dad jokes...

I would also like to say a few words about the people at IFCA. Especially Silvia, for being our shield against all the unnecessarily complicated bureaucracy. Also, all my peers in the cosmology group; particularly Patri, who opened the doors of IFCA for me, and the not-too-old-yet-not-young duo of Ana and José. They are always a pleasure to be around and have helped me with all the completely random queries I've thrown at them.

An honourable mention to my office mates; Elena, Guille, and Patricia, with whom I've spent more time than with my own family, and who in some sense became a second family. To all those ever-suffering PhD students like me, who I can now call friends and whom I wish all the best in their research and personal lives: Celia, Mikel, Rubén and especially Christian and Sergio, also my flatmates, whose eyes have witnessed my cringiest moments. Alcohol and early travels don't always mix well—they know what I mean.

II

Now, switching to my mother tongue:

Quiero agradecer a mis amigos de toda la vida, los del grupo de Cabe; Ángel R., Ángel S., Chani, Dani, Diego, Isa, por estar siempre ahí y hacerme olvidar de las preocupaciones.

También a la gente del grupo de rugby, que desde luego, saben cómo pasárselo bien, y con quienes ya me he recorrido media Europa en viajes rugbísticos que iban más de alcohol y fiesta que de rugby. (Corzos on tour).

Terminando, a Marina, quien no sé cómo me aguanta, pero sin la cual tampoco sabría qué hacer. Por todo, gracias.

Y, cómo no, a mi familia; mis padres y mi hermana, quienes siempre me han apoyado y estado ahí para mí, aunque no terminen de entender qué es lo que hago. Solo me apoyan y me demuestran que están para lo que necesite.

*Last but not least, I wanna thank
me*

Snoop Dogg

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Abstract

Since the dawn of civilisation, we have looked up at the sky with the same question in mind: where did all this come from?

Cosmology is one of humanity's oldest intellectual endeavours. What is the origin of the Universe we live in? What is it made of? How is it structured? What was it like long ago, and ultimately, how will it evolve? These questions have echoed through human thought for millennia. Yet it wasn't until Einstein introduced his theory of General Relativity in 1915 that cosmology began to take shape as a scientific discipline.

Modern cosmology is, in fact, a relatively young science, barely more than a century old. However, over the course of the second half of the twentieth century, it witnessed an extraordinary transformation. Observations of the cosmic microwave background, along with measurements of the large scale structure of the Universe and newly discovered astrophysical phenomena, have allowed us to answer some of the most fundamental questions about the cosmos. Simultaneously, breakthroughs in high energy particle physics have expanded our understanding of the Universe across its smallest scales. Together, the standard cosmological model, Λ CDM, and the Standard Model of particle physics now constitute the theoretical pillars of modern physics; powerful, elegant, and yet still incomplete and filled with uncertainties.

From the age-old queries like "Where do we come from?" or "What surrounds us?", our focus has shifted to far more intricate puzzles: How did the elements we observe come into existence? What processes led to the formation of galaxies and smaller-scale structures? And perhaps most intriguingly, why does such a large fraction of the Universe seem to be composed of two dark components whose nature remains elusive, known to us only through their subtle gravitational fingerprints on the tiny portion of the cosmos we can actually "see"?

We are now experiencing the dawn of a new era in modern cosmology, the era of high precision, driven by a wealth of high-quality cosmological and astrophysical data collected by the most sophisticated experiments to date. It is our duty as researchers to use this data to shed light on the current cosmological uncertainties and to narrow the gap between our present understanding of the Universe and its true nature.

This thesis contains my humble contribution to the field, with a particular focus on gravitational lensing as a tool that uses gravity itself to probe the constituents of the Universe. It is a compendium of three first-author publications: two on extragalactic stellar microlensing, ranging from its statistical behaviour to its application on lensed arcs for characterising stellar populations when the Universe was almost half its current age; and one exploring the effects of different dark matter models on the strong lensing features produced by galaxy-scale lenses. These publications are briefly introduced, along with a short summary of their conclusions, in the second and third part of this thesis.

This thesis is organised as follows. Chs. 1 and 2 provide brief introductions to the standard model of cosmology and a short review of dark matter, respectively. Gravitational lensing is then discussed in depth in Ch. 3. A compendium of the works presented in this thesis is compiled in Ch 4. Finally, the main conclusions of these studies are summarised in Ch. 5 (together with its Spanish counterpart, Ch. 6), wrapping up the thesis.

Resumen

Desde los albores de la civilización, hemos alzado la vista al cielo con la misma pregunta en mente: ¿de dónde viene todo esto?

La cosmología es una de las cuestiones intelectuales más antiguas de la humanidad. ¿Cuál es el origen del Universo en el que vivimos? ¿De qué está hecho? ¿Cómo está estructurado? ¿Cómo era en el pasado lejano y, en última instancia, cómo evolucionará? Estas preguntas han resonado en el pensamiento humano durante milenios. Sin embargo, no fue hasta que Einstein introdujo su teoría de la Relatividad General en 1915 que la cosmología empezó a tomar forma como una disciplina científica.

La cosmología moderna es, en realidad, una ciencia relativamente joven, apenas supera el siglo de existencia. Aun así, a lo largo de la segunda mitad del siglo XX, ha experimentado una transformación extraordinaria. Las observaciones del fondo cósmico de microondas, junto con las mediciones de la estructura a gran escala del Universo y el descubrimiento de nuevos fenómenos astrofísicos, nos han permitido responder a algunas de las preguntas más fundamentales sobre el cosmos. Paralelamente, los avances en física de partículas de altas energías han ampliado nuestra comprensión del Universo en sus escalas más pequeñas. En conjunto, el modelo cosmológico estándar, Λ CDM, y el Modelo Estándar de física de partículas constituyen hoy en día los pilares teóricos de la física moderna; modelos potentes, elegantes, pero aún incompletos y llenos de incertidumbres.

De aquellas preguntas ancestrales como “¿De dónde venimos?” o “¿Qué nos rodea?”, hemos pasado a enigmas mucho más complejos: ¿Cómo se formaron los elementos que observamos? ¿Qué procesos dieron lugar a las galaxias y estructuras a menor escala? Y quizás lo más desconcertante: ¿por qué una fracción tan grande del Universo parece estar compuesta por dos componentes oscuras cuya naturaleza desconocemos,

y que sólo logramos inferir a través de su sutil huella gravitatoria sobre la diminuta porción del cosmos que realmente podemos “ver”?

La cosmología moderna se encuentra actualmente en el amanecer de una nueva era: la era de la alta precisión, impulsada por una abundancia de datos cosmológicos y astrofísicos de alta calidad obtenidos por los experimentos más sofisticados hasta la fecha. Como investigadores, tenemos la responsabilidad de aprovechar estos datos para arrojar algo de luz sobre las actuales incertidumbres cosmológicas y reducir la brecha entre nuestro conocimiento actual del Universo y su verdadera naturaleza.

Esta tesis contiene mi humilde aportación al campo, con un enfoque particular en el efecto de lente gravitacional como herramienta para estudiar, mediante la propia gravedad, los componentes fundamentales del Universo. Se trata de un compendio de tres publicaciones lideradas por mí: dos sobre efecto de microlente en estrellas extragalácticas, desde su comportamiento estadístico hasta su aplicación en arcos gravitacionalmente amplificados para caracterizar poblaciones estelares cuando el Universo tenía aproximadamente la mitad de su edad actual; y una sobre los efectos de diferentes modelos de materia oscura en las propiedades del lente gravitacional fuerte en lentes a escala galáctica. Estas publicaciones se presentan brevemente, junto con un resumen de sus principales conclusiones, en la segunda parte de esta tesis.

Esta tesis se organiza de la siguiente manera. Los Caps. 1 y 2 introducen al modelo estándar de cosmología y una breve revisión sobre la materia oscura, respectivamente. El efecto de lente gravitatoria se discute en detalle en el Cap. 3. Un compendio de los trabajos presentados en esta tesis se recoge en el Cap. 4. Finalmente, las principales conclusiones de estos estudios se resumen en el Cap. 5 (junto con su homólogo en español, el Cap. 6), cerrando así la tesis.

Part I

Introduction

Chapter 1

Standard model of cosmology

This thesis is deeply entangled in the use of gravitational lensing to probe cosmology itself and to better understand the astrophysical phenomena we observe, with a particular focus on dark matter (DM). It is natural then to begin with an introduction to cosmology. Cosmology is a relatively young and constantly evolving science, and the picture we have today differs significantly from that of Einstein and his peers at the time he developed the general theory of relativity ([Einstein 1916](#)) (GR). It is equally likely that the picture we will have in 50 or 100 years will be drastically different.

With that in mind, in this chapter we briefly introduce the so-called concordance model of cosmology (Λ CDM): our best current understanding of the origin, evolution, and constituents of the Universe at the time of writing this thesis. Λ CDM stands for a Universe dominated in the latest times by dark energy (DE), Λ , and filled with a cold dark matter, CDM, component.

1.1 Brief outline of the Λ CDM Universe

This section gives a short overview of the Universe's evolution according to the Λ CDM model. It does not aim to be exhaustive or technically rigorous, but rather to outline our current cosmological picture and provide some context for the topics covered in this thesis.

There is solid evidence that the Universe is currently expanding, as first measured by Edwin Hubble in 1929 ([Hubble 1929](#)). He observed that galaxies are moving away from

Earth, and that their recession velocity is proportional to their distance from us. Nearly a century later, astronomers have significantly improved distance measurements and refined the determination of the expansion rate of the Universe, reaching a precision just above 1% (Riess et al. 2022; Planck Collaboration et al. 2020a).

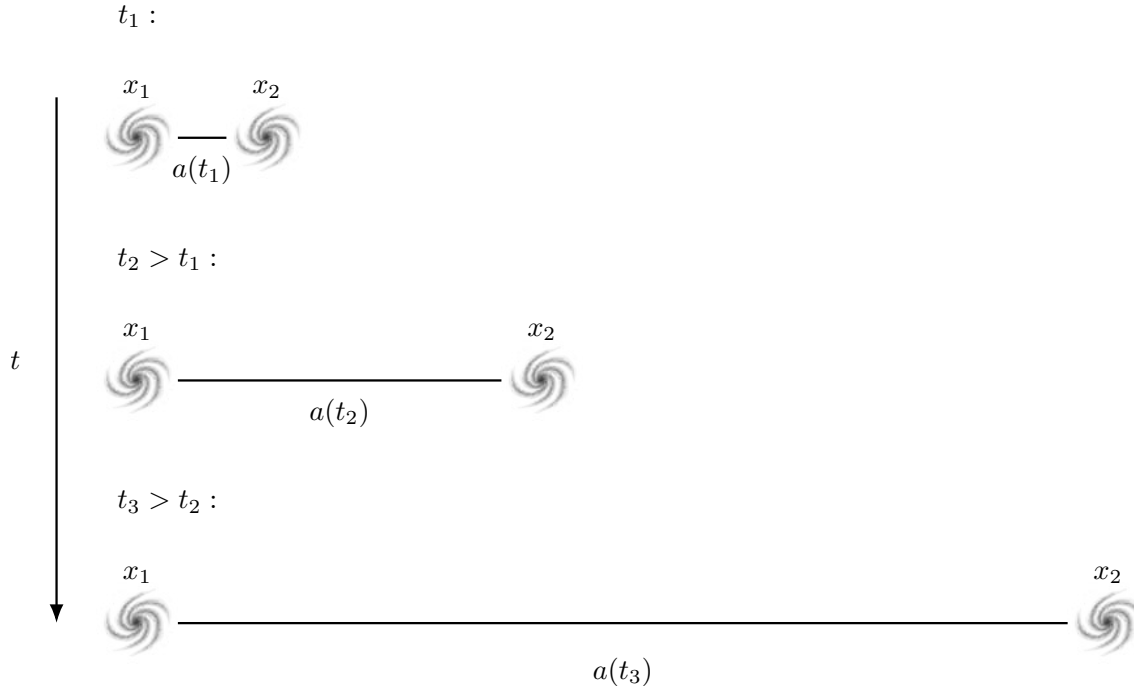


FIGURE 1.1: Schematic illustration of cosmic expansion. The comoving distance between galaxies located at x_1 and x_2 remains constant as the Universe expands. The physical distance corresponds to the comoving distance multiplied by the scale factor $a(t)$, and increases with time.

It is useful to introduce the concept of comoving distance: the distance between objects in a coordinate system that expands uniformly with the Universe. In this frame, the coordinates of galaxies remain fixed, while the physical distances between them grow as the Universe expands. This leads us to define the scale factor $a(t)$, which encodes how the size of the Universe evolves with time. The physical distance between two points is then simply the comoving distance multiplied by the scale factor. In this way, the effect of cosmic expansion is entirely absorbed into the evolution of $a(t)$.

This representation is illustrated schematically in Fig. 1.1. By convention, the scale factor is set to unity at the present time, and had smaller values in the past, meaning galaxies were closer together than they are today.

As a direct consequence, the physical wavelength of light emitted by a distant source at time t_1 is stretched by the expansion of the Universe and appears redshifted when observed at a later time t_2 . The amount of redshift is directly proportional to the ratio of the scale factors at the two times:

$$\lambda_{\text{obs}} = \lambda_{\text{em}} \frac{a(t_2)}{a(t_1)} \equiv \lambda_{\text{em}}(1 + z). \quad (1.1)$$

Another key aspect is the spatial geometry of the Universe. It can be flat (Euclidean), open, or closed. In a flat Universe, particles moving along parallel trajectories remain parallel; in a closed Universe, they eventually converge, similar to paths on the surface of a sphere; whereas in an open Universe, the same trajectories diverge. This behaviour is governed by the curvature parameter k , which takes the values $+1$, 0 , or -1 for closed, flat, and open Universes, respectively.

As we will see later, this geometric property affects how we define the comoving distances introduced earlier. It is still uncertain whether our Universe is exactly flat, open, or closed. The most recent cosmological constraints, using the cosmic microwave background (CMB) as a probe, suggest that if any curvature exists, it must be small ([Planck Collaboration et al. 2020a](#)). Current uncertainties still allow all three scenarios to remain viable, although a flat Universe is assumed within the concordance model of cosmology.

Another important fact is that the expansion of the Universe is not static; it evolves with time and is currently accelerating, as first pointed out by [Riess et al. \(1998\)](#) and [Perlmutter et al. \(1999\)](#). This accelerated expansion is believed to be driven by the so-called DE, a mysterious component that accounts for about 68% of the total energy density of the Universe.

An additional 27% is made up of DM, specifically CDM: a hypothetical non-relativistic particle that interacts only through gravity. While alternative candidates have been proposed, many involve extensions or modifications of the Standard Model of particle physics, and will be discussed in more detail in Ch. 2.

The remaining 5% corresponds to ordinary, baryonic matter described by the Standard Model, the particles and interactions that make up stars, galaxies, planets, nebulae, and even ourselves.

A still debated aspect of modern cosmology is the inflationary era: a theoretical mechanism through which the early Universe underwent a phase of rapid, exponential expansion, increasing its size by many orders of magnitude in an extremely short time. This idea, known as inflation, was first proposed by Guth & Tye (1980) and Sato (1981), and later refined by Linde (1982) and Albrecht & Steinhardt (1982).

Inflation was originally introduced to address several fundamental problems in cosmology: the observed flatness of the Universe; the horizon problem, where distant regions appear to have been in thermal equilibrium despite being causally disconnected; and the absence of magnetic monopoles predicted by grand unified theories. Moreover, inflation provides a compelling mechanism for generating the initial perturbations observed in the early Universe, via quantum fluctuations of the inflaton field, perturbations that later seeded the large scale structure (LSS) we observe today.

Despite these theoretical successes, inflation remains unconfirmed. To date, no direct observational evidence has been found that definitively proves its occurrence. While its predictions are consistent with current observations, especially the near scale-invariant spectrum of primordial perturbations seen in the CMB, the most distinctive signatures of inflation, such as a stochastic background of primordial gravitational waves (GWs), remain undetected. Nevertheless, inflationary initial conditions are often assumed within the standard cosmological framework to describe the early Universe.

The quantitative description of cosmic expansion and its dependence on the energy content of the Universe will be introduced in the next section through the Friedmann equations.

1.2 Friedmann equations

According to Einstein's GR, the dynamics of the Universe and its interaction with the matter and energy it contains are governed by the Einstein field equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1.2)$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, and Λ is the cosmological constant, which accounts for the effect of DE¹. On the right-hand side, G is Newton's gravitational constant, and $T_{\mu\nu}$ is the energy–momentum tensor that encodes the properties of matter and radiation.

Physical distances in spacetime are determined by the metric via the line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1.3)$$

Here $x^\mu = (ct, x^i)$ denotes the spacetime coordinates, with Latin indices referring only to the spatial components. The line element is an invariant quantity common to all observers. The metric tensor is a 4×4 object that describes how time and space are connected at each point in spacetime.

Assuming the cosmological principle, the Universe is homogeneous and isotropic on large scales, whose metric is the Friedmann–Lemaître–Robertson–Walker (FLRW). In spherical coordinates, the line element becomes:

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2/R_0^2} + r^2 d\Omega^2 \right], \quad (1.4)$$

where $a(t)$ is the previously introduced scale factor that encodes the expansion of the Universe, k is the spatial curvature (with values $+1$, 0 , or -1 for closed, flat, or open geometry), and R_0 is a reference curvature radius.

We also define the Hubble parameter as $H(t) \equiv \dot{a}(t)/a(t)$, with $H_0 = H(t_0)$ being its current value. The physical distance of an object from the origin is given by $r_{\text{phys}} = a(t)r$, and its time derivative yields the radial velocity:

$$v_{\text{phys}} = \dot{r}_{\text{phys}} = H(t)r_{\text{phys}} + v_{\text{pec}}, \quad (1.5)$$

where $v_{\text{pec}} = a(t)\dot{r}$ is the peculiar velocity with respect to the comoving frame.

To account for time intervals independent of the expansion, we define the conformal

¹Since the term $\Lambda g_{\mu\nu}$ does not alter the conservation of the energy–momentum tensor, it is standard practice to absorb it into the definition of the energy–momentum tensor by moving it to the right-hand side of the equation, with $T_{\mu\nu}^{(\Lambda)} = -\Lambda c^4/(8\pi G) g_{\mu\nu}$.

time $d\eta = dt/a(t)$. Using conformal coordinates, the FLRW line element becomes:

$$ds^2 = a^2(\eta) \left[-c^2 d\eta^2 + d\chi^2 + S_k^2(\chi) d\Omega^2 \right], \quad (1.6)$$

where $d\chi = dr/\sqrt{1 - kr^2/R_0^2}$ is the comoving radial coordinate, and $S_k(\chi)$ is the comoving angular distance, which depends on curvature:

$$S_k(\chi) = R_0 \begin{cases} \sin(\chi/R_0) & \text{if } k = +, \\ \chi/R_0 & \text{if } k = 0, \\ \sinh(\chi/R_0) & \text{if } k = -1. \end{cases} \quad (1.7)$$

The FLRW metric is invariant under the rescaling $a(t) \rightarrow \zeta a(t)$, $r \rightarrow r/\zeta$, and $R_0 \rightarrow R_0/\zeta$. This freedom allows us to conventionally set $a(t_0) = 1$ at the present time. From Eq. 1.1, this implies $z(t_0) = 0$.

Solving Eq. 1.2 with the FLRW metric leads to the Friedmann equations:

$$\begin{aligned} H^2 &= \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2 R_0^2}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} \left(\rho + \frac{3\mathcal{P}}{c^2} \right) + \frac{\Lambda}{3}, \end{aligned} \quad (1.8)$$

where ρ is the total energy density of the Universe, and $\mathcal{P}$ is its pressure. The total energy density can be decomposed into its major components:

$$\rho = \rho_\Lambda + \rho_m + \rho_r, \quad (1.9)$$

corresponding to DE, matter, and radiation. The matter term further splits into baryonic and CDM: $\rho_m = \rho_b + \rho_c$, while the radiation term includes photons and relativistic neutrinos: $\rho_r = \rho_\gamma + \rho_\nu$.

Combined with the continuity equation:

$$\dot{\rho} + 3H \left(\rho + \frac{\mathcal{P}}{c^2} \right) = 0, \quad (1.10)$$

²In a flat Universe, r and χ are equivalent.

and an equation of state of the form $\omega = \mathcal{P}/(\rho c^2)$ for each component, one finds:

$$\frac{\dot{\rho}}{\rho} = -3(1 + \omega)\frac{\dot{a}}{a}, \quad (1.11)$$

which leads to the scaling:

$$\rho(a) \propto a^{-3(1+\omega)}. \quad (1.12)$$

This implies that different components evolve differently with expansion:

- **Matter** ($\omega = 0$) $\rightarrow \rho_m \propto a^{-3}$

A gas of non-relativistic particles whose energy is dominated by their rest mass. The pressure $\mathcal{P}$ is negligible compared to the energy density, leading to an equation of state $\omega = 0$. As a result, matter dilutes with cosmic expansion as $\rho_m \propto a^{-3}$. Today, matter constitutes roughly $\sim 30\%$ of the total energy content of the Universe, of which about 85% is CDM and 15% baryonic matter.

- **Radiation** ($\omega = 1/3$) $\rightarrow \rho_r \propto a^{-4}$

A gas of relativistic particles, where the kinetic energy dominates the total energy density. Its equation of state is $\omega = 1/3$, leading to a scaling $\rho_r \propto a^{-4}$. In addition to dilution due to expansion, radiation loses energy through redshifting ($E \propto a^{-1}$). At present, only photons remain relativistic, being massless, while in the early hot Universe, other particles also behaved as radiation. Today, radiation contributes less than 0.01% of the total energy density of the Universe.

- **DE: Cosmological constant** ($\omega = -1$) $\rightarrow \rho_\Lambda = \text{const.}$

Observations suggest that DE is a fluid with negative pressure $\mathcal{P}$, required to explain the current accelerated expansion of the Universe. The corresponding equation of state is $\omega = -1$, implying a constant energy density: $\rho_\Lambda \propto a^0$. Dark energy dominates the present-day Universe, contributing approximately $\sim 70\%$ of the total energy budget.³ Whether ω is exactly -1 , slightly different, or changes over time remains an open question.

As a result, the Universe has gone through different domination eras, from radiation domination (RD) in the early Universe, to matter domination during structure formation (MD), and DE domination at late times (AD). The evolution of the scale factor and its influence based on the dominating component is shown in Fig. 1.2.

³The relative fractions of matter, radiation, and DE quoted in this section are based on the cosmo-

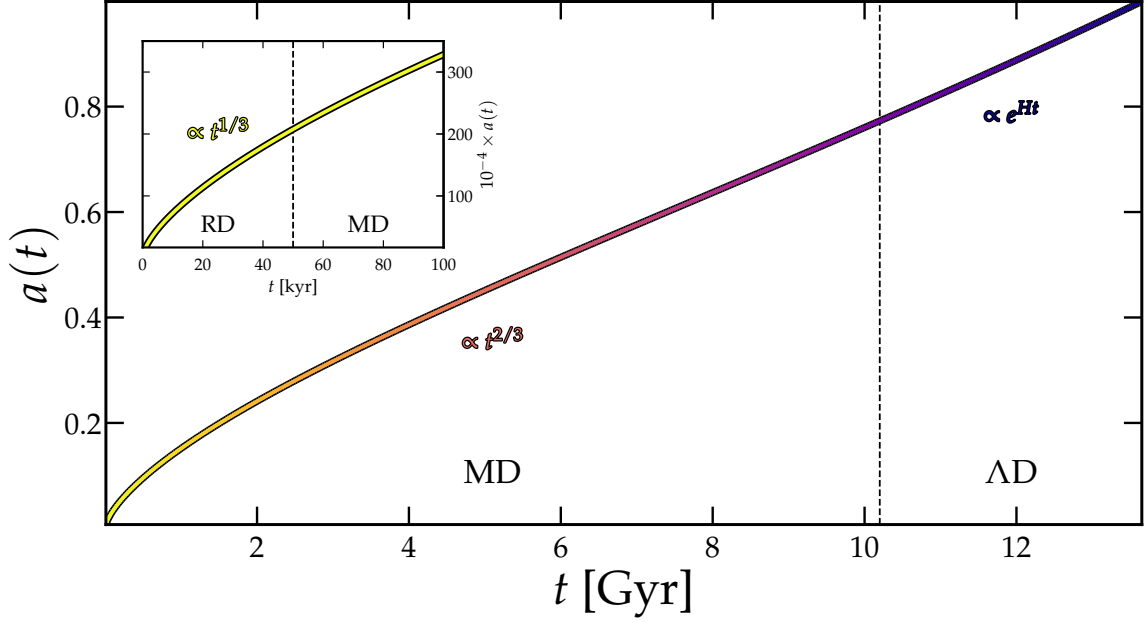


FIGURE 1.2: Evolution of the scale factor across cosmic time. At different epochs its evolution has been driven mainly by radiation (RD) until t_{eq} at ~ 50 kyrs, where matter took the leading role (MD) until DE overcame its contribution (Λ D) at $t_{\Lambda D} \approx 10$ Gyrs.

We can rewrite the first Friedmann equation in terms of the critical density

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} = 8.62 \times 10^{-27} \text{ kg m}^{-3}, \quad (1.13)$$

where the subscript ‘0’ denotes the present value. Rewriting all densities relative to this critical value, we have the following dimensionless density parameters:

$$\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{\text{crit},0}}, \quad i = r, m, k, \Lambda. \quad (1.14)$$

Where Ω_k is the curvature “density” parameter defined as $\Omega_k \equiv -kc^2/(R_0 H_0)^2$, and evolves with time as $\propto a^{-2}$. The Friedmann equations can be read then as

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda. \quad (1.15)$$

logical results from [Planck Collaboration et al. \(2020a\)](#).

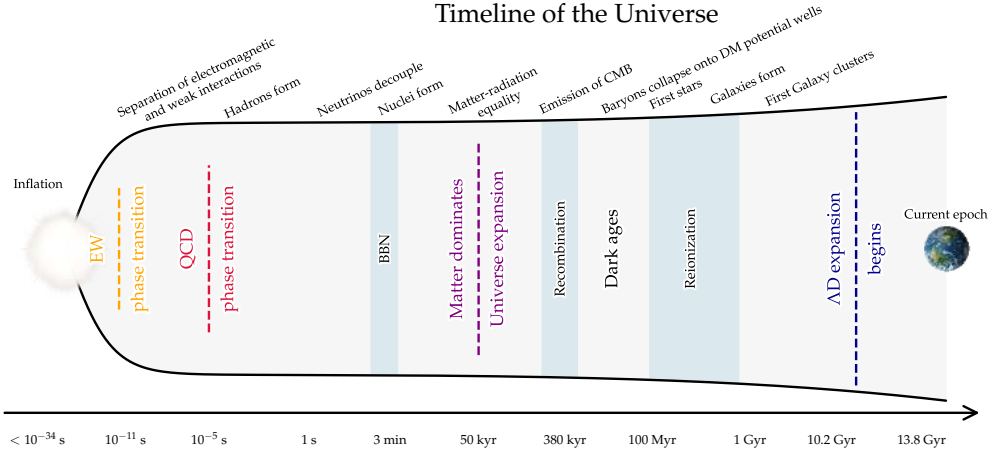


FIGURE 1.3: Schematic timeline of the Universe's main events from inflation to the current era.

Since $a(t_0) \equiv 1$, we have that

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k = \Omega_0 + \Omega_k.^4 \quad (1.16)$$

Since $(1 + z) = a(z)$, this can be read as:

$$\frac{H^2}{H_0^2} = \Omega_r(1 + z)^{-4} + \Omega_m(1 + z)^{-3} + \Omega_k(1 + z)^{-2} + \Omega_\Lambda. \quad (1.17)$$

In general, the Friedmann equation is a non-linear differential equation that may not have an exact solution. Under certain conditions, exact solutions are possible. In our Universe we have the values listed in Table 1.1 (Planck Collaboration et al. 2020a).

Parameter	Meaning	Value
H_0	expansion rate	$67.74 \pm 0.46 \text{ km s}^{-1} \text{ Mpc}^{-1}$
ρ_{crit}	critical density	$(8.62 \pm 0.12) \times 10^{-27} \text{ kg m}^{-3}$
Ω_r	radiation density	$(9.02 \pm 0.21) \times 10^{-5}$
Ω_m	matter density	0.3153 ± 0.0073
Ω_Λ	dark energy density	0.6847 ± 0.0073
$ \Omega_k $	spatial curvature	< 0.005

Table 1.1: Observed values of key parameters of our Universe.

We can integrate the first Friedmann equation to get the relation between time and

⁴Note that for a flat Universe, $\Omega_k = 0$.

scale factor:

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_r a^{-2} + \Omega_m a^{-1} + \Omega_\Lambda a^2 + \Omega_k}}. \quad (1.18)$$

The times at which the contents of matter and radiation were equal, or the moment at which DE became the leading factor in the expansion rate of our Universe, can be obtained from $a_{\text{eq}} = \Omega_r/\Omega_m$ and $a_{m\Lambda} = \Omega_m/\Omega_\Lambda$. Evaluating Eq. 1.18 at both a_{eq} and $a_{m\Lambda}$, we find:

$$\begin{aligned} t_{\text{eq}} &\approx 50\,000 \text{ yrs}, \\ t_{m\Lambda} &\approx 10.2 \text{ Gyrs}. \end{aligned} \quad (1.19)$$

Evaluating at $a(t_0) = 1$ we can obtain the age of the Universe:

$$t_0 \approx 13.8 \text{ Gyrs}. \quad (1.20)$$

1.3 Distances

The measurement of distances in cosmology and astrophysics is notoriously difficult. Distances as they appear in the metric are not directly observable, and the physical distance $d_{\text{phys}} = a(t)\chi$ cannot be measured since it refers to the separation between two points at a fixed time, and we cannot gather such information faster than the speed of light. A proper definition of distance should include the finite speed of light and the effect of the Universe's expansion.

There are two standard distances used in astrophysics within the framework of GR: luminosity distances and angular diameter distances. The former relates to the apparent dimming of an object of known luminosity at larger distances, while the latter is based on how objects of known size appear smaller the further away they are, with a twist introduced by an expanding Universe.

1.3.1 Luminosity distance

Assuming an object with known luminosity L , which is the energy emitted per unit time, and measuring its observed flux F (energy per unit time per unit area), we can derive its luminosity distance if we know the rate at which the Universe is expanding.

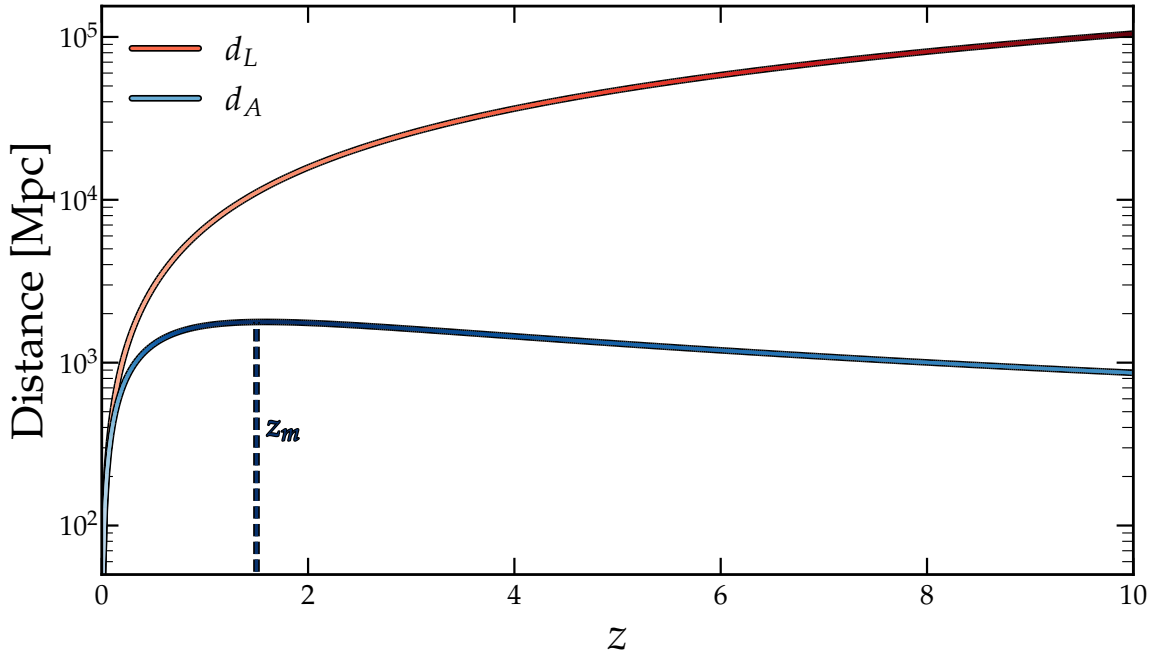


FIGURE 1.4: Evolution of distances in our Universe, as a function of redshift.

The comoving distance of an object at redshift z is:

$$\chi(z) = \int_{t_1}^{t_0} \frac{dt}{a(t)} = c \int_0^z \frac{dz}{H(z)}, \quad (1.21)$$

where the evolution of the Hubble parameter with redshift depends on the matter, radiation, and energy content as shown in Eq. 1.17.

If the source emits its radiation isotropically, in a Euclidean Universe the energy would spread over a sphere of area $4\pi\chi^2$, so the flux received by a telescope of fixed area A would decrease as $A/4\pi\chi^2$. The flux–luminosity relation would read as:

$$F = \frac{L}{4\pi\chi^2}. \quad (1.22)$$

In our expanding Universe, when the light reaches us at time t_0 , the area of the sphere with radius χ would be:

$$4\pi \times a^2(t_0) d_M^2, \quad (1.23)$$

where $d_M \equiv S_k(\chi)$ is the previously defined transverse comoving distance. We also have to account for the arrival rate of photons, which is reduced by a factor $a(t_1)/a(t_0) =$

$1/(1+z)$, and so is the observed flux. Another factor of $1/(1+z)$ arises because the redshifted photons carry less energy, further reducing the flux. These combined effects result in a flux–luminosity relation in our expanding Universe:

$$F = \frac{L}{4\pi d_M^2 (1+z)^2}. \quad (1.24)$$

Comparing Eqs. 1.22 and 1.24, we can define the luminosity distance as:

$$d_L(z) = (1+z)d_M(z), \quad (1.25)$$

where the metric distance d_M for a source at redshift z depends on the cosmological parameters.

1.3.2 Angular diameter distance

Similar to flux, the observed angular size of distant objects depends on their intrinsic size and their distance from us. Assuming an object of physical size D at comoving distance χ , whose photons were emitted at t_1 , in a non-expanding Euclidean Universe the observed angular size would be:

$$\delta\theta = \frac{D}{\chi}, \quad \text{if } \delta\theta \ll 1 \text{ (in radians)}. \quad (1.26)$$

In an expanding Universe, this relation becomes:

$$\delta\theta = \frac{D}{a(t_q)d_M}, \quad (1.27)$$

where $d_M \equiv S_k(\chi)$. Similarly to what we showed for the luminosity distance, we can compare Eqs. 1.26 and 1.27, and since $(1+z) = 1/a(t_1)$ we define the angular diameter distance as:

$$d_A \equiv \frac{d_M(z)}{1+z}. \quad (1.28)$$

Unlike in a non-expanding Universe, the angular diameter distance starts to decrease around $z_m \sim 1.5$. Beyond this redshift, objects appear larger on the sky with increasing redshift. This counter-intuitive behaviour arises from the expansion: when the light was emitted, the object was closer to us as spacetime was compressed, leading to a larger observed angular size.

It can be immediately shown by comparing Eqs. 1.24 and 1.27 that the relation between luminosity and angular diameter distances is:

$$d_L(z) = (1 + z)^2 d_A(z). \quad (1.29)$$

1.3.3 Standard objects

Measuring distances remains a challenging task. Even with the newly defined luminosity and angular diameter distances, we still require knowledge of either the intrinsic luminosity or the true physical size of the object under consideration. Alternatively, the same quantities must be known for a nearby object, or ideally, for another object within the same host galaxy, if we are dealing with smaller astrophysical systems.

To overcome this, we can rely on so-called standard objects: astrophysical sources with known and (presumably) universal luminosities or sizes that do not evolve significantly with redshift. In this way, we can calibrate their intrinsic properties locally, where independent and more direct methods allow precise measurements, and then use them across cosmic distances. These objects are known as standard candles and standard rulers.

Standard candles

Hubble used Cepheid variables to infer the expansion of the Universe. These stars exhibit periodic variations in their luminosity, with the pulsation period directly related to their intrinsic brightness, a relation commonly known as the Leavitt Law (Leavitt 1907; Leavitt & Pickering 1912), which can be expressed in terms of luminosity as:

$$\log_{10} L = \alpha \log_{10} P + \beta \quad \Rightarrow \quad L = L_0 P^\alpha, \quad (1.30)$$

where L is the stellar luminosity, P is the pulsation period (in days), α is an empirically determined slope, and $L_0 = 10^\beta$ is a normalisation constant calibrated from local Cepheids. This empirical relation allows the intrinsic luminosity of a Cepheid to be inferred directly from its period, making them powerful standard candles for extragalactic distance measurements.

It is more common, however, to express this relation in terms of magnitudes. The absolute magnitude, M , of an astronomical object is defined as the apparent magnitude

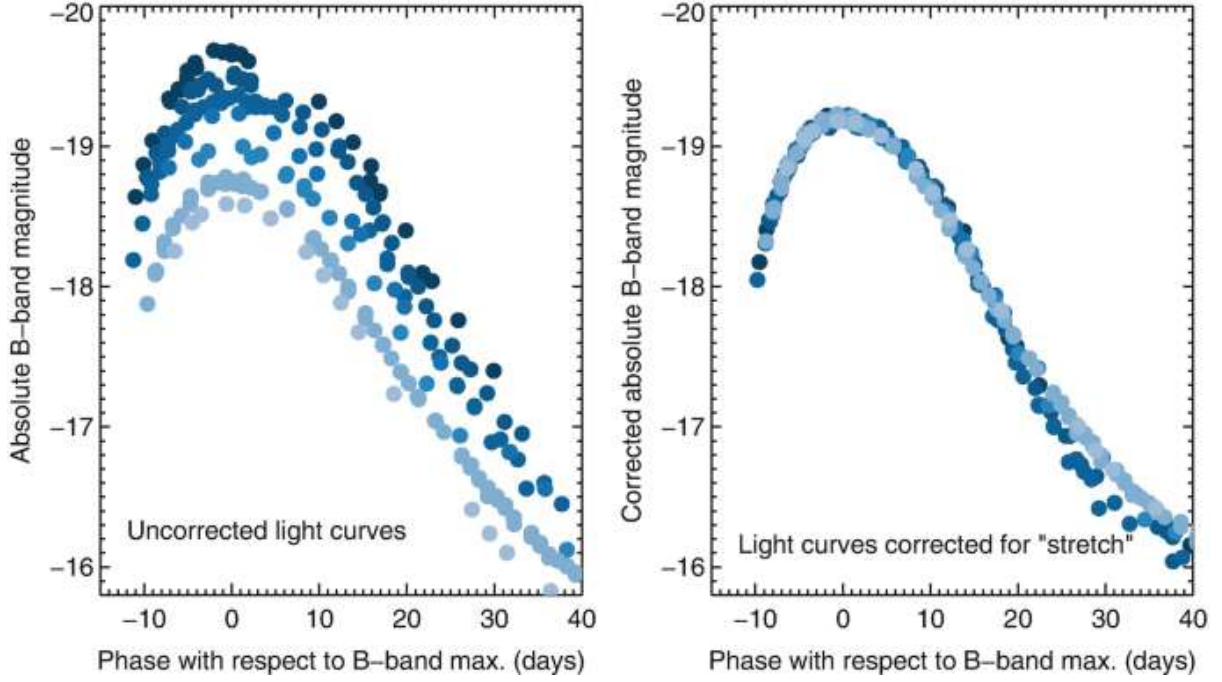


FIGURE 1.5: Different type Ia SNe before and after stretch corrections. The result is a standard luminosity curve that serves as a distance indicator. Figure from [Alsabti & Murdin \(2017\)](#).

it would have if placed at a distance of 10 parsecs from the observer. It provides a distance-independent measure of an object's intrinsic brightness. The connection between absolute magnitude M , apparent magnitude m , and luminosity distance d_L (in parsecs) is given by the distance modulus:

$$\mu = m - M = 5 \log_{10} \left(\frac{d_L}{10 \text{ pc}} \right). \quad (1.31)$$

With this definition, the Leavitt Law takes the form:

$$M = a \log_{10}(P) + b, \quad (1.32)$$

where a and b are the empirical slope and intercept, respectively, calibrated from nearby Cepheids using independent distance measurements.

Another class of standard candles is provided by Type Ia supernovae (SNe). These are thermonuclear explosions of white dwarfs in binary systems, triggered when the white dwarf accretes sufficient mass from its companion to exceed the Chandrasekhar limit ([Chandrasekhar 1931](#)). Although relatively rare, about one per century per Milky

Way-like galaxy, Type Ia SNe are extremely luminous, outshining entire galaxies and thus observable across cosmological distances. Because the explosion occurs under similar conditions, their intrinsic luminosities are nearly uniform. Residual variations do exist, but can be empirically corrected.

The observed peak magnitude of a Type Ia SN can be standardised using correlations with its light-curve shape and colour. The corrected distance modulus is typically parametrised as:

$$\mu = m_B^* - M + \alpha x_1 - \beta c, \quad (1.33)$$

where m_B^* is the peak apparent magnitude in the rest-frame B -band, M is the standardised absolute magnitude of a fiducial SN Ia, x_1 characterises the light-curve stretch (related to its width), and c is the colour parameter. The coefficients α and β are nuisance parameters, typically determined from large supernova samples in combination with cosmological fits.

Beyond Cepheids and SNe, several other classes of objects have been proposed as standard candles, although they are generally less precise or more model-dependent than the classical examples.

Tip of the red giant branch (TRGB) (McQuinn et al. 2019):

TRGB stars mark the onset of helium fusion in low-mass stars and display a relatively sharp cut-off in luminosity, particularly in the near infrared. This makes the TRGB a useful distance indicator for old stellar populations, especially in galaxies lacking Cepheids.

RR-Lyrae (Muraveva et al. 2018):

RR-Lyrae variables are pulsating horizontal branch stars with relatively short periods (typically less than a day) and a well-established period–luminosity relation in the near infrared. While fainter than Cepheids, they are valuable for measuring distances within the Local Group and to globular clusters.

Quasars (QSOs) (Dainotti et al. 2023):

Certain classes of QSOs have also been explored as high-redshift standardisable candles. For instance, correlations between their X-ray and UV luminosities, or between variability and luminosity, have been proposed. However, these relations suffer from larger intrinsic scatter and stronger dependence on the physical models, limiting their current precision for cosmological use.

Gravitational waves (Holz & Hughes 2005):

In the near future, GWs could also be used to measure luminosity distances via the so-called standard sirens. The wavefronts and amplitudes of GW signals can be correlated, as predicted by GR, with the parameters of the emitting sources, and some systems might be standardisable.

Standard rulers

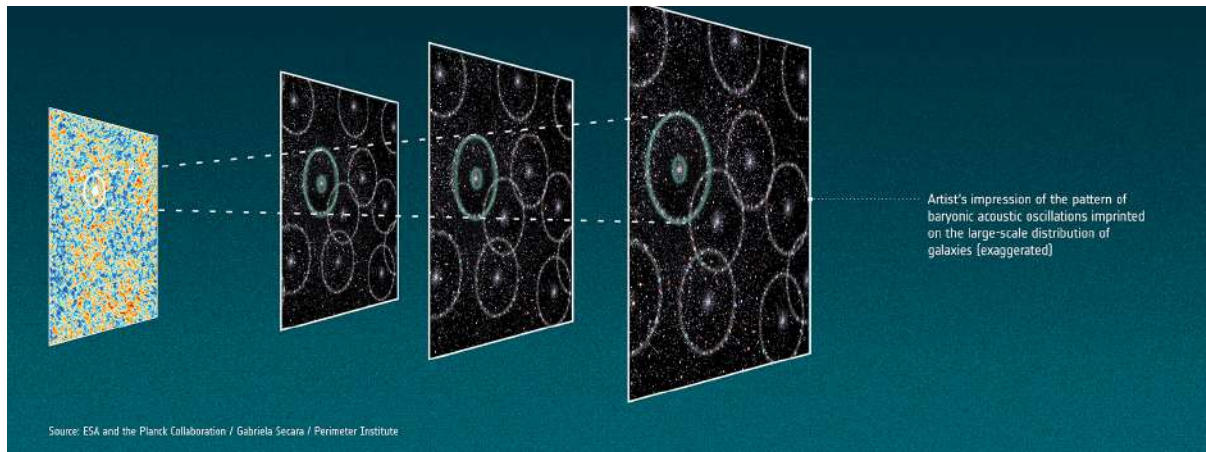


FIGURE 1.6: Pattern of BAOs imprinted on the LSS of the Universe (highly exaggerated). Galaxies are more likely to be separated by the distance represented by the rings in the image. In reality, the effects of the BAOs combined at different locations produce a more complex picture where galaxy clustering is not directly visible; it is a subtle feature that appears as a bump in the galaxy–galaxy correlation function at the scale of the sound horizon. From left to right, different cosmic epochs are shown, from the recombination epoch (CMB photons) to the current galaxy clustering. Credit: ESA and the Planck Collaboration / Gabriela Secara.

Standard rulers are physical objects or features whose intrinsic size is known or can be reliably calibrated. By measuring their apparent angular size on the sky, one can infer the angular diameter distance to the object via simple geometric projection. This provides a complementary approach to distance measurements based on standard candles, and is particularly valuable at high redshifts where apparent luminosities become difficult to measure accurately.

Baryon acoustic oscillations (BAOs) (Percival 2017):

The most widely used standard ruler in modern cosmology is the characteristic scale imprinted by BAOs. These acoustic waves propagated through the photon–baryon plasma in the early Universe before recombination, leaving a preferred physical scale in the clustering of matter. This scale, known as the sound horizon, corresponds to

the maximum comoving distance a sound wave could travel before the decoupling of photons and baryons, and is measured to be approximately $r_s \sim 147$ Mpc. It appears as a slight excess in the galaxy correlation function at this scale and can be measured in both the transverse and radial directions to constrain angular diameter distances and Hubble parameters at different redshifts.

In addition to BAOs, other proposed standard rulers include the sizes of galaxy clusters (Wagoner et al. 2021), the characteristic scale of cosmic voids (Lavaux & Wandelt 2012), and the position of features in the CMB anisotropy spectrum (Eisenstein et al. 1998). While some of these probes are still under active development or subject to modelling uncertainties, they offer potential routes to extending standard ruler techniques across wider redshift ranges and physical regimes.

When used in combination with standard candles, standard rulers provide an independent and powerful way to test cosmological models and constrain key parameters such as the expansion history, spatial curvature, and the nature of DE.

1.3.4 Distance ladder

We use these standardised objects to measure distances. However, since these objects cannot be observed at all distances, astronomers have built a hierarchical method combining these standard objects observed in the same locations. For instance, brighter standard candles like Type Ia SNe can be calibrated with fainter objects such as Cepheids. This methodology is known as the distance ladder. Each step of this ladder depends on the calibration of the previous step, and the total accuracy at the largest distances depends on the accumulated precision.

At its base, the ladder begins with geometric methods, such as trigonometric parallax (see Fig. 1.8), which require no astrophysical assumptions whatsoever. By measuring the apparent displacement of nearby stars as the Earth orbits the Sun, we can directly infer their distances with exquisite precision. Modern astrometric missions like Gaia have dramatically expanded the effective range of this method, calibrating the next step: the standard candles.

Among these, Cepheid variables play a central role. Their period–luminosity relation, established locally via parallax distances, allows us to determine the absolute magnitude of Cepheids in nearby galaxies. These in turn calibrate other distance indicators

such as the Tully–Fisher relation (Tully & Fisher 1977), which links the rotational velocity of spiral galaxies to their intrinsic luminosity. Once calibrated with Cepheid distances, the Tully–Fisher relation enables distance estimates to more remote galaxies for which individual stars are unresolved. A similar method applies to elliptical galaxies via the Faber–Jackson relation (Faber & Jackson 1976), which correlates luminosity with central stellar velocity dispersion. Both relations extend the reach of the distance ladder into the tens to hundreds of megaparsecs range.

Beyond that, Type Ia SNe serve as powerful standardisable candles. Their relatively uniform peak luminosities, once corrected for light-curve shape and colour, allow for accurate distance estimates to high-redshift galaxies, anchoring the expansion history of the Universe out to $z \sim 1.5$.

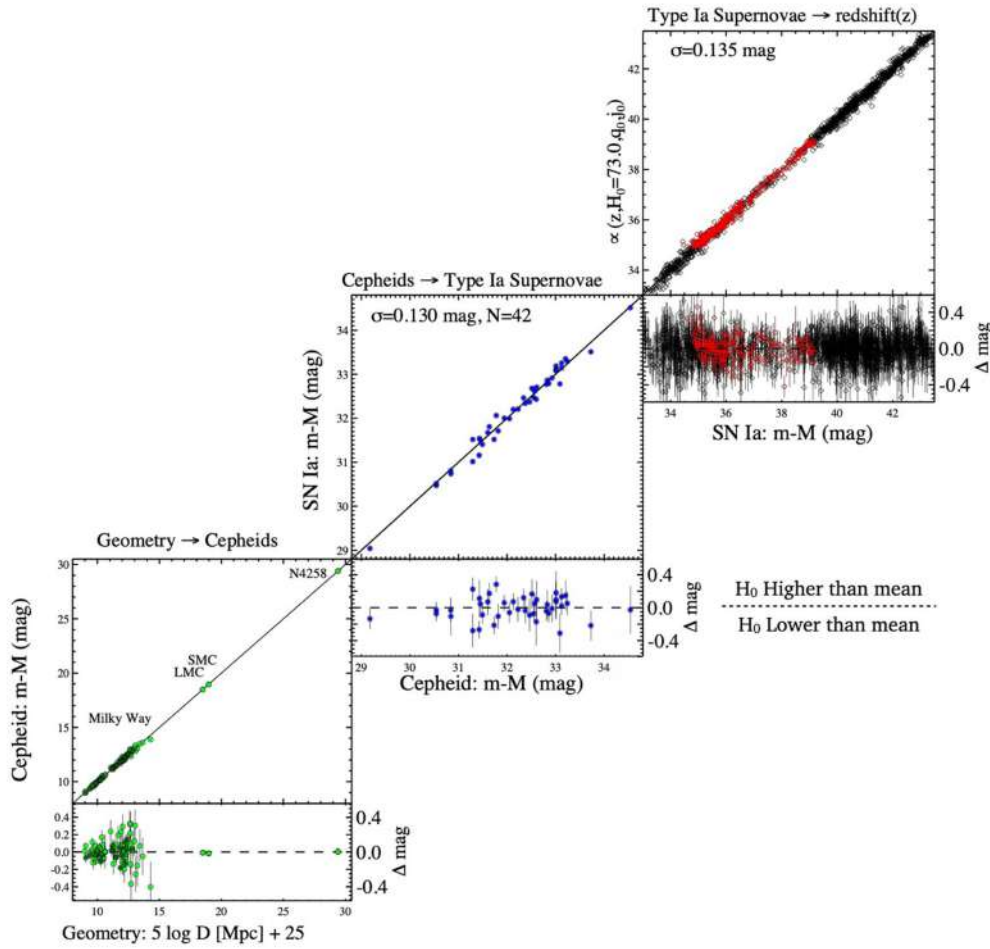


FIGURE 1.7: Distance ladder by Riess et al. (2024), combining geometrical parallax for the first rung, Cepheids in the second, and SNe in the third one. Figure adapted from Riess et al. (2024).

Finally, by correlating the distances inferred from SNe with their redshifts, we can probe the expansion history of the Universe and constrain cosmological parameters. This last step often involves fitting a cosmological model to the distance–redshift relation, and thus transitions from purely observational to model-dependent cosmology.

Each rung of the ladder builds upon the previous one. While this method is powerful, it is also cumulative in its uncertainties and vulnerable to systematics in the early steps. Nonetheless, it remains one of the most robust and widely used frameworks for anchoring the cosmic distance scale and validating our understanding of the Universe’s large-scale structure and dynamics. An example of such analysis (Riess et al. 2024) is given in Fig. 1.7.

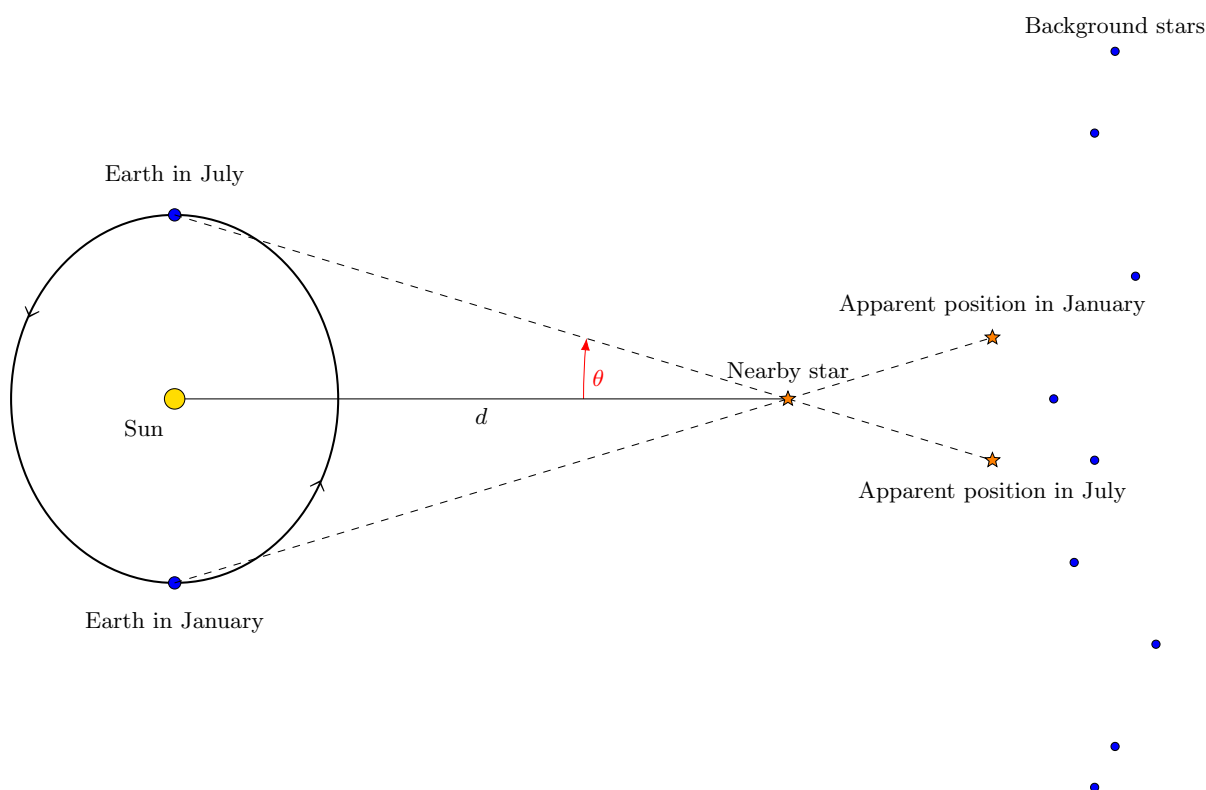


FIGURE 1.8: Schematic of the trigonometric parallax method for distance measurements of nearby objects. As the Earth orbits the Sun, a nearby star appears to shift its position against the background stars. This apparent motion, observed six months apart from two opposite points in Earth’s orbit, defines the parallax angle θ , which is inversely related to the distance d to the star. This geometrical method is the first step of the cosmic distance ladder.

1.4 Problems in Λ CDM

Λ CDM stands as our best interpretation, within the framework of GR, of how our Universe behaves, how it evolved from its infancy, and what its future might look like. Nonetheless, the model is incomplete. It does not address key open questions, most notably the nature of DM and DE, or whether or not there was an inflationary epoch. Understanding these components remains a central goal of modern cosmology and fundamental physics.

On top of these conceptual gaps, several observational tensions have emerged in recent years. These arise from the increasing precision of independent cosmological probes and the unprecedented quality of recent datasets. In particular, discrepancies between early- and late-time determinations of key cosmological parameters have reached statistical significance and challenge the internal consistency of the model. Below I summarise some of the most relevant tensions and open questions that motivate ongoing research toward a more complete cosmological framework.

1.4.1 Tensions

- **Hubble tension**

The Hubble constant, the current expansion rate of the Universe, exhibits a persistent discrepancy at the $4\text{--}6\sigma$ level between early-time predictions (from CMB and BAO) and late-time, model-independent measurements (e.g., using type Ia SNe (Zitrin et al. 2014) or strong lensing (SL) time delays (Acebron et al. 2023)). Recent estimates place the early-time value at $H_0 \approx 67 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (Planck Collaboration et al. 2020a; Louis et al. 2025), while late-time probes suggest a faster expansion rate of $H_0 \approx 73 \text{ km s}^{-1} \text{ Mpc}^{-1}$ (Breuval et al. 2024; Wong et al. 2020). The tension has reached $\sim 5\sigma$, with a wide range of proposed solutions, including systematics, new physics, or a breakdown of the standard cosmological model (Di Valentino et al. 2021).

- **S_8 tension**

The amplitude of matter clustering in the late-time Universe, conditioned on the primordial fluctuations, is often quantified by⁶ $S_8 = \sigma_8 \sqrt{\Omega_m/0.3}$, where σ_8 is the

⁵https://github.com/shsuyu/H0LiCOW-public/tree/master/H0_tension_plots

⁶Alternative parametrisations exist, such as $S_8 = \sigma_8(\Omega_m/0.3)^\alpha$, where α is an empirical parameter fitted from the data.

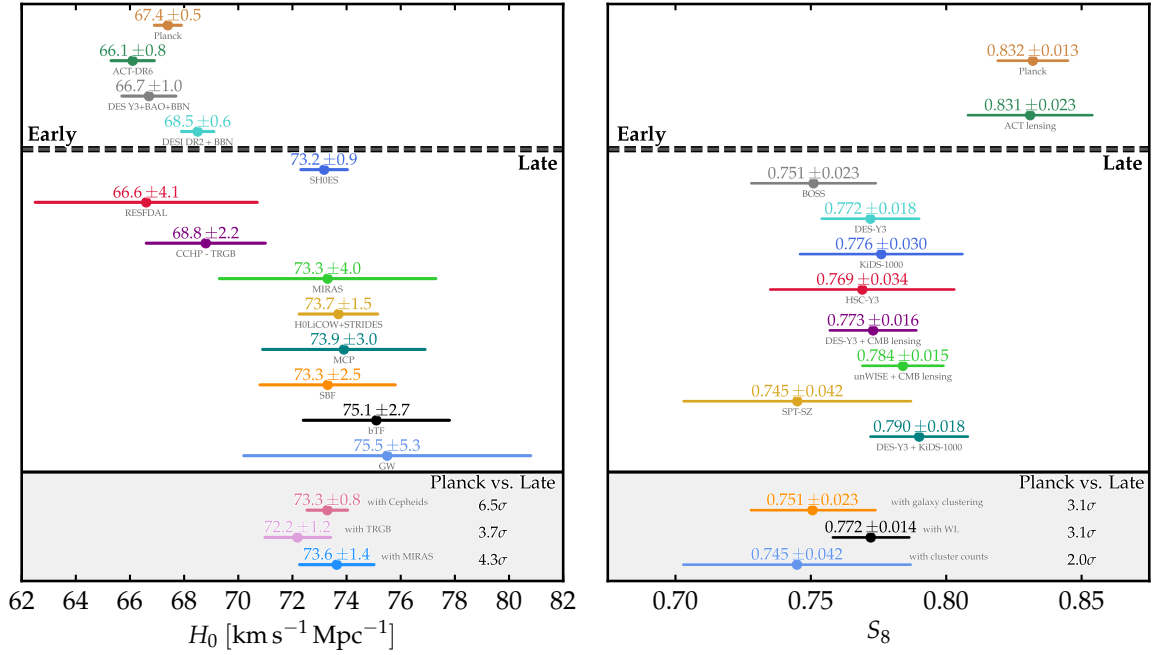


FIGURE 1.9: Overview of H_0 and S_8 tensions and current measurements. Left: H_0 early-time values from CMB anisotropies (Planck Collaboration et al. 2020a; Louis et al. 2025) and joint analyses of galaxy clustering, WL, BAOs and BBN (Abbott et al. 2018; DESI Collaboration et al. 2025). Late-time estimates span a variety of model-independent probes: distance ladder calibrations (SHOES (Breuval et al. 2024), CCHP (Freedman et al. 2025), MIRAS (Huang et al. 2020)), SL time-delay cosmography (HOLICOW+STRIDES (Wong et al. 2020; Millon et al. 2020)), a lensed supernova (Kelly et al. 2023), mega-masers (MCP (Pesce et al. 2020)), surface brightness fluctuations (SBF (Blakeslee et al. 2021)), Tully–Fisher relation (bTF (Schombert et al. 2020)), and gravitational waves (GW (Abbott et al. 2017)). Right: S_8 early-time values from CMB and lensing potential (ACT (Madhavacheril et al. 2024)), and late-time values from galaxy clustering (BOSS (Philcox & Ivanov 2022)), WL (DES-Y3 (Secco et al. 2022), KiDS-1000 (Li et al. 2023a), HSC-Y3 (Li et al. 2023b)), and their cross-correlations with CMB lensing (DES-Y3+CMB (Abbott et al. 2023), unWISE+CMB (Krolewski et al. 2021)). The bottom part of each panel shows the combined predicted value and statistical discrepancy in sigma values. Figure made using an adapted version of the code by V. Bonvin and M. Millon⁵.

variance of matter perturbations on scales of $8 h^{-1} \text{Mpc}$:

$$\sigma_8^2 = \int_0^\infty \frac{k^2}{2\pi^2} [3j_1(8k)]^2 \mathcal{P}_m(k) dk, \quad (1.34)$$

with $j_1(x)$ the spherical Bessel function of order one, and $\mathcal{P}_m(k)$ the matter power spectrum.

Early-time measurements from the CMB (Planck Collaboration et al. 2020a; Mad-

havacheril et al. 2024) yield $S_8 \approx 0.832$, while late-time observations based on galaxy clustering (Philcox & Ivanov 2022), weak lensing (Secco et al. 2022; Li et al. 2023a,b) (WL), and cross-correlations with CMB lensing (Abbott et al. 2023; Krolewski et al. 2021) prefer a lower value, around $S_8 \approx 0.756$. This discrepancy lies at the $\sim 3\sigma$ level. However, recent results from the KiDS-Legacy cosmic shear survey (Stölzner et al. 2025) show improved internal consistency and yield S_8 values closer to Planck’s, reducing the discrepancy below 1σ in that specific combination. Nonetheless, the overall tension remains under discussion.

- **Cosmic lithium problem**

The abundance of ${}^7\text{Li}$ predicted by standard Big Bang Nucleosynthesis (BBN) calculations within ΛCDM (Cyburt et al. 2016; Pitrou et al. 2018) exceeds by a factor of ~ 3 the abundance inferred from metal-poor stars in the galactic halo (Spite & Spite 1982; Sbordone et al. 2010). This tension, known as the lithium problem, persists even when the baryon density is fixed to the CMB value (Planck Collaboration et al. 2020a). Possible explanations include underestimated nuclear reaction uncertainties, biases in the observed primordial abundances, or physics beyond standard BBN (Pospelov & Pradler 2010).

These three constitute the most prominent and best-studied tensions between ΛCDM predictions and current observations. However, several other open issues remain, such as anomalies in the low multipoles of the CMB (including lack of power and quadrupole–octupole alignment), hints of parity violation or cosmic birefringence, or discrepancies in measurements of the growth rate $f\sigma_8$.

Whether these tensions point toward new physics or are the result of subtle systematics remains to be fully understood, and addressing them constitutes one of the key challenges for present and future cosmological surveys.

1.4.2 Open questions

Inflation

While not formally part of the ΛCDM model, most modern cosmological scenarios assume an inflationary epoch preceding the hot Big Bang. I will not delve into the specifics of inflationary physics, as it falls outside the scope of this thesis, but for the sake of completeness, I briefly introduce the basic idea here.

The Big Bang model could, in principle, account for the observed flatness and large-scale homogeneity of the Universe. However, doing so requires a very specific set of finely tuned initial conditions. These conditions must be imposed by hand, with no particular justification or underlying rationale. The inflationary mechanism, a period of accelerated expansion in the very early Universe preceding the hot Big Bang, can dynamically produce the initial conditions required by Λ CDM from a broad and generic set of starting points. Inflation provides an elegant solution to several long-standing problems: it explains the observed homogeneity and flatness of the Universe, and even accounts for the correlation of small-scale inhomogeneities across causally disconnected regions. Despite its appeal, however, there is still no direct observational evidence for inflation. Its most promising signatures, yet to be confirmed, include the detection of primordial B -mode polarisation in the CMB (Kamionkowski & Kovetz 2016), and a stochastic gravitational wave background (Caprini & Figueroa 2018).

Dark energy

Matter and radiation are not enough to describe the evolution of the observed Universe. Nowadays its expansion must be driven by a mysterious component with negative pressure $\mathcal{P} = -\rho c^2$, which leads to a constant energy density:

$$\rho_\Lambda \propto a^0. \quad (1.35)$$

This constant energy density implies that energy is effectively created as the Universe expands, which does not violate energy conservation as long as the continuity equation holds:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{\mathcal{P}}{c^2}\right) = 0. \quad (1.36)$$

A natural candidate for DE would be the energy associated with space itself, since as the Universe expands, more space is created and thus the total energy increases in proportion to the volume. In quantum field theory (QFT), this is known as vacuum energy. However, QFT predicts this vacuum energy to be around sixty orders of magnitude larger than observed, this is known as the cosmological constant problem and is perhaps the most severe crisis in modern theoretical physics.

A possible solution is to consider a DE component whose equation of state is not exactly

$\omega = -1$, or one where it varies with time, commonly parametrised as:

$$\omega(a) = \omega_0 + w_a(1 - a), \quad (1.37)$$

where recent results from the DESI DR2 BAO analysis (DESI Collaboration et al. 2025) may hint towards deviations from $\omega = -1$.⁷ The nature of DE is, nonetheless, barely understood: where it originates from or even how to parametrise it remains one of modern cosmology's greatest open questions.

Dark matter

Most of the matter in the Universe is in the form of the elusive DM. There is evidence for DM from many different observational probes, from the rotation curves of galaxies (Corbelli & Salucci 2000) to the WL effect in galaxy clusters (Hoekstra et al. 2002).

The nature of DM is yet unknown. Whether it is composed of new particle species or astrophysical objects is unclear. It is also uncertain if these particles interact other than gravitationally. What we do know is that if DM interacts, for instance, via the weak force, it must be an extremely weak interaction. On large scales, DM is well described as a pressureless fluid. This CDM is a keystone foundation of our cosmological model. However, on small scales (galactic and sub-galactic), the predictions of CDM are challenged by observations.

Among these, the cusp–core problem (Nguyen et al. 2021) refers to the mismatch between the cuspy inner density profiles predicted by CDM-only simulations and the flatter, cored profiles favoured by galaxy rotation curves. The diversity problem (Flores & Primack 1994) highlights the large scatter observed in galaxy rotation curves for systems with similar halo masses (much greater than what simulations predict). On smaller scales, the too-big-to-fail problem (Boylan-Kolchin et al. 2011, 2012) arises from the apparent absence of massive satellites in the Milky Way, which simulations suggest should exist and host stars. While baryonic physics and observational limitations have mitigated some of these tensions, they remain active areas of research and a source of motivation for exploring alternative DM scenarios.

⁷This parametrisation is motivated by many DE models (Chevallier & Polarski 2001; Linder 2003, 2024), but more complex evolutions are plausible.

The question remains: what is DM, and why, even though CDM matches observations perfectly on large scales, do we see so many discrepancies on smaller scales? This thesis aims to shed light on the true nature of DM using lensing as a tool. Therefore, we dedicate Ch. 2 to a detailed introduction of the topic, its current status, and some of the possible candidates.

Chapter 2

Dark matter

As mentioned at the end of Ch. 1, DM is one of the open questions in the current cosmological landscape. In this chapter, I will briefly introduce the topic, the experimental evidence, and the current caveats of the preferred CDM model, as well as mentioning the most popular candidates at present, highlighting their strengths and weaknesses and their current exclusion rates based on ongoing research. I will also briefly mention the alternatives to DM and some prospects for the future. Certain aspects or models of DM are not covered in this chapter; we refer the reader to the extensive reviews of [Bertone et al. \(2005\)](#) and [Cirelli et al. \(2024\)](#), which inspired this chapter.

2.1 Evidence for dark matter

The existence of DM is supported by a wide range of astrophysical and cosmological observations that consistently reveal discrepancies between the visible baryonic content and the total mass inferred from gravitational effects. This evidence spans multiple scales, from individual galaxies to the large-scale structure of the Universe, and includes both kinematic measurements and relativistic phenomena such as gravitational lensing. Although the microscopic nature of DM remains unknown, its gravitational influence has been measured with high precision, and its inclusion is essential for reconciling observations with theoretical models. The following subsections present the main observational arguments supporting the existence of DM.

2.1.1 Galactic scale

Much of the supporting evidence for DM has been found at galactic scales. These include, for instance, the rotation curves of spiral and dwarf galaxies, which suggest mass-to-light ratios typically around 100 or even larger. Numerical simulations also require the presence of a dark halo to ensure the long-term stability of spiral discs (Athanasoula 2002; Benson 2010). Additional support comes from galaxy–galaxy lensing (dell’Antonio & Tyson 1996; Brainerd et al. 1996; Hoekstra et al. 2004; Vegetti et al. 2010), where the distortion of background objects induced by lensing depends on the mass of the lens. Observations reveal that the inferred mass is significantly larger than the visible (baryonic) mass, consistent with the presence of DM. In this subsection, we will detail the first case as perhaps the simplest and most straightforward example of the necessity of a DM component in galaxies.

Rotation curves

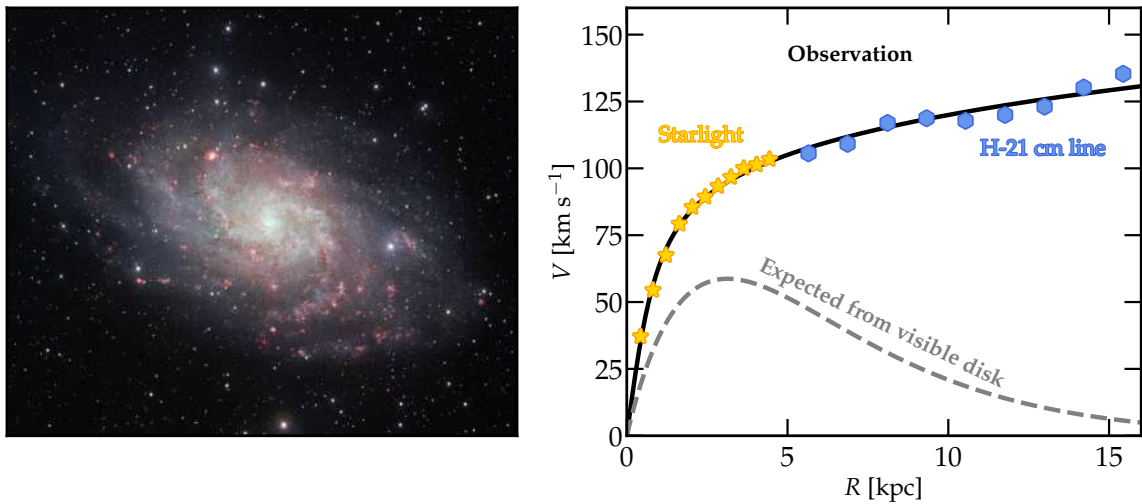


FIGURE 2.1: Rotation curve of galaxy M33. Left panel: Optical image of M33 by OmegaCAM at VLT. Right panel: Rotation curve measurements of M33 from starlight (yellow star markers) and 21 cm emission (blue hexagons) data adapted for representation purposes from Corbelli & Salucci (2000). The solid black line represents the best fit of the observed rotation curve, while the dashed grey line represents the expected rotation curve from the visible disc.

Spiral galaxies¹ rotate around an axis. The Doppler shift of atomic lines can be used to trace the circular velocity of different tracers (e.g., stars, masers, hydrogen clouds) at

¹Galaxies are usually classified according to their observed shape: Spiral, Barred Spiral, Elliptical, or Irregular galaxies that are not well described by the other categories.

various distances from the galactic centre (the position of the rotation axis), yielding a galactic rotation curve. Newtonian physics ($F = ma$) predicts the circular velocity V of a particle of mass m at a distance r from the centre (assuming spherical symmetry) as

$$V(r) = \sqrt{\frac{GM(r)}{r}}, \quad (2.1)$$

where $M(r)$ is the enclosed mass. For a typical spiral galaxy, most of the visible mass is concentrated in the bulge, a central dense region that extends typically to $\mathcal{O}(10)$ kpc. At large enough distances, the visible mass within a radius r can be assumed constant M , and the velocity is expected to decay as $V \propto r^{-1/2}$ (dashed line in Fig. 2.1). However, observations show that rotation curves remain flat (approximately constant) even at large distances from the galactic centre where no visible mass is located (solid black line in Fig. 2.1). Hence, according to Newtonian physics, a large additional invisible mass must be present to explain such observations. Assuming a dark matter halo with density $\rho(r) \propto r^{-2}$ at large r , the dependence on r vanishes in Eq. 2.1, matching the observations. This density is expected to decline at even larger radii so that V eventually drops again as $V \propto r^{-1/2}$, but no tracers are found at those distances.

The first precise measurement of this effect was obtained by [Rubin & Ford \(1970\)](#) in the Andromeda galaxy (M31), tracing hydrogen clouds, finding the flattening at ~ 22 kpc and later corroborated by many others, showing a flat curve even up to 50 kpc. This problem of missing matter was among the first and most compelling examples of a DM component.

2.1.2 Galaxy cluster scales

Velocity dispersion of cluster members

The first claim on the existence of DM came in 1933 by Fritz Zwicky ([Zwicky 2009, 1937b](#)), although the term “dark matter” can be traced back to H. Poincaré, W. Thomson, E. Öpik, J. Kapteyn, K. Lundmark, J. Oort, and S. Smith. Zwicky studied the velocity dispersion of the Coma galaxy cluster, finding that a larger mass was required to keep the galaxies gravitationally bound (See [Gavazzi et al. \(2009\)](#) for a lensing-based mass measurement of the Coma cluster). A similar study was carried out a few years later by [Smith \(1936\)](#) in the Virgo galaxy cluster. Galaxy clusters typically contain hundreds to thousands of galaxies, extend over several Mpc, and constitute the largest

gravitationally bound systems, making them excellent probes of the average properties of the Universe.

Based on the virial theorem, Zwicky estimated the mass of the Coma cluster. The virial theorem links the average kinetic and potential energies through $\langle K \rangle = -\frac{1}{2} \langle V \rangle$. For a system with $N \gg 1$ objects of mass m at equal distances r , interacting only through gravity, the total mass mN can be obtained by measuring their velocity dispersion $\langle v^2 \rangle$ and the size of the system R :

$$mN = \frac{2R\langle v \rangle^2}{G}. \quad (2.2)$$

Even with this simplistic approximation, when applied to the Coma cluster the mass obtained was much larger than the visible mass². This interpretation was not immediately accepted and took several decades to gain widespread acceptance as new probes and more detailed measurements became available.

X-ray emission

In the 1980s, a new method for assessing the matter content of clusters emerged: X-ray observations (Sarazin 1986). The hot gas in the intracluster medium is filled with ionised hydrogen and helium. As this gas collapses into the gravitational potential well of the cluster, it experiences a shock and adiabatic compression, heating up to temperatures of $T_{\text{gas}} \sim m_p v_{\text{esc}}^2 \sim 10^8$ K. These high temperatures are typical of nuclear reactions, but due to the low density, no fusion occurs even at such temperatures. The hot gas emits X-rays through thermal bremsstrahlung.

Assuming hydrostatic equilibrium and spherical symmetry, the equilibrium equation relating the gradient of the gas pressure, $\mathcal{P}_{\text{gas}}$, to the gradient of the potential, ϕ , reads

$$\frac{1}{\rho_{\text{gas}}(r)} \frac{d\mathcal{P}_{\text{gas}}}{dr} = -\frac{d\phi}{dr} = -\frac{GM(r)}{r^2}, \quad (2.3)$$

where $\rho_{\text{gas}}(r)$ and $M(r)$ are the gas density and the total gravitating mass within radius r . The total mass includes hydrogen/helium gas and dark matter. Assuming an ideal gas with a mixture of $\sim 75\%$ hydrogen and 25% helium, the pressure and density of the gas are related via $\mathcal{P}_{\text{gas}} = \rho_{\text{gas}} k_B T_{\text{gas}} / \mu m_p$, where for this mixture the mean molecular

²Nowadays, from more precise measurements in typical clusters, we know that the mass of the galaxies represents about $\sim 1\%$ of the total mass, while the gas in the intracluster medium represents 5–15% of the total. The missing mass is interpreted as DM.

weight μ is approximately 0.6. Both the density ρ_{gas} and the temperature T_{gas} can be measured from the X-ray spectrum, allowing the total cluster mass to be reconstructed from Eq. 2.3. These results confirm that the hot gas constitutes only a fraction of the total mass within the cluster, implying the presence of additional DM.

Weak lensing

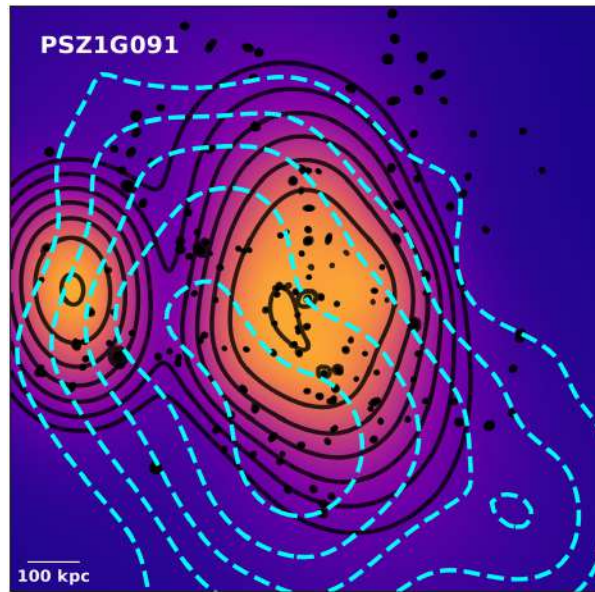


FIGURE 2.2: Total projected surface mass density for the galaxy cluster PSZ1G091. The black lines show the logarithmically spaced mass contours derived from lensing analyses. The cyan dashed lines show the equivalent mass contours obtained from X-ray analysis of Chandra observations. Both contours show a clear offset indicating the collision and dynamical state of the cluster. Source: (Cerny et al. 2025)

Similarly to galaxy–galaxy lensing, the lensing effects on background sources by galaxy clusters suggest a mass much larger than that inferred from visible components. Of particular interest are WL shear measurements. Lensing, as we will explain in Ch. 3, distorts the shape of background objects, akin to astigmatism. The measured ellipticity and the distortion direction in weakly lensed galaxies can be linked to the reduced shear, g , averaged over a sufficiently large galaxy sample. This shear field can then be used to trace the convergence, i.e., the mass distribution of the cluster.

Deep imaging of galaxy cluster fields yields a galaxy density of $n \approx 30\text{--}50$ per square arcminute, sufficient for reconstructing the shear profile and thus the scaled surface mass density. Mass maps reconstructed from WL analyses consistently show a mass density larger than that inferred from the visible matter, indicating the presence of

DM. Furthermore, comparing mass distributions derived from WL and X-ray measurements provides insight into the dynamical state of clusters. Colliding clusters, such as the Bullet Cluster (Clowe et al. 2006), provide strong evidence for DM. The bulk of baryonic mass, in the form of hot gas, is traced via X-rays, while the total mass (both visible and dark) is traced via WL. In merging systems, these mass distributions show an offset. If traced back in time, the two components would initially overlap, but during the collision, baryonic matter, being self-interacting, experiences shocks and lags behind, whereas DM haloes, with negligible self-interaction, pass through each other largely unaffected. A recent example of such colliding clusters (PSZ1G091) is shown in Fig. 2.2 (Cerny et al. 2025).

2.1.3 Cosmic shear

At scales larger than those of galaxy clusters, we encounter the so-called cosmic shear, another key piece of evidence for DM (Van Waerbeke et al. 2000; Bacon et al. 2000). Cosmic shear is a manifestation of the WL effect, previously introduced, but now arising from large-scale mass distributions such as filaments and diffuse DM clumps, rather than from individual galaxy or cluster haloes. It is measured statistically across samples of millions of distant galaxies observed over wide sky areas and wavelength ranges by various instruments (DES (Secco et al. 2022), KiDS (Li et al. 2023a), HSC (Li et al. 2023b)), with current measurements yielding a DM density of $\Omega_{\text{DM}} \approx 0.25$. The upcoming stage-4 surveys, such as Euclid (Euclid Collaboration et al. 2025) and Rubin-LSST (LSST Science Collaboration et al. 2009), are expected to refine these results significantly. Cosmic shear is not only essential for quantifying the DM content, but also provides a direct window into the formation and evolution of the large-scale structure.

Similarly to cosmic shear, the so-called CMB lensing refers to the distortions imprinted on the cosmic microwave background by intervening matter distributions. Since the CMB is the most distant light source observable, CMB lensing probes DM structures at very high redshifts ($z \gtrsim 2$), as detected by Planck and other CMB experiments (Planck Collaboration et al. 2020a; Hanson et al. 2010; Lewis & Challinor 2006).

2.1.4 Large scale structure

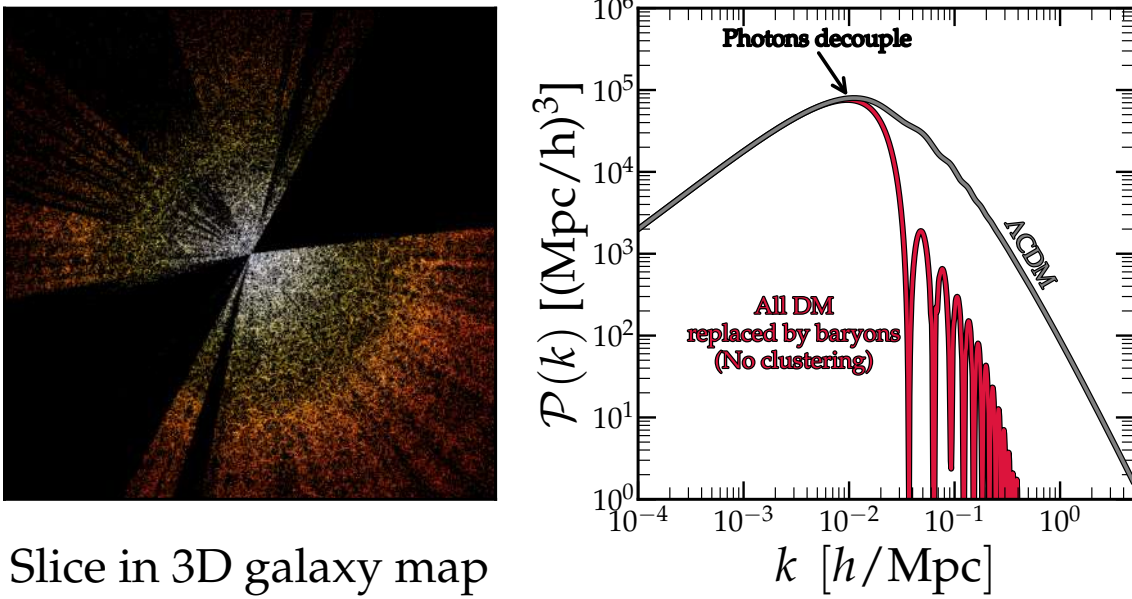


FIGURE 2.3: Left panel: Slice of the 3D galaxy mapping by DESI. Credit: D. Schlegel/Berkeley Lab. Acknowledgment: M. Zamani (NSF NOIRLab). Right panel: Matter power spectrum generated by the CLASS code (Blas et al. 2011). The gray line represents the matter power spectrum according to the concordance model of cosmology Λ CDM, while the red curve is a similar cosmology but where all DM has been transformed to baryonic matter.

Large scale: Structure formation

The most convincing and precise evidence for the existence of DM comes from the largest scales Planck Collaboration et al. (2020a); DESI Collaboration et al. (2025), i.e., from the entire observable Universe. Matter alone cannot form the structures we observe today; the presence of DM is required to explain both the current and past large-scale structure of the Universe. The present Universe is highly inhomogeneous, as shown by 3D galaxy mapping (see Fig. 2.3), which reveals walls, filaments, voids and other structures at various scales.

This information is encoded in the matter power spectrum $\mathcal{P}(k)$, shown in the right panel of Fig. 2.3. It is defined as $\langle \delta_k \delta_{k'} \rangle = (2\pi)^3 \mathcal{P}(k) \delta^3(\mathbf{k} - \mathbf{k}')$, where $\delta(\mathbf{r})$ is the density contrast and δ_k its Fourier transform. Here, $\langle \rangle$ denotes an average over $\mathbf{k}$ orientations, and $\delta^3(\mathbf{k} - \mathbf{k}')$ is the Dirac delta function, indicating that modes with $\mathbf{k} \neq \mathbf{k}'$ are statistically independent. Since $\mathcal{P}(k)$ is the Fourier transform of the density autocorrelation function, it describes the matter distribution in Fourier space. Large values of $\mathcal{P}(k)$ correspond to a strong presence of structures with characteristic scale

$1/k$, while low values indicate fewer structures. A dimensionless variance is defined as $\Delta^2(k) = k^3 \mathcal{P}(k)/(2\pi^2)$, where $\Delta^2 \gg 1$ indicates large overdensities, and $\Delta^2 \sim 1$ corresponds to densities comparable to the mean.

Under the assumption of an inflationary era, primordial inhomogeneities with tiny amplitudes, $\delta \sim 10^{-5}$, are expected. This agrees with CMB observations. The key question is how such small primordial fluctuations evolved into the large inhomogeneities observed in the late Universe. The simplest explanation is that the growth of structure, starting from these small perturbations, has been driven by the presence of DM.

To understand this process, we consider a Universe filled with matter. The general-relativistic evolution of density perturbations can be approximated by its Newtonian limit³. This fluid is fully described by its density, $\rho(\mathbf{r}, t)$, its velocity field, $\mathbf{v}(\mathbf{r}, t)$, and the equation of state for the pressure $\mathcal{P}$. The gravitational potential Φ completes the system, which evolves according to a set of non-linear differential equations known as the Euler equations:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, & \text{continuity Eq.,} \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla \varphi}{\rho} - \nabla \mathcal{P}, & \text{Newton 2 law } \mathbf{a} = \mathbf{F}/m, \\ \nabla^2 \Phi = 4\pi G \rho, & \text{Poisson.} \end{cases} \quad (2.4)$$

We expand these quantities to first-order perturbations around a homogeneous background:

$$\rho = \rho_0(t) + \rho_1(\mathbf{x}, t), \quad \mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1, \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \quad \Phi = \Phi_0 + \Phi_1. \quad (2.5)$$

Assuming a static Universe, the Euler equations reduce to

$$\begin{cases} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0, \\ \frac{\partial \mathbf{v}_1}{\partial t} + \frac{v_s^2}{\rho_0} \cdot \nabla \rho_1 + \nabla \Phi_1 = 0, & \text{(Static)} \\ \nabla^2 \Phi = 4\pi G \rho_1, \end{cases} \quad (2.6)$$

³This approximation is valid for non-relativistic matter at scales smaller than the horizon.

where $v_s = \partial \mathcal{P}_1 / \partial \rho_1$ is the sound speed of the fluid. These equations can be combined to obtain the Jeans equation:

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 = 4\pi G \rho_0 \rho_1. \quad (2.7)$$

The left-hand term represents pressure, which competes with gravitational collapse on the right-hand side. The Jeans length $\lambda_J = \sqrt{v_s^2 / (4\pi G \rho_0)}$ sets the scale at which both effects balance: structures with $\lambda > \lambda_J$ collapse, while those with $\lambda < \lambda_J$ are supported by pressure. The associated Jeans mass is $M_J = 4\pi \rho_0 \lambda_J^3 / 3$.

In an expanding Universe, solving the background quantities yields

$$\rho_0(t) = \frac{\rho_0(t_0)}{a^3}, \quad \mathbf{v}_0 = H \mathbf{r}, \quad \nabla \Phi_0 = \frac{4\pi G \rho_0}{3} \mathbf{r}. \quad (2.8)$$

Here the scale factor a has an undefined time dependence. Solving the Euler equations for a homogeneous fluid determines the evolution of $a(t)$, which, in a matter-dominated Universe, evolves as $a \propto t^{2/3}$ (see Ch. 1). In the presence of radiation or DE, a full solution of the relativistic Friedmann equations is required.

To study the evolution of density perturbations, we expand the Euler equations to first order in an expanding Universe, obtaining the standard continuity, Euler and Poisson equations. Combining them yields the Jeans equation in Fourier space:

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2H \frac{\partial \delta_k}{\partial t} + \left(\frac{v_s^2 k^2}{a^2} - 4\pi G \rho_0 \right) \delta_k = 0. \quad (2.9)$$

This equation has three key contributions: a pressure term ($\propto v_s^2$) that counteracts gravitational collapse, a gravitational term ($\propto 4\pi G \rho_0$) driving the growth of perturbations, and a “friction” term ($2H \dot{\delta}_k$) arising from the expansion of the Universe. The Jeans wave-number,

$$k_J = a \sqrt{\frac{4\pi G \rho_0}{v_s^2}}, \quad (2.10)$$

marks the critical scale below which pressure suppresses collapse and above which density perturbations can grow.

Up to this point we have considered a generic matter fluid, without further specification on which kind of matter we are dealing with.

Baryonic matter: Non-clustering

Before recombination, the baryonic matter is tightly coupled to photons, resulting in a matter fluid with a relativistic sound speed $v_s \approx c/\sqrt{3}$, where c is the speed of light⁴. Inputting this into the relativistic Jeans equation for density perturbations, and given the large pressure provided by photons since $v_s \sim \mathcal{O}(1)$, which dominates over the gravitational term, one can expect that the baryonic fluid does not cluster. Explicitly solving the Jeans equation in a matter-dominated Universe, where we have $\rho_0 = 3H^2/8\pi G$, $a(t) = \left(\frac{3}{2}tH_0\right)^{2/3}$, and $H = 2/3t$, we arrive at

$$\frac{\partial^2 \delta_k}{\partial t^2} + \frac{4}{3t} \frac{\partial \delta_k}{\partial t} + \frac{v_s^2 k^2}{\left(\frac{3}{2}H_0 t\right)^{4/3}} = 0, \quad \text{solving} \quad \delta_k = \frac{c_1 \cos(c_0 k t^{1/3}) + c_2 \sin(c_0 k t^{1/3})}{k t^{1/3}}. \quad (2.11)$$

The coefficients c_0, c_1 , and c_2 depend on v_s and H_0 , but are not specified here for simplicity. This results in an oscillatory solution, i.e., pressure waves that can be observed as peaks in the CMB power spectrum, damped over time. Since this solution yields a Jeans mode $k_J \sim aH/v_s$, which is close to the horizon (given that $v_s \sim \mathcal{O}(1)$), we can conclude that baryonic matter alone does not cluster at early times without DM, and only begins to fall into potential wells at late times, after decoupling from photons.

Dark matter: Clustering

“Dark” matter is particularly interesting because it does not interact with photons and can be easily described as a simple matter fluid with a non-relativistic sound speed, $v_s \sim 0$. This key feature distinguishes DM from baryonic matter: in this case, DM behaves as a pressureless fluid. Simply solving the Jeans equation in Fourier space under the assumption of a matter-dominated Universe, we find that

$$\frac{\partial^2 \delta_k}{\partial t^2} + \frac{4}{3t} \frac{\partial \delta_k}{\partial t} - \frac{2\delta_k}{3t^2} = 0, \quad \text{yielding} \quad \delta_k = c_{\text{grow}} t^{2/3} + c_{\text{decay}} t^{-1}. \quad (2.12)$$

Thus, the primordial matter fluctuations have a decaying mode that fades over time, and a growing mode that evolves positively as a power of time.

During the RD epoch, where $a \propto t^{1/3}$, the Jeans equation loses the pressure term $v_s^2 k^2 \delta_k / a^2$ because $v_s \sim 0$, and also the $4\pi G \rho_0 \delta_k$ term, since δ_k would correspond to radiation, which does not cluster and therefore does not create perturbations in the

⁴Here $c = 1$ in natural units.

gravitational potential. The Jeans equation reduces to

$$\frac{d^2\delta_k}{dt^2} + \frac{1}{t} \frac{d\delta_k}{dt} = 0, \quad \text{solving} \quad \delta_k \propto \ln t + \text{const.} \quad (2.13)$$

Dark matter perturbations therefore grow only logarithmically during the radiation domination era. Similarly, it can be shown that DM does not cluster efficiently during the Λ D epoch.

Summing up, the Universe becomes matter-dominated at the equality time when $a = a_{\text{eq}} \sim 1/3400$. At this time, the primordial inhomogeneities ($\delta_k \approx 10^{-5}$) grow, driven by DM as $t^{2/3}$, proportional to the scale factor, on scales $1/k$ that are 3400 times smaller than the horizon at that time. Larger scales begin clustering later, when they become smaller than the horizon. In the matter-dominated era, a mode k enters the horizon when the scale factor equals $a_k \approx H_0^2/k^2$. Nowadays we have

$$\delta_k \approx 10^{-5} (a_0/a_k) \approx 10^{-5} (k/H_0)^2, \quad (2.14)$$

which exceeds unity for scales $H_0/k \lesssim 10^{-5/2}$, about three orders of magnitude smaller than the present horizon.

Normal matter, being tightly coupled to photons (which formed pressure waves), could not cluster. Later, at recombination ($a = a_{\text{recomb}} \approx 1/1100$), photons decoupled and baryons began to fall into the gravitational potential wells created by DM.

This whole picture has been recorded in the matter power spectrum $\mathcal{P}(k)$, or equivalently in the dimensionless variance $\Delta^2(k)$. Without DM, the Universe would exhibit far less power on small scales and much more pronounced oscillations (see the red curve in the right-hand panel of Fig. 2.3). The growth of perturbations we have briefly outlined here is known as linear growth, valid when $\delta_k \ll 1$. Once an over-density reaches $\delta_k \approx 1$, the linear approximation breaks down. In this non-linear regime, smaller-scale inhomogeneities (larger k) collapse first and subsequently merge gravitationally into larger systems: sub-haloes, galaxies, and clusters. This is the so-called hierarchical structure formation. Solving the Euler–Poisson system now requires numerical simulations, since non-linear effects are essential. In this regime, DM acts as the scaffolding that organises the distribution of visible matter.

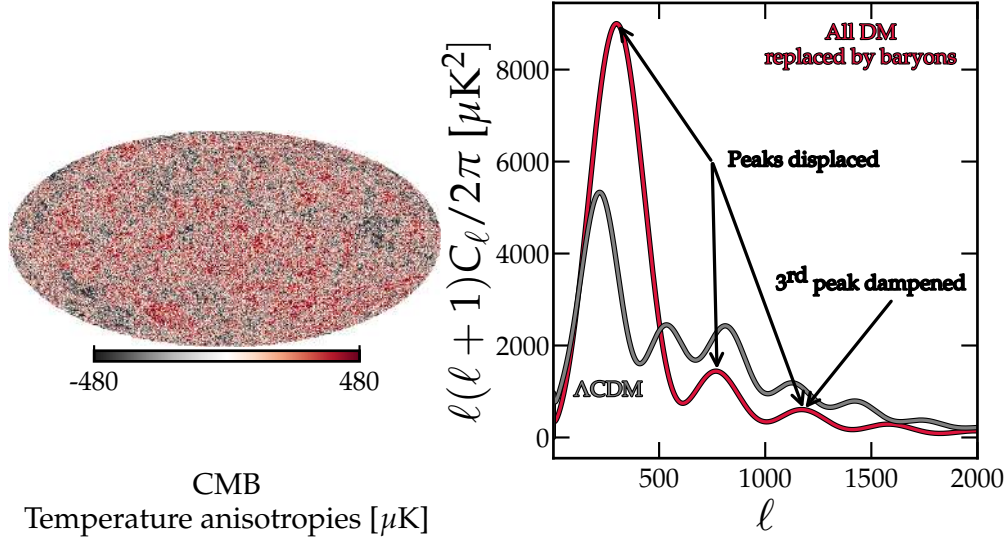


FIGURE 2.4: Left panel: Temperature anisotropies of the CMB according to the concordance model of cosmology. Right panel: Angular power spectrum of the CMB generated by the software CAMB (Lewis et al. 2000; Howlett et al. 2012). The grey line represents the angular power spectrum according to the concordance model of cosmology ΛCDM , while the red curve corresponds to a similar cosmology where all DM has been transformed into baryonic matter. The exotic cosmological model power spectrum has been normalised by a factor $\sim 1/2$ for visualisation purposes. Here the peaks are shifted and the relative amplitudes have changed, with the 3rd peak being especially suppressed, whereas CDM boosts it.

Cosmic microwave background: Acoustic peaks

The CMB provides one of the most important probes of the primordial Universe and its properties, and is therefore a key field of study in cosmology. Since its discovery in 1965 by Penzias & Wilson (1965), the CMB has undergone several observational milestones, the first being its confirmation as a perfect blackbody with a temperature of $T_{\text{CMB}} \sim 2.73 \text{ K}$, which validated the hot Big Bang scenario (Fixsen et al. 1996). Later, small variations (anisotropies) in the CMB temperature field due to the photon–baryon fluid state during recombination were hinted at by COBE and subsequently confirmed with greater accuracy by WMAP (Jarosik et al. 2003) and further detailed by Planck in the following decade (Planck Collaboration et al. 2014, 2016, 2020a). These missions measured the CMB angular power spectrum of anisotropies. The “hot-and-cold” spots trace the location of matter density perturbations at that epoch, shaping the angular power spectrum of the CMB. Dark matter affects both the positions and amplitudes of the acoustic peaks in the CMB angular power spectrum. These studies allow for precise determination of the cosmological parameters and establish ΛCDM as the concordance

model, providing the most accurate current determination of the DM density (Planck Collaboration et al. 2020a).

Here we briefly outline the physics behind this analysis and the features imprinted by DM in the CMB angular power spectrum. A more detailed treatment can be found in Dodelson (2003) and Baumann (2022).

The angular power spectrum of the CMB is conceptually similar to the matter power spectrum defined in the previous section, but now focuses on photon anisotropies. The amplitude of anisotropy at an angular scale $\theta \sim \pi/\ell$ is represented by C_ℓ . An angular scale θ corresponds to a wave-number $k \sim \ell H_0$. The power at smaller angular scales (higher ℓ) declines due to photon diffusion and the damping of small-scale structures. The measured 3rd peak is roughly similar in amplitude to the 2nd, suggesting that the 3rd was initially higher, as expected from the damping effects.

A photon fluid inhomogeneity at position $\mathbf{r}$ and time t is described by the temperature fluctuation $\Theta(\mathbf{r}, t, \hat{\mathbf{p}}) \equiv \delta T/T$, where $\hat{\mathbf{p}}$ denotes the direction of photon propagation. This fluctuation evolves according to the Boltzmann equation, which, in Fourier space, contains terms accounting for photon propagation, gravitational redshifting, and scattering with baryons via Thomson interactions.

In the tightly coupled regime before recombination, photons and baryons behave as a single fluid, and the evolution of perturbations is dominated by the first few multipoles, particularly the monopole (Θ_0) and dipole (Θ_1). The monopole evolves as

$$\dot{\Theta}_0 + \dot{\Psi}_k - \frac{k}{a}\Theta_1 = 0, \quad (2.15)$$

where Ψ_k is the metric curvature perturbation and Θ_1 encodes the velocity perturbation of the photon–baryon fluid. Physically, this equation reflects how density perturbations in the photon fluid are sourced by gravitational compression and balanced by pressure support and scattering with baryons.

After recombination, photons decouple from baryons and begin to free-stream, making higher-order multipoles increasingly relevant for describing the angular structure of the CMB anisotropies.

By rewriting the photon equations in terms of the conformal time η and combining the monopole and dipole equations, we obtain a second-order differential equation for the

monopole Θ_0 :

$$\Theta_0'' + \frac{a'}{a} \frac{R}{1+R} \Theta_0' + \frac{k^2}{3(1+R)} \Theta_0 = -\frac{k^2}{3} \Phi_k - \frac{a'}{a} \frac{R}{1+R} \Psi_k' - \Psi_k'', \quad (2.16)$$

where R is the baryon-to-photon ratio defined as $R = 3\rho_b/4\rho_\gamma$, with ρ_b and ρ_γ the baryon and photon densities, respectively. This resembles a damped, forced harmonic oscillator, where damping arises from the expansion of the Universe and the forcing terms originate from gravitational potentials Φ_k . Neglecting the forcing terms, the homogeneous solution reads

$$\Theta_0 = c_1 \cos\left(\frac{k\eta}{v_s}\right) + c_2 \sin\left(\frac{k\eta}{v_s}\right), \quad (2.17)$$

where the sound speed of the photon–baryon fluid is $v_s = 1/\sqrt{3(1+R)}$. These oscillations produce the acoustic peaks in the CMB, with their spacing set by R .

Including the gravitational forcing term shifts the baseline of the oscillations:

$$\Theta_0 = c_1 \cos\left(\frac{k\eta}{v_s}\right) + c_2 \sin\left(\frac{k\eta}{v_s}\right) - (1+R)\Phi_k. \quad (2.18)$$

This offset enhances compressional phases, leading to the observed asymmetry between odd and even peaks in the CMB power spectrum.

Free streaming regime

After recombination, photons decoupled from baryons and began to free-stream. We can write the Boltzmann equation as

$$\Theta' - (ik\mu - \tau')\Theta = \hat{S}, \quad (2.19)$$

where $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$ is the cosine of the angle between the photon propagation direction and the wave-vector, τ' is the conformal time derivative of the optical depth⁵, and $\hat{S}$ is the source function, given by $\hat{S} = -ik\mu\Phi - \tau'\Theta_0$, where we have temporarily neglected the time dependence of the gravitational potentials and the velocity perturbations of baryonic matter. Following [Dodelson \(2003\)](#), the solution in conformal time can be

⁵The optical depth is defined as the integral across conformal time of the electron number density times the Thomson scattering cross section and the scale factor.

expressed through a Legendre expansion:

$$\Theta_\ell(k, \eta_0) = \int_0^{\eta_0} g(\eta) [\Theta_0(k, \eta) + \Phi(k, \eta)] j_\ell[k(\eta_0 - \eta)] d\eta. \quad (2.20)$$

Here, j_ℓ is the spherical Bessel function arising from the Legendre expansion, and $g(\eta) = -\tau' e^{-\tau(\eta)}$ is the visibility function, which gives the probability that a photon last scattered at conformal time η . This function peaks at $\eta = \eta_{\text{recomb}}$. At earlier times, scattering was very likely, and after recombination the Universe became transparent, so in both limits $g(\eta) \rightarrow 0$. As a result, the multipoles can be approximated by

$$\Theta_\ell(k, \eta_0) \approx [\Theta_0(k, \eta_{\text{recomb}}) + \Phi(k, \eta_{\text{recomb}})] j_\ell[k(\eta_0 - \eta_{\text{recomb}})]. \quad (2.21)$$

Since photon perturbations do not grow after recombination, this solution effectively provides a snapshot of the last scattering surface. The CMB temperature field reflects both kinetic and potential energy contributions at recombination, including the fact that photons had to climb out of the gravitational potential wells, which were set by the dark matter perturbations δ . The Bessel function $j_\ell[k(\eta_0 - \eta_{\text{recomb}})]$ determines the contribution of each wave-number k to the anisotropy at an angular scale $\sim 1/\ell$. Asymptotically, $j_\ell(x) \rightarrow (x/\ell)^{\ell-1/2}/\ell$, so only modes with $\ell \sim k\eta_0$ contribute significantly. Given that $\eta_{\text{recomb}} \ll \eta_0$, it can be neglected in the argument.

The angular power spectrum C_ℓ , previously defined as the variance of temperature anisotropies at scales π/ℓ , can be written as

$$C_\ell = \frac{2}{\pi} \int_0^\infty k^2 \mathcal{P}(k) \left| \frac{\Theta_\ell(k)}{\delta(k)} \right|^2 dk, \quad (2.22)$$

where, as before, $\mathcal{P}(k)$ is the matter power spectrum and $\delta(k)$ the matter perturbation. The quantity C_ℓ can thus be interpreted, qualitatively, as the average squared amplitude of the multipoles Θ_ℓ .

Eq. 2.21 is a simplification, and several corrections must be taken into account. Including higher-order multipoles, which are smaller and out of phase with the monopole, increases $|\Theta_\ell|^2$ and leads to a suppression of subsequent peaks. The time evolution of the gravitational potential boosts the values of C_ℓ at low multipoles (up to $\ell \sim 200$); this is the so-called integrated Sachs–Wolfe effect. On small scales, photon diffusion damps the anisotropies, reducing power at high ℓ .

Although this description simplifies many aspects, it captures the key physics needed to understand the imprint of dark matter on the angular power spectrum of the CMB. The relative heights of neighbouring acoustic peaks, particularly the second and third, encode information about both the matter coupled to photons (baryons) and the matter responsible for generating the gravitational potential (Φ). The fact that the third peak is not suppressed with respect to the second points clearly to the presence of additional, non-baryonic matter. In a Universe made only of baryons, the third peak would appear strongly damped, contrary to what we observe (see Fig. 2.4).

2.2 Alternatives to dark matter

We have discussed many observational probes of DM, but there is a line of thought that attempts to explain these phenomena without invoking DM. Here we give a brief overview of the topic for completeness, although the current state of the art seems to favour DM over alternative theories.

In the 1840s, a similar issue arose to the one we face today with DM. The orbit of Uranus appeared to deviate from Newtonian predictions. Later, the discovery of Neptune solved the problem. In fact, it was the other way around: Le Verrier proposed the existence of a fictitious planet to account for Uranus's orbit, and Johann Galle found Neptune almost exactly where Le Verrier predicted. Shortly after, a similar anomaly appeared with the orbit of Mercury. Again, a hypothetical planet, Vulcan, was introduced to explain the anomaly, but it was neither found nor was the problem resolved under Newtonian gravity. The issue was eventually solved by introducing a relativistic analysis.

We now face a similar question regarding DM: do we need to add an extra component (in this case, DM), as with Uranus and Neptune, or do we need a modified version of our gravitational theory, as with Mercury and the rise of GR in the early 1900s? So far, we have discussed only the first alternative, an additional form of matter with no electromagnetic interactions and perhaps weak interactions, much weaker than those of ordinary particles. But some physicists have tried the other “Mercury-like” approach: what if GR is not sufficient, and we need revised or alternative theories of gravity?

The rotation curves of disc galaxies seem to flatten rather than fall off at large radii,

contrary to expectations from visible matter alone. We discussed this effect earlier when introducing the observational probes of DM. An alternative explanation emerged in 1983 with [Milgrom \(1983\)](#), who proposed a “modification of the Newtonian dynamics” (MOND), a theory that alters Newton’s second law $F = ma$. In that work, Milgrom postulated that below a critical acceleration a_* , this law changes to $F = ma^2/a_*$. This simple modification can explain the observations. It can be shown that the asymptotic rotation velocity at large radii scales with the galaxy’s mass as

$$v_\infty = M_b^{1/4}, \quad (2.23)$$

where M_b is the total baryonic mass of the galaxy. This relation is known as the baryonic Tully-Fisher relation⁶. Measurements of more than 200 galaxies, including faint dwarf galaxies where a larger dispersion was expected, showed that they follow this relation. This was a significant success for MOND and generated considerable interest.

Despite MOND’s initial success in explaining both flat rotation curves and the Tully-Fisher relation, it faces several discrepancies with other observations, namely:

- Some globular clusters (e.g., Palomar 14 and NGC 2419), located in the outer regions of the Milky Way where MOND should apply, do not follow the low-acceleration predictions of MOND ([Zhao 2005](#)).
- In galaxy clusters, the typical accelerations are of the order of a_* , and MOND should apply. However, MOND can only partially account for the “missing mass” in clusters ([Sanders 1999](#); [Gavazzi 2002](#)).
- The biggest challenge comes from the Bullet Cluster (and other merging galaxy clusters). These clusters show an offset between the bulk of the mass and the visible matter, not simply tied to baryons as MOND would suggest. Some have even proposed 2 eV neutrinos to account for the missing mass, but this has also been ruled out ([Natarajan & Zhao 2008](#)).
- Another important failure for MOND is the existence of DM-deficient galaxies. In such systems, normal matter does not follow MOND predictions, since no modification at distances $a \lesssim a_*$ is observed. The main challenge is to find such DM-deficient galaxies within the MOND regime.

⁶Note that this is not the Tully-Fisher relation that links the luminosity of a galaxy with its rotational velocity, sometimes used in the distance ladder.

- Milgrom’s MOND was tuned for the galaxy rotation curve problem, but the most compelling evidence for DM comes from the largest scales, namely LSS and CMB anisotropies. A major challenge for MOND is to reproduce structure formation as effectively as DM does (Pardo & Spergel 2020).

From a more conceptual perspective, note that the phenomenological MOND postulate ($F = ma^2/a_*$ at large radii) does not constitute a consistent theory. As an empirical prescription, it violates basic principles such as momentum conservation: the force experienced by two masses is no longer equal and opposite, since the lighter mass undergoes a larger acceleration and therefore a stronger MOND correction. Over the years, various internally consistent frameworks have been developed to recover MOND-like behaviour in specific limits. Below, we briefly outline the main theoretical proposals inspired by MOND.

- Emergent gravity⁷ (Verlinde 2011, 2017).
- Non-relativistic field theory for MOND (Bekenstein & Milgrom 1984).
- MOND as tensor–vector–scalar gravity (Bekenstein 2004; Skordis & Złośnik 2019, 2021).
- MOND as superfluid Dark Matter (Silverman & Mallett 2002).
- Conformal gravity (Mannheim & Kazanas 1989; Mannheim 1993).
- Cotton gravity (Harada 2021; Mo et al. 2025)

Given that MOND and other modified gravity approaches cannot fully explain the various observations in a consistent way, we will focus on explanations in terms of exotic matter content. After all, the purpose of this thesis is to study the effects of CDM on lensing phenomena and to help shed light on its nature and abundance.

2.3 Dark matter properties

In the previous sections, we have presented some of the strongest arguments for the existence of DM, with the most compelling ones coming from the largest scales,

⁷Note that emergent gravity and other models in this list are not strictly MOND theories, but share a similar motivation: to extend our gravitational theories beyond GR in order to explain effects traditionally attributed to DM without requiring an additional matter component.

namely LSS and CMB observations. We have also discussed some alternatives that aim to explain these observations without DM. However, observational fits favour a simple model of non-relativistic or “cold” dark matter, with negligible self-interactions and Gaussian adiabatic perturbations, which successfully accounts for the data. Here, we briefly summarise the main properties of this best-fit DM model. A more detailed review can be found in [Taoso et al. \(2008\)](#), [Cirelli et al. \(2024\)](#), or [Bertone et al. \(2005\)](#).

Non-relativistic

Cold Dark Matter refers to a class of non-relativistic, collisionless particles that moved at negligible velocities compared to the speed of light at the time when structures began to form in the Universe. This non-relativistic nature ensured that CDM particles could cluster efficiently under gravity, preserving small-scale density perturbations that later evolved into galaxies and sub-galactic structures. In contrast to hot or warm dark matter scenarios, CDM does not experience significant free-streaming effects, allowing the survival of fluctuations down to very small scales. This behaviour underpins the bottom-up, hierarchical formation of structure predicted by the Λ CDM model, where small objects form first and later merge to build larger systems such as galaxies and galaxy clusters.

Distribution

In the CDM paradigm, the distribution of dark matter follows a hierarchical structure across cosmic scales⁸. On galactic scales, CDM forms extended, roughly spherical haloes that dominate the mass budget beyond the luminous regions, with density profiles typically following a cuspy inner slope and an outer decline proportional to r^{-3} . In galaxy clusters, CDM constitutes the majority of the total mass ($\sim 90\%$), shaping deep gravitational potential wells that bind hot X-ray emitting gas and member galaxies, and producing measurable gravitational lensing effects. On cosmological scales, CDM traces the large-scale structure of the Universe, forming a cosmic web of filaments, sheets, and voids. This distribution reflects the growth of density perturbations over time, from initial, nearly Gaussian fluctuations imprinted in the cosmic microwave background to the highly non-linear structures observed today.

Mass

The plausible mass range for CDM particles spans several orders of magnitude, re-

⁸The largest structures form from the merging of smaller-scale structures formed at earlier times. Consequently, large-scale haloes contain a web of embedded smaller sub-haloes.

flecting the wide variety of theoretical candidates. Traditional weakly interacting massive particles (WIMPs) typically lie in the GeV to multi-TeV range, motivated by extensions of the Standard Model and thermal freeze-out scenarios. Axions, another well-motivated CDM candidate, can have much lower masses, down to 10^{-22} eV⁹ for ultralight models such as fuzzy dark matter (FDM). The defining property across these candidates is not their specific mass scale but their non-relativistic behaviour at the onset of structure formation. Regardless of their mass, CDM particles must satisfy constraints from the cosmic microwave background, large-scale structure, and small-scale observations, which collectively rule out candidates that would erase primordial fluctuations through excessive free-streaming. Other candidates, such as Massive Astrophysical Compact Halo Objects (MACHOs), where CDM consists of macroscopic objects of mass M , can reach several solar masses up to $10^{15} M_{\odot}$, although these extremely large values are disfavoured and highly constrained (Wilkinson et al. 2001; Kassiola et al. 1991; Brandt 2016).

Charge

Cold Dark Matter particles are expected to be electrically neutral¹⁰ and to interact weakly, if at all, with Standard Model particles other than through gravity. The absence of electric charge is necessary to avoid coupling with photons and baryons in the early Universe, which would leave observable imprints in the cosmic microwave background anisotropies and alter BBN predictions. Similarly, strong or electromagnetic interactions would lead to excessive energy loss and thermal coupling, contradicting large-scale structure observations. While CDM candidates such as WIMPs may interact via the weak nuclear force, such interactions must remain sufficiently suppressed to evade detection in current laboratory experiments and direct detection searches. Overall, observational evidence requires CDM particles to be effectively collisionless on astrophysical scales, with gravitational interactions dominating their dynamics across cosmic history.

Self interaction

In the standard CDM scenario, dark matter particles are assumed to be effectively collisionless, with negligible self-interactions beyond gravity. This assumption is supported

⁹At lower masses, the resulting wave-like effects become significant on galactic scales, and the density profiles of haloes in dwarf galaxies cannot be matched. This is further discussed in the axion subsection.

¹⁰They could, in principle, have a very small charge (or “millicharge”), see McDermott et al. (2011), and constraints from the EDGES 21 cm signal (Kovetz et al. 2018).

by observations of large-scale structure and galaxy cluster dynamics, which are well reproduced by models where dark matter interacts only gravitationally. However, some alternative models introduce the possibility of non-gravitational self-interactions (Carlson et al. 1992), often characterised by a self-interaction cross-section per unit mass. Such models could alleviate small-scale structure tensions, such as the core–cusp problem (Nguyen et al. 2021) or the missing satellites issue (Tulin & Yu 2018). Current observational constraints, derived from systems like the Bullet Cluster and other galaxy cluster mergers, limit the self-interaction cross-section to values below approximately $1 \text{ cm}^2/\text{g}$, ensuring that CDM remains largely non-collisional on astrophysical scales.

2.4 Dark matter candidates

In the previous sections, we have seen evidence for a non-baryonic matter component, i.e., DM, which indeed makes up the main contribution to the total matter budget of the Universe. It is a natural follow-up question to ask: what is DM? What are the constituents of this dark sector? As we have just discussed, the best-fit models favour a cold (non-relativistic), non-self-interacting, and electrically neutral candidate. Any weak interaction, if present, must be extremely suppressed, and the particle’s mass range remains broadly open. Within this framework, we focus on three important candidates that meet these criteria: WIMPs, axions, and primordial black holes (PBHs). These three popular DM models are also those considered in the papers of this thesis. In this section, we briefly describe these candidates and summarise the current state of their searches.

2.4.1 Weakly Interacting Massive Particles

Weakly Interacting Massive Particles Arcadi et al. (2018) are a well-motivated DM candidate, although no confirmed detection has been reported to date. WIMPs have been thoroughly searched for through various experiments: collider searches, where Standard Model (SM) particles may produce DM after collisions; indirect searches, where DM particles annihilate or decay into SM particles; and direct detection experiments, where DM particles interact with SM particles, producing measurable recoils. From the lensing perspective, WIMPs behave as standard CDM, with haloes described as smooth distributions of DM, perturbed only by substructure.

Formation mechanism

Let us assume a state where a generic DM particle χ interacts with Standard Model particles and remains in thermal equilibrium with the primordial plasma at temperature T . The evolution of its number density is governed by the Boltzmann equation,

$$\frac{dn_\chi}{dt} + 3H(T)n_\chi = -\langle\sigma v\rangle(n_\chi^2 - n_{\chi,\text{eq}}^2), \quad (2.24)$$

where $\langle\sigma v\rangle$ is the thermally averaged annihilation cross-section for the process $\chi\chi \rightarrow \text{SM SM}$, and $H(T)$ is the Hubble expansion rate. The term $n_{\chi,\text{eq}}$ corresponds to the equilibrium number density, which decreases exponentially when the temperature drops below the DM particle mass.

In the early Universe, when interactions are efficient, the annihilation rate exceeds the expansion rate, and the DM remains in thermal equilibrium. At later times, as the temperature falls and the density drops, annihilations become inefficient. The DM abundance then effectively freezes out and becomes constant. To simplify the analysis, it is convenient to define the comoving number density

$$Y_\chi = \frac{n_\chi}{s}, \quad \text{with} \quad s = \frac{2\pi^2}{45} h_{\text{eff}}(T) T^3, \quad (2.25)$$

where s is the entropy density and $h_{\text{eff}}(T)$ the number of effective degrees of freedom contributing to the entropy at temperature T .

With this change of variables, the Boltzmann equation becomes

$$\frac{dY_\chi}{dt} = \frac{ds}{dt} \frac{\langle\sigma v\rangle}{3H} Y_\chi^2 \left(1 - \frac{Y_{\chi,\text{eq}}}{Y_\chi}\right). \quad (2.26)$$

This equation can be interpreted qualitatively as follows: when the DM interaction rate is high enough, the solution tracks the equilibrium distribution $Y_\chi \approx Y_{\chi,\text{eq}}$. At lower temperatures, $Y_{\chi,\text{eq}}$ is suppressed by a Boltzmann factor, and the DM abundance departs from equilibrium. This departure marks the so-called thermal freeze-out¹¹, and the remaining DM abundance is referred to as a cold relic.

¹¹An alternative scenario involves a first-order phase transition that erases the pre-existing WIMP abundance and triggers a sudden mass shift. After the phase transition, WIMPs are generated through a freeze-in mechanism, despite having relatively strong interactions, contrasting with the standard freeze-out picture (Wang & Xie 2023).

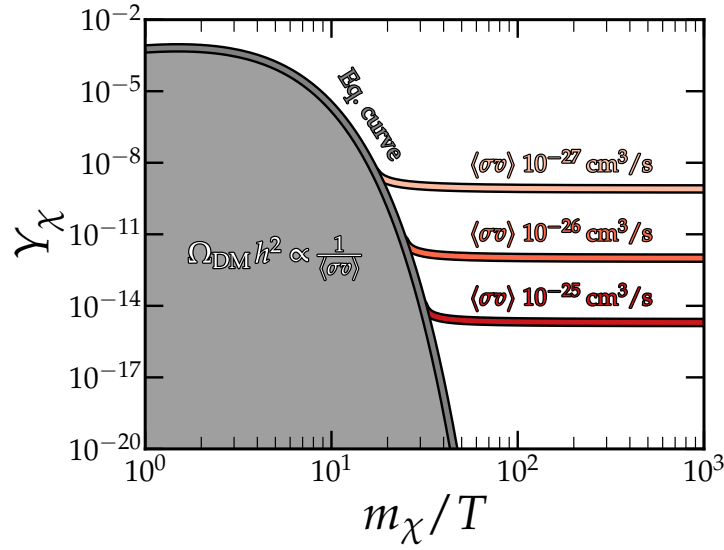


FIGURE 2.5: Comoving number density of WIMPs as a function of the ratio of the WIMP mass to the plasma temperature. The grey curve shows the thermal equilibrium distribution, while the coloured curves represent the freeze-out behaviour for different values of $\langle\sigma v\rangle$. The final DM abundance in each scenario is inversely proportional to the annihilation cross-section.

The final relic abundance of dark matter depends on the value of the annihilation cross-section and can be estimated by integrating the Boltzmann equation numerically. Under standard cosmological assumptions, a thermally averaged annihilation cross-section of the order $\langle\sigma v\rangle \sim 10^{-26} \text{ cm}^3 \text{ s}^{-1}$ reproduces the observed dark matter density $\Omega_{\text{DM}} h^2 \approx 0.12$ (Planck Collaboration et al. 2020a). This coincidence, between the required cross-section and the typical scale of weak interactions, is often referred to as the WIMP miracle.

The term reflects the appealing fact that a stable particle with a mass in the electroweak range ($m_\chi \sim 100\text{--}1000 \text{ GeV}$), interacting with a strength comparable to the weak force, naturally leads to a relic abundance consistent with cosmological observations. For interactions with cross-sections scaling like $\langle\sigma v\rangle \sim g^4/m_\chi^2$, with $g \sim \mathcal{O}(1)$, the predicted density falls within the correct order of magnitude. Although not a prediction of the standard cosmological model, this scenario provides a well-motivated benchmark for both theoretical models and experimental searches.

Searches

Direct searches

Direct detection experiments aim to observe WIMPs through their elastic scattering off nuclei¹² in underground detectors. The signal would be the tiny recoil energy transferred to a nucleus after interacting with a WIMP from the galactic halo. The expected scattering rate depends on the local DM density, the WIMP mass, the velocity distribution of the WIMPs in the detector frame, and the cross-section for WIMP–nucleus interactions.

The most recent results from the XENONnT experiment, based on an exposure of 3.1 ton-years, reported no significant excess over background expectations. As a consequence, XENONnT has set one of the most stringent exclusion limits to date on the spin-independent WIMP–nucleon cross-section,

$$\sigma_{\chi-N} < 1.17 \times 10^{-47} \text{ cm}^2, \quad (2.27)$$

for a WIMP mass above 10 GeV at 90% confidence level ([Aprile et al. 2025](#)).

Indirect searches

Indirect detection looks for the products of WIMP annihilations, such as gamma rays, neutrinos or cosmic rays, which could appear as excesses over astrophysical backgrounds at galactic or extragalactic scales. For example, the flux of gamma rays from DM annihilation depends on the WIMP annihilation cross-section, the energy spectrum of the emitted photons, and the distribution of dark matter in the target region.

A typical expression for the gamma-ray flux involves the line-of-sight integral over the squared DM density, as the annihilation process involves two DM particles. While this flux is in principle measurable, its interpretation relies heavily on assumptions about the WIMP velocity dependence, the final state of the annihilation products, and the shape of the DM density profile.

If the annihilation cross-section is velocity dependent, it becomes suppressed at low velocities, making indirect signals increasingly faint. In such cases, indirect searches lose constraining power. On the other hand, if the cross-section is velocity independent, it can be directly linked to the relic abundance and constrained accordingly.

¹²Also electron–DM scattering ([Essig et al. 2012](#)).

The most stringent limits come from observations of dwarf spheroidal galaxies (Basu et al. 2021), which are dark matter dominated and relatively free of astrophysical backgrounds. Current data exclude cross-sections above

$$\langle\sigma v\rangle < 10^{-26} \text{ cm}^3\text{s}^{-1} \quad \text{for } m_\chi < 100 \text{ GeV}. \quad (2.28)$$

This value coincides with that required to explain the observed relic abundance. Therefore, WIMPs with masses below 100 GeV are ruled out in this scenario, unless the annihilation cross-section has a strong velocity dependence.

Collider searches

Collider searches aim to produce WIMPs in high-energy particle collisions, typically at the Large Hadron Collider (LHC). Since WIMPs are neutral and weakly interacting, they escape the detector without leaving a trace. Their presence is inferred from events with missing transverse momentum, usually accompanied by a visible Standard Model particle such as a jet or a photon¹³.

These events can indicate the production of a stable, neutral particle, but cannot confirm its nature. It could be a WIMP, or simply a Standard Model particle, like a neutrino, that decays outside the detector. However, if one assumes that WIMPs couple to quarks in the same way as in direct detection scenarios, collider experiments can be used to constrain the WIMP–nucleon scattering cross-section.

Recent analyses by the CMS (CMS Collaboration 2021) and ATLAS (Atlas Collaboration et al. 2021) collaborations at the LHC have placed upper limits on this cross-section using mono-jet events with large missing energy. The strongest bounds currently available reach

$$\sigma_{\chi-N} < 9 \times 10^{-45} \text{ cm}^2, \quad (2.29)$$

for spin-independent interactions and a WIMP mass of around 30 GeV. In the case of spin-dependent interactions, the bounds are significantly weaker, roughly three orders of magnitude above this value (De Roeck 2024).

¹³This type of signature is commonly referred to as a mono-X search

2.4.2 Axions

The term axion refers both to the QCD axion and to more general axion-like particles (ALPs). The QCD axion was originally introduced to solve the strong CP problem¹⁴.

Formation mechanism and interactions

Under the Peccei–Quinn mechanism (Peccei & Quinn 1977), a scalar field ϕ with mass m_ψ and potential $V(\phi)$ has a Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \quad (2.30)$$

Spontaneous symmetry breaking gives rise to a massless Goldstone boson, which acquires a small mass through non-perturbative effects, becoming a pseudo-Goldstone boson: the QCD axion. The axion is a real angular field because ϕ and $\phi + 2\pi f$ are identified, meaning that ϕ/f is an angle. Here, f is the periodicity scale, also known as the axion decay constant. For the QCD axion, both the mass and the couplings to gluons and photons are related to f , a relation motivated by the solution of the strong CP problem. In contrast, ALPs share the general properties of axions but have no fixed relation between their mass and couplings.

Axions contribute to the energy density of the Universe through the so-called misalignment mechanism (Kolb & Turner 1990). The homogeneous scalar field obeys the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0. \quad (2.31)$$

At early times, when the expansion rate H is large, the Hubble friction term freezes the field at its initial value. As the Universe expands and H drops below the axion mass m_ψ , the field starts to roll towards the minimum of the potential and oscillates around it with a decaying amplitude. In this regime, the potential can be approximated by

$$V(\phi) \simeq \frac{1}{2}m_\psi^2\phi^2, \quad (2.32)$$

and the energy density $\rho_\psi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m_\psi^2\phi^2$ redshifts as a^{-3} , behaving as cold dark matter.

¹⁴The CP violation in QCD is governed by a term proportional to θ , which is constrained by neutron electric dipole moment measurements to be smaller than 10^{-10} , suggesting that no CP violation is present. This extreme fine-tuning, however, has no underlying physical explanation.

The present-day relic abundance can be expressed as (Hui et al. 2017)

$$\Omega_\psi \sim 0.1 \left(\frac{f}{10^{17} \text{ GeV}} \right)^2 \left(\frac{m_\psi}{10^{-22} \text{ eV}} \right)^{1/2}, \quad (2.33)$$

where f is the axion decay constant, typically close to 10^{17} GeV, a value motivated by string theory (Svrcek & Witten 2006). After inflation, the breaking of the Peccei–Quinn symmetry can also lead to the formation of topological defects, such as cosmic strings and domain walls, whose decay produces additional axions that contribute to the dark matter density.

Axions interact very weakly due to their large decay constant f . The most relevant couplings include self-interactions, interactions with photons¹⁵, and interactions with fermions, whose Lagrangians read as

$$\mathcal{L}^{\text{self}} \sim \frac{m_\psi^2}{f^2} \phi^4, \quad \mathcal{L}^\gamma \sim \frac{\phi}{f} F^{\mu\nu} \tilde{F}_{\mu\nu}, \quad \mathcal{L}^\Psi \sim \frac{\partial_\mu \phi}{f} \bar{\Psi} \gamma^\mu \gamma_5 \Psi. \quad (2.34)$$

These interactions are negligible during structure formation, but they play a key role in laboratory searches and in certain astrophysical processes such as stellar cooling or photon–axion conversion in magnetic fields.

Axions as wave dark matter

We now focus on wave dark matter (ψ DM), i.e., axions with masses below ~ 30 eV. Such particles have a de Broglie wavelength larger than the typical inter-particle separation in galactic haloes ($\rho_\psi \sim 0.4 \text{ GeV cm}^{-3}$ in the Milky Way), so they can be described as a coherent classical wave. This regime is of particular interest for this thesis. Wave-induced density perturbations at scales $\lambda_{\text{dB}} = 10\text{--}100$ pc produce unique magnification features in ψ DM haloes, which we study in the third paper (Palencia et al. 2025). The scale of these density perturbations is given by (Schive et al. 2014)

$$\lambda_{\text{dB}} = 150 \left(\frac{10^{-22} \text{ eV}}{m_\psi} \right) \left(\frac{M_h}{10^{12} M_\odot} \right)^{-1/3} \text{ pc}. \quad (2.35)$$

Starting from Eq. 2.30 with the potential approximated by Eq. 2.32, the axion field

¹⁵There is a similar interaction with the gluon field for the QCD axion only.

satisfies the Klein–Gordon equation

$$-\square\phi + m_\psi^2\phi = 0, \quad \text{where} \quad \square = g^{\mu\nu}\nabla_\mu\nabla_\nu. \quad (2.36)$$

This formalism also applies to other scalar or pseudo-scalar fields dominated by gravity, although axion self-interactions can be relevant in specific regions of parameter space.

To study structure formation, it is convenient to move to the non-relativistic limit and introduce a complex scalar field ψ :

$$\phi = \frac{1}{\sqrt{2m_\psi}} \left(\psi e^{-im_\psi t} + \psi^* e^{im_\psi t} \right). \quad (2.37)$$

If ψ varies slowly ($|\ddot{\psi}| \ll m_\psi |\dot{\psi}|$), the Klein–Gordon equation reduces to the Schrödinger equation

$$i\partial_t\psi = -\frac{\nabla^2}{2m_\psi}\psi + m_\psi\Phi\psi, \quad (2.38)$$

coupled to the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho_\psi, \quad \rho_\psi = m_\psi|\psi|^2. \quad (2.39)$$

The field can be written as $\psi = \sqrt{\rho_\psi/m_\psi} e^{i\theta}$, where the phase θ defines a velocity field $\mathbf{v} = \nabla\theta/m_\psi$. This leads to an Euler-like equation with a quantum pressure term:

$$\partial_t\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Phi + \frac{1}{2m_\psi^2}\nabla \left(\frac{\nabla^2\sqrt{\rho_\psi}}{\sqrt{\rho_\psi}} \right). \quad (2.40)$$

The last term suppresses gravitational collapse below the Jeans scale

$$k_J = \left(16\pi G\bar{\rho}_\psi \right)^{1/4} m_\psi^{1/2}. \quad (2.41)$$

For $m_\psi \sim 10^{-22}$ eV, this corresponds to $k_J \sim 7 \text{ Mpc}^{-1}$.

On large scales ($k \ll k_J$), ψ DM behaves like CDM, but at smaller scales, quantum pressure suppresses the growth of structure. A detailed analysis (Hui 2021) shows that the linear matter power spectrum is reduced by 50% relative to CDM at

$$k_{1/2} = 4.5 \left(\frac{m_\psi}{10^{-22} \text{ eV}} \right)^{4/9} \text{ Mpc}^{-1}. \quad (2.42)$$

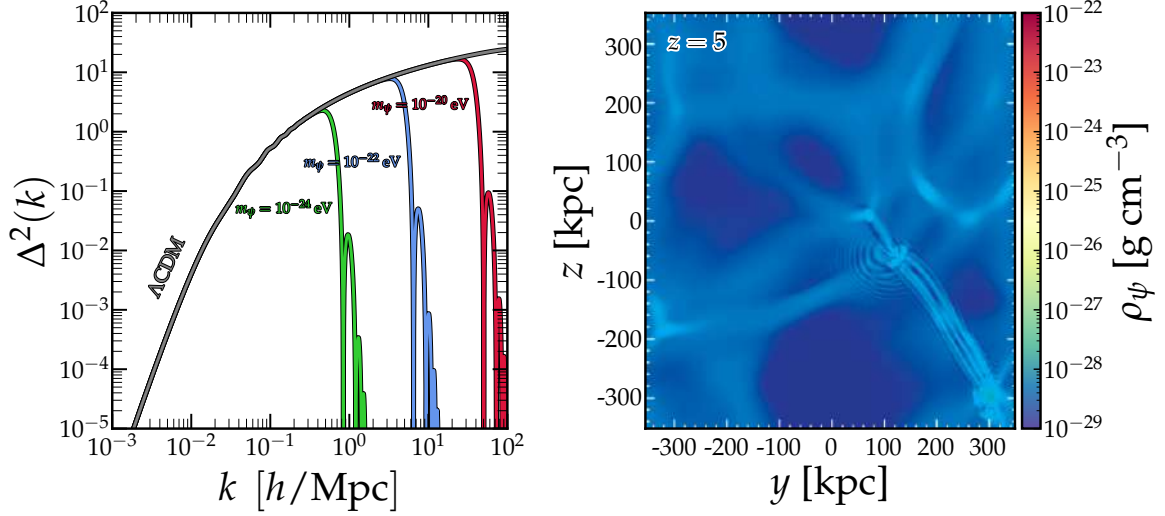


FIGURE 2.6: Left panel: Dimensionless linear matter power spectrum for different axion masses (coloured lines) compared to standard CDM (grey curve). Right panel: Snapshot of a FDM simulation at $z = 5$, spanning 700 kpc per side. The colour represents ψ DM density in g cm^{-3} . Right panel image from Hui (2021), credit: Xinyu Li.

Beyond this scale, the spectrum drops sharply, approximately as k^{-16} . This behaviour is illustrated in Fig. 2.6.

Non-linear evolution generates interference patterns and solitonic cores, ground states where quantum pressure balances gravity (Schive et al. 2014). Their size scales inversely with mass as

$$R \sim 100 \text{ pc} \frac{10^9 M_\odot}{M} \left(\frac{10^{-22} \text{ eV}}{m_\psi} \right)^2, \quad (2.43)$$

a natural outcome for FDM haloes. For sufficiently massive objects ($M \gtrsim 1/(Gm_\psi)$), relativistic corrections become relevant, potentially leading to black hole formation. Other bound structures exist, but the soliton remains a distinctive signature of ψ DM.

Current searches and constraints

Axion dark matter can be constrained through a combination of cosmological observations, astrophysical probes, and laboratory experiments. Here we focus on FDM in the ultralight axion regime, while briefly mentioning constraints relevant for larger masses and ALPs.

The quantum pressure of ψ DM suppresses structure formation below the Jeans scale.

This leaves an imprint on the linear matter power spectrum, visible in tracers such as the Lyman- α forest. Current analyses of absorption features in quasar spectra set a conservative bound of $m_\psi \gtrsim 3 \times 10^{-21}$ eV (Iršič et al. 2017; Kobayashi et al. 2017; Armengaud et al. 2017). These constraints, however, depend on assumptions about the ionising background and the thermal history of the intergalactic medium. They also neglect non-linear effects and interference-induced small-scale fluctuations, which could partially relax these bounds.

The inner density profiles of galaxies offer a complementary test. Cosmological simulations of ψ DM predict solitonic cores at the centres of haloes, with sizes that scale with the axion and halo masses ($M_{\text{sol}} \propto m_\psi M_{\text{halo}}^{1/3}$ (Schive et al. 2014)). Observations of dwarf galaxies and Milky Way satellites show good agreement with FDM in the mass range $m_\psi \sim 10^{-22}$ – 10^{-21} eV (Calabrese & Spergel 2016; Chen et al. 2017). On the other hand, analyses of stellar kinematics in disc galaxies place tension on the presence of large solitons (Bar et al. 2018), though recent studies suggest that baryonic effects and halo-to-halo variations may reconcile these results (Pozo et al. 2024).

Additional constraints on FDM arise from phenomena such as the suppression of dynamical friction (Hui et al. 2017), the subhalo mass function (Safarzadeh & Spergel 2020), and the presence of interference-induced substructures (Mishra-Sharma 2022). These effects can influence the survival of globular clusters, the abundance of satellite galaxies, and SL flux anomalies. However, a major limitation in accurately constraining FDM arises from the computational challenge of simulating ψ DM on large scales: the need to resolve the tiny de Broglie wavelength while capturing cosmological volumes makes fully consistent simulations extremely demanding. As a result, many current constraints rely on semi-analytical methods or small-box simulations, which may not capture the full non-linear dynamics of wave interference.

Table 2.1: Some constraints in the literature on FDM. Taken from Hui (2021).

Method	Constraint	Systematic Uncertainties	References
Ly α forest	$m_\psi > 3 \times 10^{-21}$ eV	Ionizing background / Temperature fluctuations	Iršič et al. (2017) Kobayashi et al. (2017) Armengaud et al. (2017)
Density profile	$m_\psi > 10^{-21}$ eV	Baryonic feedback Central black hole	Bar et al. (2018)
Satellite mass	$m_\psi > 6 \times 10^{-22}$ eV	Tidal stripping	Safarzadeh & Spergel (2020)
Satellite abundance	$m_\psi > 2.9 \times 10^{-21}$ eV	Subhalo mass function	Nadler et al. (2021)

Photon–axion interactions

Axions can interact with the electromagnetic field (see Eq. 2.34), inducing a rotation of the polarisation of photons (birefringence) as they propagate through an axion field. Searches for cosmic birefringence in the CMB have reported hints of a non-zero rotation angle at the $2\text{--}3\sigma$ level (Minami & Komatsu 2020; Diego-Palazuelos et al. 2022; Eskilt & Komatsu 2022), which favours axions as DM.

Axion detection experiments measure the axion field through its coupling to photons (or fermions, see Eq. 2.34, rightmost term). By rewriting ϕ in terms of ψ in Eq. 2.34, two timescales of interest emerge: the Compton timescale $t_{\text{Comp.}} \sim m_\psi^{-1}$ and the de Broglie timescale $t_{\text{dB}} \sim (m_\psi v^2)^{-1}$, the latter being natural for wave-induced fluctuations. This

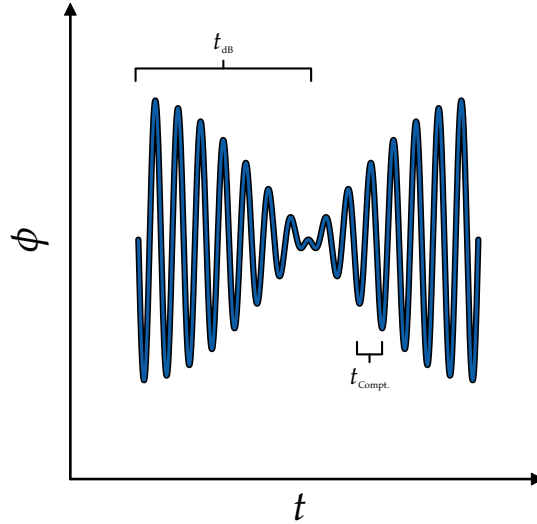


FIGURE 2.7: Schematic representation of the time dependence of the axion scalar field at a fixed spatial location $\phi(t)$. Two modulations oscillate around a mean value $\bar{\phi}$: $t_{\text{Comp.}}$, a short timescale modulation set by the axion mass m_ψ , and t_{dB} , a longer timescale modulation dependent on both m_ψ and the square of the DM velocity v^2 .

means that the value of the axion field at a fixed spatial location changes over time as a superposition of two waves with different frequencies: one (short) associated with the Compton timescale and another (long) due to spatial fluctuations on the de Broglie scale. There is indeed a non-zero probability for the density to fluctuate to lower values (even down to zero), so axion experiments that sample the entire distribution of ρ_ψ should take this into account when measuring the coupling constants, especially if timescales longer than t_{dB} are accessible.

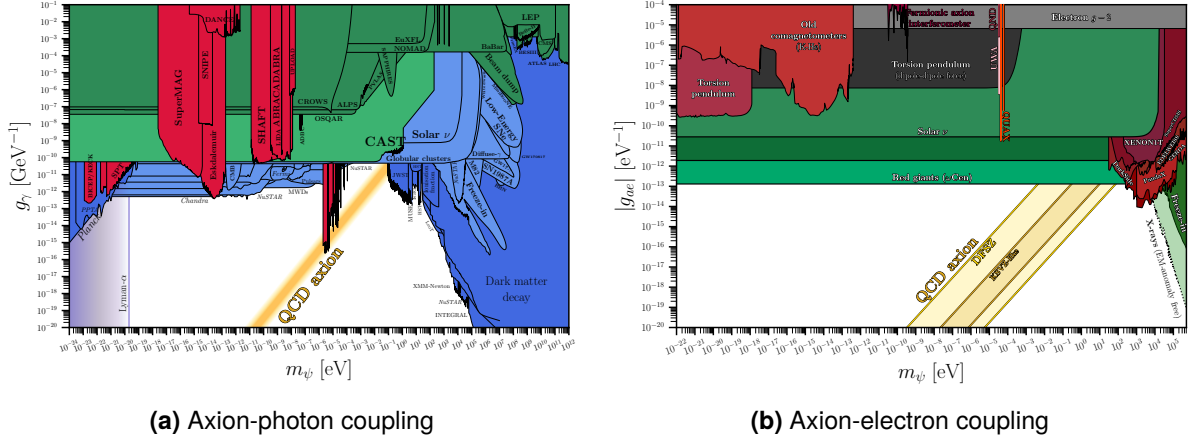


FIGURE 2.8: Current ψ DM constraints as a function of the axion mass m_ψ and the photon and electron coupling constants, g_γ and g_e , respectively. The QCD axion predictions are denoted in yellow. (For the fermion coupling, the QCD expectation of the coupling constant to electrons is model dependent, hence the different bands KSVZ and DFSZ). Figures obtained using a modified version of O’Hare (2020).

Laboratory searches exploit this coupling using resonant cavities and strong magnetic fields. Experiments such as ADMX (Du et al. 2018), CAST, and ABRACADABRA (Ouellet et al. 2019) probe g_γ by detecting photon conversion or induced electromagnetic signals. Other setups, such as CASPER (Budker et al. 2014), target axion–fermion couplings through spin-precession measurements. Current bounds on g_γ and g_e are summarised in Fig. 2.8, with future experiments aiming to fully explore the QCD axion parameter space.

2.4.3 Primordial black holes

We will conclude the set of popular current DM candidates with PBHs. Back in the 1960s and 70s, the idea of black holes forming from the collapse of large overdensities in the primordial Universe emerged (Zel’dovich & Novikov 1967; Hawking 1971). Researchers quickly realised their potential as a DM candidate (Hawking 1971; Chapline 1975), since their formation would pre-date BBN, which makes them non-baryonic. PBHs with masses larger than $M_{\text{PBH}} \gtrsim 5 \times 10^{14} \text{ g}$ would have lifetimes longer than the current age of the Universe (Page 1976), while those with lower masses were originally thought to have evaporated via Hawking radiation (Hawking 1974, 1976). However, as we will discuss later, the recently proposed memory burden effect might allow some of these light PBHs to survive up to the present day (Dvali 2018). PBHs behave like particle DM on cosmological scales, while on smaller scales their finite size introduces

different observational consequences that can be relevant for their detection.

PBHs gained popularity in the mid-90s when the MACHO collaboration released its 2-year microlensing (ML) analysis of the Large Magellanic Cloud (Alcock et al. 1997). The results indicated an excess of ML events compared to expectations from stellar populations alone. In fact, they suggested that this excess was compatible with half of the Milky Way halo being in the form of $0.5 M_{\odot}$ compact objects, although a later reanalysis of the 5.7-year MACHO data (Alcock et al. 2000) reduced this fraction to about 20%. Subsequently, the EROS-2 project lowered this number even further, to roughly 8% (Tisserand et al. 2007).

Interest in PBHs was revived in 2016 with the LIGO–Virgo results, which reported the detection of GWs from the merger of black holes with masses of several tens of solar masses (Abbott et al. 2016), values difficult to explain with standard astrophysical black holes¹⁶. Since then, PBHs have become a hot topic in the DM community. Nowadays, a wide range of experiments and observational probes are being pursued to test the PBH scenario, along with new theoretical developments that improve our understanding of these compact objects.

2.4.4 PBH formation mechanisms

In this subsection, we briefly discuss the main formation mechanisms of PBHs. The most common scenario assumes that PBHs form through the collapse of large adiabatic density perturbations produced during an inflationary epoch. We will examine both formation during the RD era, the most popular mechanism, and finally outline other possible formation mechanisms.

Collapse of adiabatic perturbations

As pointed out by Carr (1975), considering the Jeans length within a Newtonian gravitational framework, if the density contrast $\delta \equiv \delta\rho/\rho$ in a comoving slice exceeds the critical value $\delta_{\text{crit}} = v_s^2$, where v_s is the sound speed of the fluid, a PBH with mass M_{PBH} would form. The mass of the PBH is of the order of the mass enclosed within the

¹⁶Astrophysical black holes form during the final stages of very massive stars, with resulting masses typically below $M_{\text{BH}} \sim 45$ (Farmer et al. 2019). Subsequent mergers and accretion can produce higher masses, but their number density is much lower than what would be expected for PBHs.

horizon at the time,

$$M_{\text{PBH}} \sim M_H \sim \frac{t}{G} \sim 10^{15} \text{ g} \left(\frac{t}{10^{-23} \text{ s}} \right). \quad (2.44)$$

This initial calculation was later supported by simulations (Nadezhin et al. 1978). Within this framework, the initial abundance of PBHs is given by

$$\beta(M_H) \equiv \frac{\rho_{\text{PBH}}}{\rho_{\text{tot}}} = \int_{\delta_c}^{\infty} \mathcal{P}(\delta) d\delta \sim \sigma(M_H) e^{-\frac{\delta_c^2}{2\sigma^2(M_H)}}. \quad (2.45)$$

Here, we have assumed that the primordial distribution of density perturbations $\mathcal{P}(\delta)$ is Gaussian with a small variance $\sigma^2(M_H) \ll \delta_c^2$. The frequency of large density peaks above δ_c , and thus the abundance of PBHs, depends entirely on the power spectrum.

It was later realised that, due to near-critical gravitational collapse, the mass of the PBH depends on the amplitude of the fluctuation that seeds its formation (Niemeyer & Jedamzik 1998, 1999):

$$M_{\text{PBH}} = \kappa M_H (\delta - \delta_c)^\gamma, \quad (2.46)$$

where κ and γ are constants that depend on the equation of state of the background and on the shape of the perturbation (Niemeyer & Jedamzik 1999; Musco et al. 2005). For instance, Musco et al. (2005), using a Mexican-hat profile for the perturbation, found $\kappa = 4.02$ and $\gamma = 0.357$. This power-law scaling has been shown to hold down to $(\delta - \delta_c)^\gamma \gtrsim 10^{-10}$ (Musco et al. 2009; Musco & Miller 2013).

The fraction of the Universe's energy density in regions dense enough to form PBHs is calculated as

$$\beta(M_{\text{PBH}}) = 2 \int_{\delta_c}^{\infty} \frac{M_{\text{PBH}}}{M_H} \mathcal{P}(\delta(R)) d\delta(R), \quad (2.47)$$

where R is the comoving scale over which the density contrast is defined. Here, we have again assumed that the probability distribution is Gaussian with mass variance $\sigma^2(R)$. The horizon mass corresponding to a comoving radius R during the RD era is (Wang et al. 2019)

$$M_H = 5.6 \times 10^{15} M_\odot \left(\frac{g_{*,i}}{106.75} \right)^{-1/6} (Rk_0)^2, \quad (2.48)$$

normalised to a comoving wave-number $k_0 = 0.05 \text{ Mpc}^{-1}$ (this wave-number corresponds to the CMB pivot scale). We also assume that the initial degrees of freedom for entropy and energy density are equal, $g_{*,i}$.

If we assume all PBHs form at the same comoving scale, they are formed at the same time and with a fixed mass, i.e., a monochromatic mass function. At the critical collapse limit, as we saw earlier, the mass also depends on the amplitude of the fluctuation, so even with a fixed comoving scale, the resulting mass function is not strictly monochromatic but extended, peaking near the horizon mass (αM_H) with a significant low-mass tail.

It is worth mentioning that the density perturbations inferred from the CMB are not large enough to form PBHs. However, these constraints apply only on large scales, and in principle, larger perturbations could exist on small physical scales (large k). Achieving such large perturbations at a particular scale, however, requires fine-tuning of the inflationary model (Cole et al. 2023).

The fraction of regions dense enough to produce PBHs can be approximated as

$$\beta(M_{\text{PBH}}) = \alpha \operatorname{erfc} \left(\frac{\delta_c}{\sqrt{2}\sigma(R)} \right), \quad (2.49)$$

with the mass variance given by (Blais et al. 2003; Josan et al. 2009)

$$\sigma^2(R) = \frac{16}{81} \int_0^\infty (kR)^4 W^2(KR) \mathcal{P}_R(k) T^2(kR/\sqrt{3}) / k \, dk, \quad (2.50)$$

where $W(kR)$ denotes the Fourier transform of the window function used to smooth the density contrast on a comoving scale R , $\mathcal{P}_R(k)$ is the power spectrum of the primordial comoving curvature perturbations, and $T(y)$ is the transfer function characterising the evolution of sub-horizon perturbations:

$$T(y) = 3 \frac{\sin y - y \cos y}{y^3}. \quad (2.51)$$

Unless the window function is a Dirac delta, perturbations will span a range of comoving scales R (Byrnes et al. 2019), which means that the mass function will no longer be monochromatic, even if we neglect the dependence on the fluctuation amplitude from the critical collapse condition.

Assuming that the PBH mass remains constant and mergers are negligible, the PBH density scales with the scale factor a as $\rho_{\text{PBH}} \propto a^{-3}$. During the MD era, the fraction of the total density in PBHs remains constant, while during the RD era it grows as a . The

present-day density parameter for PBHs, assuming a monochromatic mass function, is

$$\Omega_{\text{PBH}} \equiv \frac{\rho_{\text{PBH},0}}{\rho_{\text{crit},0}} = \frac{\beta(M_H)}{1.1 \times 10^{-8}} \left(\frac{h}{0.7} \right)^{-2} \left(\frac{g_{*,i}}{106.75} \right)^{-1/4} \left(\frac{M_H}{M_\odot} \right)^{-1/2}. \quad (2.52)$$

A proper calculation of PBH accretion, evaporation, and merger rates must be included to place realistic constraints on the present-day value of Ω_{PBH} .

If we consider a primordial power spectrum with a finite-width peak, PBHs can form at different times and therefore at different horizon masses. In practice, the power spectrum is often divided into bins corresponding to horizon masses. A monochromatic mass function is computed for each bin, and the total extended mass function is obtained by summing all contributions. This final distribution is well approximated by a log-normal function (Green 2016).

Another important aspect to consider is that during phase transitions, the pressure is reduced, which decreases the sound speed and thus lowers δ_c (Carr 1975). This makes the formation of PBHs easier, boosting their abundance. For example, during the QCD phase transition, the formation of solar-mass PBHs is significantly enhanced (Jedamzik 1997; Jedamzik & Niemeyer 1999; Carr et al. 2019).

Up to this point, we have assumed Gaussian adiabatic perturbations seeding the formation of PBHs, specifically from the high-amplitude tail of the perturbation distribution. However, non-Gaussian adiabatic perturbations, which have not yet been confirmed (Planck Collaboration et al. (2020b) constrain the non-Gaussianity parameter to $f_{\text{NL}} \sim 0 \pm 5$), can significantly boost the probability of large-amplitude fluctuations, increasing the number of PBHs formed from the collapse of over-dense regions within the horizon.

Primordial black holes formed from the collapse of high-density peaks have spins close to zero. These peaks are nearly spherically symmetric, which results in very small angular momentum for the collapsing matter, often parametrised by the dimensionless spin parameter $a_{\text{spin}} = |\mathbf{S}|/(GM_{\text{PBH}}^2)$, where $\mathbf{S}$ is the spin vector. Typical values are of the order $a_{\text{spin}} \sim 0.01$ (Mirbabayi et al. 2020; De Luca et al. 2019). Note, however, that as PBHs merge with other astrophysical objects or accrete gas, their spin can increase (De Luca et al. 2020).

There is a line of thought that considers the possibility of a short MD era between

inflation and the onset of the RD era. In this scenario, the formation picture described above changes significantly (Kokubu et al. 2018; Harada et al. 2016). The main result is that if $\sigma \lesssim 0.5$, more PBHs are produced during the MD era than during the RD era.

Angular momentum plays an important role in the MD era, meaning that PBHs formed during this time are expected to have large spins ($a_{\text{spin}} \gtrsim 0.5$). The exact spin value depends both on the duration of this primordial MD era and on the mass variance σ .

Other formation mechanisms

Beyond the collapse of large density perturbations, other mechanisms can produce PBHs. Here, we briefly discuss PBH formation via bubble collisions, cosmic string loops or cusps, and domain wall collapse.

A first-order phase transition would create bubbles of the new phase enclosed within regions of the old phase (Hawking et al. 1982). It is believed that in the early Universe, when the temperature dropped to the grand unified energy scale or during the electroweak transition (Weinberg–Salam (Weinberg 1967; Salam & Ward 1994)), these bubbles could have formed. If two of these bubbles happened to collide, the local over-density could exceed the horizon density threshold, leading to PBH formation (Hawking et al. 1982). For this mechanism to yield a non-negligible abundance of PBHs, a fine-tuned process is required in which collisions occur but the phase transition is not instantaneous (Kusenko et al. 2020).

A cosmic string is a topological defect expected to form during very early Universe phase transitions (Kibble 1976). A network of these strings would emerge, with loops smaller than the Hubble radius being constantly produced through string interactions. These loops oscillate, and if they contract below their Schwarzschild radius, they collapse into PBHs (Hawking 1989), with a mass proportional to the typical loop mass, which in turn is proportional to the horizon mass. Since this formation is time-independent, the resulting mass function is extended and follows $M_{\text{PBH}}^{-5/2}$ (MacGibbon et al. 1998).

Similarly to bubbles, second-order phase transitions at the end of inflation can produce domain walls. The collapse of large domain walls can also result in PBHs (Rubin et al. 2001). The resulting mass function depends on the parameters of the field undergoing the second phase transition. Typically, scalar fields with sufficiently flat potentials form

lumps such as Q-balls or oscillons. These dense lumps can collapse into PBHs (Cotner & Kusenko 2017; Cotner et al. 2019). PBHs formed this way are typically lighter and can have higher spin values compared to PBHs formed via the collapse of adiabatic perturbations in the RD era.

2.4.5 Constrains on PBH abundance

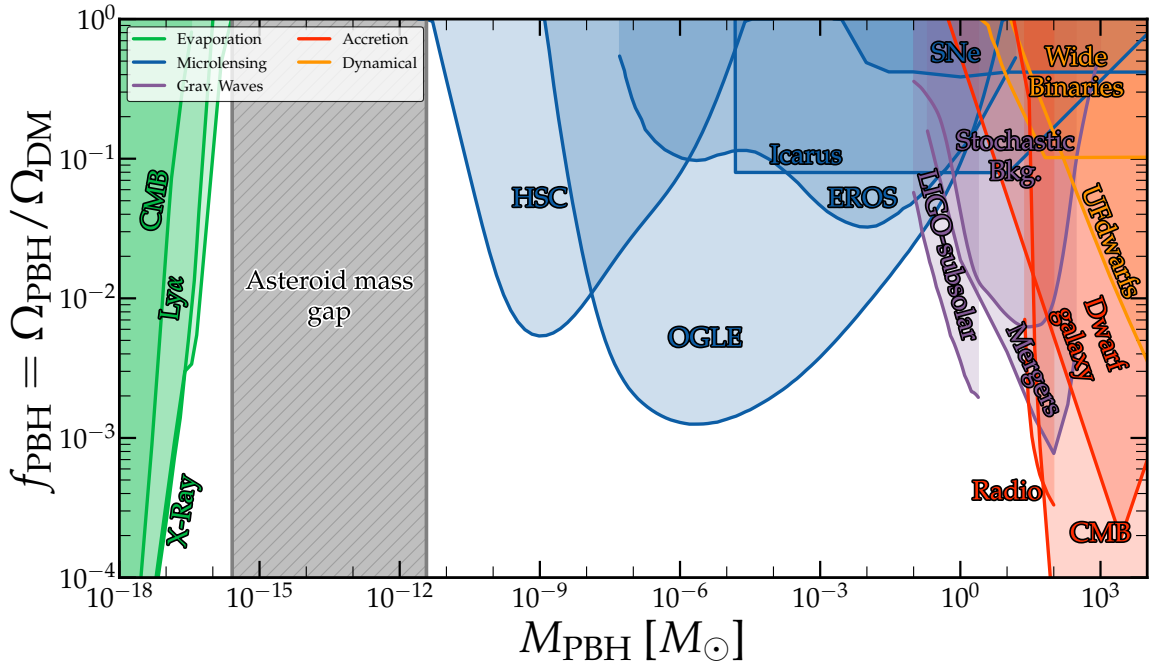


FIGURE 2.9: Simplified landscape of the viability of PBHs as DM. We show some of the most recent and stringent constraints on PBHs as DM, coming from evaporation measurements (green), which do not account for the memory burden effect, ML by compact objects (blue), GWs (purple), radiation effects from gas accretion (red), and dynamical effects on dwarf galaxies and wide binaries (orange). The asteroid mass gap, where we have no reliable constraints yet, is shaded in grey. The plot was obtained from a modified version of the PBHBound code by B. J. Kavanagh (Kavanagh 2019).

As we have seen, PBHs can form over a wide range of masses, and their observational signatures vary dramatically depending on the mass scale. As a result, a variety of probes are required to constrain the fraction of PBHs across different mass ranges. This abundance is typically expressed as the fraction of DM composed of PBHs, $f_{\text{PBH}} = \Omega_{\text{PBH}}/\Omega_{\text{DM}}$. In this subsection, we summarise some of the current constraints on the PBH contribution to DM.

Evaporation

The first set of constraints arises from PBH evaporation. As discussed earlier, PBHs with masses larger than $\sim 10^{15}$ g have lifetimes exceeding the current age of the Universe. However, lighter PBHs would have evaporated at some point in the past via Hawking radiation (Hawking 1974, 1976). The extragalactic gamma-ray background, after accounting for contributions from known astrophysical sources, can be used to limit the abundance of low-mass PBHs, since their evaporation would produce additional gamma-ray emission. The expected fraction is

$$f_{\text{PBH}} \lesssim 2 \times 10^{-8} \left(\frac{M_{\text{PBH}}}{M_{\odot}} \right)^{3+\epsilon}, \quad (2.53)$$

where ϵ is the energy dependence of the observed gamma-ray spectrum ($I_{\text{obs}} \propto E_{\gamma}^{-(1+\epsilon)}$) (Carr et al. 2010). Energy injection into the CMB during recombination would also damp its anisotropies (Poulin et al. 2017), as well as heat neutral hydrogen, with signatures observable in the 21-cm absorption line (Clark et al. 2018) (EDGES), or in the interstellar medium of dwarf galaxies (Kim 2021).

Similar constraints arise from the electron/positron flux measured by Voyager 1 (Boudaud & Cirelli 2019). Positrons emitted by PBH evaporation can also annihilate and produce X-rays, which have been searched for with SPI/INTEGRAL (DeRocco & Graham 2019). A diffuse neutrino background is also expected, with constraints coming from Super-Kamiokande (Dasgupta et al. 2020).

A proposed update to PBH evaporation theory is the so-called memory burden effect (Dvali 2018). This effect refers to the stabilisation of systems that store a large amount of information. Systems with maximal microstate degeneracy, known as “saturons”, experience a slowdown in their evaporation process. For PBHs, this would halt Hawking evaporation after roughly half of their initial mass is lost, allowing light PBHs, previously thought to have fully evaporated, to survive until today. Unlike classical stabilisation mechanisms such as spin or charge, memory burden is a purely quantum effect. The lifetime of a PBH that has undergone partial evaporation scales as S_{PBH}^{1+k} , where S_{PBH} is the PBH entropy and k is a positive integer parameter, still unconstrained. Even for the most conservative case ($k = 1$), PBHs lighter than 10^{15} g could still be present today, as their evaporation would have been slowed by this effect. For a recent analysis of its implications on PBHs, see Dvali et al. (2024).

Microlensing

Moving to larger masses, the next key observational probe for PBHs is ML. When a massive compact object aligns between a distant source and the observer, strong gravitational lensing effects appear, producing multiple images of the source. When the mass of the intervening lens is small, typically below a few tens of solar masses, the angular separation of the images is too small to be resolved, and we only observe a single, magnified image. This phenomenon is known as ML (Paczynski 1986).

If the lens and source move relative to each other and the observer, the magnification changes over time, producing a transient event: a characteristic achromatic light curve where the flux rises, reaches a peak, and then declines¹⁷. This effect, discussed in more detail in Ch. 3, is one of the key phenomena studied in the first two papers of this thesis (Palencia et al. 2024; Palencia, J. M. et al. 2025).

The duration of a ML event depends on the mass of the compact object acting as a lens, following $t_{\text{micro}} \propto M_{\text{CO}}^{1/2}$. Smaller and lighter lenses therefore produce shorter events that require higher cadence observations to detect. The EROS-2 collaboration, which observed the Large Magellanic Cloud, constrained the PBH fraction in the mass range $10^{-6} \lesssim M_{\text{CO}}/M_{\odot} \lesssim 30$, with significantly weaker constraints for $M_{\text{CO}} > 1 M_{\odot}$ (Tisserand et al. 2007). With even higher cadence observations of M31, the Subaru HSC team (Niikura et al. 2019) probed PBHs and other compact objects down to masses of $10^{-12} M_{\odot}$. The tightest constraints currently come from OGLE data on the Galactic bulge (Mróz et al. 2024b,a).

When ML occurs in stars hosted in a strongly lensed galaxy by a galaxy cluster, the magnification effects of macro- and ML combine, allowing for the detection of individual stars at cosmological distances (e.g., Icarus (Kelly et al. 2018)). As shown in Palencia et al. (2024), the statistics and likelihood of such events depend on both the lens model and the abundance of microlenses, whose baryonic component can often be traced independently, enabling direct constraints on PBHs and other dark compact objects (as tentatively suggested by Oguri et al. (2018) for Icarus).

Similarly, lensed QSOs are also subject to ML by compact objects in the lensing galaxy or cluster, which can be exploited to search for PBHs or other compact objects. With

¹⁷If the alignment of the source and lens remains constant, ML still occurs, but without flux variability it is difficult to detect.

24 lensed QSOs, [Mediavilla et al. \(2017\)](#) found that approximately a fifth of the lens galaxy mass could be made up of compact objects in the range $0.05 < M_{\text{CO}}/M_{\odot} < 0.45$, consistent with stellar populations.

The lightest PBHs accessible to ML searches are asteroid-mass PBHs, $5 \times 10^{17} \lesssim M_{\text{CO}}/\text{g} \lesssim 10^{20}$. In this regime, the image separation is on the femto arcsecond scale, and the effect is known as femtolensing. [Barnacka et al. \(2012\)](#) suggested that femtolensing of gamma-ray bursts (GRBs) could be detected through interference patterns in Fermi GRB data, but as noted by [Katz et al. \(2018\)](#), GRBs are too extended to be treated as point sources, and wave optics effects must also be considered. As a result, there are currently no robust constraints on PBHs from femtolensing. A similar idea, nanolensing of fast radio bursts, could in principle constrain planetary-mass PBHs, but no strong limits have yet been obtained.

Note that while PBHs are a natural target for these techniques, ML constraints apply to any population of undetected compact objects, including other DM candidates, free-floating planets, or astrophysical black holes.

Gravitational waves

The next relevant mass range for PBHs is tens to hundreds of solar masses. Many constraints in this regime come from GWs observations. [Nakamura et al. \(1997\)](#); [Ioka et al. \(1998\)](#) proposed that nearby solar-mass PBH binaries could form in the early Universe. When two PBHs are close enough, their gravitational attraction overcomes the cosmic expansion, causing them to form a bound pair. A third-body interaction can then add angular momentum, producing a highly eccentric binary. If such binaries survive to the present day, their coalescence would produce GW signals detectable by LIGO.

The merger rate observed by LIGO–Virgo is much smaller than what would be expected if PBHs made up all of the DM, so strong constraints have been placed in the $10\text{--}300 M_{\odot}$ range ([Hayasaki et al. 2016](#); [Ali-Haïmoud et al. 2017](#)). Lower-mass mergers, in the range $10^{-1}\text{--}0.2 M_{\odot}$, have also been constrained by LIGO–Virgo ([Abbott et al. 2019](#); [Nitz & Wang 2022](#)). Furthermore, by combining the first three runs of LIGO–Virgo–KAGRA, searches for a stochastic GW background from compact binary coalescences have placed additional constraints ([Boybeyi et al. 2024](#)).

If PBHs constitute only a fraction of the DM, they are expected to accrete DM particles during the MD era, producing steep density profiles (Mack et al. 2007). However, the effect of these mini-haloes on the PBH binary merger rate is expected to be small for equal-mass binaries (Kavanagh et al. 2018; Jangra et al. 2023). On the other hand, three-body interactions within PBH clusters can drastically alter the number of binaries that survive after matter–radiation equality, especially if PBHs represent a large fraction (or all) of the DM density (Vaskonen & Veermäe 2020; Jedamzik 2020, 2021).

The constraints shown in Fig. 2.9 assume neither clustering nor significant disruption of binaries, since the effects of these processes on the merger rate are not yet fully understood.

Accretion and dynamical effects

The most massive PBHs ($M_{\text{PBH}} \gtrsim 10\text{--}100 M_{\odot}$) are constrained by both accretion and dynamical effects. Gas accreted by PBHs is heated and emits radiation. Gas accreted before recombination, and its associated radiation, would leave an imprint on the CMB anisotropies (Carr 1981; Ricotti et al. 2008). Later, its impact on the 21-cm spectrum can also yield constraints, as observed by EDGES with an absorption excess at $z \sim 17$ (Hektor et al. 2018).

In the present-day Milky Way, gas accreted by PBHs would emit X-rays and radio waves (Gaggero et al. 2017). Chandra and VLA measurements of the Galactic centre have provided tight constraints in the $30\text{--}100 M_{\odot}$ range (Manshanden et al. 2019). Measurements of compact X-ray objects have been used to place similar constraints up to $10^7 M_{\odot}$ PBHs (Inoue & Kusenko 2017). Heating of interstellar gas in dwarf galaxies also constrains PBHs up to $10^3 M_{\odot}$ (Lu et al. 2021).

On the dynamical side, two major constraints arise: from ultra-faint dwarf galaxies (UFDGs) and from wide binary systems. Two-body interactions between PBHs and stars increase the kinetic energy of the stellar component. This effect is stronger in UFDGs due to their high mass-to-light ratios, leading to weak constraints in the mass range $10\text{--}10^4 M_{\odot}$ (Brandt 2016), though uncertainties remain regarding the initial stellar radius and velocity dispersion.

Wide binary stellar systems can also be disrupted by three-body interactions with PBHs or other compact objects (Bahcall et al. 1985). The observed distribution of wide

binary separations can thus serve as a probe to constrain the abundance of compact objects (Yoo et al. 2004). The main limitation arises from radial velocity measurements, which are required to confirm true binaries; otherwise, spurious associations can bias the inferred constraints (Quinn et al. 2009).

Many of the constraints discussed above (assuming a monochromatic mass function) are summarised in Fig. 2.9. Other limits, as well as alternative scenarios based on different PBH formation mechanisms, may alter the exclusion landscape. Even the same observational probes, when affected by astrophysical uncertainties, can yield different bounds. In this subsection we have provided only a brief overview of the topic. For a detailed and up-to-date review, see Green & Kavanagh (2021) and references therein.

Chapter 3

Gravitational lensing

This final chapter concludes the introduction of this doctoral thesis. Here we introduce gravitational lensing, a particular effect of radiation moving at speed c in a curved space-time. It is one of the most valuable astrophysical and cosmological tools, barely 100 years old. It is commonly applied to light perturbation under mass densities, but is also applicable to other sources of radiation such as GWs. The goal of this thesis is to use gravitational lensing to measure mass distributions, particularly focusing on the detection of DM, introduced in Ch. 2, but also as a magnifying glass probing distant sources otherwise too faint to be studied properly without lensing.

Here we will briefly recap the history of gravitational lensing, then move into its proper description and the immense variety of effects on background sources, probing the intervening matter distribution, from the largest scales as observed in CMB lensing or cosmic shear analysis, down to galaxy cluster lensing, galaxy lensing, and ending with sub-galactic scale lensing. Most of this chapter is inspired by the classic books of [Schneider et al. \(1992\)](#) and [Narayan & Bartelmann \(1996\)](#), but we also take inspiration from the more modern review by [Meneghetti \(2022\)](#). Dedicated sections such as WL or SL also draw inspiration from modern reviews on the topics like [Bartelmann & Maturi \(2017\)](#) or [Saha et al. \(2024\)](#), while others like ML, especially ML of small sources by compact structures in galaxy clusters or strongly lensed galaxies, the main topic of this thesis, are mostly based on the works presented in this thesis, those in which the author has participated as co-author, and the previous works that laid their foundation.

Pre-General Relativity era

As of today, we understand gravitational lensing as light rays following geodesics, the trajectories that minimise the action under a curved space-time, similar to why an aeroplane flies north when crossing the Atlantic rather than following a straight line in the Mercator projection. This explanation, however, requires GR, but a century before Einstein's revolution, Soldner (Soldner 1921), following the discussions of Newton, Cavendish, and Laplace among others, was able to derive a deflection angle due to the Sun with only a factor of 2 missing.

This calculation assumed that light was made of material particles, and using only Newtonian gravitation Soldner arrived at a deflection angle based on the object's mass M and the particle velocity v that goes as

$$\alpha = \frac{2GM}{v^2 r}, \quad (3.1)$$

where r is the distance of the particle to the centre of mass of the lensing object. For a light ray grazing the surface of the Sun, Soldner derived a deflection angle of $\alpha_{\text{Soldner}} = \alpha_{\text{Einstein}}/2 = 1''.17/2$, a factor of 2 missing yet remarkably accurate considering the time of the calculation.

First steps as science: 1919-1937

After GR was published, Einstein, using the equivalence principle, revisited Soldner's calculations on the deflection angle of the Sun and obtained a value twice as large as that previously predicted, but in good agreement with what Dyson, Eddington, and Davidson measured during the total solar eclipse in 1919 (Dyson et al. 1920). Soon after, Eddington was the first to suggest that, under a proper alignment of stars (lens and source), multiple images could appear (Eddington 1920), although the magnification factors he computed were incorrect. Later, in 1924, Chwolson reached a similar result¹, even pointing out that under perfect alignment a ring-shaped image would appear (Chwolson 1924). Nowadays, we know this effect as Einstein rings, although it was Chwolson who first suggested it. He still did not calculate the correct magnification factors; it was Einstein himself who obtained the correct value.

Einstein originally thought that, given the small angular separation between images, this effect would be impossible to detect. It took the out-of-the-box thinking of Fritz Zwicky² to point out that this effect would, in fact, be detectable. As he explained

¹Interestingly, there is a project to use the Sun as a lens to study distant planets (Turyshev et al. 2020).

²The same scientist who argued that DM must be present in galaxy clusters to explain their virialised

in his paper Zwicky (1937a), nebulae³ should be much more massive than previously thought (and far more massive than stars). People believed that galaxies had masses of the order of $10^9 M_\odot$, while Zwicky, by applying the virial theorem to the Virgo and Coma clusters, estimated their masses to be about $4 \times 10^{11} M_\odot$. Thus, their deflection angles would be much larger than those of stars or the underestimated galaxy masses of the time. He followed up with another paper in the same year Zwicky (1937c), estimating that about 1 in 400 photographic images of the sky would contain lensing by galaxies. He was also the first to point out that this effect could be used to study much more distant objects, thanks to the magnification boost provided by lensing, the so-called magnification bias⁴, and that lensing would enable mass measurements of lenses. Yet, researchers did not pay much attention to this topic until the next milestone came in the 1960s.

Formalism establishment: 1963-1979

The lensing field grew in popularity again with the birth of radio astronomy in 1963. The first QSOs were detected, offering prime targets for lensing due to their point-like nature, incredible brightness, and prominent spectral features. Three major milestones occurred around this time: i) Klimov's paper (Klimov 1963) considered the lensing of galaxies by other galaxies, suggesting ring-shaped images under proper alignment and the multiplicity of images. ii) Liebes (Liebes 1964) studied lensing of stars by stars, stars by globular clusters in the Milky Way, stars by other stars within the same globular cluster, and stars in the Andromeda galaxy by Milky Way stars; similar to how HSC constrained PBHs (see Ch. 2, PBHs constraints). Liebes estimated the probability of these events and highlighted the importance of space-based telescopes for their detection. iii) Refsdal (Refsdal 1964b) (and Liebes too) studied the properties of lensing by point-like lenses. He determined the time delay between the two images, and in a second paper (Refsdal 1964a) he proposed using this time difference to measure the Hubble constant, a widely used method to determine the current expansion rate of the Universe, as discussed in Ch. 1.

In the subsequent years, several advances in lensing theory were made, with significant development of the mathematical formalism and discussion of potential applications. However, no confirmed observations of these effects were found, and the field remained

high velocities.

³Galaxies as they were known back then.

⁴An over estimation of bright sources

partially dormant, with much speculation.

Boom of Gravitational Lensing: Period after 1979

Despite the theoretical advancements and suggestions regarding its feasibility, gravitational lensing was long considered merely an interesting effect of GR, without much attention given to its potential applications. Researchers working on lensing often did not receive the recognition they deserved. This changed dramatically after the first lensed systems were discovered. In 1979, Walsh, Carswell, and Weymann published the discovery of Q0957+561, a doubly lensed QSO (Walsh et al. 1979), whose ML effects by foreground stars were pointed out by Chang & Refsdal (1979). This led to a debate on whether it was a true lensed QSO or just a twin QSO. A year later, it was confirmed to be a lensed QSO, ending the debate (Young et al. 1980). Soon after, a second lensed candidate was identified (Weymann et al. 1980).

A second type of lensed object was announced in 1987 (Lynds & Petrosian 1986), with the discovery of the first gravitationally lensed arc, now known as the “Dragon” arc. A third and final type, confirming previous theoretical predictions, was the observation of ring-like images from radio data (Hewitt et al. 1988; Langston et al. 1989).

Following these discoveries, gravitational lensing was firmly established as a major field of research. Observations increased rapidly, leading to some of the most precise mass measurements of lenses via lensing analysis. Many new sources at astonishing redshifts have been discovered thanks to lensing magnification. Microlensing has been used to probe compact objects and test DM, while lensed type Ia SNe have been used to measure H_0 independently. With instruments such as the Hubble Space Telescope and now the James Webb Space Telescope (HST and JWST), our ability to study distant sources and lenses has reached unprecedented precision. Soon, stage-4 wide surveys like Euclid, the Rubin-LSST, and Roman will deliver tens of thousands of new systems, introducing an era of high-precision lensing analyses and potentially new discoveries that will deepen our understanding of the Universe.

3.1 Introduction to lensing formalism

Light propagation in a curved space-time is, in most cases, a complex theoretical problem. Fortunately, most real-life applications can be approximated by assuming an FLRW metric and treating lenses as local perturbations of the metric. The light

propagation can then be split into three parts: an unperturbed propagation through the FLRW metric from the source to the lens, a second part through the perturbed space-time where all the deflection occurs, and a final unperturbed propagation from the lens to the observer. For the deflection, i.e., the second part, we can assume a Minkowski flat metric weakly perturbed by a Newtonian gravitational potential Φ . This approximation remains valid as long as $\Phi \ll c^2$ and the peculiar velocity of the lens satisfies $v \ll c$. For multiple lens-plane systems, each segment can again be divided into these three zones, where the images from one lens act as sources for the next.

The conditions mentioned above hold for nearly all cases of interest. However, for a full relativistic treatment of the problem, we refer the reader to [Schneider et al. \(1992\)](#), chapter 3: “Optics in curved spacetime”, where the complete problem is addressed from first principles within a GR framework.

3.1.1 Effective refractive index of gravitational fields

Under the assumptions described in the previous section, we can describe light propagation in a local Minkowski space-time perturbed by the gravitational potential of the lens to first post-Newtonian order. The curvature of the metric acts on the light path as if it were refraction (see Fig. 3.1), and this effect can be expressed in terms of an effective refractive index of the potential, n , given by

$$n = 1 + \frac{2}{c^2}|\Phi|. \quad (3.2)$$

Note that, by definition, the potential Φ is negative and approaches zero at infinity. In conventional optics, a refractive index $n > 1$ corresponds to a propagation velocity slower than c . In fact, the effective velocity in terms of the refractive index is

$$v = \frac{c}{n} \approx c - \frac{2}{c}|\Phi|. \quad (3.3)$$

The time delay due to the reduction of light speed under the influence of the gravitational potential of the lens can be obtained by integrating along the light path from the observer to the source:

$$\Delta t = \int_{x_{\text{source}}}^{x_{\text{obs}}} \frac{2}{c^3}|\Phi|dl. \quad (3.4)$$

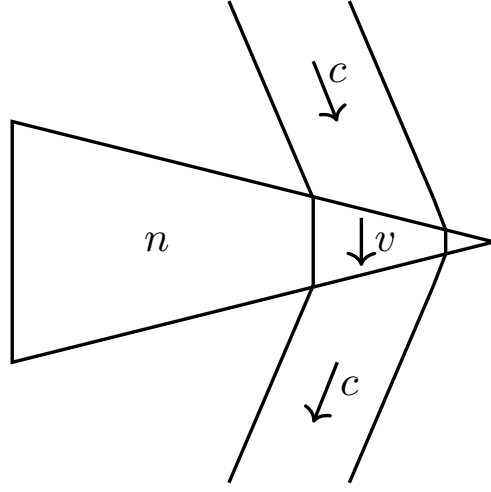


FIGURE 3.1: Schematic of light refraction under a prism. In our approximation, the Newtonian gravitational potential slows light propagation, deflecting the light-ray paths. Under the gravitational potential influence, the light speed in the “medium” is $v = c/n$, where $n = 1 + \frac{2}{c^2}|\Phi|$.

This time delay is commonly known as the Shapiro delay (Shapiro 1964). In the case of multiple images, each image has its own Shapiro delay.

The deflection angle arises from integrating the gradient of the refractive index (i.e., the gradient of the potential) along the light-ray path, in the direction perpendicular to the light propagation:

$$\hat{\alpha} = - \int_{x_{\text{source}}}^{x_{\text{obs}}} \nabla_{\perp} n \, dl = \frac{2}{c^2} \int_{x_{\text{source}}}^{x_{\text{obs}}} \nabla_{\perp} \Phi \, dl. \quad (3.5)$$

In most relevant cases, the deflection angle is very small, so we can make a further approximation⁵ by integrating the gradient along the unperturbed light path with the same impact parameter b instead of along the true deflected path.

For demonstration purposes, we now compute the deflection angle produced by a point-like lens of mass M on a light ray with impact parameter b , as depicted in Fig. 3.2. The Newtonian potential, at distance b from the lens to the unperturbed light ray, and distance z from the point of closest approach along the unperturbed path, is

$$\Phi(b, z) = - \frac{GM}{(b^2 + z^2)^{1/2}}. \quad (3.6)$$

⁵When multiple lenses are involved, additional considerations must be taken into account.

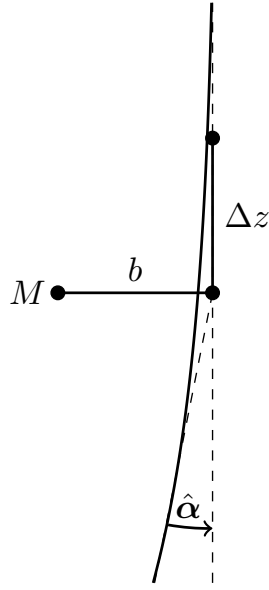


FIGURE 3.2: Deflection of a light-ray by a point-like lens of mass M . The unperturbed light ray crosses the region of influence of the lens with impact parameter b . Most of the deflection occurs along the $\pm\Delta z$ segment from the closest point of approach in the unperturbed path. Note that $\Delta z \sim b$.

The gradient perpendicular to the light propagation is then

$$\nabla_{\perp}\Phi(b, z) = \frac{GM\mathbf{b}}{(b^2 + z^2)^{3/2}}, \quad (3.7)$$

where $\mathbf{b}$ points towards the point-like lens. From Eq. 3.5, the deflection angle is

$$\hat{\alpha} = \frac{2}{c^2} \int_{-b}^{+b} \nabla_{\perp}\Phi \, dz = \frac{4GM}{c^2 b}. \quad (3.8)$$

This is twice the Schwarzschild radius of the point mass divided by the impact parameter, or twice the value obtained by Soldner in 1801 (Soldner 1921). Using the Sun's parameters for a light ray grazing its surface, the deflection angle is $1''.7$.

3.1.2 Thin screen approximation

As shown in Fig. 3.2, which can also be inferred from Eq. 3.5, most of the perturbation occurs within a length of approximately $\pm b$ along the unperturbed trajectory. Since this scale is astronomically small compared to the total path taken by the light rays, we can introduce an additional approximation that simplifies the calculation of deflection angles, the so-called thin screen approximation. This approximation states that the

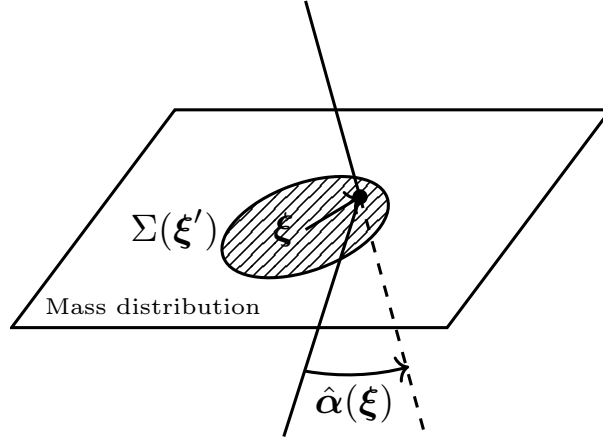


FIGURE 3.3: Illustration of the deflection angle of a light ray under the thin screen approximation. For a given surface mass density $\Sigma(\xi')$, the deflection angle at a position ξ in the lens plane is $\hat{\alpha}(\xi)$.

entire 3D mass distribution of the lens can be replaced by an equivalent mass sheet perpendicular to the line of sight. The 2D mass distribution is simply the projection of the lens's mass density along the line of sight:

$$\Sigma(\xi) = \int_{\text{l.o.s.}} \rho(\xi, z) dz, \quad (3.9)$$

where Σ is the surface mass density (i.e., the 2D mass distribution), ξ is a 2D vector in the lens plane whose origin is at the lens, z is the line-of-sight coordinate, and ρ is the 3D mass density of the lens.

The total deflection angle is the linear superposition of the contributions from all point-mass elements in the 2D distribution:

$$\hat{\alpha}(\xi) = \frac{4G}{c^2} \int_{\text{Lens plane}} \frac{(\xi - \xi')\Sigma(\xi')}{|\xi - \xi'|^2} d^2\xi'. \quad (3.10)$$

Here, ξ is the location in the lens plane where the deflection angle is evaluated, and ξ' is the position of the mass element in the projected mass distribution. The entire scenario is illustrated in Fig. 3.3.

3.1.3 Lens equation

Fig. 3.4 illustrates a generic gravitational lens system. Light rays leave the source and are deflected by a lens by an angle $\hat{\alpha}$. While the true source position is located at

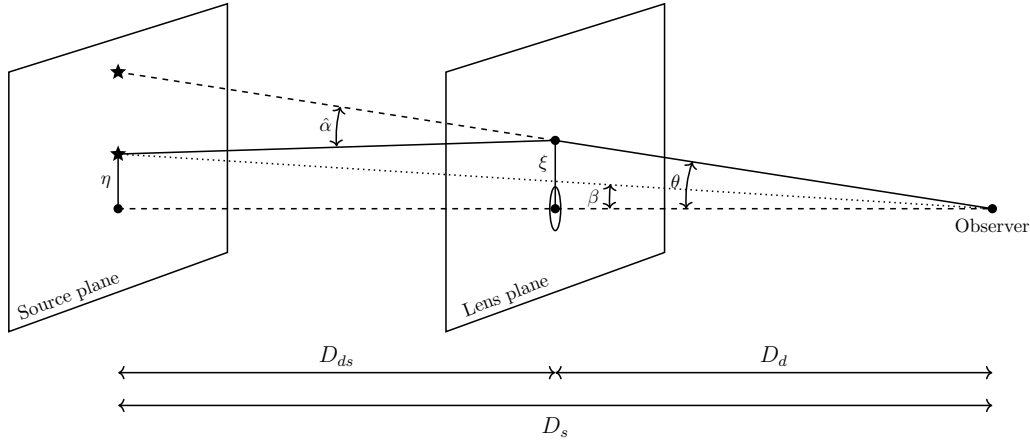


FIGURE 3.4: Schematic of a lens system. A light ray travels from a point at distance η from the origin, is deflected by an angle $\hat{\alpha}$ by the lens, and appears at a position ξ from the origin (the optical axis). The image of the source is observed at an angular distance θ , while the true position of the source is at β . The distances D_s , D_d , and D_{ds} are the angular diameter distances to the source, to the lens (deflector), and between the lens and the source, respectively. These distances are cosmology dependent.

an angle β from the observer (with respect to an arbitrary optical axis), the image appears at an angular position θ . The distances between the source and the observer D_s , between the lens and the observer D_d , and between the source and the lens D_{ds} are angular diameter distances⁶ introduced in Ch. 1. It is convenient to define the reduced deflection angle,

$$\alpha = \frac{D_{ds}}{D_s} \hat{\alpha} \quad (3.11)$$

From the geometry in Fig. 3.4, $\theta D_s = \beta D_s - \hat{\alpha} D_{ds}$. Combined with the reduced deflection angle, we can further simplify this equality to derive the lens equation⁷:

$$\beta = \theta - \alpha(\theta). \quad (3.12)$$

This equation is generally non-linear, since α depends on θ , and relates the true source position β to the observed image position θ , given the deflection field $\alpha(\theta)$. This equation is trivial to derive from the general lensing scheme shown in Fig. 3.4, since

⁶Note that these distances are redshift dependent and fixed by the assumed cosmology.

⁷This equation is also known as the ray-tracing equation, but more often the lens equation terminology is used.

from an Euclidean space we know that

$$\text{angle} = \frac{\text{separation}}{\text{distance}}. \quad (3.13)$$

In other words, the angle subtended between two points is exactly the distance between points over their distance from the observer. We work, nonetheless, on a curved space-time. It can be shown that for a distance such that Eq. 3.13 holds, this relation remains true. Such distances are the previously introduced angular diameter distances, from Ch. 1. The only difference now is that $D_{\text{ds}} \neq D_{\text{s}} - D_{\text{d}}$. We have thus reduced the problem to two ingredients: i) the surface mass density of the lens $\Sigma(\theta)$, and ii) the distances (or equivalently, the redshifts) of both the source and the lens. Together these allow us to compute $\alpha(\theta)$ and solve the lens equation.

As an example, consider a circularly symmetric lens with constant surface mass density Σ . At a distance ξ from the centre, the enclosed mass is $\Sigma\pi\xi^2$. By Gauss's theorem, the potential produced by the enclosed mass is equivalent to that of a point mass $M = \Sigma\pi\xi^2$ at the centre. Using Eq. 3.10, one finds

$$\alpha(\theta) = \frac{D_{\text{ds}}}{D_{\text{s}}} \frac{4G}{c^2 \xi} (\Sigma\pi\xi^2). \quad (3.14)$$

Here, we have used the circular symmetry to reduce the problem to one dimension. Again, similarly to how we derive the lens equation from Fig. 3.4, and, using $\xi = D_{\text{d}}\theta$,

$$\alpha(\theta) = \frac{4\pi G}{c^2} \frac{D_{\text{d}} D_{\text{ds}}}{D_{\text{s}}} \Sigma \times \theta. \quad (3.15)$$

Hence $\alpha(\theta) = \text{const} \times \theta$, so the lens equation is linear in this case ($\beta \propto \theta$) and admits an analytical solution. Defining the critical surface mass density as the inverse of the two fraction terms in the right hand side of Eq. 3.15,

$$\Sigma_{\text{cr}} = \frac{c^2}{4\pi G} \frac{D_{\text{s}}}{D_{\text{ds}} D_{\text{d}}} = 0.35 \text{ g cm}^{-2} \left(\frac{D}{1 \text{ Gpc}} \right)^{-1}, \quad (3.16)$$

with the effective distance

$$D = \frac{D_{\text{ds}} D_{\text{d}}}{D_{\text{s}}}. \quad (3.17)$$

we see that if $\Sigma = \Sigma_{\text{cr}}$ then $\alpha(\theta) = \theta$ and the lens equation implies $\beta = 0$ for all θ : the lens has as a perfectly-defined focal length. Realistic gravitational lenses, however,

do not share this property: rays with different impact parameters cross the optical axis at vastly different distances behind the lens plane, so real lenses exhibit all optical aberrations except chromatic ones⁸. The critical density also marks a natural threshold for SL: lenses with $\Sigma > \Sigma_{\text{cr}}$ are supercritical and can produce multiple images.

Now consider a circularly symmetric lens with an arbitrary mass profile. The lens equation becomes

$$\beta(\theta) = \theta - \frac{4GM(\theta)}{c^2\theta} \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}}, \quad (3.18)$$

where $M(\theta)$ is the mass enclosed within θ . If the lens is supercritical, a source located on the optical axis ($\beta = 0$) is imaged into a ring. Setting $\beta = 0$ gives the ring radius,

$$\theta_{\text{E}} = \left[\frac{4GM(\theta_{\text{E}})}{c^2} \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}} \right]^{1/2}. \quad (3.19)$$

This circular image is known as the Einstein ring. Its angular size depends not only on

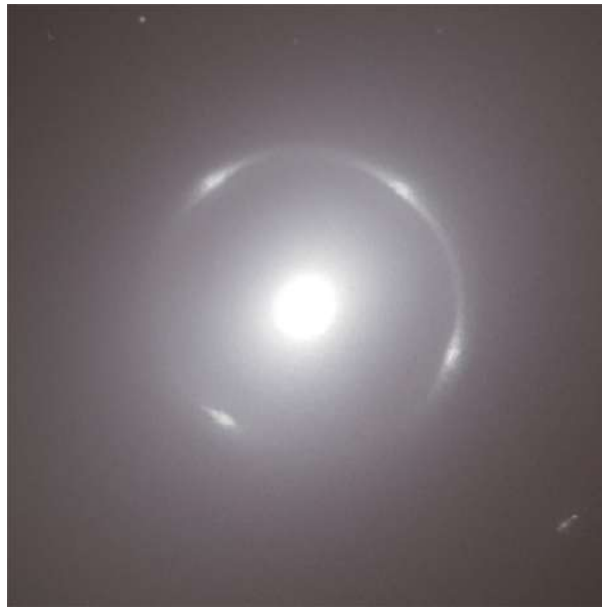


FIGURE 3.5: Centre of the galaxy NGC 6505, featuring a prominent Einstein ring encircling its core, as observed by ESA's Euclid space telescope. Credit: ESA/Euclid/Euclid Consortium/NASA. Image processing by J.-C. Cuillandre, G. Anselmi, T. Li.

the lens but also on cosmology via the distances in Eq. 3.19. Perfect Einstein rings are

⁸Under special conditions, sources may exhibit apparent chromaticity (e.g., QSO ML: the accretion disc emits different wavelengths at different radii, so bluer photons originate from a smaller emitting region and suffer stronger ML). Moreover, when wave optics is required instead of the geometric optics limit, the wavelength matters and chromatic effects arise.

rare owing to lens inhomogeneities, ellipticities, and imperfect alignments, but very recently the Euclid collaboration reported one of the most perfect examples observed to date (Fig. 3.5). The Einstein radius sets a natural lensing scale: i) in multiple-image systems, the separation is typically $\sim 2\theta_E$, and ii) it roughly delineates the transition between strong- and weak- lensing regimes; sources within θ_E from the optical axis are usually strongly lensed (multiple images, high magnification), whereas those outside are weakly magnified and singly imaged. It can also be shown that the average surface mass density inside θ_E is exactly Σ_{cr} .

3.1.4 Point-lens imaging

For a point-like lens of mass M , the lens equation can be expressed in terms of the Einstein radius as

$$\beta = \theta - \frac{\theta_E^2}{\theta}, \quad (3.20)$$

where the Einstein radius is

$$\theta_E = \left(\frac{4GM}{c^2} \frac{D_{\text{ds}}}{D_d D_s} \right)^{1/2}. \quad (3.21)$$

This lens equation is non-linear, but it admits analytical solutions. The non-linear nature leads to multiple solutions (i.e., multiple images). In this case, there are two solutions:

$$\theta_{\pm} = \frac{1}{2} \left(\beta \pm (\beta^2 + 4\theta_E^2)^{1/2} \right). \quad (3.22)$$

A generic source is thus imaged twice by a point-like lens: one image (I_-) inside the Einstein ring, and another (I_+) outside. As the source moves away from the optical axis, I_- becomes increasingly faint and approaches the lens position, while I_+ shifts towards the true source position with a magnification approaching unity.

In the previous discussion we have already introduced a key lensing effect: magnification. According to Liouville's theorem, lensing preserves surface brightness but changes the apparent size of the image. This leads to a change in observed flux. The magnification μ is defined as the ratio of the image solid angle to that of the source,

$$\mu = \frac{\Omega_I}{\Omega_S}, \quad (3.23)$$

where Ω_I and Ω_S are the apparent solid angles of the image and source, respectively.

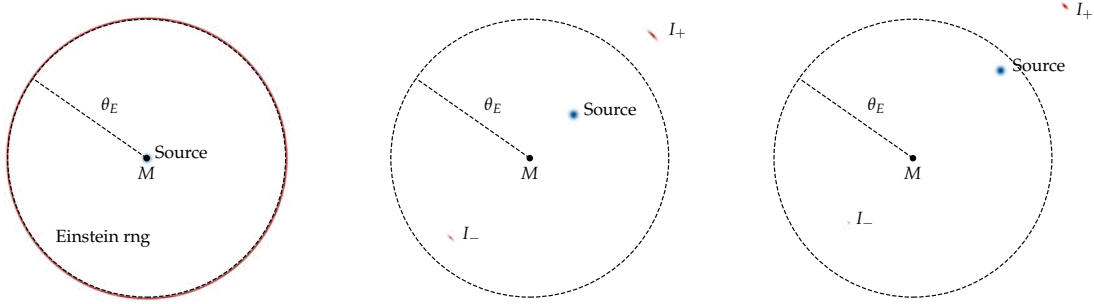


FIGURE 3.6: Lensing of a Gaussian source at different positions by a point-like lens of mass M . The first panel shows the case of perfect alignment, forming an Einstein ring. The other panels correspond to sources located progressively farther from the lens, producing two images: a negative-parity image inside the Einstein ring with lower magnification, and a positive-parity image outside the ring with higher magnification. The blue Gaussian denotes the source, and the red arcs represent the images. As the source-lens separation increases, the inner image becomes fainter and approaches the lens position, while the outer image approaches the source position and its magnification tends to unity. In the no-lensing limit, the negative image vanishes and the positive image coincides with the source position.

Thus, if the image area is twice that of the source, $\mu = 2$, while an image half as large is demagnified with $\mu = 0.5$. Fig. 3.6 illustrates these effects for a Gaussian source lensed by a point mass.

For a circularly symmetric lens, the magnification factor can be written as

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta}. \quad (3.24)$$

In the case of a point lens, using the lens equation we obtain the magnifications of the two images:

$$\mu_{\pm} = \left[1 - \left(\frac{\theta_E}{\theta_{\pm}} \right)^4 \right]^{-1} = \frac{u^2 + 2}{2u(u^2 + 4)^{1/2}} \pm \frac{1}{2}, \quad (3.25)$$

where $\theta_{\pm}$ are the location of the positive and negative parity images, and $u = \beta/\theta_E$ is the source-lens separation in units of θ_E . The negative sign of μ_- indicates that I_- has reversed parity relative to the source.

3.1.5 Effective lensing potential

For more complex lenses, it is convenient to introduce the effective lensing potential, i.e., the 2D projection on the lens plane of the 3D Newtonian potential:

$$\hat{\Psi}(\boldsymbol{\theta}) = \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}} \frac{2}{c^2} \int_{\text{l.o.s.}} \Phi(D_{\text{d}}\boldsymbol{\theta}, z) dz. \quad (3.26)$$

A dimensionless version is defined as

$$\Psi = \frac{D_{\text{d}}^2}{\xi_0} \hat{\Psi}, \quad (3.27)$$

where ξ_0 is an arbitrary length scale on the lens plane⁹ and $\boldsymbol{x} \equiv \boldsymbol{\xi}/\xi_0$. We will also use the dimensionless source plane vector $\boldsymbol{y} \equiv \boldsymbol{\eta}/\eta_0$, with $\eta_0 = \xi_0 D_{\text{s}}/D_{\text{d}}$.

Derivatives of the lensing potential have simple and very useful properties:

- Its gradient gives the (dimensionless) deflection angle:

$$\nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}) = \boldsymbol{\alpha}(\boldsymbol{x}). \quad (3.28)$$

- Its Laplacian is twice the convergence:

$$\nabla^2 \Psi(\boldsymbol{x}) = 2\kappa(\boldsymbol{x}). \quad (3.29)$$

where the convergence (the dimensionless surface mass density) is

$$\kappa(\boldsymbol{x}) = \frac{\Sigma(\xi_0 \boldsymbol{x})}{\Sigma_{\text{cr}}}. \quad (3.30)$$

3.1.6 Magnification and shear

As anticipated, gravitational lensing distorts the shapes of extended sources, producing images with sizes and shapes different from those of the unlensed source. This arises because different source elements experience different deflection angles. Locally (i.e., if the lensing potential varies slowly across the source), the mapping between source

⁹Often chosen to match a characteristic scale of the adopted parametric model, e.g., the scale radius r_s of an NFW profile.

and image planes can be linearised and written in terms of the Jacobian matrix

$$\mathbf{A} \equiv \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left(\delta_{ij} - \frac{\partial \alpha_i(\mathbf{x})}{\partial x_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \Psi(\mathbf{x})}{\partial x_i \partial x_j} \right), \quad (3.31)$$

where i, j denote the Cartesian components on the lens plane and δ_{ij} is the Kronecker delta. For simplicity we will use the following notation

$$\Psi_{ij} \equiv \frac{\partial^2 \Psi(\mathbf{x})}{\partial x_i \partial x_j}. \quad (3.32)$$

We can obtain a trace-free matrix simply by splitting off the isotropic part of the Jacobian

$$\begin{aligned} \Gamma &= - \left(\mathbf{A} - \frac{1}{2} \text{tr} \mathbf{A} \cdot \mathbf{I} \right)_{ij} = \Psi_{ij} - \delta_{ij} + \frac{1}{2} (1 - \Psi_{11} + 1 - \Psi_{22}) \delta_{ij} \\ &= \Psi_{ij} - \frac{1}{2} (\Psi_{11} + \Psi_{22}) \delta_{ij} \\ &= \begin{pmatrix} \frac{1}{2} (\Psi_{11} - \Psi_{22}) & \Psi_{12} \\ \Psi_{12} & -\frac{1}{2} (\Psi_{11} - \Psi_{22}) \end{pmatrix}. \end{aligned} \quad (3.33)$$

This matrix is known as the shear matrix, a trace free, antisymmetric matrix that quantifies the shape distortion of the source. It is convenient to define the shear pseudo-vector $\gamma = (\gamma_1, \gamma_2)$ with

$$\begin{aligned} \gamma_1(\mathbf{x}) &= \frac{1}{2} (\Psi_{11} - \Psi_{22}), \\ \gamma_2(\mathbf{x}) &= \Psi_{12} = \Psi_{21}. \end{aligned} \quad (3.34)$$

and $\gamma \equiv (\gamma_1^2 + \gamma_2^2)^{1/2}$. The shear matrix can then be written as

$$\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} = \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}, \quad (3.35)$$

where ϕ is the orientation angle of the shear.

The trace part of the Jacobian, $1/2 \text{tr}(\mathbf{A})$, is related to the Laplacian of the potential and thus to the convergence,

$$\frac{1}{2} \text{tr} \mathbf{A} = \left[1 - \frac{1}{2} (\Psi_{11} + \Psi_{22}) \right] \delta_{ij} = \left(1 - \frac{1}{2} \nabla^2 \Psi \right) \delta_{ij} = (1 - \kappa) \delta_{ij}. \quad (3.36)$$

With these two definitions, we can write the Jacobian in terms of the shear matrix (anisotropic component) and the convergence (isotropic component)

$$\begin{aligned} A &= (1 - \kappa)I - \Gamma = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix} \\ &= (1 - \kappa) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}. \end{aligned} \quad (3.37)$$

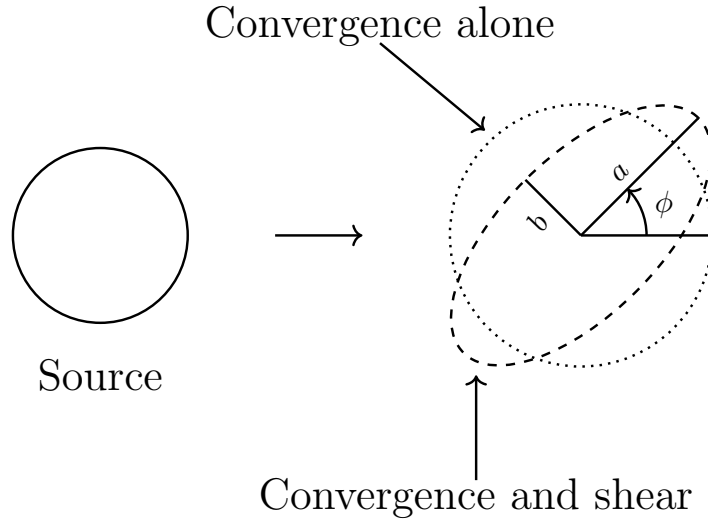


FIGURE 3.7: Effects of convergence and shear on a circular background source. The convergence κ isotropically changes the size (focusing/defocusing) while preserving the shape. The shear γ introduces an anisotropic distortion, mapping the circle into an ellipse with semi-axes a and b and orientation ϕ .

From Eq. 3.37 one can understand intuitively the effects of κ and γ . κ produces an isotropic focusing (size change, shape preserved), whereas γ shears the image anisotropically (shape change). For a circular source, the image is an ellipse with semi-axes

$$\begin{aligned} a &= \frac{1}{1 - \kappa - \gamma}, \\ b &= \frac{1}{1 - \kappa + \gamma}. \end{aligned} \quad (3.38)$$

rotated by an angle ϕ .

The total magnification is the inverse of the determinant of the Jacobian (often called the inverse magnification tensor):

$$\mu = \frac{1}{\det \mathbf{A}} = \det \mathbf{M} = \frac{1}{[(1 - \kappa)^2 - \gamma^2]}. \quad (3.39)$$

The eigenvalues of the magnification tensor $\mathbf{M} \equiv \mathbf{A}^{-1}$ (i.e., the local stretch factors) are the inverse of those of $\mathbf{A}$, and correspond to the tangential and radial magnifications:

$$\begin{aligned} \mu_t &= \frac{1}{1 - \kappa - \gamma}, \\ \mu_r &= \frac{1}{1 - \kappa + \gamma}. \end{aligned} \quad (3.40)$$

In geometric optics, locations on the lens plane where either $1 - \kappa - \gamma = 0$ or $1 - \kappa + \gamma = 0$ correspond to formally infinite magnification¹⁰. These locations are the tangential and radial critical curves (CCs), respectively. Sources near caustics produce highly magnified and strongly distorted images. Images formed around a tangential CC are distorted mostly around the tangential direction of the CC, while those formed at the radial CCs are distorted mainly perpendicular to the CC. Mapping the CCs back to the source plane via the lens equation yields the caustics. Therefore a caustics define a region on the source plane where a source would have to be located at to achieve the images with a maximum magnification. For example, a point-mass lens has a single CC, the Einstein ring, and a singular caustic, at the location of the lens (See Fig. 3.6, left panel).

The formalism above assumed locally constant κ and γ , i.e., a locally linear mapping. In general, one simply lets $\mathbf{A} = \mathbf{A}(\boldsymbol{\theta})$, allowing for spatially varying fields.

3.1.7 Second order lensing

The Jacobian mapping (first order lensing) above linearises the relation between source and image positions locally. For highly distorted images (e.g., near CCs) or for small-scale lenses with significant internal structure, higher-order terms can be important.

¹⁰Finite source size and, in some regimes, wave optics effects regularise the divergence.

A second-order expansion around the origin yields

$$y_i \simeq \frac{\partial y_i}{\partial x_i} x_j + \frac{1}{2} \frac{\partial^2 y_i}{\partial x_j \partial x_k} x_j x_k. \quad (3.41)$$

Define

$$D_{ijk} = \frac{\partial^2 y_i}{\partial x_k \partial x_j} = \frac{\partial A_{ij}}{\partial x_k}. \quad (3.42)$$

Then Eq. 3.41 becomes

$$y_i \simeq A_{ij} x_j + \frac{1}{2} D_{ijk} x_j x_k, \quad (3.43)$$

i.e., a first-order (Jacobian) mapping plus a quadratic correction governed by D . This tensor can be written in terms of derivatives of the shear:

$$\begin{aligned} D_{ij1} &= \begin{pmatrix} -2\gamma_{1,1} - \gamma_{2,2} & -\gamma_{2,1} \\ -\gamma_{2,1} & -\gamma_{2,2} \end{pmatrix}, \\ D_{ij2} &= \begin{pmatrix} -\gamma_{2,1} & -\gamma_{2,2} \\ -\gamma_{2,2} & -2\gamma_{1,2} - \gamma_{2,1} \end{pmatrix}, \end{aligned} \quad (3.44)$$

where $\gamma_{i,j} \equiv \partial \gamma_i / \partial x_j$. These are third derivatives of the lensing potential.

It is common to define the first and second flexions as the complex quantities

$$\begin{aligned} F &= F_1 + iF_2 = (\gamma_{1,1} + \gamma_{2,2}) + i(\gamma_{2,1} - \gamma_{1,2}), \\ G &= G_1 + iG_2 = (\gamma_{1,1} - \gamma_{2,2}) + i(\gamma_{2,1} + \gamma_{1,2}). \end{aligned} \quad (3.45)$$

Flexion captures the leading-order curvature and higher-order anisotropic distortions of images beyond shear. Fig 3.8 shows the effects of κ , γ , and the two flexions on a Gaussian source. The first flexion vector is simply the gradient of the convergence:

$$\mathbf{F} = \nabla \kappa, \quad (3.46)$$

i.e.,

$$\begin{aligned} F_1 &= \frac{\partial \kappa}{\partial x_1}, \\ F_2 &= \frac{\partial \kappa}{\partial x_2}. \end{aligned} \quad (3.47)$$

This property allows, in principle, for a reconstruction of the convergence field from first-flexion measurements.

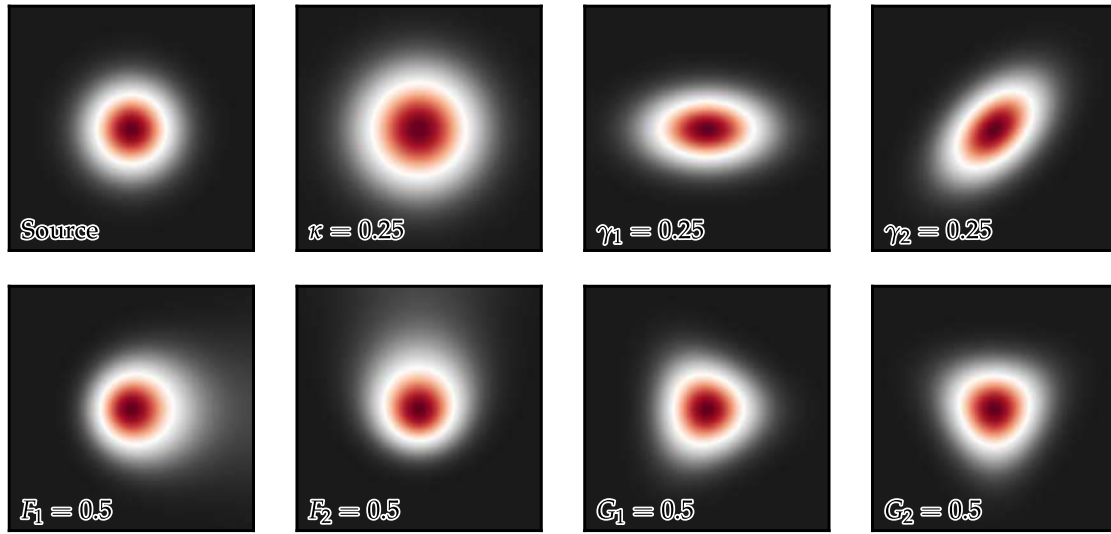


FIGURE 3.8: Gaussian source and its mapping under first- and second-order lensing effects. Top row: unlensed source, followed by convergence and shear. Bottom row: first flexion (first two panels) and second flexion (last two panels).

3.1.8 Lensing from Fermat's principle

Within the geometric-optics regime, lensing can be derived from Fermat's principle. Rewriting the (dimensionless) lens equation in the form

$$(\mathbf{x} - \mathbf{y}) - \nabla \Psi(\mathbf{x}) = 0, \quad (3.48)$$

which can be written as the following gradient

$$\nabla \left[\frac{1}{2}(\mathbf{x} - \mathbf{y})^2 - \Psi(\mathbf{x}) \right] = 0. \quad (3.49)$$

Define the (dimensional) time-delay function

$$t(\mathbf{x}) = \frac{(1 + z_d)}{c} \frac{D_s \xi_0^2}{D_d D_s} \left[\frac{1}{2}(\mathbf{x} - \mathbf{y})^2 - \Psi(\mathbf{x}) \right] = t_{\text{geom}} + t_{\text{grav}}, \quad (3.50)$$

where t_{geom} is the geometrical delay (extra path length) and t_{grav} is the gravitational (Shapiro) delay¹¹, with an extra $(1 + z_d)$ factor accounting for time dilation. Comparing Eqs. 3.49 and 3.50, we obtain the image condition $\nabla t(\boldsymbol{\theta}) = 0$: light rays follow stationary

¹¹Eq. 3.4

points of the arrival-time surface (Fermat's principle). In GR language, light follows null geodesics in the perturbed metric.

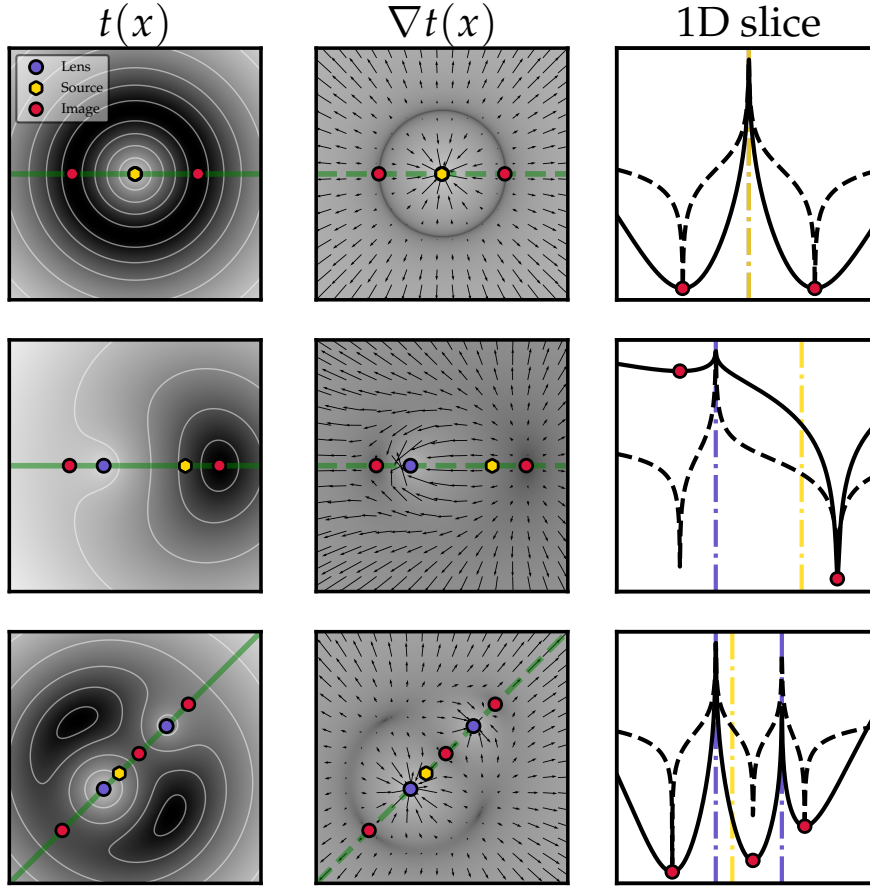


FIGURE 3.9: Illustration of lensing via Fermat's principle for point-mass lenses. Each row shows a different configuration (purple dot indicates the lens). Left: arrival-time surface with contours. Middle: gradient field (arrows indicate direction and magnitude; the colour scale shows the modulus). Right: time delay (solid) and its gradient (dashed) along the green cut. Yellow dashed-dotted lines (and circles) mark the lens position; red circles indicate image positions along the cut, i.e., where the gradient vanishes (dark regions in the middle panels). For clarity, $\log_{10}$ values are displayed.

A simple illustration regarding lensing through Fermat's principle and time delay surfaces is shown in Fig. 3.9, where three different configurations of lensing by point sources are shown. The relative difference in height at stationary points of the time delay surface, indicates the relative time delay between the image mapped in each point, with lower points in the surface experiencing changes first, then after the time difference the same change would be observed at the subsequent images.

The Hessian of the time-delay surface is

$$\mathbf{T} \equiv \frac{\partial^2 t(\mathbf{x})}{\partial x_i \partial x_j} \propto (\delta_{ij} - \Psi_{ij}) = \mathbf{A}, \quad (3.51)$$

so its eigenvalues at stationary points characterise the nature of the images. There are three types of stationary points (and thus images):

- **Type I:** minima of the time-delay surface. Both eigenvalues of $\mathbf{T}$ are positive, i.e., $\det \mathbf{A} > 0$ and $\text{tr} \mathbf{A} > 0$. The image has positive parity ($\mu > 0$).
- **Type II:** saddle points. The eigenvalues have opposite signs, i.e., $\det \mathbf{A} < 0$. The image has negative parity ($\mu < 0$).
- **Type III:** maxima. Both eigenvalues are negative, i.e., $\det \mathbf{A} > 0$ and $\text{tr} \mathbf{A} < 0$. The image has positive parity ($\mu > 0$).

As we have mention before, a negative magnification is interpreted as an image with parity flipped.

The curvature of the time-delay surface (given by $\mathbf{T} \propto \mathbf{A}$) measures the inverse magnification: small curvature implies large magnification, and vice versa. The curvature vanishes as two images merge; thus, the largest magnifications occur when images form near CCs, where $\det \mathbf{A} = 0$ and the magnification formally diverges¹².

For sources far from the lens, the time-delay surface has a single stationary point, a minimum (Type I), yielding a single positive-parity image. New extrema appear in pairs (a maximum and a saddle) as the source crosses a caustic; hence, the total number of images is typically odd. When a caustic is crossed again (from inside to outside), the image pair merges and disappears.

3.2 Lens modelling

Gravitational lensing opens the door to numerous astrophysical and cosmological analyses, but to obtain testable and accurate results, a reliable lens model is essential. Lens modelling is therefore one of the most important tasks in this field. The goal is to reconstruct, from the available data (lens members and background lensed sources),

¹²As previously mentioned, finite source size and wave-optics effects regularise the divergence.

the surface mass density Σ that best reproduces the observables, i.e., the source image positions, distortions, and relative time delays.

For this task, three key components are required: the identification and characterisation of the lens or lenses, the sources, and their distances, with the latter being cosmology-dependent. In addition, astrophysical phenomena such as luminosity variations, transient events (e.g., SNe or ML), or absorption effects must also be carefully taken into account.

Once the data are collected, the next step is to generate a mock lens system with predictions to compare against the observations. These predictions, such as flux ratios (related to magnification), relative time delays, or image positions and distortions, are compared with the data. The mismatch between predictions and observations is then used to iteratively refine the mock lens system until a convergence criterion is met, at which point a lens model as close as possible to the observations is obtained. The construction of the mock lens system, as well as the convergence algorithms and criteria, depends on the chosen model reconstruction technique.

Once we have the data, the next step is to generate a mock lens system with predictions to compare against the observations. These predictions are then compared with the data, and the offset between them is used to update the mock lens system with new predictions. This process is guided by a statistical estimator, often a χ^2 , which is iteratively computed until a convergence criterion is met. At that point, we can provide a lens system that is as close as possible to the observations.

For example, in a typical strong lens model, the main observable is the position of the lensed images, which can be obtained through visual inspection of the data¹³. For a mock lens, we can compute the source position of each image, β_i . We know that all images belonging to the same family (i.e., originating from the same background source) must share the same source position, β , which is an unknown parameter. Therefore, we define the statistic

$$\chi^2 = \sum_i \left(\frac{\beta - \beta_i}{\sigma_i} \right)^2, \quad (3.52)$$

¹³When the angular resolution and depth of the imaging are sufficiently high, we can even detect the substructure of the source, such as bright knots within the image (e.g., globular clusters, transient events like SNe, central bright regions such as bulges, or QSOs). These features provide additional constraints that can be exploited to fine-tune the model around the image regions.

where σ_i are the astrometric uncertainties, known only in the lens plane complicating the method. The concept is simple: the χ^2 statistic should be close to zero and below a tolerance threshold ε . When this criterion is satisfied, the model correctly maps all images of the same background source to the same location in the source plane. This procedure is performed independently for all families of images, and similarly for other observables such as time delays or flux ratios.

Since we generally do not know the astrometric uncertainties in the source plane, the fitting can instead be performed in the lens plane:

$$\chi^2 = \sum_i \left(\frac{\theta_i(\beta) - \theta_i}{\sigma_i} \right)^2, \quad (3.53)$$

where the fitted parameter remains the source position β , but the statistic is built on the lens plane, accounting for each image. This approach is numerically more expensive, but the uncertainties are known.

There are two major approaches to lens modelling: parametric and non-parametric. In parametric modelling, lenses are described using analytical profiles with a set of parameters that are explored within a defined parameter space until convergence is reached. Non-parametric models, also referred to as free-form models, divide the field of view into pixels, where either the mass of each pixel or an analytical profile (e.g., a Gaussian) assigned to each pixel acts as the free parameter in the fitting procedure. A third approach, known as hybrid modelling, combines the advantages of both. Hybrid methods exploit the linearity of the deflection angle, meaning that the lens deflection and potential can be expressed as a linear combination of deflection (and potential) fields from individual components. In practice, hybrid models decompose the lens into two contributions: a component described by parametric profiles (representing known lenses) and a diffuse component, such as the main DM halo, constructed via a free-form approach.

Generally, the number of observational constraints for the fitting is low, usually limited to the positions of a few images. The complexity of the lens models, however, can be really high. For instance, in the case of a galaxy cluster, each member will have several parameters to constrain: its position, ellipticity, mass, orientation, etc. This turns the problem into a set of $N \times n$ parameters, where N is the number of known lens members and n is the number of free parameters for each member. In free-form modelling, we

similarly have the number of pixels multiplied by the parameters of each pixel.

This total number of parameters, plus some additional parameters accounting for the environment (e.g., an external shear, as we will discuss later), cannot be successfully constrained with a much smaller number of observables k , since $N \times n \gg k$. Thus, we face more degrees of freedom than constraints, making the problem susceptible to strong degeneracies and large uncertainties.

To address this, we can apply several strategies. For example, we can remove the source positions from the equation by simply minimising the dispersion among the images of the same family. We can also fix the positions, ellipticities, and orientations of the lenses. Their individual masses can be tied to known mass-to-light relations, which are assumed constant for members with similar properties. Typically, the brightest members are left free, while the others are grouped together, reducing the number of free parameters. If some member galaxies are clearly distinct (e.g., red-sequence galaxies), these are assigned their own relation. Therefore, with the aid of these assumptions, the number of degrees of freedom is strongly reduced, drastically lowering the problem's complexity and approaching the number of constraints.

For a comparison of different lens modelling techniques applied to cluster lenses from the Hubble Frontier Fields (Lotz et al. 2017), we refer the reader to Zitrin et al. (2010); Meneghetti et al. (2017); Acebron et al. (2017); Priewe et al. (2017); Raney et al. (2020).

In this section, we will introduce some widely used lens models, which serve as the foundation for parametric modelling and can also be employed in free-form analyses to compare the resulting mass distribution with specific parameters. These parameters can later be used for new parametrisations, model comparisons, or even other scientific cases. We will also discuss further extensions of these models, designed to capture the actual complexity of real lenses.

3.2.1 Lens models

Point-like lens

As previously introduced, a point-like lens of mass M has a deflection angle

$$\hat{\alpha} = \frac{4GM}{c^2 D_d} \frac{1}{\theta}, \quad (3.54)$$

pointing towards the lens.

The corresponding lensing potential is

$$\hat{\Psi} = \frac{4GM}{c^2} \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}} \ln |\boldsymbol{\theta}|. \quad (3.55)$$

The lens equation is then

$$\beta = \theta - \frac{4GM}{c^2 D_{\text{d}}} \frac{1}{\theta} \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}}, \quad (3.56)$$

or, using the Einstein radius defined in Eq. 3.19,

$$\beta = \theta - \frac{\theta_{\text{E}}^2}{\theta} \rightarrow y = x - \frac{1}{x}, \quad (3.57)$$

where we have used $\xi_0 = \theta_{\text{E}}$ to obtain the dimensionless form on the right.

Its Jacobian determinant is

$$\det \mathbf{A} = \frac{y}{x} \frac{\partial y}{\partial x} = \left(1 - \frac{\alpha}{x}\right) \left(1 - \frac{\partial \alpha}{\partial x}\right) = 1 - \left(\frac{1}{x}\right)^4. \quad (3.58)$$

The magnification, as the inverse of the Jacobian determinant, is therefore

$$\mu = \left[1 - \left(\frac{1}{x}\right)^4\right]^{-1}. \quad (3.59)$$

Axially symmetric lenses

The point lens is just a special case of axially symmetric lenses, where the surface mass density is independent of the angle and depends only on the distance to the centre. If we choose the lens centre to lie on the optical axis, the problem reduces to a one-dimensional case.

The deflection angle is

$$\hat{\alpha} = \frac{4GM(\xi)}{c^2 \xi}, \quad (3.60)$$

or, in dimensionless units,

$$\alpha(x) = \frac{m(x)}{x}, \quad (3.61)$$

where $m(x)$ is the dimensionless mass defined as

$$m(x) = \frac{M(\xi_0 x)}{\pi \xi_0^2 \Sigma_{\text{cr}}}. \quad (3.62)$$

The dimensionless lens equation is then

$$y = x - \frac{m(x)}{x}. \quad (3.63)$$

If we write the deflection angle as a vector pointing towards the lens centre, and use $\mathbf{x} = (x_1, x_2)$, we have

$$\boldsymbol{\alpha}(\mathbf{x}) = \frac{m(\mathbf{x})}{x^2} \mathbf{x}. \quad (3.64)$$

The Jacobian is

$$\mathbf{A} = \mathbf{I} - \frac{m(x)}{x^4} \begin{pmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix} - \frac{dm(x)}{dx} \frac{1}{x^3} \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}. \quad (3.65)$$

With some algebra, we can obtain the following expressions for κ and γ :

$$\begin{aligned} \kappa(x) &= \frac{1}{2x} \frac{dm(x)}{dx}, \\ \gamma_1(x) &= \frac{1}{2}(x_2^2 - x_1^2) \left(\frac{2m(x)}{x^4} - \frac{dm(x)}{dx} \frac{1}{x^3} \right), \\ \gamma_2(x) &= x_1x_2 \left(\frac{dm(x)}{dx} \frac{1}{x^3} - \frac{2m(x)}{x^4} \right), \\ \gamma(x) &= \frac{m(x)}{x^2} - \kappa(x) = \bar{\kappa}(x) - \kappa(x), \end{aligned} \quad (3.66)$$

where $\bar{\kappa}(x)$ is the mean surface convergence within radius x .

The determinant of the Jacobian is

$$\begin{aligned} \det \mathbf{A} &= \frac{y}{x} \frac{dy}{dx} = \left(1 - \frac{m(x)}{x^2} \right) \left[1 - \frac{d}{dx} \left(\frac{m(x)}{x} \right) \right] \\ &= \left(1 - \frac{\alpha(x)}{x} \right) \left(1 - \frac{d\alpha(x)}{dx} \right). \end{aligned} \quad (3.67)$$

We now present some widely used axially symmetric lens models.

Singular Isothermal Sphere (SIS)

This mass profile is derived under the assumption that the matter content behaves as an ideal gas, confined within a sphere by its gravitational potential, and in thermal and hydrostatic equilibrium. Its 3D density profile is

$$\rho(r) \frac{\sigma_v^2}{2\pi G r^2}, \quad (3.68)$$

where r is the radial distance from the centre of the sphere, and σ_v is the velocity dispersion of the particles. Integrating ρ along the line of sight yields the surface mass density

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi}. \quad (3.69)$$

Note the singularity at $\xi = 0$. We can define our angular length scale ξ_0 as

$$\xi_0 = 4\pi \left(\frac{\sigma_v}{c} \right)^2 \frac{D_d D_{ds}}{D_s}. \quad (3.70)$$

With this particular choice, the dimensionless surface mass density simplifies to

$$\Sigma(x) = \frac{1}{2x} \Sigma_{cr}, \quad (3.71)$$

and the convergence is

$$\kappa(x) = \frac{1}{2x}. \quad (3.72)$$

The lensing potential takes a simple form,

$$\Psi(x) = |x|, \quad (3.73)$$

and the deflection angle, given by its gradient, reads

$$\alpha(x) = \frac{x}{|x|}, \quad (3.74)$$

which is constant in magnitude and points towards the centre. The dimensionless lens equation can then be written as

$$y = x - \frac{x}{|x|}. \quad (3.75)$$

The derivatives of the lensing potential are

$$\begin{aligned}\Psi_{11} &= \frac{x^2 - x_1^2}{x^3} = \frac{x_2^2}{x^3}, \\ \Psi_{12} &= -\frac{x_1 x_2}{x^3} = \Psi_{21}, \\ \Psi_{22} &= \frac{x^2 - x_2^2}{x^3} = \frac{x_1^2}{x^3}.\end{aligned}\tag{3.76}$$

The shear components are

$$\begin{aligned}\gamma_1 &= \frac{1}{2}(\Psi_{11} - \Psi_{22}) = \frac{1}{2} \frac{x_2^2 - x_1^2}{x^3} = \frac{1}{2} \frac{\sin^2 \phi - \cos^2 \phi}{x} = -\frac{1}{2} \frac{\cos 2\phi}{x}, \\ \gamma_2 &= \Psi_{12} = -\frac{x_1 x_2}{x^3} = -\frac{\cos \phi \sin \phi}{x} = -\frac{1}{2} \frac{\sin 2\phi}{x}.\end{aligned}\tag{3.77}$$

Note that $\gamma(x) = (\gamma_1^2 + \gamma_2^2)^{1/2} = \frac{1}{2x} = \kappa(x)$.

From Eq. 3.39, the magnification is

$$\mu = \frac{|x|}{|x| - 1}.\tag{3.78}$$

Softened Isothermal Sphere

As mentioned above, the SIS model has a central singularity. This can be avoided by introducing a core of radius r_c . In dimensionless units, $x_c \equiv r_c/\xi_0$, the lensing potential becomes

$$\Psi = \sqrt{x^2 + x_c^2}.\tag{3.79}$$

The corresponding deflection angle, in terms of the image position vector $\mathbf{x}$, is

$$\boldsymbol{\alpha}(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{x^2 - x_c^2}}.\tag{3.80}$$

The modified convergence and shear are then

$$\begin{aligned}\kappa &= \frac{x^2 + 2x_c^2}{2(x^2 + x_c^2)^{3/2}}, \\ \gamma_1 &= -\frac{x^2}{2(x^2 + x_c^2)^{3/2}} \cos 2\phi, \\ \gamma_2 &= -\frac{x^2}{2(x^2 + x_c^2)^{3/2}} \sin 2\phi,\end{aligned}\tag{3.81}$$

so that the shear amplitude satisfies

$$\gamma(x) = \frac{x^2}{2(x^2 + x_c^2)^{3/2}} = \bar{\kappa}(x) - \kappa(x).\tag{3.82}$$

Navarro-Frenk-White (NFW)

The NFW profile (Navarro et al. 1996, 1997) is of special interest, as it has been shown to describe the preferred CDM mass distribution of many haloes across a wide range of masses in Λ CDM numerical simulations. Its 3D density profile is

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2},\tag{3.83}$$

where r_s is the characteristic scale radius and ρ_s the corresponding characteristic density of the halo. These parameters are often expressed in terms of the virial mass M_{200} , defined as the mass enclosed within the radius r_{200} where the mean density equals $200 \rho_{\text{cr}}$:

$$r_{200} = 1.63 \times 10^{-2} \left(\frac{M_{200}}{h^{-1} M_{\odot}} \right)^{1/3} \left(\frac{\Omega_0}{\Omega(z)} \right)^{-1/3} (1+z)^{-1} h^{-1} \text{ kpc}.\tag{3.84}$$

Another widely used parameter is the concentration $c \equiv r_{200}/r_s$, not to be confused with the speed of light. The characteristic density can then be written as

$$\rho_s = \frac{200}{3} \rho_{\text{cr}} \frac{c^3}{[\ln(1+c) - c/(1+c)]}.\tag{3.85}$$

Choosing $\xi_0 = r_s$ as the angular scale, the surface mass density is obtained by integrat-

ing $\rho(r)$ along the line of sight:

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2 - 1} f(x), \quad \text{where} \quad f(x) = \begin{cases} -\frac{2}{\sqrt{x^2-1}} \arctan \sqrt{\frac{x-1}{x+1}}, & x > 1, \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}}, & x < 1, \\ \frac{2}{3} \rho_s r_s, & x = 1. \end{cases} \quad (3.86)$$

The lensing potential is

$$\Psi(x) = 4 \frac{\rho_s r_s}{\Sigma_{\text{cr}}} g(x),$$

$$\text{where} \quad g(x) = \frac{1}{2} \ln^2 \frac{x^2}{2} + \begin{cases} 2 \arctan^2 \sqrt{\frac{x-1}{x+1}} & x > 1, \\ -2 \operatorname{arctanh}^2 \sqrt{\frac{1-x}{1+x}} & x < 1, \\ 0 & x = 1. \end{cases} \quad (3.87)$$

The corresponding deflection angle reads

$$\alpha(x) = 4 \frac{\rho_s r_s}{\Sigma_{\text{cr}}} g(x), \quad \text{where} \quad h(x) = \ln \left(\frac{x}{2} \right) + \begin{cases} \frac{2}{\sqrt{x^2-1}} \arctan \sqrt{\frac{x-1}{x+1}}, & x > 1, \\ \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}}, & x < 1, \\ 1, & x = 1. \end{cases} \quad (3.88)$$

The convergence can be expressed using $f(x)$ from Eq. 3.86:

$$\kappa(x) = \frac{2\rho_s r_s}{\Sigma_{\text{cr}}} \frac{f(x)}{x^2 - 1}. \quad (3.89)$$

The dimensionless mass enclosed within x is

$$m(x) = \frac{4\rho_s r_s}{\Sigma_{\text{cr}}} h(x). \quad (3.90)$$

Examples of lensing by SIS and NFW models are depicted in Fig. 3.10. The point-lens case can be seen, for instance, in Fig. 3.6.

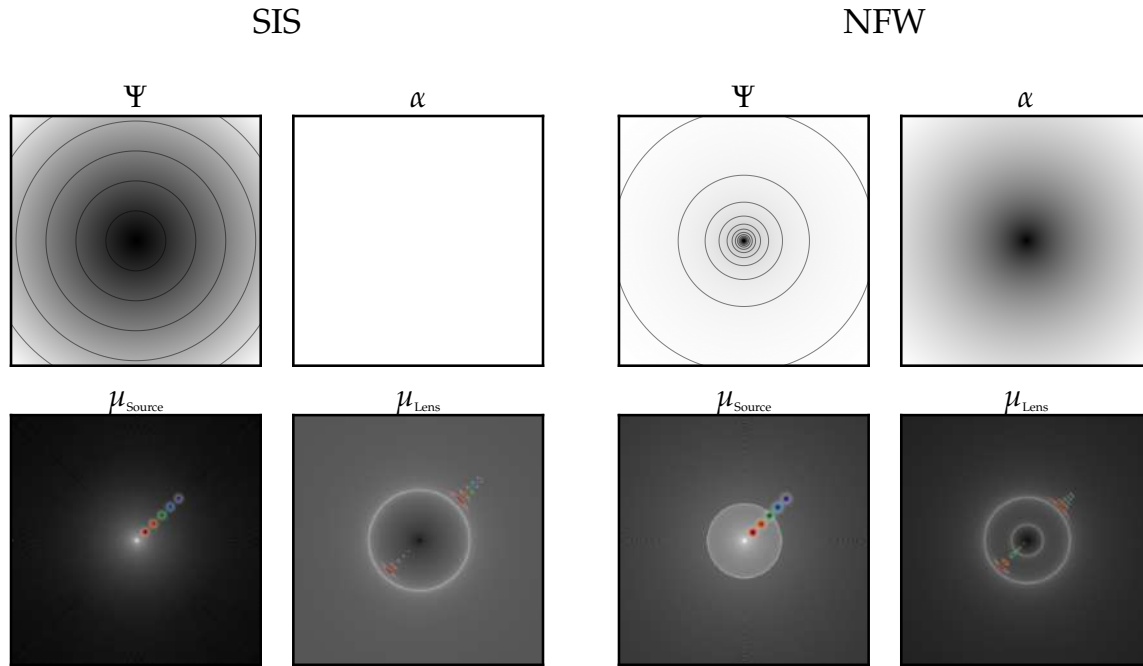


FIGURE 3.10: Lensing by different parametric models, with the SIS shown in the left panels and the NFW in the right panels. In each case, the top-left panel displays the lensing potential, and the top-right panel the reduced deflection angle. The bottom panels show, on the left, the magnification in the source plane (including caustics), and on the right, the corresponding lens-plane view with the CCs. The source plane contains several Gaussian sources, each represented with a different colour, while their image configurations (same colours) are shown in the lens-plane panel. Note that the SIS deflection angle is constant, with only a tangential CC (this changes for the softened SIS, where a core is included). The NFW profile, on the other hand, has a secondary CC, the inner radial CC, and allows for additional images as well as radial image configurations.

External shear

In addition to the parametric lens models described above, we must also account for external perturbations, commonly referred to as external shear. The environment surrounding a lens can contribute to the total convergence and shear fields, and these contributions are not always easily described by the aforementioned parametric models.

We can describe the external shear through a lensing potential Ψ_γ , which satisfies the

following conditions:

$$\begin{aligned}\kappa &= \frac{1}{2}(\Psi_{11} + \Psi_{22}) = \text{constant}, \\ \gamma_1 &= \frac{1}{2}(\Psi_{11} - \Psi_{22}) = \text{constant}, \\ \gamma_2 &= \Psi_{12} = \text{constant}.\end{aligned}\tag{3.91}$$

The first and third conditions imply that $\Psi_{11} \pm \Psi_{22}$ are constant, which means that each term must be constant individually. Therefore, the lensing potential can be written as

$$\Psi_\gamma = Ax_1^2 + Bx_2^2 + Cx_1x_2 + D.\tag{3.92}$$

This form satisfies

$$\begin{aligned}\kappa &= \frac{1}{2}(\Psi_{11} + \Psi_{22}) = A + B, \\ \gamma_1 &= \frac{1}{2}(\Psi_{11} - \Psi_{22}) = A - B, \\ \gamma_2 &= \Psi_{12} = C.\end{aligned}\tag{3.93}$$

If we set $\kappa = 0$, we obtain

$$A = -B \quad \Rightarrow \quad A = \frac{\gamma_1}{2}.\tag{3.94}$$

The external shear potential can then be expressed as

$$\Psi_\gamma = \frac{\gamma_1}{2}(x_1^2 - x_2^2) + \gamma_2 x_1 x_2.\tag{3.95}$$

Similarly, we can add a uniform mass sheet with constant convergence κ , which induces no shear (as no direction is preferred). Using the same conditions from Eq. 3.93, we find

$$\Psi_\kappa = \frac{\kappa}{2}(x_1^2 + x_2^2).\tag{3.96}$$

Any lens model can therefore be expressed as the linear sum of its intrinsic lensing potential plus an external shear potential (either Ψ_γ , Ψ_κ , or both), whose parameters become additional free parameters in the lens model reconstruction.

Ellipticity

Up to this point, we have introduced axially symmetric models and their external perturbations. This, however, remains a highly idealised scenario. Gravitational interactions, as well as baryonic physics processes, give rise to more complex shapes. In

practice, lenses are more often elliptical than spherical. For instance, galaxy-scale lenses frequently correspond to elliptical galaxies acting as lenses for background sources.

Despite being elliptical, the axially symmetric models presented earlier still provide a strong starting point. These lenses can be accurately modelled by introducing two additional parameters: the ellipticity and the position angle, which define the shape and orientation of the lens.

Let us start with an axially symmetric surface mass density $\Sigma(x)$, which can be “ellipticalised” through the variable change

$$x \rightarrow X = \sqrt{\frac{x_1^2}{1-e} + x_2^2(1-e)}. \quad (3.97)$$

Here, e is the ellipticity, defined as $e \equiv 1 - b/a$, where a and b are the major and minor semi-axes of the ellipse¹⁴. This transformation ensures that the mass enclosed within a circle of radius X equals that enclosed within a radius x in the original symmetric model. The position angle is simply a rotation of the ellipse around the optical axis, with the major axis initially aligned with the x_2 direction.

The lensing potential of an elliptical mass distribution can be similarly obtained from its axially symmetric counterpart under the same variable change.

The components of the deflection angle along the x_1 and x_2 directions now read

$$\begin{aligned} \alpha_1 &= \frac{\Psi}{\partial x_1} = \frac{x_1}{(1-e)X} \alpha_{\text{symm}}(X), \\ \alpha_2 &= \frac{\Psi}{\partial x_2} = \frac{x_2(1-e)}{X} \alpha_{\text{symm}}(X), \end{aligned} \quad (3.98)$$

where $\alpha_{\text{symm}}(X)$ is the deflection angle of the axially symmetric lens at a distance X from the centre.

An example of lensing by elliptical lenses is shown in Fig. 3.11. An NFW lens with different ellipticities and the resulting image configurations is displayed, as well as the corresponding CCs and caustics produced by these transformations.

¹⁴It is also common to use $q \equiv b/a$, the axis ratio, which satisfies $q = 1 - e$. Both definitions are widely used.

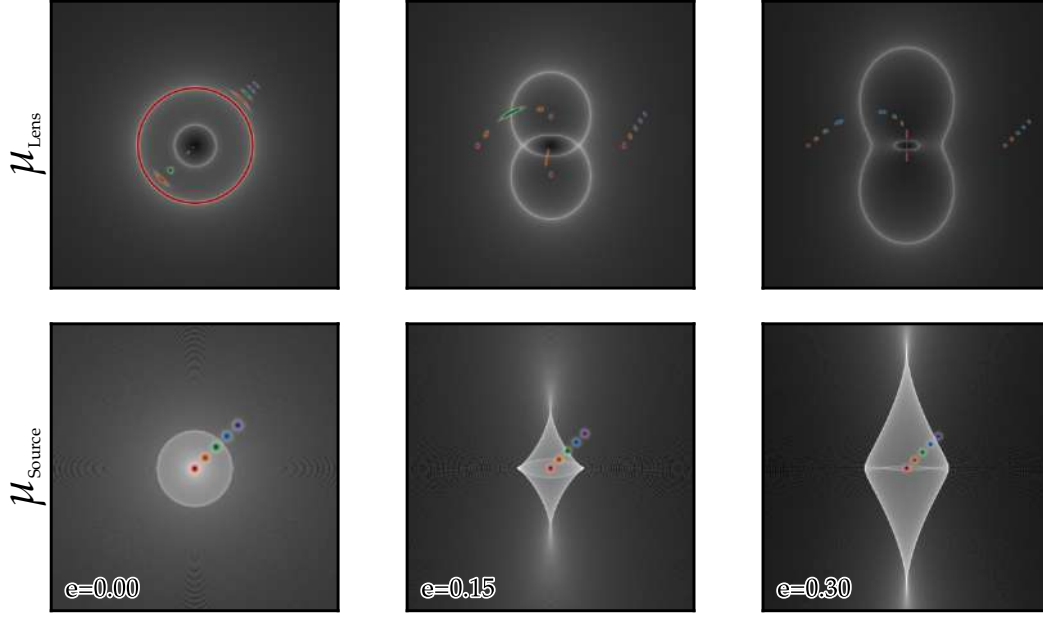


FIGURE 3.11: Example of elliptical lenses. From left to right, the same NFW lens model is shown with increasing ellipticities. The bottom panels display the magnification in the source plane, with the caustics and several Gaussian sources at slightly different positions. The top panels show the magnification in the lens plane, with the CCs and the image configurations corresponding to the sources in the bottom panels. Each colour corresponds to a different source.

3.3 Strong lensing

The SL regime occurs in the central regions of lenses, where the lens is supercritical, $\Sigma(\xi) \geq \Sigma_{\text{cr}}$. This region is delimited by CCs, formed where the Jacobian determinant drops to zero:

$$\begin{aligned}\lambda_t &= 1 - \kappa - \gamma = 0, \\ \lambda_r &= 1 - \kappa + \gamma = 0.\end{aligned}\tag{3.99}$$

The first condition defines the tangential CC, and the second the radial CC. Extended mass distributions depend on a large number of parameters that allow for the strong-lensing conditions to be met locally. These parameters include:

- Galaxies and galaxy clusters are built upon a main DM halo and baryonic matter clumps (gas and stars) that follow different spatial distributions, especially if the system is unrelaxed. The lens is thus composed of a superposition of surface mass density layers, with different scales. The total convergence can therefore be

expressed as a multipole expansion:

$$\kappa(\mathbf{x}) = \kappa_0(\mathbf{x}) + \sum_{m=1}^{\infty} \kappa_m(\mathbf{x}) e^{im\phi}, \quad (3.100)$$

where κ_0 , the monopole, describes the axially symmetric component. The ellipticity comes from the quadrupole $m = 2$. In the multipole expansion, the first three multipoles ($m = 0, 1, 2$) are the most relevant terms.

- The density profile $\rho(r)$ plays a crucial role in the lensing configurations. As we saw earlier, the different models, such as SIS or NFW profiles, predict drastically different lensing effects, from the image configurations to their multiplicity.
- External perturbations affect density profiles to different degrees. The NFW profile, for instance, given its shallower inner density slope, is more easily perturbed by an external shear than an SIS. The same applies to multiple-lens-plane lensing (lensing by matter structures in front of the main lens plane).
- Substructures, which are described by higher multipoles in Eq. 3.100, modify the local density profile. This effect is again more pronounced in shallower profiles, and its impact is enhanced near the main lens CCs.

We have shown the different lens perturbations relevant for SL analyses in Fig. 3.12. In this figure, a standard NFW model is displayed alongside its independent perturbations: ellipticity, external shear, and substructure. Finally, a configuration combining all these perturbations is presented to illustrate the complexity of real SL systems, which is particularly critical for accurate modelling of galaxy clusters.

Observables

Strong lensing systems leave a distinct imprint on background sources, such as multiple images, magnification (which allows us to observe faint objects otherwise undetectable), and time delays. These observables provide the necessary data to constrain the lens mass, its distribution, and offer valuable tests for the cosmological model.

These observables are:

- Relative positions of both the images and the lens objects, which constitute the

¹⁵This works as a toy model. Real subhalos usually have extended profiles such as NFWs.

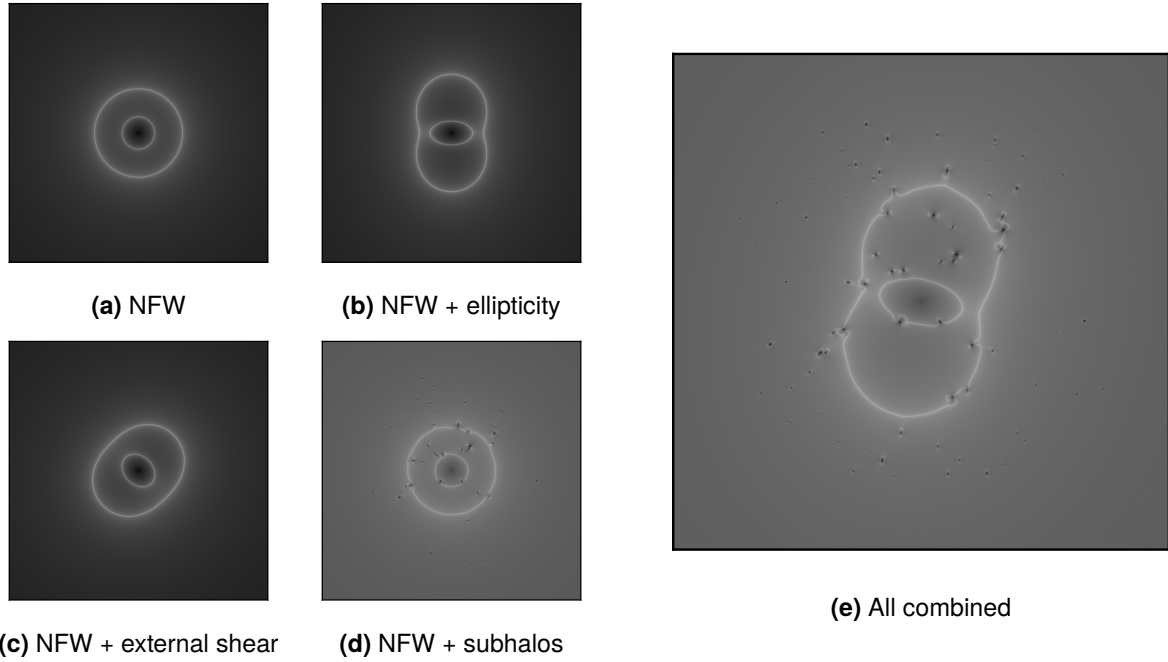


FIGURE 3.12: Example of NFW lensing configurations. The left sub-panels show an NFW lens model with different perturbations. Panel (a) shows an NFW with no perturbations. Panel (b) shows the same NFW with an ellipticity $e = 0.1$, while panel (c) adds an external shear oriented along the diagonal direction ($\gamma_e = 0.1$). Panel (d) displays the effect of adding 100 point lenses¹⁵ with random masses, positions, and ellipticities. The rightmost panel (e) shows all the aforementioned perturbations combined, resembling more closely the complexity of real lenses.

most important constraint for SL analyses.

- Relative fluxes of the images, traced to the same background source (same family). Intrinsic variability of the source can hinder this information.
- Relative time delays between the multiple images.
- Intrinsic properties of the lens, such as its spectrum and light distribution.
- When multi-epoch observations are available, light curves can also be reconstructed, which is particularly useful for studying ML.

Problems

There are several well-known issues in SL analyses. These are important to keep in mind, since, even if the lens observables are well reproduced, unaccounted-for systematics can bias the modelling and lead to incorrect inferences. Below we briefly

summarise some of the most relevant sources of bias.

Lensing near cusps

The image configuration predicted for sources located close to the cusps of tangential caustics has not been observationally confirmed yet. Two images near a CC will eventually merge together when the source lies exactly on the caustic, forming a fold. It is a long (but standard) exercise to show that, if the source is instead very close to a cusp (a pointed region of the caustic), three highly magnified images in a close triangular configuration should appear, satisfying

$$\begin{aligned}\mu_1 + \mu_2 + \mu_3 &= 0, \\ \mu_2 &= \mu_3, \\ \mu_2 &= -\frac{1}{2}\mu_1.\end{aligned}\tag{3.101}$$

Here the indices 1, 2, 3 refer to the first, second, and third images, respectively. The magnification of the first image (inside the CC) can be written analytically in terms of the distance to the cusp and the third derivative of the Fermat potential, while the remaining two follow from Eq. 3.101. A fourth image, farther from the CC and less magnified, also exists. When the source crosses the caustic outwards, these images merge and disappear.

In principle, the three brightest images of a quad configuration should obey these relations; however, no observed system fulfils them. For example, in the quad-lensed QSO B1422–231 (Patnaik et al. 1992), VLBA observations found

$$\begin{aligned}\frac{\mu_1}{\mu_2} &\sim 0.9, \\ \frac{\mu_2}{\mu_3} &\sim 2, \\ \frac{\mu_1}{\mu_3} &\sim 2.\end{aligned}\tag{3.102}$$

Mao & Schneider (1998) suggested that such anomalies could be explained by substructure perturbations, but later Mao et al. (2004) found from Λ CDM simulations that the fraction of substructure at typical image positions is of order 0.5%, seemingly insufficient to explain the observed flux anomalies. Other systems (e.g., B2045+265 (Fassnacht et al. 1999), B0712+472 (Jackson et al. 1998), B1933+503 (Sykes et al. 1998)) exhibit similar discrepancies. This “cusp relation problem” remains unresolved.

Microensing

Microensing is inherently a SL phenomenon in which the angular separation between micro-images is smaller than the instrumental resolution. When present, the observed fluxes are boosted (or suppressed) relative to what is expected from macrolensing alone, biasing flux-ratio constraints. If the relative motion between the source and the micro-caustic network is sufficiently large, the ML-induced magnification varies with time; thus, multi-epoch imaging can help identify and correct for these events. This effect is especially relevant for compact sources such as QSOs, whose typical magnitude variations lie around -0.3 to -0.5 mag¹⁶, although larger excursions are possible with lower probability and over shorter timescales. These additional uncertainties should be propagated into the SL analysis, enlarging the error bars of the inferred parameters.

Intrinsic variability

Similarly to ML, intrinsic variability of the background source can bias flux ratios when significant time delays exist between images. If image A (e.g., associated with a minimum of the time-delay surface) varies in brightness, the corresponding images $B, C, \dots$ will exhibit the same variation only after a delay equal to their relative time delays, which can be of order years for massive lenses. If not properly accounted for, this can bias lens reconstructions. As in the ML case, multi-epoch follow-up is essential. Knowing the nature of the source is also helpful; for instance, Cepheids have well-characterised light curves that can be exploited to refine the modelling.

Mass-sheet degeneracy

Since mass distributions are inferred by fitting lensing observables, we are inherently subject to transformations of the mass model that preserve those observables. The mass-sheet degeneracy is a classic example: adding a uniform mass sheet on top of the lens leaves the image positions and shapes invariant under

$$\kappa(\boldsymbol{\theta}) \rightarrow \lambda\kappa(\boldsymbol{\theta}) + (1 - \lambda), \quad (3.103)$$

with λ a scalar factor. Lensing measures differences in potential, so such a transformation cannot be distinguished using lensing data alone. The degeneracy is especially relevant in non-parametric (free-form) reconstructions and must be broken using additional information (e.g., time delays) or assumptions about the intrinsic size/brightness of the source.

¹⁶A magnification variation $\Delta\mu$ corresponds to a magnitude variation $\Delta m = -2.5 \log_{10}(\Delta\mu)$.

Multiple-source-plane lensing can also help break this degeneracy (Abdelsalam et al. 1998; Saha 2000; Normann et al. 2024). Since $\kappa = \Sigma/\Sigma_{\text{cr}}$ and Σ_{cr} depends on the source redshift, a uniform convergence sheet affects sources at different redshifts differently. This is particularly useful for cluster lenses, which often lens multiple sources at distinct redshifts. Although some galaxy lenses also show multiple source planes (Gavazzi et al. 2008), such systems are much rarer than in clusters.

3.4 Weak lensing

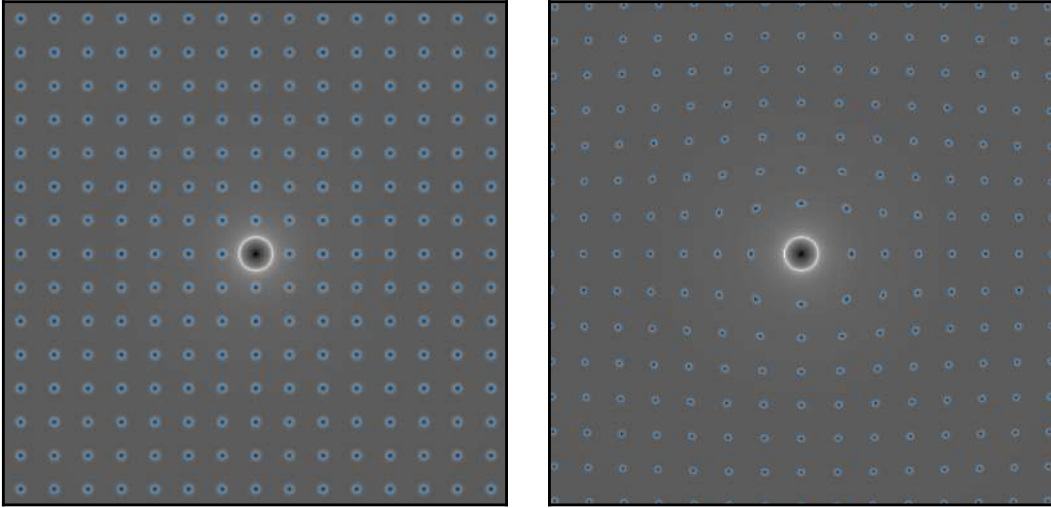


FIGURE 3.13: Effect of an SIS lens profile on a background source population. Left: Gaussian sources on an equispaced grid. Right: their corresponding ellipses after lensing. This illustrates WL distortions on sources located outside the CCs: at large distances the induced ellipticity is small and comparable to the local shear. Note that, for clarity, the sources here have no intrinsic ellipticities, whereas in real data galaxies exhibit random intrinsic shapes.

Sources outside the CCs of massive lenses are not in supercritical regions, and thus the usual SL effects do not occur. However, the lens potential, albeit weaker, still distorts these sources. The resulting small shape distortions, taken individually, carry little information. Moreover, we cannot tell at first glance whether an observed elliptical galaxy is intrinsically elliptical or has been weakly lensed. Massive lenses, particularly galaxy clusters, the most massive known gravitational lenses (Zitrin et al. 2009; Acebron et al. 2020), imprint a coherent WL signal on thousands of background sources. By stacking these effects, we can average down the noise due to the sources' intrinsic ellipticities and recover the shear field, which constrains the lens mass distribution,

predominantly in the outskirts (the inner parts are affected by SL, and lens members contaminate the WL signal).

If we take a large number of background galaxies¹⁷ and assume that their intrinsic ellipticities are randomly oriented, the mean intrinsic ellipticity vanishes in the absence of lensing. Since the individual shapes and sizes can be measured from the data, this programme is feasible observationally.

Returning to Fig. 3.7, a circular source is mapped, at first order, according to the Jacobian matrix¹⁸. A weakly lensed circular source of radius r is transformed into an ellipse with semi-axes

$$\frac{r}{1 - \kappa - \gamma}, \quad \text{and} \quad \frac{r}{1 - \kappa + \gamma}. \quad (3.104)$$

If we assume $\kappa, \gamma \ll 1$, we can then define the ellipticity

$$\varepsilon = \frac{a - b}{a + b} = \frac{2\gamma}{2(1 - \kappa)} = \frac{\gamma}{1 - \kappa} \approx \gamma, \quad (3.105)$$

with a and b the semi-axes of the ellipse, given by Eq. 3.104¹⁹. Hence, in the WL regime, the (observed) ellipticity directly traces the shear. Writing the ellipticity in components,

$$\begin{aligned} \varepsilon_1 &= \varepsilon \cos 2\phi, \\ \varepsilon_2 &= \varepsilon \sin 2\phi, \end{aligned} \quad (3.106)$$

where ϕ is the position angle of the ellipse.

If the intrinsic ellipticity is ε_s , the total (observed) ellipticity in the WL regime is

$$\varepsilon_{\text{tot}} = \varepsilon_s + \gamma = \varepsilon_s \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \quad (3.107)$$

¹⁷Typically, ground-based observations yield ~ 30 – 40 galaxies per arcmin², while deep space-based imaging can reach ~ 100 galaxies per arcmin².

¹⁸Flexion may be relevant in some cases, but we restrict ourselves here to the linear mapping by the Jacobian.

¹⁹Note that a and b are used for the major and minor semi-axis of the ellipse, respectively. The radii given by Eq. 3.104, represent the stretch in the tangential and radial directions, with respect to the CC, respectively. Near a radial-CC the radial-stretched radius would be assigned to a , and the tangential to b . Near a tangential-CC, these roles are swapped.

Averaging over many sources, the intrinsic term drops to zero and

$$\langle \varepsilon \rangle = \langle \gamma \rangle. \quad (3.108)$$

Thus, by reconstructing the average shear at many sky positions, one can build a shear map and from it infer the projected mass distribution.

As introduced earlier,

$$\begin{aligned} \kappa &= \frac{1}{2}(\Psi_{11} + \Psi_{22}), \\ \gamma_1 &= \frac{1}{2}(\Psi_{11} - \Psi_{22}), \\ \gamma_2 &= \Psi_{12}, \end{aligned} \quad (3.109)$$

where subscripts denote partial derivatives. In Fourier space, Eq. (3.109) becomes

$$\begin{aligned} \hat{\kappa} &= -\frac{1}{2}(k_1^2 + k_2^2)\hat{\Psi}, \\ \hat{\gamma}_1 &= -\frac{1}{2}(k_1^2 - k_2^2)\hat{\Psi}, \\ \hat{\gamma}_2 &= k_1 k_2 \hat{\Psi}, \end{aligned} \quad (3.110)$$

where “ $\hat{}$ ” denote Fourier transforms and $\mathbf{k} = (k_1, k_2)$ is the wave vector conjugate to $\boldsymbol{\theta}$. Eliminating $\hat{\Psi}$, we write

$$\begin{bmatrix} k^{-2} \begin{pmatrix} k_1^2 - k_2^2 \\ 2k_1 k_2 \end{pmatrix} \end{bmatrix} \begin{bmatrix} k^{-2} \begin{pmatrix} k_1^2 - k_2^2 & 2k_1 k_2 \end{pmatrix} \end{bmatrix} = 1, \quad (3.111)$$

writing

$$\begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2} \begin{pmatrix} k_1^2 - k_2^2 \\ 2k_1 k_2 \end{pmatrix} \hat{\kappa}, \quad (3.112)$$

so that

$$\hat{\kappa} = k^{-2} \begin{pmatrix} k_1^2 - k_2^2 & 2k_1 k_2 \end{pmatrix} \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} = k^{-2} \left[(k_1^2 - k_2^2) \hat{\gamma}_1 + 2k_1 k_2 \hat{\gamma}_2 \right]. \quad (3.113)$$

Using the convolution theorem,

$$(F_1 * F_2) = \hat{F}_1 \hat{F}_2, \quad (3.114)$$

and taking the inverse Fourier transform, we obtain

$$\kappa(\boldsymbol{\theta}) = \frac{1}{\pi} \int [D_1(\boldsymbol{\theta} - \boldsymbol{\theta}')\gamma_1 + D_2(\boldsymbol{\theta} - \boldsymbol{\theta}')\gamma_2] d\boldsymbol{\theta}', \quad (3.115)$$

where the auxiliary kernels are

$$\begin{aligned} D_1(\boldsymbol{\theta}) &= \frac{\theta_2^2 - \theta_1^2}{\theta^4}, \\ D_2(\boldsymbol{\theta}) &= \frac{2\theta_1\theta_2}{\theta^4}. \end{aligned} \quad (3.116)$$

Hence, there is a direct way to infer the mass distribution from the measured shear field in the WL regime (i.e., in the lens outskirts).

If we wish to be more rigorous, note that galaxy ellipticities measure the reduced shear g , not γ :

$$\langle \varepsilon \rangle = \langle g \rangle = \left\langle \frac{\gamma}{1 - \kappa} \right\rangle. \quad (3.117)$$

Nonetheless, we can still use Eq. (3.115) by substituting $\gamma \rightarrow g(1 - \kappa)$:

$$\kappa(\boldsymbol{\theta}) = \frac{1}{\pi} \int [D_1(\boldsymbol{\theta} - \boldsymbol{\theta}')g_1(1 - \kappa) + D_2(\boldsymbol{\theta} - \boldsymbol{\theta}')g_2(1 - \kappa)] d\boldsymbol{\theta}'. \quad (3.118)$$

The mass-sheet degeneracy also affects WL studies. The transformation in Eq. 3.103 preserves shapes while changing sizes. By adding information on galaxy sizes (i.e., magnification), one can use the inverse magnification to help break this degeneracy. In the WL regime,

$$R = \frac{1}{\mu} = (1 - \kappa)^2 - \gamma^2 \approx 1 - 2\kappa, \quad (3.119)$$

so we can define a χ^2 to retrieve the mass distribution via minimisation:

$$\chi^2 = \sum_{i=0}^{N_{\text{pix}}} \left\{ \frac{[\gamma_{1,i} - \gamma_{1,i}(\Psi)]^2 + [\gamma_{2,i} - \gamma_{2,i}(\Psi)]^2}{2\sigma_{\gamma,i}^2} + \frac{[R - R(\Psi)]^2}{2\sigma_{R,i}^2} \right\}, \quad (3.120)$$

where quantities in parentheses are model predictions, those without are observations, and the σ 's are measurement uncertainties. Once R is included, χ^2 is no longer invariant under the mass-sheet transformation, and the minimum (within uncertainties) corresponds to the true mass distribution.

The formalism above applies to massive lenses in front of a population of background sources. However, for very high-redshift sources, all intervening mass along the line of sight contributes to lensing. Measuring the final, lensed picture of these sources allows us to extract information about the matter distribution integrated along the path, which carries crucial cosmological information. For instance, cosmic shear refers to the WL of galaxies by the LSS in front of them (see [Kilbinger 2015](#) for a recent review). Another key example is the WL of the CMB, which broadens the distribution of temperature anisotropy scales, damps the acoustic peaks, and, critically, induces a leakage of *E*-modes into *B*-modes. Correcting for this effect is essential because the last-scattering surface is not expected to produce primordial *B*-modes; thus, their detection would constitute smoking-gun evidence for inflation (see [Lewis & Challinor 2006](#) for a comprehensive review). These two probes can also be cross-correlated to further increase their constraining power.

3.5 Microlensing

Let us consider lensing by a point-like lens, as shown in Fig. 3.6. If the source lies within the Einstein radius (the CC of that system), the angular separation between images is $\sim 2\theta_E$, scaling with the square root of the lens mass. Typically, lenses lighter than $\sim 10^6 M_\odot$ in our Galaxy produce Einstein radii smaller than a milli-arcsecond²⁰. For reference, the short-wavelength filters on JWST/NIRCam have a pixel scale of ~ 31 milli-arcseconds. Thus, the multiple images are unresolved, and a single point-like image with the combined flux is detected. In other words, ML is essentially SL in the limit where multiple images cannot be resolved. In the following subsections, we briefly review different types of ML.

3.5.1 Microlenses in our neighbourhood

Stars in the Milky Way, or its satellites, can act as lenses of background stars in the same haloes or in nearby galaxies. Here, the distances are typically of order $\mathcal{O}(\text{kpc})$, so the curvature of the background metric can be neglected and $D_s = D_d + D_{ds}$. Given the typical stellar sizes, both source and lens can be safely treated as point-like. In this

²⁰Recall that the Einstein radius depends not only on the lens mass but also on the geometry (distances) of the system.

regime, the axially symmetric deflection angle is

$$\hat{\alpha} = \frac{4GM}{c^2 b}, \quad (3.121)$$

where M is the mass of the lensing star and b the impact parameter, i.e., the angular separation between source and lens. If b is smaller than the lens stellar radius, the light-rays are absorbed. The peculiar velocities of the Earth, lens, and source can be combined into an effective velocity v_{eff} that describes the relative motion of the source with respect to a fixed caustic network. Since we do not resolve the multiple images, the only observable is the combined flux. The typical timescale over which the magnification varies is then of order days to weeks. An unlensed star approaches the lens caustic, its total magnification rises to a maximum, and then decays as it moves away. This flux variation is measured through the light curve (see Fig. 3.14). The

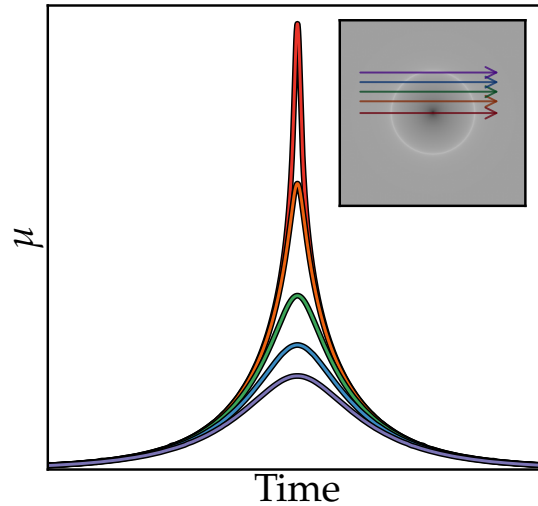


FIGURE 3.14: Light curves of point sources microlensed by a point-like lens. The inset shows the magnification map on the lens plane and several source trajectories. The corresponding magnification curves (proportional to the observed flux) are shown in the main panel with matching colours.

characteristic crossing time (Einstein time) is

$$t_E = \frac{D_d \theta_E}{v_{\text{eff}}}, \quad (3.122)$$

i.e., the time taken by the source to traverse an angular distance θ_E on the lens plane. The (point-source, point-lens) light curve can be described by four parameters: the

unlensed flux L , the time of closest approach t_0 , the dimensionless closest approach y_0 , and the crossing time t_E . The unlensed flux can be measured before or after the event (though the source may be undetectable without lensing). The maximum is reached at t_0 and, in principle, the distance of closest approach can be estimated if the lens position is known (unusual). The parameter t_E contains the physical information of the system (distances, lens mass, v_{eff}). Assuming D_s is known, t_E is degenerate in three unknowns: D_d , M , and v_{eff} . Thus, a single event alone provides limited constraining power on each of them. Moreover, if the source does not outshine the lens, the lens flux L_d enters as an additional parameter, further complicating the inference.

While a single ML event is weakly informative, large samples enable powerful statistical studies of lens populations. Consider an ensemble of microlenses in our Galaxy and an ensemble of sources in, e.g., the Magellanic Clouds. The number of detected events depends on the abundance of sources, microlenses, and the lens masses (more massive lenses yield longer events). In this picture, the sources are background stars and the lenses are compact objects in the range $10^{-6} \lesssim M_{\text{CO}}/M_{\odot} \lesssim 10^6$. Lighter lenses produce too-short, too-faint events for current facilities; heavier ones produce resolvable multiple images (SL). The ML optical depth (the probability that a source is being microlensed) over a solid angle $\delta\Omega$ is

$$\tau = \frac{1}{\delta\Omega} \int n(D_d) \pi \theta_E^2 dV, \quad (3.123)$$

where $n(D_d)$ is the microlens number density and $\pi \theta_E^2$ acts as a cross-section. The volume element is $dV = D_d^2 \delta\Omega dD_d$. Using $\theta_E^2 \propto (4GM/c^2) D_{\text{ds}}/(D_d D_s)$, Eq. (3.123) becomes

$$\begin{aligned} \tau &= \frac{1}{\delta\Omega} \int_0^{D_d} D_d^2 \delta\Omega n(D_d) \pi \frac{4GM}{c^2} \frac{D_{\text{ds}}}{D_d D_s} dD_d \\ &= \frac{4\pi G}{c^2} \int_0^{D_d} \frac{D_{\text{ds}}}{D_d D_s} \rho(D_d) dD_d. \end{aligned} \quad (3.124)$$

where $\rho(D_d) = M n(D_d)$ is the microlens mass density (assuming a single mass for simplicity). On kpc scales, we can assume $D_s = D_d + D_{\text{ds}}$. Introducing $x \equiv D_d/D_s$, we obtain

$$\tau = \frac{4\pi G}{c^2} D_s^2 \int_0^1 x(1-x) \rho(D_s x) dx. \quad (3.125)$$

If the microlens density is approximately constant along the line of sight, the integral yields

$$\tau \approx \frac{2}{3} \frac{\pi G}{c^2} \rho D_s^2. \quad (3.126)$$

Continuing from Eq. 3.126, let us estimate ρ in terms of the Galactic rotation curve. For a galaxy with circular velocity v_{rot} at radius r ,

$$v_{\text{rot}}^2 \approx \frac{GM_{\text{gal}}}{r}, \quad (3.127)$$

so, assuming (for an order-of-magnitude estimate) that the halo is entirely made of compact objects and that the mean density within r is

$$\rho \approx \frac{3}{4} \frac{M_{\text{gal}}}{\pi r^3} = \frac{3}{4\pi G} \left(\frac{v_{\text{rot}}}{r} \right)^2. \quad (3.128)$$

Plugging this into Eq. 3.126 we obtain

$$\tau \approx \frac{1}{2} \left(\frac{v_{\text{rot}}}{rc} \right)^2 D_s^2. \quad (3.129)$$

For the Milky Way, taking $r \sim D_s$ and $v_{\text{rot}} \sim 200 \text{ km s}^{-1}$, one finds $\tau \sim 10^{-6}$: roughly one in a million stars is, at any given time, being microlensed.

The probability to detect an event depends as well on its characteristic duration. For a microlens at $D_d \sim 10 \text{ kpc}$ and an effective transverse speed $v_{\text{eff}} \sim 200 \text{ km s}^{-1}$, the Einstein time-scale is

$$t_E \approx 0.2 \text{ yr} \left(\frac{M}{M_\odot} \right)^{1/2}. \quad (3.130)$$

In practice, source trajectories are neither perpendicular to the caustics nor uniform in speed; a broad distribution of t_E is therefore expected. Large-area, high-cadence surveys such as EROS (Tisserand et al. 2007) and OGLE (Mróz et al. 2017) have exploited these facts, small optical depth but measurable time-scales, to place statistical constraints on the abundance and mass spectrum of compact objects in the Milky Way halo.

In summary, a single Galactic ML light curve strongly constrains the combination of (M, D_d, v_{eff}) through t_E , but is intrinsically degenerate. Statistical ensembles of events, however, allow one to infer (or constrain) the population properties of compact objects along the line of sight.

3.5.2 Binary lenses

Up to now we have considered ML by a single point mass. In reality, many potential lenses belong to binary (or higher-multiplicity) systems, which break the axial symmetry and lead to more complex caustic structures and light curves.

The total deflection angle at a position ξ on the lens plane is the linear sum of the deflections produced by each component:

$$\hat{\alpha}_{\text{tot}} = \hat{\alpha}_A + \hat{\alpha}_B = \frac{4G}{c^2} \left[\frac{M_A}{|\xi - \xi_A|^2} (\xi - \xi_A) + \frac{M_B}{|\xi - \xi_B|^2} (\xi - \xi_B) \right], \quad (3.131)$$

where M_A and M_B are the lens masses and ξ_A, ξ_B their projected positions on the lens plane. Note that this can be generalised an N number of microlenses.

The main parameters describing a binary-lens system are:

- The mass ratio $q = M_A/M_B$.
- The projected separation d between the lenses, often expressed in units of the Einstein radius of the total mass $M_{\text{tot}} = M_A + M_B$.
- The trajectory of the source relative to the line joining the two lenses, given by the angle ϕ .
- The orbital motion of the binary: if the rotation timescale is comparable to the event duration, the caustic network evolves as the source crosses it, distorting the light curve.

Fig. 3.15 illustrates the caustics and CCs of a binary lens with equal masses for varying separations. When the lenses are far apart, each behaves nearly independently. As d decreases, the caustics merge and form intricate shapes. In the limit $d \rightarrow 0$, the binary behaves effectively as a single lens of mass M_{tot} .

The corresponding ML light curves depend strongly on the source trajectory. Fig. 3.16 shows a set of example trajectories and the resulting light curves for the configuration in the third column of Fig. 3.15.

To complicate the science case even more, the source can also be a binary system, then each source would have different crossings, and intrinsic luminosities, therefore differ-

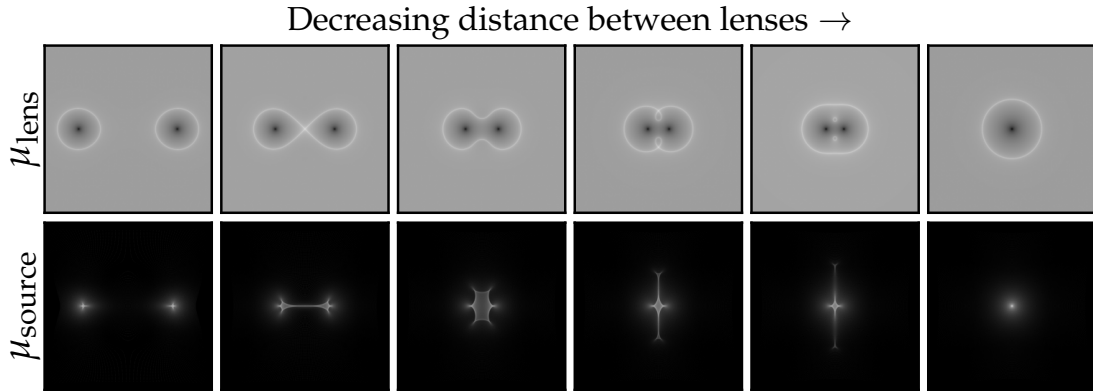


FIGURE 3.15: Critical curves (top row) and caustics (bottom row) of a binary lens composed of two equal-mass point lenses, as a function of their projected separation d (decreasing from left to right). For wide separations, the lenses act independently; at intermediate d , the caustics merge into a single extended shape; for close pairs, inner CCs appear with changed parity inside. In the limit when the separation is almost zero, with respect to the scale of the source, both microlenses behave effectively as a microlens of mass $M_{\text{tot}} = M_A + M_B$, in this case $M_{\text{tot}} = 2M$. The position of the lenses correspond to the minima in the top row panels, corresponding to the black circles in the magnification map. Note that the point-like caustic of the isolated point-like microlens, now shows cusps due to the perturbation of its companion.

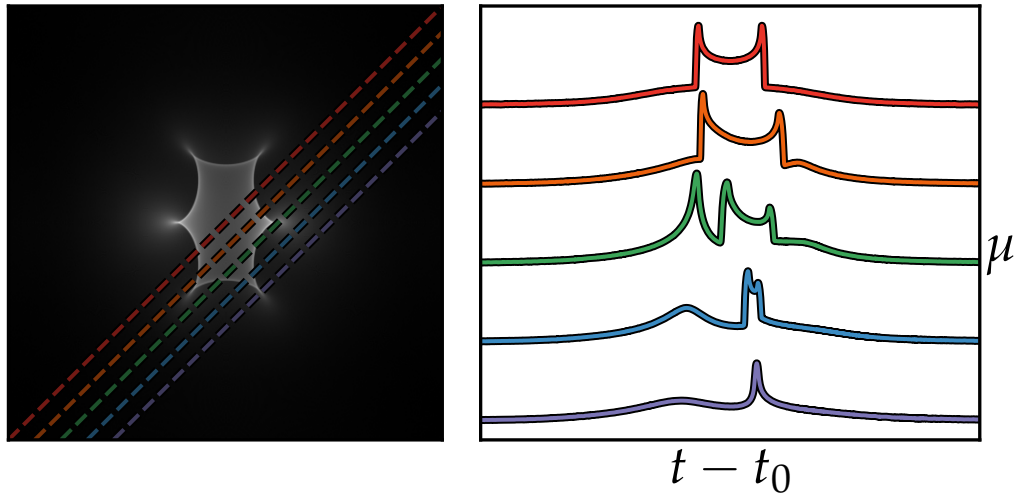


FIGURE 3.16: Microlensing light curves (right panel) corresponding to different source trajectories (left panel) across the caustics of a binary lens with $q = 1$. The curves are arbitrarily normalised for visual clarity. This lens configuration corresponds to the third column of Fig. 3.15.

ent light curves, while we measure their combine effect. For this a multi-wavelength analysis can help disentangle the problem. Extra parameters arise, like the separation between the binary source, their rotation parameters, or each luminosity at different wavelengths.

When extended sources are considered, the magnification is finite, as the point-like divergence at the caustic is smoothed by the source surface brightness profile. For a source of radius r_s , the peak magnification scales as (Miralda-Escude 1991)

$$\mu_{\max} \propto 1/\sqrt{r_s}, \quad (3.132)$$

with the maximum reached when the source centre is about $0.65 r_s$ away from the caustic (Miralda-Escude 1991). Fig. 3.17 illustrates this effect for different source sizes.

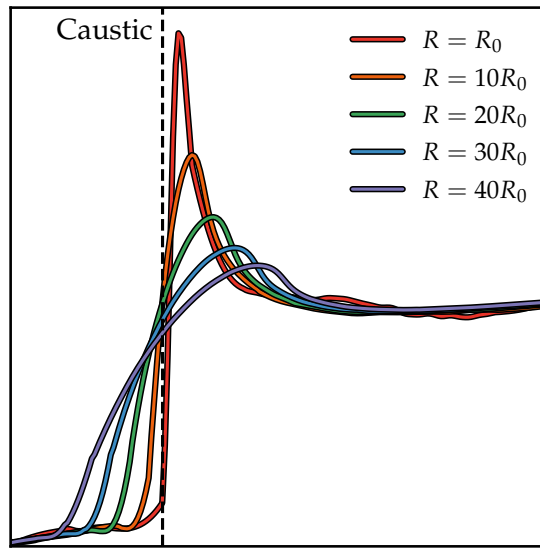


FIGURE 3.17: Effect of source size on ML peaks. Smaller sources yield higher and sharper peaks, while larger sources produce broader and lower maxima. In the limit $r_s \rightarrow 0$, the peak magnification diverges.

3.5.3 Microlenses in external fields

So far we have focused on microlenses in the Milky Way (or its satellites), where the simple Euclidean relation $D_s = D_d + D_{ds}$ holds and the macro-environment can usually be ignored. At cosmological distances, however, microlenses often live inside a macro-lens (a galaxy or galaxy cluster capable of producing resolvable strong-lensing multiple images). In that case, the small-scale deflections produced by microlenses are superposed on top of the large-scale deflection field of the macro-lens, and the resulting magnifications can reach $\mu \sim 10^2\text{--}10^4$ (and occasionally higher), giving rise to extragalactic ML.

A convenient way to account for this is to split the convergence into two parts:

$$\kappa'(\boldsymbol{\theta}) \equiv \kappa_{\text{macro}}(\boldsymbol{\theta}) - \kappa_{\star}(\boldsymbol{\theta}), \quad (3.133)$$

where κ_{macro} is the surface density inferred from the macro model (e.g., from SL constraints), and $\kappa_{\star}$ is the fraction contributed by compact objects. The macro deflection field is then computed from κ' (plus the macro shear γ_{macro}), while the microlenses are added explicitly as point masses whose combined surface density equals $\kappa_{\star}$. This avoids double-counting the mass in stars. Since the microlenses are (to first order) randomly oriented, their net large-scale shear averages to zero, so the macro shear is not modified.

Fig. 3.19 illustrates the micro-CCs produced by microlenses embedded in a macro potential near a macro CC. On the positive-parity side, the micro-CC is stretched tangentially to the macro-CC; on the negative-parity side, it is elongated along the macro-CC and typically splits into two disconnected loops with a demagnified region in between. As the microlens approaches the macro-CC, the micro-CC inflates and becomes increasingly asymmetric: the part closer to the macro-CC is pulled towards it. In effect, the local macro-magnification behaves like an “effective mass boost” for the microlens. Microlenses sufficiently close to the macro-CC can also perturb its position, re-tracing the regions of highest macro-magnification (Oguri et al. 2018; Diego et al. 2018).

Originally, this framework was developed to interpret flux-ratio anomalies and uncorrelated variability in strongly lensed QSOs (e.g., Schmidt & Wambsganss 2010; Vernardos et al. 2024; Mediavilla et al. 2024). QSOs are intrinsically variable, but after correcting for the macro time delays and magnifications, residual, uncorrelated variability between images is attributable to ML. Moreover, since QSO accretion discs are hotter (bluer) in their inner, smaller regions, the effective source size depends on wavelength, leading to chromatic ML. This chromaticity is generally absent for single stars (whose SED originates in a single photosphere), but can arise in binaries if one component lies closer to a caustic (Zheng et al. 2025).

A major leap came with the discovery of highly magnified individual stars at cosmological distances, starting with Icarus (Kelly et al. 2018), followed by many more cases (Rodney et al. 2018; Chen et al. 2019; Diego et al. 2022, 2023; Fudamoto et al. 2025). Early analytic estimates suggested extremely high peak magnifications ($\mu \sim 10^6$) but

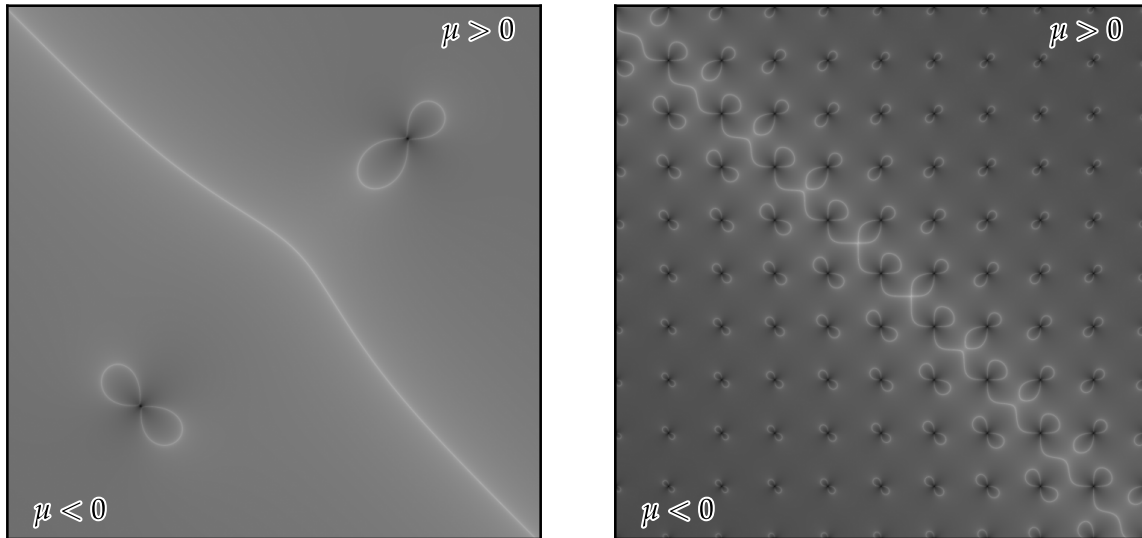


FIGURE 3.18: Critical curves of microlenses near the CC of a macro-halo. Positive parity is on the top-right part of each panel, bisected by the macro-CC, and negative is on the bottom-left. The left panel shows a larger microlens near the CC, on each parity side, while the right panels shows an equispaced grid of smaller microlenses near the same macro-CC. The negative parity micro-CC is elongated along the direction of the macro-CC, while the positive micro-CC is stretched in the tangential direction to the macro-CC. Note how the resulting micro-CC embedded in the macro-potential is similar to the resulting CCs of two near point-like lenses (See Fig. 3.15, second column). The closer the microlens is to the macro-CC, the larger the micro-CCs, thus the effect of the macro-magnification (distance to the unperturbed macro-CC) is that equivalent to an increment in mass of the microlenses.

FIGURE 3.19: Critical curves of microlenses located close to a macro-CC. The positive-parity side lies in the upper-right of each panel, separated from the negative-parity side (lower-left) by the macro-CC. Left: a single, relatively massive microlens placed on each side of the macro-CC. Right: an equispaced grid of lower-mass microlenses. On the positive-parity side, micro-CCs are stretched tangentially; on the negative-parity side, they elongate along the macro-CC and split into two loops with a demagnified region between them.

very low rates (one per cluster per century; [Miralda-Escude 1991](#)). Subsequent work showed that the stellar microlens population near macro-CCs both increases the rate and reduces the typical peak magnification to $\mu \sim 10^3\text{--}10^4$. Today, dozens of events are known thanks to HST and JWST. A recent work on the Dragon arc found more than 40 possible microlensed stars [Fudamoto et al. \(2025\)](#).

These stellar ML events are transient: as the source (or the micro-caustic network) moves, the observed flux changes on timescales of days to months. A practical detection strategy is therefore to difference multi-epoch images taken under comparable

conditions; the positions of microlensed stars appear as positive (or negative) residuals depending on whether the source brightened (or faded) between epochs (see [Fudamoto et al. 2025](#) for a detailed methodology).

Beyond enabling the study of stellar populations at high redshift (including potential Pop. III candidates at $z \sim 10\text{--}12$), extragalactic ML provides a novel constraint on the compact-object content of lenses. The expected surface density of baryonic microlenses (stars and remnants) can be estimated independently; any suggested excess on the microlens population may then be attributable to a population of dark compact objects (e.g., PBHs), allowing constraints on compact DM scenarios ([Oguri et al. 2018](#)). Moreover, local deviations from the smooth macro model can be induced by dark-matter sub-haloes, as already hinted in systems such as Mothra ([Diego et al. 2023](#)) and Godzilla ([Diego et al. 2022](#)).

Once we detect many events and their corresponding magnifications, we need to determine how likely they are given a set of macro- and micro-lens model parameters. If, for a given parameter set, we can compute the magnification probability distribution function (PDF), we can compare the observed events with this PDF and fit for the best parameters. This task, however, is highly challenging and computationally expensive, as it requires high-resolution simulations for each parameter set and over sufficiently large fields of view to ensure reliable statistics. This difficulty arises from the non-linearity of the lens equation, since $\mu(\theta)$ has no analytical expression.

To address this problem, we developed a statistical description of the magnification PDFs over a wide parameter space, based on high-resolution simulations of extragalactic ML. This effort led to the creation of a publicly available code for the fast generation of accurate magnification PDFs, `M_SMiLe`²¹, presented in the first paper of this thesis ([Palencia et al. 2024](#)). We then tested the expected number of microlensed events per observation in a strongly lensed arc, dubbed Warhol, located behind the massive galaxy cluster MACS-0416. We derived the expected number of events under different observational configurations, finding good agreement with previous detections. We also explored the potential contribution of compact DM, showing that while the total number of detected events remains essentially unchanged close to the cluster core, where intracluster stars dominate the convergence, the spatial distribution of microlensed events, which we analysed for the first time, is in fact modified by varying

²¹https://github.com/ChemaPalencia/M_SMiLe

fractions of the total convergence in microlenses. These results were published in the second paper of this thesis (Palencia, J. M. et al. 2025).

We will conclude this chapter by emphasising the power of lensing to detect and study DM. Since lensing is caused by the total matter distribution, it is insensitive to the nature of the matter; DM affects lensing in the same way as ordinary matter. With this in mind, lensing can be used to weigh lenses, study the LSS of the Universe, and probe the small-scale matter distribution within lenses. All of this can be compared to the expectations from baryons alone or from various realisations of DM models, enabling their discrimination.

Moreover, its unique magnification effects offer an unparalleled opportunity to observe faint background sources with higher angular resolution, beyond the current capabilities of state-of-the-art instruments. These detections allow us to perform further tests of cosmological model predictions, as well as to investigate astrophysical processes across different epochs of the Universe. We can therefore confidently state that lensing stands as a unique phenomenon, serving as a powerful tool for modern, precise, and meaningful cosmology and astrophysics.

Part II

Main thesis contributions

Chapter 4

Included publications

This chapter presents a compendium of the publications included in this thesis, all of which were carried out in the context of DM detection and characterisation through SL. The first and second papers explore the impact of compact microlenses near galaxy cluster CCs and their potential for detecting compact DM objects, such as PBHs, as well as milli-lenses arising from DM sub haloes. The third paper shifts the framework but not the underlying goal: here, we investigate the detectability of FDM in galaxy lenses by examining its distinctive magnification signatures inside radial CCs, in contrast to the effects expected from particle-like CDM candidates, such as WIMPs.

This compilation contains the following publications:

P.I: Palencia, J.M.; Diego, J.M.; Kavanagh, B. J.; & Martínez-Arrizabalaga, J. Statistics of magnification for extremely lensed high redshift stars. *Astronomy & Astrophysics, Volume 687, A81*, July 2024.

P.II: Palencia, J.M.; Diego, J.M.; Dai, L.; Pascale, M.; Windhorst, R.; Koekemoer, A. M.; Li, Sung Kei; Kavanagh, B. J.; Sun, Fengwu; Alfred, Amruth; Meena, Ashish K.; Broadhurst, Thomas J.; Kelly, Patrick L.; Perera Derek; Williams, Hayley; & Zitrin, Adi.. Microlensing at cosmological distances: Event rate predictions in the Warhol arc of MACS 0416. *Astronomy & Astrophysics, Volume 699, A295*, July 2025.

P.III: Palencia, J.M.; Morilla, Paloma.; Li, Sung Kei; Diego, J.M.; Alfred, Amruth; Broadhurst, Thomas J.; Kavanagh, B. J.; & Lim, Jeremy. Signatures of Fuzzy Dark

Matter Inside Radial Critical Curves . *Astronomy & Astrophysics* (Accepted for publication), July 2025.

In P.I, we address one of the key challenges in extragalactic microlensing: the computational cost of accurately modelling the probability of extreme magnifications ($\mu > 1000$) for stars embedded in strongly lensed galaxies. Such high magnification events, have opened a new observational window to detect individual stars at cosmological distances ($z > 1$). These events not only allow for unprecedented studies of stellar populations at high redshift but also provide a unique probe of the small-scale structure of the lens plane, which is sensitive to the presence of DM substructure and compact DM candidates.

We carried out state-of-the-art large-scale microlensing simulations under various macro-model magnification and microlens surface density scenarios. From these simulations, we derived analytical scaling relations that accurately reproduce the magnification probability distribution functions (PDFs) in the regime of extreme magnifications, effectively bypassing the need for computationally expensive ray-tracing simulations. These analytical tools are essential for efficiently interpreting the growing number of detections of highly magnified stars, enabling statistical analyses of microlensing populations and constraining the contribution of compact dark matter to the total convergence.

This work thus provides a crucial step forward in building fast and reliable models to interpret extragalactic microlensing observations, offering direct implications for both the detection of individual stars at cosmic distances and for testing DM scenarios through their impact on microlensing statistics.

Building on the analytical framework and scaling relations developed in the P.I, the work in P.II, applies our methodology to real observational data, demonstrating its effectiveness and efficiency in predicting and interpreting microlensing events. We focus on the Warhol arc (at $z = 0.94$) behind the MACS 0416 galaxy cluster (at $z = 0.396$), where more than a dozen highly magnified stars have been detected thanks to microlensing by intracluster stars near the critical curves. Using JWST-NIRCam observations across eight filters, combined with three HST-ACS filters, we fit the spatially resolved spectral energy distribution (SED) of the arc, enabling accurate 2D predictions of the number and properties of microlensed stars.

With this approach, we can generate wavelength-dependent 2D maps of microlensing expectations against limiting magnitudes for well-resolved arcs, providing a powerful tool to optimise and interpret cadenced observational campaigns with JWST and HST. These predictions are directly comparable to current detections, validating the analytical approximations derived in the first study. Furthermore, the method enables us to probe the initial mass function (IMF) of intracluster stars and investigate the impact of DM substructure on microlensing statistics. By confronting our model with observed event distributions, we can test the presence of compact DM candidates or sub-haloes, extending the role of microlensing as a probe of both baryonic and non-baryonic small-scale structures.

In P.III, we shift our focus from microlensing by compact objects to the impact of FDM on the strong lensing regime of galaxy-scale halos, while maintaining the goal of using gravitational lensing as a probe of DM. Using high-resolution simulations, we explore the magnification patterns and CC structures generated by FDM halos across a range of axion masses and halo configurations. We find that FDM, due to its wave-like nature, produces distinctive features such as enhanced central magnifications and the emergence of secondary radial CC; signatures that are difficult to reproduce with standard particle-CDM, even when accounting for substructure.

Our results show that for axion masses in the range $m_{\psi} \sim 10^{-22} - 10^{-21}$ eV, galaxy-mass halos exhibit magnification distributions that differ significantly from those predicted by particle-CDM models. These unique lensing imprints offer a new observational avenue to discriminate between DM models. In particular, highly magnified compact sources, such as quasars, or SNe, represent ideal laboratories for testing the presence of FDM.

Statistics of Magnification for Extremely Lensed High-Redshift Stars

J.M. Palencia, J.M. Diego, B. J. Kavanagh, & Martínez-Arrizabalaga, J.

A&A, 687, A81 (2024)

<https://doi.org/10.1051/0004-6361/202347492>

Statistics of magnification for extremely lensed high redshift stars[★]

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Received 18 July 2023 / Accepted 2 November 2023

ABSTRACT

Micro-lensing of stars in strongly lensed galaxies can lead to temporary extreme magnification factors ($\mu > 1000$), enabling their detection at high redshifts. Following the discovery of Icarus, several stars at cosmological distances ($z > 1$) have been observed using the *Hubble* Space Telescope (HST) and the *James Webb* Space Telescope (JWST). This emerging field of gravitational lensing holds promise to study individual high redshift stars. It also offers the opportunity to study the substructure in the lens plane with implications for dark matter models, as more lensed stars are detected and analysed statistically. Due to the computational demands of simulating microlensing at large magnification factors, it is important to develop fast and accurate analytical approximations for the probability of magnification in such extreme scenarios. In this study, we consider different macro-model magnification and microlensing surface mass density scenarios and study how the probability of extreme magnification factors depends on these factors. To achieve this, we created state-of-the-art large simulations of the microlensing effect in these scenarios. Through the analysis of these simulations, we derived analytical scaling relationships that can bypass the need for expensive numerical simulations. Our results are useful to interpret current observations of stars at cosmic distances which are extremely magnified and under the influence of microlenses.

Key words. gravitation – gravitational lensing: strong – gravitational lensing: micro – dark matter

1. Introduction

The discovery of Icarus (Kelly et al. 2018), a strongly magnified star at redshift $z=1.49$, recently opened the door to a new branch of gravitational lensing, the study of distant stars at extreme magnification factors ($\mu > 1000$). Icarus was being magnified by a large factor (of several hundred) by a galaxy cluster. In addition to the magnification from the cluster, a microlens in the cluster (most likely a star) boosted the amplification by another factor $O(10)$, bringing the net magnification to several thousand. This large magnification was maintained for only a few weeks, but it allowed for Icarus to be identified as a varying source, and for it to be monitored further with additional *Hubble* Space Telescope (HST) observations that allowed for its nature to be confirmed. After Icarus, a series of blue luminous stars have been discovered with similar magnification factors with HST (Chen et al. 2019; Kurov et al. 2019; Welch et al. 2022a; Diego et al. 2022; Meena et al. 2023a), culminating with the discovery of Earendel, the farthest star ever observed at $z \approx 6$ (Welch et al. 2022a,b). More recently, the *James Webb* Space Telescope (JWST) has expanded this list with additional blue supergiants (Chen et al. 2022; Meena et al. 2023b), and the first red supergiants, which are the most luminous in the infrared (IR) bands beyond the reach of HST (Diego et al. 2023). Dedicated programmes with HST to search for these extremely magnified stars near cluster critical curves, and undergoing microlensing events, such as

Flashlights (Kelly et al. 2019), have recently proven to be very successful with the discovery of over a dozen new stars at cosmic distances (Kelly et al. 2022). New programmes with JWST, such as the Prime Extragalactic Areas for Reionization and Lensing Science (PEARLS; Windhorst et al. 2023), are taking over, and are finding new examples of extremely magnified stars, specially colder ones ($T < 10\,000$ K). With the number of individual stars beyond redshift $z = 1$ rapidly increasing, we are approaching the point where the statistical significance of such detections may be sufficient to discriminate between different models of the following: (i) the underlying population of luminous stars at cosmic distances, and (ii) the amount of a small-scale substructure in the lens plane that is responsible for the microlensing fluctuations.

Future observations, especially with JWST, will soon increase the number of strongly lensed stars to $O(100)$, allowing for studies that constrain not only the nature of the distant star, but also the intervening population of microlenses. Of particular interest is the study of the most distant lensed stars, since at large redshifts the critical curves of gravitational lenses move outwards, where the role of stellar microlenses is smaller. However, the role of possible compact dark matter (DM) candidates – such as primordial black holes (PBHs) – is, in relative terms, greatest. In interacting clusters, critical curves can form in regions in between the different cluster components, where the stars responsible for the intracluster light have the smallest lensing effects. In addition, if the cluster is relatively at large redshifts ($z > 0.6$), the intracluster medium is still in its early stages of formation resulting in a relative pristine medium (in terms of stellar microlensing contamination), and hence offering

[★] A public code based on our results for analytically computed magnification PDFs can be found here: https://github.com/ChemaPalencia/M_SMiLe.

a unique opportunity to constrain models of compact DM. High redshift galaxies that are strongly lensed produce counter images in regions relatively far away from the centre of galaxy clusters, and where the abundance of stellar microlenses is smallest compared with the contribution from DM. Interacting clusters offer a unique opportunity to constrain models of compact dark matter due to the lower contamination from stellar microlensing in the region between the two clusters, thus offering a clearer view of the DM component. Such arcs already exist in the literature, for instance the giant arc at $z = 2.83$ nicknamed La Flaca, which is strongly magnified by the merging galaxy cluster El Gordo at $z = 0.87$ (Diego et al. 2023). At least one lensed star candidate has been identified in this arc. If confirmed, its interpretation will demand an accurate modelling of the probability of magnification on a wide range of scenarios that involve different contributions to the microlensing signal from compact DM models.

To extract useful information (with cosmological significance) from these events, it is necessary to understand the role that microlenses play in the magnification for objects discovered near the critical curves of gravitational lenses. This has been done so far by means of simulating the lens plane after populating it with microlenses from the gravitational lens. These microlenses include stars and remnants form the lens itself but also sometimes more exotic microlenses such as primordial black holes (Diego et al. 2018). Given the extreme values of the magnification factors, a target-simulated area A in the source plane needs to be simulated first in an area a factor μ times larger in the image plane. Combined with the fact that the source to be resolved (a star) is very small in nature, this usually results in simulations that are very computationally demanding. If one wants to perform a statistical study of the detected stars, a wide range of magnifications and microlens parameters needs to be explored making this task a daunting one if performed through simulations. A more efficient approach is to approximate the probability of magnification by analytical forms that are flexible and accurate enough so they can reproduce a wide range of scenarios. The present work aims to find these analytical forms which will set the basis for future studies on the probability of finding these stars. This probability involves several ingredients such as the number density of luminous stars, the probability of having magnification factors greater than 100 (by a galaxy or galaxy cluster)¹, and the probability of magnification when microlenses are present in a region magnified over a factor 100 by a galaxy or galaxy cluster. This paper hence focusses on this last point, and we leave the much more complex full calculation of the probability of observing a particular type of star and at a given redshift for future work.

One direct application of our work is the study of certain models of DM that could act as microlenses. Among these, primordial black holes (PBHs) have gained significant attention as a potential candidate for DM (see Green & Kavanagh 2021 for a recent review). These black holes are believed to have formed through the gravitational collapse of mass overdensities in the early Universe, predating the epoch of matter-radiation equality. Importantly, PBHs are non-baryonic and did not play a role in primordial nucleosynthesis. PBHs can potentially interact with baryonic matter through two mechanisms. Firstly, the lightest PBHs could undergo evaporation via Hawking radiation, leading to the release of energy and particles. Secondly, the heaviest PBHs could interact with baryons through the transfer

of energy from their accretion disks. PBHs with initial masses $M_{\text{PBH}} \gtrsim 10^{11}$ kg are expected to have completely evaporated by the present time (MacGibbon et al. 2008). On the other hand, massive PBHs ($M_{\text{PBH}} \gtrsim 100 M_{\odot}$) would have left an imprint on the cosmic microwave background (Serpico et al. 2020) or interact with their surroundings heating the medium (Lu et al. 2021). In the range of a few tens of $M_{\odot}$, PBHs could offer a natural explanation for the abundance of massive black hole mergers found by gravitational wave experiments (The LIGO Scientific Collaboration 2023). If PBHs with these masses exist in relatively large numbers, they contribute to the microlensing signal. Current constraints based on microlensing signatures set an upper limit on the abundance of PBHs at the 1% level of the total amount of DM. Future observations of distant lensed stars are expected to lower this upper limit even further.

This paper is organised as follows: in Sect. 2, we describe the basic properties of microlensing embedded in a region of high magnification. In Sect. 3, we introduce the methodology to follow and present the simulation procedure. Section 4 presents the results for isolated lens simulations and a study on how a simple parameter scale in terms of the macro- and micro-model parameters. Section 5 generalises this to a set of N microlenses and presents our numerical simulations. We present in Sect. 6 the modelling of the magnification probability. In Sect. 7, we develop the methodology to constraint compact DM with caustic crossing events (CCEs) using our newly obtained analytical tool. Our results and some prospects on the use of the approximations derived here are discussed in Sect. 8, and we conclude in Sect. 9.

Unless noted otherwise, we adopt a flat cosmology with $\Omega_m = 0.3$, $\Lambda = 0.7$, and $h = 0.7$ ($100 \text{ km s}^{-1} \text{ Mpc}^{-1}$). We use the term macrolens or macromodel when we refer to a lens with the mass of a galaxy or a galaxy cluster. The term microlens or micromodel is used to refer to much smaller lenses with masses comparable to stellar masses. The caustics associated with microlenses are referred to as microcaustics.

2. Simple notions of gravitational lensing

In this section, we provide a brief overview of the formalism of gravitational lensing, a fundamental tool in studying many astrophysical phenomena. We focus specifically on the gravitational lensing effect caused by the perturbation of a set of point-like lenses located in a region of high magnification near a critical curve (or CC). This regions of extreme magnification map into the source plane into similar regions of extreme magnification known as caustics. A source with infinitesimal size has divergent magnification when is placed at the exact position of the caustic. Stars are very small but have a limited size. In this case, the maximum magnification scales as $1/\sqrt{R_*}$, where R_* is the star radius.

As mentioned in the introduction, we refer to the components of the microlensing, the lensing effects caused by the point-like lenses, with the prefix ‘micro-’ (e.g. micro-images, micro-caustics, etc.). In contrast, we use the prefix ‘macro-’ for those effects associated with the galaxy cluster.

The position of a lensed image θ , and its actual position in the sky β are related through the lens equation (Schneider et al. 1992; Narayan & Bartelmann 1996)

$$\beta = \theta - \alpha(\Sigma, \theta), \quad (1)$$

where α is the deflection angle at the position θ produced by a lens of surface mass density $\Sigma(\theta)$. As α is a function of θ ,

¹ In general, all lensed stars involve magnification factors, μ , from galaxies or clusters of galaxies above $\mu = 100$.

Eq. (1) is generally non-linear leading to multiple solutions θ for the same β , and usually forbidding an analytical solution. The deflection angle can be obtained from the derivatives of the effective lensing potential

$$\psi(\theta) = \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}} \frac{2}{c^2} \int \phi(D_{\text{d}}\theta, z) dz, \quad (2)$$

where D_{d} , D_{s} and D_{ds} are the angular diameter distances to the lens, the source, and between the lens and the source, respectively, and ϕ is the Newtonian potential of the lens. The deflection angle is the gradient of ψ with respect to θ

$$\alpha = \nabla_{\theta}\psi, \quad (3)$$

while its Laplacian is related to $\Sigma(\theta)$ as

$$\nabla_{\theta}^2\psi = 2 \frac{D_{\text{d}}D_{\text{ds}}}{D_{\text{s}}} \frac{4\pi G}{c^2} \Sigma(\theta) = 2 \frac{\Sigma(\theta)}{\Sigma_{\text{crit}}} \equiv 2\kappa(\theta), \quad (4)$$

where κ is known as the convergence and is the ratio between the surface mass density and the critical surface mass density, Σ_{crit} .

The lensing distortion can be described by the Jacobian matrix

$$\mathbf{A} \equiv \frac{\partial \beta}{\partial \theta} = \left(\delta_{ij} - \frac{\partial \alpha_i(\theta)}{\partial \theta_j} \right) = \left(\delta_{ij} - \frac{\partial^2 \psi(\theta)}{\partial \theta_i \partial \theta_j} \right) = (\delta_{ij} - \psi_{ij}) = \mathbf{M}^{-1}, \quad (5)$$

which is the inverse of the magnification tensor $\mathbf{M}$. Since the convergence is half of the Laplacian of ψ , we have that

$$\kappa(\theta) = \frac{1}{2}(\psi_{11} + \psi_{22}) = \frac{1}{2}\text{tr} \psi_{ij}. \quad (6)$$

Another important quantity is the shear tensor $\gamma(\theta)$, whose components can be also expressed as linear combinations of ψ_{ij} ,

$$\gamma_1(\theta) = \frac{1}{2}(\psi_{11} - \psi_{22}), \quad (7)$$

$$\gamma_2(\theta) = \psi_{12} = \psi_{21},$$

the modulus of the shear tensor is $\gamma^2 = (\gamma_1^2 + \gamma_2^2)^{1/2}$, where γ is known as the shear. We can rewrite the Jacobian matrix in terms of the convergence and the shear

$$\mathbf{A} = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}. \quad (8)$$

Gravitational lensing preserves the surface brightness of an object while altering its apparent size. The magnification, which represents the ratio between the total observed flux of a lensed image and that of the unaltered image, is determined by the ratio between the solid angles of the image and the original, unlensed source. The CC of a lens system refers to a collection of positions in the lens plane where the magnification diverges. By utilising Eq. (1), we can trace the origins of these critical curves back to the source plane, thereby obtaining their corresponding complementary caustic curves.

Combining Eqs. (5) and (8), the magnification μ is

$$\mu = \mu_t \mu_r = \det \mathbf{M} = \frac{1}{\det \mathbf{A}} = \frac{1}{(1 - \kappa)^2 - \gamma^2}, \quad (9)$$

where μ_t and μ_r are the tangential and radial components of the magnification, $\mu_t^{-1} = 1 - \kappa - \gamma$, and $\mu_r^{-1} = 1 - \kappa + \gamma$. We note that

similar to $\mathbf{A}$ and $\mathbf{M}$, μ is a function of the position θ in the lens plane.

Despite the non-linearity of Eq. (1), the deflection angle remains linear. Thus, we can express α as the sum of the individual components of the lens system. In our case of interest

$$\alpha = \sum_{i=0}^N \alpha_{*,i} + \alpha_{\text{m}} = \alpha_* + \alpha_{\text{m}}, \quad (10)$$

where $\alpha_{*,i}$ is the deflection angle caused by each of the N point-like lenses, α_* is their combined deflection angle, and α_{m} is the contribution from the galaxy cluster. For a point-like lens substituting the corresponding Newtonian potential into Eq. (2), the deflection angle for a lens of mass M_i at the position θ_i is

$$\alpha_{*,i} = \frac{D_{\text{ds}}}{D_{\text{d}}D_{\text{s}}} \frac{4GM_i}{c^2} \frac{\theta - \theta_i}{|\theta - \theta_i|^2}. \quad (11)$$

To obtain α_{m} , we consider a region near the tangential CC of the cluster. By definition a tangential CC forms where $1 - \kappa - \gamma = 0$ and $|1 - \kappa + \gamma| > 0$. For simplicity we assume $\gamma_2 = 0$ and so $\gamma = \gamma_1$. This is equivalent to considering a portion of the CC where the deflection field is aligned in a direction perpendicular to the CC, which is a common situation in many real lenses. The high magnification arises from the tangential component of the macro-magnification $\mu_t \gg \mu_r$. Depending on which side of the CC we consider we might have a positive or negative parity. The convergence and shear values of the macro-lens are given by

$$\gamma_{\text{m}} = \frac{1}{2} \left(\frac{1}{\mu_r} \mp \frac{1}{\mu_t} \right), \quad (12)$$

$$\kappa_{\text{m}} = 1 - \gamma_{\text{m}} \mp \frac{1}{\mu_t},$$

where the minus and plus sign refer to the positive and negative parities respectively. If $(1 - \kappa)^2 - \gamma^2 > 0$, the parity is positive, otherwise if this factor is negative we are in a negative parity region. This subtle change in the form of the convergence and the shear allows for completely different phenomena occurring at each parity. The deflection angle caused by a smooth potential such as that of the macrolens increases linearly as we approach the macro-CC, thus we have that the deflection angles from the macro model in the tangential and radial directions is

$$\alpha_{\text{m},t} \propto (\tilde{\kappa} + \gamma_{\text{m}}), \quad (13)$$

$$\alpha_{\text{m},r} \propto (\tilde{\kappa} - \gamma_{\text{m}}).$$

It is important to note that we have not used the macro-convergence but its modified value $\tilde{\kappa} = \kappa_{\text{m}} - \kappa_*$, taking into account the modification of the point-like lenses on the macropotential (Diego et al. 2018). As the point-like lenses are isotropically (and randomly) distributed they do not contribute a shear, so there is no need for a correction in the macro-shear. Including Eqs. (11) and (13) in Eq. (10), we obtain the total deflection angle, thus allowing us to solve numerically any lens problem related to our given configuration.

3. Numerical simulations

Extreme magnification near a CC gives rise to very crowded source planes. The microlens surface mass density, Σ_* , is effectively re-scaled by the large macromodel magnification μ , resulting in a large effective surface mass density $\Sigma_{\text{eff}} = \mu_t \Sigma_*$. The

value of the effective surface mass density often surpasses the critical density, leading to optically thick lenses where continuous microlensing events occur. Earlier work has relied on expensive simulations to study the statistical properties of the observed flux (Diego et al. 2018; Venumadhav et al. 2017). A recent paper by Meena et al. (2022) presents a new simulation method to circumvent the large computing time required for inverse ray shooting. This method provides a new and faster approach to obtain magnification in the source plane, and thus the light curves of compact sources under microlensing. However, the aim of this work is to obtain statistics of magnification bypassing the simulation step which is the most resource consuming. These two methods are complementary. If only statistics of magnification are required, our approach is faster; however, if light curves are required, our method cannot provide them, while that of Meena et al. (2022) can.

To approximate the changes in flux properties over time for an extremely lensed source, caused by a specific microlens population with surface mass density Σ_* at a given distance from a tangential macro-CC ($\mu_m = \mu_t \mu_r$), a series of simulations must be conducted by varying these parameters. Subsequently, the $d(\text{area})/d\mu$ or probability distribution function of the total magnification needs to be modelled, which will be discussed in more detail in Sect. (6). The model function's parameters will be studied in relation to the physical parameters, establishing their underlying relation. Once this step is completed, it becomes possible to bypass the numerical simulations that currently consume the majority of time in these analysis, thereby eliminating this bottleneck. Previous works have obtained analytical approximations for the magnifications (Deguchi & Watson 1987; Seitz & Schneider 1994; Seitz et al. 1994; Neindorf 2003; Tuntsov et al. 2004). However, these methods fail at large macro-magnifications and high surface mass densities, where this work is focussed. Thus the need to derive analytical methods in this regime.

The computation of magnification in the source plane is accomplished using the inverse ray-shooting technique (Kayser et al. 1986). This method uses the lens equation as shown in Eq. (1) to trace back an image observed in the lens plane to its original position in the source plane. By discretising the lens plane over a sufficiently large area and performing this tracing, a reasonably accurate approximation of the magnification in the source plane can be obtained. The accuracy of this magnification probability with respect to pixel size and other resolution effects is extensively discussed in Appendix A.

In order to utilise the inverse ray-shooting technique effectively, it is necessary to determine the deflection angle at each pixel of the simulated lens plane, considering both the contributions from the micro and macro components. Initially, we compute the deflection angle generated by the microlensing population, denoted as α_* . To accomplish this, we randomly distribute a set of point-like lenses within a circular region with a radius of N_x , where N_x corresponds to the size in pixels of our simulated lens plane in the tangential direction of the macro-magnification. The masses of these lenses are randomly picked from a Spera initial-final mass function (Spera et al. 2015), with the constraint that they do not exceed $1.5 M_\odot$. We only compute the deflection angle from the lenses within a circle of radius N_x . By utilising Eq. (11) and considering the position and mass of each microlens, we can calculate the corresponding deflection angle $\alpha_{*,i}(\theta)$ for each individual microlens. Next, we proceed to calculate the corresponding deflection angle associated with the galaxy cluster $\alpha_m(\theta)$. To begin, we calculate the convergence and shear of the macro-model based on the macro-magnification, as

described in Eq. (12). The specific choice of parity will depend on the simulation being conducted. Subsequently, we proceed to correct the convergence, κ , by considering the influence of the microlenses, that is the effective convergence is $\kappa_{\text{eff}} = \kappa - \kappa_*$. Although this correction may be negligible in cases where the surface mass density of microlenses is small, it becomes crucial in crowded regions characterised by large stellar populations. In such scenarios, the effect of microlenses becomes significant and must be accounted for in the analysis. Using Eqs. (13) and (10), we can proceed to calculate the total deflection angle at each position required for the inverse ray-shooting method. These equations allow us to combine the contributions from the macromodel and the microlenses to obtain the complete deflection angle necessary for the analysis. Since the total deflection angle is the sum of independent contributions from each component, we can exploit this property by utilising the same deflection angle obtained from a set of microlenses with different macromodel configurations. This approach allows us to save both time and computational resources by avoiding redundant calculations of the microlens deflection angles for each specific macromodel configuration. In this study, we conducted simulations using a computational grid consisting of approximately $N_x \approx 3 \times 10^6$ pixels in the tangential direction and $N_y \approx 0.15 \times 10^6$ pixels in the radial direction. This area gets remapped into a region $(3 \times 10^6/\mu_t) \times (0.15 \times 10^6/\mu_r)$ in the source plane. Each pixel in the simulation corresponds to a size of 2 nas. Consequently, the total size of the simulation area amounted to $6 \times 0.3 \text{ mas}^2$, with a lens plane redshift of $z_d = 0.7$ and a source plane redshift of $z_s = 1.3$. For this particular system, the critical surface mass density was estimated to be approximately $\Sigma_{\text{crit}} \approx 3.1 \times 10^3 M_\odot/\text{pc}^2$. We conducted 39 distinct simulations for each parity, where we varied the values of Σ_* , μ_t , and μ_r . As a result, we obtained a range of effective surface mass densities spanning from 0.01 to 66.5 times the critical density. Furthermore, in order to gain a deeper understanding of the statistics related to the negative parity regime and investigate additional effects associated with low effective surface mass densities, we conducted a few additional realisations alongside the main simulations. On average, it takes 32 CPU-hours to simulate the deflection angle for a single microlens of the aforementioned size and resolution that occupies 1.4 TB of disk space. Complex simulations with thousands of microlenses can reach hundreds of thousands of CPU-hours. Detailed explanations of these supplementary simulations can be found in Sect. 6 and Appendix B. As far as we are aware, these state-of-the-art simulations represent the largest scale simulations conducted at this resolution level for the purpose of studying microlensing by compact lenses around a cluster-CC. The substantial size of the simulations provides us with a significant amount of statistical data, enabling a comprehensive analysis of the probability of magnification for background sources across a wide range of values.

To simplify the modelling of magnification probabilities and ease a direct comparison between different simulations, the final magnification is normalised by a factor of $1000/\mu_m$. Unless specifically stated otherwise, this normalisation has been consistently applied throughout our analysis. As a result, a pixel in our simulations with a magnification value of μ will now be represented as $\tilde{\mu} = 1000 \times \mu/\mu_m$. The PDF of the magnification after being re-scaled this way peaks around values close to $\mu = 1000$, but not exactly at this value as we describe below. For ease of notation, we continue to refer to the re-scaled magnification as μ in our discussions.

The inverse ray-shooting technique is a computationally intensive numerical method that relies on the total number of

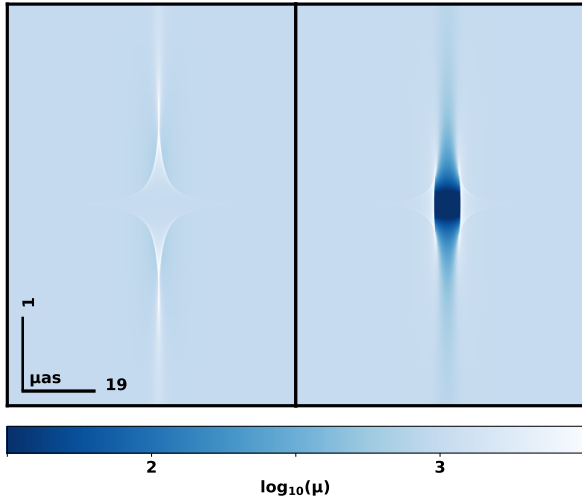


Fig. 1. Magnification in the source plane of a single microlens near a macro-CC where $\mu_m = 1000$. The darkest blue indicates a lower magnification, while the whites show where the magnification diverges (i.e. the micro-caustics). *Left:* single microlens on the positive parity side of the CC. The caustics have the familiar diamond shape. *Right:* same, but in the negative parity region, crossing the macro-CC. Now the shape of the micro-caustics is broken into two distorted triangles with a demagnification area in between, but increasing the maximum achievable magnification at the tips of these triangles. Outside the caustics, the average magnification is that of the macro model μ_m . The vertical direction has been scaled down a factor ~ 10 for better visualisation. The probability of magnification for both examples is shown in Fig. 2.

pixels (a combination of the simulated area and pixel size) and the number of microlenses. Simulating scenarios with high surface mass densities can be exceptionally expensive, necessitating the development of tools that offer an alternative to the costly numerical simulations. The main objective of this work is to develop and present a tool that serves as a shortcut from the lens parameters to the magnification probability. Providing an alternative approach that eliminates the requirement for time-consuming and computationally expensive simulations. By offering this tool, researchers can efficiently estimate the magnification probability based on the lens parameters, streamlining their analyses and reducing the reliance on extensive numerical simulations.

It is crucial to emphasise that despite the development of shortcut tools, the spatial distribution of the micro-caustic network remains an important factor in many analyses. Therefore, numerical simulations continue to play a vital role in understanding and studying the intricate details of the micro-caustic structure.

Given the large size of the simulations, we used a computer cluster. To facilitate the development of this work, a parallelised version of the inverse ray-shooting technique was employed, thus allowing us to fully exploit the resources of the computer cluster ALTAMIRA.

4. Microlenses near a CC

The primary objective of this work is to model the statistical properties of the flux emitted by a background source as it traverses the micro-caustic network created by microlenses, such as stars from the intracluster light (ICL) and compact DM candidates like PBHs, in a region of high magnification near

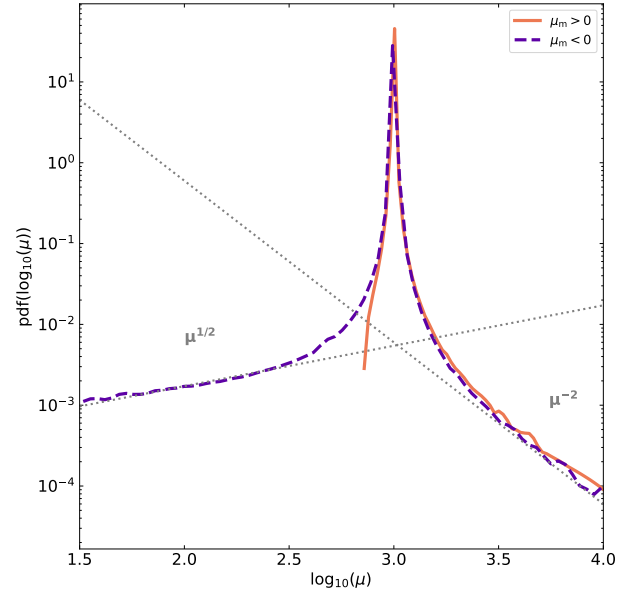


Fig. 2. Probability of magnification (in $\log_{10}$ bins) for the two single lenses. The purple dashed line represents the microlenses on the negative parity side of the macro-CC, while the orange solid line represents the positive parity case on the right and left panels of Fig. 1, respectively. The dotted lines represent the power-laws that always appear for isolated microlenses near macro-CCs, and for N microlenses with low Σ_{eff} . The variation in the value of the μ^{-2} power-law amplitude is shown in Fig. 3.

a CC associated with a galaxy cluster. This problem is inherently complex due to the non-linear nature of the lens equation. To gain insights into the flux statistics arising from the perturbation of the galaxy cluster's smooth potential by numerous microlenses, it is beneficial to analyse the case of an isolated microlens near a CC.

4.1. Single microlens case

This analysis of a single microlens near a CC provides valuable information for scenarios characterised by low optical depths, where a microlens can be treated as an isolated entity. This was the case for Icarus and its counter image Iapyx (Kelly et al. 2018), and generally applies for images not too close to the macro-CC and within regions of low Σ_* ($\Sigma_{\text{eff}} \ll \Sigma_{\text{crit}}$). In addition, the single microlens case can be used to test limitations of the numerical simulation, such as the limited spatial resolution, or pixel size.

The outcome of the inverse ray-shooting technique applied to a single microlens situated near a macro-CC is presented in Fig. 1. This illustration showcases the results for both positive and negative parities. To exemplify the magnification probability for a single microlens within positive and negative macro-model regions, we refer to Fig. 2.

This visualisation offers insights into several key observations. Firstly, the size of the critical curves is scaled down by factors μ_t and μ_r in the tangential and radial directions, respectively (refer to Oguri et al. 2018 for further details). Furthermore, both parities exhibit a prominent peak in the probability of magnification around the value of the macro-model magnification μ_m . As magnification increases beyond this peak, the probability

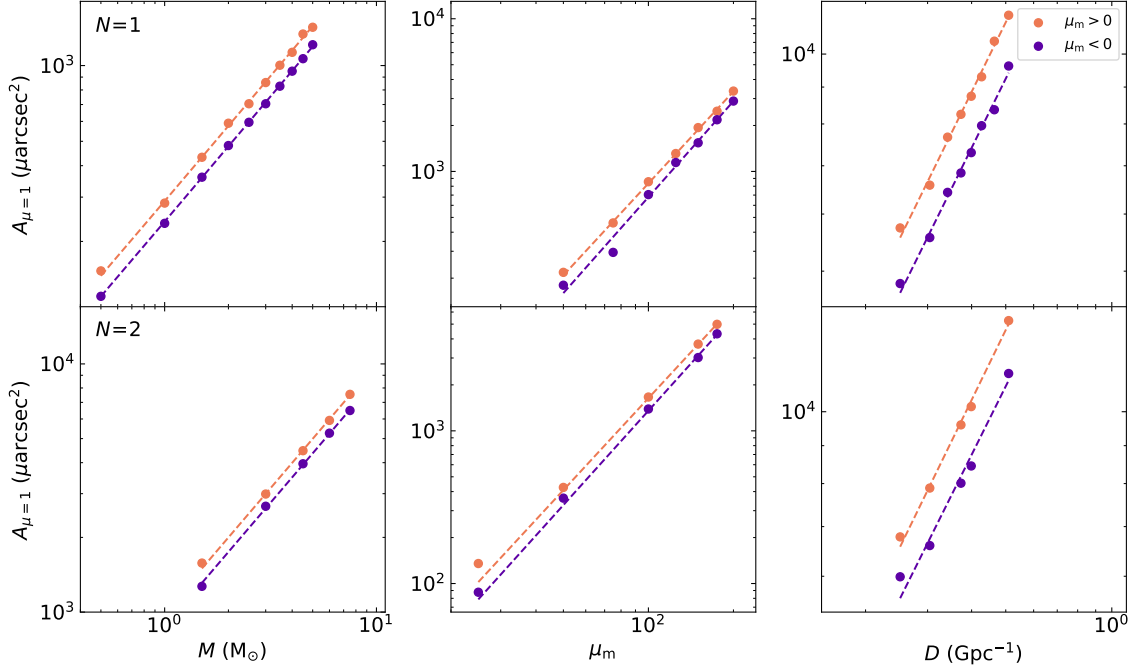


Fig. 3. Extrapolated value of the power-law $\propto \mu^{-2}$ at $\mu = 1$ from the probability distribution function of a system of N microlenses in the vicinity of a macro-CC. The case of one microlens is shown in the top row. The case for two microlenses is shown in the bottom row. We have changed the total mass of the microlenses, the magnification from the macro model and the reduced distance in the first, second and third columns respectively, while keeping the other parameters fixed. Note that for both systems the amplitude scales at the same rate for each parameter. All these lens systems have $\Sigma_{\text{eff}} \ll \Sigma_{\text{crit}}$. Dashed lines represent the best fit to a powerlaw obtained by χ^2 minimisation. We found that the amplitude scales linearly with the reduced distance and it scales cuadratically with the macro-magnification and the mass of the lenses.

decreases following a power-law scaling of μ^{-3} (or μ^{-2} if logarithmic bins are considered, as in our analysis). This behaviour is well-established in the literature (see Rauch 1991) and serves as a means to assess the resolution of our simulations (see Appendix A). When the probability of magnification no longer adheres to this power-law decline, it indicates the limit of our simulation's resolution. Additionally, the negative parity introduces a demagnification relative to μ_m , with the probability of demagnification exhibiting a decreasing slope of $-1/2$ for lower magnifications. These findings shed light on the behaviour and characteristics of magnification probabilities associated with a single microlens near a macro-CC at both positive and negative parities.

This straightforward scenario provides valuable insights into the impact of various parameters on the magnification probability. Parameters such as the lens mass, the macro-model magnification, and the reduced distance $D = D_{\text{ds}}/(D_s D_d)$ play crucial roles in shaping the magnification probability. To examine these effects, we perform a fitting procedure to the power-law function, μ^{-2} , on the right tail of the PDF of the magnification, and investigate the relationship between the amplitude of the power-law and these variables. The results of this analysis are illustrated in Fig. 3. The amplitude of the power-law (dotted lines in Fig. 3) increases cuadratically with the mass and the macromagnification, and it increases linearly with the reduced distance.

4.2. Double microlens case

Before delving into the realistic scenario of a distribution of N microlenses, it is insightful to examine the case of two

microlenses. This analysis will elucidate the fundamental scalings that will be further explored and elucidated throughout the remainder of this paper. We consider a second microlens with half the mass of the first microlens, centred at a distance equal to half the size of the original microcaustic in each direction, exactly on top of the microcaustic, maximising the overlapping (if placed closer it would behave as a single microlens whose mass is the combination of each sub-mass).

The scenario of two nearby microlenses in a high magnification region exhibits similarities to the single lens scenario. The linearity shown in Fig. 3 persists even for larger number of microlenses as long as the effective surface mass density remains below $0.1 \Sigma_{\text{crit}}$. However, once the micro-caustics start to overlap, resulting in Σ_{eff} values beyond this threshold, the magnification probability undergoes a transformation. At this point, the power-law decrease in the probability for higher magnifications vanishes, and a log-normal behaviour starts to emerge. In the most extreme scenarios of $\Sigma_{\text{eff}} \gg 1$, the log-normal shape of the PDF of the magnification gets narrower the larger the value of Σ_{eff} . This phenomenon, known as the “more is less effect”, has been previously observed and studied (see Welch et al. 2022a; Diego et al. 2018) and is studied in more detail in Sect. 6. The scalings of the probability of magnification are exactly the same as the ones observed for the case of a single microlens. These scalings are shown in the bottom row of Fig. 3.

This simple test demonstrates that increasing the mass of the lens or lenses, increasing the magnification of the macro-model (i.e. reducing the distance to the macro-CC), or increasing the reduced distance all lead to an increase in the probability of magnification at higher values.

Building upon these insights, a similar procedure is applied to the general case of N microlenses in Sect. 6, which is the main objective of this work.

5. Generalisation to N microlenses

Here, we consider a more realistic scenario that is more likely to be found in real observations. In this scenario, we consider a large population of microlenses with different surface mass densities and macro-model magnifications. The simulations consist of nearly 90 cases with different effective surface mass densities and parities, and the computational resources required for these simulations amount to approximately 0.5×10^6 CPU-hours.

A significant finding is that the magnification statistics exhibit distinct behaviours depending on whether the ratio $\Sigma_{\text{eff}}/\Sigma_{\text{crit}}$ is smaller or greater than one. We refer to these cases as the low surface mass density and medium-high surface mass density scenarios, respectively. These two cases display different characteristics in terms of the magnification statistics. Figures 4–7 show some of the simulated magnification maps and statistics at the source plane for each surface mass density regime, low and high respectively, increasing the surface mass density towards the right at each parity (column).

To increase the effective surface mass density, there are two approaches. Firstly, one can increase the surface mass density of the microlenses by either including more microlenses or by having more massive ones. This results in larger micro-caustics, which in turn contribute to an increase in Σ_{eff} . Secondly, one can approach the macro-CC more closely, effectively increasing the macro-magnification. This mapping of micro-caustics into a smaller region enhances the likelihood of overlap between micro-caustics, thereby increasing the probability of a microlensing event due to caustic crossings.

The overlapping micro-caustics distorts the probability of magnification in the high-end. A simple way to understand this is by realising that locally, each microlens can perturb the magnification acting over nearby microlenses, perturbing the magnification from the macromodel. Alternatively, the maximum magnification is inversely proportional to the smoothness of the deflection field. Very corrugated deflection fields naturally result in smaller maximum magnifications. In the negative parity case, there is still a tail of demagnification, but as the surface mass density increases, this tail disappears, and the magnification probability becomes similar to that of the positive case. This effect is evident in Fig. 6, where for an effective surface mass density 48 times larger than the critical value, the micro-caustics for the negative parity are indistinguishable from those of the positive case. In these high-density scenarios, the areas of demagnification have been completely eliminated. This more is less effect leads to a decrease in the probability of higher magnifications. The effects of micro-caustic overlap and the more-is-less effect can be observed in Figs. 4 and 6.

5.1. Low surface mass density

In the low surface mass density case, where the combined effect of the macro-model and microlenses does not exceed the critical surface mass density, the micro-caustics do not heavily overlap. As a result, they maintain a similar shape to the isolated microlens case, and the magnification probability resembles that shown in Fig. 2. As the effective surface mass density increases, both positive and negative parity cases exhibit a broadening of the main peak and a shift towards smaller magnification values.

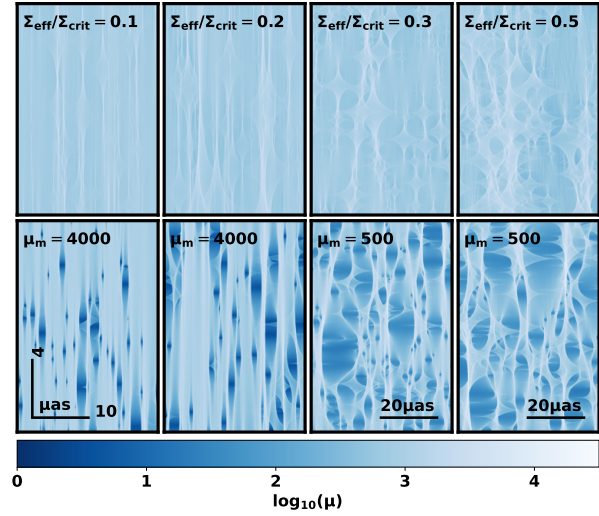


Fig. 4. High resolution view of the magnification at the source plane. *Top panels:* positive parity side of the macro-CC. *Bottom panels:* negative parity side of the macro-CC. The effective surface mass density increases from left to right. The value of the macro magnification and the ratio of the effective surface density to the critical value are shown for each column. Note that the y -direction has been compressed a factor 10 for better visualisation. For the same reason, the two rightmost columns have been compressed a factor 2 in the x -direction.

Additionally, the power-law tails of the probability distributions increase in amplitude and begin to transition to a log-normal distribution at smaller magnifications due to the increased overlap of micro-caustics. In the case of the negative parity, there is an excess probability of lower magnifications compared to the macro-magnification value. This demagnification effect follows a power-law distribution, with an increasing amplitude and a broadening peak structure as the surface mass density increases. This behaviours can be observed in Fig. 5 for the positive (right panel) and negative (left panel) macro-magnification parity cases.

5.2. Medium-high surface mass density

In the medium-high surface mass density case, the effective surface mass density exceeds the critical value, leading to the inevitable overlap of micro-caustics. The microlenses affect the macro-model magnification considerably and form a complex web of deformed micro-caustics that covers the entire plane, as depicted in Fig. 6. The magnification probabilities for the medium-high surface mass density simulations are shown in Fig. 7 for the positive (right panel) and negative (left panel) macro-magnification parity cases.

As Σ_{eff} increases, the probability distribution of the re-scaled magnification becomes more concentrated within a smaller range and around smaller magnification values². Both parities exhibit a peak-shaped distribution that saturates to a constant width. In the negative parity case, there is still a tail of demagnification, but as the surface mass density increases, this tail becomes less relevant and eventually it disappears. For the most extreme values of Σ_{eff} , the magnification probability becomes similar to that of the positive case. This effect is evident in

² We recall that all magnification values in this plot have been re-scaled by the factor $1000/\mu_m$.

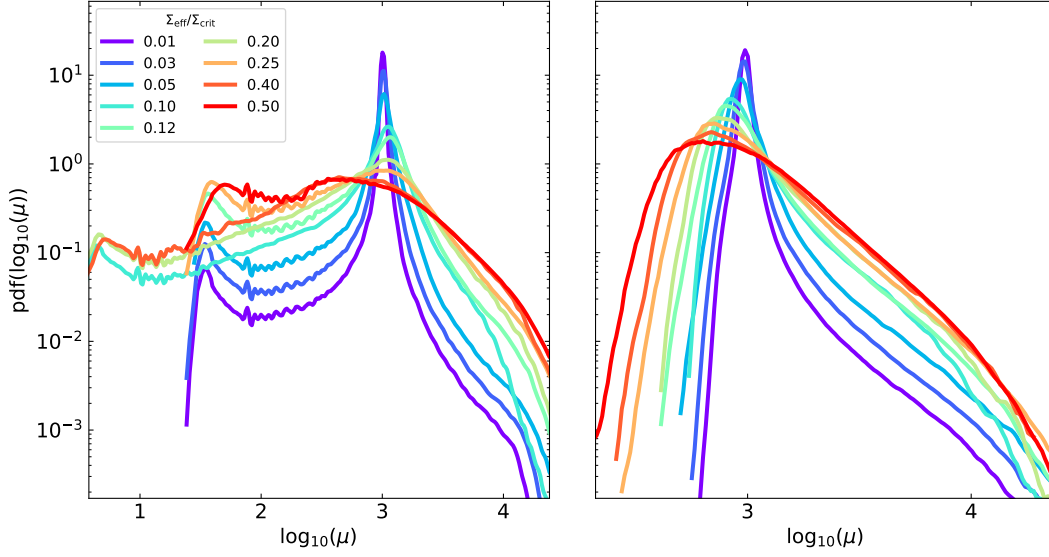


Fig. 5. Probability of magnification for a set of simulations with different Σ_{eff} in the low surface mass density regime. *Left panel:* negative parity side of the macro-CC. *Right panel:* positive parity side of the macro-CC.

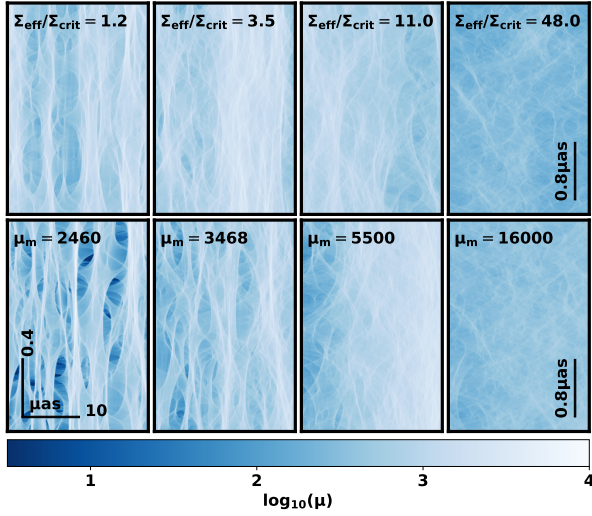


Fig. 6. High resolution view of the magnification at the source plane. *Top panels:* positive parity side of the macro-CC. *Bottom panels:* negative parity side of the macro-CC. The effective surface mass density increases from left to right. The value of the macro magnification and the ratio of the effective surface density to the critical value are shown for each column. Note that the y-direction has been compressed a factor 25 (50 for the last column) for better display.

Fig. 6, where for an effective surface mass density 48 times larger than the critical value, the micro-caustics for the negative parity are indistinguishable from those of the positive case. In these high-density scenarios, the areas of demagnification have been completely eliminated.

6. Analytical modelling

In this section, we describe the analytical functions employed to fit the magnification probabilities in each simulation. We differentiate between the low and medium-high surface mass density

regimes as well as the positive and negative parities. Additionally, we demonstrate how each model parameter scales in relation to the physical properties of the lens system. To account for resolution effects, as described in Appendix A, we have carefully considered them in each simulation. All best-fit data points are presented in Appendix C.

The process we follow is straightforward. For each case study, we fitted the simplest analytical functions that not only offer a good approximation to the magnification probability distribution but also exhibit function parameters that demonstrate a discernible trend with respect to the lens model parameters. These trends can be easily extrapolated. We fitted the model parameters to simple models via χ^2 minimisation. The model parameter value according to the best fit can be found in a Github repository³. As a result, we can work in the inverse direction: by utilising the lens characteristics, we can determine their corresponding model parameters. These parameters can then be incorporated to obtain an analytical approximation of the magnification probability distribution. This approach allows us to bypass the need for extensive numerical simulations, saving us time and computational resources.

Before delving into the analytical modelling that links the properties of the microlenses and the macro-model to the magnification probability, it is important to acknowledge a degeneracy present in the magnification probability with respect to Σ_{eff} . Specifically, two different simulations with the same Σ_{eff} but different values of Σ_s and μ_t can yield the same magnification probability (see Fig. 8).

Under certain circumstances, as described in Appendix B, this degeneracy can be broken. Additionally, if we undo the previous normalisation, different values of μ_t can also break this degeneracy. Furthermore, we observe that, in general, μ_t does not significantly influence the magnification probability. In the most extreme cases of surface mass density, and for high magnifications regardless of the surface mass density, this parameter

³ https://github.com/ChemaPalencia/M_SMiLe

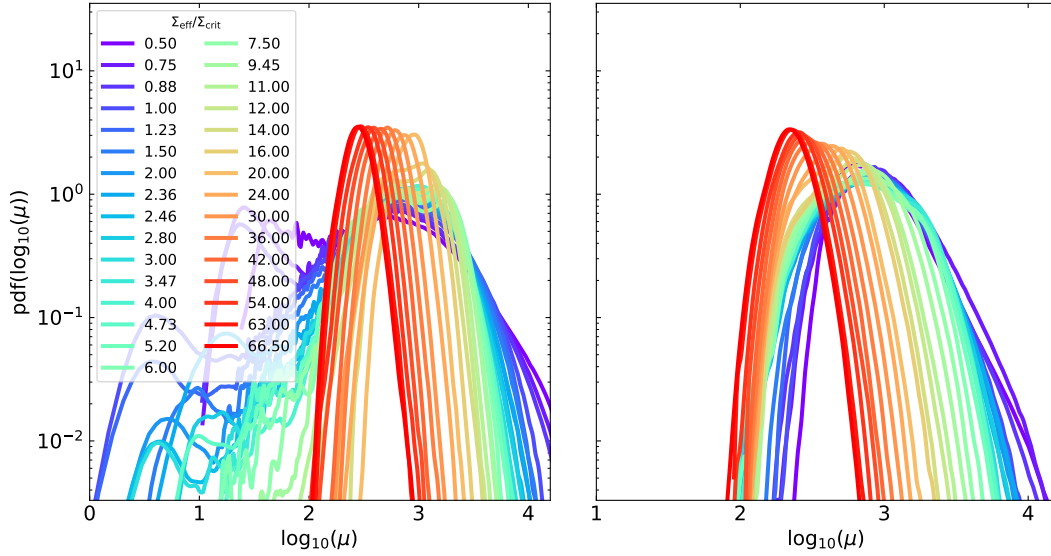


Fig. 7. Probability of magnification for a set of simulations with different Σ_{eff} in the medium-high surface mass density regime. *Left panel:* negative parity side of the macro-CC. *Right panel:* positive parity side of the macro-CC.

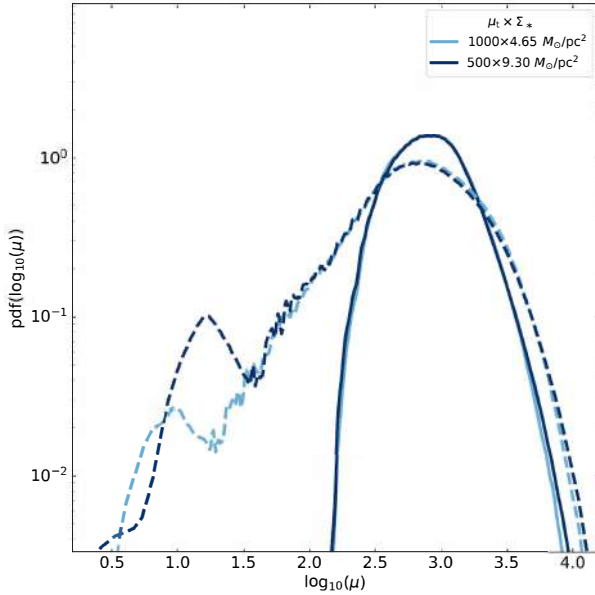


Fig. 8. Probability of magnification for two simulations with different Σ_{eff} and μ_t but the same Σ_{eff} . Positive parities are shown as solid lines, while dashed lines represent negative parities. A difference in the position of the probability excess in the low magnification regime is due to a difference in μ_t between the simulations.

does not have a significant impact on the probability of attaining such magnifications.

6.1. Low effective surface mass density

This case is similar to the isolated microlens studied previously. It describes the probability of magnification, which is important in scenarios with a low density of microlenses, important for high redshift stars whose CC will be in the outermost parts of the

cluster, where stellar microlenses are reduced. It also describes sources at lower macro-magnifications, or larger distances to the macro-CC. Furthermore, it serves as a useful bridge, linking the single microlens case to the high-density regime scenarios that are computationally more demanding. In this regime, we observe slight differences in magnifications between the positive and negative parities. As mentioned earlier, we now present the modelling of each case to provide a comprehensive understanding of the magnification behaviour in this regime.

The magnification probability functions in this study are characterised by a combination of power-law and log-normal functions. To facilitate the understanding of the parameter scalings in each case, we adopt the following notation:

- Capital letters are used to represent the parameters of the log-normal functions.
- Lowercase letters are used to denote the parameters of the power-law functions.
- The alphabetical ordering of the parameters corresponds to their appearance in the probability distribution function from lower to higher magnification values, progressing from left to right in the PDF figures.
- A superscript “+” or “−” is added to indicate the positive or negative parity case, respectively.

By employing this notation, we can succinctly represent and communicate the parametrisation of the magnification probability functions in a clear and organised manner.

6.1.1. Positive parity

A positive parity is obtained when $(1 - \kappa)^2 - \gamma^2 > 0$, indicating that the image retains the same parity as the source. This regime has been extensively investigated in previous studies. However, the understanding of probabilities at the highest magnification factors has been limited by the constraints of past simulations. The large field of view and high resolution of our simulations enable us to provide a precise description of the highest magnification factors, where the more-is-less effect comes into play.

The modelling of the magnification probability in this regime is straightforward and resembles that of a single microlens embedded in a highly magnified region. The probability distribution exhibits a peak structure centred around values similar to the magnification of the macro-model at that specific distance from the macro-CC. With our re-scaling of the magnification, these peaks are centred at $\mu = 1000$. This central peak smoothly transitions into the well-known power-law decay that characterises the probability distribution over the remaining range of magnification values. However, the presence of overlapping micro-caustics leads to a decrease in the probability at the highest magnification values. This decrease occurs more rapidly than the power-law decay, resulting in a distinctive behaviour in the tail of the magnification probability distribution.

As the effective surface mass density increases, several notable changes occur in the magnification probability distribution. Firstly, the peaks in the distribution decrease in amplitude, become broader, and shift towards values smaller than the macromodel magnification. Additionally, these initially symmetrical peaks begin to exhibit a skewness towards larger magnification values.

Conversely, the power-law components of the distribution experience an increase in amplitude as the effective surface mass density increases. However, the range of magnifications described by these power-laws gets reduced, and the suppression of extreme magnification factors starts at smaller magnification values compared to lower surface mass density regimes.

To effectively model these magnification probabilities, we have divided the probability density functions into three distinct magnification regimes: (i) the low magnification regime, which is characterised by the presence of peaks, including the main one at the mode of the distribution; (ii) an intermediate region, which is described by the classic μ^{-2} power-law; (iii) the saturation extreme regime, where the power-law gets suppressed by microlenses. By breaking down the PDFs into these three regimes, we can provide a comprehensive and accurate representation of the magnification probabilities across the entire range of magnification values.

- Low magnification regime: For the main peak of the PDF, we utilise a skewed log-normal distribution to capture the shape and characteristics of the magnification probabilities (part of the Eq. (14) with normalisation factor B^+). In addition to the skewed log-normal distribution for the peak, we incorporate another log-normal distribution to account for the slight excess of probability at magnification values smaller than the peak (A^+ term in the same equation).

$$\begin{aligned} \text{PDF}(\log_{10}(\mu)) = & A^+ \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_A^+))^2}{2\sigma_A^{+2}}\right) \\ & + B^+ \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_B^+))^2}{2\sigma_B^{+2}}\right) \\ & \times \left[1 + \text{erf}\left(\frac{\log_{10}(\mu) - \log_{10}(\mu_B^+)}{\sqrt{2}\sigma_B^+}\right)\right]. \end{aligned} \quad (14)$$

By combining these analytical functions, we can effectively model the peak structure and the slight excess of probability at lower magnification values observed in the PDFs for the lowest surface mass densities. From now on, in the context

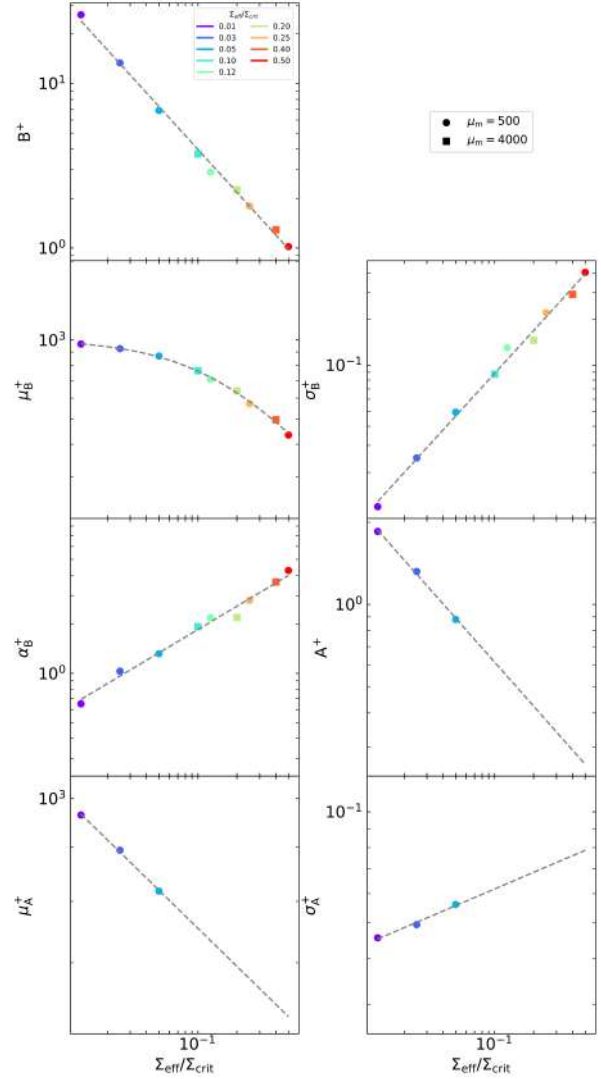


Fig. 9. Scaling of the model parameters for the low magnification regime at the low surface mass density and positive parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

of this paper, a capital letter “X” refers specifically to the amplitude of the X-log-normal distribution, where X stands for A, B, C, etc. μ_X represents the centre of distribution without skewness, σ_X represents its width, and α_X represents its skewness amplitude. In Fig. 9, we present the scaling of these model parameters with respect to the effective surface mass density.

- Intermediate magnification regime: In the intermediate magnification regime, we model the magnification probability using a combination of two power-law functions with a fixed relation between their amplitudes. These power-laws capture the smooth transition from the peak region to the power-law decay with a fixed index of -2 . The amplitude of the free index power-law is obtained by fitting the ratio of the amplitudes from a first fit of two independent power-laws. We use this model function to reduce the number of

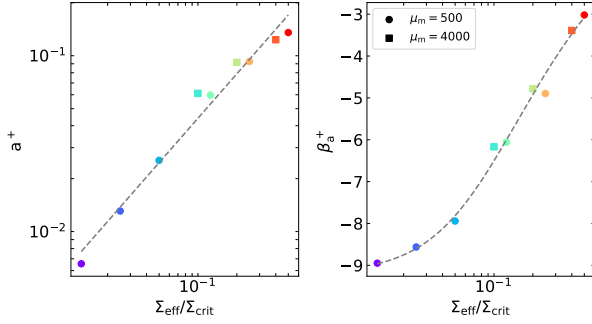


Fig. 10. Scaling of the model parameters for the intermediate magnification regime at the low surface mass density and positive parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

parameters. This procedure is repeated for the power-laws in the negative parity case.

$$\text{PDF}(\log_{10}(\mu)) = a^+ \left(\frac{\sqrt{\Sigma_{\text{eff}}}}{591} \left(\frac{\mu}{10^{3.5}} \right)^{\beta_a^+} + \left(\frac{\mu}{10^{3.5}} \right)^{-2} \right). \quad (15)$$

This analytical model provides a convenient way to approximate the magnification probability in the intermediate magnification regime. Similarly to our previous notation for log-normal functions, we adopt a notation for power-law distributions. The lowercase letter “x” represents the amplitude of the x-power-law distribution, while β_x denotes the index or exponent of the power-law. Figure 10 shows the scaling of these model parameters with Σ_{eff} .

- Extreme magnification regime: The suppression of the magnification probability at extreme magnification factors due to the microlenses can be described by a log-normal,

$$\text{PDF}(\log_{10}(\mu)) = C^+ \exp \left(\frac{-\left(\log_{10}(\mu) - \log_{10}(\mu_C^+) \right)^2}{2\sigma_C^{+2}} \right). \quad (16)$$

These parameter scaling are shown at Fig. 11.

An example of a magnification probability curve fitted using this model is shown in Fig. 12. The figure showcases the agreement between the fitted model (dot-dashed line) and the simulated data (solid line). This exemplifies how our analytical approach can accurately capture the magnification probability distribution over a wide range of magnifications.

6.1.2. Negative parity

The negative magnification regime, where $(1 - \kappa)^2 - \gamma^2 < 0$, corresponds to cases where the magnification μ is negative, resulting in an image with reversed parity compared to the source. While this regime has often been overlooked in the literature, it plays a significant role in understanding certain features observed in caustic crossing events. The unique characteristics of the negative parity regime can offer a better explanation for some observed events (such as the non-observation of Icarus’ counter-image), making a thorough modelling of this regime crucial for a comprehensive understanding of extreme magnification events.

In the negative parity regime, the magnification probability exhibits similar characteristics to the positive parity case at the highest magnifications. However, there are unique features observed at lower magnifications. We still observe a peak

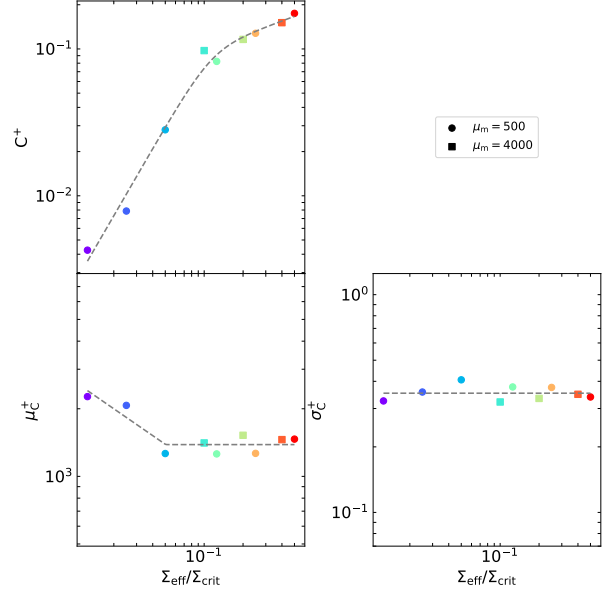


Fig. 11. Scaling of the model parameters for the extreme magnification regime at the low surface mass density and positive parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

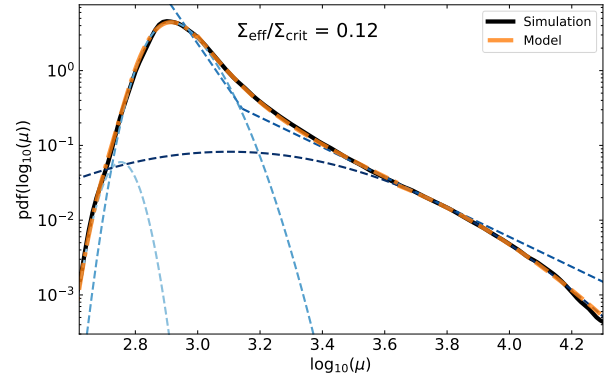


Fig. 12. Magnification probability function for a simulation within the low surface mass density regime and positive parity. The dashed lines represent each of the fitting model components and their total contribution.

close to the macromodel value ($\mu = 1000$ after re-scaling), followed by a power-law decrease that transitions to a log-normal behaviour at the highest magnification values. At the lowest magnifications, we observe a mild increase in the probability from a power-law component, which is much less steep compared to the power-law at higher magnifications. Additionally, there is a “bump” or excess in the probability of the lowest magnifications. Interestingly, the position of this bump does not seem to follow a specific dependence on Σ_{eff} . These characteristics of the magnification probability in the negative parity regime provide important insights into the unique features and behaviours of caustic crossing events in gravitational lensing.

In the negative parity regime, similar to the positive regime, the width of the log-normals increases with increasing effective surface mass density. The range of the power-law component decreases, and the amplitude of the central peak decreases while

the amplitudes of the other log-normals increase, similar to the amplitudes of the power-law components. However, there is a slight difference in the behaviour of the central peaks. Initially, they shift towards higher magnification values, but as the surface mass density further increases, they start moving towards lower magnification values, mimicking the trend observed in the positive parity case. This behaviour, alongside the demagnification components, reflects the unique characteristics of the negative magnification regime and provides important insights into the magnification probability in caustic crossing events with negative parities.

Here we also break the modelling into three regimes based on magnification values. Thus, providing a comprehensive description of the magnification probability distribution in the negative magnification regime: i) The low magnification or demagnification regime featuring the main differences with respect to the positive parity; ii) The intermediate magnification, covering the central peak; iii) The high magnification regime, spawning the power-law decrease and log-normal decrease at the highest magnification factors.

- Low magnification regime: In the negative magnification regime, we model the probability distribution using a combination of power-law functions and a skewed log-normal distribution. An interesting finding is the presence of a constant index of 0.5 in the low magnification regime power-law, similar to the index of -2 observed in the high magnification regime. This modelling approach allows us to capture the unique characteristics of the negative magnification regime, including a smooth transition to the peaks. The skewed log-normal appears as a bump or excess of probability on top of the power-law at the lowest magnifications. To the left of this excess, the decrease is only that of the log-normal. Analytically this model is described as follows:

$$\begin{aligned} \text{PDF}(\log_{10}(\mu)) = & a^- \left[\left(\frac{\Sigma_{\text{eff}}}{1928} \right)^{1.7678} \left(\frac{\mu}{10^{2.4}} \right)^{\beta_a^-} + \left(\frac{\mu}{10^{2.4}} \right)^{0.5} \right] \\ & + A^- \exp \left(- \frac{(\log_{10}(\mu) - \log_{10}(\mu_A^-))^2}{2\sigma_A^{-2}} \right) \\ & \times \left[1 + \text{erf} \left(\alpha_A^- \frac{\log_{10}(\mu) - \log_{10}(\mu_A^-)}{\sqrt{2}\sigma_A^-} \right) \right]. \end{aligned} \quad (17)$$

These parameter scalings are shown in Fig. 13. We have observed that unlike other parameters, the characteristics of this excess in the probability distribution, that appear at low magnification factors for the negative parity only, do not scale directly with Σ_{eff} . This distinction arises because the position of the excess is solely determined by the μ_r parameter, independent of Σ_{eff} . Consequently, the normalisation of the distribution places this peak at different positions relative to μ_m . This observation becomes apparent when examining Fig. 8, where the negative parity probability distributions are identical for the entire range of magnifications except for the lowest factors where these bumps occur. To gain further insight into these peaks at low magnification factors, we conducted an additional set of 12 simulations with varying μ_t , μ_r , and Σ_* , in addition to the previous simulations. This comprehensive data-set helps us better understand the characteristics of these peaks. We proceed to fit the position of the peak before normalisation as a function of μ_r :

$$\tilde{\mu}_A^- = \mu_A^- \frac{\mu_m}{1000}. \quad (18)$$

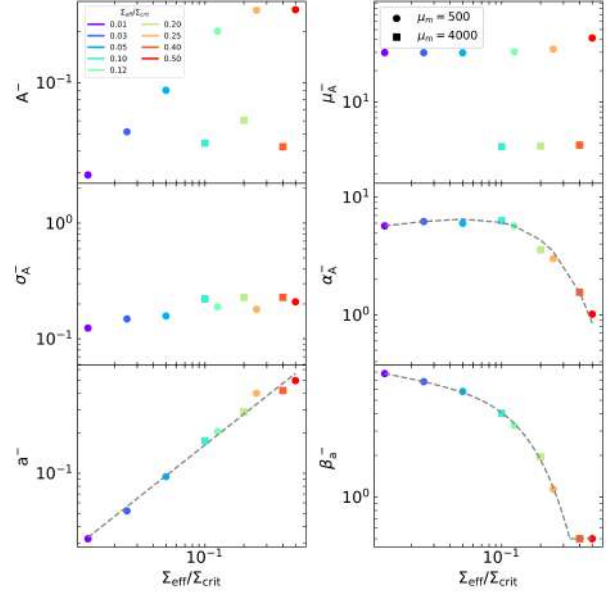


Fig. 13. Scaling of the model parameters for the low magnification regime at the low surface mass density and negative parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

Similarly, we perform a fitting procedure to determine the width of the peak at each Σ_{eff} and μ_r . We use the same scaling of the width with respect to Σ_{eff} , but with different scaling factors determined by μ_r (see Fig. 14). We define this scaling factor in terms of μ_r as follows:

$$R_{\sigma_A}(\mu_r) = \frac{\sigma_A^-(\mu_r)}{\sigma_A^-(4)}. \quad (19)$$

The dependence of this quantity with respect to μ_r is shown in Fig. 15. Furthermore, we observe that the amplitude does not exhibit a simple scaling with Σ_{eff} , regardless of the peak position corrections. To characterise the amplitude, we introduce a new quantity:

$$\tilde{A}^- = A^- \mu_m, \quad (20)$$

that shows a clear trend with Σ_{eff} . Figure 14 illustrates the scaling of the peak parameters, with the corrections applied, as a function of Σ_{eff} .

- Intermediate magnification regime: This regime encompasses the central peak, which can be effectively modelled as a log-normal distribution:

$$\text{PDF}(\log_{10}(\mu)) = B^- \exp \left(- \frac{(\log_{10}(\mu) - \log_{10}(\mu_B^-))^2}{2\sigma_B^{-2}} \right). \quad (21)$$

The scaling for these parameters is illustrated in Fig. 16.

- High magnification regime: This regime can be modelled similarly to the positive parity case, with two power-laws and a log-normal. However, in this case, the log-normal is added to the power-laws instead of being a piecewise

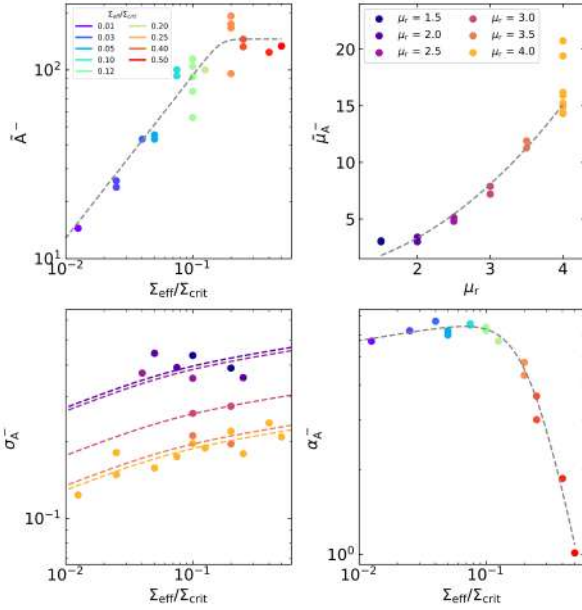


Fig. 14. Scaling of the model parameters for the low magnification regime at the low surface mass density and negative parity scenario. As in Fig. 13 point rainbow colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

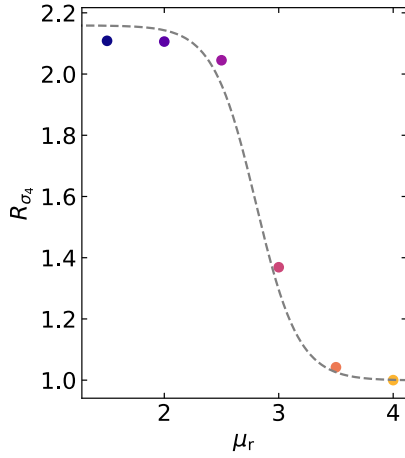


Fig. 15. Ratio between the width of the peak (at low magnifications for low surface mass density and negative parity) at a given μ_r and its corresponding width at $\mu_r = 4$. Point colours represent the μ_r over the critical value as shown in Fig. 14.

function. At the right end of the log-normal, the contribution of the power-laws becomes negligible. This analytical model is:

$$\text{PDF}(\log_{10}(\mu)) = b^- \left(\left(\frac{\Sigma_{\text{eff}}}{995} \right)^2 \left(\frac{\mu}{10^{3.5}} \right)^{\beta_b^+} + \left(\frac{\mu}{10^{3.5}} \right)^{-2} \right) + C^- \exp \left(- \frac{(\log_{10}(\mu) - \log_{10}(\mu_C^-))^2}{2\sigma_C^{-2}} \right). \quad (22)$$

Figure 17 shows the scaling of this parameters with respect to Σ_{eff} .

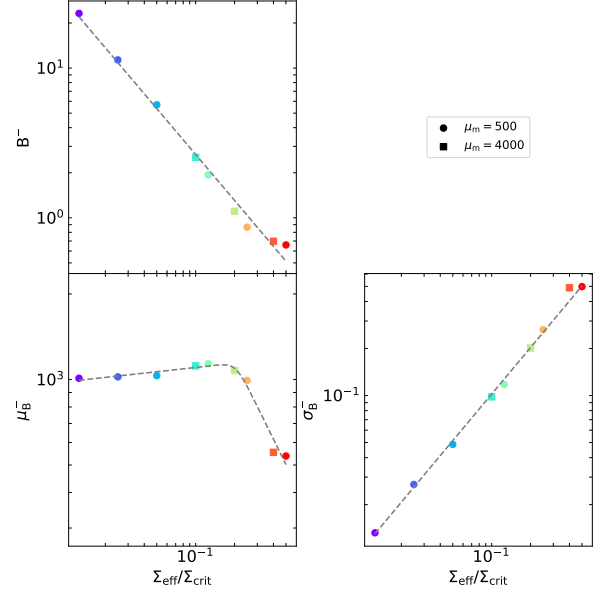


Fig. 16. Scaling of the model parameters for the intermediate magnification regime at the low surface mass density and negative parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

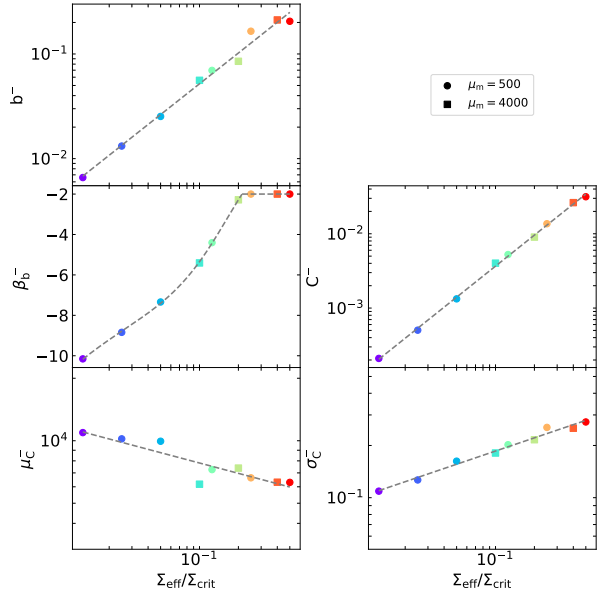


Fig. 17. Scaling of the model parameters for the extreme magnification regime at the low surface mass density and negative parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 5.

Figure 18 showcases the modelling of the magnification probability in the negative parity regime, as previously developed.

6.2. High effective surface mass density

This regime is expected to be more prevalent in realistic scenarios, where the combination of factors such as the distance to the macro-CCs, the surface mass densities of stars derived from the ICL, and the presence of a hypothetical population of compact

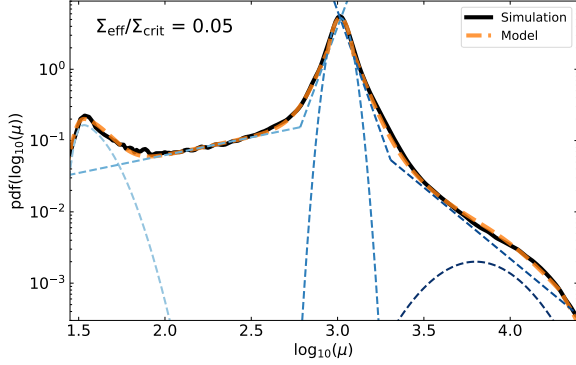


Fig. 18. Magnification probability function for a simulation within the low surface mass density regime and negative parity. The dashed lines represent each of the fitting model components and their total contribution.

DM can lead to higher values of Σ_{eff} . These elevated values often exceed the critical surface mass density, making the occurrence of this regime more likely.

In this regime, the magnifications for positive and negative parity differ, similar to what is observed in the low surface scenario. However, as the effective surface mass density increases, the magnification probabilities for both parity regimes tend to become more similar. In fact, for larger values of surface mass density, the modelling for both parity scenarios becomes identical. Following the previous approach, we now explain the modelling for both parity scenarios.

It is worth mentioning that even though this scenario is relatively straightforward in terms of modelling, the scaling of parameters exhibits more complexity and displays distinct regime behaviours. Nonetheless, we have successfully captured the entire range of effective surface mass densities included in our simulations. Moreover, we observe a saturation regime in the magnification probability at the highest densities, allowing for straightforward extrapolation to larger values. This enables us to fully describe the magnification statistics and behaviour across the entire range of effective surface mass densities in our study.

6.2.1. Positive parity

In this regime, we observe a transition from the low surface mass density regime, where the power-law distribution is dominant, to a regime characterised by a high-magnification log-normal distribution. The magnification probability in this regime can be effectively modelled as the sum of two log-normal distributions, with both distributions exhibiting skewness. Analytically, this can be represented as the sum of two skewed log-normal distributions:

$$\begin{aligned} \text{PDF}(\log_{10}(\mu)) = & A^+ \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_A^+))^2}{2\sigma_A^{+2}}\right) \\ & \times \left[1 + \text{erf}\left(\alpha_A^+ \frac{\log_{10}(\mu) - \log_{10}(\mu_A^+)}{\sqrt{2}\sigma_A^+}\right)\right] \\ & + B^+ \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_B^+))^2}{2\sigma_B^{+2}}\right) \\ & \times \left[1 + \text{erf}\left(\alpha_B^+ \frac{\log_{10}(\mu) - \log_{10}(\mu_B^+)}{\sqrt{2}\sigma_B^+}\right)\right]. \end{aligned} \quad (23)$$

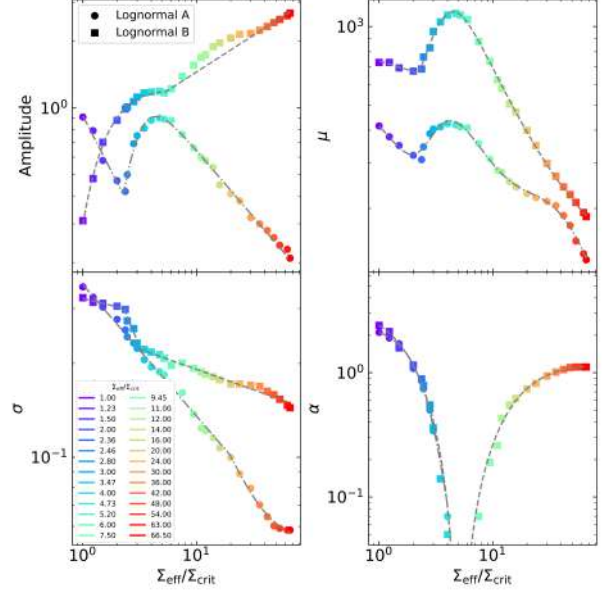


Fig. 19. Scaling of the model parameters at the high surface mass density and positive parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 7.

The scaling of the parameters in this model with respect to Σ_{eff} , as well as an example of a probability density function obtained from our simulations and fitted using this model, are depicted in Figs. 19 and 20, respectively. These figures provide a visual representation of how the model parameters evolve with increasing Σ_{eff} and demonstrate the individual components of the fitted compound distribution.

6.2.2. Negative parity

In this regime, the probability distribution at the highest magnification factors is conceptually similar to that of the positive parity regime. The power-law distribution is no longer an accurate description of the probability, and instead, the sum of two log-normal distributions provides a more appropriate magnification probability approximation. However, at the lowest magnification factors, the power-law distribution still persists. The crucial difference is that the power-law index is no longer fixed at 0.5. As the effective surface mass density increases, the power-law index grows, while the amplitude of the distribution remains constant. This variation in the power-law index reflects the changing behaviour of the magnification probability distribution compared to the low surface mass density regime. The range of magnification values for which the power-law distribution remains a valid description covers all values up to the magnification at which the log-normal and power-law distributions yield the exact same probability. However, as Σ_{eff} approaches a value of approximately $5 \Sigma_{\text{crit}}$, the power-law distribution returns extremely low probabilities, rendering its contribution negligible. In contrast to the positive parity regime, the most suitable model that we have found to fit our simulations in this regime consists of three log-normal distributions, which are fully symmetrical and do not exhibit any skewness.

Additionally, we have observed a peak at the lowest magnification factors in this regime, which is likely the same

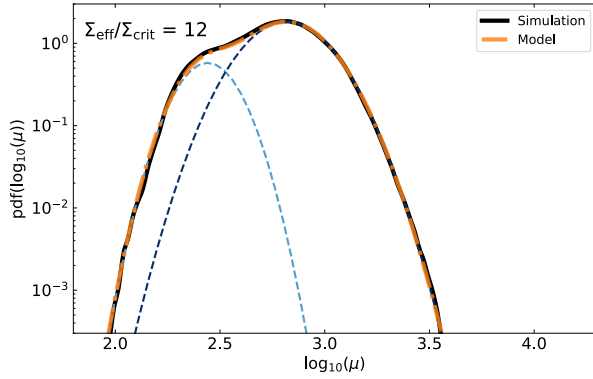


Fig. 20. Magnification probability function for a simulation within the high surface mass density regime and positive parity. The dashed lines represent each of the fitting model components and their total contribution.

excess that we observed in the low density regime. However, due to limitations in our simulation statistics, we do not have sufficient data to fully describe this peak in detail. Given that the excess peak at the lowest magnification factors is unlikely to be observed due to the known luminosity functions of stars, high redshift considerations, and limitations of current telescopes, we can omit the modelling of this peak and approximate the magnification as a power-law distribution in this regime. Qualitatively, we have observed that these peaks exhibit growth in both width and amplitude as well as a shift towards larger magnification values. Additionally, there is a correlation between the position of the peak and μ_r , similar to what was observed in the low density regime. Further investigations and simulations would be required to better understand and characterise this feature at the lowest magnification values, we left this task for a future work. The analytical model we have employed for the probability distribution is expressed as follows:

$$\begin{aligned} \text{PDF}(\log_{10}(\mu)) = & A^- \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_A^-))^2}{2\sigma_A^{-2}}\right) \\ & + B^- \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_B^-))^2}{2\sigma_B^{-2}}\right) \\ & + C^- \exp\left(-\frac{(\log_{10}(\mu) - \log_{10}(\mu_C^-))^2}{2\sigma_C^{-2}}\right). \end{aligned} \quad (24)$$

The power-law distribution, which is valid only for lower magnification factors until it becomes overtaken by the log-normal distributions, is described by the equation:

$$\text{PDF}(\log_{10}(\mu)) = a^- \left(\frac{\mu}{10^{2.4}}\right)^{\beta_a^-}. \quad (25)$$

Regarding the negative parity regime, we have also depicted the dependence of the model parameters on Σ_{eff} in Fig. 21, and we have also provided an example of a simulation magnification probability fitted according to this models in Fig. 22.

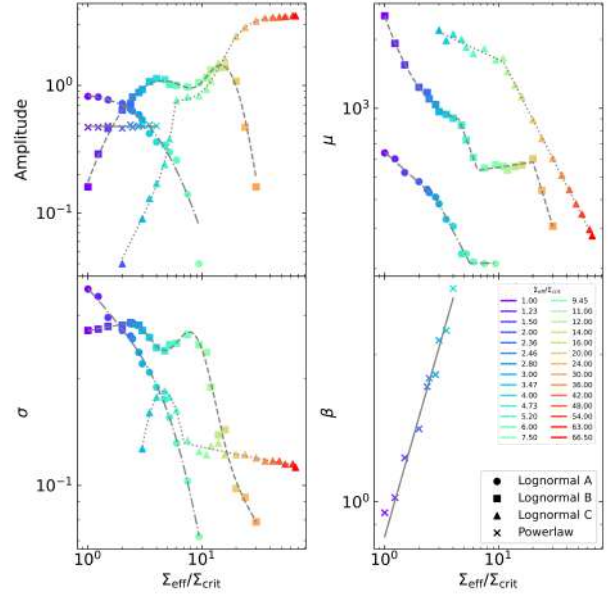


Fig. 21. Scaling of the model parameters at the high surface mass density and negative parity scenario. Point colours represent the Σ_{eff} over the critical value as shown in Fig. 7.

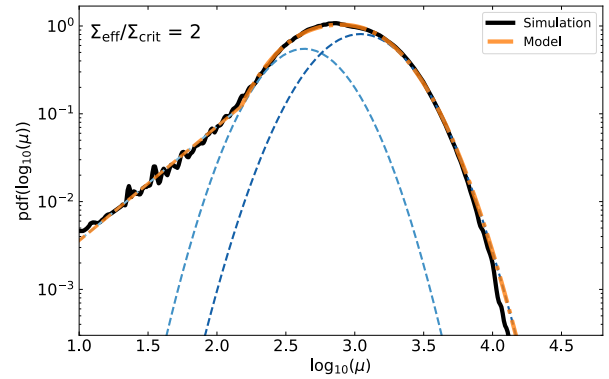


Fig. 22. Magnification probability function for a simulation within the high surface mass density regime and negative parity. The dashed lines represent each of the fitting model components and their total contribution.

7. Constraining compact DM with CCE

During caustic crossing events, the flux of the background stars can increase by factors of several thousands, to tens of thousands. This translates in magnitude changes of $O(10)$ compared to fluxes without magnification. This change in their apparent magnitude allows the detection of these sources, as well as a characterisation of the underlying microlensing population causing such events. Here we outline how the tools we have developed can be used to constrain compact DM with caustic crossing events, and provide an illustration of the expected magnification as a function of time, which illustrates how we can estimate the magnification probability.

Microlensing of small sources within a strongly lensed galaxy is responsible for the extremely lensed events such as Icarus, Earendel, or Godzilla (Kelly et al. 2018; Welch et al. 2022a; Diego et al. 2022). These lensing events result in a

significant enhancement of the observed fluxes from these objects, enabling their detection and facilitating detailed studies of their properties. This scientific scenario in itself presents an intriguing subject of investigation and has led to the development of dedicated scientific programmes like Flashlights (Kelly et al. 2019). However, these extremely lensed events hold particular importance in the field of DM studies. They offer unique opportunities to probe the nature and distribution of DM, as the lensing effects depend on the underlying DM distribution within the lensing system (Oguri et al. 2018).

As seen in the previous section the magnification probability in microlensing events is primarily determined by two factors: Σ_* and μ_t . The latter can be obtained from the strong-lensing models of the cluster, taking into account the distance of the image to the macro-CC. But even when the magnification cannot be accurately described, the scaling of the magnification with distance, d , to the CC, $\mu \propto d^{-1}$ allows to study the relative detection rate as a function of distance. On the other hand, Σ_* can be further divided into two components: Σ_b , which accounts for baryonic matter such as stars, and Σ_{DM} , which represents the contribution from DM.

The estimation of Σ_b can be derived from the ICL, which provides valuable information about the distribution of stars within the cluster. The remaining unknown parameter is Σ_{DM} , which represents the contribution of DM. This parameter can be inferred from the observed rate of microlensing events in a statistical sense, after a sufficiently large number of lensed stars has been observed. Of particular interest are lensed stars observed in regions where the contribution from Σ_b is lowest. This includes the CC forming between two merging clusters, specially during the first core passage, since the intermediate region in between the two clusters is expected to have very small contributions from stars in the intracluster light.

Monitoring the change of flux over time in the magnified images as the source moves in the source plane provides valuable information about the magnification probability. By comparing these observed flux variations with the analytical models derived from the previous section, we can perform statistical inference to estimate the parameter Σ_{DM} that best fits the data. This statistical analysis allows us to obtain an approximation of the density of compact DM in the microlensing population. Specifically, we can update the fraction of DM made up of our model of interest, denoted as “X”, by determining the ratio of its density to the total density of DM, represented by $f_X = \Omega_X / \Omega_{DM}$. This provides insights into the contribution of the specific model of interest to the overall DM content in the microlensing population.

The characteristic timescales of microlensing events are influenced by several factors, including the distance to the source and its transverse motion relative to the micro-caustics. To illustrate this, we can consider a simulated caustic pattern in which the source is located at a redshift of $z_s = 1.3$. The transverse velocity of a star with respect to the caustics at this distance can be estimated using the formula (see Miralda-Escudé 1991)

$$v_{\perp} = \left| \frac{v_s - v_o}{(1 + z_s)} - \frac{D_s(v_d - v_o)}{D_d(1 + z_d)} \right|. \quad (26)$$

The transverse velocity depends on the distances to the lens plane, and the source plane through their angular diameter distances, and depends also on the transverse velocities of the cluster, the source, and the observer, v_d , v_s , and v_o , respectively. To fix the transverse velocity $v_{\perp}$, a typical value

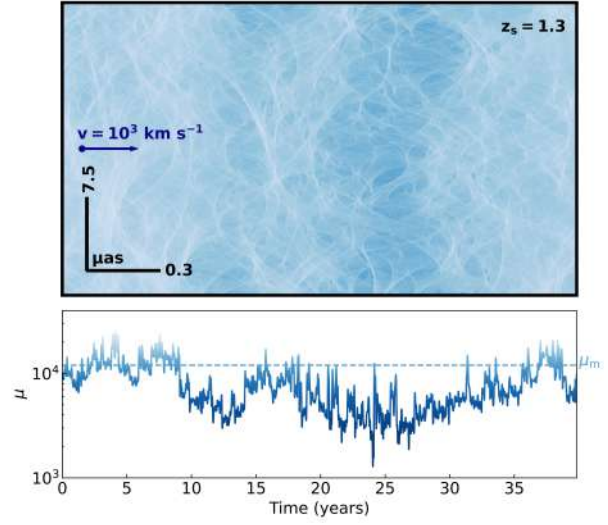


Fig. 23. Magnification of a source at redshift 1.3 and angular size 2 nas sweeping the micro-caustic web at a constant speed of 1000 km s^{-1} . *Top panel:* magnification pattern in the source plane. *Bottom panel:* magnification over time. The dashed line represents the magnification from the macro model.

of $\sim 1000 \text{ km s}^{-1}$ can be used as suggested by (Kelly et al. 2018). Finally, the angular transverse velocity at the source plane (see Oguri et al. 2018) is

$$u_s = \frac{v_s}{1 + z_s} \frac{1}{D_s}. \quad (27)$$

Plugging the transverse velocity value of $\sim 1000 \text{ km s}^{-1}$ into the equation, assuming a distance corresponding to a source at a redshift ~ 1.3 , we can calculate the transverse velocity in angular units. This yields a value of approximately 52 nas per year . In the context of our simulation, this corresponds to a crossing time of approximately 40 yr. Figure 23 illustrates an example of such a microlensing event, showing the evolution of the magnification over time due to the motion of the source.

Monitoring an arc close to a macro-CC allows us to observe flux changes in the sources within the arc, which can be attributed to microlensing events. By characterising the source properties, such as its intrinsic flux, we can translate the observed flux changes into magnification values. With longer exposure times, we can monitor the flux changes of more sources over extended periods, which improves the accuracy of the magnification probability estimation. By combining the flux changes from multiple sources, we can obtain a better estimate of the magnification probability distribution. Using the analytical approximations that we have established in the previous section and the estimated magnification probabilities, we can quickly obtain an estimate of the magnification probability in terms of parameters such as Σ_* , μ_t , and μ_r . This approach allows us to bypass the need for time-consuming simulations, which would require significant computational resources and subsequent analysis to obtain the magnification probability. Furthermore, by employing Markov chain Monte Carlo methods, we can perform parameter estimation and identify the most favoured combination of parameters that best explain the observed data. This enables us to derive

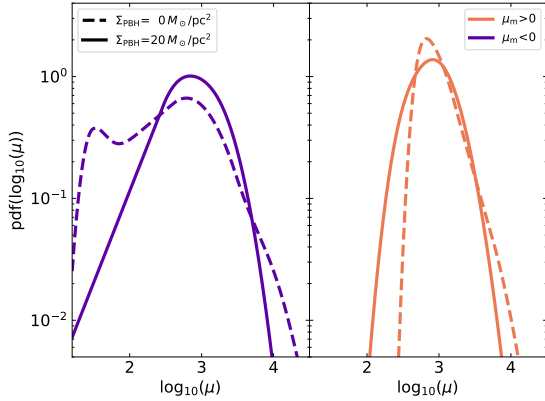


Fig. 24. Magnification probability for a case with $\Sigma_{\text{eff}}/\Sigma_{\text{crit}} = 2$, in both parity regimes. This mimics the magnification probability of a hypothetical microlensing population consisting of $5 M_{\odot}/\text{pc}^2$ stars and $20 M_{\odot}/\text{pc}^2$ PBHs. The critical density is $\Sigma_{\text{crit}} = 2500 M_{\odot}/\text{pc}^2$. $\mu_t = 200$ and $\mu_r = 4$. *Left panel:* negative macro-magnification. *Right panel:* positive macro-magnification. The dashed curve represents the magnification probability for the same system with no PBHs.

constraints on the physical properties of the lensing system, such as the surface mass density of the microlenses and the macro-magnification factors, by comparing the observed flux variations with the estimated magnification probabilities. Independent estimations of the macro-magnification factors μ_t and μ_r , as well as the baryon surface mass density Σ_b constrain the parameter space reducing the number of free parameters to one, Σ_{DM} , the contribution from compact DM.

Studying the outskirts of clusters is a promising approach for probing the role of DM in microlensing effects. In these regions, the contribution of stellar populations to the microlensing population is diminished, allowing the potential dominance of DM objects, if present. This offers an opportunity to investigate the properties of DM in the form of compact objects. Arcs farther from the cluster centre are associated with source planes at larger redshifts. To probe larger redshifts, it becomes necessary to observe intrinsically brighter objects. By comparing the observed probabilities of magnification to what would be expected solely from the stellar populations, we can set lower limits on the contribution of compact DM. Setting lower limits on the fraction of compact DM, f_X , requires careful analysis and comparison between observed magnification probabilities and our analytical predictions. By constraining the minimum amount of compact DM necessary to explain the lack of higher magnification events, we gain valuable insights into the presence and properties of DM in the form of compact objects within galaxy clusters.

Figure 24 shows the magnification probability from a hypothetical population of microlenses made of $5 M_{\odot}/\text{pc}^2$ in the form of stars, and $20 M_{\odot}/\text{pc}^2$ made of PBHs which corresponds to approximately a 1% of the total DM in realistic scenarios. For completeness the magnification probability for the same scenario with no PBHs is displayed as a dashed line. The macro-model tangential magnification, μ_t , is 200 which makes $\Sigma_{\text{eff}} = 2\Sigma_{\text{crit}} = 5000 M_{\odot}/\text{pc}^2$. Being the macro-images at a distance of approximately 0.2 arcseconds from a macro-CC, a powerful telescope like the JWST would have the capability to resolve both the positive parity image and the negative parity image. By comparing the flux differences between these two images, we can estimate the probability distribution functions of magnification

at both parities. If we can observe both parities we have even more constraining power, usually the negative parity image is more constraining. In this particular scenario, the observed flux differences can be used to infer the presence of an underlying surface mass density of approximately $20 M_{\odot}/\text{pc}^2$ composed of primordial black holes (PBHs) or other compact objects. This analysis can be carried out with only one image, but adding a second image of the same source in the reverse parity regime greatly increases the constraining power on the surface mass density of the microlensing population.

8. Discussion

In this paper, we have used the inverse ray-shooting technique to generate very large simulations of extragalactic microlensing phenomena caused by compact objects within galaxy clusters, and near the caustic region. These microlensing effects occur alongside the strong lensing effects generated by the gravitational field of the cluster, resulting in a combined lensing scenario. This emerging field of gravitational lensing provides a unique opportunity to investigate and study background sources. Furthermore these microlensing events allow us to constrain the surface mass density of the microlenses and provide valuable information about their luminous and non-luminous constituents. The non-luminous components are particularly interesting as they are directly linked to the presence and properties of compact DM.

In our study, we have focussed on the simulation of microlensing near tangential CCs, where $\mu_t \gg \mu_r$. This choice is motivated by the fact that radial CCs are located closer to the centre of the cluster, where the influence of DM compact objects is comparatively smaller. In contrast, tangential CCs offer a more favourable scenario for studying the effects of DM, as the population of microlenses in this region is expected to be predominantly composed of DM. Furthermore, higher redshift sources provide an additional advantage in probing the DM population associated with microlensing. As the redshift of the source increases, the tangential CCs move further outwards from the centre of the cluster. Consequently, the contribution from the cluster's light becomes less significant, enhancing the relative impact of the DM population. This makes higher redshift sources more effective in isolating and studying the effects of DM compact objects within the microlensing framework.

In our simulations, we replicate the magnification effects experienced by stars or compact objects passing through an arc that intersects the perturbed gravitational potential of the cluster. This occurs due to the presence of a population of microlenses near a macro-caustic, considering both parity configurations. We have obtained the magnification probabilities for each combination of μ_t , μ_r , and Σ_* simulated for both parity configurations, considering scenarios where Σ_{eff} is below the critical value and scenarios where it is equal to or greater than this value. In the presence of increasing surface mass density and a stronger influence of perturbations from microlenses, the most frequently observed magnification deviates from the macro-model's magnification at the simulated distance to the macro-CC. Specifically, it tends to shift towards smaller values as the disruption of the cluster's smooth potential becomes more dominated by the perturbation effects of microlenses. We have found previously unknown effects such as the power-law decay in magnification at negative parities, along with the bump or excess at the lowest magnifications. Using our newly developed tool, we have the capability to generate probability density functions for magnification within the range of effective surface

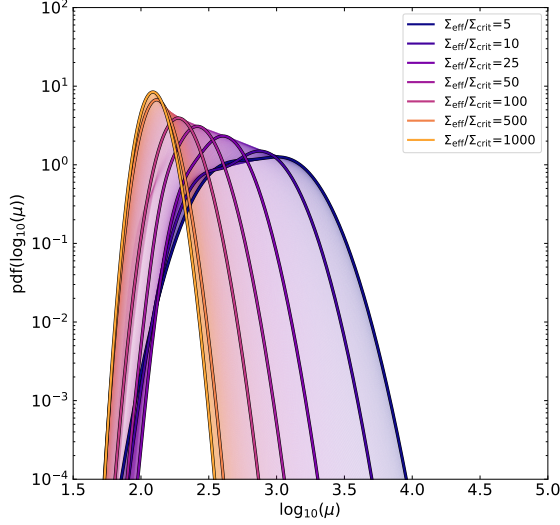


Fig. 25. Analytical approximation for the probability of magnification for a fixed $\mu_m = 1000$, and varying Σ_* . By increasing Σ_{eff} the more-is-less and saturation effects appear, obtaining a thicker log-normal centred at lower magnifications.

mass densities Σ_{eff} spanning from 0.01 to 66.5 $M_\odot \text{ pc}^{-2}$. Furthermore, we can extrapolate these magnification probabilities for even larger effective surface mass densities as needed.

8.1. Saturation regime and log-normal behaviour

The extreme saturation regime, characterised by $\Sigma_{\text{eff}}/\Sigma_{\text{crit}} \gg 1$, poses significant computational demands and thus it is the more interesting to describe with analytical models. Remarkably, we observe that in this regime, the well-established power-law behaviour that typically characterises the magnification probability for sparsely overlapping caustics gets completely eradicated. Instead, we find that the magnification probability follows a log-normal distribution. While the log-normal behaviour has been previously noted in the literature (Diego et al. 2018, Welch et al. 2022a), its complete understanding or comprehensive explanation has remained elusive due to limitations in microlensing simulations. Our simulations provide a sufficiently large area in the source plane with high resolution, enabling us to place confidence in the statistical properties of magnification in this regime.

Figure 25 shows how increasing Σ_{eff} results in the more-is-less effect, which reduces the spread of variability in flux when more and more microlenses are added. Such small variability in flux has been already observed in highly magnified stars such as Earendel (Welch et al. 2022a,b). The lack of variability due to microlensing in Earendel can be explained if the probability of magnification is already well into the log-normal regime.

Log-normal distributions are common in nature, in particular in multiplicative processes that transform into small additive increments in log-scale. The superposition of many microcaustics mimics a multiplicative process in which each microcaustic multiplies the flux of the background source by some amount. An increase in Σ_{eff} can be attributed to two factors: either an increase in μ_t or in Σ_* . At higher macro-magnifications, the lens plane is mapped onto smaller regions in the source plane, so the probability of caustics overlapping increases. To better

understand the log-normal behaviour and the saturation regime, it is useful to understand how the micro-caustics are shaped and how increasing Σ_* and μ_m affects in the probability of overlap. In their paper, Oguri et al. (2018) derive an analytical approximation for the shape of a micro-caustic by solving the equation where the inverse of the magnification equals zero. They determined the sizes of a micro-caustic located near the tangential CC, in the positive regime, in both the tangential and radial directions to be

$$\begin{aligned} \beta_{\text{caus},t} &\approx \frac{\sqrt{\mu_t}}{\mu_r} \theta_E, \\ \beta_{\text{caus},r} &\approx \frac{\theta_E}{\sqrt{\mu_r}}, \end{aligned} \quad (28)$$

respectively. For the negative parity the caustic size is similar but reduced by a factor of $2\sqrt{2}$ in the tangential direction, and it exhibits a demagnification region in the centre of the caustics. θ_E is the Einstein radius of the lens defined as

$$\theta_E = \left(\frac{4GM}{c^2} \frac{D_{\text{ds}}}{D_s D_d} \right). \quad (29)$$

A similar solution for the sizes of the micro-CCs was also obtained by Diego et al. (2018).

Increases in μ_m result in larger caustics concentrated in smaller regions, while larger Σ_* results from more microlenses, more massive microlenses resulting in larger micro-caustics, or a combination of both effects. These aforementioned effects promote the overlapping of micro-caustics, particularly at the tips of the caustics where higher magnification is concentrated. Consequently, these regions are the first to experience distortion as a result of overlapping. This phenomenon explains why the more-is-less effect becomes prominent at larger values of Σ_{eff} . The log-normal behaviour of the magnification probability can be understood as the manifestation of the central limit theorem in the logarithmic domain, arising from the overlapping of caustics.

Figure 26 depicts the presence of a saturation regime for $\Sigma_{\text{eff}}/\Sigma_{\text{crit}} \geq 30$ for the positive parity and $\Sigma_{\text{eff}}/\Sigma_{\text{crit}} \geq 25$ for the negative parity. Within this regime, the magnification probability follows a log-normal distribution, with a shift in its central value as Σ_{eff} increases (the changes in width and amplitude are almost negligible). It should be noted that the centre of the log-normal distribution requires subsequent correction, taking into account the value of μ_m . At the highest simulated effective surface mass densities, both parities exhibit the same log-normal distribution shifted towards higher magnifications for the negative parity. This can be observed in Fig. 7.

8.2. Constraints on PBHs

In Sect. 7, we considered a hypothetical scenario where the microlenses, consisting of 25% stars and 75% primordial black holes (PBHs), constituted 1% of the system's critical density. The macro-model predicted a magnification of $\mu_m = 200$, resulting in an effective surface mass density twice the critical density value. In this setup, background sources would undergo microlensing events following the probability distributions illustrated in Fig. 24. We would observe these events when the product of the intrinsic luminosity of the source and the magnification ($L\mu$) exceeds the detection threshold of our telescopes.

The HST and the JWST have already observed numerous microlensing events, and the number of such observations will

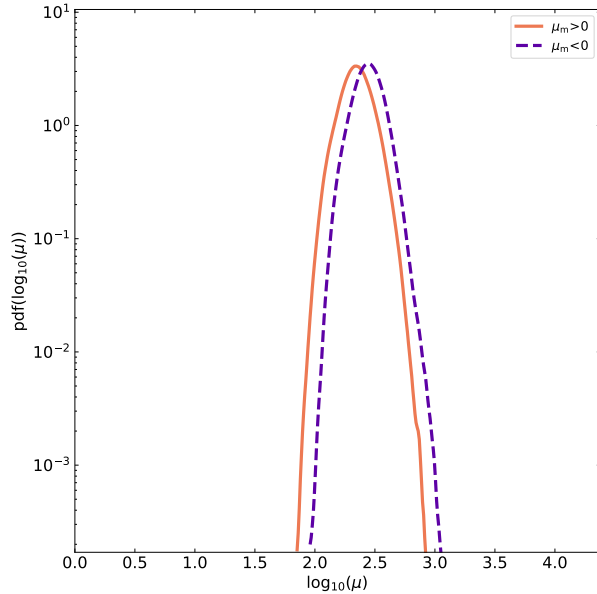


Fig. 26. Probability of magnification for two simulations with the same $|\mu_l|$, μ_r , Σ_* , and different parities in the saturation regime $\Sigma_{\text{eff}}/\Sigma_{\text{crit}} = 66.5$.

continue to increase over time. Continuous monitoring of arcs located in the outskirts of large galaxy clusters, will eventually accumulate sufficient data to perform statistical inference on the surface mass density of microlenses in these regions. Additionally, photometric measurements of the ICL in the same area will allow us to estimate the contribution of stars to the population of microlenses. Our current methodology provides a rapid estimation of the magnification probability, which can be compared against observational data. This allows us to determine the most likely value of Σ_* based on the experimental results. Moreover, since Σ_b can be measured independently, the DM surface mass density (Σ_{DM}) can be obtained simply by taking the difference between the inferred total density (from microlensing flux fluctuations) and the contribution from baryonic matter (from SED fitting of the observed photometry).

The data release by the LIGO-Virgo-KAGRA collaboration in 2016 (Abbott et al. 2016) has brought renewed interest in the in primordial black holes as a potential candidate for DM. While observational searches for the extra-galactic γ -ray background (Page & Hawking 1976), have ruled out the mass range of PBHs with initial masses $M_{\text{PBH}} \lesssim 10^{-17} M_\odot$, CMB and dynamical constraints have ruled out massive PBHs ($M_{\text{PBH}} \gtrsim 100 M_\odot$; Serpico et al. 2020; Lu et al. 2021). The remaining window of PBH masses, where they could potentially constitute a significant fraction of the DM content in the Universe, is an elusive range where only gravitational interactions play a significant role in possible observations.

Several methods have been suggested to constrain the abundance of PBHs in this open window (see Carr et al. 2021 and Green & Kavanagh 2021 for an extended review), in particular gravitational microlensing. The EROS-2 and MACHO surveys (Tisserand et al. 2007; Alcock et al. 2000), which focussed on, microlensing observations of stars in the Magellanic Clouds, and the OGLE survey (Niikura et al. 2019b), which conducted microlensing studies of the galactic bulge, have provided valuable constraints on the mass range of primordial black holes (PBHs) within the Milky Way. These studies have

placed constraints on the presence of PBHs in the mass range of $10^{-6} \lesssim M_{\text{PBH}} \lesssim 10 M_\odot$ at the (1–0.1)% level around $1 M_\odot$. The Subaru Hyper Suprime-Cam has played a crucial role in further constraining the mass range of primordial black holes within the Milky Way and the Andromeda galaxy (M31; Niikura et al. 2019a). In particular, the Subaru Hyper Suprime-Cam has extended the constraints to even lower PBH masses, specifically for $M_{\text{PBH}} \gtrsim 10^{-11} M_\odot$, through the microlensing observations of stars from M31.

Quasar microlensing involves a similar mechanism to extreme magnification events, where compact objects within a galaxy act as lenses that magnify the flux of background quasars images, resulting in flux changes in these images. Mediavilla et al. (2017) used optical and X-ray data from 24 lensed quasars. The results indicated that $\sim 20\%$ of the mass of the lens galaxies consisted of compact objects within the mass range of $0.05 M_\odot \lesssim M \lesssim 0.45 M_\odot$. Similarly to quasar microlensing, extremely lensed events offer a novel approach to constraining the amount of compact DM present in galaxy clusters. In 2018, Oguri et al. (2018) utilised the Icarus data (Kelly et al. 2018) to constrain the abundance of compact DM in the presence of the intracluster light.

By combining the results obtained from the aforementioned studies and incorporating future data collected by space telescopes such as the JWST, it will be possible to establish new constraints on the fraction of DM composed of PBHs, potentially even setting lower limits on their abundance. These ongoing investigations hold promise for advancing our understanding of the nature and contribution of PBHs to the overall DM content of the Universe.

8.3. Early stars, GWs, and high redshift SNe

The analytic approximation of the magnification probability presented in this paper not only provides a valuable method for constraining compact DM, but it also opens up possibilities for constraining the abundances of lensed sources at higher redshifts. In a recent study Diego et al. (2024) quantified the probability of observing a supergiant star due to the crossing of a micro-caustic, which amplifies the star's intrinsic luminosity above the detection threshold of the HST. To achieve this, the authors employed both a macro-model and a micro-model of the lensing system, along with the characterisation of the background galaxy, the Spock arc. This, and previous investigations relied on computationally expensive numerical simulations to account for the variability of the macro-model magnification and the contribution of microlenses within the arc. However, with the development of our new tool, these types of analyses can be significantly accelerated, providing faster and more efficient estimations. In their study, Diego et al. (2024) also presented forecasts for the constraints on these stars using the JWST. By incorporating the improved magnification probability obtained from our analytic approach, future observations with the JWST will enable more precise constraints on the presence and characteristics of lensed supergiant stars.

The rate of extreme magnification events depends on both the number of background sources and the probability of magnification that produces detectable events. Since we can now infer the later, by observing the number of events over a certain period of time, we can extract information about the abundance of the background sources.

Among the sources whose combined intrinsic luminosity and maximum magnifications achievable can lead to extreme magnification events, we discuss the following: high redshift

stars, high redshift supernovae (SNe), gravitational waves (GWs) from binary black hole systems, and population III stars.

The discussion of microlensing effects on quasars (QSOs) will not be addressed here, as they have already been extensively investigated, and their typical sizes surpass those of the previously described sources, thereby limiting the maximum magnification to the order of $O(10)$. However, it is worth noting that certain regions within QSOs exhibit smaller sizes, which can enhance the maximum magnification. The detection of such variations may be possible through multi-wavelength studies of these objects.

8.3.1. Luminous stars at cosmological distances

Although the probability of extreme magnification events for individual stars may be low, the vast number of stars in the Universe makes them ideal candidates to undergo such events.

The probability of a source intersecting a micro-caustic and experiencing extreme magnification increases with the number of background sources available. When considering cosmological distances ($z > 1$), it becomes challenging to detect stars unless they are particularly luminous. Telescopes such as the HST and JWST are capable of observing the most massive and bright stars at these distances.

In general, the mass-luminosity relation for main sequence stars follows a relationship, where the luminosity L is correlated with the stellar mass M as follows:

$$\frac{L}{L_{\odot}} = A \left(\frac{M}{M_{\odot}} \right)^{\alpha}, \quad (30)$$

where the slope α typically ranges from 3 to 4, with slightly lower values observed for the most massive stars. This mass-luminosity relation indicates that the most massive stars are capable of attaining the highest levels of luminosity within the main sequence. Stars outside the main sequence exhibit different mass-luminosity relations. However, the population of stars outside the main sequence is generally smaller in number compared to main sequence stars. Therefore, they are expected to contribute to a lower rate of extremely lensed events.

Smaller stars have a higher maximum magnification due to their smaller radius, but their intrinsic luminosity is also lower compared to larger stars. The luminosity of a star is generally correlated with its radius, and the exponent a in the scaling relation $L \propto R^a$ is typically greater than 1. As a result, the maximum luminosity $L_{\max}$ scales with the radius as $L_{\max} \propto R^{a-0.5}$. This implies that larger stars are more likely to be observed in extreme magnification events because their intrinsic luminosity exceeds the decrease in maximum magnification with respect to smaller stars. Therefore, as a general rule, larger and more massive stars are more commonly observed in caustic crossing events due to their higher intrinsic luminosity.

8.3.2. Population III stars

The existence and properties of Pop. III stars (Bromm et al. 2001) are of great interest in astrophysics and cosmology. These hypothetical stars, which are believed to have formed in the early universe and have very low or negligible metallicity, could provide crucial insights into the early stages of cosmic evolution. In their paper Windhorst et al. (2018) discuss the luminosity-mass relation for Pop. III stars as well as a similar population called “Pop. II.5” stars, which have a modest metallicity. They found that this relation is close to the Eddington limit, where the luminosity is proportional to the mass ($L \propto M$).

Due to their short lifespan, Pop. III stars are primarily expected to exist at high redshifts ($z \gtrsim 6-7$). The study of caustic crossing events with the JWST may lead to the first direct observations of Pop. III stars and significantly advance our understanding of the early Universe, including the re-ionisation process and the formation of the first galaxies.

8.3.3. GWs from binary black holes

Gravitational waves can also experience deflections in their paths as they propagate through regions with large distributions of matter. This gravitational lensing effect on GWs can lead to magnification or demagnification of the waves, similar to the lensing effect on light.

Broadhurst et al. (2018) proposes that a subset of the low-frequency GWs detected by the LIGO-Virgo collaboration could be attributed to GWs originating from sources at redshifts greater than one ($z > 1$) and magnified by a factor greater $\mu > 100$. These magnified GWs are hypothesised to arise from binary black hole systems with chirp masses around $30 M_{\odot}$. At these magnifications microlensing events will only be important for small sources, but the large wavelengths of the GWs suggest that wave optics might need to be considered.

8.3.4. High redshift SNe

Supernovae are extremely compact and luminous objects prone to microlensing effects. If the progenitor is located within a few astronomical units (AU) from a micro-caustic, the expansion of its photosphere can lead to extreme magnification increasing over time (typical rate of expansion ~ 6 AU/day). As the photosphere expands, it can intersect the micro-caustic, resulting in a maximum increase in the observed brightness of the SNe. However, as the photosphere continues to expand and move away from the micro-caustic, the magnification gradually decreases. This effect will leave an in-print on the light curve of the SNe.

All SNe can be observed through this mechanism, but we focus on type Ia SNe. These SNe are extremely important in the field of cosmology, as they serve as standard candles for measuring astronomical distances. The origin of the initial Type Ia SNe could potentially be traced back to the initial generation of white dwarfs. Consequently, their abundance at high redshifts ($z \gtrsim 6$) remains uncertain, given the time required for the formation of binary systems. The JWST has the capability to provide data of Type Ia SNe at redshifts ranging from $z = 6$ to 12, which would considerably augment our understanding of the abundance of these entities as well as their properties at high redshifts.

Readers can refer to Diego (2019) for a more extensive analysis on the methodologies employed for measuring these phenomena and their corresponding likelihood of occurrence.

9. Conclusions

Giant arcs intersecting a cluster’s CC provides an exceptional scenario for caustic crossing events to arise. If small and bright sources happen to lay close to the cluster caustic, they undergo extreme magnification boosting their fluxes by factors of hundreds or even thousands. This magnification can momentarily be even larger when the star crosses a microcaustic, providing a boost of $O(10)$ magnitudes and making a luminous star at redshift $z > 1$ detectable with current telescopes. This new branch of gravitational lensing opens the door to answer some questions about the nature and abundances of the magnified sources, providing a natural telescope to detect them. Moreover, monitoring

these arcs allows for the characteristics of the underlying population of microlenses to be inferred, constraining their surface mass density and possibly their nature. This is particularly interesting in the context of DM, where certain DM candidates, such as primordial black holes, can contribute to the microlensing signal.

This type of analysis requires the probability distribution function of the magnification to be known. Previous works have relied on expensive numerical simulations, with one simulation per set of parameters to be tested. Furthermore, accurate approximations of the magnification probability require large areas A in the source plane (increased by a factor μ_m in the lens plane) and small pixel scales. These three effects combined result in thousands of CPU hours per simulation, making it extremely difficult to carry out such an analysis. We have developed an approximation tool that enables a straightforward and efficient estimation of magnification probabilities by considering the scaling properties associated with the lens parameters. To develop our magnification probability approximation tool, we conducted an extensive series of simulations on microlensing near macro-CCs, involving approximately 0.5 million CPU hours. These simulations explored the parameter space for both macro- and micro-models. We obtained the magnification probability distributions and subsequently derived analytical models to represent these distributions. We derived the scaling relations between the model parameters and physical parameters such as the effective surface mass density of microlenses. These scaling relations allowed us to quickly compute the probability of magnification in a wide range of scenarios. As a result, we can now determine the magnification probability directly from the system's physical parameters, eliminating the need for extensive simulations, offering a viable alternative to time-consuming numerical simulations. This analytical approximation can be obtained in less than one second, thereby enabling the aforementioned studies.

During our modelling process, we observed that in extreme scenarios where $\Sigma_{\text{eff}} \gg 1$ the magnification probability deviates from the well-known power-law distribution with an index of -3 (-2 when considering probabilities in logarithmic bins). Instead, it exhibits a log-normal behaviour when the micro-caustics overlap in the source plane. This effect becomes prominent when either the surface mass density of the microlenses increases or when approaching the cluster-CC as the microlens magnification, μ_m , grows. Interestingly, we discovered a degeneracy in the magnification probability between the surface mass density of microlenses and the tangential magnification. Many parameters scale in terms of the effective surface mass density, $\Sigma_{\text{eff}} = \Sigma_* \times \mu_t$. The aforementioned degeneracy can be broken by the radial magnification, denoted as μ_r . In particular, we find that two simulations with the same Σ_{eff} but different Σ_* and μ_t have the same shape but are shifted relative to each other. Furthermore, our simulations confirm the more-is-less effect discussed by Welch et al. (2022a), as well as the concept of the effective microlens mass, $\mu_t M$, mentioned by Diego et al. (2018).

By delving into the study of the negative parity case, which has often been overlooked in the literature, we have made intriguing observations. We have found that this regime exhibits similarities to the positive parity regime, but with two distinct differences: (1) A demagnification effect emerges relative to the most likely magnification value. This probability of demagnification follows a negative power-law scaling ($\mu^{-1/2}$), which can even be observed for isolated lenses. Additionally, there is a probability excess at the lower end of the magnification distribution, and its position is solely determined by the radial magnification, μ_r . (2) Although the probability of magnification at the most likely

factor is smaller compared to its positive parity counterpart, the probability at the largest magnification values is slightly higher in the negative parity regime.

We have identified two distinct regimes based on the effective surface mass density. When Σ_{eff} is smaller than the critical value of the system, the magnification probability follows a similar model as that of isolated lenses near macro-CCs. The only additional component is the power-law transition to a log-normal distribution at the high magnification end of the probability distribution. When Σ_{eff} is comparable to or significantly greater than its critical value, the magnification probability functions are modelled as the sum of two log-normal distributions for the positive parity case. In this regime, the negative parity exhibits a power-law demagnification but no longer follows a fixed slope as observed in the low- Σ regime. As the effective surface mass density increases, the contribution of the power-law demagnification becomes negligible. In the negative parity regime, when $3 < \Sigma_{\text{eff}}/\Sigma_{\text{crit}} < 9$, the magnification probability is effectively modelled by three log-normal distributions.

When the effective surface mass density exceeds a certain threshold of approximately 30–40 times the critical surface mass density, the magnification probability enters the saturation regime. In this regime, the probability distribution is predominantly described by a single log-normal distribution. Interestingly, for values of $\Sigma_{\text{eff}}/\Sigma_{\text{crit}}$ above approximately 55, the parameters of the log-normal distribution coincide for both the positive and negative parity regimes. However, there is a slight shift towards higher magnifications in the negative parity side of the main-CC. At this point the perturbation from the microlenses have completely erased the main-CC and both parities are virtually indistinguishable.

Acknowledgements. J.M.P. acknowledges financial support from the Formación de Personal Investigador (FPI) programme, ref. PRE2020-096261, associated with the Spanish Agencia Estatal de Investigación project MDM-2017-0765-20-2. J.M.D. acknowledges the support of project PGC2018-101814-B-I00 (MCIU/AEI/MINECO/FEDER, UE) Ministerio de Ciencia, Investigación y Universidades. B.J.K. acknowledges funding from the Ramón y Cajal Grant RYC2021-034757-I, financed by MCIN/AEI/10.13039/501100011033 and by the European Union "NextGenerationEU"/PRTR. We acknowledge Santander Supercomputación support group at the University of Cantabria who provided access to the supercomputer Altamira Supercomputer at the Institute of Physics of Cantabria (IFCA-CSIC), member of the Spanish Supercomputing Network, for performing simulations.

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Appendix A: Resolution effects

The inverse ray-shooting technique, despite its effectiveness, is not without limitations. In this appendix, we address the impact of resolution effects on our estimation of the magnification probability. We found three key factors that have an impact on the derived magnification probability from the simulations. These factors include the pixel size, the masses of the microlenses, and the limited statistics at the lowest magnification factors.

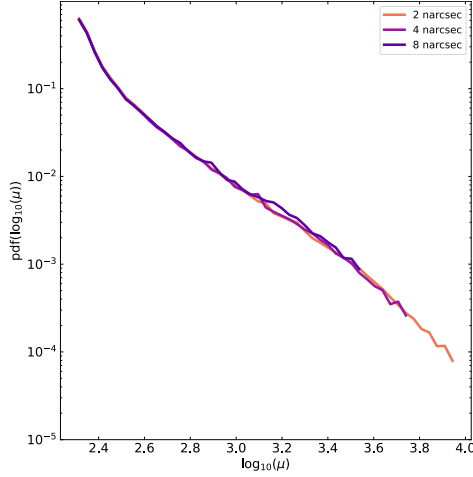


Fig. A.1. Magnification probability approximation for an inverse ray tracing simulation of an isolated microlens of $1M_{\odot}$ near a macro-CC. Each curve represents a different pixel size. Larger pixels reach smaller maximum magnifications.

A.1. Pixel size

When the pixel size in our simulations is increased, it imposes limitations on accurately representing the intricate structures within the micro-caustics web. A larger pixel size mimics a larger source, which affects the maximum magnification experienced. As discussed earlier, the maximum magnification is inversely proportional to the source size ($\mu_{\max} \propto 1/\sqrt{R}$), meaning larger sources experience lower maximum magnifications. Most of the peak magnification is mostly produced by the narrow tips of the micro-caustics. With larger pixels, these specific regions cannot be properly sampled, smoothing out the extreme magnification values near the caustics. Consequently, the magnification values provided by larger pixels tend to reflect an average rather than capturing the enhanced magnification in the localised tips of the micro-caustics. We expect to observe a decrease in the maximum magnification obtained in our simulations as the pixel size increases.

To quantitatively evaluate this effect, we conducted multiple simulations using identical parameters while varying only the pixel size. By obtaining the approximated magnification probability for each simulation, we can compare the results at the high magnification end to analyse the impact of the pixel size. In our study, we focussed on simulating an isolated microlens that follows the well-established power-law behaviour, $A > \mu \propto \mu^{-2}$. This choice allowed us to investigate the effect of pixel size and its influence on the resulting magnification probability.

Fig. (A.1) illustrates the magnification probability obtained for three distinct pixel sizes. It is evident that smaller pixel sizes yield better approximations of the magnification probability,

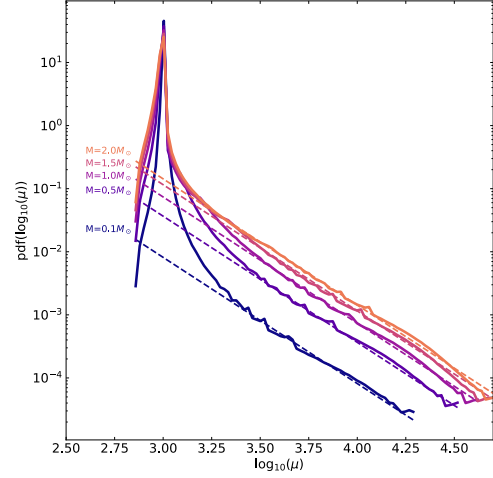


Fig. A.2. Magnification probability approximation for an inverse ray-shooting simulation of an isolated microlens near a macro-CC. Each curve represents a different microlens mass. For a fixed pixel size (2 nas), the maximum magnification we can measure increases with the mass of the microlens. The dashed lines represent the μ^{-2} power-law that governs the magnification probability at high magnification factor in this regime. The amplitude of this power-law increases quadratically as seen in Sec. (4).

particularly at higher magnifications. For our purposes, the most relevant finding is the fact that increasing the pixel size reduces the maximum magnification while maintaining the power-law behaviour up to the maximum magnification. This means that the power-law cutoff observed at extreme magnification values when Σ_{eff} approaches 1 is not a resolution effect, but a real property of the probability of magnification.

It is important to consider the computational time required for numerical simulations, which scales as $O(N^2)$, where N represents the number of pixels in one of the dimensions of the simulated region. Reducing the pixel size by a factor of 1/2 increases the computational time by a factor of 4. Therefore, a trade-off exists between the desired simulation time and the accuracy of the magnification probability. After careful analysis, we have determined that a pixel size of 2 nas is well-suited for our specific task, of microlenses at cosmological distances and in realistic scenarios, striking a balance between computational efficiency and the required level of accuracy for the magnification estimates. A pixel size of 2 nas represents well the high magnification tail of the magnification PDF up to magnifications where the probability is negligible, while smaller pixels require much larger simulation times and disk space. For larger pixel sizes that are less resource consuming to simulate we observe resolution effects at non-negligible probabilities.

A.2. Mass of the microlenses

An additional resolution effect arises from the mass of the microlens itself. Smaller microlenses exhibit smaller micro-caustics, as the size of the caustics scales linearly with the Einstein angle (θ_E), which is proportional to the mass (M) of the lens. Consequently, lighter lenses have smaller caustics, and for a fixed pixel size, the maximum magnification achievable from these lenses is reduced. In our simulations, we consider microlens masses obtained from a distribution derived by

Spera et al. (2015), which mimics an old stellar population with masses not exceeding $1.5 M_{\odot}$, while remnants and ordinary stars are not lighter than $0.1 M_{\odot}$. Our primary focus is on the smallest lenses within this distribution. Similar to our previous analysis, we simulate isolated masses ranging from 0.1 to $2 M_{\odot}$ and analyse their respective magnification probabilities. The results, presented in Fig. (A.2), confirm our expectations, with larger masses exhibiting higher maximum magnification values. However, it is worth noting that even in the case of the smallest lenses from our simulations, this resolution effect does not significantly impact our findings. The figure shows the scaling discussed in the main paper, where the amplitude of the power-law scales quadratically with the mass of the microlens.

A.3. Magnification approximation at the lowest values

The inverse ray-shooting technique employed in our study approximates the magnification in the source plane pixels based on the number of photons traced from the lens plane. By utilising the deflection angle in the lens plane and solving the lens equation, we can determine the corresponding position in the source plane for each position in the lens plane. This approach yields accurate results when the lens plane has a sufficiently large area and good resolution. However, a limitation of this method arises from the fact that we can only obtain integer values for the magnification, while in reality, the magnification can be any real number. Therefore, this technique effectively rounds the magnification value to the closest integer. This limitation does not significantly impact our results when the magnification values differ by large factors (e.g. 10^4 versus $10^4 + 1$). However, it becomes more significant when the magnification values are of order 1, as the rounding leads to a noisier estimate of the probability of magnification.

In our positive parity simulations, the magnifications fall within the range of $\gtrsim \mu_m$. When the effective surface density increases, there is a slight reduction in this range. However, since μ_m is typically large, the minimum magnifications in our observations are not affected by this effect.

In simulations with negative parity, a demagnification effect relative to μ_m appears. This demagnification exhibits also a power-law behaviour, and its parameters can be constrained by studying larger magnifications where this effect is negligible. Additionally, a distinctive feature in the form of a skewed log-normal bump emerges in the magnification distribution at the lowest magnification values. The position of this bump decreases as μ_r (as shown in Fig. (14)), and it is influenced by this demagnification effect. It is important to note that characterising these peaks in the magnification distribution is challenging, and our accuracy in this regard may not be as precise as in other parts of the probability density functions. Nevertheless, our fitting procedure successfully captures the main elements of this portion of the model. In the super-critical simulations with negative parity, the combination of the demagnification effect and reduced statistics due to a smaller area in the source plane (scaled by a factor of μ_m , which is larger for these simulations) has hindered our ability to accurately model the low magnification excess observed in these specific cases. Despite this limitation, we have observed an interesting trend: the fixed position of these peaks begins to shift towards larger magnification values as Σ_{eff} increases.

Appendix B: Sub-structure in the central peak

In our simulations, we have discovered an intriguing feature characterised by the emergence of substructure around the mode of the magnification probability. This effect is observed primarily at low values of Σ_{eff} , and does not significantly impact the probability distribution at the largest magnifications, which is the driving force our current study. However, it is important to note that this substructure may have implications in other cases or scenarios. While we do not provide a detailed modelling of this substructure in the present paper, we offer a brief qualitative description.

In contrast to the previous description of a single peak characterised by a log-normal distribution, our recent observations reveal the presence of two distinct peaks around the expected value μ_m . Moreover, the width, amplitude, and centre of these newly discovered peaks do not exhibit a clear scaling relationship with Σ_{eff} as observed in previous analyses, although we observe that this effect manifests itself only when $\Sigma_{\text{eff}} < 0.3$.

To gain a better understanding of the observed substructure in the magnification probability, we conducted a series of simulations where two out of the three main model parameters, namely μ_t , μ_r , and Σ_* , were fixed while varying the third parameter. The results of these simulations are presented in Fig. (B.1).

Our main findings can be summarised as follows:

- The parameter μ_t plays a crucial role in determining the positions of the peaks in the magnification probability. Increasing μ_t leads to a shift of both peaks towards each other. However, the lower magnification peak experiences a larger shift compared to the other peak. The ratio between the heights or amplitudes of the peaks remains constant, although the amplitude of both peaks decreases as a result of the increased Σ_{eff} .
- The parameter Σ_* is correlated with the width of the peaks. Larger Σ_{eff} values result in broader peaks. The centres of the peaks remain constant, and the ratio of amplitudes also remains constant. Similar to the case of varying μ_t , the increase in Σ_{eff} leads to a reduction in the amplitude of both peaks.
- The ratio of amplitudes is controlled by the parameter μ_r . For lower radial magnifications, the peak with higher magnification dominates. The ratio between the amplitude of this peak and its lower magnification counterpart decreases as μ_r increases. At $\mu_r \approx 1.5$, the ratio is approximately 1. For radial magnifications larger than 2.5, only one peak is observed. The position of the peaks may also be slightly influenced by μ_r .

The observed substructure in the magnification probability manifests itself primarily for low radial magnifications and low Σ_{eff} values, typically of the order of $\lesssim 10\%$ of the critical surface density. In our simulations, where the critical surface density is approximately $\Sigma_{\text{crit}} \approx 3100 M_{\odot}/\text{pc}^2$, we found that this effect is at it highest for $\mu_t = 1000$, $\Sigma_* = 0.31 M_{\odot}/\text{pc}^2$, and $\mu_r = 1.5$. For the sake of simplicity, we have presented the observed substructure effect only in the context of negative parity simulations. However, it is important to note that we have also observed this effect in simulations with both positive and negative parity.

To the best of our knowledge, the substructure effect that we have observed in our simulations represents a newly discovered phenomenon whose modelling is left for future work.

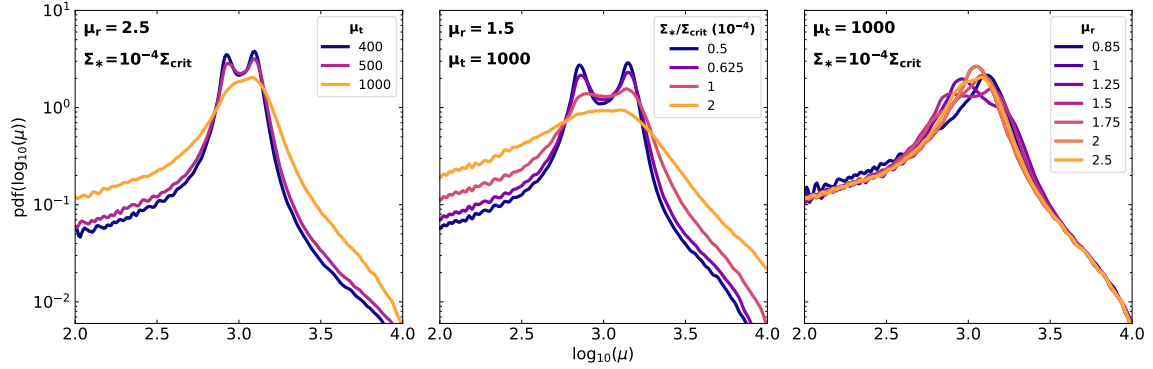


Fig. B.1. Zoom in on the magnification probability around the mode. We show the substructure of this peak and its changes as we fixed two parameters and allowed the remaining one to change. μ_t (left panel) controls the position of each peak, while its width is only scaled with Σ_* (middle panel). The ratio of peak amplitudes depends on μ_r (Right panel).

Appendix C: Fitting data points

Table C.1. Best-fit parameters' low effective surface mass density and positive parity as described in Eqs. (14 - 16).

$\Sigma_{\text{eff}}/\Sigma_*$	A^+	μ_A^+	σ_A^+	B^+	μ_B^+	σ_B^+	α_B^+	a^+	β_a^+	C^- (10^{-2})	μ_C^+	σ_C^+
38.75	0.902	2.953	0.039	24.753	2.991	0.015	0.140	0.007	-8.948	0.43	3.355	0.325
77.50	1.270	2.959	0.052	16.291	2.981	0.022	0.072	0.013	-8.562	0.79	3.316	0.357
155.00	1.800	2.965	0.067	9.483	2.964	0.035	0.120	0.025	-7.942	2.81	3.102	0.406
310.00	-	-	-	3.737	2.882	0.087	1.931	0.061	-6.168	9.75	3.149	0.321
387.50	-	-	-	2.893	2.850	0.130	2.200	0.060	-6.060	8.23	3.100	0.377
620.00	-	-	-	2.255	2.806	0.145	2.200	0.092	-4.781	11.61	3.183	0.334
775.00	-	-	-	1.800	2.757	0.220	2.800	0.093	-4.897	12.78	3.102	0.375
1240.00	-	-	-	1.289	2.696	0.290	3.633	0.123	-3.390	15.10	3.164	0.349
1550.00	-	-	-	1.020	2.637	0.403	4.276	0.135	-3.020	17.47	3.166	0.339

Notes. μ_A^+ , μ_B^+ , and μ_C^+ are given in $\log_{10}$ units.

This appendix contains the best-fit parameters obtained via χ^2 minimisation of the models described in section 6 against the simulated data. The data points shown in tables (C.1 - C.4) are fitted to simple models easily extrapolated (powerlaws, log-normals, constants...). More info can be found in the Github repository⁴.

⁴ https://github.com/ChemaPalencia/M_SMiLe

Table C.2. Best-fit parameters' low effective surface mass density and negative parity as described in Eqs. (17, 21, 22).

$\Sigma_{\text{eff}}/\Sigma_*$	A^-	μ_A^-	σ_A^-	α_A^-	a^-	β_a^-	B^-	μ_B^-	σ_B^-	b^-	β_b^-	C^- (10^{-3})	μ_C^-	σ_C^-
38.75	0.03	1.48	0.124	5.7	0.032	7.79	23.20	3.004	0.013	0.007	-10.16	0.21	4.039	0.109
77.50	0.05	1.48	0.149	6.2	0.052	6.81	11.34	3.009	0.027	0.013	-8.84	0.50	4.009	0.127
155.00	0.09	1.47	0.158	6.0	0.094	5.79	5.69	3.014	0.049	0.025	-7.34	1.33	3.998	0.162
310.00	0.04	0.57	0.222	6.3	0.175	4.02	2.54	3.048	0.098	0.056	-5.41	4.22	3.791	0.181
387.50	0.20	1.48	0.189	5.7	0.206	3.28	1.93	3.055	0.118	0.070	-4.40	5.25	3.861	0.202
620.00	0.06	0.57	0.228	3.6	0.289	1.95	1.11	3.031	0.201	0.085	-2.29	9.03	3.868	0.216
775.00	0.27	1.51	0.180	3.0	0.399	1.14	0.87	2.997	0.264	0.165	-2.00	13.65	3.821	0.254
1240.00	0.04	0.58	0.228	1.6	0.418	0.50	0.70	2.744	0.490	0.211	-2.00	26.25	3.800	0.251
1550.00	0.27	1.62	0.209	1.0	0.496	0.50	0.66	2.731	0.498	0.205	-2.00	31.50	3.799	0.273

Notes. μ_A^- , μ_B^- , and μ_C^- are given in $\log_{10}$ units.

Table C.3. Best-fit parameters' high effective surface mass density and positive parity as described in Eq. (23).

$\Sigma_{\text{eff}}/\Sigma_*$	A^+	μ_A^+	σ_A^+	α_A^+	B^+	μ_B^+	σ_B^+	α_B^+
0.50	1.03	476	0.369	2.67	0.013	2256	0.380	2.88
0.75	0.99	476	0.371	2.57	0.090	1148	0.362	2.38
0.88	1.00	454	0.355	2.31	0.140	946	0.349	2.50
1.00	0.91	415	0.352	2.11	0.310	728	0.325	2.40
1.23	0.79	377	0.326	1.90	0.480	728	0.314	2.14
1.50	0.58	349	0.303	1.71	0.700	693	0.312	1.58
2.00	0.47	320	0.277	1.08	0.880	674	0.305	1.15
2.36	0.42	308	0.256	0.86	0.990	687	0.298	0.89
2.46	0.50	345	0.243	0.74	1.010	764	0.274	0.79
2.80	0.69	388	0.233	0.50	1.070	828	0.259	0.55
3.00	0.75	404	0.223	0.34	1.120	957	0.233	0.36
3.47	0.81	412	0.206	0.14	1.170	1042	0.222	0.14
4.00	0.88	424	0.195	0.07	1.180	1107	0.217	0.05
4.73	0.90	419	0.188	-	1.190	1125	0.213	-
5.20	0.89	412	0.184	-	1.170	1107	0.208	-
6.00	0.88	408	0.172	-	1.220	1063	0.197	-
7.50	0.76	356	0.161	-	1.390	876	0.201	0.07
9.45	0.66	310	0.138	-	1.560	736	0.193	0.19
11.00	0.60	280	0.124	-	1.690	624	0.188	0.26
12.00	0.58	276	0.121	-	1.740	578	0.184	0.43
14.00	0.54	254	0.116	-	1.890	504	0.181	0.55
16.00	0.45	242	0.107	-	1.970	470	0.177	0.62
20.00	0.41	228	0.100	-	2.070	399	0.172	0.74
24.00	0.38	220	0.088	-	2.150	349	0.172	0.82
30.00	0.32	213	0.080	-	2.170	298	0.170	0.94
36.00	0.30	205	0.070	-	2.240	266	0.169	1.01
42.00	0.28	188	0.064	-	2.330	242	0.163	1.06
48.00	0.26	172	0.060	-	2.430	226	0.159	1.10
54.00	0.24	150	0.059	-	2.510	211	0.153	1.11
63.00	0.23	134	0.058	-	2.610	193	0.147	1.11
66.50	0.21	127	0.058	-	2.680	186	0.144	1.11

Table C.4. Best-fit parameters' high effective surface mass density and negative parity as described in Eqs. (24 - 25).

$\Sigma_{\text{eff}}/\Sigma_*$	A^-	μ_A^-	σ_A^-	B^-	μ_B^-	σ_B^-	C^-	μ_C^-	σ_C^-	a^-	β_a^-
1.00	0.82	639	0.355	0.16	2532	0.272	-	-	-	0.47	0.95
1.23	0.81	602	0.338	0.29	1920	0.274	-	-	-	0.47	1.02
1.50	0.77	525	0.295	0.46	1545	0.279	-	-	-	0.48	1.23
2.00	0.72	479	0.272	0.64	1235	0.281	-	-	-	0.46	1.41
2.36	0.65	443	0.262	0.72	1172	0.286	-	-	-	0.49	1.72
2.46	0.63	429	0.258	0.81	1100	0.284	-	-	-	0.47	1.79
2.80	0.59	410	0.236	0.88	1038	0.281	-	-	-	0.48	1.82
3.00	0.54	383	0.219	0.93	967	0.272	0.09	2190	0.127	0.48	2.14
3.47	0.42	328	0.207	1.07	941	0.260	0.13	1974	0.160	0.49	2.24
4.00	0.36	306	0.189	1.13	905	0.243	0.17	2104	0.177	0.48	2.73
4.73	0.34	232	0.174	1.11	843	0.238	0.24	1996	0.184	-	-
5.20	0.30	232	0.155	1.02	727	0.248	0.38	1843	0.177	-	-
6.00	0.26	214	0.131	1.00	611	0.250	0.76	1735	0.162	-	-
7.50	0.14	210	0.103	0.97	549	0.265	0.81	1822	0.134	-	-
9.45	0.04	210	0.072	1.05	567	0.248	0.83	1627	0.124	-	-
11.00	-	-	-	1.15	558	0.236	0.93	1650	0.122	-	-
12.00	-	-	-	1.32	538	0.189	1.08	1442	0.129	-	-
14.00	-	-	-	1.37	557	0.139	1.49	1261	0.131	-	-
16.00	-	-	-	1.39	567	0.143	1.61	1124	0.122	-	-
20.00	-	-	-	1.08	602	0.098	2.42	890	0.122	-	-
24.00	-	-	-	0.47	438	0.093	2.84	738	0.122	-	-
30.00	-	-	-	0.16	306	0.079	3.19	605	0.119	-	-
36.00	-	-	-	-	-	-	3.38	511	0.117	-	-
42.00	-	-	-	-	-	-	3.41	442	0.117	-	-
48.00	-	-	-	-	-	-	3.43	383	0.117	-	-
54.00	-	-	-	-	-	-	3.46	346	0.115	-	-
63.00	-	-	-	-	-	-	3.51	296	0.115	-	-
66.50	-	-	-	-	-	-	3.51	278	0.112	-	-

Microlensing at Cosmological Distances: Event Rate Predictions in the Warhol Arc of MACS 0416

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A&A, 699, A295 (2025)

<https://doi.org/10.1051/0004-6361/202555447>

Microlensing at cosmological distances: Event rate predictions in the Warhol arc of MACS 0416

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Received 8 May 2025 / Accepted 9 June 2025

ABSTRACT

Highly magnified stars ($\mu > 100$) are now routinely identified as transient events at cosmological distances thanks to microlensing by intra-cluster stars near the critical curves of galaxy clusters. Using the *James Webb* Space Telescope (JWST) in combination with the *Hubble* Space Telescope, we outline here an analytical framework that is applied to the Warhol arc (at $z = 0.94$) in the MACS 0416 galaxy cluster (at $z = 0.396$), where over a dozen microlensed stars have been detected to date. This method is general and can be applied to other lensed arcs. Within this lensed galaxy, we fit the spatially resolved spectral energy distribution spanned by eight JWST-NIRCam filters combined with three ACS filters, for accurate lensed star predictions in 2D. With this tool we can generate 2D maps of microlensed stars for well-resolved arcs in general, incorporating wavelength dependence and limiting apparent magnitude. These maps can be directly compared with planned cadenced campaigns from JWST and Hubble, offering a means to constrain the initial mass function and the level of dark matter substructure.

Key words. gravitational lensing: strong – gravitational lensing: micro – methods: observational – stars: abundances – stars: AGB and post-AGB – galaxies: high-redshift

1. Introduction

Galaxy clusters are the most powerful gravitational lenses in the Universe. Comprising hundreds to thousands of galaxies, hot gas, and dark matter, these clusters distort space-time, bending the paths and amplifying the light of distant galaxies behind them. This well-known gravitational lensing effect results in a range of phenomena for background sources, such as altering their apparent position and shape, amplifying their brightness, and creating multiple images with different arrival times (Schneider et al. 1992).

Galaxies near the critical curves (hereafter CCs) of galaxy clusters form elongated images, commonly known as lensed arcs, which are typically magnified by a factor of $\mathcal{O}(10)$. CCs map onto caustics in the source plane with a well-defined dependency of the magnification, μ , with respect to the distance, d , to the caustic, $\mu = \mu_o / \sqrt{d}$, where for clusters, μ_o usually takes values of ~ 10 – 30 when d is expressed in arcseconds. Sources with radius, R , (such as compact star-forming regions where $R \lesssim 0.1''$) within a lensed galaxy crossing a caustic, can achieve

maximum magnifications of $\mu_{\max} = \mu_o / \sqrt{R} \approx 50$. Even smaller objects, such as stars $R \lesssim 10^{-8}$ arcsec, can theoretically achieve extreme magnification factors, $\mu_{\max} \sim 10^6$ (Miralda-Escude 1991). Such large magnification is possible if the distribution of mass is smooth and the lensed image lies extremely close to the CC. However, microlenses (such as stars from the intracluster medium) and small-scale structures within the cluster (dwarf galaxies, globular clusters, sub-haloes predicted in abundance by massive particle cold dark matter (CDM), or pervasive density modulations predicted in ultra-light particle CDM or Wave DM) perturb the gravitational potential, which reduces this maximum achievable magnification of the order of tens of thousands (Venumadhav et al. 2017; Diego et al. 2018; Weisenbach et al. 2024). This reduction in magnification near the CC is compensated (as demanded by flux conservation) by larger magnification factors around the microlenses and small-scale substructures farther away from the CC. These small-scale structures create a web of micro- and milli-caustics near the cluster caustic.

When a background star moves through this web of micro-caustics the combined lensing effect from different microlenses produces many unresolved images (microimages) of the same

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star, which results in a combined magnification typically of the order of thousands, hence the term ‘extremely magnified stars’. The angular separation between these images is typically on the scale of microarcseconds (Venumadhav et al. 2017), hence far smaller than the resolution capabilities of current telescopes that operate with resolutions of tens of milliarcseconds for the most powerful optical and infrared telescopes. As a result, we observe a single image of the background star, which represents the combined magnification (or flux) of all the microimages.

Due to the relative motion between the source and the web of microcaustics, this magnification varies over time, and peaks when the source crosses one of the multiple microcaustics and then gradually decreases until the star is no longer detectable. These events can last from a few days to several years, depending primarily on the redshift of the source, its luminosity, the mass of the microlens, and more importantly the relative velocity and direction of motion with respect to the web of microcaustics.

Microlenses can also be made of compact dark matter objects, such as primordial black holes (PBHs) (Venumadhav et al. 2017; Diego et al. 2018; Carr & Kühnel 2020; Green & Kavanagh 2021; Palencia et al. 2024). However, current studies are consistent with a fully stellar microlens scenario (Oguri et al. 2018; Müller & Miralda-Escudé 2025). The discovery of Icarus (Kelly et al. 2018), a lensed star at redshift $z = 1.49$, marked the beginning of this field. Since then, $O(100)$ of these lensed high-redshift stars have been discovered with both the *Hubble* Space Telescope (HST) and the *James Webb* Space Telescope (JWST).

Notable discoveries include Earendel, the farthest known star with a redshift of $z \approx 6$ (Welch et al. 2022a,b) (see Ji & Dai (2025) for caveat on the source size constraint), and the Dragon Arc in Abell 370, with 46 detected events in a single galaxy (Fudamoto et al. 2025). These discoveries open a new frontier in gravitational lensing, and enable the study of stars in the early Universe, shortly after reionization, their evolution, and potentially even the first generation of Population III stars (Zackrisson et al. 2024). They also offer new ways to investigate the nature of dark matter, either through its presence as compact objects within galaxy clusters (Diego et al. 2018; Oguri et al. 2018), or via small-scale fluctuations in the dark matter density, as predicted by wave dark matter models (Amruth et al. 2023; Broadhurst et al. 2025). High-cadence light curves of caustic-crossing events can serve as sensitive probes of dark matter microstructures formed in the early Universe (Dai & Miralda-Escudé 2020).

MACS J0416.1-2403 (Ebeling et al. 2001; Mann & Ebeling 2012), commonly referred to as MACS 0416, at redshift $z = 0.396$, is one of six galaxy clusters studied by HST as part of the *Hubble* Frontier Fields programme (HFF, Lotz et al. 2017). Later observed by JWST, MACS 0416 holds the record for the lens with the largest number of spectroscopically confirmed lensed galaxies (Rihtaršič et al. 2025). Several of the lensed galaxies cross the CC, hence with portions of these galaxies reaching maximum magnification.

This cluster has been extensively studied in the optical and infrared (IR) bands, with data from both telescopes being used by experts to derive best-fit lens models. Prior to JWST observations of this cluster, some of the earliest lensed stars at cosmological distances ($z \gtrsim 1$) were discovered in this cluster by HST (Chen et al. 2019; Kaurov et al. 2019). Later on, Flashlights, a HST programme for taking extremely deep images of lensing galaxy clusters, identified about a dozen additional lensed stars (Kelly et al. 2022) in two background galaxies, named Spock and Warhol, both at redshift $z \approx 1$. Future deeper

observations of these galaxies are expected to yield yet more microlensing events.

The modest redshift of $z \approx 1$ for Spock and Warhol, with a distance modulus of 44.10, increases the detection rates, since we can observe fainter stars. Typically, observations are biased towards the brightest stars, as they require a smaller (and more likely) magnification, in order to be detectable. This makes these two galaxies prime targets for studying the abundance of different star types at redshift $z = 1$, specifically probing the high-mass end of the initial mass function (IMF), as we are limited to the most luminous and hence brightest stars.

In this paper, we focus on Warhol and provide a detailed forecast for the number of detectable events across eight NIRCам filters. We also present the first spatial distribution forecast for the number of events. Previous studies have made similar predictions for Spock (Diego et al. 2024b). Warhol and Spock are not the only galaxies in MACS 0416 that host microlensing events. Mothra (Diego et al. 2023b), a lensed star at redshift $z = 2.09$, was also discovered in a galaxy behind MACS 0416; it provides interesting constraints on dark matter, as it demands the presence of a small millilens near the lensed star in order to explain its peculiar magnification.

Future dedicated observations of this galaxy cluster are expected to lead to many more detections, which would offer valuable insights into the nature of dark matter within galaxy clusters, as well as the formation and evolution of stars at high redshift. A larger number of events will allow us to test the results derived from this paper.

This paper is structured as follows. Section 2 details the data used in our analysis. In Section 3, we outline the methodology used to transition from data analysis to forecasting the rate of lensed events. The results, which cover both spatial and integrated event counts across eight NIRCам filters, are presented in Section 4. In Section 5, we discuss the prospects for detecting lensed high-redshift stars in MACS 0416 and similar arcs. Finally, our conclusions are summarised in Section 6.

Unless stated otherwise, we assume a flat cosmological model with $\Omega_m = 0.3$, $\Lambda = 0.7$, and $h = 0.7$ ($100 \text{ km s}^{-1} \text{ Mpc}^{-1}$). All magnitudes are given in the AB system. The Warhol arc is at redshift $z = 0.94$, and within the adopted cosmological model, 1 arcsec corresponds to $\sim 15.3 \text{ kpc}$.

2. Data

In this section, we describe the high-resolution HST and JWST mosaics of MACS 0416 that we used, as well as the scientific programmes from which the data were obtained. Finally we present the calibrated lens model used in this work.

To perform all photometric spectral energy distributions (SED) fitting, we used a combination of 11 photometric mosaic images of MACS 0416, using public data from HST (described further below) and the JWST Prime Extragalactic Areas for Reionisation and Lensing Science (PEARLS) program (Windhorst et al. 2023). We focus on a 24-arcsecond region centred on Warhol, using mosaics at a pixel scale of 30 mas per pixel.

For the lensing model of MACS 0416, we used the free-form lens model from Diego et al. (2024a) based on the latest JWST data. For the contribution of microlenses at the position of Warhol, we assumed a mean surface mass density of microlenses, $\Sigma_* = 59.39 \text{ M}_\odot \text{ pc}^{-2}$ (Kaurov et al. 2019), which is subsequently modulated by the intra-cluster light (ICL). The rapidly declining brightness near the brightest cluster galaxy

(BCG) suggests that the surface mass density varies along the arc, rather than remaining constant.

2.1. HST

We used HST mosaics that were constructed by PEARLS team members following the techniques first described by Koekemoer et al. (2011, 2013). These mosaics include public archival HST data on MACS 0416, specifically data from the Hubble Frontier Fields (HFF, Lotz et al. 2017), and the Beyond Ultra-deep Frontier Fields and Legacy Observations program (BUFFALO, Steinhardt et al. 2020), as well as other public archival HST programs available in the STScI MAST Archive¹. For this paper, the mosaics that we used include all the HST Advanced Camera for Surveys (ACS) observations on this cluster in the F435W, F606W, and F814W filters, which cover a wavelength range of approximately 0.35 to 0.96 microns.

2.2. JWST

We used JWST mosaics constructed by PEARLS team members, using data from the PEARLS program obtained in eight filters from the NIRCam instrument: F090W, F115W, F150W, F200W, F277W, F356W, F410M, and F444W, which cover wavelengths from 0.795 to 4.981 microns. The corresponding magnitude depths, at which a point-source signal-to-noise ratio of 5 is achieved for fake sources injected (Williams, in preparation), are 29.7, 29.5, 29.5, 29.5, 29.5, 29.6, 29.0, and 29.3 m_{AB} , respectively. The HST and JWST mosaics that we used were all produced in a way that aligns them to a common astrometric grid, at a pixel scale of 30 mas per pixel. See Windhorst et al. (2023) for further details about the construction of the mosaics that we use here.

2.3. Lens model

We utilised the free-form code WSLAP+ (Diego et al. 2005, 2007) to derive the mass distribution and magnification of the macromodel. WSLAP+ is a hybrid method that allows the combination of weak and strong lensing constraints. For MACS0416, we only used strong lensing constraints. The method exploits the linearity (with mass) of the deflection field, which can be expressed as a linear combination of functions of the mass. It describes the mass distribution as the combination of two components: (i) a superposition of Gaussians located either on a regular grid or a adaptive grid, with additional Gaussians or increased resolution in specific regions of the cluster where higher mass is located, effectively acting as a prior for the mass distribution. This represents the smooth global component of the cluster lens. (ii) A compact component that accounts for the individual masses of cluster members, assumed to be proportional to their flux in a certain reference band. For this component, there is only one free parameter: the mass scaling factor or mass-to-light ratio of the cluster members. Different layers can be introduced to account for variations in the mass-to-light ratio among different types of galaxies within the cluster. The strong- and weak-lensing problem is then formulated as a system of linear equations:

$$\Theta = \Gamma X, \quad (1)$$

where the array X contains the free parameters: the grid points for the Gaussian approximation of the smooth component, the

mass-to-light scalings, and the identified sources positions. The lensing observables, including the positions of strongly lensed sources (and when available the two shear components), are arranged in the Θ array. The Γ matrix is known and depends on the fiducial grid and redshifts of the lensed galaxies. The solution X is obtained by inverting a scalar function constructed from the system of linear equations. The lens model for MACS0416 is derived from a very large number of constraints (343) and is described in detail in Diego et al. (2024a).

3. Methodology

The objective of this paper is to derive a realistic forecast for the expected number of microlensing events on the Warhol arc. These transient events originate from stars within the lensed galaxy whose apparent magnitude, m , is typically well below the detection threshold, which makes them undetectable in most observational epochs. However, a temporary magnification boost, μ_{micro} , arises when the source approaches a microcaustic, which brings their apparent magnitude, m_{obs} , above the detection threshold:

$$m_{\text{obs}} = m - 2.5 \log_{10}(\mu_{\text{micro}}) \lesssim m_e, \quad (2)$$

where m_e represents the detection threshold. The average number of detected stellar transient events depends on two main factors: the micro-magnification probability distribution function (PDF) and the background stellar population. The former describes the micro-magnification as a random variable that depends on the tangential component of the macrolens, μ_t , the radial component, μ_r , and the surface mass density of microlenses, Σ_* (see Palencia et al. 2024, for further details). Furthermore, local perturbations in the macrolens caused by millilenses can reshape the local macro-magnification map, thereby altering the magnification PDFs. The latter is influenced by various parameters, such as the age of the population, its metallicity, environmental constraints like dust attenuation, and the total stellar mass, among others. These parameters collectively shape the SED of the population, though some degeneracies exist, such as the dust and star formation history (SFH) degeneracy, where dusty and older populations both produce similar features with attenuated UV-rest-frame emission. Consequently, the number and properties of the source stars can be inferred from the total SED measured from the galaxy.

In this section, we present our methodology. We used a Markov chain Monte Carlo (MCMC) for the estimation of the stellar population parameters from SEDs across different regions of Warhol, which allowed us to construct a set of source stellar populations. We then integrated local magnification PDFs that reflect the peculiarities of the lens model, and applied them to the source stars to forecast the average expected number of events in various NIRCam filters at multiple detection thresholds. The results are presented and further discussed in Sections 4 and 5, respectively.

3.1. SED fitting

The first step in our analysis involves retrieving the SED of the Warhol arc. To achieve this, we carefully removed the contaminating foreground light from the galaxy cluster, specifically the ICL. We utilised SExtractor (Bertin & Arnouts 1996), a publicly available code that enables source detection, deblending of overlapping or nearby sources, and image background estimation. Additionally, we used SExtractor to delineate the

¹ <https://archive.stsci.edu>

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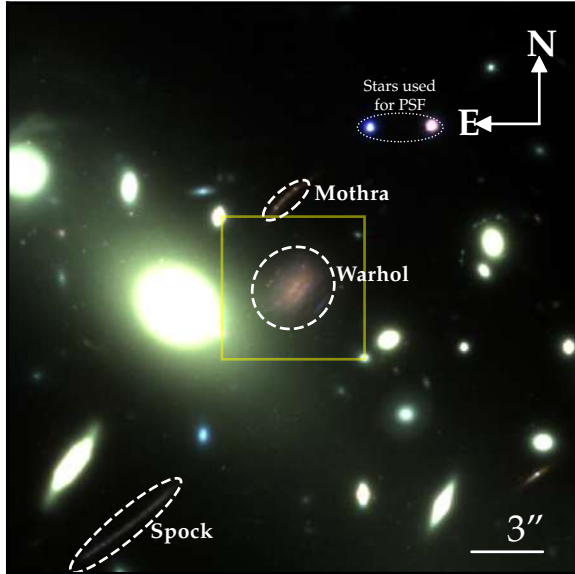


Fig. 1. MACS 0416 colour image showing a $24'' \times 24''$ region centred on the Warhol arc. Other prominent caustic crossing galaxies where microlensing transients have been identified are also marked. The colour image was created by combining filters from two instruments: NIRCcam (F444W and F356W for the red channel; F277W, F200W, and F150W for the green channel; F115W and F090W for the blue channel) and WFC3 (F814W, F606W, and F435W for the blue channel). The yellow square outlines the central region analysed in this study.

region of pixels that corresponds to the Warhol arc. The next step involved subtracting the remaining part of the BCG emission, which **SExtractor** identifies as a source and treats independently from the image background. In our specific case, the BCG is almost directly in front of Warhol, so its subtraction is crucial for recovering an accurate representation of Warhol's SED. For this task, we relied on **PetroFit** (Geda et al. 2022), a Python package designed for tasks such as galaxy light profile fitting. We modelled the main BCG using a Sérsic profile (Sérsic 1963), which was then subtracted from the images. This removal of both foreground contaminants was performed separately for each filter image, and resulted in a set of contamination-free images across different filters for the Warhol arc.

The next step is to match the point spread function (PSF) of our images to prevent biases and ensure consistent flux measurements at the pixel level, thereby guaranteeing accurate photometry. Since broader PSFs spread the light of sources more, matching the images to the broadest PSF among all the filters ensures that we achieve bias-free photometry in each pixel of the arc. To accomplish this, we first need to determine the PSF for each filter. Foreground stars, being point-like sources (significantly smaller than the instrument's response), directly reflect the shape of the instrument's PSF in their images. We used two stars located north-west of Warhol (see Fig. 1) to construct an empirical PSF using **Photutils** (Bradley et al. 2024). Next, we calculated a convolution kernel for each image. This kernel is the inverse Fourier transform of the ratio between the Fourier transforms of the PSFs, specifically the narrower PSF over the broader one. We then convolved the images with narrower PSFs using the corresponding kernels, which resulted in a set of

PSF-matched, contaminant-free images that were ready for accurate photometric estimation.

The most accurate estimation of the background stellar population would involve fitting the SED at each pixel independently. However, as our method relies on MCMC to find the set of stellar population parameters that best reproduces the SEDs, it would be extremely computationally demanding. Instead, we grouped pixels with similar SEDs. To refine this process, we first masked the known brightest globular clusters (GCs) and potential transient events in our images; these appear as black dots in Fig. 2.

Globular clusters in MACS 0416 present challenges for observing microlensing transients behind them, but their local modifications to the macromodel and enhanced microlens density can significantly affect the rate of events around such millilenses. As for pixels containing the brightest transients in Warhol, we masked them because their SEDs do not represent the underlying stellar population; the transient's flux outshines the background, which prevents accurate SED determination. Both GCs and bright transients appear as local maxima, which makes their identification straightforward. We removed an area equivalent to twice the σ of an effective Gaussian centred at each maximum. Since the masked areas are relatively small compared to the size of the arc, we do not anticipate a significant underestimation of the total number of transient events expected in the arc, though recovering the full spatial distribution of events within the arc remains challenging.

Once transients and GCs are masked, we applied various methods to group pixels with similar SEDs. First, we performed a principal component analysis (PCA) to reduce the dimensionality of our dataset from 11 filters to three principal components, which capture 96.6% of the colour variability in our data. Additionally, we conducted a colour-colour analysis by plotting the differences between pairs of filters. Both PCA and colour-colour analysis can be combined with clustering techniques such as K-means, DBSCAN, or by applying linear cuts. We clustered the pixels based on two criteria: cuts performed in the reduced data spaces from PCA and colour-colour analysis, and spatial correlation to ensure that regions are spatially connected. The final regions identified through clustering, along with their median SEDs and 1σ contours, are shown in Fig. 2.

To estimate the photometric error, we used aperture photometry with different apertures around the Warhol arc. By analysing the observed variability in flux over area measurements around the arc, we can infer an approximate photometric error per square arcsecond. This error can then be extrapolated to the Warhol regions with known areas.

With accurate photometric SEDs for specific regions, calculated as the total contribution from all pixels within that region, and estimates of the flux errors for each filter, we can now fit these SEDs to reference models. For this purpose, we used the Flexible Stellar Population Synthesis (FSPS) library (Conroy et al. 2009, 2010; Conroy & Gunn 2010; Johnson et al. 2024). FSPS is a sophisticated tool for modelling the SED of stellar populations and offers flexible input options that allow users to specify various parameters, including IMFs, metallicities, masses, redshifts, ages, and more. It also allows for the incorporation of environmental conditions, such as dust attenuation, dust and nebular emission, or even emission from active galactic nuclei (AGNs) with different torus optical depths, which shape the AGN's contribution to the total SED.

One of the many outputs that FSPS generates is the photometric fluxes measured in the filters we are using. This enables us to compare model predictions to our data and explore the parameter space to find the best-fit parameters that produce an

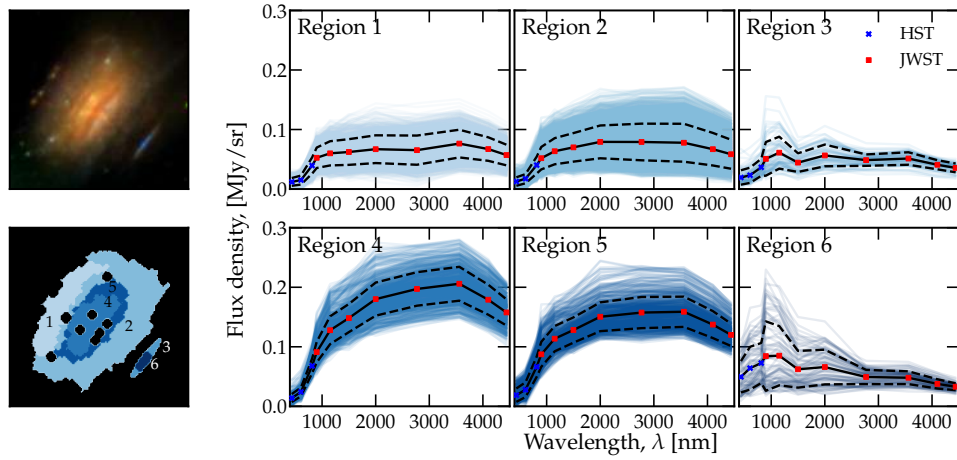


Fig. 2. Similar SED regions after clustering. Top-left panel: Colour image of Warhol. Bottom-left panel: regions of similar SED pixels on Warhol. Black dots indicate masked regions, globular clusters from the intracluster medium, or transient events whose SEDs do not represent the underlying stellar population. Second, third, and fourth columns: SEDs for each pixel in the different regions. The solid black line represents the median SED, while the dashed black lines indicate the 1σ contour. Violet crosses and red squares indicate the photometric measurements obtained using HST and JWST filters, respectively, all introduced in Section 2.

SED that minimises a χ^2 with our observations. We performed this fitting for all our regions, assuming different SFH models, which influence the quantity and types of stars in our selected regions.

The fitting process was carried out using modified packages from the PixeFit (Abdurro'uf et al. 2021) library, which utilises FSPS to generate the models. Additionally, we employed emcee (Foreman-Mackey et al. 2013), a Python implementation of the Affine Invariant MCMC Ensemble sampler (Goodman & Weare 2010), to explore the parameter space and find the best-fit solutions. Once the MCMC fitting was completed, we reconstructed the SFH of the stellar population based on the assumed model and best-fit parameters. This provides the star formation rate, i.e., the total amount of stellar mass formed per year from the beginning of the region, the mass distribution of the galaxy at its redshift, $z \sim 1$. By integrating the SFH, we can determine the total amount of stars within various age bins. Since we used a region-integrated SED and assumed a homogeneous population of stars throughout the region, the mass distribution across the region was derived by weighting the total mass per age bin for that specific region by the flux distribution.

Once this process is completed for all six regions, we can produce a map that contains the mass distribution for each age bin (see Fig. 3 for the integrated value over all SFH). As the total mass distribution is weighted by the flux distribution, the mass obtained is biased by lensing magnification. To correct for this, we divided the mass in each pixel by its magnification. Since higher magnification corresponds to a smaller area in the source plane, this correction naturally reduces the mass in regions of high magnification. Conversely, higher magnifications make it easier to detect events in these regions. The corrected version of the mass distribution is shown in Fig. 3.

3.2. Microlensing magnification

By using FSPS, we can not only reconstruct the SFH based on the best-fit parameters from the MCMC, but also generate a set of isochrones using different stellar libraries. In our case, we employed the MIST library (Dotter 2016; Choi et al. 2016;

Paxton et al. 2011, 2013, 2015). FSPS outputs a file containing various stellar properties, such as age, type, luminosity, temperature, and surface gravity. All stars with the same age form an isochrone. Each star has a weight parameter (fraction by mass), which represents the fraction of stars with similar characteristics per solar mass formed. Since we know the total stellar mass formed per pixel and age, we can now determine the total number of stars in each pixel along with their luminosity, temperature, and other required stellar parameters. See Fig. 4 for an output example of FSPS isochrones.

The final step before incorporating microlensing is to determine the apparent magnitude of each star in the different NIR-Cam filters. This was achieved using the luminosity and temperature of each star. The apparent magnitude of a source is given by

$$m = -2.5 \log_{10}(F_\nu) - 48.6, \quad (3)$$

where F_ν is the flux received in the selected filter, corrected for the filter response $S(\lambda)$, and redshift dependence (Bessell & Murphy 2012):

$$F_\nu = (1+z) \frac{\int f_\lambda \left(\frac{\lambda}{1+z} \right) S(\lambda) \lambda d\lambda}{\int S(\lambda) (c/\lambda) d\lambda}. \quad (4)$$

The specific flux per unit frequency interval is given by

$$f_\lambda(\lambda) = \frac{L}{4\pi D_l(z)^2} \frac{\text{SED}(\lambda)}{\int \text{SED}(\lambda) d\lambda}, \quad (5)$$

where L is the stellar bolometric luminosity, $D_l(z)$ is the luminosity distance, and the stellar SED per wavelength is represented by a blackbody spectrum without loss of generality. Using Eqs. (3), (4), and (5), we can calculate the apparent magnitude of each star in the isochrones. For each star in the isochrone, we have their luminosity as shown in Fig. 4. We repeated this for each of the NIRCam filters.

Now that we have all the necessary information from the perspective of the sources (i.e. the stars), we can proceed to the second half of the process: microlensing. For this task, we gathered

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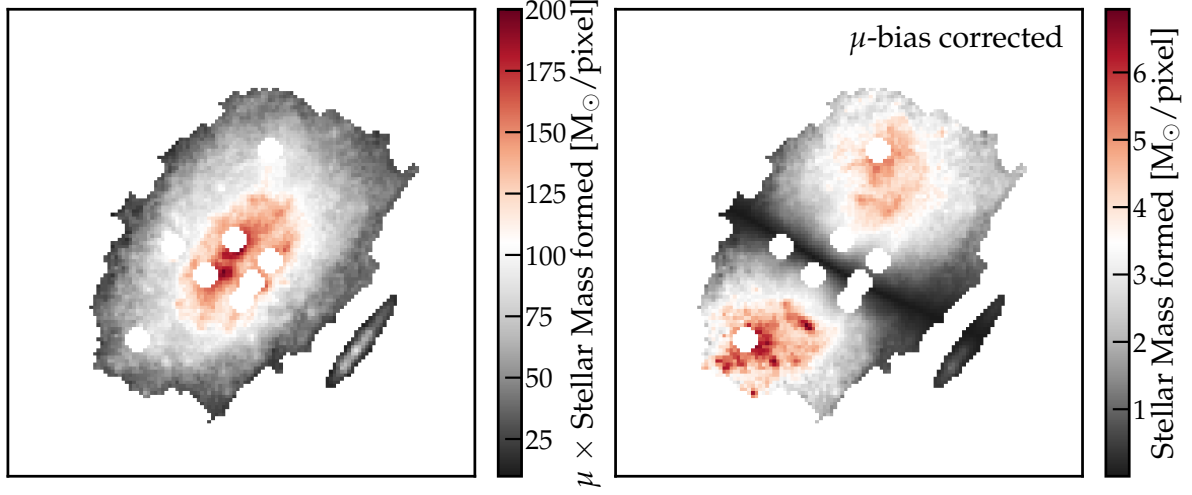


Fig. 3. Distribution of total stellar mass formed per pixel in Warhol throughout the entire SFH shown before and after accounting for magnification bias correction, in the left and right panels respectively. Brighter pixels may suggest a larger stellar mass, but magnification can make smaller stellar populations appear brighter, which would introduce a bias. To correct for this effect, we divide the stellar mass formed in each pixel by μ_m , which accounts for the change in source-plane area relative to the lens-plane pixel area due to magnification. Near the critical curve, the number of stars in the source plane imaged in this region is lower than in areas with lower magnification, but the magnification is extremely large, which enables the detection of individual fainter stars but within a reduced source area. One pixel corresponds to 32 mas, or 490 pc on the source plane if $\mu = 1$.

information from the macromodel and microlenses, and calculated the magnification statistics. Depending on these elements, a source will experience a flux boost, μ_{micro} , such that its perceived flux is $\mu_{\text{micro}} \times F_f(\nu)$. Following Eq. (3), its new apparent magnitude becomes

$$m_{\text{micro}} = m - 2.5 \log_{10}(\mu_{\text{micro}}). \quad (6)$$

The value of μ_{micro} is drawn from a magnification PDF estimated using `M_SMiLe` (Palencia et al. 2024). The PDF varies from pixel to pixel as the macrolensing and microlensing models change. `M_SMiLe` computes a semi-analytical magnification PDF based on three input elements: tangential macro-magnification, μ_t , radial macro-magnification, μ_r , and microlens surface mass density, Σ_* . The output at each pixel, i , is

$$p_i(\log_{10}(\mu_{\text{micro}}); \mu_{t,i}, \mu_{r,i}, \Sigma_{*,i}), \quad (7)$$

which we carefully mapped to $p_i(\mu)$ and normalised to unity. At first order, this probability density function peaks at $\mu_m \approx \mu_t \times \mu_r$, while $\Sigma_{\text{eff}} = \mu_t \times \Sigma_*$ controls the width of the PDF. As Σ_{eff} increases, higher μ_{micro} values become more likely, until $\Sigma_{\text{eff}} \gg \Sigma_{\text{crit}}$, where the ‘more is less’ effect begins (Diego et al. 2018; Dai & Pascale 2021; Welch et al. 2022a; Palencia et al. 2024; Kawai & Oguri 2024). At this point, extreme values become less likely until the PDF converges to an attractor universal form, and no further differences are observed even at higher Σ_{eff} or μ_m . For a more detailed discussion, see Diego et al. (2018), Palencia et al. (2024).

We used the WSLAP+ mass distribution, κ , and the shear, γ , to estimate the macrolens model parameters. Both macro-magnification components (see Fig. 5) can be easily obtained as

$$\begin{aligned} \mu_t &= 1/(1 - \kappa - \gamma), \\ \mu_r &= 1/(1 - \kappa + \gamma). \end{aligned} \quad (8)$$

The Σ_* value was derived from ICL measurements from (Kaurov et al. 2019). We assumed a pivot value of $59.39 M_\odot/\text{pc}^2$, which was modulated according to the ICL variation over the arc, as shown in Fig. 6.

To obtain a more realistic picture of the lensing scheme, we also included GCs as millilenses. As seen in Diego et al. (2024a), one can expect ~ 70 GCs in the $6'' \times 6''$ area around Warhol. Only a fifth of them will be detectable through photometry, but all of them will locally perturb the macromodel shown in Fig. 5. To achieve this, we randomly placed a single realisation of 72 sub-haloes in the 200×200 pixels around Warhol with masses ($10^6 M_\odot \lesssim M_{\text{milli}} \lesssim 10^7 M_\odot$) following the distributions obtained by the MOKA simulations (Giocoli et al. 2012, 2016) in a galaxy cluster at the MACS 0416 redshift. For each position and mass, we placed a Gaussian mass with a full width at half maximum equal to one pixel (corresponding to 239 pc at the redshift of the lens), computed the deflection angle of each lens in the pixels of the field of view, added them linearly to those obtained from WSLAP+, and recalculated κ and γ . Following Eq. (8), we obtained μ_t and μ_r . In order to investigate the possible impact of compact dark matter (such as primordial black holes), we also assumed two additional scenarios, where we included an extra 3% and 10% of the total κ as microlenses that do not contribute to the ICL, i.e. compact dark matter. For reference, the critical density in the MACS 0416 and Warhol lensing system is $2970 M_\odot/\text{pc}^2$, and $\kappa = \Sigma/\Sigma_{\text{crit}}$, where Σ is the surface mass density in the lens plane.

By inputting $\mu_{t,i}$, $\mu_{r,i}$, and $\Sigma_{*,i}$ into `M_SMiLe`, we obtained the magnification probability density function in each pixel that we combined with the estimated number of stars, along with their apparent magnitudes, which we retrieved from our MCMC fitting. Integrating the PDF above a given minimum magnification, set by the star luminosity, dust attenuation, filter of choice,

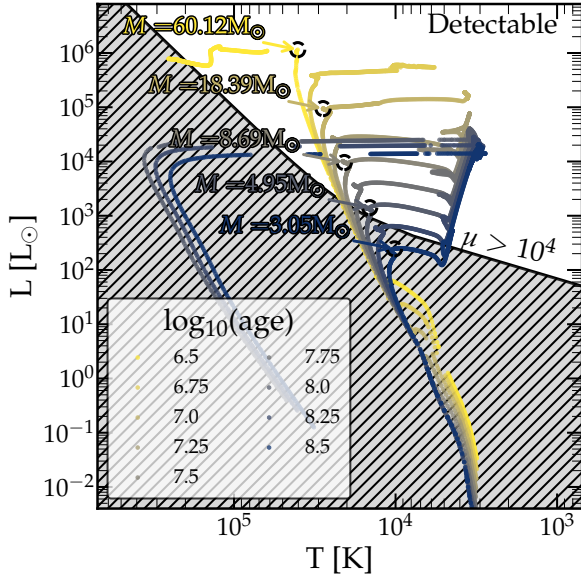


Fig. 4. Set of isochrones outputted by FSPS from the MIST stellar library at different stellar ages in region 1 and best-fit metallicity $Z = 0.31 Z_{\odot}$. Each colour represents the same population as it evolves with time, with older populations growing dimmer and redder. The solid black line delimits the regions of the diagram in which a magnification larger than a factor of 10^4 would be necessary to observe the star in at least one of the eight NIRCcam filters used in this work at a limiting magnitude of 30. The shaded region is thus the stars that cannot be observed in any filter with these conditions. The arrows mark the turn-off points, where stars leave the main sequence. The corresponding masses indicate the stellar mass at which this transition occurs for a subset of the isochrones shown in the plot.

its redshift, and limiting magnitude of the observations, provides the probability of observing a star, j , in the corresponding pixel, i .

$$P_{i,j} = \int_{\mu_{\min,j}}^{\infty} p_i(\mu; \mu_{t,i}, \mu_{r,i}, \Sigma_{*,i}) d\mu. \quad (9)$$

Summing over all stars, weighted by their expected abundance in that pixel, $w_{i,j}$, yields the expected number of events in that pixel:

$$n_i = \sum_{j=1}^{N_{\text{star}}} P_{i,j} w_{i,j}. \quad (10)$$

The quantity $\mu_{\min}$ represents the minimum magnification required for the observed magnitude, following Eq. (6), to be above a given detection threshold, such that $m_{\text{micro}} < m_e$. Naturally, $\mu_{\min}$ depends on the stellar apparent magnitude and the adopted threshold, $\mu_{\min} = \mu_{\min}(m, m_e)$. Summing over all pixels yields the average number of stars detected across the entire arc above the specified threshold:

$$\langle N \rangle = \sum_{i=1}^{N_{\text{pix}}} n_i. \quad (11)$$

The primary source of uncertainty in this calculation arises from the SFH. The uncertainties in the total stellar mass formed affect

the number of stars susceptible to microlensing. By repeating the calculation performed with the median SFH, but now using the 16th and 84th percentiles obtained from the MCMC chains, we derived a pixel-level uncertainty in the number of events, σ_{n_i} . The variance in the expected number of events is then given by

$$\sigma^2 \langle N \rangle = \sum_{i=1}^{N_{\text{pix}}} \sigma_{n_i}^2. \quad (12)$$

For a given telescope pointing towards the arc with a limiting depth or magnitude threshold, there is an associated Poissonian uncertainty in the expected number of events, which can be approximated as

$$\sigma_{\text{Poisson}} = \sqrt{\langle N \rangle}. \quad (13)$$

Finally, the total uncertainty in the number of events in the arc was obtained by summing all sources of error in quadrature:

$$\sigma_{\text{total}} = \sqrt{\sigma_{\langle N \rangle}^2 + \sigma_{\text{Poisson}}^2}. \quad (14)$$

4. Results

After removing contaminating foregrounds and applying linear cuts in both the PCA dimensions and colour-colour spaces, we identified six distinct regions, as shown in Fig. 2. The second to fourth columns display the normalised SEDs of the pixels, alongside the median SED with the corresponding 1σ contours.

Fig. 7 presents the best SED fitting results obtained through MCMC analysis for each region, alongside the residuals for the 11 filters used in the photometric fitting (see Appendix A for median values and 1σ limits). The extracted SFHs, modelled as exponentially decaying SFHs, are shown in Fig. 8. By integrating the SFH over cosmic time, we derive the stellar mass formed in each region at different age bins. This mass is distributed across the region's pixels by normalising it according to the flux in each pixel, as illustrated in the left panel of Fig. 3. However, this flux is biased by lensing magnification, which necessitates a correction by dividing each pixel's mass by the macro-magnification predicted at its position within our lens model. The right panel of Fig. 5 displays this magnification, and the corrected stellar mass distribution, free from magnification bias, is shown in the right panel of Fig. 3.

Summing across all regions, we find that the total stellar mass formed in the multiply lensed region of Warhol over its full SFH is $1.27 \pm_{0.03}^{0.41} \times 10^6 M_{\odot}$, or half that value if we consider that we have two connected images of the same background galaxy. The oldest stellar population, at 794 Myr (whose SFR is constant for the most recent times), is found in region 4, while the youngest, at 60 Myr, corresponds to region 3. All six regions exhibit sub-solar metallicities, with regions 1 and 2 having the highest values at 0.31 and 0.23 $Z_{\odot}$, respectively, while the remaining regions show metallicities at approximately 10% of the solar value.

After generating the isochrones that correspond to the stellar populations that best reproduce the observed photometric spectra in Warhol, we applied Eq. (9) to each star in the isochrones, and weighted them by the total stellar mass formed in that pixel at the corresponding isochrone age. By combining this information via Eq. (10), we obtained the probability density of microlensing events across the Warhol arc at different fiducial observational thresholds, as depicted in Fig. 9. This figure illustrates the distribution of events at various magnitude

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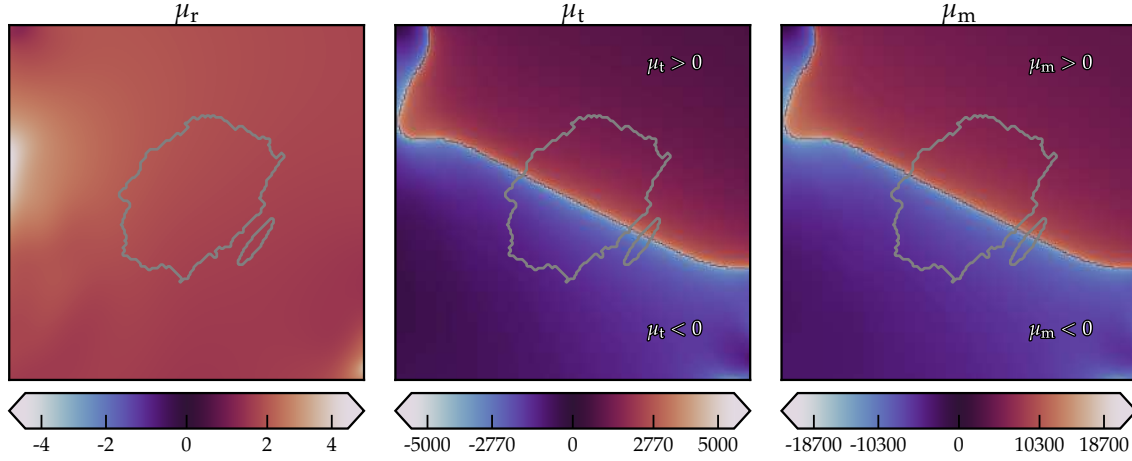


Fig. 5. Magnification model from WSLAP+ around Warhol. The left panel shows the smooth radial component. The middle panel shows the rapidly varying tangential magnification, with the divergent critical curve near the diagonal. The right panel shows the combined macromodel value, $\mu_m = \mu_t \times \mu_r$. Both μ_m and μ_t are shown in log scale for better visualisation. The region shown is the central 200×200 pixels in Fig. 1, enclosed in the yellow square around the Warhol arc. The grey line marks the position of Warhol.

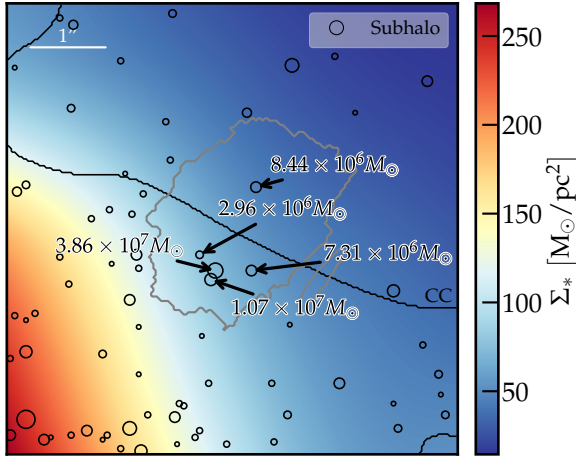


Fig. 6. Map of Σ_* modulated by the ICL, relative to a pivotal value of $\sim 60 M_\odot/\text{pc}^2$. Open black circles indicate the positions of a random realisation of DM sub-haloes, with their location probability following the ICL distribution. The sub-haloes exhibit a mass distribution derived from the small-scale results of N -body simulations of dark matter within galaxy clusters at the MACS 0416 redshift. Larger masses are represented by larger circles. The masses of the sub-haloes that intersect the arc are indicated in the figure. This region shows the central 200×200 pixels in Fig. 1.

thresholds in the JWST NIRCам F090W filter. By integrating the expected event rates across all pixels using Eq. (11), we derive the expected number of events per pointing in F090W as a function of limiting magnitude, shown in Fig. 10. Figs. 9 and 10 incorporate a lens model that includes microlensing by sub-haloes, whose mass distribution follows the results obtained from the MOKA simulations (Giocoli et al. 2012, 2016) in a galaxy cluster at MACS 0416 redshift. The positions of these

sub-haloes are marked by open black circles in Fig. 6, and their localised impact on the event distribution is evident in Fig. 9, particularly at shallower detection thresholds where their relative influence is greater. Fig. 11 presents the integrated number of expected events per pointing in F090W under the assumption of no millilenses. The overall impact of millilenses on the total event count is at the level of 1%, well below our uncertainties, which are dominated by Poisson statistics and SFH uncertainties from the SED fitting. However, their effect on the spatial distribution of events is more pronounced, as seen in Fig. 12. This figure shows the ratio of pixel-wise event rates for a 27-magnitude detection threshold in F090W. At the location of a millilens, the event rate is enhanced, whereas the surrounding regions exhibit a localised suppression. This results in a total integrated expectation that remains largely unchanged, as previously seen in Fig. 10. In the southern part of the arc, the combined influence of two neighbouring millilenses produces a similar effect to that observed in the northern region, albeit with a more pronounced suppression in the intermediate region between them.

The different colours in Figs. 10 and 11 correspond to varying fractions of microlenses added to the constrained value from the ICL fitting. These additional microlenses, assumed to be compact dark matter objects such as PBHs, are parametrised as a fraction of the smooth surface mass density at the Warhol location. We find that the total expected number of events across the arc remains largely unchanged. Higher concentrations of compact dark matter result in a steeper slope, which indicates an increased sensitivity of event rates to limiting magnitude. This can be understood as due to the reduction in maximum magnification near the critical curves as more microlenses (PBHs) are added (Venumadhav et al. 2017). The effect should be compensated in the faint end of the luminosity function, where more events are expected farther away from the critical curve and associated to the increase in number of compact dark matter microlenses. This scenario is confirmed when we look at the spatial distribution of events when PBHs are added, which varies

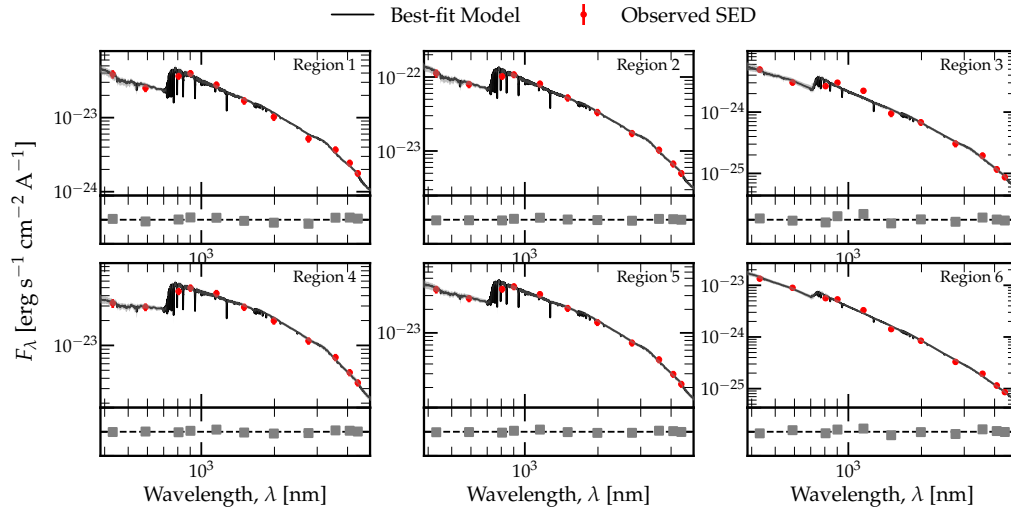


Fig. 7. SED of Warhol regions and best-fit model from the MCMC realizations. The observed data points are showed in red and the solid black line represents the best model from the Flexible Stellar Population Synthesis code. The grey shaded area represents the 1σ contours. The residuals are showed in the bottom sub-panels for each region.

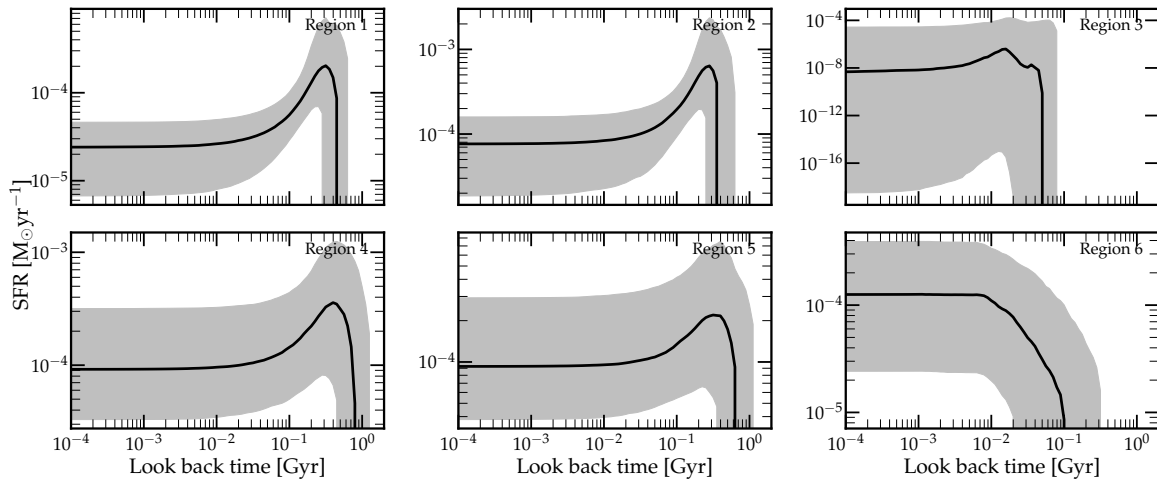


Fig. 8. SFHs of Warhol regions from the best fit in our MCMC realizations. The solid black lines represent the median SFR at each lookback time while the grey shaded contour shows the 1σ contour.

with the addition of extra microlenses, as shown in Fig. 13, where we plot the event ratio for a scenario with 10% additional κ in microlenses compared to a purely baryonic microlens model. We observe a reduction in the number of events near the critical curve due to the ‘more is less’ effect (Diego et al. 2018; Palencia et al. 2024), while the event rate is boosted at larger distances. The turnover point in this behaviour depends on the added fraction of microlenses: for an additional 10% in κ , we find turnover radii of $1.98''$ and $0.33''$ on the positive and negative parity sides of the arc, respectively, while for a 3% increase in κ , these values shift to $0.96''$ and $0.15''$, approximately half the previous distances.

We conclude that, after introducing millilenses into the macromodel and varying the fraction of microlenses in our model, both modifications imprint distinct patterns on the event rate distribution, while their effect on the integrated transient rate remains low or nearly undetectable. Millilenses cluster the event rate probabilities at their location in the lens plane, which simultaneously reduces the rate in the surrounding regions. In contrast, increasing the microlens abundance in Warhol, which is already large due to the baseline value extracted from ICL analysis, further decreases the event rate probability near high magnification regions such as the cluster CC, while enhancing the rates in areas with lower macromagnifications.

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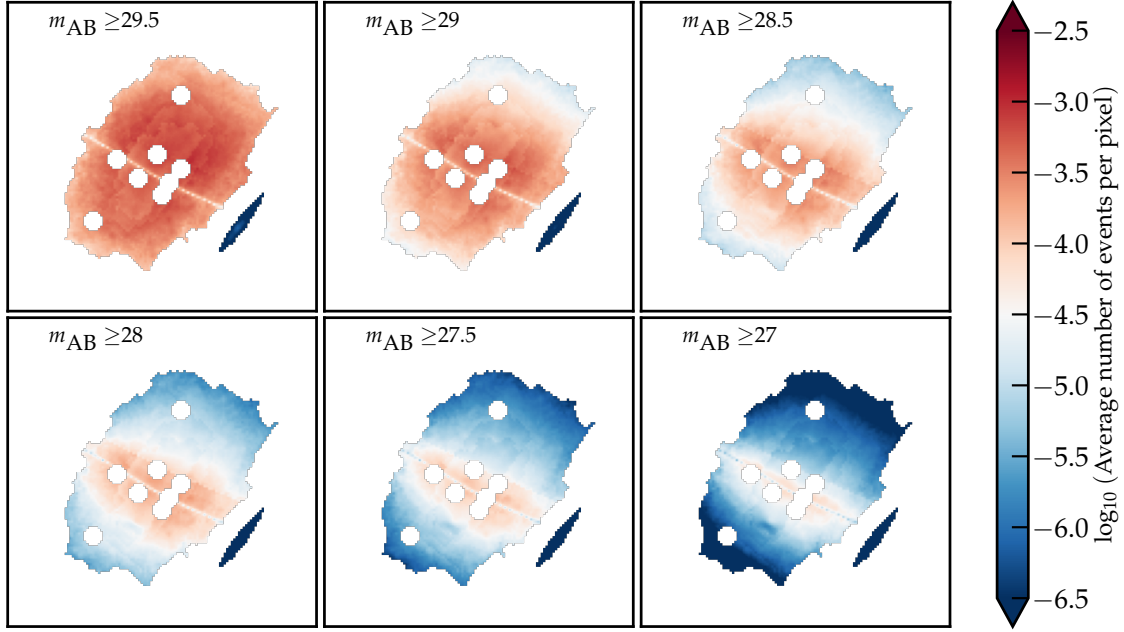


Fig. 9. Spatial distribution of event probability predicted for different thresholds in F090W. Each pixel value represents the expected average number of detections above a certain threshold per pixel. The total number of pixels included in the galaxy mask and after masking out the globular clusters and possible events is roughly 5000. One pixel length corresponds to 32 mas, or 490 pc on the source plane if $\mu = 1$. In shallow observations we only expect to see events at the critical curve. In deep observations, we expect to see events more uniformly distributed across the arc. Lower thresholds predict fewer events, which are asymmetrically distributed, favouring a negative parity. This is due to the increased Σ_+ closer to the BCG and the statistical properties of the negative parity regime. As the detection threshold increases, this asymmetry diminishes.

We predict an expected number of events per pointing at a limiting magnitude of 29.5 in eight NIRCcam filters of order unity (See Table 1). This result is consistent with Yan et al. (2023), who report seven events in Warhol across four epochs, which yields a mean of 1.75 events per pointing at similar limiting magnitudes. The predicted event rates for all filters and magnitude thresholds are provided in Table 1.

5. Discussion

In this section we discuss the results obtained on the predicted microlensing event rates on eight NIRCcam filters on Warhol, a lensed galaxy at redshift $z = 0.94$. First we explain the importance of the chosen models, both lensing models and stellar population models. We then evaluate the strengths and weaknesses of our forecasting pipeline.

Our results depend on two key factors: the lens modelling, both macro and micro, and the assumed stellar properties, including priors on stellar population parameters, dust attenuation models, and IMFs. The choice of these models directly impacts the predicted event rates. Here, we discuss our selections and their consequent effects on our results.

5.1. Macrolens

The choice of macrolens affects our predictions in two primary ways. Firstly, the local values of shear and convergence and their derivatives determine the macromagnification, which serves as an input for our code `M_SMiLe`, and influence the statistical properties of microlensing magnification. Secondly, different models

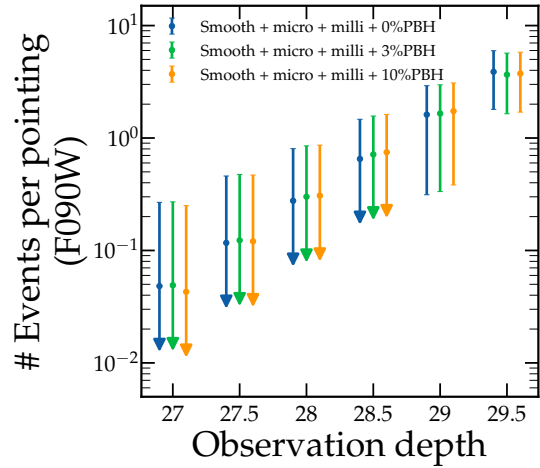


Fig. 10. Total number of events observed in Warhol for different thresholds in F090W, adding millilenses. Each value is the integrated value of all the pixels in the arc as shown in Fig. 9.

predict slightly varying positions for the critical curve, thereby altering the spatial extent of the parity regimes, which in turn affects the statistical behaviour of micromagnification.

In the case of Warhol, its distinct morphology and bright knots, as shown in Fig. 2 top left panel, define critical points

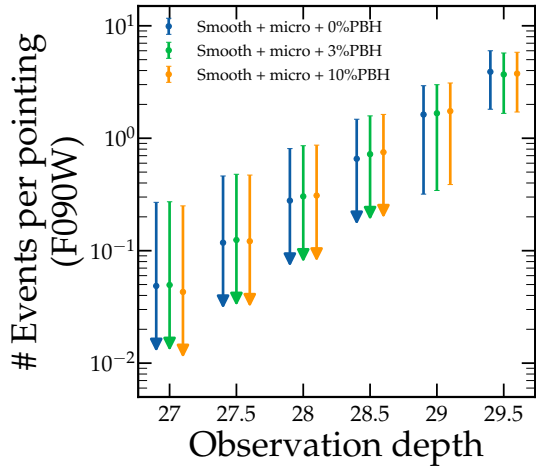


Fig. 11. Total number of events predicted in Warhol for different thresholds in F090W. Each value is the integrated value of all the pixels in the arc similar to Fig. 9.

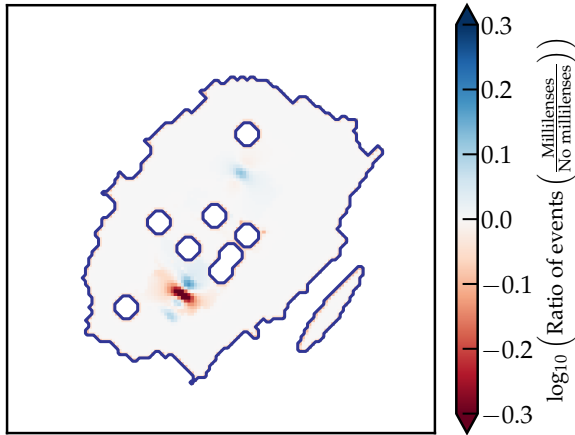


Fig. 12. Ratio of expected stars magnified above 27.5 mag, assuming millilenses at the positions marked by black circles in Fig. 6, compared to the model without millilenses.

mirrored in both parity images. The critical curve precisely bisects the arc (Broadhurst et al. 2025), which leaves little margin for error in its location at the resolution of our telescopes. Thus, only minor variations in local macromagnification are expected at the arc’s location, which implies a minimal impact on the total expected event rate. Moreover, MACS 0416 is one of the best-studied galaxy cluster lenses and hosts the largest number of spectroscopically confirmed sources and hundreds of multiply lensed images. This results in some of the most accurate lens models available for any galaxy cluster. However, different lens models may produce the critical curve at the right position yet have different slopes on the potential, which affects the scaling of the magnification with distance to the critical curve, $\mu \propto d^{-1}$. Overall, the entire problem is degenerate in $\Sigma_{\text{eff}} = \Sigma_s \times \mu$, so uncertainties in the macromodel and the ICL contribution can be reduced to uncertainties in Σ_{eff} .

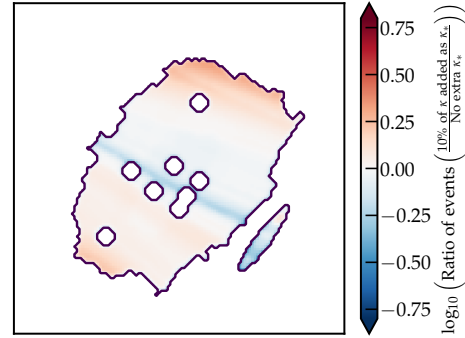


Fig. 13. Ratio of expected stars magnified above 27.5 magnitudes. Expected number of events with the Σ_s derived from the ICL over the events expected with a 10% increased Σ_s . This mimics the effect of a hypothetical PBH (or other compact dark matter candidate) in the cluster medium.

However, this level of precision regarding the location of the critical curve in Warhol does not extend to all arcs. For example, Spock, another arc in MACS 0416, is located near a cluster member galaxy and lacks multiple critical points, which leads to greater uncertainty in its macrolens model and, consequently, in its predicted event rates (Li et al. 2025). Other arcs, such as the Dragon arc in Abell 370, which holds the record for the largest number of detected microlensed stars at cosmological distances (Fudamoto et al. 2025), are also more challenging to model than Warhol. These examples highlight the necessity of high-quality spectroscopic and photometric data in galaxy clusters to refine mass distribution models and improve constraints on lensing efficiency.

5.2. Micro and Millilens

The second major component of the lens model are the small-scale perturbations, including both millilenses and microlenses. As illustrated in Fig. 12, the spatial distribution of events is influenced by the presence of millilenses. These structures locally enhance the cluster’s lensing efficiency by increasing both the macromagnification and, if the millilens is a globular cluster, the number of microlenses. This results in a higher event rate at the millilens position, albeit at the cost of a slightly reduced rate in the surrounding regions.

Given that tens of globular clusters and possibly DM sub-haloes are expected to intersect arcs like Warhol, accurately modelling their effects is crucial. Additionally, they can alter the local parity of the macro+milli model, which could potentially lead to long-duration events in negative parity regions, as is suspected in the case of Mothra (Diego et al. 2023a). While our study primarily focuses on the detection of events, light curves also play an essential role in a comprehensive analysis of microlensed stars in galaxy clusters.

Besides globular clusters, the actual CDM paradigm predicts the existence of non-luminous millilens-scale haloes (Dai et al. 2018, 2020; Williams et al. 2024; Ji & Dai 2025), which act as millilenses of 10^6 – $10^8 M_\odot$. The spatial distribution of microlensing events can serve as a probe for these small-scale structures, as has been demonstrated for the case of wave DM (Broadhurst et al. 2025), particularly when combined with event duration data, although the latter is partially degenerate with the microlens mass.

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Table 1. Predicted event rate in the Warhol arc across multiple NIRCcam filters and limiting magnitudes.

Limiting magnitude	F090W	F115W	F150W	F200W	F277W	F356W	F410M	F444W
29.5	$3.9^{+2.1}_{-2.1}$	$5.1^{+2.4}_{-2.4}$	$5.6^{+2.6}_{-2.6}$	$5.2^{+2.5}_{-2.4}$	$4.01^{+2.13}_{-2.12}$	$2.7^{+1.7}_{-1.7}$	$2.0^{+1.5}_{-1.5}$	$1.8^{+1.4}_{-1.4}$
29.0	$1.6^{+1.3}_{-1.3}$	$2.1^{+1.5}_{-1.5}$	$2.2^{+1.5}_{-1.5}$	$2.1^{+1.5}_{-1.5}$	$1.6^{+1.3}_{-1.3}$	$1.07^{+1.05}_{-1.05}$	$0.8^{+0.9}_{-0.9}$	$0.8^{+0.9}_{-0.9}$
28.5	$0.7^{+0.8}_{-0.8}$	$0.9^{+0.9}_{-0.9}$	$0.9^{+1.0}_{-1.0}$	$0.8^{+0.9}_{-0.9}$	$0.6^{+0.8}_{-0.8}$	$0.5^{+0.7}_{-0.7}$	$0.4^{+0.6}_{-0.6}$	$0.3^{+0.6}_{-0.6}$
28.0	$0.3^{+0.5}_{-0.5}$	$0.4^{+0.6}_{-0.6}$	$0.4^{+0.6}_{-0.6}$	$0.4^{+0.6}_{-0.6}$	$0.3^{+0.5}_{-0.5}$	$0.2^{+0.4}_{-0.4}$	$0.1^{+0.4}_{-0.4}$	$0.1^{+0.4}_{-0.4}$
27.5	$0.1^{+0.3}_{-0.3}$	$0.2^{+0.4}_{-0.4}$	$0.2^{+0.4}_{-0.4}$	$0.1^{+0.4}_{-0.4}$	$0.1^{+0.3}_{-0.3}$	$0.1^{+0.3}_{-0.3}$	$0.1^{+0.2}_{-0.2}$	$0.1^{+0.2}_{-0.2}$
27.0	$0.04^{+0.22}_{-0.22}$	$0.1^{+0.3}_{-0.3}$	$0.1^{+0.3}_{-0.3}$	$0.1^{+0.2}_{-0.2}$	$0.04^{+0.21}_{-0.21}$	$0.03^{+0.17}_{-0.17}$	$0.02^{+0.15}_{-0.15}$	$0.02^{+0.14}_{-0.14}$

Microlenses, a key component in our statistical framework M_{SMiLE} , also play a critical role in shaping the total event rate. A larger microlens population allows for events to occur farther from the critical curve, while simultaneously suppressing events closer to it due to the ‘more is less’ effect. This results in an overall reduction of event rates at shallower thresholds, whereas deeper pointings exhibit an increase in total expected events. These effects are illustrated in Figs. 10 and 13. We propose leveraging these effects to infer the dark matter abundances at microlens scales in galaxy clusters. This method is expected to be particularly effective in less crowded cluster regions, at higher redshifts where the critical curve moves outwards, or in merging clusters such as El Gordo (Diego et al. 2023a; Caminha et al. 2023; Frye et al. 2023), where the intracluster medium remains relatively pristine.

5.3. Star formation history

The SFH of a galaxy describes how much stellar mass has formed over time. Different galaxies exhibit a variety of SFHs, ranging from continuous star formation to single bursts or multiple episodes of star formation. To model SFHs, we generally adopt one of two approaches: a non-parametric method, where the galaxy’s lifetime is divided into age bins of chosen widths and the total stellar mass formed in each bin is fitted independently, or a parametric approach, where the SFH is assumed to follow a predefined functional form governed by a set of parameters.

In general, non-parametric SFH modelling is more flexible and offers a better chance of capturing the true SFH due to a larger number of free parameters. However, having more free parameters (one per age bin) increases the computational complexity, particularly in MCMC-based models, where exploring high-dimensional parameter spaces can hinder convergence and can also increase the risk of over-fitting. Parametric models, on the other hand, are less flexible since they impose a predefined SFH shape, but they require fewer parameters, which significantly improves the efficiency of MCMC sampling. Nevertheless, incorrect assumptions about the SFH shape can introduce systematic biases in the inferred SFHs.

In this work, we adopt a simple, exponentially decaying SFH, also known as a τ -model, where the star formation rate evolves as

$$\Psi(t) = A e^{-\frac{t-t_0}{\tau}} \Theta(t - t_0), \quad (15)$$

where t_0 represents the onset of star formation, τ controls the decay rate, and A is a normalisation factor.

Other commonly used parametric SFHs include the delayed exponential decay, double power-law models, constant star formation, multi-burst scenarios, and combinations of these models. The exponentially decaying SFH is a reasonable assumption, as it successfully reproduces the observed SED of many galaxies. Similarly, Li et al. (2025) conducted an extensive model comparison for Spock, and found that a non-parametric fit also favoured an exponentially decaying SFH. We tested different parametric SFH models in each of the six regions of Warhol: constant SFH, delayed exponentially decaying, double power law, and exponentially decaying, and we found a preference for the latter model in each region. However, since we did not perform non-parametric modelling or test a combination of parametric models, we may be missing a more complex SFH. It is possible that the true SFH is primarily characterised by an exponential decline but includes short bursts of star formation that are not captured in our analysis. Despite this limitation, analysing discrete regions within Warhol allows us to better account for variations in SFH across the arc.

Disentangling more complex SFHs would require higher quality data. We conclude that 11 photometric filters provide a reasonable understanding of the SFH in these arcs, but a more detailed approach could be achieved, particularly for the most recent SFH epochs, through spectroscopic analysis. Emission lines such as H_α and Pa_α are well-known tracers of recent star formation, and future studies should incorporate them to improve the accuracy of SFH reconstruction.

5.4. IMFs

The stellar IMF describes the number of main sequence stars formed as a function of stellar mass (Salpeter 1955). Many fundamental properties of stellar populations, including their SFH, metallicity, luminosity function, and stellar evolution, are directly influenced by the IMF. Measuring the IMF through resolved photometry has only been possible in nearby galaxies such as the Small and Large Magellanic Clouds (SMC and LMC) (Massey et al. 1995; Hunter et al. 1995, 1997; Sirianni et al. 2000, 2002; Da Rio et al. 2009), M31 (Weisz et al. 2015), and the Milky Way (Chabrier 2003). In all these cases, the observed IMF is typically modelled as a broken power law, with a commonly accepted slope of 2.3 for high-mass stars ($>1.4 M_\odot$) and a shallower slope for lower-mass stars. While the 2.3 slope for massive stars is widely agreed upon, different models vary in

their treatment of the low-mass end of the IMF. The most commonly adopted IMFs include those of Kroupa (Kroupa 2001), Salpeter (Salpeter 1955), and Chabrier (Chabrier 2003), among others.

Transient events typically arise from the brightest members of the stellar populations, including O and B-type main sequence stars, red super-giants (RSGs), and possibly the tip of the red giant branch if the detection threshold is at a sufficiently high magnitude and the distance modulus of the arc is not too large. These stars are not only the most luminous but also the most massive, meaning that transient detections are primarily sensitive to the high-mass tail of the IMF. While lower-mass stars are not directly observed in these events, their presence influences the evolution of nearby massive stars, and due to their shallower IMF slope, they are significantly more abundant.

Whether the IMF evolves with redshift remains an open question (Davé 2011; Li et al. 2023). Complementary studies, such as Li et al. (2025), or Williams (in preparation), have investigated this issue, finding that at $z \sim 1$, a power-law slope of 2.3 is favoured. However, since our focus is on implementing M_{SMiLe} to assess its ability to spatially resolve transient event rate distributions rather than directly constraining the IMF, we adopt a Kroupa IMF (Kroupa 2001) for our analysis. Additionally, introducing the high-mass end slope of the IMF as a free parameter would significantly increase the computational time required for MCMC fitting.

5.5. Source plane modelling

In this work, we have modelled the Warhol arc in the lens plane. This approach was chosen over source-plane modelling to remain independent of any specific lens model. A lens-plane SED fitting yields a stellar population that directly represents the observed data in the arc, whereas source-plane modelling would require delensing the arc based on a particular lens model. Since gravitational lensing is achromatic, the SED of a lensed portion of the arc corresponds to that of the background source, with magnification only affecting the total brightness. This amplification would impact the inferred stellar mass, which could later be associated with a given lens model. In contrast, lens-plane modelling remains tied to the initially assumed model throughout the analysis.

One advantage of source-plane modelling is that it allows mirror-imaged pixels to be combined, effectively averaging out foreground contamination due to imperfect subtraction of the BCG, ICL, or unmodelled GCs in the lens plane. However, for an arc like Warhol, where the critical curve is well constrained and there is little dispersion among available lens models, the lens-plane modelling advantage has a limited impact. The dominant source of uncertainty in our estimates of the event rate remains the degeneracy in the SED fitting, which arises from the intrinsic complexity of stellar population synthesis models. Nevertheless, we argue that since our goal is to establish a general pipeline to estimate microlensing rates in lensed arcs, a lens-plane approach is more robust, particularly in cases like the Spock arc, where the lens models differ significantly, unlike in Warhol.

6. Conclusions

In this work, we presented a detailed forecast for the detection of highly magnified stars in the Warhol arc at redshift $z = 0.94$ lensed by the galaxy cluster MACS 0416, utilising a combination of strong lensing models, stellar population synthesis, and microlensing statistical techniques. Our analysis, based on photometric data from JWST and HST, provides insights into the

spatial distribution and expected number of transient events in different filters and magnitude thresholds.

Our results indicate that the predicted microlensing event rates in the JWST NIRCам filters are in good agreement with current observations (Yan et al. 2023). We have characterised the spatial distribution of microlensing events in Warhol and shown that the highest event rates are not necessarily located at the positions of peak macromagnification, i.e., directly on the CC. Instead, we find that regions with milder macromagnification values report the highest number of expected events. This is consistent with the balance between the increased density of sources at lower magnifications or larger areas in the source plane and the statistical effects of microlensing in highly magnified regions, where the ‘more is less’ effect suppresses the number of observable transients.

Additionally, we have explored the impact of small-scale structures, including globular clusters, primordial black holes (PBHs), and millilenses, on the spatial distribution of microlensing events. Our results demonstrate that, while the total number of detected events remains largely unchanged, the presence of millilenses modifies the clustering of events, which enhances the microlensing efficiency in their immediate vicinity while slightly suppressing it in surrounding regions. This effect is particularly evident at shallow detection thresholds. The presence of dark matter in the form of PBHs and other millilens-scale sub-structures further alters the spatial pattern of events, creating distinct signatures that could be used to probe the small-scale distribution of dark matter within galaxy clusters.

Despite the limitations inherent to our modelling assumptions, our work demonstrates the power of microlensing as a statistical tool for studying lensed stellar populations at high redshift. Future spectroscopic follow-up observations, particularly targeting emission lines such as H α and Pa α , will be crucial in refining SFH models and breaking degeneracies between dust and age effects.

With ongoing and upcoming JWST observations, we expect a growing sample of detected lensed stars across multiple galaxy clusters. The methodology developed in this study provides a robust framework for interpreting these discoveries and advancing our understanding of both stellar populations at cosmological distances and the nature of dark matter on sub-galactic scales.

Acknowledgements. J.M.P. acknowledges financial support from the Formación de Personal Investigador (FPI) programme, ref. PRE2020-096261, associated with the Spanish Agencia Estatal de Investigación project MDM-2017-0765-20-2. J.M.D. acknowledges the support of project PGC2018-101814-B-I00 (MCIU/AEI/MINECO/FEDER, UE) Ministerio de Ciencia, Investigación y Universidades. B.J.K. acknowledges funding from the Ramón y Cajal Grant RYC2021-034757-I, financed by MCIN/AEI/10.13039/501100011033 and by the European Union “NextGenerationEU”/PRTR. S.K.L. acknowledges support from the Research Grants Council (RGC) of Hong Kong through the General Research Fund (GRF) 17312122. This research is based on observations made with the NASA/ESA *Hubble* Space Telescope and *James-Webb* Space Telescope obtained from the Space Telescope Science Institute, which is operated by the Association of Universities for Research in Astronomy, Inc., under NASA contract NAS 5–26555. These observations are associated with programs GO-13496, GO-15936, and GO-16278, as well as by GTO-1176 and GTO-1208. The data are available at MAST: <https://doi.org/10.17909/gvxn-w422>. We acknowledge Santander Supercomputación support group at the University of Cantabria who provided access to the supercomputer Altamira Supercomputer at the Institute of Physics of Cantabria (IFCA-CSIC), member of the Spanish Supercomputing Network, for performing simulations. We made use of the following software packages: Python, NumPy, SciPy, FSPS, PixiedFit, Matplotlib, SExtractor, Petrolfit, Astropy, Photutils, and DS9.

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Appendix A: SED Fitting results

Table A.1. SED MCMC-fitting results.

Region	$\log(M_*) [M_\odot]$	$Z/Z_\odot$	Age [Gyr]	τ	A_V
1	$5.15^{+0.07}_{-0.07}$	$0.31^{+0.18}_{-0.12}$	$0.44^{+0.25}_{-0.15}$	$0.12^{+0.11}_{-0.06}$	$0.33^{+0.14}_{-0.12}$
2	$5.61^{+0.07}_{-0.07}$	$0.23^{+0.14}_{-0.09}$	$0.40^{+0.25}_{-0.14}$	$0.10^{+0.12}_{-0.05}$	$0.38^{+0.17}_{-0.12}$
3	$3.8^{+0.4}_{-0.2}$	$0.08^{+0.12}_{-0.06}$	$0.06^{+0.03}_{-0.03}$	$0.001^{+0.009}_{-0.001}$	$0.74^{+0.8}_{-0.5}$
4	$5.7^{+0.2}_{-0.1}$	$0.09^{+0.12}_{-0.07}$	$0.8^{+0.6}_{-0.3}$	$0.31^{+0.30}_{-0.16}$	$0.9^{+0.6}_{-0.4}$
5	$5.4^{+0.2}_{-0.1}$	$0.08^{+0.30}_{-0.07}$	$0.7^{+0.5}_{-0.3}$	$0.37^{+0.30}_{-0.19}$	$0.8^{+0.6}_{-0.4}$
6	$4.1^{+0.2}_{-0.2}$	$0.07^{+0.32}_{-0.05}$	$0.10^{+0.23}_{-0.08}$	$0.33^{+0.34}_{-0.17}$	$1.0^{+0.3}_{-0.6}$

Notes. The fitted parameters are the median values and 1σ percentiles.

Signatures of Fuzzy Dark Matter Inside Radial Critical Curves

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A&A, 701, A24 (2025)

<https://doi.org/10.1051/0004-6361/202555778>

Signatures of fuzzy dark matter inside radial critical curves

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Received 2 June 2025 / Accepted 16 July 2025

ABSTRACT

We investigated the strong gravitational lensing properties of fuzzy dark matter (FDM) haloes, focussing on the magnification properties near radial critical curves (CCs). Using simulated lenses, we computed magnification maps for a range of axion masses and halo configurations. We show that FDM produces enhanced central magnification and secondary CCs that are not easily reproduced by standard cold dark matter (CDM), even when subhaloes are included. The strength and scale of these effects depend primarily on the de Broglie wavelength, which is governed by the axion and halo masses. We find that axion masses in the range $m_\phi \sim 10^{-22}$ – 10^{-21} eV in galaxy-mass haloes lead to distinctive magnification distributions. Our results suggest that observations of highly magnified, compact sources near radial arcs, such as quasars or supernovae, could serve as a powerful test for the presence of FDM.

Key words. galaxies: halos – dark matter

1. Introduction

The lambda-cold dark matter cosmological model (Λ CDM) provides the most successful description of our Universe to date (Planck Collaboration VI 2020). In this framework, the energy density of the Universe consists predominantly of mysterious dark energy ($\sim 70\%$), which drives the accelerated expansion (observed primarily at $z < 1$). The remaining $\sim 30\%$ corresponds to matter, of which only about 5% is the familiar baryonic matter, while the remaining $\sim 25\%$ is attributed to the elusive cold dark matter (CDM). Although the gravitational effects of CDM have been evident for decades, its true nature remains a mystery in modern cosmology. Although the Λ CDM model accurately reproduces the large-scale structure of the universe, it still faces significant challenges on small scales (Del Popolo & Le Delliou 2017), particularly at scales $\lesssim 1$ Mpc. These small-scale problems include, among others, the core-cusp problem (Moore 1994; Moore et al. 1999; Flores & Primack 1994; Oh et al. 2011; Gentile et al. 2004), the missing satellites problem (Moore et al. 1999; Klypin et al. 1999), and the too-big-to-fail problem (Boylan-Kolchin et al. 2011, 2012).

Some progress has been made in addressing the core-cusp tension through high-resolution cosmological simulations of disc galaxies that incorporate strong baryonic feedback, particularly supernova-driven outflows, and high star formation thresholds (Guedes et al. 2011; McCarthy et al. 2012; Brook et al. 2012). However, the extent to which these mechanisms can alleviate other small-scale issues, such as core formation or satellite abundances, remains debated (Schaller et al. 2015).

The fundamental nature of dark matter remains unknown. Alternative scenarios to CDM, such as self-interacting dark matter (Tulin & Yu 2018) or warm dark matter (Viel et al. 2013), have been proposed to account for persistent discrepancies between the predictions of standard CDM and observations on small scales. Modifications to the Λ CDM paradigm have also been proposed to address these small-scale issues. Among the leading CDM candidates are weakly interacting massive particles (WIMPs) (Arcadi et al. 2018), which are fermionic and behave as discrete particles, and wave dark matter (ψ DM), a bosonic alternative composed of ultralight particles with masses below 10 eV (Hui 2021; Ferreira 2021). This latter model reflects the particle-wave duality of quantum mechanics, as the de Broglie wavelength of the particles exceeds their average separation in galaxies, allowing dark matter to be treated effectively as a classical wave.

In particular, fuzzy dark matter (FDM) is a kind of ψ DM model that is non-self-interacting, non-relativistic, and extremely light, with a mass in the range 10^{-23} – 10^{-20} eV (Hu et al. 2000; Hui et al. 2017). One of the key advantages of FDM is its ability to address two of the aforementioned problems: the core-cusp problem, through the formation of a solitonic core in galaxies (Schive et al. 2014; Mocz et al. 2017), and the missing satellite problem, via the suppression of structure formation below the de Broglie scale (Robles et al. 2015; Schive et al. 2016; Kulkarni & Ostriker 2021). This suppression is reminiscent of warm dark-matter models. However, FDM is still CDM and is indistinguishable from standard CDM predictions on large scales (Schive et al. 2014; Hui 2021).

The FDM prediction of solitonic cores and the suppression of small-scale structure in the linear regime (Hui 2021) make this model highly testable and subject to numerous current constraints. Observations of the early Universe (Zhang et al. 2024),

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axion detection experiments (Aja et al. 2022; Leung et al. 2019), and searches for astrophysical-related phenomena (Eberhardt et al. 2024; Pinetti 2025) have been used to place general limits on the existence of ψ DM. In the case of FDM, the usual mass range, around 10^{-22} eV, has been constrained by the small-scale spatial fluctuations measured via the Lyman- α forest (Iršič et al. 2017; Kobayashi et al. 2017; Armengaud et al. 2017), by galaxy dynamics and internal structure (Nadler et al. 2021) and by the sizes and stellar radial velocities of some ultra-faint dwarf galaxies (Dalal & Kravtsov 2022). These constraints disfavour an axion mass near the commonly adopted value of 10^{-22} eV, yet they are subject to systematic uncertainties. These arise from limited data and poorly understood baryonic physics (Hui 2021) or from the lack of full-wave simulations that could capture higher-order effects (Dalal & Kravtsov 2022). Properly accounting for these factors could potentially relax the current bounds. Regardless of the strength of these constraints, it is crucial to develop complementary approaches based on independent methods to better assess and mitigate systematic uncertainties.

Gravitational lensing presents itself as an excellent tool to probe axion masses and test the viability of FDM as a dark matter candidate (Laroche et al. 2022; Amruth et al. 2023; Powell et al. 2023; Broadhurst et al. 2025). Since lensing traces the underlying projected mass distribution, it is sensitive to substructure on different scales, as predicted by various dark-matter models. Axion masses of around 10^{-22} eV in FDM are particularly notable for producing de Broglie wavelengths that give rise to mass density fluctuations on parsec to kiloparsec scales in galaxy clusters and galaxy-scale lens systems, respectively. These fluctuations lead to distinctive patterns in the mass distribution compared with the conventional smooth global profiles expected from standard CDM. Such differences have already been proposed as solutions for discrepancies between observational data and the best-fitting canonical CDM-based lensing models (Amruth et al. 2023), such as the long-standing flux-ratio anomalies (Keeton et al. 2003; Goldberg et al. 2010; Xu et al. 2015; Shajib et al. 2019), the position anomalies in radio observations (Spingola et al. 2018; Hartley et al. 2019), or the asymmetry in microlensed stars in galaxy clusters (Broadhurst et al. 2025).

In this paper, we develop a lensing framework tailored to FDM distributions in galaxy-scale systems, with a particular focus on deviations in their magnification patterns compared with the case of standard CDM haloes. We focus on radial critical curves (CCs), which are often neglected in the literature, as smooth models predict demagnified central images, and radial images are also less common than their tangential counterparts. However, certain small-scale objects (smaller than the de Broglie wavelength), such as active galactic nuclei, quasi-stellar objects (QSOs), or supernovae, can be bright enough to be observed even at modest magnification factors and be sensitive to magnification changes due to FDM fluctuations. However, perhaps the most important fact is that near the CCs, the differences between standard CDM and FDM can be accentuated. The radial critical region corresponds to the portion of the lens where $(1 - \kappa) + \gamma = \varepsilon \approx 0$, where κ is the convergence, γ is the shear, and ε is an arbitrarily small number. In this region, since $\gamma > 0$ always, it must follow that $\kappa > 1$. In the classical CDM scenario, adding a substructure increases the value of κ , thereby making the term $(1 - \kappa)$ more negative and, in most cases, leading to a reduction in magnification ($\mu \propto \varepsilon^{-1}$). In contrast, in FDM, negative mass fluctuations (with respect to the mean) relax the condition for criticality and increase the probability of larger

magnification (smaller $|\varepsilon|$). In other words, this simple reasoning leads to the expectation of a higher number of highly magnified objects in the vicinity of the radial CC region in FDM models compared to standard CDM. In this work, we provide a quantitative assessment of the differences between canonical CDM and FDM near radial CCs in terms of their magnification statistics. Here, we demonstrate that FDM predicts a magnification distribution near the centre of haloes, specifically around radial CCs, that differs significantly from that predicted by standard CDM, even when including subhaloes as small-scale perturbers. This behaviour represents an effect unique to FDM, providing a promising avenue to test FDM models as an alternative to canonical CDM.

This paper is structured as follows. Section 2 introduces the basics of gravitational lensing, with a particular emphasis on magnification, the observable we propose to use for distinguishing an FDM universe from one governed by the classical representation of CDM. In Section 3, we describe the various mass profiles implemented in our simulations, which are later compared in terms of their magnification properties. In Section 4, we present the methodology followed in this work. The results of our analysis, highlighting the differences among the models and the tests employed, are presented in Section 5. In Section 6, we discuss the implications of our findings and the potential of magnification as a tool for discriminating between dark matter models. Finally, our main conclusions are summarised in Section 7. We assume the Planck 18 cosmological model (Planck Collaboration VI 2020) with $\Omega_m = 0.31$, $\Lambda = 0.69$, and $h = 0.676$ ($100 \text{ km s}^{-1} \text{ Mpc}^{-1}$). In this work, we study the differences between FDM and the standard CDM description within the Λ CDM framework. Without loss of generality, we refer to the latter simply as CDM throughout the paper, even though FDM is itself a form of CDM.

2. Lensing formalism

In this section, we briefly introduce the gravitational lensing formalism (Schneider et al. 1992), focussing on the gravitational lensing effect caused by galaxy lenses. These lenses are typically described by a Navarro–Frenk–White (NFW) dark matter halo (Navarro et al. 1996, 1997), consistent with both observations and hydrodynamic simulations (in CDM). A Sérsic profile was adopted for the contribution from baryons. Finally, for the FDM mass profiles, we included the soliton structure and density fluctuations on top of an NFW profile.

The formalism presented here is, nonetheless, mass-model independent and can be readily applied to any deflecting structure, including, but not limited to, NFW or Sérsic profiles. In addition, when working with complex mass models, we can take advantage of the deflection angle linearity, i.e. the total deflection angle can be expressed as the sum of the deflection angle of each component. Specifically, if the total mass distribution can be expressed as

$$\Sigma_{\text{tot}}(\theta) = \sum_{i=1}^N \Sigma_i(\theta), \quad (1)$$

then the total deflection angle is

$$\alpha_{\text{tot}}(\theta) = \sum_{i=1}^N \alpha_i(\theta), \quad (2)$$

where $\alpha_i(\theta)$ is the deflection angle at the position θ generated by the mass distribution Σ_i .

The position of a lensed image β and the corresponding source position θ are connected through the lens equation

$$\beta = \theta - \alpha(\Sigma, \theta), \quad (3)$$

where α is the deflection angle induced at θ by a lens with a surface mass density $\Sigma(\theta)$. Since α depends on θ , the equation is generally non-linear, often admitting multiple image positions for a given source location and lacking an analytical solution in most cases.

The deflection angle is derived from the effective lensing potential:

$$\psi(\theta) = \frac{D_{\text{ds}}}{D_{\text{d}} D_s} \frac{2}{c^2} \int \phi(D_{\text{d}} \theta, z) dz, \quad (4)$$

where D_{d} , D_s , and D_{ds} represent the angular diameter distances to the lens, to the source, and between the lens and the source, respectively. The function ϕ denotes the Newtonian potential of the lens. The deflection is then given by

$$\alpha = \nabla_{\theta} \psi, \quad (5)$$

and the Laplacian of ψ is related to the surface mass density through

$$\nabla_{\theta}^2 \psi = 2 \frac{D_{\text{d}} D_{\text{ds}}}{D_s} \frac{4\pi G}{c^2} \Sigma(\theta) = 2 \frac{\Sigma(\theta)}{\Sigma_{\text{crit}}} \equiv 2\kappa(\theta), \quad (6)$$

where G is the gravitational constant, and κ is the convergence, defined as the dimensionless surface mass density relative to the critical value Σ_{crit} .

Gravitational lensing also modifies the shape and size of background sources. These distortions are encoded in the Jacobian matrix

$$\mathbf{A} \equiv \frac{\partial \beta}{\partial \theta} = \delta_{ij} - \frac{\partial \alpha_i}{\partial \theta_j} = \delta_{ij} - \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j} = \delta_{ij} - \psi_{ij} = \mathbf{M}^{-1}, \quad (7)$$

which is the inverse of the magnification tensor $\mathbf{M}$.

3. Mass profiles

In the previous section, we introduced the lensing formalism for a generic mass distribution $\Sigma(\theta)$, which is the 2D projection of a generic $\rho(\theta, z)$ 3D profile, obtained by integrating the density along the line of sight z . In this section, we summarise the mass distributions used in this work for both the particle description of CDM and the wave-like FDM models.

3.1. Navarro-Frenk-White profile

Simulations have shown that, under the Λ CDM paradigm, the power spectrum of the density perturbations, combined with the collisionless nature of CDM, point to cuspy dark haloes with density profiles scaling as $\rho_{\text{DM}}(r) \propto r^{-1}$. These profiles are well described by the NFW parameterisation (Navarro et al. 1996, 1997), from very massive galaxy clusters ($M \sim 10^{15} M_{\odot}$) to galaxies with masses around $M \sim 10^{11} - 10^{12} M_{\odot}$. Simulations of FDM haloes also show an NFW profile modulated by a central solitonic core and density fluctuations characteristic of its wave nature (Schive et al. 2014).

The NFW profile is a radial function given by

$$\rho_{\text{NFW}}(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}, \quad (8)$$

where the two parameters, ρ_s and r_s , are the characteristic density and the scale radius of the halo. Lower-mass haloes are usually well described by NFW profiles as well, but the core-cusp problem arises: the central density tends to an almost constant value rather than rising steeply at the innermost radii. Fuzzy dark matter (FDM) offers a natural solution to this discrepancy through solitonic structures, as seen in numerical simulations (Schive et al. 2014; Liao et al. 2024). The characteristic density, ρ_s , can also be expressed as a function of the concentration parameter

$$\rho_s = \frac{200}{3} \rho_{\text{crit}} \frac{c^3}{\left[\ln(1+c) - \frac{c}{1+c} \right]}. \quad (9)$$

The concentration parameter, c , is simply the ratio between the radius, r_{200} , and the scale radius of the halo ($c = r_{200}/r_s$), and scales with the mass of the halo roughly as $M^{-0.1}$ (Dutton & Macciò 2014). The radius r_{200} encloses an average density 200 times that of the critical density:

$$r_{200} = \frac{1.63 \times 10^{-2}}{(1+z)h} \left(\frac{M_{200}}{h^{-1} M_{\odot}} \right)^{1/3} \left[\frac{\Omega_0}{\Omega(z)} \right]^{-1/3} \text{ kpc}. \quad (10)$$

Here, h is the dimensionless Hubble parameter, and M_{200} is the mass enclosed within the sphere of radius r_{200} , also commonly used to parametrise the halo, analogously to ρ_s and r_s . Defining $x \equiv r/r_s$, the simplicity of the NFW profile, specifically its radial symmetry, allows an analytical derivation of its surface mass density:

$$\Sigma_{\text{NFW}}(x) = \frac{2\rho_s r_s}{x^2 - 1} \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \arctan \sqrt{\frac{x-1}{x+1}}, & x > 1 \\ 1 - \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}}, & x < 1. \end{cases} \quad (11)$$

The limit at $x = 1$ is finite: $\Sigma_{\text{NFW}}(1) = \frac{2}{3} \rho_s r_s$. The deflection angle is then given by

$$\alpha_{\text{NFW}}(x) = \frac{4\rho_s r_s \Sigma_{\text{crit}}^{-1}}{x} h(x), \quad (12)$$

with the auxiliary function $h(x)$ defined as

$$h(x) = \ln\left(\frac{x}{2}\right) + \begin{cases} \frac{2}{\sqrt{x^2-1}} \arctan \sqrt{\frac{x-1}{x+1}}, & x > 1 \\ \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh} \sqrt{\frac{1-x}{1+x}}, & x < 1 \\ 1, & x = 1. \end{cases} \quad (13)$$

The mass profile presented here is valid for CDM haloes and subhaloes according to CDM predictions. This profile serves as the central baseline model for FDM; further distinctions are outlined in the following subsection.

3.2. FDM profile

Simulations show that FDM halo profiles are composed of three distinct components. First, a baseline NFW-like profile, as described in the previous subsection. Second, a central solitonic core that produces a flat inner density profile and naturally addresses the core-cusp problem. Together, these two components constitute the smooth mass distribution of the halo. Third, small-scale wavelike density fluctuations, arising from the quantum nature of the field, introduce granularity beyond the smooth profile. The first component was discussed previously; here, we focus on the solitonic core and small-scale wave-like density fluctuation, which are unique features of FDM.

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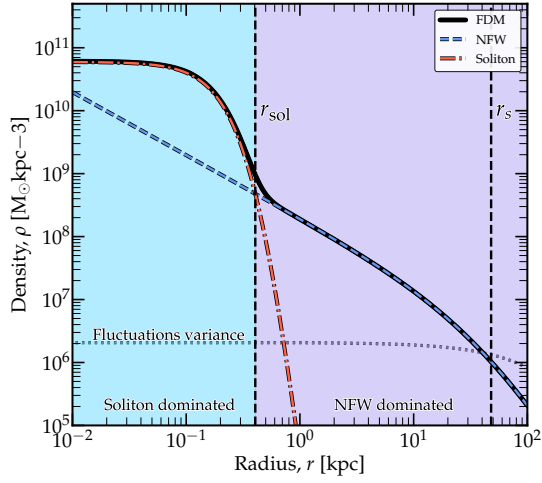


Fig. 1. Radial profile of an FDM halo ($M_h = 7 \times 10^{11} M_\odot$ and $m_\psi = 10^{-22}$ eV at $z = 0.9$), neglecting quantum fluctuations (solid black line). The inner profile of the halo is dominated by the soliton structure (dot-dashed red line). At $r = r_{\text{sol}}$, the profile transitions to a standard NFW dark matter halo (dashed blue line). The radial variance of the fluctuations, given by Eq. (21) (dot-dashed grey line) is shown in arbitrary units.

3.2.1. Soliton

Solitons are stationary, spherically symmetric ground-state solutions of the Schrödinger–Poisson equation. They appear as a central flat density, as opposed to the cuspy inner density of NFW haloes. Solitons exhibit a time-variable centre offset (1 kpc) referred to as the soliton random walk, which arises from wave interference (Schive et al. 2020). This random walk has a timescale of approximately 100 Myr (Schive et al. 2020), which is too large to affect the lensing systems that we observe. Solitons are typically characterised by a single parameter, M_s , the soliton mass. The so-called soliton–halo relation expresses the soliton mass in terms of its host halo and redshift. This relation was first identified in cosmological simulations (Schive et al. 2014). More recent studies (Liao et al. 2024) confirm that it takes the form $M_s \propto m_\psi^{-1} (1 + z_h)^{1/2} M_h^{1/3}$, where m_ψ is the axion mass, M_h is the halo mass, and z_h is its redshift. Although this relation has been confirmed by other studies, some results suggest different relations, such as $M_s \propto M_h^{5/9}$ (Mocz et al. 2017). A large scatter in the soliton–halo relation has been reported, suggesting that the relation may not be universal and remains an open question. For the remainder of this work, we assume a soliton–halo relation as found by Schive et al. (2014) ($M_s \propto M_h^{1/3}$), specifically in the form reported by Liao et al. (2024), since their simulations cover higher halo masses that had not been explored in previous works.

The soliton mass density profile is given by

$$\rho_{\text{sol}}(r) = \frac{\rho_c}{\left(1 + 0.091 \left(\frac{r}{r_c}\right)^2\right)^8}, \quad (14)$$

where

$$\rho_c = 0.019 \left(\frac{mc^2}{10^{-22} \text{ eV}}\right)^{-2} \left(\frac{r_c}{\text{kpc}}\right)^{-4} M_\odot \text{ pc}^{-3}. \quad (15)$$

The radius r_c is defined as the distance at which the density drops to half its central value.

The smooth component of the FDM model mass density profile, as shown in Fig. 1, is then

$$\rho_{\text{FDM}}(r) \simeq \begin{cases} \rho_{\text{sol}}(r) & r < r_{\text{sol}} \\ \rho_{\text{NFW}}(r) & r > r_{\text{sol}}, \end{cases} \quad (16)$$

where r_{sol} is the transition radius between models, which can be found numerically from the condition $\rho_{\text{sol}}(r_{\text{sol}}) = \rho_{\text{NFW}}(r_{\text{sol}})$. It is typically a few times r_c : $2.5 \lesssim r_{\text{sol}}/r_c \lesssim 3.5$ (Mocz et al. 2017; Chiang et al. 2021; Furlanetto et al. 2025).

Finally, the 2D projection of the mass density profile (without quantum fluctuations) can be obtained by numerically integrating along the line of sight:

$$\Sigma_{\text{FDM}}(r) = 2 \int_0^\infty \rho_{\text{FDM}}(r, z) dz. \quad (17)$$

Here, r is defined as the radial distance within the lens plane and z is the tangential direction to the plane.

3.2.2. FDM fluctuations

In the preceding subsections, we described the smooth component of the FDM profile, which is composed of a main NFW halo in the outskirts and a soliton profile in the central region. In addition, FDM exhibits density clumps roughly separated by half the de Broglie wavelength, which depends on the axion and halo masses:

$$\lambda_{\text{dB}} = 150 \left(\frac{10^{-22} \text{ eV}}{m_\psi}\right) \left(\frac{M_h}{10^{12} M_\odot}\right)^{-1/3} \text{ pc}. \quad (18)$$

Following Amruth et al. (2023), Kawai et al. (2022), Dalal et al. (2021), we assumed that the integrated column density fluctuations along the line of sight can be well approximated as a Gaussian random field (GRF). This GRF can be drawn as a random realisation from the power spectrum

$$P(k) = \frac{4\pi}{3r_h(x)} \left(\frac{\lambda_{\text{dB}}}{2}\right)^3 \exp\left(-\frac{\lambda_{\text{dB}}^2 k^2}{4}\right), \quad (19)$$

where

$$r_h(x) = \frac{\left(\int_Z dz \rho_h(r)\right)^2}{\int_Z dz \rho_h^2(r)}, \quad (20)$$

is the effective halo size, which contains information on the density dispersion along the line of sight, Z . Here, ρ_h is the average local mass density of the halo. This field is modulated by a radial variance function that depends on x :

$$\sigma^2(x) \approx \begin{cases} \frac{\kappa_s^2}{r_s \lambda_{\text{dB}}} \left[\frac{\pi}{x} - \frac{1}{(x^2 - 1)^3} \left(\frac{6x^4 - 17x^2 + 26}{3} + \frac{2x^6 - 7x^4 + 8x^2 - 8}{\sqrt{1 - x^2}} \text{sech}^{-1}(x) \right) \right], & x < 1 \\ \frac{\kappa_s^2}{r_s \lambda_{\text{dB}}} \left[\frac{\pi}{x} - \frac{1}{(x^2 - 1)^3} \left(\frac{6x^4 - 17x^2 + 26}{3} + \frac{2x^6 - 7x^4 + 8x^2 - 8}{\sqrt{x^2 - 1}} \sec^{-1}(x) \right) \right], & x > 1 \end{cases}, \quad (21)$$

where the limit at $x \equiv r/r_s = 1$ is $\frac{\kappa_s^2}{r_s l_{\text{dB}}}(\pi - 64/21)$ and $\kappa_s \equiv \rho_s r_s / \Sigma_{\text{crit}}$ is the characteristic convergence.

The surface mass density of the FDM can then be drawn as a random realisation given $P(k)$ in Eq. (19) modulated by the radial variance given by Eq. (21). Neither the soliton nor the GRF density fluctuations admit analytical expressions for their deflection angles and must therefore be computed numerically.

3.3. Sérsic profile

The profiles shown so far describe dark matter haloes that are in good agreement with the data. Such haloes account for approximately 90% of the total mass of the galaxies, while the other 10% consists of baryons, whose mass can be traced from the light distribution. The effect of the baryons is a dampening of the FDM mass surface density fluctuations, as found by Amruth et al. (2023). In the present study, we considered only a modest dampening factor of ~20% to show the differences between CDM and FDM.

The surface brightness of elliptical galaxies, bulges, and discs of spiral galaxies is best fitted by a Sérsic profile ($r^{1/n}$), where n is a free parameter known as the Sérsic index. Although not a 3D mass profile, the Sérsic profile is a good approximation of the 2D projection of the mass distribution, assuming a homogeneous light-to-mass ratio. Specifically, this circularly symmetric mass profile is expressed as

$$\Sigma(r) = \Upsilon I_e \exp \left\{ -b(n) \left[\left(\frac{r}{r_e} \right)^{1/n} - 1 \right] \right\}, \quad (22)$$

where Υ is the light-to-mass ratio of the galaxy, I_e is the luminosity density at the effective radius r_e , and $b(n)$ is a constant defined so that the luminosity enclosed within r_e equals half of the total luminosity. The Sérsic index, n , determines the concentration of the profile, with lower values corresponding to shallower inner slopes and steeper outer fall-offs. Typical values include $n \approx 1$ for exponential discs in spiral galaxies, and $n \approx 4$ for bulges and elliptical galaxies, corresponding to the de Vaucouleurs profile.

As in the NFW profile, we defined a dimensionless quantity $x = (r/r_e)^{1/n}$. As shown in Eq. (6), the convergence is half the Laplacian of the lensing potential. The lensing potential, ψ , can be obtained by solving that equation, and the deflection angle is then the gradient of the lensing potential. To derive an analytical expression for the Sérsic profile, we followed the procedure of Cardone (2004). The deflection angle takes the form

$$\alpha(x) = 2\alpha_e x^{-n} \left[1 - \frac{\Gamma(2n, bx)}{\Gamma(2n)} \right], \quad (23)$$

where α_e is the deflection angle at $r = r_e$, whose value is given by

$$\alpha_e = nr_e \kappa_e b^{-2n} e^b \Gamma(2n), \quad (24)$$

in arcsec units. The quantity κ_e is defined as $\kappa_e = \Upsilon I_e / \Sigma_{\text{crit}}$, where $\Gamma(a, z)$ is the incomplete gamma function and $\Gamma(a)$ is the actual gamma function. The parameter $b(n)$ can be found from the equation

$$\Gamma(2n, b) = \Gamma(2n)/2. \quad (25)$$

Tabulated values of $b(n)$ for $n \in \{1, \dots, 15\}$ can be found in Mazure & Capelato (2002).

4. Methodology

In this work, we assumed different mock lenses by varying their halo masses and ellipticities, and by adding various small-scale perturbations to study the differences in radial magnification arcs between CDM and FDM for different axion masses. For each case, we computed the deflection angles numerically or, when possible, analytically. We then added them linearly and computed the magnification in the lens plane following Eq. (7):

$$\det A = \left(1 - \frac{\partial \alpha_x}{\partial \theta_x} \right) \left(1 - \frac{\partial \alpha_y}{\partial \theta_y} \right) - \frac{\partial \alpha_x}{\partial \theta_y} \frac{\partial \alpha_y}{\partial \theta_x}. \quad (26)$$

The deflection angles for the Sérsic and NFW profiles were obtained analytically (Eqs. (12) and (23)), significantly speeding up the process. The soliton and FDM fluctuations were computed numerically, according to Eqs. (1) and (2). Each pixel was treated as a point lens, and the total deflection angle was estimated via a convolution between the mass and distance kernels using the fast Fourier transform (Cooley & Tukey 1965). For each case, we simulated a field of view (FOV) ranging from three to seven times the Einstein radius of the lens to fully cover the CCs. This variable FOV is motivated by the ellipticity of the lens, which stretches the CCs along one axis while compressing them along the other. We assumed that all lenses were circularly symmetric when computing the deflection angle and added ellipticity afterwards as

$$\begin{aligned} F_{x_{\text{elliptical}}} &= F \cdot \frac{x}{(1-e)r} \\ F_{y_{\text{elliptical}}} &= F \cdot \frac{y(1-e)}{r}, \end{aligned} \quad (27)$$

where F is a circularly symmetric function, x and y are the Cartesian coordinates of the lens plane, $r \equiv \sqrt{x^2 + y^2}$ is the radial distance within the plane, and $e \equiv 1 - b/a$ is the ellipticity, with a and b representing the major and minor semi-axes of the ellipse.

First, for a mock lens, we constructed the smooth CDM lens model by combining an NFW dark matter halo with a Sérsic profile representing the baryonic component of the galaxy. We then computed the deflection angles for each component, added them linearly, and obtained the resulting magnification map. Additionally, we assumed a fiducial axion mass. For practical reasons, we selected m_ψ from three values, $m_\psi \in 0.4, 1, 10 \times 10^{-22}$ eV: the lower limit was motivated by the conservative constraint on the axion mass from Chiang et al. (2023). The middle value was chosen for historical reasons, as it is the most commonly adopted in ultralight axion FDM studies and is found by many studies to reproduce the observed properties of multiply lensed images (Amruth et al. 2023; Laroche et al. 2022) or the innermost kinematics of dwarf galaxies (Broadhurst et al. 2020). The upper value lies in a regime in which current constraints remain weak.

We considered three different halo masses, M_{200} : a fiducial mass of $7 \times 10^{11} M_\odot$ as in Amruth et al. (2023), a smaller halo of $3 \times 10^{11} M_\odot$, and a massive halo of $4 \times 10^{12} M_\odot$. The combination of three axion masses and three halo masses yields a total of nine models, as shown in Table 1. We also adopted the lens parameters from Amruth et al. (2023) for the HS 0810+2554 system, using $z_l = 0.89$, $z_s = 1.51$, $r_s = 50$ kpc, and $c = 9$ for $M_{200} = 7 \times 10^{11} M_\odot$, scaling them with mass as $c \propto M^{-0.1}$ and $r_{200} \propto M^{1/3}$, where $r_s = r_{200}/c$.

To test the ellipticity effect, we adopted the values $e = 0$ (a perfectly axisymmetric system), $e = 0.2$ as in Amruth et al. (2023), and $e = 0.4$ (an extremely elliptical configuration) for

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Table 1. List of models.

Model ID	M_h [$10^{12} M_\odot$]	m_ψ [10^{-22} eV]	λ_{dB} [pc]
11	0.3	0.4	560
12	0.3	1	224
13	0.3	10	22
21	0.7	0.4	422
22	0.7	1	169
23	0.7	10	17
31	4	0.4	236
32	4	1	95
33	4	10	9

Notes. The model ID consists of two indices representing the halo mass and axion mass, respectively, with values increasing from 1 to 3. For example, the most massive halo with the medium axion mass would be identified as 32, while the lighter halo with the heaviest axion would be indexed as 13.

one of the simulated lenses. We first computed the analytical deflection quantities for the circular case and then applied the ellipticity transformation described in Eq. (27).

For each case, once the magnification was computed, we compared the magnification statistics in the lens plane from the FDM model against the CDM model, both without subhaloes (smooth case) and with small-scale perturbers. We adopted the same contour radius given by the isomagnification contours from the smooth CDM model (as shown in Fig. 2) and obtained the magnification histograms of the pixels enclosed within them, as shown in Fig. 3. These isomagnification contours trace the shape of the radial critical curve in the smooth CDM model and were normalised by the factor r/r_{CC} , where r is the distance from the centre to the contour and r_{CC} is the distance to the critical curve. In cases where ellipticity was introduced, r and r_{CC} were measured with respect to the closest point on the critical curve to the centre. Once we had the histograms for the magnification inside the isomagnification contours, we estimated the probability of having high magnification ($\mu > 10$) inside the enclosed area $P(\mu \geq \mu_e | \theta_{\text{model}})$, i.e. the p -value associated with the model parameters θ_{model} . For an accurate depiction of the magnification statistics in the presence of FDM density fluctuations, we ensured that the pixel size provided a resolution of at least 10 pixels across the span of λ_{dB} , as defined in Eq. (18). We also performed several GRF realisations to obtain a robust estimate of the magnification statistics and to mitigate possible systematic biases arising from a single realisation. For the soliton parameters, we used the publicly available code SHR¹, adopting the updated model by Liao et al. (2024), which is based on the original formulation of Schive et al. (2014).

5. Results

In this section, we present the main results from the analysis of magnification distributions within radial CCs in FDM-simulated lenses (see Fig. 4) and compare them with those of their CDM counterparts. We also examine the impact of ellipticity in both scenarios and, finally, explore the effects of adding NFW subhaloes to the CDM model compared to the FDM case. For reference, the magnification maps corresponding to the lens models listed in Table 1 are shown in Appendix A.

¹ <https://github.com/calab-ntu/fdm-soliton-halo-relation>

The key distinguishing feature of FDM (and ψ DM in general), compared to other dark matter models, is that destructive interferences give rise to negative density fluctuations with respect to the NFW baseline, which in turn generate high-magnification regions inside the CCs (see Fig. 5). Within these curves, the critical condition is fulfilled: $1 - \kappa - \gamma \approx 0$ for tangential CCs, and $1 - \kappa + \gamma \approx 0$ for radial CCs. Adding mass in the form of small-scale perturbers could modify the local magnification distribution; however, on average, it does not significantly affect the global statistics, except for small background sources located close to the perturbers, where the local PDF can change substantially. In contrast, the wave-like fluctuations inherent to FDM can produce extended high-magnification regions, provided that the associated de Broglie wavelength is sufficiently large relative to the lensing scale. These regions are intrinsically different from those produced by CDM subhaloes or other small-scale structures. In the regime where the fluctuation scale is much smaller than the scale of the lens, given by its Einstein radius (such as in galaxy clusters), FDM behaves effectively as a population of millilenses (Diego et al. 2024; Perera et al. 2025).

The two main parameters that govern the FDM fluctuations, and thus the changes in magnification statistics, are the halo mass, M_h , and the axion mass, m_ψ . The de Broglie wavelength scales with both parameters as shown in Eq. (18), where λ_{dB} increases as $\lambda_{\text{dB}} \propto m_\psi^{-1}$ and $\lambda_{\text{dB}} \propto M_h^{-1/3}$. This makes the axion mass the most important factor in the growth of the density fluctuation scale. The size of the CCs—or, in other words, the lensing area of influence for strong lensing effects—increases with the halo mass approximately proportional to M .

Models 11 and 21 feature a λ_{dB} much larger than the typical lens scale. In such cases, the resulting perturbations in magnification patterns are excessively strong and would likely already have been detected through image position anomalies. Moreover, no Einstein rings would form in these scenarios, and the deviations far exceed existing tensions. Model 12 behaves similarly, although the effects are slightly milder than in models 11 and 21.

Models 13, 22, and 31 show a large effect without completely disrupting the CCs, as occurs in models 11, 21, and 12. These models present a central demagnification caused by the soliton, corresponding to larger halo and axion masses. All of these models exhibit the largest p -values, as shown in Fig. 4.

Models 23 and 32 are similar to each other and show smaller perturbations confined to regions very close to the CCs, resembling the millilensing or microlensing regime. Their p -values also indicate an increased probability of high-magnification regions ($\mu \geq 10$) near the centre, due to the presence of a secondary CC created by the soliton, at the cost of enhanced demagnification further in.

Finally, model 33 illustrates the regime in which FDM approaches standard CDM within the resolution limits of this study. At this resolution, the only distinction between FDM and CDM is the presence of the solitonic core, which increases the probability of high magnification near the halo centre. At much higher resolution, the small-scale density fluctuations would still be visible and their effects would resemble those of microlenses.

5.1. Ellipticity

In this study, we have considered only axisymmetric lenses up to this point. However, real lenses typically exhibit some degree of ellipticity. This ellipticity ranges from 0, corresponding to perfectly circular profiles, up to approximately $e = 0.5$, which is considered extremely large, as higher values are rare in nature.

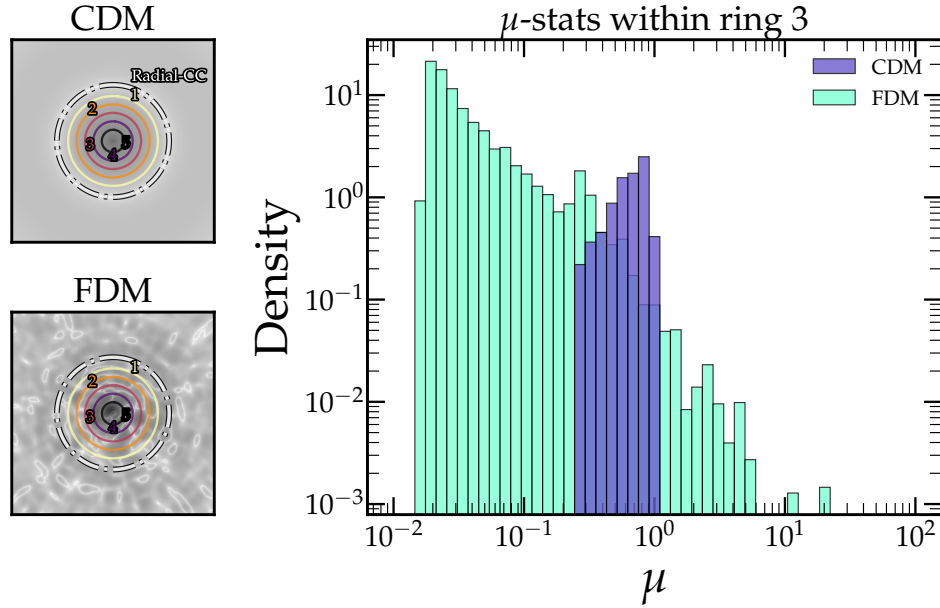


Fig. 2. Illustration of the methodology employed in this work. Upper left: Zoom-in of the central halo region magnification map. The dot-dashed white line shows the position of the radial CC, where maximum magnification is achieved (alongside the tangential CC, not shown here). Bottom left: As in the upper left panel, but for the FDM case with a soliton structure at the centre, which further demagnifies the central region, along with wave-like mass density fluctuations that are both positive and negative. Negative fluctuations ($\delta\kappa < 0$) create islands of high magnification inside the CCs, whereas adding more mass ($\delta\kappa > 0$), both in FDM and in CDM, produces the opposite effect and increases demagnification. In both cases, the coloured rings show isomagnification contours (defined in the symmetric CDM case as the CC shape and scaled down by a factor r/r_{CC}) inside the radial CC. These contours enclose regions where we collect magnification statistics of the simulated pixels within each curve for later comparison. Right: Example magnification statistics. Both histograms show the magnification of pixels within the third ring (dark pink) of the left panels. The purple histogram shows the CDM prediction, where $\mu \lesssim 1$, as expected for an NFW profile. Light blue bins represent the FDM prediction, where both the positive and negative fluctuations in $\delta\kappa$ reduce and enhance the magnification compared to CDM. This results in a high probability of both demagnified and highly magnified images. These simulations correspond to a perfectly symmetrical lens ($e = 0$), a halo mass of $M_{200} = M_h = 7 \times 10^{11} M_\odot$, and an axion mass of $m_\psi = 10^{-22}$ eV.

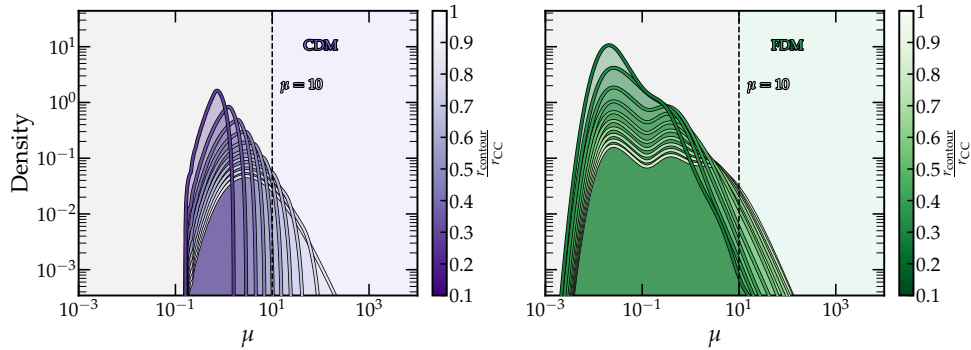


Fig. 3. Probability of magnification estimated within smooth CDM isomagnification contours for the CDM profile (left) and corresponding FDM simulations (right). Brighter colours represent larger contours, with the outermost matching the radial CC, while darker colours correspond to smaller contours closer to the centre of the halo. The dashed black vertical line marks a magnification value of ten, which we set as the threshold for comparing p -values.

For ellipticity, we adopted three nominal values: 0, 0.2, and 0.4, and repeated the same analysis in each case, studying the magnification distributions within the isomagnification contours inside the radial CC. For simplicity, we restricted ourselves to a single axion mass and halo mass, corresponding to those of

model 22. These magnification maps are shown in Fig. 6. We observe a decrease in the area enclosed by the radial CC, which moves progressively closer to the centre of the halo. Initially, the shape stretches along the direction of the minor semi-axis, but at higher ellipticities it also shrinks along that axis. As

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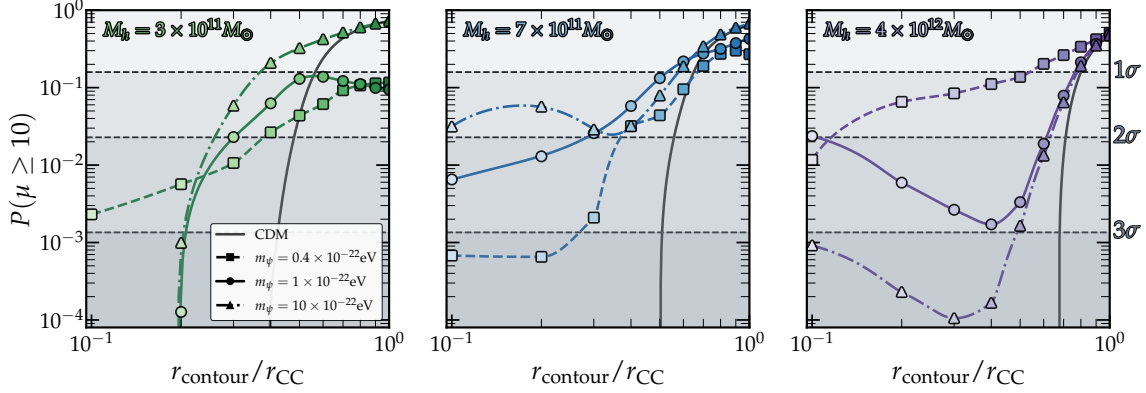


Fig. 4. p -values for a magnification factor equal to or larger than ten for each of the simulated lenses. The left panel shows the p -values for the lightest halo mass ($3 \times 10^{11} M_\odot$) for all axion masses. Middle and right panels show the equivalent p -values for the $7 \times 10^{11} M_\odot$ and $4 \times 10^{12} M_\odot$ haloes, respectively. The grey lines represent the complementary p -values for the CDM model at each halo mass. Halo masses increase from left to right with green, blue, and purple colours. The squares, circles, and triangles represent different axion masses, increasing in this order. Dashed, horizontal lines indicate the significance of finding an image with magnification equal to or greater than ten for each model. For the smooth model, it is virtually impossible to achieve such magnification at a distance to the centre of the halo equal to half that of the CC.

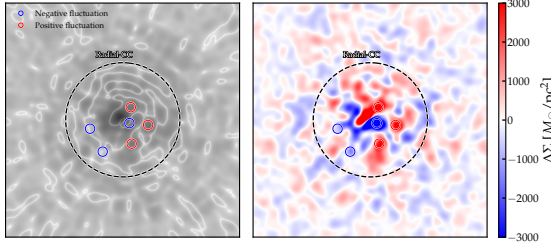


Fig. 5. Magnification patterns inside the radial CC (left) and mass density fluctuations (right). Open blue circles mark the positions of negative mass density fluctuations, while open red circles indicate positive fluctuations. Positive fluctuations in the mass (both present in CDM and FDM) inside the radial CC lead to demagnification, whereas negative mass fluctuations (exclusive in FDM) can result in new critical regions. As we approach the centre of the halo, the negative fluctuations in FDM can compensate the increase in κ from the underlying NFW + Sérsic, fulfilling again the radial criticality condition, $1 - \kappa + \gamma \approx 0$.

shown in Fig. 7, for $e = 0.2$, the p -values are higher than in the circular case. However, at the highest ellipticity, the area of high magnification becomes significantly smaller. Images within the smaller radial CC are more likely to be demagnified; thus, the p -values are smaller than the two other cases. Nevertheless, some regions of high magnification exist between CCs along the minor semi-axis, although these are not covered in this work.

5.2. CDM subhaloes

The main objective of this paper is to demonstrate that the enhanced magnification statistics observed in the central regions of FDM haloes (containing both negative and positive mass fluctuations relative to the underlying NFW) are difficult to reproduce within classical Λ CDM models, where mass offsets relative to the NFW baseline can only be positive. These enhanced statistics appear to be a unique feature of the negative fluctuations in FDM (or potentially other wave-like dark matter models

not considered here). To explore this, we focussed again on halo model 2 ($M_h = 7 \times 10^{11} M_\odot$) and introduced subhaloes at various positions and with different masses inside the region enclosed by the radial CC.

We considered three subhalo masses: $10^6 M_\odot$, comparable to the most massive globular clusters; $10^7 M_\odot$; and $10^8 M_\odot$, representative of low-mass dwarf galaxies. Different subhalo positions were explored to account for the enhanced lensing effect when a lens lies near the CC. In such cases, a point-mass lens of mass M located close to a CC with magnification μ behaves as if it has an effective mass of $M_{\text{eff}} = \mu \times M$. We modelled subhaloes with axisymmetric NFW profiles, adopting concentration parameters scaling as $M^{-0.1}$ (Dutton & Macciò 2014) and scale radii scaling as $M^{1/3}$. We adopted reference values of $c = 45.7$ and $r_s = 0.18$ kpc for a $10^6 M_\odot$ subhalo, scaling these quantities accordingly for the other masses.

When including subhaloes, we find a slight increase in the magnification p -values (for $\mu \geq 10$) compared to the smooth CDM case, as shown in Figs. 8 and 9. However, this enhancement is smaller than that produced by FDM (see Fig. 4). Interestingly, when the subhalo is too massive ($10^8 M_\odot$), it can even reduce the p -values with respect to the base model without subhaloes, when considering a large radius for the contour of the studied region, as illustrated in Fig. 9. This can be understood as an increase in the curvature of the deflection field of the otherwise smooth model. Since magnification is inversely proportional to the curvature of the deflection field, adding more (positive) substructure reduces the magnification. Alternatively, this can be understood as the term $\mu_r^{-1} = 1 - \kappa + \gamma$ deviating further from ≈ 0 (and becoming more negative) when κ increases inside the radial CC.

The inclusion of subhaloes induces two distinct effects: (i) they can form small secondary CCs at their positions, resulting in higher μ , but locally reducing magnification near them. However, the p -values in these regions remain low, and highly magnified central images remain statistically rare; (ii) if located near the radial CC and sufficiently massive, a subhalo can distort and pull the main CC inward. Even then, the probability of achieving high magnifications stays below that expected from FDM.

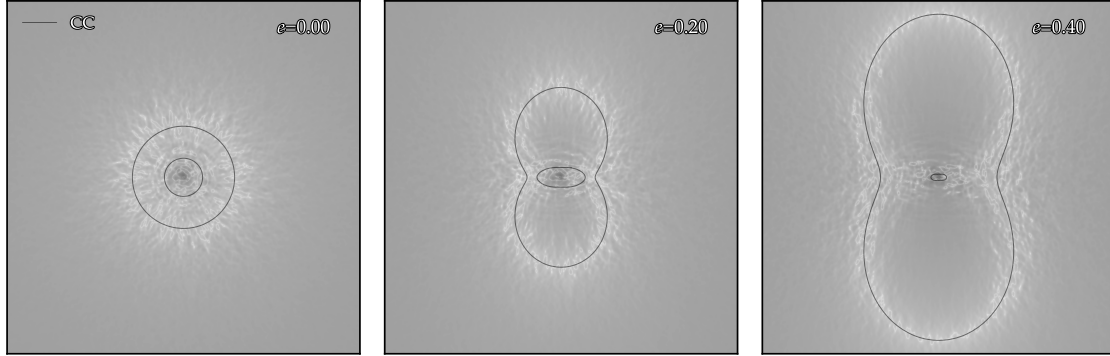


Fig. 6. Ellipticity effects on magnification. From left to right, the lens ellipticity increases from none to high. The lens parameters are those of model 22 from Table 1 ($M_h = 7 \times 10^{11} M_\odot$ and $m_\psi = 10^{-22}$ eV). The smooth CCs stretch along the major axis and compress along the minor axis, forming an hourglass-shaped configuration. Fluctuations in FDM follow the elliptical mass distribution of the lens; as a result, fluctuations located farther from the centre are attenuated, while those closer in are enhanced. Inside the radial CC, however, as the CC approaches the halo centre, the density fluctuations overlap with the soliton region and are consequently suppressed. In contrast, the interface between the tangential and radial CCs experiences enhanced magnification. The black lines correspond to the CCs of the smooth CDM model.

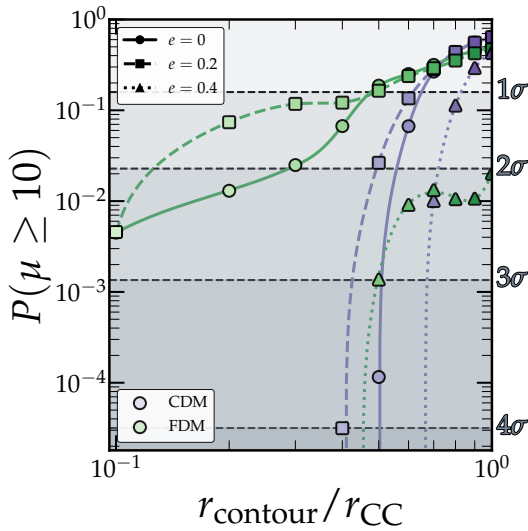


Fig. 7. p -values for magnification values equal to or greater than ten in both the smooth CDM model (purple), and the wave-like FDM model (green). Each point represents the p -value calculated for pixels inside the scaled CC contours, with a size expressed as a fraction of the radial CC size (r/r_{CC}). Different marker symbols denote the p -values corresponding to different ellipticity realizations. As in Fig. 4, the p -value is the probability within the given model of attaining a magnification equal to or greater than ten. Dashed, horizontal dashed lines indicate the significance contours. The halo and axion masses correspond to model 22 in Table 1, with $M_h = 7 \times 10^{11} M_\odot$ and $m_\psi = 10^{-22}$ eV.

When subhaloes are placed closer to the halo centre, their impact on the global magnification statistics within the radial CC region diminishes, as shown in Fig. 8. In the limit where a subhalo lies exactly at the centre, its behaviour becomes nearly indistinguishable from that of the smooth CDM case. This effect is partially explained by the magnification of the macro model (i.e. the galaxy halo) at the subhalo's location. Subhaloes near the radial CC, where $\mu > 1$, thus behave as if they were more

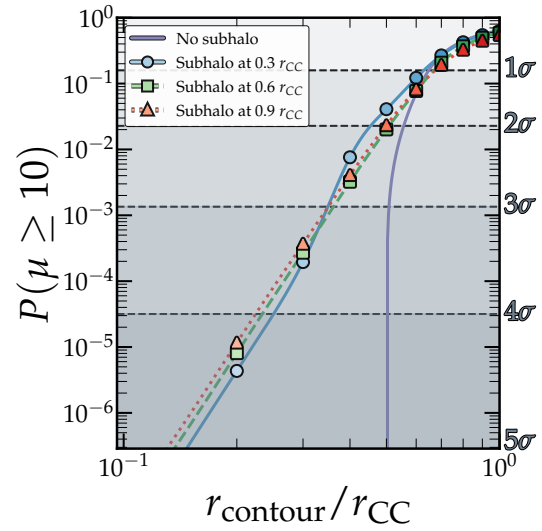


Fig. 8. p -values for magnification values equal or greater than ten in the smooth CDM model (purple) and adding a subhalo of $10^7 M_\odot$ at different positions relative to the centre of the main halo. As in Figs. 4 and 7, the p -value is the probability of obtaining an image with magnification equal to or greater than ten given the models. The significance values are indicated by the dashed horizontal lines. The halo and axion masses correspond to model 22 in Table 1, with $M_h = 7 \times 10^{11} M_\odot$ and $m_\psi = 10^{-22}$ eV.

massive, effectively shifting the radial CC inward and increasing the p -values within the smaller contours near the centre. Conversely, subhaloes located closer to the centre, where $\mu < 1$, behave as if they had lower effective masses. In these cases, any increase in p -values is primarily driven by the formation of secondary CCs, similar to those produced by a solitonic core; however, these secondary CCs can also demagnify the region inside them. Since the central macro magnification is lowest, the enhancement in p -values for centrally placed subhaloes remains minimal.

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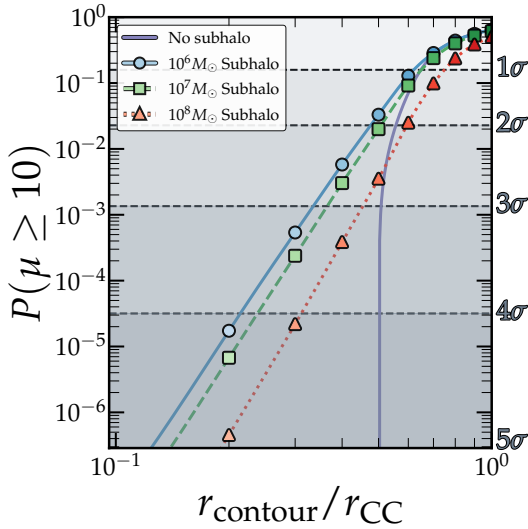


Fig. 9. Same as Fig. 8, showing p -values for magnifications equal to or greater than ten in the smooth CDM model and after adding subhaloes. In this case, the subhalo mass varies between realisations, as colour-coded in the legend, while its position is fixed at half the radius of the radial critical curve from the centre of the main halo. The halo and axion masses correspond to model 22 in Table 1, with $M_h = 7 \times 10^{11} M_\odot$ and $m_\psi = 10^{-22}$ eV.

When the subhalo position is fixed (see Fig. 9), the behaviour simplifies: more massive subhaloes, being stronger overdensities, more effectively demagnify their central region and reduce the higher magnification area. Subhaloes closer to the radial CC remain more efficient at perturbing and shifting the CC inward.

While subhaloes alone cannot reproduce the magnification statistics of FDM, they may still contribute to positional anomalies, similarly to perturbations introduced by small satellite galaxies or low-mass cluster members.

6. Discussion

The results presented in this work highlight a key observational distinction between standard CDM and FDM in the context of strong gravitational lensing: FDM gives rise to a higher probability of magnification within the radial CC as a consequence of interference-driven density patterns. In particular, the negative mass fluctuations in FDM can compensate for the increase in κ inside the radial CC, resulting in new critical regions around the negative fluctuations. Meanwhile, standard CDM models, even when including subhaloes, fail to reproduce similar levels of central magnification. This behaviour, unique to FDM scenarios, offers a promising avenue for constraining the nature of dark matter through precise lensing measurements.

Specifically, we find that extended high-magnification regions near the centre of lenses, associated with soliton-induced secondary CCs and fluctuations on de Broglie scales, are difficult to mimic with conventional CDM substructure. This result suggests that magnification statistics within radial CCs, over a large span of well-modelled lenses, can provide a complementary probe of FDM. With the next generation of surveys, including the ongoing *EUCLID* mission, the upcoming *Rubin-LSST*, and *Roman*, thousands of galaxy-scale lensing systems are expected to be discovered. These systems can then be followed up with

specialised instruments that are better suited for strong lensing and/or spectroscopic analysis.

The magnification statistics presented here characterise regions of high magnification but do not capture the spatial distribution of magnifications. Extended sources would require a follow-up analysis incorporating convolved magnification maps with specific source shapes. Nonetheless, point-like sources are better suited to test this statistical framework. Given the resolution of our simulation, we argue that bright, small sources such as quasars (QSOs) are well described by our results and represent an ideal observational test of the predictions presented here. Subparsec, compact but luminous star clusters, which are larger sources, are even more suitable, as they are less affected by microlensing than QSOs.

While some axion mass ranges near 10^{-22} eV have been excluded by complementary constraints, it has been argued that these limits remain weak due to modelling assumptions or noisy data. Here, we present a complementary lensing-based method to constrain these ranges.

The smallest axion masses (10^{-23} eV) are unlikely, as the large distortions they produce in the critical curves should already have been observed. The intermediate range, around 10^{-22} – 10^{-21} eV, yields plausible effects. However, we argue that the largest axion mass considered may provide the most compelling case. While smaller axion masses produce deviations from the smooth CDM model that are too strong to have gone unnoticed, this higher mass yields lensing effects that remain significant when applied to lower-mass haloes but consistent with current observational constraints in more massive haloes, where the behaviour closely resembles that of CDM. Interestingly, these smaller haloes are more challenging to detect observationally, suggesting that this axion mass still offers a viable and testable window for future studies.

Some unmodelled effects remain, such as the presence of a central supermassive black hole. We argue that such a component would act as a point-like lens, slightly shifting the smooth CC outwards and introducing a small inner CC, while leaving the main results of this work effectively unchanged. However, its impact on the solitonic structure is less clear and is beyond the scope of this study; therefore, a detailed analysis is deferred to future work. It is also important to note that the true global density profiles of real galaxies remain uncertain. Current models represent the best approximations based on available data and theoretical expectations. Any revision to our understanding of the smooth mass distribution would directly affect the resulting lensing predictions and constraints.

A source of uncertainty arises in the detection of central bright images, due to the difficulty in disentangling them from the emission of the lens galaxy itself. In this regard, spectroscopic follow-up observations would be valuable to distinguish between the two, since complementary images of the lensed source provide an expected spectrum that can be used to separate it from the lens galaxy. The application of this methodology to observational data, along with the associated challenges, will be addressed in future studies. With the advent of new facilities conducting large, deep surveys, we anticipate a wealth of data—particularly for central images of lensed QSOs—that will enable a detailed statistical study of CDM versus FDM.

7. Conclusion

We explored the impact of wave-like dark matter fluctuations on gravitational lensing magnification patterns, with a focus on radial critical curves (CCs). Our analysis reveals observational

signatures that distinguish FDM from classical CDM, particularly in compact lensing configurations. The main conclusions of this work are as follows.

- We have shown that wave-like density fluctuations inherent to FDM can enhance magnification near radial CCs in a manner that is difficult to reproduce with CDM, even when including substructure. In particular, the negative fluctuations in FDM (relative to the NFW profile), which are not present in standard CDM, can produce islands of high magnification in the region interior to the radial critical curve. Such islands cannot be reproduced by classical CDM models.
- Our results indicate that axion masses in the range 10^{-22} – 10^{-21} eV, especially within low-mass haloes, produce effects distinguishable from CDM and merit further investigation.
- Statistical differences in magnification, particularly for point-like sources such as QSOs or compact star clusters near the centres of lenses, offer a promising observational signature for constraining ultralight axion dark matter. This work motivates targeted searches for these compact, highly magnified sources within radial arcs as potential probes of the FDM parameter space.
- While enhanced central images are more probable in the FDM scenario, they remain plausible, though less probable, in standard CDM models with subhaloes. To place robust constraints on the axion mass, a statistically significant sample of lenses with anomalous radial images is required. Each system would necessitate a tailored lens model, given the highly non-linear uncertainties associated with the observational data. A joint likelihood analysis combining multiple systems would strengthen such constraints. Upcoming wide-field surveys, such as *Euclid*, *Rubin-LSST*, and *Roman* are expected to detect numerous lensed systems, providing a promising dataset for pursuing this approach. The number of systems with anomalous radial images required to place a robust constraint on the axion mass within the FDM framework is beyond the scope of this paper, as each system would require a tailored lens model and propagate the highly non-linear uncertainties inherent in the observational data.

Acknowledgements. We sincerely thank Hsi-Yu Schive and Pin-Yu Liao for insightful discussions regarding the soliton-halo relation and galaxy parameters. JMP acknowledges financial support from the Formación de Personal Investigador (FPI) programme, ref. PRE2020-096261, associated with the Spanish Agencia Estatal de Investigación project MDM-2017-0765-20-2. J.M.D. acknowledges the support of project PID2022-138896NB-C51 (MCIU/AEI/MINECO/FEDER, UE) Ministerio de Ciencia, Investigación y Universidades. SKL and JL acknowledge the General Research Fund under grant RGC/GRF 17312122, which is issued by the Research Grants Council of Hong Kong S.A.R. PM and TJB are supported by the Spanish Grant PID2023-149016NB-I00 (MINECO/AEI/FEDER,UE), as well as the Basque government Grant No. IT1628-22. PM also acknowledges financial support from fellowship PIF22/177 (UPV/EHU). We acknowledge Santander Supercomputación support group at the University of Cantabria who provided access to the supercomputer Altamira Supercomputer at the Institute of Physics of Cantabria (IFCA-CSIC), member of the Spanish Supercomputing Network, for performing simulations. We made use of the following software: Python, NumPy, SciPy, Matplotlib, Astropy, and Powerbox.

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Appendix A: Simulated lenses

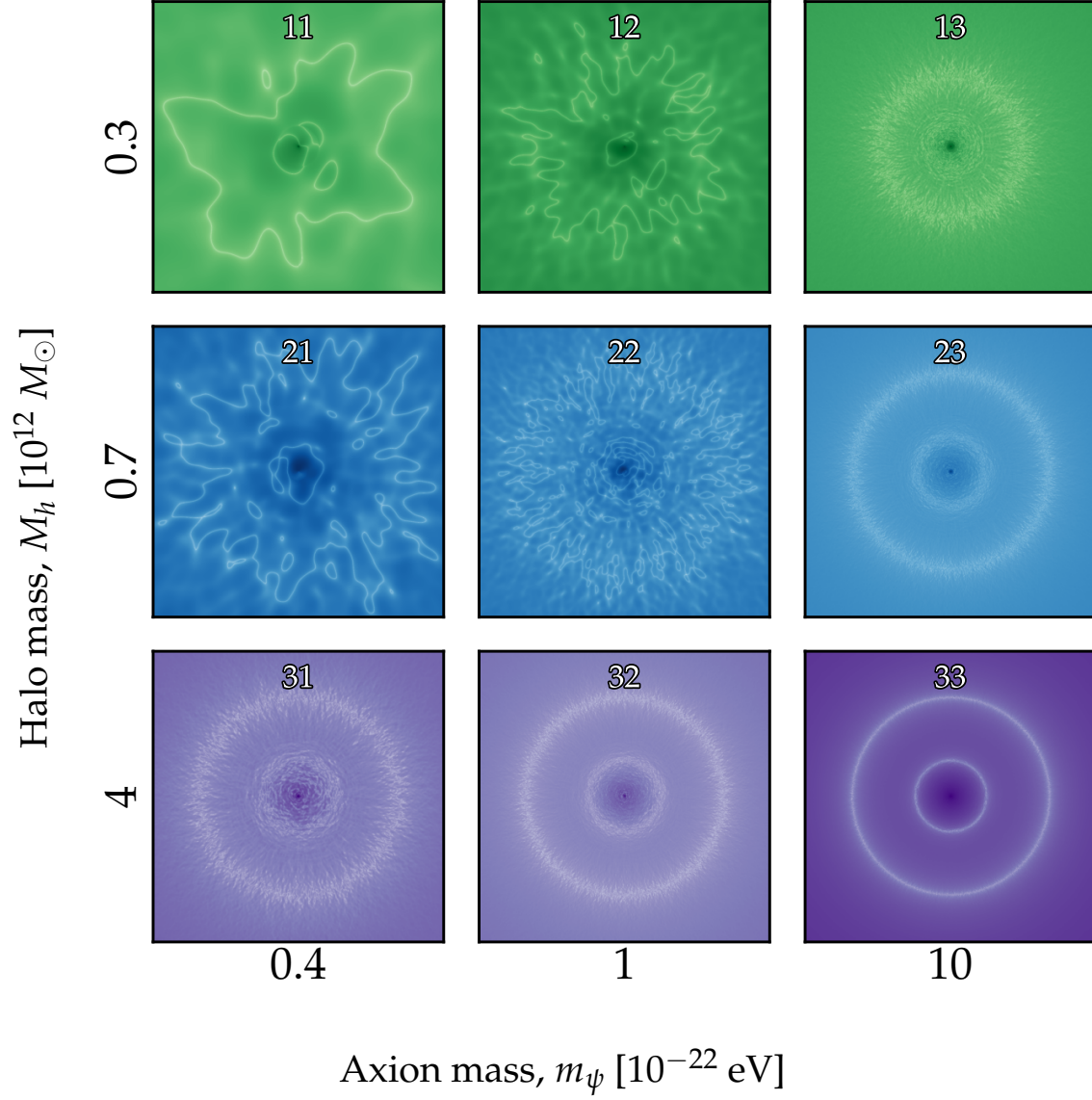


Fig. A.1. Magnification maps of simulated lenses according to Table 1. Axion mass increases from left to right, and halo mass increases from top to bottom.

Appendix B: Image position anomalies

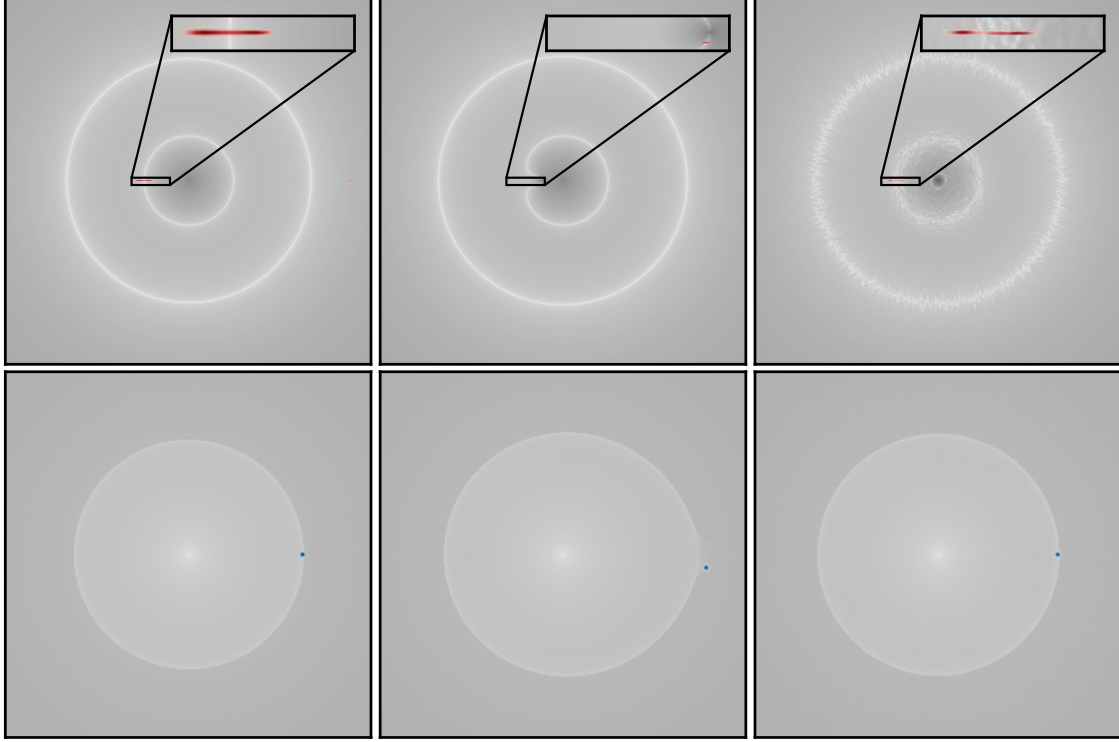


Fig. B.1. Radial image configurations for three lens models: a smooth particle-like CDM model (left panel), the same model with a $10^7 M_\odot$ subhalo placed halfway between the centre and the radial CC (central panel), and an FDM model corresponding to an axion mass of 5×10^{-22} eV (right panel). The top row shows the CCs, while the bottom row shows the corresponding caustics. The blue dots in the bottom panels represent a Gaussian source with a width of 0.8 pixels, placed at the radial caustic to produce bright radial images on the CC. The resulting arcs or images appear in red in the top panels. In the smooth CDM case, the arcs are symmetric and located on top of the radial CC. When a subhalo is added, a similar configuration arises, though the image appears closer to the centre and with lower magnification. In the FDM case, the arcs are asymmetric and exhibit magnification fluctuations on the scale of the axion's de Broglie wavelength. The two merged radial images seen in the CDM case are now split into distinct components by the fluctuations. Although the caustics are broader in this case, we can confidently say that for an FDM lens, if the source size is comparable to the de Broglie scale, significant image position asymmetries and brighter images closer to the centre can arise compared to the particle-CDM model.

Part III

Concluding Remarks

Chapter 5

Conclusions

Gravitational lensing has been used for decades to probe the abundance and distribution of DM. Since its establishment as a key tool in astrophysics and cosmology in the 1970s, lensing has played a central role in testing the cosmological model. WL measurements have provided some of the most precise mass estimates for massive structures such as galaxy clusters, clearly revealing that a significant fraction of their mass, around 90%, is in the form of this elusive component. The spatial distribution of DM does not necessarily follow that of baryons, especially in merging or unrelaxed systems such as the Bullet Cluster, which remains one of the strongest pieces of evidence for the existence of DM. On larger scales, lensing of the LSS, through observations like cosmic shear and CMB lensing, has delivered accurate measurements of the matter distribution and clustering in the late Universe. On smaller scales, SL has been used to study the internal structures and sub-halo populations of lenses, while time delays in transient events (notably type Ia supernovae) have been employed to measure the current expansion rate of the Universe (H_0), providing independent support for the existence of DE driving cosmic acceleration. ML, on the other hand, has been crucial for exploring the smallest mass scales, revealing the presence of solar- and sub-solar-mass lenses, which can be used to probe the compactness of DM. Lensing is now entering a high-precision era, with space-based instruments such as JWST and upcoming Stage-IV surveys like Euclid and the Rubin-LSST, which promise an unprecedented large data revolution.

Beyond cosmology, lensing opens a unique window into the study of high-redshift

sources, enabling the observation of astrophysical phenomena that would otherwise remain inaccessible. With this in mind, the goals of this thesis have been clear: to push forward the theoretical understanding of gravitational lensing observables, in order to provide robust analyses of both the properties of high-redshift sources and the nature of DM.

At the onset of this thesis, extremely magnified events were considered rare, early detections of individual stars at cosmological distances were regarded as serendipitous. This changed rapidly, as dozens of such events were subsequently discovered. These detections have allowed us to probe the properties of stellar populations at high redshift, comparing them to those of stars in the local Universe. With the prospect of many more events to come, accompanied by exciting astrophysical phenomena, a major challenge emerged: extracting statistically meaningful information about both the microlenses and the sources requires understanding the likelihood of such extreme magnification events, and how this likelihood depends on the lensing environment. This, in turn, requires large microlensing simulations which, despite significant algorithmic progress during the course of this thesis, remain computationally expensive and time-consuming.

To overcome this, our first paper ([Palencia et al. 2024](#)) is dedicated to characterising the magnification PDFs, their dependence on lensing parameters, and the impact of small-scale perturbations such as milli-lenses (sub-haloes) and compact DM objects. We identified several key effects:

1. a strong dependence on macro-parity, with events on opposite sides of the CC exhibiting completely different behaviours;
2. the saturation regime, where microlenses very close to the macro-CC disrupt the smooth macro-critical curve structure, erasing the parity distinction;
3. the more-is-less effect, where adding additional microlenses can actually suppress the highest-magnification events; and
4. the magnification wall, where the PDFs converge to a universal shape where the probability of magnification becomes independent of its proximity to the macro-CC.

All these results were combined into an analytical tool (`M_SMiLe`) capable of generating

accurate PDFs rapidly, bypassing the need for costly simulations, a tool that has now become state-of-the-art and is widely used in microlensing studies (Palencia, J. M. et al. 2025; Li et al. 2025; Williams et al. 2025; Broadhurst et al. 2025; Diego et al. 2024).

Building on this foundation, our second paper (Palencia, J. M. et al. 2025), together with two companion works (Li et al. 2025; Williams et al. 2025), validates this framework by applying it to real data. We predicted, independently, the expected number of microlensing events in two strongly lensed arcs, achieving good agreement with the observed rates. While the companion papers focused on constraining the IMF of high-redshift stars, we concentrated on deriving the spatial distribution of events as a function of the macro-model parameters, obtaining for the first time a probability map for microlensing events across strongly lensed arcs. We also explored the contribution of compact DM objects. While their impact is negligible near bright cluster members at low redshift, where stellar microlenses dominate, it becomes significant for higher-redshift arcs, where baryonic microlenses are less abundant. We also assessed the role of sub-haloes: while the total number of events remains unchanged within current uncertainties, the spatial distribution is strongly affected, with events clustering around these substructures. This offers a unique observational signature for detecting DM sub-haloes.

In our third paper (Palencia et al. 2025), we shift our focus to FDM. This ultra-light axion-like form of DM introduces wave-like density fluctuations, altering the smooth halo profiles expected from particle-like CDM (e.g., WIMPs). The scale of these fluctuations is set by the axion mass and halo mass, the latter of which can be constrained through lensing. Importantly, negative fluctuations below the baseline smooth profile create high-magnification features within radial CCs, regions where standard CDM models predict low or even de-magnification. These features are difficult to mimic with sub-haloes or compact structures typical of other CDM models, meaning that their systematic detection could point directly towards FDM, or else set strong constraints on its parameter space. Our work highlights that this effect inside radial CCs, which has been largely overlooked, and ongoing and future surveys will be well placed to exploit this signature.

In summary, the results of this thesis provide two unique pathways through which gravitational lensing continues to prove itself as a powerful tool to probe both the nature of DM and the early Universe. Follow-up studies are already building on these

results to exploit the signatures we have uncovered, with the aim of shedding new light on this otherwise “dark” component that dominates the cosmic mass budget.

Chapter 6

Conclusiones

El efecto de lente gravitacional se ha utilizado durante décadas para estudiar la abundancia y distribución de la materia oscura (DM). Desde que se consolidó en los años 70 como una herramienta clave en astrofísica y cosmología, su papel en la validación del modelo cosmológico ha sido central. Las mediciones de lente débil (WL) han proporcionado algunas de las estimaciones de masa más precisas para estructuras masivas como los cúmulos de galaxias, revelando de forma inequívoca que una fracción significativa de su masa, alrededor del 90%, está compuesta por este componente elusivo. La distribución espacial de la DM no siempre sigue a la de los bariones, especialmente en sistemas en fusión o no relajados como el cúmulo “Bullet”, que sigue siendo una de las pruebas más sólidas de su existencia. A escalas mayores, la lente gravitacional del entramado cósmico, a través de observaciones de “cosmic shear” y del CMB, ha permitido mediciones precisas de la distribución de materia en el Universo tardío. A escalas más pequeñas, la lente fuerte (SL) ha servido para estudiar las estructuras internas de las lentes y la población de subhalos, mientras que los retardos temporales en eventos transitorios (como supernovas Ia) se han empleado para estimar la tasa de expansión del Universo (H_0), aportando evidencia independiente de la energía oscura (DE) que impulsa la aceleración cósmica. Por otro lado, la microlente (ML) ha permitido explorar las escalas de masa más pequeñas, revelando la presencia de lentes de masa estelar o subestelar, útiles para indagar en la posible naturaleza compacta de la DM. Actualmente, el efecto de lente gravitacional está entrando en una era de alta precisión, con instrumentos espaciales como JWST y los futuros proyectos de cuarta generación como Euclid o el Rubin-LSST, que inaugurarán una era de grandes datos

sin precedentes.

Más allá de la cosmología, el efecto de lente gravitacional ofrece una ventana única para estudiar objetos a alto corrimiento al rojo, haciendo accesibles fenómenos astrofísicos que de otra forma permanecerían ocultos. Con esta perspectiva, los objetivos de esta tesis han sido claros: avanzar en la comprensión teórica de los observables de lente gravitacional para realizar análisis robustos tanto de las propiedades de fuentes a gran distancia como de la naturaleza de la DM.

Al inicio de esta tesis, los eventos extremadamente magnificados eran considerados raros; las primeras detecciones de estrellas individuales a distancias cosmológicas se interpretaban como hallazgos fortuitos. Sin embargo, en poco tiempo este panorama cambió drásticamente, con decenas de nuevos eventos descubiertos. Estas detecciones han permitido caracterizar poblaciones estelares en alto corrimiento al rojo y compararlas con las poblaciones locales. Ante la perspectiva de detectar muchos más eventos, junto a fenómenos astrofísicos aún inexplorados, surgió un reto importante: extraer información estadísticamente significativa tanto de las microlentes como de las fuentes requiere entender la probabilidad de alcanzar factores de magnificación tan extremos y cómo estos dependen del entorno del lente. Para ello son necesarias simulaciones de microlente a gran escala que, aunque han evolucionado con nuevos algoritmos durante el desarrollo de esta tesis, siguen siendo extremadamente costosas en tiempo y recursos computacionales.

Para solventar este problema, el primer trabajo ([Palencia et al. 2024](#)) se centró en caracterizar las funciones de distribución de probabilidad (PDFs) de magnificación, su dependencia con los parámetros del lente, y el impacto de perturbaciones a pequeña escala, como milli-lentes (subhalos) u objetos compactos de DM. Identificamos varios efectos clave:

1. una fuerte dependencia con la paridad del macro-lente, con comportamientos radicalmente distintos a cada lado de la curva crítica (CC);
2. el régimen de saturación, en el cual microlentes muy cercanas a la CC principal destruyen su estructura suave y borran la distinción de paridad;
3. el efecto “más es menos”, donde añadir más microlentes puede reducir la frecuencia de eventos de máxima magnificación; y

4. la llamada “pared de magnificación”, donde las PDFs convergen hacia una forma universal donde la probabilidad de magnificación se vuelve independiente de su proximidad a la CC macro.

Todos estos resultados se integraron en una herramienta analítica (M_SMiLe) capaz de generar PDFs precisas de forma rápida, eliminando la necesidad de simulaciones costosas. Esta herramienta se ha convertido en un referente actual para el análisis de microlentes ([Palencia, J. M. et al. 2025](#); [Li et al. 2025](#); [Williams et al. 2025](#); [Broadhurst et al. 2025](#); [Diego et al. 2024](#)).

Basándonos en este trabajo, el segundo artículo ([Palencia, J. M. et al. 2025](#)), junto con dos estudios complementarios ([Li et al. 2025](#); [Williams et al. 2025](#)), validó este marco aplicándolo a datos reales. Predijimos, de forma independiente, el número esperado de eventos de microlente en dos arcos fuertemente magnificados, obteniendo una excelente concordancia con las tasas observadas. Mientras los trabajos complementarios se centraron en restringir la función inicial de masa (IMF) de estrellas a alto redshift, nuestra contribución principal fue derivar, por primera vez, un mapa de probabilidad espacial de eventos de microlente en arcos fuertemente magnificados. También exploramos la contribución de objetos compactos de DM. Si bien su influencia es mínima cerca de miembros brillantes de cúmulos a bajo redshift, donde dominan las microlentes estelares, se vuelve más relevante en arcos a mayor redshift, donde las microlentes bariónicas son menos abundantes. Analizamos además el efecto de los subhalos, descubriendo que aunque el número total de eventos no cambia de forma significativa, su distribución espacial sí lo hace, mostrando un patrón de agrupamiento en torno a estas estructuras. Este efecto ofrece una forma observacional única para detectar subhalos de DM.

En el tercer trabajo ([Palencia et al. 2025](#)), centramos la atención en la DM difusa (FDM). Este modelo de axiones ultraligeros introduce fluctuaciones ondulatorias en la densidad, alterando los perfiles suaves de halo que se esperan en modelos de DM de tipo partícula como CDM. La escala de estas fluctuaciones depende de la masa del axión y de la masa del halo, siendo este último parámetro accesible mediante estudios de lente gravitacional. Un aspecto clave es que las fluctuaciones negativas respecto al perfil de referencia generan regiones de alta magnificación dentro de las CC radiales, donde los modelos estándar de CDM predicen baja magnificación o incluso desmagnificación. Estos efectos son difíciles de replicar con subhalos o estructuras compactas típicas de

otros modelos de DM, lo que convierte su detección sistemática en una posible evidencia de FDM o, en su defecto, en una vía para restringir su espacio de parámetros. Nuestro estudio subraya que este efecto dentro de las CC radiales, que ha sido en gran medida ignorado, combinado con las campañas actuales y futuras podrán explotarlo de forma eficaz.

En conclusión, los resultados de esta tesis ofrecen dos vías únicas mediante las cuales el efecto de lente gravitacional se reafirma como una herramienta excepcional para explorar tanto la naturaleza de la DM como el Universo joven. Estudios de continuación ya están ampliando estas ideas para aprovechar las señales descubiertas, con el objetivo de arrojar nueva luz sobre este componente “oscuro” que domina el presupuesto de masa cósmica.

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