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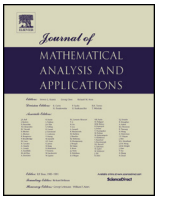

Regular Articles

Approximation schemes for path integration on Riemannian manifolds

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ABSTRACT

Truth is much too complicated to allow anything but approximations.
[John von Neumann]

In this paper, we prove a finite dimensional approximation scheme for the Wiener measure on closed Riemannian manifolds, establishing a generalization for L^1-functionals, of the approach followed by Andersson and Driver on [1]. We follow a new approach motived by the categorical concept of colimit.

1. Introduction

In 1920, N. Wiener, based on Daniell's interpretation of integral [7–9], defined in [25] an integral for bounded and continuous functionals $F : \mathcal{C}_{\mathbf{x}_0}[0,1] \to \mathbb{R}$, where the notation $\mathcal{C}_{\mathbf{x}_0}[0,1]$ stands for the space of continuous functions $\mathbf{u} : [0,1] \to \mathbb{R}$ with base point $\mathbf{u}(0) = \mathbf{x}_0$. In later papers [25–30], he connected this notion to that of Brownian motion and he defined the so-called Wiener process. In posterior works, he generalized these results defining a probability measure $\mu_{\mathbf{x}_0}$ on the measurable space $(\mathcal{C}_{\mathbf{x}_0}[0,1], \mathcal{B}_{\mathbf{x}_0})$, where $\mathcal{B}_{\mathbf{x}_0}$ stands for the Borel σ-algebra of $\mathcal{C}_{\mathbf{x}_0}[0,1]$ endowed with the uniform convergence topology. This measure is characterized by the following property: For each finite partition $\mathcal{T} = \{t_1 < t_2 < \cdots < t_n\} \subset (0,1]$ and each family $(B_t)_{t\in\mathcal{T}}$ of Borel subsets of $\mathbb{R}$, the identity

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$$\mu_{\mathbf{x}_0}\left(\pi_{\mathcal{T}}^{-1}(B_t)_{t\in\mathcal{T}}\right) = \int_{B_{t_1}} \overset{(n)}{\cdots} \int_{B_{t_n}} \prod_{i=1}^{n} p_{t_i-t_{i-1}}(x_i, x_{i-1}) \prod_{i=1}^{n} dx_i, \quad t_0 = 0,\ x_0 = \mathbf{x}_0, \tag{1}$$

holds, where $p_t(x,y)$ is the heat kernel of $\mathbb{R}$ and $\pi_{\mathcal{T}} : \mathcal{C}_{\mathbf{x}_0}[0,1] \to \mathbb{R}^{\mathcal{T}}$ is the projector defined by

$$\pi_{\mathcal{T}} : \mathcal{C}_{\mathbf{x}_0}[0,1] \longrightarrow \mathbb{R}^{\mathcal{T}}, \quad \pi_{\mathcal{T}}(\mathbf{u}) := (\mathbf{u}(t))_{t\in\mathcal{T}}.$$

At first, it seems there is no easy way to compute the integral of an arbitrary measurable functional $F : \mathcal{C}_{\mathbf{x}_0}[0,1] \to \mathbb{R}$. Nevertheless, Wiener proved in [25] an analogue of Jessen's formula [15,23] for the measure $\mu_{\mathbf{x}_0}$. More explicitly, he proved that given a bounded and continuous functional on $\mathcal{C}_{\mathbf{x}_0}[0,1]$ and a partition $\mathcal{T} = \{\mathcal{T}_n\}_{n\in\mathbb{N}}$, $\mathcal{T}_n = \{t_i^n\}_{i=1}^n$ of $[0,1]$ with mesh zero,

$$\lim_{n\to\infty} \max_{2\le i\le n} |t_i^n - t_{i-1}^n| = 0,$$

the integral of F can be computed though finite dimensional integrals via

$$\int_{\mathcal{C}_{\mathbf{x}_0}[0,1]} F(\mathbf{u})\ d\mu_{\mathbf{x}_0}(\mathbf{u}) = \lim_{n\to\infty} \int_{\mathbb{R}^n} F_n(x_1, x_2, ..., x_n) \prod_{i=1}^{n} p_{t_i^n - t_{i-1}^n}(x_i, x_{i-1}) \prod_{i=1}^{n} dx_i,$$

where the functions $F_n : \mathbb{R}^n \to \mathbb{R}$ are defined by

$$F_n(x_1, x_2, ..., x_n) := F(\mathbf{u}_{(x_1,x_2,...)}),$$

where $\mathbf{u}_{(x_1,x_2,...)}$ denotes the linear interpolation of the points $x_1, x_2, ..., x_n$, for each $n \in \mathbb{N}$. In [23], the author generalized this formula to every L^1-functional proving that for each $F \in L^1(\mathcal{C}_{\mathbf{x}_0}[0,1], \mu_{\mathbf{x}_0})$, there exists a finite dimensional functional sequence $(F_n)_{n\in\mathbb{N}} \in \bigtimes_{n\in\mathbb{N}} L^1(\mathbb{R}^n, \mu_{\mathbf{x}_0}^n)$ such that

$$\int_{\mathcal{C}_{\mathbf{x}_0}[0,1]} F\ d\mu_{\mathbf{x}_0} = \lim_{n\to\infty} \int_{\mathbb{R}^n} F_n\ d\mu_{\mathbf{x}_0}^n \quad \text{where} \quad d\mu_{\mathbf{x}_0}^n = \prod_{i=1}^{n} p_{t_i^n - t_{i-1}^n}(x_i, x_{i-1}) \prod_{i=1}^{n} dx_i. \tag{2}$$

A similar discussion can be done for the category of Riemannian Manifolds. Given a compact connected Riemannian manifold (M,g) (closed Riemannian manifold for short) of dimension m, we can construct, analogously to the case of $[0,1]$, the measure space $(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$. Here, the notation $\mathcal{C}_{\mathbf{x}_0}(M)$ denotes the space of continuous curves $\gamma : [0,1] \to M$ beginning at $\mathbf{x}_0$ and $\mu_{\mathbf{x}_0}$ the Wiener measure on $\mathcal{C}_{\mathbf{x}_0}(M)$, i.e., a measure satisfying an analogue of equation (1) for this setting (see section 2 for further details). Similar versions of Jessen type formula have been developed for the category of Riemannian manifolds in [1]. In that article, Andersson and Driver proved that given a bounded and continuous functional $F : \mathcal{C}_{\mathbf{x}_0}(M) \to \mathbb{R}$, the identity

$$\int_{\mathcal{C}_{\mathbf{x}_0}(M)} F\ d\mu_{\mathbf{x}_0} = \lim_{n\to\infty} \int_{H_{\mathcal{T}_n}(M)} F(\sigma)\ d\nu_{\mathcal{T}_n}(\sigma) \tag{3}$$

holds, where $(H_{\mathcal{T}_n}(M), \nu_{\mathcal{T}_n})$ is a finite dimensional measure space based on the geometrical data of (M,g) and $\mathcal{T} = \{\mathcal{T}_n\}_{n\in\mathbb{N}}, \mathcal{T}_n = \{t_i^n\}_{i=1}^n$, is a partition of $[0,1]$ with mesh zero. Roughly, $H_{\mathcal{T}_n}(M)$ is the space of piecewise geodesics paths in M, $\sigma : [0,1] \to M$, which change direction only at the partition points $\mathcal{T}_n = \{t_i^n\}_{i=1}^n$. The precise definition of the pair $(H_{\mathcal{T}_n}(M), \nu_{\mathcal{T}_n})$ will be given in section 6.

After Andersson and Driver, several developments and generalizations have been done by several authors, for instance this scheme has been generalized to heat kernels on vector bundles in [2,3] and further developed

in [16–19]. This type of finite dimensional approximation of path integrals has a great impact in theoretical physics and in particular are extremely useful for the Feynman path integral approach to quantum field theory [10,11].

The aim of this article is to establish a generalization of equation (3) for every integrable functional $F : \mathcal{C}_{\mathbf{x}_0}(M) \to \mathbb{R}$, not necessarily bounded and continuous in the vein of the analogous result (2) for the classical Wiener measure proved in [23]. To obtain this result, we use a categorical point of view. We prove that for $1 \leq p < \infty$, the space $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ is the colimit of certain diagram consisting on L^p-spaces of finite dimensional data associated with M.

This generalization and structural result has many applications in stochastic analysis. For instance, this scheme allows to embed the notion of the Stratonovich stochastic integral

$$\int_0^1 f(X_t) \circ dX_t \in L^2(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}),$$

in our approximation techniques, where X denotes the Wiener process on $(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$. Indeed, since the Stratonovich stochastic integral is not, in general, a continuous and bounded functional, we cannot apply the approximation (3) directly, but we can apply the techniques of this article.

The paper is organized as follows. In section two, we present the abstract results about colimits that will be used throughout this article and we recall the definition and construction of the Wiener measure on closed Riemannan manifolds. In section three, we prove that for $1 \leq p < \infty$, the space $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ is the colimit of a diagram consisting on L^p-spaces of Cartesian products of a finite number of M-factors. In section four, we derive a finite dimensional approximation formula of type (2) for integrable functionals $F \in L^1(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, via a particular realization of the colimit of the diagram defined in the preceding section. In section five, we apply our approximation scheme to Stratonovich stochastic integrals. The end of this article, section six, is devoted to recall Andersson and Driver's scheme and to adapt the results developed in sections three and four to their framework. In particular, we obtain a generalization of (3) for every integrable functional.

It is convenient to remark that all the results obtained in this article can be easily adapted to the category of Riemannian manifolds with boundary and to the case of continuous paths with fixed initial and end point, the so-called Pinned Wiener spaces. See for instance [4] for the definition and construction of the Wiener measure on these spaces.

2. Preliminaries

In this short section, we will recall some basic facts about colimits and the Wiener measure on Riemannian manifolds that will be used through this article. It will be also useful to fix notation.

2.1. Basic notions regarding colimits

Let $\mathfrak{C}$ be a fixed category and I a directed set. A diagram is a functor $F : I \to \mathfrak{C}$. It can be represented as (X_i, φ_{ij}) for a family of objects indexed by I, $\{X_i : i \in I\}$, and for each $i \leq j$ a morphism $\varphi_{ij} : X_i \to X_j$ such that φ_{ii} is the identity on X_i and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for each $i \leq j \leq k$. In general, we can consider diagrams as functors $F : \mathfrak{J} \to \mathfrak{C}$ indexed by a general category $\mathfrak{J}$, however, diagrams indexed by directed sets are enough for our purposes. A cocone is a pair (X, ϕ_i) where X is an object of $\mathfrak{C}$ and $\phi_i : X_i \to X$ is a morphism of $\mathfrak{C}$ such that $\phi_i = \phi_j \circ \varphi_{ij}$ for every $i \leq j$. The class of cocones of a given diagram forms itself a category. A colimit of the diagram (X_i, φ_{ij}) is defined to be a cocone (L, ψ_i) characterized by the following universal property: For any cocone (X, ϕ_i), there exists a unique morphism $\varphi_X : L \to X$ making the diagram of Fig. 1 commutative.

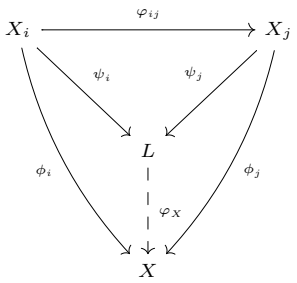

Fig. 1. Diagram of the Universal Property.

The colimit of a given diagram can be also characterized as the initial object in the category of cocones. The colimit of a given diagram does not necessarily exist, however if the colimit exists, it is unique up to a unique isomorphism in the category of cocones. Any given isomorphism class of the colimit is called a *realization*. The colimit of a given diagram (X_i, φ_{ij}) is commonly denoted by $L \equiv \varinjlim X_i$.

In this article, we will consider the category $\mathfrak{Ban}$ whose objects are Banach spaces and whose morphisms are linear isometries. The proof of the existence of the colimit for every diagram on $\mathfrak{Ban}$ can be found in [24, App. L] or in [5] and references therein.

Though this article, we will make use of the following simple Lemma.

Lemma 2.1. *Let (X_i, φ_{ij}) be a diagram on $\mathfrak{Ban}$. Then a cocone (X, ϕ_i) is a realization of the colimit of (X_i, φ_{ij}) if and only if $\bigcup_{i\in I} \phi_i(X_i)$ is dense in X.*

Proof. Suppose that (X, ϕ_i) defines a realization of the colimit of (X_i, φ_{ij}). Consider the cocone (Y, ρ_i) where

$$Y := \overline{\bigcup_{i\in I} \phi_i(X_i)} \subset X,$$

and where $\rho_i : X_i \to Y$ are the canonical inclusions. By the universal property there exists a unique isometry $\varphi_Y : X \to Y$ making the diagram of Fig. 1 commutative. It is straightforward to prove that the morphism φ_Y is in fact an isomorphism, hence $\bigcup_{i\in I} \phi_i(X_i)$ is dense in X. On the other hand, take a cocone (X, ϕ_i) with $\bigcup_{i\in I} \phi_i(X_i)$ dense in X. For any other cocone (Y, ρ_i), define the morphism $\varphi_Y : X \to Y$ by

$$\varphi_Y \circ \phi_i := \rho_i, \quad i \in I.$$

Then it is easy to see that the morphism φ_Y is the unique morphism making the diagram of Fig. 1 commutative. Hence (X, ϕ_i) is a colimit. This concludes the proof. □

2.2. Wiener measure on Riemannian manifolds

Along this article, (M, g) denotes a compact connected Riemannian manifold (closed Riemannian manifold for short) with a fixed base point $\mathbf{x}_0 \in M$. The notation $\mathcal{C}_{\mathbf{x}_0}(M)$ stands for the space of continuous paths $\gamma \in \mathcal{C}([0,1], M)$ satisfying $\gamma(0) = \mathbf{x}_0$, $\mathcal{B}_{\mathbf{x}_0}$ for the Borel σ-algebra of $\mathcal{C}_{\mathbf{x}_0}(M)$ with respect to the uniform convergence topology given by the induced metric of M and $\mu_{\mathbf{x}_0}$ for the Wiener measure on M with base point $\mathbf{x}_0$.

We recall the definition of the measure space $(\mathcal{C}_{\mathbf{x}_0}(M), \mathcal{B}_{\mathbf{x}_0}, \mu_{\mathbf{x}_0})$. Consider in (M, g) the measure $\mu : \mathcal{B}_M \to [0, +\infty]$ induced by the metric g, where $\mathcal{B}_M$ denotes the Borel σ-algebra of M. This measure is locally given by the expression

$$d\mu = \sqrt{\det(g_{ij})_{ij}}\; dx_1 \wedge \cdots \wedge dx_m$$

where m is the dimension of M and $(g_{ij})_{ij}$ is the matrix of g in a local chart. For each closed Riemannian manifold (M, g), there exists a heat kernel $p_t(x, y)$, for $t > 0$, $x, y \in M$, i.e., the Schwartz kernel of the selfadjoint operator $e^{t\Delta}$ on $L^2(M, \mu)$, where Δ denotes the Laplace-Beltrami operator on (M, g). The proof of the existence of this map can be found in [4,13]. A well-known consequence of the Kolmogorov extension Theorem [31, Th. 6.1], is the existence of a probability measure

$$\mu_{\mathbf{x}_0} : \mathcal{B} \longrightarrow [0, +\infty]$$

on $(M^{[0,1]}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra of $M^{[0,1]}$ with respect to the product topology, satisfying the identity

$$\mu_{\mathbf{x}_0}\left(\pi_{\mathcal{T}}^{-1}(B_t)_{t\in\mathcal{T}\setminus\{0\}}\right) = \int_{B_{t_1}} \overset{(n)}{\cdots} \int_{B_{t_n}} \prod_{j=1}^{n} p_{t_j - t_{j-1}}(x_j, x_{j-1}) \prod_{j=1}^{n} d\mu(x_j), \quad x_0 = \mathbf{x}_0, \tag{4}$$

for each finite partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n\} \subset [0, 1]$ and each family of Borel subsets $(B_t)_{t\in\mathcal{T}\setminus\{0\}} \subset \mathcal{B}_M$. Here and in the sequel, the notation $\pi_{\mathcal{T}}$ stands for the projector defined by

$$\pi_{\mathcal{T}} : M^{[0,1]} \longrightarrow M^{\mathcal{T}\setminus\{0\}}, \quad \pi_{\mathcal{T}}(\gamma_t)_{t\in[0,1]} := (\gamma_t)_{t\in\mathcal{T}\setminus\{0\}}. \tag{5}$$

Since (M, g) is compact, it is, in particular, stochastically complete (see for instance [13]), and therefore

$$\int_M p_t(x, y) d\mu(y) = 1$$

for each $t > 0$ and $x \in M$. This fact implies that the measure $\mu_{\mathbf{x}_0}$ is of probability, that is, $\mu_{\mathbf{x}_0}(M^{[0,1]}) = 1$.

On the other hand, by [4, Cr. 2.19], the measure $\mu_{\mathbf{x}_0}$ satisfies the identity $\mu_{\mathbf{x}_0}(\mathcal{C}^{\alpha}_{\mathbf{x}_0}(M)) = 1$ for each $\alpha \in (0, 1/2)$ where $\mathcal{C}^{\alpha}_{\mathbf{x}_0}(M)$ stands for the subset of $M^{[0,1]}$ consisting of Hölder continuous paths $\gamma : [0, 1] \to M$ of exponent $\alpha \in (0, 1)$ satisfying $\gamma(0) = \mathbf{x}_0$. Therefore, since $\mathcal{C}_{\mathbf{x}_0}(M)$ is a Borel subset of $M^{[0,1]}$ (see [12, Th. 10.28]) containing $\mathcal{C}^{\alpha}_{\mathbf{x}_0}(M)$ and since

$$\mathcal{B} \cap \mathcal{C}_{\mathbf{x}_0}(M) := \{B \cap \mathcal{C}_{\mathbf{x}_0}(M) : B \in \mathcal{B}\},$$

coincides with the Borel σ-algebra $\mathcal{B}_{\mathbf{x}_0}$ (see [20, Prop. 2.2]), we can consider the restricted probability space $(\mathcal{C}_{\mathbf{x}_0}(M), \mathcal{B}_{\mathbf{x}_0}, \mu_{\mathbf{x}_0})$. The restricted measure $\mu_{\mathbf{x}_0}$ is called the Wiener measure of M with base point $\mathbf{x}_0$. The proof of these facts can be found for instance in [4,13] and references therein.

3. Approximation scheme for the Wiener measure

In this section, we will prove that the spaces $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, $1 \leq p < \infty$, are realizations of the colimit of the diagram $(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}), \pi^*_{\mathcal{T}\mathcal{T}'})$. Let us start by defining the diagram $(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}), \pi^*_{\mathcal{T}\mathcal{T}'})$. Consider the directed set $\mathcal{P}$ consisting on partitions of $[0, 1]$

$$\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n\},$$

partially ordered by inclusion. For notational simplicity, we will use the notation $M^{\mathcal{T}}$ to denote

$$M^{\mathcal{T}} := \bigtimes_{t\in\mathcal{T}\setminus\{0\}} M.$$

For a partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n\}$ in $\mathcal{P}$, consider the probability space $(M^{\mathcal{T}}, \bigotimes_{i=1}^n \mathcal{B}_M, \mu_{\mathbf{x}_0}^{\mathcal{T}})$ where $\mu_{\mathbf{x}_0}^{\mathcal{T}}$ is the measure defined by

$$d\mu_{\mathbf{x}_0}^{\mathcal{T}} = \prod_{i=1}^n p_{t_i - t_{i-1}}(x_{t_i}, x_{t_{i-1}}) \prod_{i=1}^n d\mu(x_{t_i}), \quad x_0 = \mathbf{x}_0,$$

and $\bigotimes_{i=1}^n \mathcal{B}_M$ denotes the product σ-algebra. For each pair of partitions $\mathcal{T}, \mathcal{T}'$ with $\mathcal{T} \subset \mathcal{T}'$, we obtain a measurable projection map

$$\pi_{\mathcal{T}\mathcal{T}'} : M^{\mathcal{T}'} \longrightarrow M^{\mathcal{T}}, \quad \pi_{\mathcal{T}\mathcal{T}'}(x_t)_{t\in\mathcal{T}'\setminus\{0\}} := (x_t)_{t\in\mathcal{T}\setminus\{0\}}.$$

We therefore obtain a diagram in the category of measure spaces, indexed by the directed set $\mathcal{P}$. For each pair of partitions $\mathcal{T}, \mathcal{T}'$ with $\mathcal{T} \subset \mathcal{T}'$, we can consider the pullback operator

$$\pi_{\mathcal{T}\mathcal{T}'}^* : L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}) \longrightarrow L^p(M^{\mathcal{T}'}, \mu_{\mathbf{x}_0}^{\mathcal{T}'}), \quad \pi_{\mathcal{T}\mathcal{T}'}^*(f) := f \circ \pi_{\mathcal{T}\mathcal{T}'}.$$

Since each $\pi_{\mathcal{T}\mathcal{T}'}$ is measure-preserving, the pullback $\pi_{\mathcal{T}\mathcal{T}'}^*$ is an isometry for each $\mathcal{T} \subset \mathcal{T}'$. By functoriality of the pullback, we obtain a diagram in $\mathfrak{Ban}$ indexed by the directed set $\mathcal{P}$, where all arrow directions are now swapped, as the pullback is contravariant. This diagram will be subsequently denoted by $(L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}), \pi_{\mathcal{T}\mathcal{T}'}^*)$.

It will be proved that the cocone $(L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \phi_{\mathcal{T}})$, where $\phi_{\mathcal{T}}$ are the morphisms defined by

$$\phi_{\mathcal{T}} : L^p(M^{\mathcal{T}}, \nu_{\mathbf{x}_0}^{\mathcal{T}}) \longrightarrow L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \quad \phi_{\mathcal{T}}(f) := f \circ \pi_{\mathcal{T}}|_{\mathcal{C}_{\mathbf{x}_0}(M)},$$

defines a realization of the colimit of $(L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}), \pi_{\mathcal{T}\mathcal{T}'}^*)$. The following Lemma will be useful for this purpose.

Lemma 3.1. *Let $1 \leq p < \infty$, then the subspace $\bigcup_{\mathcal{T}\in\mathcal{P}} \phi_{\mathcal{T}}(L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}))$ is dense in $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$.*

Proof. As we have recalled in section 2.2, the following equality of σ-algebras

$$\mathcal{B}_{\mathbf{x}_0} = \mathcal{B} \cap \mathcal{C}_{\mathbf{x}_0}(M), \tag{6}$$

holds. By definition, the product σ-algebra $\mathcal{B}$ is generated by the cylinder sets and therefore by (6), it follows that

$$\mathcal{B}_{\mathbf{x}_0} = \sigma(\mathcal{R}), \quad \mathcal{R} := \{\pi_{\mathcal{T}}^{-1}(B_t)_{t\in\mathcal{T}} : (B_t)_{t\in\mathcal{T}} \subset \mathcal{B}_M, \mathcal{T} \subset (0,1] \text{ finite}\} \cap \mathcal{C}_{\mathbf{x}_0}(M).$$

Since $(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ has finite measure, by [6, Lem. 3.4.6], the subspace $\operatorname{Span}\{\chi_R : R \in \mathcal{R}\}$, where χ_R denoted the characteristic function of R, is dense in $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$. From the inclusion

$$\{\chi_R : R \in \mathcal{R}\} \subset \bigcup_{\mathcal{T}\in\mathcal{P}} \phi_{\mathcal{T}}(L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}})),$$

it follows that $\bigcup_{\mathcal{T}\in\mathcal{P}} \phi_{\mathcal{T}}(L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}))$ is dense in $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$. This concludes the proof. □

Finally, we prove the main result of this section.

Theorem 3.2. *Let $1 \leq p < \infty$, then the cocone $(L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \phi_{\mathcal{T}})$ defines a realization of the colimit of $(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}), \pi^*_{\mathcal{T}\mathcal{T}'})$.*

Proof. By Lemma 3.1, the subspace

$$\bigcup_{\mathcal{T}\in\mathcal{P}} \phi_{\mathcal{T}}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})) \subset L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$$

is dense in $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$. Therefore, we can apply Lemma 2.1 to $(L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \phi_{\mathcal{T}})$ obtaining the result. □

4. Derivation of the limit formula

In this section, we will prove that the integral of a given integrable functional can be expressed as the limit of finite dimensional integrals. To relate the colimit structure obtained in the last section with the integration procedure, we have to provide a particular realization of the colimit of the directed system $(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}), \pi^*_{\mathcal{T}\mathcal{T}'})$. To define this realization, we need the following definition.

An element $(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \in \bigtimes_{\mathcal{T}\in\mathcal{P}} L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$ is said to be co-Cauchy if for each $\varepsilon > 0$, there exists $\mathcal{R} \in \mathcal{P}$ such that

$$\|\pi^*_{\mathcal{T}\mathcal{T}'}(f_{\mathcal{T}}) - f_{\mathcal{T}'}\|_{L^p_{\mathcal{T}}} < \varepsilon, \text{ for all } \mathcal{T}, \mathcal{T}' \in \mathcal{P} \text{ with } \mathcal{R} \subset \mathcal{T} \subset \mathcal{T}'.$$

For notational simplicity we denote $L^p \equiv L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ and $L^p_{\mathcal{T}} \equiv L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$.

We define the space $\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}))$ by

$$\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})) := \left\{(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \in \bigtimes_{\mathcal{T}\in\mathcal{P}} L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}) : (f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \text{ is co-Cauchy}\right\} \Big/ \sim, \tag{7}$$

where we relate $(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \sim (g_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}$ if

$$\lim_{\mathcal{T}} \|f_{\mathcal{T}} - g_{\mathcal{T}}\|_{L^p_{\mathcal{T}}} = 0.$$

We define in $\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}))$ the norm

$$\|(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}\|_{\mathfrak{L}} := \lim_{\mathcal{T}} \|f_{\mathcal{T}}\|_{L^p_{\mathcal{T}}}.$$

All the limits involved are considered as limits of nets. For further considerations, it is imperative to obtain the completeness of the space $\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}))$, which is not obvious from the definition. For this reason, we identify it directly with $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ via the following result.

Theorem 4.1. *The following spaces are isometrically isomorphic for $1 \leq p < \infty$*

$$\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})) \simeq L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}).$$

Proof. Let us prove that the map

$$\mathfrak{I}_p : \mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})) \longrightarrow L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \quad (f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \mapsto \lim_{\mathcal{T}} \phi_{\mathcal{T}}(f_{\mathcal{T}})$$

defines an isometric isomorphism. The map $\mathfrak{I}_p$ is well defined since the net $(\phi_{\mathcal{T}}(f_{\mathcal{T}}))_{\mathcal{T}\in\mathcal{P}}$ converges in $L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0})$, a fact that follows from the identity

$$\begin{aligned}\|\phi_{\mathcal{T}}(f_{\mathcal{T}})-\phi_{\mathcal{T}'}(f_{\mathcal{T}'})\|_{L^p} &= \|(\phi_{\mathcal{T}'}\circ\pi^*_{\mathcal{T}\mathcal{T}'})(f_{\mathcal{T}})-\phi_{\mathcal{T}'}(f_{\mathcal{T}'})\|_{L^p}\\ &= \|\pi^*_{\mathcal{T}\mathcal{T}'}(f_{\mathcal{T}})-f_{\mathcal{T}'}\|_{L^p_{\mathcal{T}'}},\quad \mathcal{T}\subset\mathcal{T}'\end{aligned}$$

and the co-Cauchy property. It is easily seen that $\mathfrak{I}_p$ is an isometry since

$$\|\mathfrak{I}_p(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}\|_{L^p}=\|\lim_{\mathcal{T}}\phi_{\mathcal{T}}(f_{\mathcal{T}})\|_{L^p}=\lim_{\mathcal{T}}\|f_{\mathcal{T}}\|_{L^p_{\mathcal{T}}}=\|(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}\|_{\mathfrak{L}}.$$

Finally, we prove that $\mathfrak{I}_p$ is onto. Let $f\in L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0})$ and take $\mathcal{D}=\{t_i\}_{i\in\mathbb{N}\cup\{0\}}$, $t_0=0$, a dense countable subset of $[0,1]$. Define the sequence of partitions $\mathcal{P}=\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ by

$$\mathcal{P}_n:=\{t_i\}_{i=0}^n,\quad n\in\mathbb{N}.$$

Clearly $\mathcal{P}_n\subset\mathcal{P}_{n+1}$, $n\in\mathbb{N}$ and $\bigcup_{n\in\mathbb{N}}\mathcal{P}_n=\mathcal{D}$. It is easily seen using the same techniques of the proof of Lemma 3.1, that $\bigcup_{n\in\mathbb{N}}\phi_{\mathcal{P}_n}(L^p(M^{\mathcal{P}_n},\mu_{\mathbf{x}_0}^{\mathcal{P}_n}))$ is dense in $L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0})$. Hence there exists a sequence

$$(f_n)_{n\in\mathbb{N}},\quad f_n\in L^p(M^{\mathcal{P}_{N_n}},\mu_{\mathbf{x}_0}^{\mathcal{P}_{N_n}})\text{ for some }N_n\in\mathbb{N},$$

such that

$$\lim_{n\to\infty}\phi_{\mathcal{P}_{N_n}}(f_n)=f\text{ in }L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0}).$$

Take a strictly increasing sequence $(M_n)_{n\in\mathbb{N}}\subset\mathbb{N}$ such that $N_n\le M_n, n\in\mathbb{N}$. Then, in particular, $\mathcal{P}_{N_n}\subset\mathcal{P}_{M_n}$ and

$$g_n:=\pi^*_{\mathcal{P}_{N_n}\mathcal{P}_{M_n}}(f_n)\in L^p(M^{\mathcal{P}_{M_n}},\mu_{\mathbf{x}_0}^{\mathcal{P}_{M_n}}),\quad n\in\mathbb{N}.$$

Define the sequence

$$(F_n)_{n\in\mathbb{N}},\quad F_n:=\begin{cases}\pi^*_{\mathcal{P}_{M_m}\mathcal{P}_n}(g_n) & \text{if } M_m\le n<M_{m+1}\\ 0 & \text{if } n<M_1\end{cases}$$

then

$$(F_n)_{n\in\mathbb{N}}\in\bigtimes_{n\in\mathbb{N}}L^p(M^{\mathcal{P}_n},\mu_{\mathbf{x}_0}^{\mathcal{P}_n})\text{ and }\lim_{n\to\infty}\phi_{\mathcal{P}_n}(F_n)=f\text{ in }L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0}).$$

Define the associated net $(F_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}$ by

$$F_{\mathcal{T}}:=\begin{cases}\pi^*_{\mathcal{P}_n\mathcal{T}}(F_n) & \text{if } \mathcal{P}_n\subset\mathcal{T}\text{ but }\mathcal{P}_{n+1}\not\subset\mathcal{T}\\ 0 & \text{else}\end{cases}$$

Finally, we have that $(F_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}\in\bigtimes_{\mathcal{T}\in\mathcal{P}}L^p(M^{\mathcal{T}},\mu_{\mathbf{x}_0}^{\mathcal{T}})$ and $\lim_{\mathcal{T}}\phi_{\mathcal{T}}(F_{\mathcal{T}})=f$, which implies that $\mathfrak{I}_p(F_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}=f$. This concludes the proof. □

It is appropriate to note the following in accordance with the proof of Theorem 4.1. One is tempted to think that in order to prove the surjectivity of $\mathfrak{I}_p$, for each $f\in L^p(\mathcal{C}_{\mathbf{x}_0},\mu_{\mathbf{x}_0})$, it is sufficient to use Lemma 2.1,

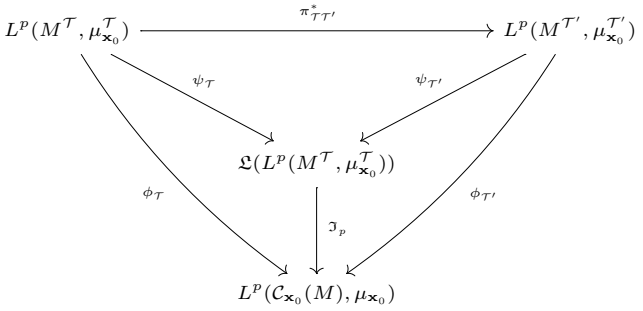

Fig. 2. Diagram of the Universal Property II.

choosing a sequence $(f_{\mathcal{T}})_{\mathcal{T}} \in \bigcup_{\mathcal{T}\in\mathcal{P}} \phi_{\mathcal{T}}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}))$ such that $\lim_{\mathcal{T}} \phi_{\mathcal{T}}(f_{\mathcal{T}}) = f$. Nonetheless, it must be observed that the sequence $(f_{\mathcal{T}})_{\mathcal{T}}$ is not necessarily contained in the Cartesian product $\bigtimes_{\mathcal{T}\in\mathcal{P}} L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$ which makes things a bit more involved.

We define the morphisms $\psi_{\mathcal{R}} : L^p(M^{\mathcal{R}}, \mu^{\mathcal{R}}_{\mathbf{x}_0}) \to \mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}))$, $\mathcal{R} \in \mathcal{P}$, through

$$\psi_{\mathcal{R}}(f_{\mathcal{R}}) = (g_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}, \quad g_{\mathcal{T}} := \begin{cases} \pi^*_{\mathcal{R}\mathcal{T}}(f_{\mathcal{R}}) & \text{if } \mathcal{R} \subset \mathcal{T} \\ 0 & \text{else} \end{cases} \tag{8}$$

By Theorem 4.1, the space $\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}))$ is a Banach space and therefore the pair $(\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})), \psi_{\mathcal{R}})$ defines a cocone. The colimit of $(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}), \pi^*_{\mathcal{T}\mathcal{T}'})$ will be identified with $(\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})), \psi_{\mathcal{R}})$.

Theorem 4.2. *The cocone* $(\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})), \psi_{\mathcal{T}})$ *defines another realization of the colimit of* $(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}), \pi^*_{\mathcal{T}\mathcal{T}'})$.

Proof. Since the colimit is unique up to a unique isomorphism on the category of cocones, it is enough to prove that $(\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})), \psi_{\mathcal{T}})$ is isomorphic to $(L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \phi_{\mathcal{T}})$ in the category of cocones. By the proof of Theorem 4.1, the map

$$\mathfrak{I}_p : \mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})) \longrightarrow L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}), \quad (f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \mapsto \lim_{\mathcal{T}} \phi_{\mathcal{T}}(f_{\mathcal{T}})$$

is an isometric isomorphism. To prove that it defines an isomorphism of cocones, we need to verify the commutativity of the diagram of Fig. 2 for each $\mathcal{T} \in \mathcal{P}$. Take $f_{\mathcal{T}} \in L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$, then by a simple computation we obtain

$$(\mathfrak{I}_p \circ \psi_{\mathcal{T}})(f_{\mathcal{T}}) = \lim_{\mathcal{Q}} h_{\mathcal{Q}}, \quad h_{\mathcal{Q}} := \begin{cases} \phi_{\mathcal{T}}(f_{\mathcal{T}}) & \text{if } \mathcal{T} \subset \mathcal{Q} \\ 0 & \text{else} \end{cases}$$

and thus $(\mathfrak{I}_p \circ \psi_{\mathcal{T}})(f_{\mathcal{T}}) = \phi_{\mathcal{T}}(f_{\mathcal{T}})$. Hence the diagram of Fig. 2 commutes and the proof is concluded. □

Here in after, as a direct consequence of Theorem 4.2, we can and we shall denote $\mathfrak{L}(L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})) \equiv \varinjlim L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$.

As a rather direct application of the isometric property of $\mathfrak{I}_p$, we obtain our integral limit approximation. Indeed, if $F \in L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, there exists an element $(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \in \varinjlim L^p(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$ such that $\|F\|_{L^p} = \|(f_{\mathcal{T}})_{\mathcal{T}}\|_{\mathfrak{L}}$. Therefore, we have

$$\int_{\mathcal{C}_{\mathbf{x}_0}(M)} |F|^p \, d\mu_{\mathbf{x}_0} = \lim_{\mathcal{T}} \int_{M^{\mathcal{T}}} |f_{\mathcal{T}}|^p \, d\mu^{\mathcal{T}}_{\mathbf{x}_0}.$$

Furthermore, if we take into account that given $F \in L^1(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, we can write it as $F = F^+ - F^-$ with F^+, F^- positive and $F^+, F^- \in L^1(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, then we get the following result.

Theorem 4.3. *Let $F \in L^1(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, then there exists $(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \in \bigtimes_{\mathcal{T}\in\mathcal{P}} L^1(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0})$ such that*

$$\int_{\mathcal{C}_{\mathbf{x}_0}(M)} F \, d\mu_{\mathbf{x}_0} = \lim_{\mathcal{T}} \int_{M^{\mathcal{T}}} f_{\mathcal{T}} \, d\mu^{\mathcal{T}}_{\mathbf{x}_0}.$$

5. Stratonovich stochastic integral

In this section, we will apply the developed theory to a particular example, the Stratonovich stochastic integral. Our approach can be used to get a finite dimensional approximation scheme for this type of integrals, in contrast to the Andersson and Driver's framework [1] that only can be used if the involved functional is bounded and continuous.

Let us firstly recall briefly some basic facts about stochastic integration. Let $(X, Y) = (\{X_t\}_{t\in[0,1]}, \{Y_t\}_{t\in[0,1]})$ be a pair of bounded $\mathbb{R}$-valued semimartingales defined in the probability space $(\Omega, \mathcal{F}, \mu)$. Then, the Stratonovich integral of X with respect to Y is defined by the relation

$$\int_0^1 X_t \circ dY_t := \lim_{L^2(\mu)} \sum_{i=1}^{n} \frac{X_{t_i^n} + X_{t_{i-1}^n}}{2} (Y_{t_i^n} - Y_{t_{i-1}^n}) \in L^2(\Omega, \mu)$$

where $\mathcal{T} = \{\mathcal{T}_n\}_{n\in\mathbb{N}}$, $\mathcal{T}_n := \{t_i^n\}_{i=0}^n$, is a fixed partition with mesh zero of $[0, 1]$. It is related to the Itô stochastic integral by the relation

$$\int_0^1 X_t \circ dY_t = \int_0^1 X_t \, dY_t + [X, Y]_t$$

where $[X, Y]_t$ denotes the covariation of the processes (X, Y) and dY_t denotes the Itô differential. In the case in which $(X, Y) = (\{X_t\}_{t\in[0,1]}, \{Y_t\}_{t\in[0,1]})$ are bounded $\mathbb{R}^N$-valued semimartingales, we define

$$\int_0^1 X_t \circ dY_t := \sum_{i=1}^{N} \int_0^1 X_t^i \circ dY_t^i.$$

It is worth to mention that the usual definition of the Stratonovich integral is under convergence in probability [21, Th. 26, Ch. V]. Since we will deal with semimartingales defined on compact manifolds, we only need the definition for bounded ones, in which case, the convergence in probability implies the L^2-convergence as a consequence of Vitali's convergence Theorem.

It must be taken into account that if (M, g) is a closed Riemannian manifold embedded in the Euclidean space $\mathbb{R}^N$, by [14, Pr. 3.2.1], the M-valued stochastic process $X = \{X_t\}_{t\in[0,1]}$ defined by the coordinate functionals of $(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, defines a $\mathbb{R}^N$-valued bounded semimartingale. This implies by [14, Pr. 1.2.7, (i)] that $\{f(X_t)\}_{t\in[0,1]}$ is a real valued semimartingale for each $f \in \mathcal{C}^\infty(M)$. Therefore the Stratonovich stochastic integral of $f(X)$ with respect to X, where $f \in \mathcal{C}^\infty(M, \mathbb{R}^N)$, $f = (f_1, f_2, ..., f_N)$, is well defined.

The main result of this section is the following. It establishes which is exactly the preimage of the functional $\int_0^1 f(X_t) \circ dX_t \in L^2(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ via the identification $\mathfrak{I}_2 : \varinjlim L^2(M^{\mathcal{T}}, \mu^{\mathcal{T}}_{\mathbf{x}_0}) \to L^2(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$.

Theorem 5.1. *Let (M,g) be a closed Riemannian manifold embedded in the Euclidean space $\mathbb{R}^N$, $f \in \mathcal{C}^\infty(M,\mathbb{R}^N)$, $f=(f_1,f_2,...,f_N)$, and $\{X_t\}_{t\in[0,1]}$ the M-valued semimartingale defined by the coordinate functionals of $(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0})$. Then*

$$\mathfrak{I}_2\left(\sum_{j=1}^{N}\sum_{i=1}^{n}\frac{f_j(\mathbf{x}_{t_i})+f_j(\mathbf{x}_{t_{i-1}})}{2}(x^j_{t_i}-x^j_{t_{i-1}})\right)_{\mathcal{T}\in\mathcal{P}}=\int_0^1 f(X_t)\circ dX_t,$$

where $\mathcal{T}=\{0=t_0<t_1<\cdots<t_n\}$ and $\mathbf{x}_{t_i}=(x^1_{t_i},\cdots,x^N_{t_i})\in M\subset\mathbb{R}^N$.

Proof. Since M is compact and embedded in $\mathbb{R}^N$, the process $X=\{X_t\}_{t\in[0,1]}$ is, in particular, a bounded $\mathbb{R}^N$-valued semimartingale and its Stratonovich integral is defined in the usual manner by

$$\begin{aligned}\int_0^1 f(X_t)\circ dX_t &:= \sum_{j=1}^{N}\int_0^1 f_j(X_t)\circ dX^j_t\\ &= \lim_{L^2(\mu_{\mathbf{x}_0})}\sum_{j=1}^{N}\sum_{i=1}^{n}\frac{f_j(X_{t^n_i})+f_j(X_{t^n_{i-1}})}{2}(X^j_{t^n_i}-X^j_{t^n_{i-1}}),\end{aligned}$$

for every partition $\mathcal{Q}=\{\mathcal{Q}_n\}_{n\in\mathbb{N}}$, $\mathcal{Q}_n:=\{t^n_i\}_{i=0}^n$ with mesh zero of $[0,1]$ satisfying $t^n_0=0$ for each $n\in\mathbb{N}$. This limit can be expressed as a convergence of nets via

$$\int_0^1 f(X_t)\circ dX_t=\lim_{\mathcal{T}}\sum_{j=1}^{N}\sum_{i=1}^{n}\frac{f_j(X_{t_i})+f_j(X_{t_{i-1}})}{2}(X^j_{t_i}-X^j_{t_{i-1}}),$$

where $\mathcal{T}=\{0=t_0<t_1<\cdots<t_n\}$. The limit is understood in L^2-convergence. On the other hand, we define the maps $F^j_{\mathcal{T}}:M^{\mathcal{T}}\to\mathbb{R}$, $j\in\{1,2,...,N\}$, by

$$F^j_{\mathcal{T}}(\mathbf{x}_{t_i})^n_{i=1}:=\sum_{i=1}^{n}\frac{f_j(\mathbf{x}_{t_i})+f_j(\mathbf{x}_{t_{i-1}})}{2}(x^j_{t_i}-x^j_{t_{i-1}}),$$

where $\mathcal{T}=\{0=t_0<t_1<\cdots<t_n\}$ and $\mathbf{x}_{t_i}=(x^1_{t_i},x^2_{t_i},\cdots,x^n_{t_i})$. Then, by the boundedness of each $F^j_{\mathcal{T}}$, it follows that $F^j_{\mathcal{T}}\in L^2(M^{\mathcal{T}},\mu^{\mathcal{T}}_{\mathbf{x}_0})$ and by the definition of the morphisms $\phi_{\mathcal{T}}:L^2(M^{\mathcal{T}},\mu^{\mathcal{T}}_{\mathbf{x}_0})\to L^2(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0})$, we obtain

$$\phi_{\mathcal{T}}(F^j_{\mathcal{T}})=\sum_{i=1}^{n}\frac{f_j(X_{t_i})+f_j(X_{t_{i-1}})}{2}(X^j_{t_i}-X^j_{t_{i-1}}).$$

Therefore, if we prove that $(F^j_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}}\in\varinjlim L^2(M^{\mathcal{T}},\mu^{\mathcal{T}}_{\mathbf{x}_0})$ for each $j\in\{1,2,...,N\}$, by the definition of $\mathfrak{I}_2$ and the linearity of $\phi_{\mathcal{T}}$, we will deduce

$$\begin{aligned}\mathfrak{I}_2\left(\sum_{j=1}^{N}F^j_{\mathcal{T}}\right)_{\mathcal{T}\in\mathcal{P}} &= \lim_{\mathcal{T}}\sum_{j=1}^{N}\phi_{\mathcal{T}}(F^j_{\mathcal{T}})\\ &= \lim_{\mathcal{T}}\sum_{j=1}^{N}\sum_{i=1}^{n}\frac{f_j(X_{t_i})+f_j(X_{t_{i-1}})}{2}(X^j_{t_i}-X^j_{t_{i-1}})\end{aligned}$$

$$= \int_0^1 f(X_t) \circ dX_t,$$

obtaining the required result. Finally, we prove that, indeed, $(F_{\mathcal{T}}^j)_{\mathcal{T}\in\mathcal{P}} \in \varinjlim L^2(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}})$ for each $j \in \{1, 2, ..., N\}$. Since each $\phi_{\mathcal{T}}$ is an isometric isomorphism, we have

$$\begin{aligned} \|\pi_{\mathcal{R}\mathcal{T}}^*(F_{\mathcal{R}}^j) - F_{\mathcal{T}}^j\|_{L_{\mathcal{T}}^2} &= \|(\phi_{\mathcal{T}} \circ \pi_{\mathcal{R}\mathcal{T}}^*)(F_{\mathcal{R}}^j) - \phi_{\mathcal{T}}(F_{\mathcal{T}}^j)\|_{L^2} \\ &= \|\phi_{\mathcal{R}}(F_{\mathcal{R}}^j) - \phi_{\mathcal{T}}(F_{\mathcal{T}}^j)\|_{L^2}, \quad \mathcal{R} \subset \mathcal{T}. \end{aligned}$$

The convergence of $\{\phi_{\mathcal{T}}(F_{\mathcal{T}}^j)\}_{\mathcal{T}\in\mathcal{P}}$ (that is justified by the existence of the integral $\int_0^1 f(X_t) \circ dX_t$) together with the last identity, implies the co-Cauchy property for $(F_{\mathcal{T}}^j)_{\mathcal{T}\in\mathcal{P}}$. Hence $(F_{\mathcal{T}}^j)_{\mathcal{T}\in\mathcal{P}} \in \varinjlim L^2(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}})$ and the proof is concluded. □

As a direct consequence of the preceding result and the isometric property of $\mathfrak{I}_2$, we deduce the following finite dimensional approximation under the same hypothesis of Theorem 5.1,

$$\int_{\mathcal{C}_{\mathbf{x}_0}(M)} \left| \int_0^1 f(X_t) \circ dX_t \right|^2 d\mu_{\mathbf{x}_0} = \lim_{\mathcal{T}} \int_{M^{\mathcal{T}}} \left| \sum_{j=1}^N \sum_{i=1}^n \frac{f_j(\mathbf{x}_{t_i}) + f_j(\mathbf{x}_{t_{i-1}})}{2} (x_{t_i}^j - x_{t_{i-1}}^j) \right|^2 d\mu_{\mathbf{x}_0}^{\mathcal{T}},$$

where $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n\}$. Finally, it is convenient to remark that all the considerations made in this section can be adapted easily to the cover the Itô stochastic integration.

6. Approximation scheme for the geometric framework

In this final section, we will give an approximation scheme based in the geometric measure introduced by Andersson and Driver in [1]. Up to here, we have proved that the spaces $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, $1 \leq p < \infty$, define a realization of the colimit of the diagram $(L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}), \pi_{\mathcal{T}\mathcal{T}'}^*)$, where the measures $\mu_{\mathbf{x}_0}^{\mathcal{T}}$ are given by

$$d\mu_{\mathbf{x}_0}^{\mathcal{T}} = \prod_{i=1}^n p_{t_i - t_{i-1}}(x_{t_i}, x_{t_{i-1}}) \prod_{i=1}^n d\mu(x_{t_i}), \quad \mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n\}.$$

Moreover, we have provided a particular realization, $\varinjlim L^p(M^{\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}})$, of the colimit that allows to stablish a finite dimensional approximation result for integrable functionals.

However, observe that this approximation scheme does not follow the philosophy of Andersson and Driver approach materialized in equation (3). For this reason, in this section, we adapt our scheme to cover the measure considered by Andersson and Driven in [1]. We start recalling some facts and definitions of [1]. Through this section, the notation $\mathcal{C}_0(\mathbb{R}^m)$ stands for the space of continuous paths $\gamma : [0, 1] \to \mathbb{R}^m$ such that $\gamma(0) = 0$.

6.1. Piecewise linear path space

Consider $H(\mathbb{R}^m) \subset \mathcal{C}_0(\mathbb{R}^m)$ the subspace consisting on finite energy paths

$$H(\mathbb{R}^m) := \{\gamma \in \mathcal{C}_0(\mathbb{R}^m) : \gamma \text{ is absolutely continuous and } E_{\mathbb{R}^m}(\gamma) < \infty\},$$

where the energy functional is defined by

$$E_{\mathbb{R}^m}(\gamma) := \int_0^1 \langle \gamma'(s), \gamma'(s) \rangle \, ds.$$

For each partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, the space of piecewise linear paths on $\mathbb{R}^m$ with respect to $\mathcal{T}$, subsequently denoted by $H_{\mathcal{T}}(\mathbb{R}^m)$, is defined through

$$H_{\mathcal{T}}(\mathbb{R}^m) := \{\gamma \in \mathcal{C}_0(\mathbb{R}^m) : \gamma \text{ is linear for } t \notin \mathcal{T}\}.$$

Clearly $H_{\mathcal{T}}(\mathbb{R}^m)$ is linearly isomorphic to $\mathbb{R}^{m\times\mathcal{T}}$, where

$$\mathbb{R}^{m\times\mathcal{T}} := \bigtimes_{t\in\mathcal{T}\setminus\{0\}} \mathbb{R}^m,$$

via the linear isomorphism

$$\Pi_{\mathcal{T}} : H_{\mathcal{T}}(\mathbb{R}^m) \longrightarrow \mathbb{R}^{m\times\mathcal{T}}, \quad \Pi_{\mathcal{T}}(\gamma) = (\gamma(t_1), \gamma(t_2), ..., \gamma(t_n)).$$

Since $H_{\mathcal{T}}(\mathbb{R}^m)$ is linear, it follows that $T_\gamma H_{\mathcal{T}}(\mathbb{R}^m) \simeq H_{\mathcal{T}}(\mathbb{R}^m)$ for each $\gamma \in H_{\mathcal{T}}(\mathbb{R}^m)$. We introduce the Riemannian metric on $H_{\mathcal{T}}(\mathbb{R}^m)$, $h_{\mathcal{T}} \in \Gamma(T^*H_{\mathcal{T}}(\mathbb{R}^m) \otimes T^*H_{\mathcal{T}}(\mathbb{R}^m))$, defined by

$$h_{\mathcal{T}}(u, v) := \int_0^1 \langle u'(s), v'(s) \rangle \, ds, \quad u, v \in T_\gamma H_{\mathcal{T}}(\mathbb{R}^m), \ \gamma \in H_{\mathcal{T}}(\mathbb{R}^m),$$

and its corresponding volume form $\mathrm{Vol}_{h_{\mathcal{T}}} \in \Gamma(\wedge^{m\times n} T H_{\mathcal{T}}(\mathbb{R}^m))$ determinated by

$$\mathrm{Vol}_{h_{\mathcal{T}}}(u_1, u_2, ..., u_{m\times n}) := \sqrt{\det(h_{\mathcal{T}}(u_i, u_j)_{ij})},$$

where $\{u_1, u_2, ..., u_{m\times n}\} \subset T_\gamma H_{\mathcal{T}}(\mathbb{R}^m)$ is an oriented basis and $\gamma \in H_{\mathcal{T}}(\mathbb{R}^m)$. Finally, we introduce a Borel measure $\mu_{\mathcal{T}}$ on $H_{\mathcal{T}}(\mathbb{R}^m)$. As usual, when we deal with a measure associated with a Riemannian structure, we choose the Borel σ-algebra associated with the topology induced by the corresponding Riemannian metric. Along this section, this will be done several times without specifying it again.

Definition 6.1. For each partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0, 1]$, we denote by $\mu_{\mathcal{T}}$ the Borel measure on $H_{\mathcal{T}}(\mathbb{R}^m)$ defined by the density

$$d\mu_{\mathcal{T}} = \frac{1}{(\sqrt{2\pi})^{mn}} \exp\left\{-\frac{1}{2} E_{\mathbb{R}^m}\right\} \mathrm{Vol}_{h_{\mathcal{T}}}.$$

6.2. Piecewise geodesic path space

Now, we define the curved analogue of the measure space $(H_{\mathcal{T}}(\mathbb{R}^m), \mu_{\mathcal{T}})$. Let (M, g) be a closed Riemannian manifold of dimension m. Consider $H(M) \subset \mathcal{C}_{\mathbf{x}_0}(M)$ to be the Hilbert manifold of finite energy paths, defined by

$$H(M) := \{\gamma \in \mathcal{C}_{\mathbf{x}_0}(M) : \gamma \text{ is absolutely continuous and } E(\gamma) < \infty\}$$

where the energy functional E is given through

$$E(\gamma) := \int_0^1 g(\gamma'(s), \gamma'(s))\, ds.$$

Recall that $\gamma \in \mathcal{C}_{\mathbf{x}_0}(M)$ is said to be absolutely continuous if $f \circ \gamma$ is absolutely continuous for all $f \in \mathcal{C}^\infty(M)$. The tangent space $T_\gamma H(M)$ to $H(M)$ at γ can be identified with the space of absolutely continuous vector fields $X : [0,1] \to TM$ along γ such that $X(0) = 0$ and $G^1(X,X) < \infty$ where

$$G^1(X,X) := \int_0^1 g\left(\frac{\nabla X(t)}{dt}, \frac{\nabla X(t)}{dt}\right)\, dt.$$

As usual, we denote

$$\frac{\nabla X(t)}{dt} := //_t(\gamma)\frac{d}{dt}\{//_t(\gamma)^{-1}X(t)\},$$

where $//_t(\gamma) : T_{\mathbf{x}_0}M \to T_{\gamma(\mathbf{x}_0)}M$ denotes the parallel translation along γ relative to the Levy-Civita covariant derivative ∇.

Let $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a partition of $[0,1]$. We define the subspace of $H(M)$,

$$H_{\mathcal{T}}(M) := \left\{\gamma \in H(M) \cap \mathcal{C}^2([0,1]\backslash\mathcal{T}, M) : \nabla\gamma'(t)/dt = 0 \text{ for } t \notin \mathcal{T}\right\},$$

consisting on piecewise geodesic paths in $H(M)$ which change directions only at the partition points. The space $H_{\mathcal{T}}(M)$ is a finite dimensional submanifold of $H(M)$ of dimension $n \times m$. For $\gamma \in H_{\mathcal{T}}(M)$, the tangent space $T_\gamma H_{\mathcal{T}}(M)$ can be identified with elements $X \in T_\gamma H(M)$ satisfying the Jacobi equations on $[0,1]\backslash\mathcal{T}$. In other words, $X \in T_\gamma H(M)$ is in $T_\gamma H_{\mathcal{T}}(M)$ if and only if

$$\frac{\nabla^2}{dt^2}X(t) = R(\gamma'(t), X(t))\gamma'(t),$$

where R is the curvature tensor of ∇. We can give to $H_{\mathcal{T}}(M)$ a Riemannian structure introducing the $\mathcal{T}$-metric. The $\mathcal{T}$-metric $g_{\mathcal{T}} \in \Gamma(T^*H_{\mathcal{T}}(M) \otimes T^*H_{\mathcal{T}}(M))$ is defined by

$$g_{\mathcal{T}}(X,Y) := \sum_{i=1}^n g\left(\frac{\nabla X(t_{i-1}+)}{dt}, \frac{\nabla Y(t_{i-1}+)}{dt}\right)\Delta_i t, \quad X, Y \in T_\gamma H_{\mathcal{T}}(M),$$

for each $\gamma \in H_{\mathcal{T}}(M)$, where the notation $\nabla X(t_{i-1}+)/dt$ is a shorthand of $\lim_{t \downarrow t_{i-1}} \nabla X(t)/dt$. We denote by $\mathrm{Vol}_{g_{\mathcal{T}}} \in \Gamma(\wedge^{m\times m}TH_{\mathcal{T}}(M))$ the volume form associated to $g_{\mathcal{T}}$. It is determinated by

$$\mathrm{Vol}_{g_{\mathcal{T}}}(X_1, X_2, ..., X_{m\times n}) := \sqrt{\det(g_{\mathcal{T}}(X_i, X_j)_{ij})},$$

where $\{X_1, X_2, ..., X_{m\times n}\} \subset T_\gamma H_{\mathcal{T}}(M)$ is an oriented basis and $\gamma \in H_{\mathcal{T}}(M)$.

Definition 6.2. For each partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0,1]$, we denote by $\nu_{\mathcal{T}}$ the Borel measure on $H_{\mathcal{T}}(M)$ defined by the density

$$d\nu_{\mathcal{T}} := \frac{1}{(\sqrt{2\pi})^{mn}} \exp\left\{-\frac{1}{2}E\right\} \mathrm{Vol}_{g_{\mathcal{T}}}.$$

This measure spaces $(H_{\mathcal{T}}(M), \mathcal{B}_{\mathcal{T}}, \nu_{\mathcal{T}})$, where $\mathcal{B}_{\mathcal{T}}$ stands for the Borel σ-algebra of $H_{\mathcal{T}}(M)$, will be the finite dimensional candidates for the approximation scheme.

6.3. Cartan's development map

In general, it is not quite easy to deal with the manifold $H_{\mathcal{T}}(M)$. Due to this fact, we will identify this space through the well known space $H_{\mathcal{T}}(\mathbb{R}^m)$ via Cartan's development map. Cartan's development map $\Phi : H(\mathbb{R}^m) \to H(M)$ is defined, for $\alpha \in H(\mathbb{R}^m)$, by $\Phi(\alpha) := \gamma$, where $\gamma \in H(M)$ is the unique solution of the ordinary differential equation

$$\gamma'(t) = //_t(\gamma)\alpha'(t), \quad \gamma(0) = \mathbf{x}_0. \tag{9}$$

The anti-development map $\Phi^{-1} : H(M) \to H(\mathbb{R}^m)$ is defined by $\Phi^{-1}(\gamma) := \alpha$ where $\alpha \in H(\mathbb{R}^m)$ is given by

$$\alpha(t) := \int_0^t //_r^{-1}(\gamma)\gamma'(r)\ dr.$$

The map $\Phi : H(\mathbb{R}^m) \to H(M)$ is bijective and smooth, hence it defines a diffeomorphism of infinite dimensional Hilbert manifolds, but it is not in general an isometry of Riemannian manifolds. The development map $\Phi : H(\mathbb{R}^m) \to H(M)$ has the property

$$\Phi(H_{\mathcal{T}}(\mathbb{R}^m)) = H_{\mathcal{T}}(M).$$

We shall denote $\Phi|_{H_{\mathcal{T}}(\mathbb{R}^m)}$ by $\Phi_{\mathcal{T}}$.

A fundamental property of Cartan's development map is that it preserves the $\mathcal{T}$-measure, in the sense that

$$\mu_{\mathcal{T}}(B) = \nu_{\mathcal{T}}(\Phi_{\mathcal{T}}(B)), \tag{10}$$

for each Borel subset B of $H_{\mathcal{T}}(\mathbb{R}^m)$.

Moreover, it can be seen that the development map relates the measure $\nu_{\mathcal{T}}$ of $H_{\mathcal{T}}(M)$ with the well known heat kernel measure in the flat space $\mathbb{R}^{m\times\mathcal{T}}$. For a partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, the heat kernel measure $\lambda_0^{\mathcal{T}}$ is the Borel measure on $\mathbb{R}^{m\times\mathcal{T}}$ defined by

$$d\lambda_0^{\mathcal{T}} := \prod_{i=1}^n p_{t_i - t_{i-1}}(x_{t_i}, x_{t_{i-1}}) \prod_{i=1}^n dx_{t_i},$$

where $x_0 = 0$ and $p_t(x, y)$ is the heat kernel of $\mathbb{R}^m$. In [1, Lem. 4.11], it is proved the identity

$$\mu_{\mathcal{T}}(\Pi_{\mathcal{T}}^{-1}(B)) = \lambda_0^{\mathcal{T}}(B) \tag{11}$$

for each Borel subset B of $\mathbb{R}^{m\times\mathcal{T}}$. Hence joining equations (10) and (11) yields

$$\nu_{\mathcal{T}}((\Phi_{\mathcal{T}} \circ \Pi_{\mathcal{T}}^{-1})(B)) = \lambda_0^{\mathcal{T}}(B).$$

As a consequence, we can reduce the structure of the space $L^p(H_{\mathcal{T}}(M), \nu_{\mathcal{T}})$ to the more familiar one $L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}})$. That is, for $1 \le p < \infty$, the following spaces are identified

$$L^p(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}) \simeq L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}),$$

via the isometric isomorphism

$$\Lambda_{\mathcal{T}} : L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right) \longrightarrow L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}), \quad \Lambda_{\mathcal{T}}(f) := f \circ \Phi_{\mathcal{T}} \circ \Pi_{\mathcal{T}}^{-1}. \tag{12}$$

It can be also introduced an almost everywhere defined stochastic extension of the development map, called the stochastic development map $\tilde{\Phi} : \mathcal{C}_0(\mathbb{R}^m) \to \mathcal{C}_{\mathbf{x}_0}(M)$ and its corresponding anti-development map $\tilde{\Phi}^{-1} : \mathcal{C}_{\mathbf{x}_0}(M) \to \mathcal{C}_0(\mathbb{R}^m)$. It can be proved that they preserve the Wiener measure in the sense that

$$\lambda_0(B) = \mu_{\mathbf{x}_0}(\tilde{\Phi}(B)),$$

for each Borel subset B of $\mathcal{C}_0(\mathbb{R}^m)$ where λ_0 is the Wiener measure on the classical Wiener space $\mathcal{C}_0(\mathbb{R}^m)$ (see for instance [23]). Hence, thanks to the stochastic development map, we can reduce the structure of the space $L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$ to the more familiar one $L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0)$ under the philosophy of (12). For $1 \leq p < \infty$, the following spaces are identified

$$L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}) \simeq L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0),$$

via the map

$$\Lambda : L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}) \longrightarrow L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0), \quad \Lambda(f) := f \circ \tilde{\Phi}. \tag{13}$$

6.4. Finite dimensional approximation scheme

To prove an analogue of Theorem 3.2 for the geometric measure $\nu_{\mathcal{T}}$, we will follow the following philosophy. Firstly we shall prove the approximation scheme for the classical Wiener measure space $(\mathcal{C}_0(\mathbb{R}^m), \lambda_0)$ and then we will make use the identifications provided by equations (12) and (13), to translate the result to the geometric framework.

Let $\mathcal{P}$ be the directed set consisting on partitions of $[0, 1]$

$$\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n = 1\},$$

partially ordered by inclusion and consider the projectors

$$\pi_{\mathcal{T}\mathcal{T}'} : \mathbb{R}^{m\times\mathcal{T}'} \longrightarrow \mathbb{R}^{m\times\mathcal{T}}, \quad \pi_{\mathcal{T}\mathcal{T}'}(x_t)_{t\in\mathcal{T}'\setminus\{0\}} := (x_t)_{t\in\mathcal{T}\setminus\{0\}}$$
$$\pi_{\mathcal{T}} : \mathbb{R}^{m\times[0,1]} \longrightarrow \mathbb{R}^{m\times\mathcal{T}}, \quad \pi_{\mathcal{T}}(x_t)_{t\in[0,1]} := (x_t)_{t\in\mathcal{T}\setminus\{0\}}.$$

In analogy with the preceding sections, we define the diagram $(L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}), \eta_{\mathcal{T}\mathcal{T}'})$ where the morphisms $\eta_{\mathcal{T}\mathcal{T}'}$, $\mathcal{T} \subset \mathcal{T}'$, are defined through

$$\eta_{\mathcal{T}\mathcal{T}'} : L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}) \longrightarrow L^p(\mathbb{R}^{m\times\mathcal{T}'}, \lambda_0^{\mathcal{T}'}), \quad \eta_{\mathcal{T}\mathcal{T}'}(f) := f \circ \pi_{\mathcal{T}\mathcal{T}'}.$$

Using the same techniques of section 4, it is easily proved that for each $1 \leq p < \infty$, the cocone $(L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0), \varphi_{\mathcal{T}})$ defines a realization of the colimit of $(L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}), \eta_{\mathcal{T}\mathcal{T}'})$, where the morphisms $\varphi_{\mathcal{T}}$, $\mathcal{T} \in \mathcal{P}$, are given by

$$\varphi_{\mathcal{T}} : L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}) \longrightarrow L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0), \quad \varphi_{\mathcal{T}}(f) := f \circ \pi_{\mathcal{T}}|_{\mathcal{C}_{\mathbf{x}_0}(\mathbb{R}^m)}.$$

This establishes the approximation scheme for the classical Wiener space $(\mathcal{C}_0(\mathbb{R}^m), \lambda_0)$. Now, we proceed to define the diagram and cocone for the geometric measure $\nu_{\mathcal{T}}$.

Let us consider the diagram $\left(L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right), \varrho_{\mathcal{T}\mathcal{T}'}\right)$ where for each $\mathcal{T} \subset \mathcal{T}'$, $\varrho_{\mathcal{T}\mathcal{T}'}$ are the morphisms defined by

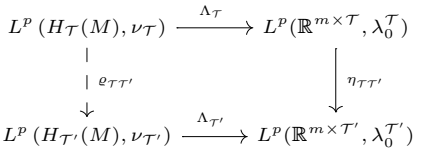

Fig. 3. Definition of the morphisms $\varrho_{\mathcal{T}\mathcal{T}'}$.

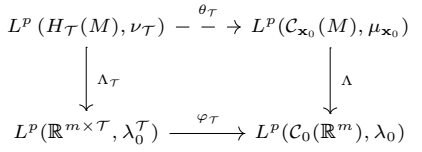

Fig. 4. Definition of the morphisms $\theta_{\mathcal{T}}$.

$$\varrho_{\mathcal{T}\mathcal{T}'}: L^p\left(H_{\mathcal{T}}(M),\nu_{\mathcal{T}}\right)\longrightarrow L^p\left(H_{\mathcal{T}'}(M),\nu_{\mathcal{T}'}\right),\quad \varrho_{\mathcal{T}\mathcal{T}'}:=\Lambda_{\mathcal{T}'}^{-1}\circ\eta_{\mathcal{T}\mathcal{T}'}\circ\Lambda_{\mathcal{T}},$$

in other words, the unique morphisms making the diagram of Fig. 3 commutative.

The main result of this subsection is to prove that the cocone $(L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0}),\theta_{\mathcal{T}})$ is a realization of the colimit of the diagram $\left(L^p\left(H_{\mathcal{T}}(M),\nu_{\mathcal{T}}\right),\varrho_{\mathcal{T}\mathcal{T}'}\right)$, where the morphisms $\theta_{\mathcal{T}}$, $\mathcal{T}\in\mathcal{P}$, are defined by

$$\theta_{\mathcal{T}}: L^p\left(H_{\mathcal{T}}(M),\nu_{\mathcal{T}}\right)\longrightarrow L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0}),\quad \theta_{\mathcal{T}}:=\Lambda^{-1}\circ\varphi_{\mathcal{T}}\circ\Lambda_{\mathcal{T}}.$$

In other words, the morphisms $\theta_{\mathcal{T}}$ are the unique morphisms making commutative the diagram of Fig. 4. This identification establishes an analogue of Theorem 3.2 for the geometrical framework. The definition of the involved morphisms $\varrho_{\mathcal{T}\mathcal{T}'},\theta_{\mathcal{T}}$ is given through the commutativity of the diagrams of Fig. 3 and 4, respectively, in order to make things natural, in the categorical meaning of the word. This will become clear in the proof of the main Theorem of this subsection, Theorem 6.3.

Theorem 6.3. *The cocone* $(L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0}),\theta_{\mathcal{T}})$ *defines a realization of the colimit of* $\left(L^p\left(H_{\mathcal{T}}(M),\nu_{\mathcal{T}}\right),\varrho_{\mathcal{T}\mathcal{T}'}\right)$.

Proof. By (12), we have a family of isometric isomorphisms indexed by $\mathcal{T}$,

$$\Lambda_{\mathcal{T}}: L^p\left(H_{\mathcal{T}}(M),\nu_{\mathcal{T}}\right)\to L^p(\mathbb{R}^{m\times\mathcal{T}},\mu_{\mathbf{x}_0}^{\mathcal{T}}).$$

By the commutativity of the diagram of Fig. 3, this family of isomorphisms $(\Lambda_{\mathcal{T}})_{\mathcal{T}}$ establishes a natural isomorphism between the diagrams $\left(L^p\left(H_{\mathcal{T}}(M),\nu_{\mathcal{T}}\right),\varrho_{\mathcal{T}\mathcal{T}'}\right)$ and $(L^p(\mathbb{R}^{m\times\mathcal{T}},\lambda_0^{\mathcal{T}}),\eta_{\mathcal{T}\mathcal{T}'})$. Hence the colimits of these two diagrams are isomorphic via a naturally defined isomorphism, see for instance the reference [22, Cor. 3.6.3] for a proof of this fact. As the colimit of $(L^p(\mathbb{R}^{m\times\mathcal{T}},\lambda_0^{\mathcal{T}}),\eta_{\mathcal{T}\mathcal{T}'})$ is $(L^p(\mathcal{C}_0(\mathbb{R}^m),\lambda_0),\varphi_{\mathcal{T}})$, to prove the Theorem it is enough to prove that the cocone $(L^p(\mathcal{C}_{\mathbf{x}_0}(M),\mu_{\mathbf{x}_0}),\theta_{\mathcal{T}})$ is isomorphic to $(L^p(\mathcal{C}_0(\mathbb{R}^m),\lambda_0),\varphi_{\mathcal{T}})$ via a naturally defined isomorphism. The commutativity of the diagram of Fig. 4, establishes that Λ is our required isomorphism. This concludes the proof. □

6.5. Derivation of the limit formula

In this final subsection, we prove an analogue of Theorem 4.3 for the geometric measure $\nu_{\mathcal{T}}$. Let us denote by $\varinjlim L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}})$ and $\varinjlim L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right)$, the analogues of the space defined by (7), under the natural changes to adapt it to $(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}})$ and $(H_{\mathcal{T}}(M), \nu_{\mathcal{T}})$, respectively. We omit the explicit description of these spaces for notational convenience. Rephrasing the arguments of section 4, it can be proved that the operator

$$\mathfrak{I}_p : \varinjlim L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}) \longrightarrow L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0), \quad (f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \mapsto \lim_{\mathcal{T}} \varphi_{\mathcal{T}}(f_{\mathcal{T}}) \tag{14}$$

is an isometric isomorphism. Thanks to this isomorphism and the identifications provided by (12) and (13), we prove the final result of this article, Theorem 6.4. As a direct consequence of Theorem 6.3 and the following Theorem 6.4, we obtain that the cocone

$$(\varinjlim L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right), \psi_{\mathcal{R}}),$$

where the morphisms $\psi_{\mathcal{R}}$ are the corresponding analogues of (8) for $(H_{\mathcal{T}}(M), \nu_{\mathcal{T}})$, defines a realization of the colimit of $(L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right), \varrho_{\mathcal{T}\mathcal{T}'})$.

Theorem 6.4. *Let $1 \leq p < \infty$, then the following spaces are isometrically isomorphic*

$$\varinjlim L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right) \simeq L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}). \tag{15}$$

In consequence, for every $F \in L^1(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0})$, there exists an element $(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} \in \bigtimes_{\mathcal{T}\in\mathcal{P}} L^1\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right)$ such that

$$\int_{\mathcal{C}_{\mathbf{x}_0}(M)} F \, d\mu_{\mathbf{x}_0} = \lim_{\mathcal{T}} \int_{H_{\mathcal{T}}(M)} f_{\mathcal{T}} \, d\nu^{\mathcal{T}}. \tag{16}$$

Proof. By (12), we have a family of isometric isomorphisms indexed by $\mathcal{T}$,

$$\Lambda_{\mathcal{T}} : L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right) \to L^p(\mathbb{R}^{m\times\mathcal{T}}, \mu_{\mathbf{x}_0}^{\mathcal{T}}).$$

It is easily seen using the definition of the morphisms $\varrho_{\mathcal{T}\mathcal{T}'}$, Fig. 3, that the operator

$$\Sigma_p : \varinjlim L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right) \longrightarrow \varinjlim L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}), \quad \Sigma_p(f_{\mathcal{T}})_{\mathcal{T}\in\mathcal{P}} := (\Lambda_{\mathcal{T}}(f_{\mathcal{T}}))_{\mathcal{T}\in\mathcal{P}}$$

defines an isometric isomorphism. Finally, the composition operator

$$\begin{array}{ccc} \varinjlim L^p\left(H_{\mathcal{T}}(M), \nu_{\mathcal{T}}\right) & \dashrightarrow & L^p(\mathcal{C}_{\mathbf{x}_0}(M), \mu_{\mathbf{x}_0}) \\ \Big\downarrow{\scriptstyle \Sigma_p} & & \Big\uparrow{\scriptstyle \Lambda^{-1}} \\ \varinjlim L^p(\mathbb{R}^{m\times\mathcal{T}}, \lambda_0^{\mathcal{T}}) & \xrightarrow{\mathfrak{I}_p} & L^p(\mathcal{C}_0(\mathbb{R}^m), \lambda_0) \end{array}$$

where the morphisms $\mathfrak{I}_p$ and Λ are given by (14) and (13), respectively, defines an isometric isomorphism. This concludes the proof of (15). Formula (16) follows from the isometric property of the induced operator. □

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References

[1] L. Andersson, B.K. Driver, Finite dimensional approximations to Wiener measure and path integral formulas on manifolds, J. Funct. Anal. 165 (2) (1999) 430–498.
[2] C. Bär, Renormalized integrals and a path integral formula for the heat kernel on a manifold, in: Analysis, Geometry and Quantum Field Theory, in: Contemp. Math., vol. 584, Amer. Math. Soc., Providence, RI, 2012, pp. 179–197.
[3] C. Bär, F. Pfäffle, Path integrals on manifolds by finite dimensional approximation, J. Reine Angew. Math. 625 (2008) 29–57.
[4] C. Bär, F. Pfäffle, Wiener measures on Riemannian manifolds and the Feynman-Kac formula, Mat. Contemp. 40 (2011) 37–90.
[5] J.M.F. Castillo, The Hitchhiker guide to categorical Banach space theory. Part I, Extr. Math. 25 (2) (2010) 103–149.
[6] D.L. Cohn, Measure Theory, Springer, New York, 2013.
[7] P.J. Daniell, A general form of integral, Ann. Math. (2) 19 (4) (June 1918) 279–294.
[8] P.J. Daniell, Integrals in an infinite number of dimensions, Ann. Math. (2) 20 (4) (July 1919) 281–288.
[9] P.J. Daniell, Functions of limited variation in an infinite number of dimension, Ann. Math. (2) 21 (1) (September 1919) 30–38.
[10] R. Feynman, Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys. 20 (2) (April 1948) 367–387.
[11] R. Feynman, A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, 1965.
[12] G.B. Folland, Real Analysis, Modern Techniques and Their Applications, second edition, John Wiley & Sons, Inc., 1999.
[13] A. Grigor'yan, Heat Kernel and Analysis on Manifolds, Studies in Advanced Mathematics, vol. 47, American Mathematical Society, 2009.
[14] E.P. Hsu, Stochastic Analysis on Manifolds, Graduate Studies in Mathematics, vol. 38, 2002.
[15] B. Jessen, The theory of integration in a space of infinite number of dimensions, Acta Math. 63 (December 1934) 249–323.
[16] A.P.C. Lim, Path integrals on a compact manifold with non-negative curvature, Rev. Math. Phys. 19 (09) (2007) 967–1044.
[17] M. Ludewig, Path integrals on manifolds with boundary, Commun. Math. Phys. 354 (2) (2016) 621–640.
[18] M. Ludewig, Heat kernel asymptotics, path integrals and infinite-dimensional determinants, J. Geom. Phys. 131 (2018) 66–88.
[19] M. Ludewig, Strong short-time asymptotics and convolution approximation of the heat kernel, Ann. Glob. Anal. Geom. 55 (2) (2019) 371–394.
[20] R. Peled, Notes on sigma algebras for Brownian motion course, Lecture Notes, http://www.math.tau.ac.il/~peledron/Teaching/Brownian_motion/Sigma_algebra_notes.pdf.
[21] P.E. Protter, Stochastic Integration and Differential Equations, 2nd edition, Springer-Verlag, 2004.
[22] E. Riehl, Category Theory in Context, Dover Publications, 2016.
[23] J.C. Sampedro, On the space of infinite dimensional integrable functions, J. Math. Anal. Appl. 488 (1) (2020).
[24] N.E. Wegge-Olsen, K-Theory and C*-Algebras, A Friendly Approach, Oxford University Press, 1993.
[25] N. Wiener, The mean of a functional of arbitrary elements, in: Annals of Mathematics, in: Second Series, vol. 22, 1920, pp. 66–72.
[26] N. Wiener, The average value of an analytic functional, Proc. Natl. Acad. Sci. USA 7 (1921) 253–260.
[27] N. Wiener, The average of an analytic functional and the Brownian movement, Proc. Natl. Acad. Sci. USA 7 (1921) 294–298.
[28] N. Wiener, Differential space, J. Math. Phys. 2 (1923) 131–174.
[29] N. Wiener, The average value of a functional, Proc. Lond. Math. Soc. (2) 22 (1924) 454–467.
[30] N. Wiener, Generalized harmonic analysis, Acta Math. 55 (1930) 117–258.
[31] Y. Yamasaki, Measures on Infinite-Dimensional Spaces, World Scientific Publishing Co., Singapore, 1985.