



Orientability through the algebraic multiplicity

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Abstract. This paper establishes some hidden connections between the theory of generalized algebraic multiplicities, χ , and the notion of orientability of vector bundles. The novel approach adopted in it facilitates the definition of several invariants closely related to the first Stiefel–Whitney characteristic class through some path integration techniques.

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1. Introduction

The classical notion of algebraic multiplicity for eigenvalues of parametric linear operators of the form $\lambda I_U - K$, where K is a compact operator in a real Banach space U and I_U stands for the identity map in U , was substantially generalized by Esquinas and López-Gómez [5, 6, 18] to cover general analytic families

$$\mathfrak{L}(\lambda) = \sum_{n=0}^{\infty} \lambda^n L_n, \quad \lambda \in \mathbb{R},$$

of Fredholm operators of index zero. Note that in the classical setting

$$L_0 = -K, \quad L_1 = I_U \quad \text{and} \quad L_j = 0 \quad \text{for all } j \geq 2.$$

The generalized algebraic multiplicity of [5, 6, 18], denoted throughout this paper by $\chi = \chi[\mathfrak{L}, \lambda_0]$, was introduced to characterize the nonlinear eigenvalues, in the context of bifurcation theory, of a wide class of C^r -curves, $\lambda \mapsto \mathfrak{L}(\lambda)$, of linear Fredholm operators of index zero (see Chapter 4 of [18]),

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and it extends the classical notion of algebraic multiplicity at λ_0 for classical operator families of the form $\lambda I_U - K$, i.e.,

$$\chi[\lambda I_U - K, \lambda_0] = \dim \operatorname{Ker} [(\lambda_0 I_U - K)^{\nu(\lambda_0)}], \quad (1.1)$$

where $\nu(\lambda_0)$ is the algebraic ascent of λ_0 .

More recently, adopting a geometrical point of view, the authors established in [22] a connection between $\chi[\mathfrak{L}, \lambda_0]$ and the notion of local intersection index of algebraic varieties, which is a central device in algebraic geometry. Essentially, it was established that the algebraic multiplicity $\chi[\mathfrak{L}, \lambda_0]$ of a curve of Fredholm operators of index zero $\mathfrak{L} : [a, b] \rightarrow \Phi_0(U)$ at a given eigenvalue $\lambda_0 \in [a, b]$ equals the intersection index of the curve $\mathfrak{L}([a, b]) \subset \Phi_0(U)$ with respect to the stratified set of singular operators $\mathcal{S}(U) \subset \Phi_0(U)$ at $\mathfrak{L}(\lambda_0)$, where we are denoting by $\Phi_0(U)$ the set of linear continuous operators $T : U \rightarrow U$ on a Banach space U that are Fredholm of index zero, i.e., such that

$$\dim \operatorname{Ker} [T] = \operatorname{codim} R[T] < \infty.$$

The main goal of this paper is to study vector bundles via topological K -Theory focusing special attention into the obstruction described by the first Stiefel–Whitney characteristic class, ω_1 . One of our main findings reveals how that obstruction can be fully described through χ . These connections are established by combining the Atiyah–Jänich index map

$$\mathfrak{Ind} : [X, \Phi_0(U)] \longrightarrow \tilde{K}\mathcal{O}(X), \quad (1.2)$$

with some spectral techniques developed by the authors in [20–22] for continuous Fredholm maps $h : X \rightarrow \Phi_0(U)$. In (1.2), $[X, \Phi_0(U)]$ stands for the set of homotopy classes of continuous maps $X \rightarrow \Phi_0(U)$, and $\tilde{K}\mathcal{O}(X)$ denotes the real reduced K -group of a compact path connected topological space X ; the K -group consists of the stable equivalence classes of real vector bundles with base space X . In particular, our new approach shows that the orientability of a given vector bundle can be characterized through the generalized algebraic multiplicity, χ .

The relationship between the obstruction associated to the first Stiefel–Whitney class and the concept of algebraic multiplicity can be described, shortly, as follows. Under the appropriate assumptions on U and X , the index map (1.2) is an isomorphism and hence, each real vector bundle $E \rightarrow X$ has an associated single parameterized family of Fredholm operators $h : X \rightarrow \Phi_0(U)$. Essentially, we will establish that, by considering $\omega_1(E)$ as a map $\pi_1(X) \rightarrow \mathbb{Z}_2$, the value $\omega_1(E)[\gamma]$ equals the sign, according to its oddity, of the total generalized algebraic multiplicity of the closed Fredholm curve

$$h \circ \gamma : \mathbb{S}^1 \longrightarrow \Phi_0(U).$$

Equivalently, $\omega_1(E)[\gamma]$ is the sign of the intersection index between $[h \circ \gamma](\mathbb{S}^1) \subset \Phi_0(U)$ and the stratified set of singular operators $\mathcal{S}(U) \subset \Phi_0(U)$. Therefore, the information provided by the first Stiefel–Whitney class can be packaged in terms of the way that a geometrical object in $\Phi_0(U)$, as, e.g., a parameterized family of Fredholm operators, intersects to $\mathcal{S}(U)$.

Thanks to the versatility of the new approach, by using some path integration techniques on Riemannian manifolds, we can introduce a new topological invariant of stable equivalence classes of real vector bundles via the integration of loops of the base space X . Namely, the *global torsion invariant*, which can be defined as follows. For any given closed Riemannian manifold and a base-point $\mathbf{x} \in M$, the *global torsion invariant* is the map $\Lambda : \tilde{K}\mathcal{O}(M) \rightarrow [-1, 1]$ defined by

$$\Lambda(E) := \int_{\mathcal{L}_{\mathbf{x}}(M)} \chi_2[\mathfrak{Jn}\mathfrak{d}^{-1}[E] \circ \gamma, \mathbb{S}^1] \, d\mu_{\mathbf{x}}(\gamma) \quad \text{for all } E \in \tilde{K}\mathcal{O}(M),$$

where $\mathcal{L}_{\mathbf{x}}(M)$ stands for the space of continuous loops $\gamma : \mathbb{S}^1 \rightarrow M$ with $\gamma(0) = \mathbf{x}$, $\mu_{\mathbf{x}}$ is the normalised Wiener measure on $\mathcal{L}_{\mathbf{x}}(M)$, and, for every $\mathfrak{L} \in \mathcal{C}(\mathbb{S}^1, \Phi_0(U))$, $\chi_2[\mathfrak{L}, \mathbb{S}^1] := (-1)^{\chi_T}$ where χ_T is the total algebraic multiplicity of $\mathfrak{L}$. The real number Λ packages the information provided by the class w_1 in a robust and compact way. Indeed, as established by Theorem 4.2, a vector bundle $E \rightarrow M$ is orientable if, and only if, $\Lambda(E) = 1$. Moreover, according to Theorem 4.3, $\Lambda(E) = \Lambda(F)$ if $E = F$ in $\tilde{K}\mathcal{O}(M)$, i.e., the map $\Lambda : \tilde{K}\mathcal{O}(M) \rightarrow [-1, 1]$ is a topological invariant of stable equivalence classes of real vector bundles over M .

Finally, we find out the value of the global torsion invariant for the circle and the n -dimensional torus. The global torsion invariant Λ for the circle is given by

$$\Lambda : \tilde{K}\mathcal{O}(\mathbb{S}^1) \longrightarrow [-1, 1], \quad \Lambda([T\mathbb{S}^1]) = 1, \quad \Lambda([\mathcal{M}]) = \frac{1}{\sqrt[4]{2}},$$

where $T\mathbb{S}^1$ is the tangent bundle of $\mathbb{S}^1$ and $\mathcal{M}$ is the Möbius bundle. Its values for the n -dimensional torus $\mathbb{T}^n := \times_{i=1}^n \mathbb{S}^1$ are

$$\Lambda(\tilde{K}\mathcal{O}(\mathbb{T}^n)) = \left\{ \left(\frac{1}{\sqrt[4]{2}} \right)^m : m \in \{1, 2, \dots, n\} \right\}.$$

This paper is organized as follows. Section 2 collects all the necessary preliminaries to read comfortably the rest of the paper. Precisely, it reviews, very briefly, the concept of generalized algebraic multiplicity, χ , and the concept of parity, σ , discussed by Fitzpatrick, Pejsachowicz and Rabier [11]. As the parity σ is a pivotal invariant to study the topology of $\Phi_0(U)$ and can be determined from χ , it establishes a bridge between the algebraic information provided by χ and some relevant topological aspects of $\Phi_0(U)$. Section 3 describes the relationship between the first Stiefel–Whitney characteristic class, ω_1 , and the algebraic multiplicity, χ . It studies also some notions of orientability of vector bundles and parameterized families of Fredholm operators. Section 4 introduces the global torsion invariant, Λ . Finally, Sect. 5 finds out the global torsion invariant of the circle and the n -dimensional torus.

2. Preliminaries

This section collects some important properties of the algebraic multiplicity χ of Esquinas and López-Gómez [5, 6, 18]. Among them, its connections with σ ,

the topological parity of Fitzpatrick and Pejsachowicz [10]. These invariants and their several relations will be used in the forthcoming sections to prove the main results of this paper. As they are scattered in a number of research papers and specialized monographs, they have been packaged in this section for the convenience of the reader. Although χ and σ are closely related to the index of intersection of algebraic varieties, i (see [22]), these links remain outside the main scope of this paper.

2.1. The generalized algebraic multiplicity χ

This section collects some important properties of the generalized algebraic multiplicity, χ , of Esquinas and López-Gómez [5, 6, 18]. Throughout it, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Ω is a subdomain of $\mathbb{K}$, and, for any given $\mathfrak{L} \in \mathcal{C}(\Omega, \Phi_0(U))$, a point $\lambda \in \Omega$ is said to be a *generalized eigenvalue* of $\mathfrak{L}$ if $\mathfrak{L}(\lambda) \notin GL(U)$. Then, the *generalized spectrum* of $\mathfrak{L}$, $\Sigma(\mathfrak{L})$, is defined through

$$\Sigma(\mathfrak{L}) := \{\lambda \in \Omega : \mathfrak{L}(\lambda) \notin GL(U)\}.$$

The next concept goes back to [6], where it was introduced to characterize the nonlinear eigenvalues of a rather general class of nonlinear Fredholm maps in the context of local bifurcation theory.

Definition 2.1. Let $\mathfrak{L} \in \mathcal{C}^r(\Omega, \Phi_0(U))$ and $1 \leq \kappa \leq r$. Then, a given $\lambda_0 \in \Sigma(\mathfrak{L})$ is said to be a κ -transversal eigenvalue of $\mathfrak{L}$ if

$$\bigoplus_{j=1}^{\kappa} \mathfrak{L}_j \left(\bigcap_{i=0}^{j-1} \text{Ker}[\mathfrak{L}_i] \right) \oplus R[\mathfrak{L}_0] = U \quad \text{with} \quad \mathfrak{L}_{\kappa} \left(\bigcap_{i=0}^{\kappa-1} \text{Ker}[\mathfrak{L}_i] \right) \neq \{0\},$$

where we are denoting $\mathfrak{L}_j := \frac{1}{j!} \mathfrak{L}^{(j)}(\lambda_0)$, $1 \leq j \leq r$. In such case, the generalized algebraic multiplicity, χ , is defined by

$$\chi[\mathfrak{L}, \lambda_0] := \sum_{j=1}^{\kappa} j \cdot \dim \mathfrak{L}_j \left(\bigcap_{i=0}^{j-1} \text{Ker}[\mathfrak{L}_i] \right). \quad (2.1)$$

In particular, when $\text{Ker}[\mathfrak{L}_0] = \text{span}[\varphi_0]$ for some $\varphi_0 \in U$ such that $\mathfrak{L}_1 \varphi_0 \notin R[\mathfrak{L}_0]$, then

$$\mathfrak{L}_1(\text{Ker}[\mathfrak{L}_0]) \oplus R[\mathfrak{L}_0] = U \quad (2.2)$$

and hence, λ_0 is a 1-transversal eigenvalue of $\mathfrak{L}(\lambda)$ with $\chi[\mathfrak{L}, \lambda_0] = 1$. The transversality condition (2.2) goes back to Crandall and Rabinowitz [4]. The following concept, going back to [18], plays a pivotal role in the sequel.

Definition 2.2. Let $\mathfrak{L} \in \mathcal{C}(\Omega, \Phi_0(U))$ and $\kappa \in \mathbb{N}$. A generalized eigenvalue $\lambda_0 \in \Sigma(\mathfrak{L})$ is said to be κ -algebraic if there exists $\varepsilon > 0$ such that

- (a) $\mathfrak{L}(\lambda) \in GL(U)$ if $0 < |\lambda - \lambda_0| < \varepsilon$;
- (b) There exists $C > 0$ such that $\|\mathfrak{L}^{-1}(\lambda)\| < \frac{C}{|\lambda - \lambda_0|^{\kappa}}$ if $0 < |\lambda - \lambda_0| < \varepsilon$;
- (c) κ is the minimal integer for which the previous property holds.

Subsequently, the set of κ -algebraic eigenvalues of $\mathfrak{L}$ will be denoted by $\text{Alg}_{\kappa}(\mathfrak{L})$, and the set of *algebraic eigenvalues* by $\text{Alg}(\mathfrak{L}) := \bigcup_{\kappa \in \mathbb{N}} \text{Alg}_{\kappa}(\mathfrak{L})$. According to Theorems 4.4.1 and 4.4.4 of [18], if $\mathfrak{L}(\lambda)$ is analytic in Ω , i.e., $\mathfrak{L} \in \mathcal{H}(\Omega, \Phi_0(U))$, then, either $\Sigma(\mathfrak{L}) = \Omega$, or $\Sigma(\mathfrak{L})$ is discrete and $\Sigma(\mathfrak{L}) \subset \text{Alg}(\mathfrak{L})$.

Subsequently, we denote by $\mathcal{A}_{\lambda_0}(\Omega, \Phi_0(U))$ the set of curves $\mathfrak{L} \in \mathcal{C}^r(\Omega, \Phi_0(U))$ such that $\lambda_0 \in \text{Alg}_{\kappa}(\mathfrak{L})$ with $1 \leq \kappa \leq r$ for some $r \in \mathbb{N}$. According to Theorems 4.3.2 and 5.3.3 of [18], for every $\mathfrak{L} \in \mathcal{C}^r(\Omega, \Phi_0(U))$, $\kappa \in \{1, 2, \dots, r\}$ and $\lambda_0 \in \text{Alg}_{\kappa}(\mathfrak{L})$, there exists a polynomial $\Phi : \Omega \rightarrow \mathcal{L}(U)$ with $\Phi(\lambda_0) = I_U$ such that λ_0 is a κ -transversal eigenvalue of the path

$$\mathfrak{L}^{\Phi} := \mathfrak{L} \circ \Phi \in \mathcal{C}^r(\Omega, \Phi_0(U)), \quad (2.3)$$

and $\chi[\mathfrak{L}^{\Phi}, \lambda_0]$ is independent of the curve of *transversalizing local isomorphisms* Φ chosen to transversalize $\mathfrak{L}$ at λ_0 through (2.3). Therefore, the next concept of multiplicity is consistent

$$\chi[\mathfrak{L}, \lambda_0] := \chi[\mathfrak{L}^{\Phi}, \lambda_0]. \quad (2.4)$$

This notion of algebraic multiplicity can be easily extended by setting $\chi[\mathfrak{L}, \lambda_0] = 0$ if $\lambda_0 \notin \Sigma(\mathfrak{L})$ and $\chi[\mathfrak{L}, \lambda_0] = +\infty$ if $\lambda_0 \in \Sigma(\mathfrak{L}) \setminus \text{Alg}(\mathfrak{L})$ and $r = +\infty$. Thus, $\chi[\mathfrak{L}, \lambda]$ is well defined for all $\lambda \in \Omega$ of any smooth path $\mathfrak{L} \in \mathcal{C}^{\infty}(\Omega, \Phi_0(U))$. In particular, for any analytical curve $\mathfrak{L} \in \mathcal{H}(\Omega, \Phi_0(U))$. The next uniqueness result of Mora-Corral [23], axiomatizes χ ; some refinements can be found in [19, Ch. 6].

Theorem 2.3. *Let U be a $\mathbb{K}$ -Banach space. For every $\lambda_0 \in \mathbb{K}$ and any open neighborhood $\Omega_{\lambda_0} \subset \mathbb{K}$ of λ_0 , the algebraic multiplicity χ is the unique map*

$$\chi[\cdot, \lambda_0] : \mathcal{C}^{\infty}(\Omega_{\lambda_0}, \Phi_0(U)) \longrightarrow [0, \infty]$$

such that

- (NP) *There exists a rank one projection $\Pi \in \mathcal{L}(U)$ such that $\chi[(\lambda - \lambda_0)\Pi + I_U - \Pi, \lambda_0] = 1$.*
 (PF) *For every pair $\mathfrak{L}, \mathfrak{M} \in \mathcal{C}^{\infty}(\Omega_{\lambda_0}, \Phi_0(U))$, $\chi[\mathfrak{L} \circ \mathfrak{M}, \lambda_0] = \chi[\mathfrak{L}, \lambda_0] + \chi[\mathfrak{M}, \lambda_0]$.*

The axiom (PF) is the *product formula* and (NP) is a *normalization property* for establishing the uniqueness of χ . From these two axioms one can derive the remaining properties of χ ; among them, that it equals the classical algebraic multiplicity when $\mathfrak{L}(\lambda) = \lambda I_U - K$ for some compact operator K . Actually, for every $\mathfrak{L} \in \mathcal{C}^{\infty}(\Omega_{\lambda_0}, \Phi_0(U))$, the following properties are satisfied (see [19] for any further details):

- $\chi[\mathfrak{L}, \lambda_0] \in \mathbb{N} \uplus \{+\infty\}$.
- $\chi[\mathfrak{L}, \lambda_0] = 0$ if, and only if, $\mathfrak{L}(\lambda_0) \in GL(U)$.
- $\chi[\mathfrak{L}, \lambda_0] < \infty$ if and only if $\lambda_0 \in \text{Alg}(\mathfrak{L})$.
- If $U = \mathbb{K}^N$, then, in any basis, $\chi[\mathfrak{L}, \lambda_0] = \text{ord}_{\lambda_0} \det \mathfrak{L}(\lambda)$.
- If $K : U \rightarrow U$ is linear and compact, for any eigenvalue, λ_0 , of K , (1.1) holds.

Therefore, χ extends the classical algebraic multiplicity. An equivalent construction of χ beginning with property (d) has been recently carried out by the authors in [22] through the Schur complement.

2.2. The topological parity, σ , and its relation with χ

In this section, we study the topology of $\Phi_0(U)$ via the *parity*, σ , which is a topological invariant of paths introduced by Fitzpatrick, Pejsachowicz and Rabier [11], and collect some relations between σ and χ .

In general, $\Phi_0(U)$ is an open non-linear subset of $\mathcal{L}(U)$. Subsequently, we denote the set of *singular operators* by

$$\mathcal{S}(U) := \Phi_0(U) \setminus GL(U) = \biguplus_{n \in \mathbb{N}} \mathcal{S}_n(U),$$

$$\mathcal{S}_n(U) \equiv \{L \in \Phi_0(U) : \dim \operatorname{Ker}[L] = n\}.$$

According to Fitzpatrick and Pejsachowicz [7], for every $n \in \mathbb{N}$, $\mathcal{S}_n(U)$ is a Banach submanifold of $\Phi_0(U)$ of codimension n^2 , which allows us to view $\mathcal{S}(U)$ as a stratified analytic set of $\Phi_0(U)$. By Theorem 2 of Kuiper [16], the space of isomorphisms, $GL(H)$, of any real or complex separable infinite dimensional Hilbert space, H , is contractible and hence path-connected. Thus, in general, it is not possible to introduce an orientation for operators in $GL(U)$, since $GL(U)$ can be path-connected. By an orientation we mean the choice of a path connected component of the space $GL(U)$ when it contains at least two. This fact reveals a fundamental difference between finite and infinite dimensional spaces, as, for every $N \in \mathbb{N}$, it is folklore that $GL(\mathbb{R}^N)$ consists of two path-connected components, $GL^\pm(\mathbb{R}^N)$. A key technical tool to overcome this shortcoming was provided by the concept of *parity* introduced by Fitzpatrick and Pejsachowicz [10]. The parity is a generalized local detector of the change of orientability of a given *admissible path*, $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$. Although one cannot expect to get a global orientation in $\Phi_0(U)$ when $GL(U)$ is path-connected, one can study the orientability as a local phenomenon through the concept of *parity*.

Subsequently, a Fredholm path $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$ is said to be *admissible* if $\mathfrak{L}(a), \mathfrak{L}(b) \in GL(U)$, and we denote by $\mathcal{C}([a, b], \Phi_0(U))$ the set of admissible paths. Moreover, for every $r \in \mathbb{N} \uplus \{+\infty, \omega\}$, we set

$$\mathcal{C}^r([a, b], \Phi_0(U)) \equiv \mathcal{C}^r([a, b], \Phi_0(U)) \cap \mathcal{C}([a, b], \Phi_0(U)),$$

$$\mathcal{H}([a, b], \Phi_0(U)) \equiv \mathcal{C}^\omega([a, b], \Phi_0(U)).$$

The geometric way to introduce the notion of parity consists in defining it for $\mathcal{C}$ -transversal paths, and then for general admissible curves through the density of $\mathcal{C}$ -transversal paths in $\mathcal{C}([a, b], \Phi_0(U))$, established by Fitzpatrick and Pejsachowicz in [7]. A continuous Fredholm path, $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$, is said to be $\mathcal{C}$ -*transversal* if

- (i) $\mathfrak{L} \in \mathcal{C}^1([a, b], \Phi_0(U))$;
- (ii) $\mathfrak{L}([a, b]) \cap \mathcal{S}(U) \subset \mathcal{S}_1(U)$ and it is finite;
- (iii) $\mathfrak{L}$ is transversal to $\mathcal{S}_1(U)$ at each point of $\mathfrak{L}([a, b]) \cap \mathcal{S}(U)$.

When $\mathfrak{L}$ is $\mathcal{C}$ -transversal, then, the (total) *parity* of $\mathfrak{L}$ in $[a, b]$ is defined by

$$\sigma(\mathfrak{L}, [a, b]) := (-1)^\kappa,$$

where $\kappa \in \mathbb{N}$ equals the cardinal of $\mathfrak{L}([a, b]) \cap \mathcal{S}(U)$. Thus, the parity of a $\mathcal{C}$ -transversal path, $\mathfrak{L}(\lambda)$, is the number of times, mod 2, that $\mathfrak{L}(\lambda)$ intersects transversally the stratified analytic set $\mathcal{S}(U)$. The fact that the set of

$\mathcal{C}$ -transversal paths is dense in the set of admissible paths, $\mathcal{C}([a, b], \Phi_0(U))$, allows us to define the parity for a general $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$ through

$$\sigma(\mathfrak{L}, [a, b]) := \sigma(\tilde{\mathfrak{L}}, [a, b]),$$

where $\tilde{\mathfrak{L}}$ is any $\mathcal{C}$ -transversal curve satisfying $\|\mathfrak{L} - \tilde{\mathfrak{L}}\|_\infty < \varepsilon$ for sufficiently small $\varepsilon > 0$.

Subsequently, an homotopy $H \in \mathcal{C}([0, 1] \times [a, b], \Phi_0(U))$ is said to be *admissible* if $H([0, 1] \times \{a, b\}) \subset GL(U)$, and given two paths, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, they are said to be *$\mathcal{A}$ -homotopic* if they are homotopic through some admissible homotopy. A fundamental property of the parity established by Fitzpatrick and Pejsachowiz [10] is its invariance under admissible homotopies. The next result, which is Theorem 4.5 of [20], shows how the parity of any admissible Fredholm path $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$ can be computed through χ .

Theorem 2.4. *Any continuous admissible path $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$ is $\mathcal{A}$ -homotopic to an admissible analytic Fredholm curve $\mathfrak{L}_\omega \in \mathcal{H}([a, b], \Phi_0(U))$. Moreover,*

$$\sigma(\mathfrak{L}, [a, b]) = (-1)^{\sum_{i=1}^n \chi[\mathfrak{L}_\omega, \lambda_i]}, \quad \text{where } \Sigma(\mathfrak{L}_\omega) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}.$$

For every $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$ and any isolated eigenvalue $\lambda_0 \in \Sigma(\mathfrak{L})$, the *localized parity* of $\mathfrak{L}$ at λ_0 can be defined by $\sigma(\mathfrak{L}, \lambda_0) := \lim_{\eta \downarrow 0} \sigma(\mathfrak{L}, [\lambda_0 - \eta, \lambda_0 + \eta])$. As a consequence of Theorem 2.4, the next result holds (see [20, Cor. 4.6]).

Corollary 2.5. *Assume $\mathfrak{L} \in \mathcal{A}_{\lambda_0}([a, b], \Phi_0(U))$, i.e., $\mathfrak{L} \in \mathcal{C}^r([a, b], \Phi_0(U))$ with $\lambda_0 \in \text{Alg}_\kappa(\mathfrak{L})$ for some integer $r \geq 1$ and $1 \leq \kappa \leq r$. Then, $\sigma(\mathfrak{L}, \lambda_0) = (-1)^{\chi[\mathfrak{L}, \lambda_0]}$.*

Corollary 2.5 establishes a sharp connection between the topological notion of parity, σ , and the algebraic concept of multiplicity, χ .

Next, we will introduce the concept of parity to closed curves, $\sigma(\cdot, \mathbb{S}^1)$, introduced by Fitzpatrick and Pejsachowiz in [8]. By considering the circle $\mathbb{S}^1$ as given from $[a, b]$ through identification of a and b , one can define

$$\sigma(\mathfrak{L}, \mathbb{S}^1) := \deg(\mathfrak{P}(a) \circ \mathfrak{P}(b)^{-1}) \quad \text{for all } \mathfrak{L} \in \mathcal{C}(\mathbb{S}^1, \Phi_0(U)),$$

where $\mathfrak{P} : [a, b] \rightarrow \Phi_0(U)$ is any parametrix of $\mathfrak{L}$ and $\deg$ stands for the Leray–Schauder degree. If $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(U))$ is closed, i.e., $\mathfrak{L}(a) = \mathfrak{L}(b)$, then $\sigma(\mathfrak{L}, [a, b]) = \sigma(\mathfrak{L}, \mathbb{S}^1)$ (see Fitzpatrick and Pejsachowiz [8], where it was also established that this new notion of parity is also homotopy invariant).

When, in addition, U is of Kuiper type (i.e., $GL(U)$ is contractible), then, the fundamental group $\pi_1(\Phi_0(U), T)$ does not depend on the chosen base point $T \in \Phi_0(U)$, i.e., $\pi_1(\Phi_0(U), T) \equiv \pi_1(\Phi_0(U))$, because in this case, $\Phi_0(U)$ is path-connected. Therefore, one can introduce the map

$$\sigma : \pi_1(\Phi_0(U)) \longrightarrow \mathbb{Z}_2, \quad \sigma([\gamma]) := \sigma(\gamma, \mathbb{S}^1), \quad (2.5)$$

which is well defined since it is invariant by homotopy, and defines a group isomorphism. Thus, $\pi_1(\Phi_0(U)) \simeq \mathbb{Z}_2$ if U is of Kuiper type. This makes apparent how the parity map describes some non trivial features of the topology of $\Phi_0(U)$.

We end this section by establishing a link between χ and the notion of parity for closed curves $\sigma(\cdot, \mathbb{S}^1)$. It follows easily combining the proof of [20, Th. 4.5] with Theorem 2.4.

Theorem 2.6. *For every $\mathcal{L} \in \mathcal{C}(\mathbb{S}^1, \Phi_0(U))$, $\sigma(\mathcal{L}, \mathbb{S}^1) = (-1)^{\sum_{i=1}^n \chi[\mathcal{L}_\omega, \lambda_i]}$, where $\mathcal{L}_\omega \in \mathcal{H}(\mathbb{S}^1, \Phi_0(U))$ is homotopic to $\mathcal{L}$ and $\Sigma(\mathcal{L}_\omega) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$.*

3. K-Theory at the light of spectral theory

In this section, which is the central one of this paper, the theory of real vector bundles will be complemented and sharpened by means of the spectral devices reviewed in Sect. 2. Specifically, we will characterize the obstruction detected by the first Stiefel–Whitney class of vector bundles through some spectral properties by means of the Atiyah–Jänich map.

We begin by recalling some basic facts to introduce the Atiyah–Jänich morphism. Two real vector bundles, E and F , over a compact path-connected topological space X , are said to be *stably equivalent* if there are $N, M \in \mathbb{N}$ such that

$$E \oplus \underline{\mathbb{R}}^N \simeq F \oplus \underline{\mathbb{R}}^M,$$

where $\underline{\mathbb{R}}^i$ denotes the trivial bundle $X \times \mathbb{R}^i$ of rank i over X for each $i \in \{N, M\}$, $\oplus$ stands for the Whitney sum of vector bundles, and $\simeq$ expresses that both real vector bundles are isomorphic. Naturally, the stable equivalence induces an equivalence relation in the set of isomorphism classes of real vector bundles over X , denoted by $\text{Vect}(X)$, whose associated quotient is the reduced Grothendieck group, $\tilde{K}\mathcal{O}(X)$. It is a group under the Whitney sum of vector bundles. Given a real Banach space U , the device linking vector bundle theory with the theory of Fredholm operators is the Atiyah–Jänich index map

$$\mathfrak{Jnd} : [X, \Phi_0(U)] \longrightarrow \tilde{K}\mathcal{O}(X) \quad (3.1)$$

introduced by Atiyah [1] and Jänich [15], which is a sort of generalization of the classical notion of index of a Fredholm operator. We are denoting by $[X, \Phi_0(U)]$ the set of homotopy classes of continuous maps $X \rightarrow \Phi_0(U)$. It is a group under the composition $\circ$ defined by

$$(\mathcal{L} \circ \mathfrak{P})(x) := \mathcal{L}(x) \circ \mathfrak{P}(x), \quad x \in X, \quad \mathcal{L}, \mathfrak{P} \in [X, \Phi_0(U)].$$

The map (3.1) is a homomorphism of groups and makes exact the sequence

$$[X, GL(U)] \xrightarrow{i_*} [X, \Phi_0(U)] \xrightarrow{\mathfrak{Jnd}} \tilde{K}\mathcal{O}(X),$$

where i_* is the canonical inclusion, i.e., $[X, GL(U)] = \text{Ker } \mathfrak{Jnd}$. Some reasonably self-contained references for these materials are Mukherjee [24, Ch. 1,2], Cohen [3], Zaidenberg et al. [31] and Husemoller [14].

As in this section we need this morphism to be an isomorphism, we begin by fixing a class of real Banach spaces U for which this property holds. For this, we need to recall some basic concepts. Given a real Banach space, U , a Schauder basis, $\{u_n\}_{n \in \mathbb{N}}$, in U , is said to be unconditional, if for every sequence of real numbers, $\{\alpha_n\}_{n \in \mathbb{N}}$, for which $\sum_{n \in \mathbb{N}} \alpha_n u_n$ converges (say, to x), this series is unconditionally convergent, i.e., regardless the permutation

$\sigma : \mathbb{N} \rightarrow \mathbb{N}$, the series $\sum_{n \in \mathbb{N}} \alpha_{\sigma(n)} u_{\sigma(n)}$ converges to x . A real Banach space U is said to admit a symmetric basis if there exists an unconditional basis $\{u_n\}_{n \in \mathbb{N}}$ on U such that:

- (1) For any two sequences of real numbers, $\{\alpha_n\}_{n \in \mathbb{N}}$, $\{\beta_n\}_{n \in \mathbb{N}}$, such that $|\beta_n| \leq |\alpha_n|$ for all $n \in \mathbb{N}$, the series $\sum_{n \in \mathbb{N}} \beta_n u_n$ converges if $\sum_{n \in \mathbb{N}} \alpha_n u_n$ converges, and

$$\left\| \sum_{n \in \mathbb{N}} \beta_n u_n \right\| \leq \left\| \sum_{n \in \mathbb{N}} \alpha_n u_n \right\|.$$

- (2) If $\{\alpha_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers such that $\sum_{n \in \mathbb{N}} \alpha_n u_n$ converges, then $\sum_{n \in \mathbb{N}} \alpha_n u_{\sigma(n)}$ also converges for every permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$, and

$$\left\| \sum_{n \in \mathbb{N}} \alpha_n u_n \right\| = \left\| \sum_{n \in \mathbb{N}} \alpha_n u_{\sigma(n)} \right\|.$$

A real Banach space, U , is said to contain a complemented infinite dimensional subspace admitting a symmetric basis if it has two closed subspaces, $V, W \subset U$, such that $U = V \oplus W$, $\dim V = +\infty$, and V admits a symmetric basis. The following result is Theorem 2.3 of Zaidenberg et al. [31].

Theorem 3.1. *Let X be a compact path-connected topological space and U be a real Banach space of Kuiper type that contains a complemented infinite dimensional subspace admitting a symmetric basis. Then the index map (3.1) is an isomorphism of groups.*

Throughout the rest of this paper, we will assume that the real Banach space U is *admissible* in the sense that it satisfies the assumptions of Theorem 3.1, i.e., it is of Kuiper type and it contains a complemented infinite dimensional subspace admitting a symmetric basis. Note that any real infinite-dimensional separable Hilbert space is admissible.

3.1. Orientability of Vector Bundles

This section collects some background on orientability, which is invoked in Sect. 3.2 to obtain some of the main findings of this paper.

A real vector bundle $E \rightarrow X$ is said to be orientable if it admits a trivializing atlas whose transition functions have positive determinant. According to, e.g., Husemoller [14, Th. 12.1], E is orientable if, and only if, the first Stiefel–Whitney class of E , $\omega_1(E) \in H^1(X, \mathbb{Z}_2)$, is zero, where $H^1(X, \mathbb{Z}_2)$ is the first cohomology group of X with coefficients in $\mathbb{Z}_2$. It is folklore that a vector bundle E is orientable if, and only if, its associated determinant line bundle, $\det E := \wedge^n E$, is trivial.

Since the first Stiefel–Whitney class, $\omega_1 : \text{Vect}(X) \rightarrow H^1(X, \mathbb{Z}_2)$, depends only on the stable equivalence class, it induces, in a natural way, a homomorphism of groups

$$w_1 : \tilde{KO}(X) \longrightarrow H^1(X, \mathbb{Z}_2).$$

Thus, it is rather natural to agree that the class $E \in \tilde{KO}(X)$ is orientable if $w_1(E) = 0$. In particular, for every continuous map $h : X \rightarrow \Phi_0(U)$, $\mathfrak{Ind}[h] \in \tilde{KO}(X)$ is orientable if $w_1(\mathfrak{Ind}[h]) = 0$.

Now, note that, for every class $E \in \tilde{K}\mathcal{O}(X)$, the first Stiefel–Whitney class can be regarded as a homomorphism $\pi_1(X) \rightarrow \mathbb{Z}_2$. Indeed, as a consequence of the universal coefficient theorem, the groups $H^1(X, \mathbb{Z}_2)$ and $\text{Hom}(\pi_1(X), \mathbb{Z}_2)$ are isomorphic. To describe this isomorphism, following [9, Sect. 2] and denoting $\pi_1(X) \equiv \pi_1(X, x_0)$, each homomorphism $\varphi : \pi_1(X) \rightarrow \mathbb{Z}_2$ sends the commutator of $\pi_1(X)$, $[\pi_1(X), \pi_1(X)]$, to zero $0 \in \mathbb{Z}_2$. Thus, it induces a homomorphism

$$\tilde{\varphi} : \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \simeq H_1(X, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2.$$

By the universal coefficient theorem, each $\tilde{\varphi} : H_1(X, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ corresponds to a unique cohomology class $w \in H^1(X, \mathbb{Z}_2)$. The inverse isomorphism $\Gamma : H^1(X, \mathbb{Z}_2) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ can be described explicitly as follows. If $w \in H^1(X, \mathbb{Z}_2)$, and $\gamma \in \pi_1(X)$ is represented by $g : \mathbb{S}^1 \rightarrow X$, then

$$[\Gamma(w)](\gamma) = \langle g^*(w), [\mathbb{S}^1] \rangle_{\mathbb{Z}_2}, \quad (3.2)$$

where $g^* : H^1(X, \mathbb{Z}_2) \rightarrow H^1(\mathbb{S}^1, \mathbb{Z}_2)$ is the induced morphism in cohomology, $[\mathbb{S}^1]$ is the generator of $H^1(\mathbb{S}^1, \mathbb{Z}_2)$, and $\langle \cdot, \cdot \rangle_{\mathbb{Z}_2} : H_1(\cdot, \mathbb{Z}_2) \times H^1(\cdot, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is the Kronecker (duality) pairing.

For every $E \in \mathfrak{Ind}([X, \Phi_0(U)]) = \tilde{K}\mathcal{O}(X)$, one can also describe the orientability of E in terms of its determinant bundle. Indeed, according to Wang [29], for every $h \in [X, \Phi_0(U)]$, the determinant bundle of $\mathfrak{Ind}[h]$ can be defined as the line bundle

$$\det \mathfrak{Ind}[h] := \wedge^{\max} \text{Ker } h \otimes (\wedge^{\max} \text{coKer } h)^*$$

where $\wedge^{\max}$ denotes the wedge product in the corresponding dimension of the vector space where we are defining the operation. It turns out that these two notions of orientability are actually the same. This is the content of the following result of Pejsachowicz [25].

Theorem 3.2. *Let U be an admissible real Banach space, X a compact path-connected space, and $h : X \rightarrow \Phi_0(U)$ a continuous map. Then, the next statements are equivalent*

- (a) $w_1(\mathfrak{Ind}[h]) = 0$ in $H^1(X, \mathbb{Z}_2)$.
- (b) $\det \mathfrak{Ind}[h]$ is a trivial line bundle.

The equivalence of these two (and some other) notions of orientability is far from evident, and it has been one of the central issues in the theory of the topological degree for Fredholm operators (see, e.g., [10–12, 26], and the references therein).

3.2. Intersection Morphism

In this section we study the obstruction detected by the first Stiefel–Whitney fundamental class and establish its relationship with the generalized algebraic multiplicity χ introduced in Sect. 2.1. Essentially, for any given vector bundle E over X , we will ascertain some of its most significative topological properties through the techniques introduced in Sect. 2 in the classifying space $\Phi_0(U)$. These findings count among the main novelties of this paper.

For any given vector bundle $E \rightarrow X$, its pre-image via the index map, $\mathcal{I}nd^{-1}[E] : X \rightarrow \Phi_0(U)$, is a parameterized family of Fredholm operators of index zero. The main goal of this section is to reformulate some of the main topological invariants of E in terms of the algebraic data provided by χ and σ (see Sect. 2 for the precise definitions). Adopting this methodology, the first Stiefel–Whitney class, ω_1 , is going to be related to the concept of algebraic multiplicity, χ .

Subsequently, for notational convenience, for every $\mathfrak{L} \in \mathcal{C}(\mathbb{S}^1, \Phi_0(U))$, we will denote

$$\chi_2[\mathfrak{L}, \mathbb{S}^1] := (-1)^{\chi_T} \quad \text{with} \quad \chi_T \equiv \sum_{\lambda_0 \in \Sigma(\mathfrak{L}^\omega)} \chi[\mathfrak{L}^\omega, \lambda_0],$$

where $\mathfrak{L}^\omega \in \mathcal{H}(\mathbb{S}^1, \Phi_0(U))$ is any admissible analytic curve homotopic to $\mathfrak{L}$.

To relate the topological properties to the spectral ones, we introduce the intersection morphism

$$\mathcal{I} : \tilde{K}\mathcal{O}(X) \longrightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2),$$

defined, for every $E \in \tilde{K}\mathcal{O}(X)$ and $[\gamma] \in \pi_1(X)$, by

$$\mathcal{I}[E]([\gamma]) := \chi_2[\mathcal{I}nd^{-1}[E] \circ \gamma, \mathbb{S}^1]. \quad (3.3)$$

Its name is motivated by the fact that χ equals the intersection index of two certain algebraic varieties (see [22]). To prove that $\mathcal{I}$ is well defined, we have to check that $\mathcal{I}[E] : \pi_1(X) \rightarrow \mathbb{Z}_2$ is a homomorphism of groups.

Lemma 3.3. *For every $E \in \tilde{K}\mathcal{O}(X)$, the map $\mathcal{I}[E] : \pi_1(X) \rightarrow \mathbb{Z}_2$ is a group homomorphism.*

Proof. Fix $x_0 \in X$ and let $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$. Then, its product on $\pi_1(X, x_0)$ is given by the loop $[\gamma_1 * \gamma_2] \in \pi_1(X, x_0)$ defined through

$$\gamma_1 * \gamma_2 : \mathbb{S}^1 \rightarrow X, \quad (\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

where we are viewing $\mathbb{S}^1$ as the interval $[0, 1]$ with endpoints identified. Then, it becomes apparent that

$$\mathcal{I}nd^{-1}[E] \circ (\gamma_1 * \gamma_2) = (\mathcal{I}nd^{-1}[E] \circ \gamma_1) * (\mathcal{I}nd^{-1}[E] \circ \gamma_2).$$

By Theorem 2.6, we get

$$\mathcal{I}[E]([\gamma_1 * \gamma_2]) = \chi_2[\mathcal{I}nd^{-1}[E] \circ (\gamma_1 * \gamma_2), \mathbb{S}^1] = \sigma(\mathcal{I}nd^{-1}[E] \circ (\gamma_1 * \gamma_2), [0, 1]).$$

Finally, by using the additivity property of the parity [10, Th. 6.6] applied to the partition $[0, 1] = [0, 1/2] \cup [1/2, 1]$, we obtain that

$$\begin{aligned} \mathcal{I}[E]([\gamma_1 * \gamma_2]) &= \sigma(\mathcal{I}nd^{-1}[E] \circ (\gamma_1 * \gamma_2), [0, 1]) \\ &= \sigma((\mathcal{I}nd^{-1}[E] \circ \gamma_1) * (\mathcal{I}nd^{-1}[E] \circ \gamma_2), [0, 1]) \\ &= \sigma(\mathcal{I}nd^{-1}[E] \circ \gamma_1, [0, 1/2]) \cdot \sigma(\mathcal{I}nd^{-1}[E] \circ \gamma_2, [1/2, 1]) \\ &= \chi_2[\mathcal{I}nd^{-1}[E] \circ \gamma_1, \mathbb{S}^1] \cdot \chi_2[\mathcal{I}nd^{-1}[E] \circ \gamma_2, \mathbb{S}^1] \\ &= \mathcal{I}[E]([\gamma_1]) \cdot \mathcal{I}[E]([\gamma_2]). \end{aligned}$$

This ends the proof. $\square$

Lemma 3.4. *The intersection morphism $\mathfrak{J} : \tilde{K}\mathcal{O}(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ defines a homomorphism of groups. Thus, it defines a morphism in the category of groups $\mathfrak{Gr}$.*

Proof. Since $\mathfrak{J}\mathfrak{nd}$ is an isomorphism of groups, it is apparent that

$$\mathfrak{J}\mathfrak{nd}^{-1} : (\tilde{K}\mathcal{O}(M), \oplus) \rightarrow ([M, \Phi_0(U)], \circ)$$

establishes a homomorphism of groups. Thus, for every $E, F \in \tilde{K}\mathcal{O}(X)$ and $[\gamma] \in \pi_1(X)$,

$$\chi_2[\mathfrak{J}\mathfrak{nd}^{-1}([E] \oplus [F]) \circ \gamma, \mathbb{S}^1] = \chi_2[(\mathfrak{J}\mathfrak{nd}^{-1}[E] \circ \mathfrak{J}\mathfrak{nd}^{-1}[F]) \circ \gamma, \mathbb{S}^1].$$

By the definition of χ_2 and Theorem 2.3(PF), we find that

$$\begin{aligned} \chi_2[(\mathfrak{J}\mathfrak{nd}^{-1}[E] \circ \gamma) \circ (\mathfrak{J}\mathfrak{nd}^{-1}[F] \circ \gamma), \mathbb{S}^1] \\ = \chi_2[\mathfrak{J}\mathfrak{nd}^{-1}[E] \circ \gamma, \mathbb{S}^1] \cdot \chi_2[\mathfrak{J}\mathfrak{nd}^{-1}[F] \circ \gamma, \mathbb{S}^1]. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathfrak{J}[E \oplus F]([\gamma]) &= \chi_2[\mathfrak{J}\mathfrak{nd}^{-1}([E] \oplus [F]) \circ \gamma, \mathbb{S}^1] \\ &= \chi_2[\mathfrak{J}\mathfrak{nd}^{-1}[E] \circ \gamma, \mathbb{S}^1] \cdot \chi_2[\mathfrak{J}\mathfrak{nd}^{-1}[F] \circ \gamma, \mathbb{S}^1] \\ &= \mathfrak{J}[E]([\gamma]) \cdot \mathfrak{J}[F]([\gamma]). \end{aligned}$$

This shows that $\mathfrak{J}[E \oplus F] = \mathfrak{J}[E] \cdot \mathfrak{J}[F]$ and ends the proof. $\square$

The next result establishes that $\mathfrak{J}$ equals the first Stiefel–Whitney class morphism $w_1 : \tilde{K}\mathcal{O}(X) \rightarrow H^1(X, \mathbb{Z}_2)$ modulo the isomorphism $\Gamma : H^1(X, \mathbb{Z}_2) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ defined in (3.2). Since it fully describes the obstruction of the first Stiefel–Whitney class by means of χ , it is one of the main findings of this paper.

Theorem 3.5. *Let U be an admissible real Banach space, and X a compact path-connected space. Then, $\mathfrak{J} = \Gamma \circ w_1$ as morphisms $\tilde{K}\mathcal{O}(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ of $\mathfrak{Gr}$, where Γ is the isomorphism between $H^1(X, \mathbb{Z}_2)$ and $\text{Hom}(\pi_1(X), \mathbb{Z}_2)$ defined by (3.2).*

Proof. By Fitzpatrick and Pejsachowicz [9, Pr. 2.7], the Stiefel–Whitney morphism w_1 can be factorized through the following diagram in $\mathfrak{Gr}$,

$$\begin{array}{ccc} \tilde{K}\mathcal{O}(X) & \xrightarrow{w_1} & H^1(X, \mathbb{Z}_2) \\ \mathfrak{J}\mathfrak{nd} \uparrow & & \downarrow \Gamma \\ [X, \Phi_0(U)] & \xrightarrow{\tilde{\sigma}} & \text{Hom}(\pi_1(X), \mathbb{Z}_2) \end{array}$$

where $\tilde{\sigma} : [X, \Phi_0(U)] \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ is the morphism defined by

$$\tilde{\sigma}[h]([\gamma]) := \sigma(h \circ \gamma, \mathbb{S}^1) \quad \text{for all } [\gamma] \in \pi_1(X).$$

To show the commutativity of the diagram, it suffices to prove that

$$\Gamma(w_1(\mathfrak{J}\mathfrak{nd}[h])) = \tilde{\sigma}[h] \quad \text{for all } h \in [X, \Phi_0(U)], \quad (3.4)$$

viewed as homomorphisms $\pi_1(X) \rightarrow \mathbb{Z}_2$. Since they are $\mathbb{Z}_2$ -valued group homomorphisms, (3.4) holds from the fact that their corresponding kernels are equal. As

$$w_1 : \tilde{K}\mathcal{O}(\cdot) \rightarrow H^1(\cdot, \mathbb{Z}_2), \quad \mathfrak{I}\mathfrak{n}\mathfrak{d} : [\cdot, \Phi_0(U)] \rightarrow \tilde{K}\mathcal{O}(\cdot),$$

are natural transformations in the category $\mathfrak{Top}$ of topological spaces and continuous maps, it follows that, for every continuous map $g : \mathbb{S}^1 \rightarrow X$,

$$g^*(w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h])) = w_1(g!(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h])) = w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h \circ g]), \quad (3.5)$$

where $g^* : H^1(X, \mathbb{Z}_2) \rightarrow H^1(\mathbb{S}^1, \mathbb{Z}_2)$ and $g! : \tilde{K}\mathcal{O}(X) \rightarrow \tilde{K}\mathcal{O}(\mathbb{S}^1)$ are the induced morphisms by the cohomology functor $H^1(\cdot, \mathbb{Z}_2)$ and the K -theory functor $\tilde{K}\mathcal{O}(\cdot)$, respectively. Pick a loop $[\gamma] \in \text{Ker}[\Gamma(w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h]))] \subset \pi_1(X)$. Then, for any representation of the loop $[\gamma]$, $g : \mathbb{S}^1 \rightarrow X$, we can deduce from (3.2) and (3.5) that

$$0 = [\Gamma(w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h]))](\gamma) = \langle g^*(w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h])), [\mathbb{S}^1] \rangle_{\mathbb{Z}_2} = \langle w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h \circ g]), [\mathbb{S}^1] \rangle_{\mathbb{Z}_2}.$$

Thus, since $H_1(\mathbb{S}^1, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \simeq H^1(\mathbb{S}^1, \mathbb{Z}_2)$, it becomes apparent that $\langle w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h \circ g]), [\mathbb{S}^1] \rangle_{\mathbb{Z}_2} = 0$ in $\mathbb{Z}_2$ if, and only if, $w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h \circ g]) = 0$ in $H^1(\mathbb{S}^1, \mathbb{Z}_2)$. On the other hand, since

$$\tilde{K}\mathcal{O}(\mathbb{S}^1) = \{[T\mathbb{S}^1], [\mathcal{M}]\} \simeq \mathbb{Z}_2,$$

where $T\mathbb{S}^1$ is the tangent bundle of $\mathbb{S}^1$ and $\mathcal{M}$ is the Möbius bundle, taking into account that $\mathcal{M}$ is not orientable, it follows that $\mathbb{S}^1$ is orientable if, and only if, is trivial. This implies that $w_1 : \tilde{K}\mathcal{O}(\mathbb{S}^1) \rightarrow H^1(\mathbb{S}^1, \mathbb{Z}_2)$ is an isomorphism. Consequently, $w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h \circ g]) = 0$ in $H^1(\mathbb{S}^1, \mathbb{Z}_2)$ if, and only if, $\mathfrak{I}\mathfrak{n}\mathfrak{d}[h \circ g]$ is the identity on $\tilde{K}\mathcal{O}(\mathbb{S}^1)$. Since $[\mathbb{S}^1, GL(U)] = \text{Ker } \mathfrak{I}\mathfrak{n}\mathfrak{d}$, necessarily $h \circ g \in [\mathbb{S}^1, GL(U)]$. Therefore, $\sigma(h \circ g, \mathbb{S}^1) = 1$, i.e.,

$$\tilde{\sigma}[h](\gamma) = \sigma(h \circ g, \mathbb{S}^1) = 1,$$

and hence, $[\gamma] \in \text{Ker}[\tilde{\sigma}[h]]$. The proof of the converse inclusion follows identical patterns. Consequently,

$$\text{Ker}[\Gamma(w_1(\mathfrak{I}\mathfrak{n}\mathfrak{d}[h]))] = \text{Ker}[\tilde{\sigma}[h]] \quad \text{for all } h \in [X, \Phi_0(U)],$$

which implies (3.4) and the commutativity of the diagram. Thus, $\Gamma \circ w_1 = \tilde{\sigma} \circ \mathfrak{I}\mathfrak{n}\mathfrak{d}^{-1}$. On the other hand, $\tilde{\sigma} \circ \mathfrak{I}\mathfrak{n}\mathfrak{d}^{-1} = \mathfrak{I}$. Indeed, for every $E \in \tilde{K}\mathcal{O}(X)$ and $[\gamma] \in \pi_1(X)$, it follows from Theorem 2.6 that

$$\mathfrak{I}[E](\gamma) = \chi_2[\mathfrak{I}\mathfrak{n}\mathfrak{d}^{-1}[E] \circ \gamma, \mathbb{S}^1] = \sigma(\mathfrak{I}\mathfrak{n}\mathfrak{d}^{-1}[E] \circ \gamma, \mathbb{S}^1) = \tilde{\sigma}[\mathfrak{I}\mathfrak{n}\mathfrak{d}^{-1}[E]](\gamma).$$

Therefore, $\mathfrak{I} = \Gamma \circ w_1$. This ends the proof. $\square$

As a consequence of the proof of Theorem 3.5, it is apparent that the intersection morphism $\mathfrak{I} : \tilde{K}\mathcal{O}(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ factorizes the diagram in $\mathfrak{Gr}$,

$$\begin{array}{ccc} \tilde{K}\mathcal{O}(X) & \xrightarrow{w_1} & H^1(X, \mathbb{Z}_2) \\ \mathfrak{I}\mathfrak{n}\mathfrak{d} \uparrow & \searrow \mathfrak{I} & \downarrow \Gamma \\ [X, \Phi_0(U)] & \xrightarrow{\tilde{\sigma}} & \text{Hom}(\pi_1(X), \mathbb{Z}_2) \end{array}$$

because

$$\tilde{\sigma} \circ \mathfrak{J}nd^{-1} = \mathfrak{J} = \Gamma \circ w_1.$$

Since $\mathfrak{J}$ describes the class w_1 in terms of the algebraic multiplicity, χ , it establishes a link between the topological information of the vector bundle E and the spectral properties of the underlying Fredholm paths.

Moreover, since $\mathfrak{J} = \Gamma \circ w_1$, $\mathfrak{J}$ defines a topological invariant of stable equivalence classes of vector bundles. In other words, $E \neq F$ in $\tilde{KO}(X)$ for any pair of vector bundles E, F over X with $\mathfrak{J}[E] \neq \mathfrak{J}[F]$. So, the next result holds.

Corollary 3.6. *The intersection morphism $\mathfrak{J} : \tilde{KO}(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ is a topological invariant of stable equivalence classes of real vector bundles with base X .*

Thanks to Theorems 3.2 and 3.5, the orientability of a given vector bundle $E \rightarrow X$ can be characterized in terms of χ . Precisely, the next result holds.

Corollary 3.7. *Let $E \rightarrow X$ be a real vector bundle over X . Then, the following conditions are equivalent:*

- E is orientable,
- $\chi_2[\mathfrak{J}nd^{-1}[E] \circ \gamma, \mathbb{S}^1] = 1$ for all $[\gamma] \in \pi_1(X)$,
- $\mathfrak{J}[E] \equiv 1$, where 1 stands for the identity in $\text{Hom}(\pi_1(X), \mathbb{Z}_2)$.

Proof. By definition, the vector bundle $E \rightarrow X$ is orientable if, and only if, $w_1(E) = 0$ in $H^1(X, \mathbb{Z}_2)$. Thus, since, due to Theorem 3.5, $\mathfrak{J} = \Gamma \circ w_1$, and Γ is an isomorphism, it is apparent that E is orientable if, and only if, $\mathfrak{J}[E] \equiv 1$. $\square$

The interest of these findings relies on the crucial fact that, although the computation of $w_1[E]$ is difficult in practice, the intersection morphism $\mathfrak{J}$ can be easily computed in many particular examples, as it will become apparent in the next sections. Another relevant consequence of these findings is the fact that the real line bundles can be completely described through their spectral properties. Precisely, the next result holds. Subsequently, we denote by $\text{Vect}_1(X)$ the set of isomorphism classes of line bundles over X . In particular $\text{Vect}_1(X)$ is a group with the tensor product $\otimes$ of line bundles.

Theorem 3.8. *The restricted intersection morphism $\mathfrak{J} : \text{Vect}_1(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ is an isomorphism, i.e., two line bundles $L, L' \in \text{Vect}_1(X)$ are isomorphic if, and only if,*

$$\chi_2[\mathfrak{J}nd^{-1}[L] \circ \gamma, \mathbb{S}^1] = \chi_2[\mathfrak{J}nd^{-1}[L'] \circ \gamma, \mathbb{S}^1] \quad \text{for all } [\gamma] \in \pi_1(X).$$

Proof. First, we show that $\mathfrak{J}$ is well defined. We should see that the isomorphism classes of the line bundles coincide with the stable equivalence ones. If $L, L' \in \text{Vect}_1(X)$ are isomorphic, they are clearly stably isomorphic. Assume that $L, L' \in \text{Vect}_1(X)$ are stably isomorphic. Then, by the properties of the Stiefel–Whitney class, $\omega_1(L) = \omega_1(L')$. Since $\omega_1 : \text{Vect}_1(X) \rightarrow H^1(X, \mathbb{Z}_2)$ is an isomorphism, necessarily $\omega_1(L) = \omega_1(L')$ implies that the line bundles

L, L' are isomorphic. This proves our claim. On the other hand, we already know that $\mathfrak{J} = \Gamma \circ \omega_1$ as maps $\text{Vect}_1(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$. Thus, since Γ and $\omega_1 : \text{Vect}_1(X) \rightarrow H^1(X, \mathbb{Z}_2)$ are isomorphisms, it follows that $\mathfrak{J}$ is also an isomorphism. This ends the proof. $\square$

As illustrated by the next result, in some circumstances the intersection morphism $\mathfrak{J}$ can classify also every stable equivalence class of real vector bundles.

Theorem 3.9. *If all orientable vector bundles are stably trivial, i.e., all orientable maps $X \rightarrow \Phi_0(U)$ are homotopic, then the intersection morphism $\mathfrak{J} : \tilde{K}\mathcal{O}(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ is an isomorphism of groups. Thus, $\mathfrak{J}$ classifies all stable equivalence classes of vector bundles, i.e., $E, F \in \text{Vect}(X)$ are stably equivalent if, and only if,*

$$\chi_2[\mathfrak{J}\mathfrak{nd}^{-1}[E] \circ \gamma, \mathbb{S}^1] = \chi_2[\mathfrak{J}\mathfrak{nd}^{-1}[F] \circ \gamma, \mathbb{S}^1] \quad \text{for all } [\gamma] \in \pi_1(X).$$

Proof. Since any orientable vector bundle is stably trivial, $\text{Ker}[w_1]$ is the identity of $\tilde{K}\mathcal{O}(X)$. Thus, $w_1 : \tilde{K}\mathcal{O}(X) \rightarrow H^1(X, \mathbb{Z}_2)$ is injective. It is surjective because $\omega_1 : \text{Vect}_1(X) \rightarrow H^1(X, \mathbb{Z}_2)$ is an isomorphism. So, w_1 is an isomorphism. Therefore, $\mathfrak{J} = \Gamma \circ w_1$. This ends the proof. $\square$

Roughly spoken, the Stiefel–Whitney class, or, equivalently, the intersection morphism

$$\mathfrak{J}[E] \equiv (\Gamma \circ w_1)[E] : \pi_1(X) \rightarrow \mathbb{Z}_2,$$

measures how the vector bundle $E \rightarrow X$ twists along a given loop of the base space $[\gamma] \in \pi_1(X)$. In the following section, we will introduce a new topological invariant of vector bundles that will encode all these values, $\mathfrak{J}[E] : \pi_1(X) \rightarrow \mathbb{Z}_2$, through a generalized analogue of the arithmetical mean, giving rise to a sort of global measure of the torsion of E . This invariant is far more comfortable to work with than with w_1 , because it deals with real numbers, instead of cohomology classes.

4. The global torsion invariant

This section introduces a new topological invariant of stable equivalence classes of real vector bundles that encodes the information given by the first Stiefel–Whitney class, w_1 . Besides characterizing the orientability of a vector bundle, w_1 also gives some useful information on non orientable bundles which can be used to classify them. This information is actually encoded in the values of the map $\mathfrak{J}[E] : \pi_1(X) \rightarrow \mathbb{Z}_2$, where $\mathfrak{J}$ is the intersection morphism constructed in Sect. 3.2. Essentially, the basic idea consists in summing up the values of this map.

For any given closed smooth manifold M , there exists a Riemannian metric, g , defined on M for which (M, g) becomes a Riemannian manifold. In the sequel, we fix this metric g and a base-point $\mathbf{x} \in M$. Then, the *global torsion invariant* $\Lambda : \tilde{K}\mathcal{O}(M) \rightarrow [-1, 1]$ can be defined by

$$\Lambda(E) := \int_{\mathcal{L}_{\mathbf{x}}(M)} \mathfrak{J}[E]([\gamma]) \, d\mu_{\mathbf{x}}(\gamma) \quad \text{for all } E \in \tilde{K}\mathcal{O}(M), \quad (4.1)$$

where $\mathcal{L}_{\mathbf{x}}(M)$ stands for the loop space of M , i.e., the space of continuous loops $\gamma : \mathbb{S}^1 \rightarrow M$ with base-point $\mathbf{x}$, i.e., $\gamma(0) = \mathbf{x}$, and $\mu_{\mathbf{x}} : \mathcal{B}_{\mathbf{x}} \rightarrow [0, +\infty]$ is the normalised Wiener measure on $\mathcal{L}_{\mathbf{x}}(M)$; $\mathcal{B}_{\mathbf{x}}$ denotes the Borel σ -algebra of $\mathcal{L}_{\mathbf{x}}(M)$ under the topology of the uniform convergence. In Appendix A we have recalled the definition of the measure $\mu_{\mathbf{x}}$.

The real number Λ packages the information provided by the class w_1 in a robust and compact way. Thanks to Theorem 3.5, Λ is easily computable in a number of cases. The next result expresses the global torsion invariant Λ in terms of the algebraic multiplicity χ introduced in Sect. 2. It is very useful for computational purposes.

Theorem 4.1. *For every $E \in \tilde{K}\mathcal{O}(M)$, $\mathfrak{I}[E](\cdot) \in L^1(\mathcal{L}_{\mathbf{x}}(M), \mu_{\mathbf{x}})$ and*

$$\Lambda(E) = \sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]). \quad (4.2)$$

Proof. Since M is a real topological manifold, by Lee [17, Th. 1.16], $\pi_1(M)$ is countable. Thus, there exists a sequence of loops, $\eta_n : \mathbb{S}^1 \rightarrow M$, $n \in \mathbb{Z}$, possibly finite, such that

$$\pi_1(M) = \bigsqcup_{n \in \mathbb{Z}} [\eta_n],$$

where $\sqcup$ denotes the disjoint union. Since $\mathfrak{I}[E]$ is a map $\pi_1(M) \rightarrow \mathbb{Z}_2$, it is constant on each homotopy class. Thus, for every $\gamma \in \mathcal{L}_{\mathbf{x}}(M)$,

$$\mathfrak{I}[E](\gamma) = \sum_{n \in \mathbb{Z}} \mathfrak{I}[E](\eta_n) \cdot \mathbf{1}_{[\eta_n]}(\gamma),$$

where $\mathbf{1}_{[\eta_n]}(\gamma) = 1$ if $\gamma \in [\eta_n]$, and $\mathbf{1}_{[\eta_n]}(\gamma) = 0$ if not. Thus, by the definition of the intersection morphism (see (3.3), if necessary),

$$\mathfrak{I}[E](\gamma) = \sum_{n \in \mathbb{Z}} \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta_n, \mathbb{S}^1] \cdot \mathbf{1}_{[\eta_n]}(\gamma).$$

As $\mathfrak{I}[E](\cdot)$ is the pointwise limit of the simple functions defined by

$$f_m(\gamma) := \sum_{n=-m}^m \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta_n, \mathbb{S}^1] \cdot \mathbf{1}_{[\eta_n]}(\gamma), \quad m \geq 1,$$

and $[\eta_n] \in \mathcal{B}_{\mathbf{x}}$, the function $\mathfrak{I}[E](\cdot) : \mathcal{L}_{\mathbf{x}}(M) \rightarrow \mathbb{R}$ is measurable. Moreover, since $|\mathfrak{I}[E](\gamma)| = 1$ for all $\gamma \in \mathcal{L}_{\mathbf{x}}(M)$, we have that

$$\int_{\mathcal{L}_{\mathbf{x}}(M)} |\mathfrak{I}[E](\gamma)| \, d\mu_{\mathbf{x}}(\gamma) = 1$$

and hence $\mathfrak{I}[E](\cdot) \in L^1(\mathcal{L}_{\mathbf{x}}(M), \mu_{\mathbf{x}})$. Finally, by the dominated convergence theorem, we find from (4.1) that

$$\begin{aligned} \Lambda(E) &= \int_{\mathcal{L}_{\mathbf{x}}(M)} \mathfrak{I}[E](\gamma) \, d\mu_{\mathbf{x}}(\gamma) = \lim_{m \rightarrow \infty} \int_{\mathcal{L}_{\mathbf{x}}(M)} f_m(\gamma) \, d\mu_{\mathbf{x}}(\gamma) \\ &= \lim_{m \rightarrow \infty} \sum_{n=-m}^m \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta_n, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta_n]) \end{aligned}$$

$$= \sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{Ind}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]).$$

This shows (4.2) and ends the proof. $\square$

The next result characterizes the orientability of the vector bundles in terms of Λ .

Theorem 4.2. *A vector bundle $E \rightarrow M$ is orientable if, and only if, $\Lambda(E) = 1$.*

Proof. Suppose E is orientable. Then, by Corollary 3.7, $\mathfrak{J}[E] \equiv 1$. Thus, $\mathfrak{J}[E](\gamma) = 1 \in \mathbb{Z}_2$ for all $[\gamma] \in \pi_1(M)$. Hence, by (3.3),

$$\chi_2[\mathfrak{Ind}^{-1}[E] \circ \gamma, \mathbb{S}^1] = 1 \quad \text{for all } [\gamma] \in \pi_1(M).$$

Therefore, (4.2) implies that

$$\Lambda(E) \equiv \int_{\mathcal{L}_{\mathbf{x}}(M)} \mathfrak{J}[E](\gamma) \, d\mu_{\mathbf{x}}(\gamma) = \sum_{[\eta] \in \pi_1(M)} \mu_{\mathbf{x}}([\eta]) = \mu_{\mathbf{x}}(\mathcal{L}_{\mathbf{x}}(M)) = 1,$$

since $\mu_{\mathbf{x}}$ is a probability measure.

Conversely, suppose $\Lambda(E) = 1$. Then, by Theorem 2.4,

$$\sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{Ind}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]) = 1. \quad (4.3)$$

Subsequently, we consider the following subsets of $\pi_1(M)$:

$$\begin{aligned} \mathcal{D} &:= \{[\eta] \in \pi_1(M) : \chi_2[\mathfrak{Ind}^{-1}[E] \circ \eta, \mathbb{S}^1] = 1\}, \\ \mathcal{N} &:= \{[\eta] \in \pi_1(M) : \chi_2[\mathfrak{Ind}^{-1}[E] \circ \eta, \mathbb{S}^1] = -1\}. \end{aligned}$$

According to (4.3), we have that

$$1 = \sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{Ind}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]) = \sum_{[\eta] \in \mathcal{D}} \mu_{\mathbf{x}}([\eta]) - \sum_{[\eta] \in \mathcal{N}} \mu_{\mathbf{x}}([\eta]).$$

On the other hand,

$$1 = \sum_{[\eta] \in \pi_1(M)} \mu_{\mathbf{x}}([\eta]) = \sum_{[\eta] \in \mathcal{D}} \mu_{\mathbf{x}}([\eta]) + \sum_{[\eta] \in \mathcal{N}} \mu_{\mathbf{x}}([\eta]).$$

Thus, by subtracting the last two identities, we find that

$$\sum_{[\eta] \in \mathcal{N}} \mu_{\mathbf{x}}([\eta]) = 0.$$

Since every path-connected component of $\mathcal{L}_{\mathbf{x}}(M)$ has a positive Wiener measure, we have that $\mu_{\mathbf{x}}([\eta]) > 0$ for all $[\eta] \in \pi_1(M)$ and hence, $\mathcal{N} = \emptyset$. Consequently,

$$\chi_2[\mathfrak{Ind}^{-1}[E] \circ \eta, \mathbb{S}^1] = 1 \quad \text{for all } [\eta] \in \pi_1(M).$$

So, by (3.3), $\mathfrak{J}[E] \equiv 1$. Therefore, by Corollary 3.7, E is orientable. This ends the proof. $\square$

The detailed proof that every path-connected component of $\mathcal{L}_{\mathbf{x}}(M)$ has positive Wiener measure will be given on the last two lines before the statement of Theorem 4.5 in Sect. 4.1 below, where some important features of the Wiener measure used in this section are collected.

According to Theorem 4.2, Λ is far from appropriate for comparing orientable vector bundles. However, Λ is extremely useful to measure the degree of non-orientability, as it will become apparent later.

We end this section by establishing that Λ is a topological invariant of stable equivalent classes of real vector bundles.

Theorem 4.3. $\Lambda(E) = \Lambda(F)$ if $E = F$ in $\tilde{K}\mathcal{O}(M)$, i.e., the map $\Lambda : \tilde{K}\mathcal{O}(M) \rightarrow [-1, 1]$ is a topological invariant of stable equivalence classes of real vector bundles over M .

Proof. Suppose that $E = F$ in $\tilde{K}\mathcal{O}(M)$. Then, by Corollary 3.6, $\mathfrak{I}[E] = \mathfrak{I}[F]$, because $\mathfrak{I}$ is a topological invariant of real vector bundles. Thus, thanks to (3.3), we have that

$$\chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta, \mathbb{S}^1] = \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[F] \circ \eta, \mathbb{S}^1] \quad \text{for all } [\eta] \in \pi_1(M).$$

Consequently,

$$\sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]) = \sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[F] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]).$$

Therefore, by Theorem 4.2, $\Lambda(E) = \Lambda(F)$. This ends the proof. $\square$

We end this section with the next additive formulae for Λ .

Proposition 4.4. For every $E, F \in \tilde{K}\mathcal{O}(M)$,

$$\Lambda([E] \oplus [F]) = \sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[F] \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]).$$

Proof. By definition (see (4.2)),

$$\Lambda([E] \oplus [F]) = \sum_{[\eta] \in \pi_1(M)} \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}([E] \oplus [F]) \circ \eta, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]).$$

Since $\mathfrak{I}\mathfrak{nd}^{-1} : (\tilde{K}\mathcal{O}(M), \oplus) \rightarrow ([M, \Phi_0(U)], \circ)$ establishes a homomorphism of groups,

$$\chi_2[\mathfrak{I}\mathfrak{nd}^{-1}([E] \oplus [F]) \circ \eta, \mathbb{S}^1] = \chi_2[(\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \mathfrak{I}\mathfrak{nd}^{-1}[F]) \circ \eta, \mathbb{S}^1].$$

Hence,

$$\Lambda([E] \oplus [F]) = \sum_{[\eta] \in \pi_1(M)} \chi_2[(\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta) \circ (\mathfrak{I}\mathfrak{nd}^{-1}[F] \circ \eta), \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\eta]).$$

Therefore, by the definition of χ_2 at the beginning of Sect. 3.2 and the product formula of the χ (see Theorem 2.3(PF)), we find that

$$\chi_2[(\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta) \circ (\mathfrak{I}\mathfrak{nd}^{-1}[F] \circ \eta), \mathbb{S}^1] = \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[E] \circ \eta, \mathbb{S}^1] \cdot \chi_2[\mathfrak{I}\mathfrak{nd}^{-1}[F] \circ \eta, \mathbb{S}^1].$$

This concludes the proof. $\square$

4.1. Decomposition of the Wiener measure

In this section, we will reduce the calculation of the global torsion invariant, Λ , to the determination of the heat kernel of the universal covering of M , which is far more easy to compute than the one of M . Note that the loop space of M , $\mathcal{L}_{\mathbf{x}}(M)$, can be expressed as the union of its path-connected components

$$\mathcal{L}_{\mathbf{x}}(M) = \bigsqcup_{\eta \in \pi_1(M, \mathbf{x})} [\eta], \quad (4.4)$$

where $[\eta]$ denotes the homotopy class of η . Let $\tilde{M}$ be the universal covering space of M with covering projection $\pi : \tilde{M} \rightarrow M$, and endow $\tilde{M}$ with the Riemannian structure given by the pull-back metric $\tilde{g} := \pi^*g$. Then, $\pi : (\tilde{M}, \tilde{g}) \rightarrow (M, g)$ is a regular Riemannian covering. According to, e.g., [27, Cor. 4, Sect. 6, Ch. 2], the fundamental group of M based on $\mathbf{x}$, $\pi_1(M, \mathbf{x})$, is isomorphic to the group of deck, or covering, transformations of the covering $\pi : \tilde{M} \rightarrow M$, subsequently denoted by $\text{Aut}_M \tilde{M}$. Actually, once chosen $\tilde{\mathbf{x}} \in \pi^{-1}(\mathbf{x})$, the isomorphism can be defined through

$$\Phi : \pi_1(M, \mathbf{x}) \longrightarrow \text{Aut}_M \tilde{M}, \quad [\eta] \mapsto \varphi_\eta,$$

where $\varphi_\eta : \tilde{M} \rightarrow \tilde{M}$ is the unique covering transformation sending $\tilde{\mathbf{x}}$ to $\tilde{\eta}(1)$, and $\tilde{\eta}$ is the unique lifting of η with $\tilde{\eta}(0) = \tilde{\mathbf{x}}$. In this way, (4.4) can be expressed as

$$\mathcal{L}_{\mathbf{x}}(M) = \bigsqcup_{\varphi \in \text{Aut}_M \tilde{M}} \mathcal{L}_{\tilde{\mathbf{x}}}^\varphi(M),$$

where $\mathcal{L}_{\tilde{\mathbf{x}}}^\varphi(M)$ stands for the path component of $\mathcal{L}_{\mathbf{x}}(M)$ containing the homotopy class $\Phi^{-1}(\varphi)$. According to [2, Th. 4.3] and [28], it is easily seen that the map

$$\Theta : \bigsqcup_{\varphi \in \text{Aut}_M \tilde{M}} \mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M}) \longrightarrow \mathcal{L}_{\mathbf{x}}(M), \quad \tilde{\eta} \mapsto \pi \circ \tilde{\eta},$$

is a homeomorphism with the uniform convergence topology, where we use the notation $\tilde{\eta}$ to emphasize that the curve is defined on $\tilde{M}$, and the spaces $\mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M})$ are defined by

$$\mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M}) := \{\gamma \in \mathcal{C}([0, 1], \tilde{M}) : \gamma(0) = \tilde{\mathbf{x}}, \gamma(1) = \varphi(\tilde{\mathbf{x}})\}.$$

Moreover, Θ preserves the Wiener measure, in the sense that, for every $B \in \mathcal{B}_{\mathbf{x}}$,

$$\lambda_{\mathbf{x}}(B) = \sum_{\varphi \in \text{Aut}_M \tilde{M}} \lambda_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}\left(\Theta^{-1}(B) \cap \mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M})\right),$$

where $\lambda_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}$ is the non-normalised Wiener measure on $\mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M})$, and $\lambda_{\mathbf{x}}$ is the non-normalised Wiener measure on $\mathcal{L}_{\mathbf{x}}(M)$ (see Appendix A, or [2], for the precise definition). As a direct consequence, setting $B = \mathcal{L}_{\mathbf{x}}(M)$, the next relationship between the heat kernel of M and the corresponding heat kernel of its universal covering space $\tilde{M}$ holds

$$p_1(\mathbf{x}, \mathbf{x}) = \sum_{\varphi \in \text{Aut}_M \tilde{M}} \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}})), \quad (4.5)$$

where $\tilde{p}_t(x, y)$ stands for the heat kernel of $\tilde{M}$. In particular, since

$$\Theta(\mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M})) = \mathcal{L}_{\mathbf{x}}^{\varphi}(M) \quad \text{for all } \varphi \in \text{Aut}_M \tilde{M},$$

the restricted map $\Theta_{\varphi} : \mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M}) \rightarrow \mathcal{L}_{\mathbf{x}}^{\varphi}(M)$ is also a homeomorphism. Moreover, for every $B \in \mathcal{B}_{\mathbf{x}} \cap \mathcal{L}_{\mathbf{x}}^{\varphi}(M)$, we have that

$$\lambda_{\mathbf{x}}(B) = \sum_{\phi \in \text{Aut}_M \tilde{M}} \lambda_{\tilde{\mathbf{x}}}^{\phi(\tilde{\mathbf{x}})} \left(\Theta^{-1}(B) \cap \mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M}) \right) = \lambda_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\Theta^{-1}(B)). \quad (4.6)$$

Thus, Θ_{φ} also preserves the Wiener measure. As according to (4.6) we have that

$$\lambda_{\mathbf{x}}(\mathcal{L}_{\mathbf{x}}^{\varphi}(M)) = \lambda_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\mathcal{C}_{\tilde{\mathbf{x}}}^{\varphi(\tilde{\mathbf{x}})}(\tilde{M})) = \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}})) > 0, \quad (4.7)$$

it becomes apparent that every path-connected component of $\mathcal{L}_{\mathbf{x}}(M)$ has a positive Wiener measure. Furthermore, as a consequence of (4.5) and (4.7), the following result holds.

Theorem 4.5. *The normalized Wiener measure of the path components of $\mathcal{L}_{\mathbf{x}}(M)$ is given by*

$$\mu_{\mathbf{x}}(\mathcal{L}_{\mathbf{x}}^{\varphi}(M)) = \frac{\tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}}))}{\sum_{\phi \in \text{Aut}_M \tilde{M}} \tilde{p}_1(\tilde{\mathbf{x}}, \phi(\tilde{\mathbf{x}}))}, \quad \varphi \in \text{Aut}_M \tilde{M}, \quad (4.8)$$

where $\tilde{p}_t(x, y)$ denotes the heat kernel of $\tilde{M}$ and $\tilde{\mathbf{x}} \in \pi^{-1}(\mathbf{x})$.

As a direct consequence of Theorems 2.4 and 4.5, one can determine the global torsion invariant of any given vector bundle $E \rightarrow M$ in terms of χ and the heat kernel of the universal covering $\tilde{M}$. Indeed, for every $E \in \tilde{K}\mathcal{O}(M)$, (4.2) and (4.8) imply that

$$\Lambda(E) = \frac{\sum_{\varphi \in \text{Aut}_M \tilde{M}} \chi_2[\mathcal{J}\text{nd}^{-1}[E] \circ \Phi^{-1}(\varphi), \mathbb{S}^1] \cdot \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}}))}{\sum_{\varphi \in \text{Aut}_M \tilde{M}} \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}}))}. \quad (4.9)$$

Moreover, as, due to Theorem 3.8, the restricted morphism $\mathcal{J} : \text{Vect}_1(X) \rightarrow \text{Hom}(\pi_1(X), \mathbb{Z}_2)$ is an isomorphism, it follows that

$$\begin{aligned} \Lambda(\tilde{K}\mathcal{O}(M)) &= \Lambda(\text{Vect}_1(X)) \\ &= \left\{ \int_{\mathcal{L}_{\mathbf{x}}(M)} \xi([\gamma]) \, d\mu_{\mathbf{x}}(\gamma) : \xi \in \text{Hom}(\pi_1(X), \mathbb{Z}_2) \right\}. \end{aligned} \quad (4.10)$$

Consequently, setting

$$q(\zeta) := \frac{\sum_{\varphi \in \text{Aut}_M \tilde{M}} \zeta(\varphi) \cdot \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}}))}{\sum_{\varphi \in \text{Aut}_M \tilde{M}} \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}}))} \quad \text{for all } \zeta \in \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2),$$

the values of the global torsion invariant are given by

$$\Lambda(\tilde{K}\mathcal{O}(M)) = \{q(\zeta) : \zeta \in \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)\}. \quad (4.11)$$

In the next section, we will show that (4.11) is useful for ascertaining the values of Λ in some practical examples of interest.

5. Examples

In this section we will compute the global torsion invariant, Λ , of the circle $\mathbb{S}^1$ and the n -dimensional torus $\mathbb{T}^n$ by using all the machinery developed in Sects. 3 and 4.

5.1. Global torsion invariant of $\mathbb{S}^1$

The aim of this subsection is computing $\Lambda : \tilde{K}\mathcal{O}(\mathbb{S}^1) \rightarrow [-1, 1]$, where the circle $\mathbb{S}^1$ is regarded as the quotient $\mathbb{R}/2\sqrt{\pi}\mathbb{Z}$; the period factor $2\sqrt{\pi}$ is chosen for computational convenience. It is well known that

$$\tilde{K}\mathcal{O}(\mathbb{S}^1) = \{[TS^1], [\mathcal{M}]\},$$

where TS^1 is the tangent bundle of $\mathbb{S}^1$ and $\mathcal{M}$ is the Möbius bundle. Viewed as groups, it is easily seen that $\tilde{K}\mathcal{O}(\mathbb{S}^1) \simeq \mathbb{Z}_2$, where TS^1 is the identity (since it is trivial) and $\mathcal{M}$ is the generator.

We begin by computing the index map. Let U be an admissible real Banach space. By Theorem 3.1, $[\mathbb{S}^1, \Phi_0(U)] \simeq \mathbb{Z}_2$. Let $\mathfrak{C} : \mathbb{S}^1 \rightarrow \Phi_0(U)$ be the constant map $x \mapsto T$, where $T \in GL(U)$ is fixed. Since $\mathfrak{Ind}$ is a group homomorphism, the identity must go to the identity and hence, $\mathfrak{Ind}([\mathfrak{C}]) = [TS^1]$. Pick a singular operator $T \in \mathcal{S}(U)$ and an open ball $B_\varepsilon(T) \subset \Phi_0(U)$ of centre T and radius $\varepsilon > 0$. Let $P, Q \in \mathcal{L}(U)$ be projections onto $\text{Ker}[T]$ and $R[T]$, respectively. Then, the following topological direct sum decompositions hold

$$U = (I_U - P)(U) \oplus \text{Ker}[T], \quad U = R[T] \oplus (I_U - Q)(U).$$

Moreover, setting

$$R[I_U - P] = (I_U - P)(U) \equiv \text{Ker}[T]^\perp, \quad R[I_U - Q] = (I_U - Q)(U) \equiv R[T]^\perp,$$

it follows from Fitzpatrick and Pejsachowicz [7, p. 286] that every $L \in \Phi_0(U)$ can be expressed as a block operator matrix

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

where

$$\begin{aligned} L_{11} &:= QL(I_U - P), & L_{12} &:= QLP, \\ L_{21} &:= (I_U - Q)L(I_U - P), & L_{22} &:= (I_U - Q)LP. \end{aligned}$$

In particular, since $TP = 0$ and $(I_U - Q)T = 0$, the operator T can be expressed as

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

with $T_{11} \in GL(\text{Ker}[T]^\perp, R[T])$. Now, consider the segment $\gamma : J_\varepsilon \equiv [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \rightarrow \Phi_0(U)$ defined by

$$\gamma(t) := \begin{pmatrix} T_{11} & 0 \\ 0 & tI_n \end{pmatrix} = T_{11} \oplus tI_n, \quad t \in J_\varepsilon,$$

where $n = \dim \text{Ker}[T]$ and I_n is the identity matrix of rank n . Note that $\gamma(t) \in GL(U)$ for every $t \in J_\varepsilon \setminus \{0\}$, and that $\gamma(J_\varepsilon) \subset B_\varepsilon(T)$. Now, we reparameterize this curve in t under an affine transformation to get a curve parameterized in $[0, \frac{1}{2}]$, denoted by $\gamma_1 : [0, \frac{1}{2}] \rightarrow \Phi_0(U)$, such that $\gamma_1(0), \gamma_1(\frac{1}{2}) \in GL(U)$. Observe that, since $GL(U)$ is contractible, it is, in particular, path-connected. Thus, there exists a curve $\gamma_2 : [\frac{1}{2}, 1] \rightarrow \Phi_0(U)$ such that $\gamma_2(\frac{1}{2}) = \gamma_1(0)$, $\gamma_2(1) = \gamma_1(\frac{1}{2})$ and $\gamma_2([\frac{1}{2}, 1]) \subset GL(U)$. Consider the curve

$$\mathfrak{L} : \mathbb{S}^1 \longrightarrow \Phi_0(U), \quad \mathfrak{L}(t) := \begin{cases} \gamma_1(t) & \text{if } t \in [0, \frac{1}{2}], \\ \gamma_2(t) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \quad (5.1)$$

Let $\gamma : [0, 1] \rightarrow \mathbb{S}^1$ be the parametrization of the circle given by $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$. Then, by the properties of the parity

$$\begin{aligned} \sigma(\mathfrak{L} \circ \gamma, [0, 1]) &= \sigma(\gamma_1, [0, \frac{1}{2}]) \sigma(\gamma_2, [\frac{1}{2}, 1]) = \sigma(\gamma_1, [0, \frac{1}{2}]) \\ &= \sigma(\gamma, [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]) = \sigma(T_{11} \oplus tI_n, [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]) \\ &= \sigma(T_{11}, [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]) \sigma(tI_n, [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]) = \sigma(tI_n, [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]) \\ &= \text{sign det}(-\frac{\varepsilon}{2}I_n) \text{sign det}(\frac{\varepsilon}{2}I_n) = -1. \end{aligned}$$

Hence, since $\sigma(\mathfrak{L}, \mathbb{S}^1) = \sigma(\mathfrak{L} \circ \gamma, [0, 1])$, we deduce that $\sigma(\mathfrak{L}, \mathbb{S}^1) = -1$. Consequently,

$$\sigma(\mathfrak{C}, \mathbb{S}^1) = 1, \quad \sigma(\mathfrak{L}, \mathbb{S}^1) = -1. \quad (5.2)$$

Thus, since the parity map $\sigma : [\mathbb{S}^1, \Phi_0(U)] \rightarrow \mathbb{Z}_2$ is an isomorphism, it becomes apparent that $[\mathfrak{C}] \neq [\mathfrak{L}]$ on $[\mathbb{S}^1, \Phi_0(U)]$. Therefore, the index map is necessarily given by

$$\mathfrak{Jnd} : [\mathbb{S}^1, \Phi_0(U)] \longrightarrow \tilde{K}\mathcal{O}(\mathbb{S}^1) \quad \begin{cases} [\mathfrak{C}] \mapsto [T\mathbb{S}^1], \\ [\mathfrak{L}] \mapsto [\mathcal{M}]. \end{cases}$$

To compute Λ , we still have to calculate $\mathfrak{J}[E] : \pi_1(\mathbb{S}^1) \rightarrow \mathbb{Z}_2$ for every $E \in \tilde{K}\mathcal{O}(\mathbb{S}^1)$.

Suppose $E = [T\mathbb{S}^1]$. Then, regarding $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$, $|z| = 1$, and setting

$$\pi_1(\mathbb{S}^1) = \{\gamma_n : \gamma_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \gamma_n(z) = z^n\} \simeq \mathbb{Z},$$

it follows that, for every $n \in \mathbb{Z}$,

$$\mathfrak{J}[E](\gamma_n) = \chi_2[\mathfrak{Jnd}^{-1}[E] \circ \gamma_n, \mathbb{S}^1] = \chi_2[\mathfrak{C} \circ \gamma_n, \mathbb{S}^1] = \chi_2[T, \mathbb{S}^1] = 1,$$

by the properties of χ discussed in Sect. 2. Thus, $\mathfrak{J}[E] \equiv 1$ and therefore

$$\Lambda([T\mathbb{S}^1]) = \int_{\mathcal{L}_x(\mathbb{S}^1)} \mathfrak{J}[E](\gamma) d\mu_x(\gamma) = \int_{\mathcal{L}_x(\mathbb{S}^1)} 1 \cdot d\mu_x(\gamma) = 1.$$

Note that this value can be also found, directly, by applying Theorem 4.2, as the trivial bundle is orientable.

Subsequently, we suppose that $E = [\mathcal{M}]$. Then, since

$$\begin{aligned} \gamma_n &= \gamma_1 \circ \overset{n}{\cdots} \circ \gamma_1, & \text{if } n \in \mathbb{Z}_{\geq 0}, \\ \gamma_n &= \gamma_{-1} \circ \overset{-n}{\cdots} \circ \gamma_{-1}, & \text{if } n \in \mathbb{Z}_{< 0}, \end{aligned}$$

by the product formula of the multiplicity, it becomes apparent that, for every $n \in \mathbb{Z}$,

$$\begin{aligned} \mathfrak{I}[E](\gamma_n) &= \chi_2[\mathfrak{I}n\mathfrak{d}^{-1}[E] \circ \gamma_n, \mathbb{S}^1] = \prod_{i=1}^{|n|} \chi_2[\mathfrak{I}n\mathfrak{d}^{-1}[E] \circ \gamma_{\pm 1}, \mathbb{S}^1] \\ &= \prod_{i=1}^{|n|} \chi_2[\mathfrak{L} \circ \gamma_{\pm 1}, \mathbb{S}^1] = \prod_{i=1}^{|n|} \chi_2[\mathfrak{L}, \mathbb{S}^1], \end{aligned}$$

where we convine the value of the empty product to be one. Next, we will determine $\chi_2[\mathfrak{L}, \mathbb{S}^1]$. By (5.2), we have that

$$\chi_2[\mathfrak{L}, \mathbb{S}^1] = \sigma(\mathfrak{L}, \mathbb{S}^1) = -1.$$

Thus, for every $n \in \mathbb{Z}$,

$$\mathfrak{I}[E](\gamma_n) = \prod_{i=1}^{|n|} \chi_2[\mathfrak{L}, \mathbb{S}^1] = (-1)^n.$$

Hence, we find that

$$\Lambda(E) = \int_{\mathcal{L}_{\mathbf{x}}(\mathbb{S}^1)} \mathfrak{I}[E](\gamma) \, d\mu_{\mathbf{x}}(\gamma) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \mu_{\mathbf{x}}(\gamma_n).$$

Now, we proceed to the computation of $\mu_{\mathbf{x}}(\gamma_n)$ through (4.8). Let us consider the circle $\mathbb{S}^1$ as the quotient $\mathbb{R}/2\sqrt{\pi}\mathbb{Z}$. It is well known that the universal covering of $M = \mathbb{S}^1$ is $\tilde{M} = \mathbb{R}$ with corresponding covering map $\pi : \tilde{M} \rightarrow M$, $x \mapsto [x]$, where $[x]$ denotes the class of $x \in \mathbb{R}$ in the quotient $\mathbb{R}/2\sqrt{\pi}\mathbb{Z}$. It is easily seen that $\text{Aut}_M \tilde{M} = \{\varphi^n : n \in \mathbb{Z}\}$, where $\varphi^n(x) = x + 2\sqrt{\pi}n$, $x \in \mathbb{R}$, for each $n \in \mathbb{N}$. Since the heat kernel of the universal covering space $\tilde{M} = \mathbb{R}$ is

$$\tilde{p}_t(x, y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}},$$

it follows from (4.8) that, for every $n \in \mathbb{Z}$,

$$\mu_{\mathbf{x}}(\mathcal{L}_{\mathbf{x}}^{\varphi^n}(\mathbb{S}^1)) = \frac{\tilde{p}_1(\tilde{\mathbf{x}}, \varphi^n(\tilde{\mathbf{x}}))}{\sum_{m \in \mathbb{Z}} \tilde{p}_1(\tilde{\mathbf{x}}, \varphi^m(\tilde{\mathbf{x}}))} = \frac{\exp\{-\pi n^2\}}{\sum_{m \in \mathbb{Z}} \exp\{-\pi m^2\}} = \frac{\Gamma(\frac{3}{4})}{\sqrt[4]{\pi}} \exp\{-\pi n^2\}.$$

To get the last identity, we have used that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2} = \theta_3(0, e^{-\pi}) = \frac{\sqrt[4]{\pi}}{\Gamma(\frac{3}{4})}, \quad (5.3)$$

where $\theta_3(z, q)$ stands for the Jacobi–Theta function (see [30], if necessary). Hence,

$$\Lambda(E) = \int_{\mathcal{L}_{\mathbf{x}}(\mathbb{S}^1)} \mathfrak{I}[E](\gamma) \, d\mu_{\mathbf{x}}(\gamma) = \frac{\Gamma(\frac{3}{4})}{\sqrt[4]{\pi}} \sum_{n \in \mathbb{Z}} (-1)^n \exp(-\pi n^2).$$

Again, a simple computation with the Jacobi–Theta function yields to

$$\sum_{n \in \mathbb{Z}} (-1)^n \exp(-\pi n^2) = \sqrt[4]{\frac{\pi}{2}} \frac{1}{\Gamma(\frac{3}{4})}, \quad (5.4)$$

which implies that $\Lambda([\mathcal{M}]) = \frac{1}{\sqrt[4]{2}}$. In particular, by Theorem 4.2, since $\Lambda([\mathcal{M}]) \neq 1$, it follows that $\mathcal{M}$ is not orientable. So, our analysis establishes a new (different) proof of this well known fact.

Therefore, we have proved that the global torsion invariant Λ of the circle is given by

$$\Lambda : \tilde{K}\mathcal{O}(\mathbb{S}^1) \longrightarrow [-1, 1], \quad \Lambda([T\mathbb{S}^1]) = 1, \quad \Lambda([\mathcal{M}]) = \frac{1}{\sqrt[4]{2}}.$$

As a direct application of the additive formula of Proposition 4.4, we can obtain $\Lambda([T\mathbb{S}^1])$ from $[\mathcal{M}]$. Since $[\mathcal{M}] \oplus [\mathcal{M}] = [T\mathbb{S}^1]$ and

$$\chi_2[\mathfrak{J}\mathfrak{n}\mathfrak{d}^{-1}[\mathcal{M}] \circ \gamma_n, \mathbb{S}^1] = (-1)^n \quad \text{for all } n \in \mathbb{Z},$$

from Proposition 4.4 it is apparent that

$$\begin{aligned} \Lambda([T\mathbb{S}^1]) &= \sum_{n \in \mathbb{Z}} \chi_2[\mathfrak{J}\mathfrak{n}\mathfrak{d}^{-1}[\mathcal{M}] \circ \gamma_n, \mathbb{S}^1] \cdot \chi_2[\mathfrak{J}\mathfrak{n}\mathfrak{d}^{-1}[\mathcal{M}] \circ \gamma_n, \mathbb{S}^1] \cdot \mu_{\mathbf{x}}([\gamma_n]) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n (-1)^n \mu_{\mathbf{x}}([\gamma_n]) = \sum_{n \in \mathbb{Z}} \mu_{\mathbf{x}}([\gamma_n]) = 1. \end{aligned}$$

5.2. Global torsion invariant of $\mathbb{T}^2$

In this subsection, we will compute the global torsion invariant $\Lambda(\tilde{K}\mathcal{O}(\mathbb{T}^2))$ of the torus considering it as the quotient

$$\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1 \equiv \mathbb{R}^2 / [2\sqrt{\pi}\mathbb{Z} \times 2\sqrt{\pi}\mathbb{Z}]. \quad (5.5)$$

It is well known that the universal covering of $M = \mathbb{T}^2$ is $\tilde{M} = \mathbb{R}^2$ with corresponding covering map

$$\pi : \tilde{M} \longrightarrow M, \quad (x, y) \mapsto [(x, y)],$$

where $[(x, y)]$ denotes the class of $(x, y) \in \mathbb{R}^2$ in the quotient (5.5). An easy computation shows that

$$\text{Aut}_M \tilde{M} = \{\varphi^{n_1, n_2} : n_1, n_2 \in \mathbb{Z}\},$$

where the morphisms $\varphi^{n_1, n_2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are defined, for every $n_1, n_2 \in \mathbb{Z}$, by

$$\varphi^{n_1, n_2}(x, y) = (x + 2\sqrt{\pi}n_1, y + 2\sqrt{\pi}n_2).$$

The group of isomorphisms is given by $\pi_1(\mathbb{T}^2) \simeq \text{Aut}_M \tilde{M} \simeq \mathbb{Z} \oplus \mathbb{Z}$. To compute the values of the global torsion invariant of $\mathbb{T}^2$, $\Lambda(\tilde{K}\mathcal{O}(\mathbb{T}^2))$, we will use (4.11). Since the heat kernel of the universal covering space $\tilde{M} = \mathbb{R}^2$ is

$$\tilde{p}_t(x, y) = \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t}}, \quad (x, y) \in \mathbb{R}^2, \quad t > 0,$$

it follows that, for every $\tilde{\mathbf{x}} \in \pi^{-1}(\mathbf{x})$,

$$\tilde{p}_1(\tilde{\mathbf{x}}, \varphi^{n_1, n_2}(\tilde{\mathbf{x}})) = \frac{1}{4\pi} \exp\{-\pi(n_1^2 + n_2^2)\}.$$

Hence, using the summability methods involving the theta function (5.3), yields to

$$\sum_{\varphi \in \text{Aut}_{\tilde{M}} M} \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}})) = \sum_{n_1, n_2 \in \mathbb{Z}} \tilde{p}_1(\tilde{\mathbf{x}}, \varphi^{n_1, n_2}(\tilde{\mathbf{x}}))$$

$$= \frac{1}{4\pi} \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2} \right)^2 = \frac{1}{4\pi} \frac{\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}.$$

Now, we will compute the group $\text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)$. Clearly, since $\text{Aut}_{\tilde{M}} M$ is generated by the transformations $\varphi^{1,0}$ and $\varphi^{0,1}$, every homomorphism $\zeta : \text{Aut}_{\tilde{M}} M \rightarrow \mathbb{Z}_2$ is determined by the values $\zeta(\varphi^{1,0}), \zeta(\varphi^{0,1}) \in \mathbb{Z}_2$. In this way, we obtain the group isomorphisms

$$\text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2) \simeq \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

As a direct consequence, given any homomorphism $\zeta \in \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)$, we can write the action of ζ on each $\varphi^{n_1, n_2} \in \text{Aut}_{\tilde{M}} M$ as

$$\zeta(\varphi^{n_1, n_2}) = [\zeta(\varphi^{1,0})]^{n_1} \cdot [\zeta(\varphi^{0,1})]^{n_2}.$$

This allows us to compute the sum

$$\begin{aligned} \sum_{\varphi \in \text{Aut}_{\tilde{M}} M} \zeta(\varphi) \cdot \tilde{p}_1(\tilde{\mathbf{x}}, \varphi(\tilde{\mathbf{x}})) &= \sum_{n_1, n_2 \in \mathbb{Z}} [\zeta(\varphi^{1,0})]^{n_1} \cdot [\zeta(\varphi^{0,1})]^{n_2} \cdot \tilde{p}_1(\tilde{\mathbf{x}}, \varphi^{n_1, n_2}(\tilde{\mathbf{x}})) \\ &= \frac{1}{4\pi} \left(\sum_{n \in \mathbb{Z}} [\zeta(\varphi^{1,0})]^n e^{-\pi n^2} \right) \left(\sum_{n \in \mathbb{Z}} [\zeta(\varphi^{0,1})]^n e^{-\pi n^2} \right), \end{aligned}$$

where $\zeta(\varphi^{1,0}), \zeta(\varphi^{0,1}) \in \mathbb{Z}_2$ depend on the chosen $\zeta \in \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)$. Therefore, substituting in (4.11), we find that

$$\begin{aligned} \Lambda(\tilde{K}\mathcal{O}(\mathbb{T}^2)) &= \left\{ \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}} \left(\sum_{n \in \mathbb{Z}} \alpha^n e^{-\pi n^2} \right) \left(\sum_{n \in \mathbb{Z}} \beta^n e^{-\pi n^2} \right) : \alpha, \beta \in \mathbb{Z}_2 \right\} \\ &= \left\{ 1, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt{2}} \right\}, \end{aligned}$$

where (5.3) and (5.4) have been used in the last step. This information has been represented in the left table of Fig. 1. According to Theorem 3.8, $\mathfrak{J}$ defines an isomorphism between $\text{Vect}_1(\mathbb{T}^2)$ and $\text{Hom}(\pi_1(\mathbb{T}^2), \mathbb{Z}_2) \simeq \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)$. Thus, $\text{Vect}_1(\mathbb{T}^2) \simeq \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)$ and therefore, each $\zeta \in \text{Hom}(\text{Aut}_{\tilde{M}} M, \mathbb{Z}_2)$ corresponds to a single isomorphism class of line bundle. Each row of the table corresponds to an isomorphism class of line bundle. So, the table describes the values of the global torsion invariant on each line bundle.

Rephrasing these computations, we can obtain the corresponding result for the n -dimensional torus $\mathbb{T}^n := \times_{i=1}^n \mathbb{S}^1$, where each factor is taken as $\mathbb{S}^1 \equiv \mathbb{R}/2\sqrt{\pi}\mathbb{Z}$. In this case, the values of the global torsion invariant are given by

$$\Lambda(\tilde{K}\mathcal{O}(\mathbb{T}^n)) = \left\{ \left(\frac{1}{\sqrt[4]{2}} \right)^m : m \in \{1, 2, \dots, n\} \right\}.$$

The corresponding results for $n = 3$ are summarized in the right table of Fig. 1, where the values of the global torsion invariant, Λ , on each line bundle of $\mathbb{T}^3$ are collected.

Author contributions All the authors have prepared the manuscript from the very beginning to the very end.

$\zeta(\varphi^{1,0})$	$\zeta(\varphi^{0,1})$	Λ	$\zeta(\varphi^{1,0,0})$	$\zeta(\varphi^{0,1,0})$	$\zeta(\varphi^{0,0,1})$	Λ
1	1	1	1	1	1	1
1	-1	$1/\sqrt[4]{2}$	-1	1	1	$1/\sqrt[4]{2}$
-1	1	$1/\sqrt[4]{2}$	1	1	-1	$1/\sqrt[4]{2}$
-1	-1	$1/\sqrt{2}$	-1	1	-1	$1/\sqrt{2}$
			1	-1	-1	$1/\sqrt{2}$
			-1	-1	1	$1/\sqrt{2}$
			-1	-1	-1	$1/\sqrt[4]{8}$

FIGURE 1. The global torsion invariant for $\mathbb{T}^2$ and $\mathbb{T}^3$, respectively

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Appendix A. The Wiener measure on loops spaces

For any given Riemannian manifold, (M, g) , let denote by $\mathcal{B}_M$ the Borel σ -algebra of M , and consider, in (M, g) , the measure $\mu : \mathcal{B}_M \rightarrow [0, +\infty]$ induced by the metric g . Locally, this measure can be expressed by

$$d\mu = \sqrt{\det(g_{ij})_{ij}} \, dx_1 \wedge \cdots \wedge dx_m$$

where m is the dimension of M and $(g_{ij})_{ij}$ is the matrix of g in a local chart. According to Bär and Pfäffle [2] and Grigor'yan [13], for any given closed Riemannian manifold, (M, g) , there exists a heat kernel $p_t(x, y)$, $t > 0$, $x, y \in M$. Namely, the Schwartz kernel of the self-adjoint semigroup $e^{t\Delta}$ on $L^2(M, \mu)$, where Δ stands for the Laplace–Beltrami operator on (M, g) .

For any given (fixed) $\mathbf{x} \in M$, the Wiener measure on the loop space

$$\mathcal{L}_{\mathbf{x}}(M) := \{\gamma \in \mathcal{C}([0, 1], M) : \gamma(0) = \gamma(1) = \mathbf{x}\}$$

is a measure $\lambda_{\mathbf{x}} : \mathcal{B}_{\mathbf{x}} \rightarrow [0, +\infty]$ on the measurable space $(\mathcal{L}_{\mathbf{x}}(M), \mathcal{B}_{\mathbf{x}})$, where $\mathcal{B}_{\mathbf{x}}$ stands for the Borel σ -algebra of $\mathcal{L}_{\mathbf{x}}(M)$ with respect to the topology of the uniform convergence, such that, for every finite subset

$$\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\} \subset [0, 1]$$

and any $(B_t)_{t \in \mathcal{T} \setminus \{0, 1\}} \subset \mathcal{B}_M$,

$$\begin{aligned} \lambda_{\mathbf{x}}(\pi_{\mathcal{T}}^{-1}(B_t)_{t \in \mathcal{T} \setminus \{0, 1\}}) &= \int_{B_{t_1}} \cdots \int_{B_{t_n}} \prod_{i=1}^{n+1} p_{t_i - t_{i-1}}(x_i, x_{i-1}) \\ &\quad \times \prod_{i=1}^n d\mu(x_i), \quad x_0 = x_{n+1} = \mathbf{x}. \end{aligned} \quad (5.6)$$

In this context, we are using the notation $\pi_{\mathcal{T}}$ to denote the projector

$$\pi_{\mathcal{T}} : M^{[0, 1]} \longrightarrow M^{\mathcal{T} \setminus \{0, 1\}}, \quad \pi_{\mathcal{T}}(\gamma_t)_{t \in [0, 1]} := (\gamma_t)_{t \in \mathcal{T} \setminus \{0, 1\}}.$$

Since

$$\lambda_{\mathbf{x}}(\mathcal{L}_{\mathbf{x}}(M)) = p_1(\mathbf{x}, \mathbf{x}) > 0, \quad (5.7)$$

the measure $\lambda_{\mathbf{x}}$ is not a probability measure, unless $p_1(\mathbf{x}, \mathbf{x}) = 1$. Nevertheless, the normalized measure

$$\mu_{\mathbf{x}} = p_1(\mathbf{x}, \mathbf{x})^{-1} \lambda_{\mathbf{x}}$$

provides us with a probability measure. Rephrasing (5.6), it is apparent that, for every finite subset

$$\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\} \subset [0, 1]$$

and each $(B_t)_{t \in \mathcal{T} \setminus \{0, 1\}} \subset \mathcal{B}_M$, setting $x_0 = x_{n+1} = \mathbf{x}$, one has that

$$\begin{aligned} \mu_{\mathbf{x}}(\pi_{\mathcal{T}}^{-1}(B_t)_{t \in \mathcal{T} \setminus \{0, 1\}}) &= \int_{B_{t_1}} \cdots \int_{B_{t_n}} p_1(\mathbf{x}, \mathbf{y})^{-1} \\ &\quad \prod_{i=1}^{n+1} p_{t_i - t_{i-1}}(x_i, x_{i-1}) \prod_{i=1}^n d\mu(x_i). \end{aligned}$$

The measure $\mu_{\mathbf{x}} : \mathcal{B}_{\mathbf{x}} \rightarrow [0, +\infty]$ is ususally refereed to as the *normalized Wiener measure*.

This construction can be easily generalized to cover pinned spaces

$$\mathcal{C}_{\mathbf{x}}^{\mathbf{y}}(M) := \{\gamma \in \mathcal{C}([0, 1], M) : \gamma(0) = \mathbf{x}, \gamma(1) = \mathbf{y}\},$$

where it is possible to construct a generalized Wiener measure, $\lambda_{\mathbf{x}}^{\mathbf{y}} : \mathcal{B}_{\mathbf{x}}^{\mathbf{y}} \rightarrow [0, +\infty]$, as well as a normalized Wiener measure, $\mu_{\mathbf{x}}^{\mathbf{y}} : \mathcal{B}_{\mathbf{x}}^{\mathbf{y}} \rightarrow [0, +\infty]$, where $\mathcal{B}_{\mathbf{x}}^{\mathbf{y}}$ stands for the Borel σ -algebra of $\mathcal{C}_{\mathbf{x}}^{\mathbf{y}}(M)$ under the topology of the uniform convergence (see Bär and Pfäffle [2] for any further required details).

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