FOLIACIONES Y VALORES DIFERENCIALES EN LA CLASIFICACIÓN ANALÍTICA DE CÚSPIDES PLANAS

FOLIATIONS AND DIFFERENTIAL VALUES ON THE ANALYTIC CLASSIFICATION OF PLANE CUSPS

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"Sus griegos, cuando emplearon el mármol, copiaron sus construcciones de madera, sin razón, porque otros lo habían hecho así. Después sus maestros del Renacimiento hicieron copias en yeso de copias de mármol de copias de madera. Ahora estamos aquí nosotros haciendo copias de acero y hormigón de copias de yeso de copias de mármol de copias de madera."

— *Ayn Rand*

Contents

Resumen

Sea (C, P) una curva irreducible, o rama, en un germen (M, P) de una superficie compleja analítica regular. El objetivo de este trabajo es dar una interpretación geométrica a invariantes analíticos de C, relacionados con el semimódulo de valores diferenciales de C. Para hacer esto, consideramos listas de foliaciones determinadas por las 1-formas que definen el semimódulo de valores diferenciales de C (bases estándar).

Nuestro trabajo está principalmente limitado al caso de cúspides singulares, esto es, ramas con un solo par de Puiseux (n, m), con $2 < n \leq m$ y $mcd(n, m) = 1$.

En esta memoria presentamos tres resultados principales:

- Interpretación geométrica del moduli analítico de ramas planas en términos de semirraíces analíticas y la Teoría de Foliaciones.
- Obtención de bases de Saito de una cúspide a partir de bases estándar.
- Descripción de un subconjunto de raíces del polinomio de Bernstein-Sato de una cúspide C en términos de su semimódulo de valores diferenciales.

Estos resultados pueden encontrarse en nuestros trabajos [12, 13, 53].

Precisemos la terminología básica para los enunciados y resultados en las tres direcciones previas.

El *semimódulo* Λ_C *de valores diferenciales* de una rama C se define como sigue

$$\Lambda_C := \{\nu_C(\omega) : \omega \text{ una 1-forma}\}.$$

El *valor diferencial* $\nu_C(\omega)$ de ω por C es igual a $\mathrm{ord}_t(\alpha) + 1$, donde $\phi^*\omega = \alpha(t)dt$ y $\phi(t)$ es una parametrización primitiva de C. El conjunto Λ_C es un Γ_C-semimódulo, siendo Γ_C el semigrupo de C. En otras palabras, $\lambda + \gamma \in \Lambda_C$, para cualquier $\gamma \in \Gamma_C$, $\lambda \in \Lambda_C$. La *base* $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ de Λ_C es la única sucesión creciente minimal que genera Λ_C como Γ_C-semimódulo, esto es, tenemos que

$$\Lambda_C = \bigcup_{i=-1}^{s} (\lambda_i + \Gamma_C), \quad \text{con } \lambda_i \notin \lambda_j + \Gamma_C, \text{ para } i \neq j.$$

Por definición, hay listas de 1-formas holomorfas $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ satisfaciendo que $\nu_C(\omega_i) = \lambda_i$, para $i = -1, 0, \ldots, s$. Estas sucesiones $\mathcal{S}$ son denominadas *bases estándar minimales* de C.

De ahora en adelante, vamos a asumir que C es una cúspide con un solo par de Puiseux (n, m). Consideremos

$$\pi : (\tilde{M}, E) \to (M, P)$$

la mínima resolución de singularidades de C, obtenida como una composición finita de transformaciones cuadráticas. Denotamos por $D \subset E$ la componente irreducible del divisor excepcional creada en la última explosión de la resolución π. Nos referimos a D como el *divisor* *cuspidal* de C. Decimos que una foliación $\mathcal{F}$ es *totalmente D-dicrítica* cuando su transformado estricto por π es regular, transversal a D, y tiene cruzamientos normales con E en todos los puntos de D.

Además del valor diferencial $\nu_C(\omega)$ de una 1-forma ω, estamos interesados en su *valor divisorial* $\nu_D(\omega)$, que corresponde con la valoración divisorial asociada a D. En sistemas de coordenadas apropiados este valor es interpretado en términos de un grado pesado en los monomios.

Sea $\mathcal{B}$ la base de Λ_C y $\mathcal{S}$ una base estándar minimal, como en los párrafos previos. Un primer resultado muestra que las 1-formas $\omega_1, \omega_2, \ldots, \omega_s$ definen foliaciones totalmente D-dicríticas.

Decir que una foliación $\mathcal{F}$ es totalmente D-dicrítica implica que $\mathcal{F}$ tiene una familia de ramas invariantes con el mismo par de Puiseux (n, m) que C. Esta observación nos conduce a uno de nuestros principales resultados:

Tomemos una de las 1-formas ω_i de la base estándar minimal $\mathcal{S}$ con $i \geq 1$, un punto no esquina $Q \in D$ de E, y denotemos por $\gamma = C_Q^{\omega_i}$ la curva invariante por ω_i teniendo a Q como punto infinitamente próximo. Entonces el semimódulo de valores diferenciales de γ es Λ_γ con base

$$(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_{i-1}).$$

En particular, Λ_γ no depende de la elección del punto $Q \in D$. Además, tenemos que $\nu_\gamma(\omega_j) = \lambda_j$, para $j = -1, 0, \ldots, i-1$, y por tanto $(\omega_{-1}, \omega_0, \ldots, \omega_{i-1})$ es una base estándar minimal de γ. Cuando el punto Q pertenece al transformado estricto de C, decimos que γ es una *semirraíz analítica* de C.

Lo anterior muestra que las semirraíces analíticas aproximan a C usando tipos analíticos más simples, en lo que se refiere al semimódulo de valores diferenciales. Además, podemos colocar cualquier cúspide como una curva invariante de una 1-forma ω_{s+1} con propiedades similares a las de una base estándar minimal. Por ello podemos jerarquizar el espacio de moduli de cúspides planas en términos de semirraíces analíticas. No obstante, mostramos que la definición que damos de semirraíz analítica no pasa bien al caso de ramas que no sean cúspides.

Detallemos el segundo punto sobre bases de Saito.

Sea $\Omega^1_{M,P}[C]$ el $\mathcal{O}_{M,P}$-módulo de gérmenes de 1-formas holomorfas con C como curva invariante. Se trata de un $\mathcal{O}_{M,P}$-módulo libre de rango dos. Cualquier base de $\Omega^1_{M,P}[C]$ es denominada *base de Saito* de C.

Mostremos como calcular una base de Saito cuando C es una cúspide, en términos de la estructura combinatoria del semimódulo de valores diferenciales.

Empecemos con una breve descripción de la combinatoria de Λ_C. Consideremos la sucesión de descomposición

$$\Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_s = \Lambda_C; \quad \Lambda_i = \cup_{j=-1}^{i} (\lambda_j + \Gamma_C), \; i = -1, 0, 1, \ldots, s.$$

Para cada índice $i = 1, 2, \ldots, s+1$, definimos los *ejes* u_i^n, u_i^m, u_i y $\tilde{u}_i$ como

- $u_i^n = \text{mín}\{\lambda_{i-1} + n\ell \in \Lambda_{i-2};\ \ell \geq 1\}$.
- $u_i^m = \text{mín}\{\lambda_{i-1} + m\ell \in \Lambda_{i-2};\ \ell \geq 1\}$.
- $u_i = \text{mín}\{u_i^n, u_i^m\} = \text{mín}((\lambda_{i-1} + \Gamma_C) \cap \Lambda_{i-2})$.
- $\tilde{u}_i = \text{máx}\{u_i^n, u_i^m\}$.

Los *valores críticos* t_i^n, t_i^m, t_i y $\tilde{t}_i$ están definidos por: $t_{-1} = n$, $t_0 = m$ y

$$\left.\begin{array}{rclrcl} t_i^n &=& t_{i-1} + u_i^n - \lambda_{i-1}, & t_i^m &=& t_{i-1} + u_i^m - \lambda_{i-1} \\ t_i &=& \text{mín}\{t_i^n, t_i^m\}, & \tilde{t}_i &=& \text{máx}\{t_i^n, t_i^m\} \end{array}\right\} \quad 1 \leq i \leq s+1.$$

Dada una base estándar minimal $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$, tenemos que t_i es el valor divisorial $\nu_D(\omega_i)$ de ω_i. Extendemos esta propiedad clave a los otros valores críticos de la siguiente manera. Dado un valor crítico T en el conjunto

$$\{\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{s+1}; t_{s+1}\},$$

obtenemos de forma algorítmica una 1-forma ω, tal que C es una rama invariante de ω y $\nu_D(\omega) = T$. Además, si consideramos ω_{s+1} y $\tilde{\omega}_{s+1}$ dos 1-formas con C invariante, $\nu_D(\omega_{s+1}) = t_{s+1}$ y $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Probamos que $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ es una base de Saito de C. Resaltamos que la 1-forma ω_{s+1} coincide con la mencionada anteriormente al hablar de semirraíces analíticas.

La construcción de bases de Saito, nos permite estudiar lo que llamamos pares de multiplicidades de Saito, siendo estos un nuevo invariante analítico de la curva, obtenidos como una generalización de invariantes dados por Y. Genzmer. Más precisamente, consideremos $\pi : (M_N, E^N) \to (M_0, P_0)$ una sucesión de transformaciones cuadráticas, con divisor excepcional $\pi^{-1}(P_0) = E^N$ y $E \subset E^N$ una componente irreducible de E^N. Definimos los dos siguientes números

$$\begin{aligned} \mathfrak{s}_E(C) &= \text{mín}\{\nu_E(\omega);\ \omega \text{ pertenece a una base Saito de } C\}, \\ \widetilde{\mathfrak{s}}_E(C) &= \text{máx}\{\nu_E(\omega);\ \omega \text{ pertenece a una base Saito de } C\}. \end{aligned}$$

La pareja $(s_E(C), \widetilde{\mathfrak{s}}_E(C))$ es el par de multiplicidades de Saito de C con respecto de E. Cabe notar que esta definición se extiende al caso en el que C no sea una cúspide.

Probamos que para el caso del divisor cuspidal D, se cumple que $(s_D(C), \widetilde{\mathfrak{s}}_D(C)) = (t_{s+1}, \tilde{t}_{s+1})$, demostrando así que esta pareja de valores está siempre determinado por el semimódulo de valores diferenciales. No obstante, si tomamos $E = E_1^1$ el divisor tras una única explosión, deja de ser cierto que el par $(s_E(C), \widetilde{\mathfrak{s}}_E(C))$ esté determinado por el semimódulo de valores diferenciales.

Ahora presentamos nuestros resultados sobre raíces del polinomio de Bernstein-Sato de C.

Consideremos el anillo no conmutativo $A = \mathbb{C}\{x_1, \ldots, x_p, \partial_1, \ldots, \partial_p\}$ de series de potencias en $2p > 0$ variables, y definamos $\mathcal{D}$ como el cociente de A por los conmutadores $[x_i, x_j] = 0$ y $[\partial_i, x_j] = \delta_{ij}$, donde δ_{ij} es la delta de Kronecker. El anillo $\mathcal{D}$ es el conjunto de operadores diferenciales en p variables, cuya acción en $\mathbb{C}\{x_1, \ldots, x_p\}$ se define considerando el elemento ∂_i como la derivada parcial con respecto a x_i.

Tomemos $\mathcal{D}[\rho]$ el anillo de polinomios en la variable ρ y coeficientes en $\mathcal{D}$. Dada una función $g \in \mathbb{C}\{x_1, \ldots, x_p\}$, podemos extender la acción de $\mathcal{D}$ a cualquier función de la forma g^ρ, poniendo $\partial_i \cdot g^\rho = \rho g^{\rho-1} \partial_i g$.

Consideremos $g \in \mathbb{C}\{x_1, \ldots, x_p\}$, y sea $\mathcal{I}$ el ideal (no nulo) de todos los posibles polinomios $B(\rho) \in \mathbb{C}[\rho]$ para los que existe un $P \in \mathcal{D}[\rho]$ satisfaciendo la ecuación:

$$P(\rho) \cdot g^{\rho+1} = B(\rho) g^\rho.$$

El generador mónico $b_g(\rho)$ de $\mathcal{I}$ es llamado el *polinomio de Bernstein-Sato* de g. Este no depende de la ecuación local g escogida de la hipersuperficie $H = (g = 0)$, por tanto podemos hablar del polinomio de Bernstein-Sato $b_H(\rho)$ de H.

En el caso de una cúspide singular C, mostramos dos enunciados:

- El valor $-\lambda/nm$ es una raíz del polinomio de Bernstein-Sato de C, para cualquier $\lambda \in (\lambda_1 + \Gamma_C) \setminus \Gamma_C$.
- Si $n \leq 4$, entonces para cualquier $\lambda \in \Lambda_C \setminus \Gamma_C$, tenemos que $-\lambda/nm$ es una raíz del polinomio de Bernstein-Sato de C.

Terminamos esta sección comentando brevemente posibles líneas de investigación que surgen a raíz de los resultados presentados.

El primer problema que se plantea es el de tratar de dar una generalización de todo lo mencionado para ramas que no sean cúspides. La línea sobre la que trabajar pasaría por dar una descripción de la combinatoria del semimódulo de valores diferenciales de una rama con varios pares de Puiseux, en términos análogos a los usados con los intervalos circulares. La evidencia mostrada por el hecho de que el concepto de semirraíz analítica, tal y como lo hemos definido, no pasa bien al caso de varios pares de Puiseux, nos hace pensar que no se trata de un objetivo sencillo.

Otro problema que abordar es el estudio de los pares de multiplicidades de Saito. Los resultados aquí mencionados, muestran que existen divisores para los que, al menos en el caso cuspidal, los pares de multiplicidades de Saito están determinados por el semimódulo de valores diferenciales (divisor cuspidal) o no lo están (divisor tras una única explosión). No hemos dado ninguna explicación sobre esta fenomenología. El objetivo sería dar una caracterización que nos permita decidir cuando los pares de multiplicidades de Saito respecto de un divisor están determinados o no por el semimódulo. En caso afirmativo, dar una fórmula cerrada para estos valores. Recordamos que los valores divisoriales respecto de divisores cuspidales se expresan, en coordenadas adaptadas, como valores monomiales con pesos, convirtiendo su estudio en un problema combinatorio.

Como última línea de investigación, sobre los polinomios de Bernstein-Sato, sería averiguar si la restricción a que la multiplicidad sea a lo sumo 4 en el resultado que presentamos es innecesaria o no.

Introduction

Let (C, P) be an irreducible curve, or branch, in a germ (M, P) of a complex analytic regular surface. The goal of this work is to give a geometrical interpretation of analytic invariants of C, related with the semimodule of differential values of C. In order to this, we consider lists of foliations determined by the 1-forms that define the semimodule of differential values of C (standard bases).

Our work is mainly restricted to the case of singular cusps, that is, branches with a single Puiseux pair (n, m), with $2 \leq n < m$ and $gcd(n, m) = 1$.

In this doctoral thesis we present three main achievements:

- Geometrical interpretation of the moduli of analytic plane cusps in terms of analytic semiroots and Foliation Theory.
- Obtaining Saito bases of a cusp from standard bases.
- Description of a subset of roots of the Bernstein-Sato polynomial of a cusp C in terms of the semimodule of differential values.

These results can be found in our works [12, 13, 53].

Let us precise the basic language required for the statements and results in the three previous directions.

The *semimodule* Λ_C *of differential values* of a branch C is defined as follows

$$\Lambda_C := \{\nu_C(\omega) : \omega \text{ a 1-form}\}.$$

The *differential value* $\nu_C(\omega)$ of ω by C is equal to $\mathrm{ord}_t(\alpha) + 1$, where $\phi^*\omega = \alpha(t)dt$ and $\phi(t)$ is a primitive parametrization of C. The set Λ_C is a Γ_C-semimodule, where Γ_C denotes the semigroup of C. In other words, $\lambda + \gamma \in \Lambda_C$ for any $\gamma \in \Gamma_C$, $\lambda \in \Lambda_C$. The *basis* $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ of Λ_C is the unique minimal increasing sequence that generates Λ_C as Γ_C-semimodule, that is, we have

$$\Lambda_C = \bigcup_{i=-1}^{s} (\lambda_i + \Gamma_C), \quad \text{with } \lambda_i \notin \lambda_j + \Gamma_C, \text{ for } i \neq j.$$

By definition, there are lists of holomorphic 1-forms $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$, satisfying that $\nu_C(\omega_i) = \lambda_i$, for $i = -1, 0, 1, \ldots, s$. Such sequences $\mathcal{S}$ are called *minimal standard bases* of C.

From now on, we assume that C is a cusp with a single Puiseux pair (n, m). Consider

$$\pi : (\tilde{M}, E) \to (M, P)$$

the minimal resolution of singularities of C, obtained as a finite composition of quadratic transformations. We denote by $D \subset E$ the irreducible component of the exceptional divisor created in the last blow-up of the resolution π. The component D is called the *cuspidal divisor* of C. We say that a foliation $\mathcal{F}$ is *totally D-dicritical* when its strict transform by π is regular, transverse to D, and has normal crossings with E at all the points of D.

Besides to the differential value $\nu_C(\omega)$ of a 1-form ω, we are interested in its *divisorial value* $\nu_D(\omega)$, which corresponds with the divisorial valuation associated to D. In appropriate systems of coordinates this value is interpreted in terms of a weighted degree of monomials.

Let $\mathcal{B}$ be the basis of Λ_C and $\mathcal{S}$ a minimal standard basis, as above. A first result is that the 1-forms $\omega_1, \omega_2, \ldots, \omega_s$ define totally D-dicritical foliations.

Saying that a foliation $\mathcal{F}$ is totally D-dicritical implies that $\mathcal{F}$ has an infinite family of invariant branches with the same single Puiseux pair (n, m) as C. This observation leads to one of our main results:

Take one of the 1-forms ω_i of the minimal standard basis $\mathcal{S}$ with $i \geq 1$, a point $Q \in D$ not a corner of E, and denote by $\gamma = C_Q^{\omega_i}$ the invariant curve by ω_i having Q as infinitely near point. Then the semimodule of differential values Λ_γ of γ is Λ_{i-1} with basis

$$(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_{i-1}).$$

In particular, Λ_γ does not depend on the choice of the point $Q \in D$. Moreover, we have that $\nu_\gamma(\omega_j) = \lambda_j$, for $j = -1, 0, \ldots, i-1$, and hence $(\omega_{-1}, \omega_0, \ldots, \omega_{i-1})$ is a minimal standard basis of γ. When the point Q belongs to the strict transform of C, we say that γ is an *analytic semiroot* of C.

Let us detail the second point about Saito bases. Let $\Omega^1_{M,P}[C]$ be the $\mathcal{O}_{M,P}$-module of germs of holomorphic 1-forms with C as invariant curve. It is a free $\mathcal{O}_{M,P}$-module of rank two. Any basis of $\Omega^1_{M,P}[C]$ is called a *Saito basis* of C.

Let us show how to compute a Saito basis when C is a cusp, in terms of the combinatorial structure of the semimodule of differential values.

Let us start with a brief description of the combinatorics of Λ_C. Consider the decomposition sequence

$$\Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_s = \Lambda_C; \quad \Lambda_i = \bigcup_{j=-1}^{i} (\lambda_j + \Gamma_C), \; i = -1, 0, 1, \ldots, s.$$

For each index $i = 1, 2, \ldots, s+1$, we define the *axes* u_i^n, u_i^m, u_i and $\tilde{u}_i$ as

- $u_i^n = \min\{\lambda_{i-1} + n\ell \in \Lambda_{i-2};\ \ell \geq 1\}$.
- $u_i^m = \min\{\lambda_{i-1} + m\ell \in \Lambda_{i-2};\ \ell \geq 1\}$.
- $u_i = \min\{u_i^n, u_i^m\} = \min((\lambda_{i-1} + \Gamma_C) \cap \Lambda_{i-2})$.
- $\tilde{u}_i = \max\{u_i^n, u_i^m\}$.

The *critical values* t_i^n, t_i^m, t_i and $\tilde{t}_i$ are defined by: $t_{-1} = n$, $t_0 = m$ and

$$\left.\begin{array}{rclrcl} t_i^n &=& t_{i-1} + u_i^n - \lambda_{i-1}, & t_i^m &=& t_{i-1} + u_i^m - \lambda_{i-1} \\ t_i &=& \min\{t_i^n, t_i^m\}, & \tilde{t}_i &=& \max\{t_i^n, t_i^m\} \end{array}\right\} \quad 1 \leq i \leq s+1.$$

Given a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$, we have that t_i is the divisorial value $\nu_D(\omega_i)$ of ω_i. We extend this key property to the other critical values in the following way. Given a critical value T in the set

$$\{\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{s+1}; t_{s+1}\},$$

we obtain, in an algorithmic way, a 1-form ω such that C is an invariant branch of ω and $\nu_D(\omega) = T$. Furthermore, if we consider ω_{s+1} and $\tilde{\omega}_{s+1}$ two 1-forms with C invariant and $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$ respectively, we prove that $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis of C.

Now we present our results about roots of the Bernstein-Sato polynomial of C. Consider the ring of non-commutative power series $A = \mathbb{C}\{x_1, \ldots, x_p, \partial_1, \ldots, \partial_p\}$ in $2p > 0$ variables, and let $\mathcal{D}$ be the quotient of A by the commutators $[x_i, x_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$, where δ_{ij} is the Kronecker's delta. The ring $\mathcal{D}$ is the set of differential operators in p variables, whose action on $\mathbb{C}\{x_1, \ldots, x_p\}$ is defined by considering the element ∂_i as the partial derivative with respect to x_i.

We take $\mathcal{D}[\rho]$ the ring of polynomials in the variable ρ and coefficients in $\mathcal{D}$. Given any function $g \in \mathbb{C}\{x_1, \ldots, x_p\}$, we can extend the action of $\mathcal{D}$ to functions of the form g^ρ just by putting $\partial_i \cdot g^\rho = \rho g^{\rho-1}\partial_i g$.

Consider $g \in \mathbb{C}\{x_1, \ldots, x_p\}$ and let $\mathcal{I}$ be the (non-zero) ideal of all possible polynomials $B(\rho) \in \mathbb{C}[\rho]$ for which there exists a $P \in \mathcal{D}[\rho]$ satisfying the equation:

$$P(\rho) \cdot g^{\rho+1} = B(\rho) g^\rho.$$

The monic generator $b_g(\rho)$ of $\mathcal{I}$ is called the *Bernstein-Sato polynomial* of g. It does not depend on the chosen local equation g of the hypersurface $H = (g = 0)$, hence we can speak about the Bernstein-Sato polynomial $b_H(\rho)$ of H.

In the case of a singular cusp C, we show two statements:

- The value $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C for any $\lambda \in (\lambda_1 + \Gamma_C) \setminus \Gamma_C$.
- If $n \leq 4$, then for any $\lambda \in \Lambda_C \setminus \Gamma_C$, we have that $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C.

The thesis is structured as follows:

Chapters 1-4 are mostly introductory. In Chapter 1 we introduce notions and notations about plane curves: primitive parametrization, Newton polygon, resolution of singularities, topological/analytic invariants, etc.

In Chapter 2 we introduce the concept of foliation in a complex analytic regular surface. We recall the existence of resolution of singularities for foliations. We also prove a combinatorial criteria about totally D-dicriticalness.

In Chapter 3 we study the combinatorial structure of semimodules appearing as semimodules of differential values of cusps.

Chapter 4 is devoted to the introduction of the computational techniques of standard bases. This theory is general and can be applied to other kinds of local algebra problems.

In Chapter 5 we present Delorme's decomposition. The computations are done either with a parametrization or an implicit equation.

In Chapter 6 we introduce the concepts of extended standard basis and standard system for a cusp. They are used in Chapters 7 and 8.

In Chapter 7 we prove hierarchy results about the moduli of analytic plane cusps. Again, we rely on the structure of the elements of a minimal standard basis seen in Chapter 5.

In Chapter 8 we use the structure of a minimal standard basis to compute a Saito basis of the cusp C. The proof is based on the combinatorial techniques from Chapter 3 and the use of Delorme's decompositions. Moreover, we define new analytic invariants of curves.

Finally, in Chapter 9 we show how to detect roots of the Bernstein-Sato polynomial for the case of cusps.

Now we proceed to summarize the content of each chapter of this work.

Plane Curve Singularities

Fix $(M_0, P_0) = (\mathbb{C}^2, \mathbf{0})$ a germ of a regular complex surface. Denote by $\mathcal{O}_{M_0,P_0}$ its ring of complex analytic functions. After choosing a local system of coordinates (x, y), we have that the ring

$\mathcal{O}_{M_0,P_0}$ coincides with the ring $\mathbb{C}\{x,y\}$ of complex convergent power series in x and y. A plane curve (C,P_0) in (M_0,P_0) is a non trivial principal ideal $(f) = I \subset \mathcal{O}_{M_0,P_0}$. We denote by f or by $f = 0$ an implicit equation of C.

Assume that C is an irreducible curve, or branch. Then the implicit equation f is irreducible as an element in $\mathbb{C}\{x,y\}$. By Newton-Puiseux Theorem (see for instance [14]), there exists a parametrization $\phi : (\mathbb{C},0) \to (M_0,P_0)$ of C, with $\phi(t) = (x(t),y(t))$, where $x(t),y(t) \in \mathbb{C}\{t\}$. Having a parametrization means that $f \circ \phi = 0$. Additionally, we are going to assume that ϕ is a primitive parametrization. In other words, the multiplicity of f at P_0, which we recall is the order of f at P_0, coincides with the value $\min\{\mathrm{ord}_t(x(t)), \mathrm{ord}_t(y(t))\}$.

We define the semigroup of C as:

$$\Gamma_C = \{\nu_C(h) : h \in \mathcal{O}_{M_0,P_0}\}; \text{ where } \nu_C(h) := \mathrm{ord}_t(h \circ \phi).$$

The semigroup of C admits a finite system of generators $\Gamma_C = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots \overline{\beta}_g \rangle$, with $\overline{\beta}_i < \overline{\beta}_{i+1}$ for $i = 0,1,\ldots,g-1$. It is a complete topological invariant. In fact, Zariski's equsingularity theory tell us that for two branches C and D the following statements are equivalent:

a) C and D are topologically equivalent, in the sense that there is an homeomorphism of the ambient spaces sending C to D.
b) C and D share the same semigroup $\Gamma_C = \Gamma_D$.
c) C and D are equisingular, that is, they have the "same" resolution of singularities: same dual graph.

We are interested in the notion of analytic invariants of a branch, that is, properties that remain constant under analytic isomorphisms of the ambient spaces.

Similarly to the semigroup, we define the semimodule Λ_C of differential values of C by

$$\Lambda_C := \{\nu_C(\omega) : \omega \in \Omega^1_{M_0,P_0}\},$$

where $\Omega^1_{M_0,P_0}$ is the $\mathcal{O}_{M_0,P_0}$-module of holomorphic 1-forms in (M_0,P_0) and $\nu_C(\omega)$ is the *differential value* of ω by C. The differential value of ω is given by the number $\mathrm{ord}_t(\alpha)+1$, with $\phi^*\omega = \alpha(t)dt$. The semimodule of differential values is an analytic invariant of C. The relevance, as analytic invariant, of Λ_C relies on the following theorem proved by A. Hefez and M.E. Hernandes in [34].

Theorem. *Let (C,P_0) be a branch whose semigroup is $\Gamma_C = \langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$. There exists a system of local coordinates such that C has a normal form parametrization defined as follows: if $\Lambda_C \setminus \Gamma_C = \emptyset$, then we put $(t^{\overline{\beta}_0}, t^{\overline{\beta}_1})$. Otherwise, if we have that $\Lambda_C \setminus \Gamma_C \neq \emptyset$, then we put*

$$\left(t^{\overline{\beta}_0}, t^{\overline{\beta}_1} + t^{\lambda_Z} + \sum_{i>\lambda_Z, i \notin \Lambda_C - \overline{\beta}_0} a_i t^i \right). \quad (1)$$

Here $\lambda_Z = \min(\Lambda_C \setminus \Gamma_C) - \overline{\beta}_0$ is the Zariski's invariant of C.

Moreover, we have that (C,P_0) is analytically equivalent to another branch (C',P_0) if and only if there exists $r \in \mathbb{C}^*$ with $r^{\lambda_Z - \overline{\beta}_1} = 1$ and $a_i = r^{i-\overline{\beta}_1} a'_i$ for every coefficient a'_i of a normal form parametrization of C'.

The previous theorem solves the analytic classification of plane branches. In [36] M.E Hernandes and M.E.R. Hernandes extend the previous result to general plane curves.

Because of the previous theorem, we decided to study the relationship of the semimodule of differential values with other analytic invariants of branches. We say that C is a cusp with Puiseux pair (n,m), when the semigroup Γ_C of C is generated by the pair (n,m), that is, $\Gamma_C = \langle n,m \rangle$, where $2 \leq n < m$ and $gcd(n,m) = 1$. In this situation the combinatorics of the semimodule of the differential values are easier and well known, see [2, 21]. These combinatorics are one of the main ingredients that we use along all our proofs.

Assume that C is a cusp with Puiseux pair (n,m). To simplify our computations, we place ourselves in a system of coordinates (x,y) adapted to C. In other words (x,y) is defined as a system of coordinates where we can find a primitive parametrization of C of the shape $(t^n, at^m + h.o.t.)$ with $a \neq 0$. Equivalently, we can take as implicit equation of C the following one

$$f = y^n + \mu x^m \sum_{ni+mj>nm} b_{ij} x^i y^j; \quad \mu \neq 0.$$

We can also define an adapted system of coordinates of C in terms of its resolution of singularities. We precise this last notion. The blow-up with center $\mathbf{0}$ of $(\mathbb{C}^2, \mathbf{0})$ is defined as the germ space of the sets

$$Bl_{U,\mathbf{0}} := \{((x,y),[\alpha_1 : \alpha_2]) \in U \times \mathbb{P}^1(\mathbb{C}) : x\alpha_2 = y\alpha_1\},$$

being $\mathbf{0} \subset U \subset \mathbb{C}^2$ an open set. The blow-up can be considered in (M_0, P_0) since this surface is isomorphic to $(\mathbb{C}^2, \mathbf{0})$. We denote by $\sigma : Bl_{M_0,P_0} \longrightarrow (M_0, P_0)$ the projection in the first coordinates. We have that $\sigma^{-1}(P_0) = E^1 \cong \mathbb{P}^1(\mathbb{C})$ is the exceptional divisor of the blow-up σ. Given $g \in \mathcal{O}_{M_0,P_0}$, we define the strict transform of g by σ as the implicit equation, in Bl_{M_0,P_0}, of the curve defined by $(g \circ \sigma) - \nu_{P_0}(g)(E^1)$, where we are using the notation of divisors as a simplification. The notion of strict transform extends, inductively, to a finite sequence of blow-ups.

Notice that since $n \geq 2$, then C is singular at P_0. Consider a singular plane curve (D, P_0) in (M_0, P_0). In virtue of the classical Theorem of Resolution of Singularities for plane curves, we can find a finite sequence of blow-ups $\pi : (M_N, E^N) \longrightarrow (M_0, P_0)$, such that the strict transform $\tilde{D}$ of D by π is a non singular curve and the intersection points of $\tilde{D}$ with the exceptional divisor $E^N = \pi^{-1}(P_0)$ are a non singular points of E^N.

We have a minimal resolution of singularities

$$\pi : (M_N, E^N) \longrightarrow (M_0, P_0),$$

with $\pi = \sigma_1 \circ \sigma_2 \circ \ldots \sigma_N$, and N minimal upon all the resolution of singularities of C. Each σ_i is a blow-up as above. Saying that (x,y) is a local system of adapted coordinates with respect to C is equivalent to saying that we can compute π just in a combinatorial way. These sequences of blow-ups are named *cuspidal sequences*. We say that a system of coordinates is adapted to π if it is also adapted to C.

If we consider $E^N = \pi^{-1}(P_0)$ the exceptional divisor of π, we can decompose this curve into its irreducible components $E^N = E^N_1 \cup E^N_2 \cup \ldots \cup E^N_N$, here the subscript indicates the order of appearance. In particular, we call E^N_N the cuspidal divisor of π. It is also defined by saying that the strict transform of C by π has no empty intersection with E^N_N. Since the divisor E^N_N only depends on C, we also say that E^N_N is the cuspidal divisor of C.

Moreover, given π a cuspidal sequence, which coincides with resolution of singularities of a cusp C with Puiseux pair (n,m), we say that (n,m) is the Puiseux pair of π.

Totally Dicritical Foliations

The definition of the semimodule of differential values of a branch C requires the use of holomorphic 1-forms in the analytic surface (M_0, P_0). For this reason, we study geometrical properties of the foliations that give rise to the semimodule of differential values.

A germ of a foliation $\mathcal{F}$ in (M_0, P_0) can be defined by a holomorphic 1-form $\omega \in \Omega^1_{M_0,P_0}$ that can be written as $\omega = Adx + Bdy$, with $gcd(A, B) = 1$, in a system of coordinates (x, y) at (M_0, P_0). We can define foliations in general surfaces or analytic spaces, not just a germ, by considering gluing conditions between the defining charts of the desired ambient space. We say that a curve C defined by $f = 0$ is an invariant curve by $\mathcal{F}$ if $\omega \wedge df = f\eta$, where df stands for the differential of f and η is a holomorphic 2-form.

Suppose that the foliation $\mathcal{F}$ defined by ω is singular at P_0, that is, $A(\mathbf{0}) = B(\mathbf{0}) = 0$. We consider the matrix

$$J(\omega) = \begin{pmatrix} -\frac{\partial B}{\partial x}(\mathbf{0}) & -\frac{\partial B}{\partial y}(\mathbf{0}) \\ \frac{\partial A}{\partial x}(\mathbf{0}) & \frac{\partial A}{\partial y}(\mathbf{0}) \end{pmatrix}.$$

Denote by λ, μ the eigenvalues of $J(\omega)$. We say that P_0 is a simple singularity of $\mathcal{F}$ if $J(\omega)$ is non-nilpotent (that is $(\lambda, \mu) \neq (0, 0)$), and one of the following two conditions is satisfied:

1. $\lambda\mu = 0$ (saddle-node case).
2. $\lambda\mu \neq 0$ and λ/μ is not a positive rational number.

We need the strongest concept of simple point with respect to a normal crossings divisor $E^0 \subset M_0$. Namely, we say that the point P_0 is a simple point of $(M_0, E^0, \mathcal{F})$ if one of two following conditions is satisfied:

a) P_0 is a simple singularity of $\mathcal{F}$, there exists an irreducible component of E^0 through P_0 and all the irreducible components of E^0 are invariant by $\mathcal{F}$.
b) The point P_0 is a regular point of $\mathcal{F}$ and $\mathcal{F}$ has normal crossings with E^0. That is to say, if L is the only invariant curve of $\mathcal{F}$ through P_0, then $E^0 \cup L$ is a normal crossings divisor.

It is possible to define the notion of strict transform of a foliation by a blow-up $\sigma : (M_1, E^1) \to (M_0, P_0)$ with center P_0. In local terms, consider $Q \in E^1$ and (x_1, y_1) a local system of coordinates of (M_1, E^1) at Q, such that $x_1 = 0$ is a local implicit equation of E^1. Given a foliation $\mathcal{F}$ defined by a 1-form ω, the strict transform of $\mathcal{F}$ by σ at Q is the foliation defined by $x_1^{-k}\pi^*\omega$. Here k is the maximum integer number such that $x_1^{-k}\pi^*\omega$ is holomorphic at Q. The notion of strict transform of a foliation extends inductively to finite sequences of blow-ups.

A. Seidenberg shows in [51] that there exists a resolution of singularities of $\mathcal{F}$ by considering a finite sequence of blow-ups, in the following sense: we say that $\pi : (M_N, E^N) \to (M_0, P_0)$ is a resolution of singularities of $\mathcal{F}$ if its strict transform $\widetilde{\mathcal{F}}$ by π has only simple points of $(M_N, E^N, \tilde{\mathcal{F}})$.

Let $\pi : (M_N, E^N) \to (M_0, P_0)$ be a sequence of blow-ups and consider an irreducible component D of the exceptional divisor E^N. Let $u = 0$ be a reduced implicit equation of D at a point $Q \in D$. The *divisorial value* $\nu_D(h)$ of a function $h \in \mathcal{O}_{M_0,P_0}$ is the number of times that u divides π^*h. The notion of divisorial value can be extended to 1-forms and 2-forms by means of adequate logarithmic presentations in the sense of K. Saito [49].

Now assume that π is a cuspidal sequence with Puiseux pair (n, m) and cuspidal divisor D. Let (x, y) be a system of adapted coordinates with respect to π. Given a function $h \in \mathcal{O}_{M_0,P_0}$, we write h as

$$h = \sum_{\alpha,\beta \geq 0} a_{\alpha\beta} x^\alpha y^\beta.$$

Since we are in a system of adapted coordinates, the divisorial value of h corresponds with $\nu_D(h) = \min\{n\alpha + m\beta : a_{\alpha\beta} \neq 0\}$ (see Proposition 2.3.2). Besides, for the case of 1-forms, we have that given $\omega = Adx + Bdy$, then $\nu_D(\omega) = \min\{\nu_D(xA), \nu_D(yB)\}$. Finally, for $\eta = gdx \wedge dy$, we obtain that $\nu_D(\eta) = \nu_D(xyg)$ (see Propositions 2.3.7 and 2.3.9).

As we have already said, a foliation $\mathcal{F}$ is *totally D-dicritical* when its strict transform by π is regular, transverse to D, and has normal crossings with E at all the points of D. We also say that 1-form ω is totally D-dicritical when the foliation defined by ω is totally D-dicritical.

In this work we give an equivalent combinatorial condition to total D-dicriticalness. Let us precise it. Consider $\omega \in \Omega^1_{M_0,P_0}$ such that $\nu_D(\omega) = q$ and write

$$\omega = \sum_{\alpha,\beta\geq 0} x^\alpha y^\beta \left\{\mu_{\alpha\beta}\frac{dx}{x} + \xi_{\alpha\beta}\frac{dy}{y}\right\}.$$

The Newton cloud of ω is $\mathcal{NC}_{x,y}(\omega) = \{(\alpha,\beta) : \mu_{\alpha\beta} \neq 0 \text{ or } \xi_{\alpha\beta} \neq 0\}$, and the initial part of ω is given by

$$\mathrm{In}_{n,m;x,y}(\omega) = \sum_{n\alpha+m\beta=q} x^\alpha y^\beta \left\{\mu_{\alpha\beta}\frac{dx}{x} + \xi_{\alpha\beta}\frac{dy}{y}\right\}.$$

We say that ω is *resonant* if there exists a non zero constant μ such that

$$\mathrm{In}_{n,m;x,y}(\omega) = \mu x^\alpha y^\beta \left\{m\frac{dx}{x} - n\frac{dy}{y}\right\}.$$

Let (b,d) be such that $dn - bm = 1$ and with the property that $0 \leq b < n$ and $0 < d \leq m$. We define the region $R^{n,m}$ by $R^{n,m} = H^{n,m}_- \cap H^{n,m}_+$, where

$$\begin{aligned} H^{n,m}_- &= \{(\alpha,\beta) \in \mathbb{R}^2;\ (n-b)\alpha + (m-d)\beta \geq 0\},\\ H^{n,m}_+ &= \{(\alpha,\beta) \in \mathbb{R}^2;\ b\alpha + d\beta \geq 0\}, \end{aligned}$$

We say that ω is *pre-basic* if $\mathcal{NC}_{x,y}(\mathrm{In}_{x,y,n,m}(\omega)) = \{(a,b)\}$ and $\mathcal{NC}_{x,y}(\omega) \subset (a,b) + R^{n,m}$. Note that $\nu_D(\omega) < nm$ is a sufficient condition to assure that ω is pre-basic, see Proposition 2.4.11.

We prove that being totally D-dicritical is equivalent to being pre-basic and resonant, see Proposition 2.5.1).

Cuspidal Semimodules

We describe structural properties of the semimodule of differential values of a cusp.

Recall that any Γ-semimodule Λ has a basis $B = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ and we have the *decomposition sequence* of Λ given by:

$$\Lambda_{-1} = (\lambda_{-1} + \Gamma) \subset \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_s = \Lambda; \quad \Lambda_j = \bigcup_{i=-1}^{j} (\lambda_i + \Gamma),\ \text{for } j = -1, 0, \ldots, s.$$

Assume that Γ is cuspidal, that is, $\Gamma = \langle n, m\rangle$. In this case Λ is called a *cuspidal semimodule*, even if Λ does not correspond with the semimodule of differential values of a cusp. The *axes* u^n_i, u^m_i, u_i and $\tilde{u}_i$, and the *critical values* $t_{-1}, t_0, t^n_i, t^m_i, t_i$ and $\tilde{t}_i$, are structural parameters of Λ, as in the case of the semimodule of differential values.

We say that Λ is an *increasing* semimodule if $\lambda_i > u_i$ for $i = 1, \ldots, s$. We have that Λ is the semimodule of differential values of a cusp C if and only if $\lambda_{-1} = n$, $\lambda_0 = m$ and Λ is increasing, see [21, 2]. Furthermore, in this case, if $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$ is a minimal standard basis of the corresponding cusp C, we have that $\nu_D(\omega_i) = t_i$ for $i = -1, 0, \ldots, s$, where D is the cuspidal divisor of C, see Theorem 5.2.10.

In order to understand the structure of Λ we need to define extra parameters associated to the semimodule. In particular we define the *bounds* k^n_i, k^m_i, the *limits* $\ell^n_{i+1}, \ell^m_{i+1}$, and the *colimits*

a_{i+1}, b_{i+1}, accordingly to the following relations (see Lemma 3.5.1):

$$\begin{aligned} u_{i+1}^n &= \lambda_i + n\ell_{i+1}^n = \lambda_{k_i^n} + mb_{i+1}, \\ u_{i+1}^m &= \lambda_i + m\ell_{i+1}^m = \lambda_{k_i^m} + na_{i+1}. \end{aligned}$$

The main technical results linking the previous parameters are the following ones:

a) The axes and critical values are ordered as follows (see Lemma 3.6.1):

1. $u_1 < u_2 < \ldots < u_{s+1} < \tilde{u}_{s+1} < \tilde{u}_s < \ldots < \tilde{u}_1$.
2. $t_1 < t_2 < \ldots < t_{s+1} < \tilde{t}_{s+1} < \tilde{t}_s < \ldots < \tilde{t}_1$.

b) The bounds k_i^n and k_i^m are determined inductively once we know if $u_i = u_i^n$ or $u_i = u_i^m$. More precisely, if $u_i = u_i^n$, we have that $k_i^n = i-1$ and $k_i^m = k_{i-1}^m$. If $u_i = u_i^m$, then $k_i^n = k_{i-1}^n$ and $k_i^m = i-1$, see Proposition 3.5.9.

c) The relationship between bounds, limits and colimits is given by (see Proposition 3.6.3):

1. If $k_i^n = i-1$, then $\ell_{i+1}^n + a_{i+1} = a_i$ and $\ell_{i+1}^m + b_{i+1} = \ell_i^m$.
2. If $k_i^m = i-1$, then $\ell_{i+1}^n + a_{i+1} = \ell_i^n$ and $\ell_{i+1}^m + b_{i+1} = b_i$.

The result a) allows to work with *initial parts* of elements of minimal standard basis. The results b) and c) are key for the presentation of different versions of Delorme's decomposition.

In the proof of above results, we have used the idea of circular intervals. Roughly speaking, it corresponds to a reordering of the intervals of length n, by considering its terms modulo n and dividing them by m modulo n.

It is worth to mention that in [3] and in [41] the authors give a different approach to study the combinatorics of these semimodules.

Standard Bases

The minimal standard bases that we consider in this work do not fit exactly with the classical standard bases concerning ideals, algebras and modules. Nevertheless, both cases can be treated with the same kind of techniques.

In this chapter, we recall the known general notions and algorithms about standard bases.

Delorme's Decompositions

In this chapter, we present several results that can be considered as avatars of the classical Delorme's Decomposition Theorem in [21]. These statements are the technical core that supports most of the proofs of the main results of this thesis. In particular, we end this chapter with a refinement, in the cuspidal case, of the usual algorithms for computing a minimal standard basis, and its use when the input is an implicit equation.

Consider Λ_C the semimodule of differential values of a cusp C with Puiseux pair (n, m) and a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$ of C. Let $\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ be the basis of Λ_C. Denote by Λ_i the intermediate semimodules of the decomposition sequence of Λ_C, and by t_i, u_i and k_i the critical values, axes and bounds of Λ_C. Let us also take a local system of coordinates (x, y) adapted to C. The main technical results in this work are the following ones:

Theorem (5.2.10)**.** *For each $1 \leq i \leq s$ we have the following statements*

1. $\lambda_i = \sup\{\nu_C(\omega) : \nu_D(\omega) = t_i\}$, here D stands for the cuspidal divisor of C.

2. *If* $\nu_C(\omega) = \lambda_i$*, then* $\nu_D(\omega) = t_i$.
3. *For each* 1*-form* ω *with* $\nu_C(\omega) \notin \Lambda_{i-1}$*, there is a unique pair* $a, b \geq 0$ *such that* $\nu_D(\omega) = \nu_D(x^a y^b \omega_i)$*. Moreover, we have that* $\nu_C(\omega) \geq \lambda_i + na + mb$.
4. *We have that* $\lambda_i > u_i$*. In particular, the semimodules* Λ_i *are increasing.*
5. *Let* $k = \lambda_i + na + mb$*, then* $k \notin \Lambda_{i-1}$ *if and only if for all* ω *such that* $\nu_C(\omega) = k$ *we have that* $\nu_D(\omega) \leq \nu_D(x^a y^b \omega_i)$.

Note that the divisorial value of an element of a minimal standard basis is fixed by the critical values.

Theorem (5.3.1 Decomposition Theorem)**.** *Consider indices* $0 \leq j \leq i \leq s$ *and denote by* $*$ *one of the elements* n *or* m*. Take* ω *a* 1*-form such that* $\nu_D(\omega) = t^*_{i+1}$ *and* $\nu_C(\omega) > u^*_{i+1}$*. There is a decomposition of the* 1*-form* ω *given by*

$$\omega = \textstyle\sum_{\ell=-1}^{j} f_\ell^{ij} \omega_\ell, \tag{2}$$

such that $v^*_{ij} = \min\{\nu_C(f_\ell^{ij}\omega_\ell);\ -1 \leq \ell < j\}$*, where* $v^*_{ij} = \nu_C(f_j^{ij}\omega_j)$ *and* $v^*_{ij} = \lambda_j + t^*_{i+1} - t_j$*. In particular, if* $j = i$*, we have that* $v^*_{ii} = \lambda_i + t^*_{i+1} - t_i = u^*_{i+1}$*. Moreover if* $-1 \leq \ell < j$*, the following holds:*

1. *If* $j < i$*, we have that* $\nu_C(f_\ell^{ij}\omega_\ell) = v^*_{ij}$ *for* $\ell = k_j$*, and* $\nu_C(f_\ell^{ij}\omega_\ell) > v^*_{ij}$*, for any* $\ell \neq k_j$.
2. *If* $j = i$*, we have that* $\nu_C(f_\ell^{ii}\omega_\ell) = v^*_{ii}$ *for* $\ell = k^*_j$*, and* $\nu_C(f_\ell^{ii}\omega_\ell) > v^*_{ii}$*, for any* $\ell \neq k^*_j$.

The previous writing in Equation (2) is what we have call in this text a Delorme's decomposition of ω.

Now let us sketch how to compute a minimal standard basis of the cusp C, when it is given in by an implicit equation $f = 0$.

The main detail is the following one: given two 1-forms ω, ω', such that $\nu_C(\omega) = \nu_C(\omega') < \infty$, we need to find the unique constant μ^+ such that $\nu_C(\omega + \mu^+\omega') > \nu_C(\omega)$. This is done in a straightforward way when starting with a primitive parametrization of C. In this chapter, we present a new method to do that from an implicit equation.

We consider the weighted monomial order $\preceq$ with respect (n, m) in $(\mathbb{Z}_{\geq 0})^2$. The order $\preceq$ is defined as: $(a, b) \prec (c, d)$ if either $na + mb < nc + md$ or $na + mb = nc + md$ with $a < c$. Write $\omega = Adx + Bdy$ and $\omega' = A'dx + B'dy$ and consider the vector fields $X_\omega = -B\partial_x + A\partial_y$ and $X_{\omega'} = -B'\partial_x + A'\partial_y$. Put h, h' "final reductions modulo $\{f\}$" of $X_\omega(f)$ and $X'_\omega(f)$ respectively. We show that $\nu_C(\omega) = \nu_C(\omega')$ implies that $lp(h) = lp(h')$, see Proposition 5.4.3. If we write $lt(h) = \mu x^a y^b$ and $lt(h') = \mu' x^a y^b$ (here lp and lt stand for the leading power and the leading term respectively). Then we have that the coefficient μ^+ we are looking for is $\mu^+ = -\mu/\mu'$.

As a consequence of this procedure, we extend some results of D. Pol in [48] and J. Briançon *et al.* in [9] relatives to the extended jacobian ideal $\mathcal{J}(f) = (f_x, f_y, f)$ of C. Namely, given $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$ a minimal standard basis of C, if we denote by h_i a final reduction of $X_{\omega_i}(f)$ modulo $\{f\}$, for $i = -1, 0, \ldots, s$, then $\{h_{-1}, h_0, \ldots, h_s\}$ is a minimal standard basis of $\mathcal{J}(f)$.

Standard Systems

In this chapter we enlarge the concept of minimal standard basis to extended standard basis and standard systems.

We say that a sequence of 1-forms $\mathcal{E} = (\omega_{-1}, \omega_0, \ldots, \omega_s, \omega_{s+1})$ is an *extended standard basis* of C when $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$ is a minimal standard basis of C and ω_{s+1} satisfies the following two conditions:

1. $\nu_D(\omega_{s+1}) = t_{s+1}$.
2. $\nu_C(\omega_{s+1}) = \infty$, that is, C is invariant by ω_{s+1} (see Lemma 2.1.1).

A *standard system* $(\mathcal{E}, \tilde{\mathcal{E}})$ for the cusp C is the data of an extended standard basis $\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1})$ and a family $\tilde{\mathcal{E}} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1})$ of 1-forms satisfying that

$$\nu_D(\tilde{\omega}_j) = \tilde{t}_j, \quad \nu_C(\tilde{\omega}_j) = \infty, \quad 1 \leq j \leq s+1.$$

We prove the existence of standard systems that complete a given minimal standard basis. Moreover, we show that ω_{s+1} and every $\tilde{\omega}_i$, except $\tilde{\omega}_1$, are basic and resonant, hence they are totally D-dicritical.

Analytic Semiroots

Let ω be a pre-basic and resonant 1-form in adapted coordinates with respect to the cusp C. By the results in Chapter 2, we know that the foliation defined by ω is totally D-dicritical, where D is the cuspidal divisor of C. Consider a non-corner point $P \in D$, there exists a unique branch C_P^{ω} invariant by ω passing through P, named an ω-cusp through P. The strict transform of C_P^{ω} in P is non singular and transverse to D at P. Thus, the resolution of singularities of C_P^{ω} is the same as the one of C. Note that two of these curves C_P^{ω} and C_Q^{ω}, with $P \neq Q$, are not in general analytically equivalent, see Example 7.8)

Consider an extended standard system $\mathcal{E} = (\omega_{-1}, \omega_0, \ldots, \omega_{s+1})$ of the cusp C. Recall that ω_i is totally D-dicritical for $i \geq 1$. We call *analytic weak $\mathcal{E}$-semiroot* of index $i \geq 1$ to any ω_i-cusp $C_P^{\omega_i}$. We say that $C_P^{\omega_i}$ is the *analytic $\mathcal{E}$-semiroot* of C of index i if P is the infinitely near point of C in D. We observe that the analytic $\mathcal{E}$-semiroot of index $s+1$ is the cusp C itself.

One of the most important results of this work is the following one:

Theorem. *For any analytic weak $\mathcal{E}$-semiroot $\gamma = C_P^{\omega_i}$ of index $1 \leq i \leq s+1$, then*

$$\mathcal{E}_i = (\omega_{-1}, \omega_0, \ldots, \omega_i)$$

is an extended standard basis of γ and the semimodule of differential values is $\Lambda_\gamma = \Lambda_{i-1}$. Moreover, we have the equality of differential values

$$\nu_C(\omega_\ell) = \nu_\gamma(\omega_\ell), \quad \textit{for} \quad -1 \leq \ell \leq i-1.$$

Let us expand $\mathcal{E}$ to a standard system

$$(\mathcal{E}, \tilde{\mathcal{E}}) = (\omega_{-1}, \omega_0, \ldots, \omega_{s+1}; \tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_{s+1}).$$

We have the following:

Theorem. *Let $\tilde{\gamma} = C_P^{\tilde{\omega}_i}$ be an $\tilde{\omega}_i$-cusp with $2 \leq i \leq s+1$. Then we have the equality of differential values*

$$\nu_C(\omega_\ell) = \nu_{\tilde{\gamma}}(\omega_\ell), \quad \textit{for} \quad -1 \leq \ell \leq i-1.$$

Moreover, we have the inclusion $\Lambda_{i-1} \subset \Lambda_{\tilde{\gamma}}$.

We give an example where the inclusion $\Lambda_{i-1} \subset \Lambda_{\tilde{\gamma}}$ is strict.

We end the chapter with an example of a branch with two Puiseux pairs that shows that it is not easy to generalize the previous results in a straightforward way.

Saito Bases and Other Analytic Invariants

Consider (C, P_0) any plane curve in (M_0, P_0). K. Saito showed in [49] that $\Omega^1_{M_0,P_0}[C]$ is a rank two free $\mathcal{O}_{M_0,P_0}$-module. We say that a basis of $\Omega^1_{M_0,P_0}[C]$ is a *Saito basis* of C.

Y. Genzmer in [25] introduced an analytic invariant for curves depending on Saito bases. More precisely, he defined the *Saito pair of multiplicities* of C as the following two numbers

$$\begin{aligned} \mathfrak{s}_{P_0}(C) &= \min\{\nu_{P_0}(\omega);\ \omega \text{ belongs to a Saito basis of } C\}, \\ \widetilde{\mathfrak{s}}_{P_0}(C) &= \max\{\nu_{P_0}(\omega);\ \omega \text{ belongs to a Saito basis of } C\}, \end{aligned}$$

where $\nu_{P_0}(\omega)$ is the multiplicity of ω at P_0. Finding a Saito basis is the most complicated part when determining the previous invariant. In this chapter we present a method to compute a Saito basis of a cusp C. It is based on the information coming from the semimodule of differential values. We build a Saito basis as follows:

Theorem (8.2)**.** *Denote by Λ_C the semimodule of differential values for the cusp C, with length $s \geq 0$. Let t_{s+1} and $\tilde{t}_{s+1}$ be the last critical values of Λ_C. Then, there are two 1-forms $\omega_{s+1}, \tilde{\omega}_{s+1}$ having C as an invariant curve and such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Moreover, for any pair of 1-forms as above, the set $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis for C.*

The existence of the 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$, satisfying the required properties, follows from the existence of standard systems proved in Chapter 6.

The proof of the previous theorem relies on the results about Delorme's decompositions of the 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$. We proceed to give a sketch of the proof.

First, according to [49], we have the following result

Lemma (Saito's Criterion)**.** *Let C' be a curve defined by the implicit equation $g = 0$. Given $\eta_1, \eta_2 \in \Omega^1_{M_0,P_0}[C']$, then $\{\eta_1, \eta_2\}$ is a Saito basis of C' if and only if*

$$\eta_1 \wedge \eta_2 = u g dx \wedge dy,$$

where $u \in \mathcal{O}_{M_0,P_0}$ is a unit, and (x, y) is the chosen coordinate system.

We use the Saito's criterion to check that Theorem 8.2 is true when the basis the semimodule of differential values of C is $B = (\lambda_{-1}, \lambda_0)$. In this case, by Theorem 1.5.2, we have that C is quasi-homogeneous, that is, we can find coordinates such that (t^n, t^m) is a primitive parametrization of C. It is a direct application of Saito's criterion that $\omega_1 = nxdy - mydx$ and $\tilde{\omega}_1 = ny^{n-1}dy - mx^{m-1}dx$ give a Saito basis of C. Note that $\nu_D(\omega_1) = t_1 = n + m$ and $\nu_D(\tilde{\omega}_1) = \tilde{t}_1 = nm$. Knowing that $\{\omega_1, \tilde{\omega}_1\}$ is a Saito basis of C, we can verify that any couple of 1-forms with C invariant and the same divisorial values as $\{\omega_1, \tilde{\omega}_1\}$ gives a Saito basis.

Now, if the basis B has more than two elements, we did not prove the theorem by checking that ω_{s+1} and $\tilde{\omega}_{s+1}$ satisfy Saito's criterion. In fact, the proof is split in two parts: first, we find a generator system of $\Omega^1_{M_0,P_0}[C]$ that contains $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$. Second, we show that the previous generator system can be reduced to just $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$.

We can generalize the Saito pair of multiplicities as follows. Let us consider $\pi' : (M_{N'}, E^{N'}) \to (M_0, P_0)$ a sequence of blow-ups and take $E \subset E^{N'}$ an irreducible component of the exceptional divisor. We define as before

$$\begin{aligned} \mathfrak{s}_{E}(C) &= \min\{\nu_{E}(\omega);\ \omega \text{ belongs to a Saito basis of } C\}. \\ \widetilde{\mathfrak{s}}_{E}(C) &= \max\{\nu_{E}(\omega);\ \omega \text{ belongs to a Saito basis of } C\}. \end{aligned}$$

The Saito pair of multiplicities $(\mathfrak{s}_E(C), \widetilde{\mathfrak{s}}_E(C))$ with respect to E is an analytic invariant. When π' is just a single blow, we recover the one defined by Y. Genzmer. We show in Theorem 8.4.3 that $(\mathfrak{s}_D(C), \widetilde{\mathfrak{s}}_D(C)) = (t_{s+1}, \tilde{t}_{s+1})$. In other words, the Saito pair of multiplicities, with respect to the cuspidal divisor D, is determined by the semimodule of differential values of the cusp C.

Nonetheless, we give an example that shows that $(\mathfrak{s}_{P_0}(C), \widetilde{\mathfrak{s}}_{P_0}(C))$ is not completely determined by Λ_C. Anyway, in [27], it is shown that we can deduce some information about $s_{P_0}(C)$ from the semimodule of differential values.

Roots of the Bernstein-Sato Polynomial

Consider C a curve in (M_0, P_0) with implicit equation $f = 0$. In [17], P. Cassou-Noguès gives algebraic conditions that the coefficients of f must satisfy to assure that a particular rational number is a root of the Bernstein-Sato polynomial.

In order to give the previous algebraic conditions, we need to use a special kind of adapted coordinates that we call nice coordinates. First we define the cuspidal sets P, J, M as

$$\begin{aligned} P &:= \{(p_1, p_2) \in (\mathbb{Z}_{\geq 0})^2 : 0 \leq p_1 < m-1, 0 \leq p_2 < n-1 \text{ and } np_1 + mp_2 > nm\}, \\ J &:= \{j = p_{1,j}n + p_{2,j}m - nm : (p_{1,j}, p_{2,j}) \in P\}, \\ M &:= \{(m - p_1 - 1, n - p_2 - 1) : (p_1, p_2) \in P\}. \end{aligned}$$

We say that (x, y) is a system of nice coordinates if in these coordinates we can find an implicit equation of C as

$$f = x^m + y^n + \sum_{j \in J} z_j x^{p_{1,j}} y^{p_{2,j}}; \quad z_j \in \mathbb{C}.$$

As we see, a nice equation does not depend on M, however, the conditions given by P. Cassou-Noguès do.

By means of those algebraic conditions and based on the computation of several examples, we proposed ourselves to prove the following statement

Conjecture (9.3). *Let C be a cusp with semigroup $\Gamma_C = \langle n, m \rangle$ and semimodule of differential values Λ_C. Then for any element $\lambda \in \Lambda_C \setminus \Gamma_C$, the rational number $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C.*

Up to this moment, we have proven the previous conjecture when $n \leq 4$ (see Theorem 9.2) and we have also showed that

Theorem (9.1). *Let C be a cusp with semigroup $\Gamma_C = \langle n, m \rangle$ and semimodule of differential values Λ_C. Assume that $\lambda_1 = \min(\Lambda_C \setminus \Gamma_C)$ exists. Then for any element $\lambda \in (\lambda_1 + \Gamma_C) \setminus \Gamma_C \subset \Lambda_C$, the rational number $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C.*

The idea for proving both theorems is the same one. We fix a semimodule of differential values Λ and we apply the algorithm for computing it, in this way we see which are the algebraic conditions that the coefficients of a nice equation of cusp must satisfy in order to have Λ as its semimodule of differential values. Once we have obtained these conditions, we just compare them with the ones given in [17]. This also explains why we did not give a general proof independently of n, and why we limit ourselves to $n \leq 4$. The calculations become more complex as the value of n increases.

1

Plane Curve Singularities

In this chapter we present the main object of study, germs of plane branches. They can be defined as the zero set of an implicit equation or via a parametrization. We mostly consider branches which are singular and the invariants that we can associate to them. In this work there are two types of invariants that we are interested in: those which only depend on the topology, and those which also relay on some analytic structure. For instance, O. Zariki showed that the equisingularity is a complete topological invariant for germs of plane curves. Concerning analytic invariants, the most relevant one presented in this chapter is the semimodule of differential values of a branch.

At the end of the chapter, we focus on the study of the family of branches with a single Puiseux pair. In this text, we call them cusps.

1.1 The Ring of Convergent Power Series

In this section, we will use [14, 30] as the main reference books. Along the whole text, we denote by $(M_0, \mathcal{O}_{M_0})$ a regular complex analytic surface. More precisely, the space $(M_0, \mathcal{O}_{M_0})$ is a ringed space in local $\mathbb{C}$-algebras, locally isomorphic to $\mathbb{C}^2$ with its structural sheaf of germs of holomorphic functions.

We are mostly interested in studying local behaviours. Given $G \subset M_0$ a compact set, we denote by (M_0, G) the germ at G of the analytic space $(M_0, \mathcal{O}_{M_0})$. When $G = \{P_0\}$, we simply write (M_0, P_0). In this last case, we have that after choosing a system of local coordinates (x, y) of M_0 at P_0, the local ring $\mathcal{O}_{M_0,P_0}$ is isomorphic to the *ring of convergent power series* in two variables $\mathbb{C}\{x, y\}$.

A *plane curve* $(C, \mathcal{O}_C)$ is a reduced analytic subspace of $(M_0, \mathcal{O}_{M_0})$ of codimension 1. At any point $P \in C$, the germ of the space (C, P) is defined by a principal ideal $(f) \subset \mathcal{O}_{M_0,P}$ different from zero or the total ring. Note that f may not be reduced, nonetheless we always consider its reduced structure.

The local ring $\mathcal{O}_{C,P}$ is isomorphic to $\mathcal{O}_{M_0,P}/(f)$. We say that f or $f = 0$ is a local *implicit equation* of the curve $(C, \mathcal{O}_C)$ at P. Most of the time we will work with germs of plane curves (C, P). When there is no confusion, we just write C without indicating the point P. We are only going to deal with plane curves, for this reason, we usually substitute the term "plane curve" by just "curve".

Remark 1.1.1. Note that we can define the notion of hypersurface in a complex analytic space of dimension $p \geq 2$ in a manner similar to the case of curves.

Now, we proceed to recall a few concepts associated to the ring of convergent power series in $p \geq 1$ variables. We say that $g \in \mathbb{C}\{x_1, \ldots, x_p\}$ is *regular of order k*≥ 1 at the variable x_p, if

$$g(\mathbf{0}) = 0,\ \frac{\partial g}{\partial x_n}(\mathbf{0}) = 0,\ \frac{\partial^2 g}{\partial x_p^2}(\mathbf{0}) = 0, \cdots, \frac{\partial^{k-1} g}{\partial x_p^{k-1}}(\mathbf{0}) = 0,\ \frac{\partial^k g}{\partial x_p^k}(\mathbf{0}) \neq 0,$$

where $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{C}^p$.

Given a polynomial $h \in \mathbb{C}\{x_1, \ldots, x_{p-1}\}[x_p]$, we say that h is a *Weierstrass polynomial* in the variable x_p if h satisfies the following two conditions: first, the director coefficient of h is 1, that is

$$h = x_p^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0, \quad a_{k-1}, a_{k-2}, \ldots, a_0 \in \mathbb{C}\{x_1, x_2, \ldots, x_{p-1}\}.$$

Second, the coefficients are zero at $\mathbf{0}$, $a_i(\mathbf{0}) = 0$ for $i = 0, 1, \ldots, k-1$.

Theorem 1.1.2. *(Weierstrass preparation Theorem) Given $g \in \mathbb{C}\{x_1, \ldots, x_p\}$ regular of order k at the variable x_p, there exists a unique unit $u \in \mathbb{C}\{x_1, \ldots, x_p\}$ such that $g = uh$ where $h \in \mathbb{C}\{x_1, \ldots, x_{p-1}\}[x_p]$ is a Weierstrass polynomial in the variable x_p.*

Because of Theorem 1.1.2, we can define locally any irreducible curve in the space (M_0, P_0), possibly after renaming the variables, by an element of $\mathbb{C}\{x\}[y]$ with unitary director coefficient. Another consequence of the Weierstrass preparation Theorem is the following.

Remark 1.1.3. The ring $\mathbb{C}\{x_1, \ldots, x_p\}$ is a noetherian unique factorization domain.

Let (C, P_0) be a plane curve defined by the implicit equation $f = 0$. The equation f can be factorized as $f = f_1^{r_1} f_2^{r_2} \cdots f_s^{r_s}$, with $f_i \in \mathbb{C}\{x, y\}$ for $i = 1, 2, \ldots, s$. An irreducible factor f_i defines an irreducible component C_i of C; each one of them is said to be a plane *branch*, or just branch, of C. When C is irreducible, we omit the subscript and we identify the curve with its only branch.

Given a curve (C, P_0) with implicit equation $f \in \mathbb{C}\{x, y\}$, we say that the curve C is *singular* at P_0 if the jacobian matrix $J(f)$ of f is zero at the point P_0. Otherwise, we say that the plane curve C is *regular* at P_0. A first approach to classify singular curves is in terms of their multiplicity. If we write $f = \sum_{k \geq 0} f_k$ with $f_k \in \mathbb{C}[x, y]$ an homogeneous polynomial of degree k. The function f, or the curve C, has *multiplicity* k_0 at P_0 if $f_k = 0$ for $k < k_0$ and $f_{k_0} \neq 0$. The cone $f_{k_0} = 0$ is called the *tangent cone* of f, or the tangent cone of the curve C. We denote by $\nu_{P_0}(f)$, or by $\nu_{P_0}(C)$, the multiplicity at P_0. Notice that $f = 0$ is singular at point P_0 if and only if the multiplicity $\nu_{P_0}(f)$ is at least 2. Additionally, the multiplicity does not depend on the system of coordinates.

Remark 1.1.4. By Hensel's lemma, we have that the tangent cone of a branch is a single straight line, that we count with multiplicity k_0.

1.2 The Newton-Puiseux Theorem

We can find parametrizations of branches using one variable power series. This is done by means of the classical Newton-Puiseux algorithm, where the reader is refered to [14, 56] for more details. The computation relies on the Newton polygon of the curve, that we proceed to define.

In general, given an element $g \in \mathbb{C}\{x_1, x_2, \ldots, x_p\}$, we can decompose it as a sum of its monomials, that is, $g = \sum_{\alpha \geq \mathbf{0}} g_\alpha x^\alpha$, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in (\mathbb{Z}_{\geq 0})^p$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_p^{\alpha_p}$. We define the *Newton cloud* of g as

$$\mathcal{NC}_{x_1, x_2, \ldots, x_n}(g) = \{\alpha : g_\alpha \neq 0\},$$

and the *Newton polytope* of g as:

$$\mathcal{NP}_{x_1,x_2,\ldots,x_n}(g) := \text{ convex hull of } \Big(\bigcup_{\alpha \in \mathcal{NC}_{x_1,x_2,\ldots,x_n}(g)} (\alpha + (\mathbb{R}_{\geq 0})^p) \Big).$$

In the two dimensional case, the Newton polytope is called the *Newton polygon*.

We include the coordinate system in the notation for both the Newton cloud and the Newton polytope, to remark their dependency on the coordinates chosen. We have introduced these notions in a general p variable case, because they reappear in Chapter 4 when talking about standard bases.

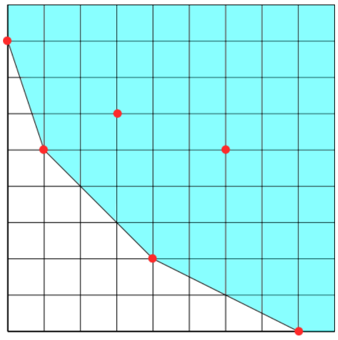

Figure 1.1: *The figure above represents the Newton cloud of* $y^8 + xy^5 + 6x^3y^6 - 7x^4y^2 + 11x^6y^5 + x^9$ *(red dots) and its Newton polygon (blue part).*

The border of a Newton polygon is always defined by two semi lines s_0, s_∞: one parallel to the horizontal axis and the other one parallel to the vertical axis; and a set (it may be empty) of segments $\{s_\ell\}_{\ell \in L}$ in $(\mathbb{R}_{\geq 0})^2$ of strictly increasing negative slopes.

By a *parametrization* of a plane branch C with implicit equation $f = 0$, we mean a map

$$\phi^\# : \mathbb{C}\{x, y\} \simeq \mathcal{O}_{M_0,P_0} \longrightarrow \mathbb{C}\{t\} \simeq \mathcal{O}_{\mathbb{C},0},$$

such that $\phi^\#(f) = 0$. A parametrization can be identified with a non constant map

$$\phi : (\mathbb{C}, 0) \longrightarrow (M_0, P_0),$$

given by $\phi(t) = (x(t), y(t))$ with $x(t), y(t) \in \mathbb{C}\{t\}$, and in particular the kernel is (f). We also say that $\phi(t)$ is a parametrization of the curve. Besides, the map $\phi^\#$ can be determined by $\phi(t)$, just by putting $\phi^\#(h) = h \circ \phi$. As stated above, we can compute parametrizations by means of the Newton-Puiseux algorithm. In fact, these parametrizations can be seen as roots of an implicit equation.

Just as a remark the reader is referred, for instance, to [29]. There it is treated a family of hypersurfaces where we can find a parametrizations similarly to the case of plane curves. They are called quasi-ordinary hypersurfaces.

Now, assume that the multiplicity of f at P_0 is $\nu_{P_0}(f) = n \geq 2$, and consider $f_n = 0$ the tangent cone of f. Up to reordering the coordinates (x, y), we can assume that $f_n = c(y + \lambda x)^n$. A parametrization $\phi(t)$ of f or C is a *primitive parametrization* if $\phi(t) = (t^n, y(t))$. Note that in this case we have that

$$y(t) = \sum_{i \geq n} a_i t^i,$$

with $gdc(n, \{i : a_i \neq 0\}) = 1$.

Theorem 1.2.1 (Newton-Puiseux Theorem)**.** *Any branch C admits a parametrization $\phi(t) = (x(t), y(t))$ with $x(t), y(t) \in \mathbb{C}\{t\}$.*

Remark 1.2.2. Not only we can guarantee the existence of a parametrization, but also, we can find a primitive one. For this reason, we will only work with primitive parametrizations.

We can recover implicit equations from primitive parametrizations as follows: consider $\phi(t) = (t^n, s(t))$ a primitive parametrization of C. An implicit equation of C is given by the formula:

$$f_\phi = \prod_{k=0}^{n-1} \left(y - s\left(\exp(2\pi i k/n) x^{1/n}\right)\right), \tag{1.1}$$

where $x^{1/n}$ is a n^{th}-root of x. It is satisfied the following:

- The curve defined by $f_\phi = 0$ is irreducible.
- Given $g \in \mathbb{C}\{x, y\}$, then $g(t^n, s(t)) = 0$ if and only if f_ϕ divides g.

If $s(t)$ is a polynomial, we have another way to compute an implicit equation of the branch C. We can consider the polynomials $x - t^n, y - s(t) \in \mathbb{C}[x, y, t]$. The resultant of $x - t^n$ and $y - s(t)$ with respect to the variable t is an implicit equation of C, see [52] Theorem 4.39.

Assume that $f = 0$ is an implicit equation of a branch C. By Weierstrass preparation Theorem 1.1.2, we can assume that f is either a Weierstrass polynomial in $\mathbb{C}\{x\}[y]$ or in $\mathbb{C}\{y\}[x]$. This is because x or y, but not both, do not divide the tangent cone of a branch C.

Example 1.2.3. We want to emphasize that any curve defined by the an implicit equation as follows is irreducible:

$$f = y^n + \mu x^m + \sum_{ni+mj>nm} a_{ij} x^i y^j; \quad \mu \neq 0,$$

where $gcd(n, m) = 1$. We make this observation, because most of the branches we are going to consider are as above.

Remark 1.2.4. A parametrization $\phi^\#$ of a branch $f = 0$ induces an isomorphism:

$$\phi^\# : \mathcal{O}_{C,P_0} \equiv \mathbb{C}\{x, y\}/(f) \longrightarrow \mathbb{C}\{x(t), y(t)\} \subset \mathbb{C}\{t\},$$

Recall that $\phi^\#$ sends x to $x(t)$ and y to $y(t)$. In this way we can see the local ring of a plane branch in terms of an implicit equation or in terms of a parametrization.

Consider $\phi(t)$ a primitive parametrization of the branch (C, P_0). As we mentioned, the parametrization can be written as:

$$\phi(t) = (x(t), y(t)) = (t^n, \sum_{i \geq n} a_i t^i). \tag{1.2}$$

The *characteristic exponents* $(\beta_0, \beta_1, \ldots, \beta_g)$ of the branch C and the list $(e_0, e_1, \ldots, e_g)$ are defined as follows

- $\beta_0 = e_0 = n$.
- For $j \geq 1$ the exponent β_j is the minimum index $i > \beta_{j-1}$ satisfying both $a_i \neq 0$ and

$$gcd(i, e_{j-1}) < e_{j-1}.$$

Next we define $e_j = gcd(\beta_j, e_{j-1}) < e_{j-1}$.

The number of characteristic exponents is finite, and the index g is called the *genus* of the branch C. We have that $gcd(\beta_0, \beta_1, \ldots, \beta_g) = 1 = e_g$. Besides, the branch is singular if and only if $g \geq 1$.

From these two lists, we can define the *Puiseux pairs* (n_i, m_i) of C, for $i = 1, 2, \ldots, g$, as $n_i = e_{i-1}/e_i$ and $m_i = \beta_i/e_i$.

We say that a curve of genus one, or with a single Puiseux pair, is a *cusp*. At the end of the chapter we will give an equivalent definition of cusp in terms of its resolution of singularities.

Primitive parametrizations allow to compute in an easy way intersection multiplicities. We recall that given two functions $f, g \in \mathbb{C}\{x, y\}$, the *intersection multiplicity* f and g at the point P_0 is defined as

$$i_{P_0}(f, g) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, g).$$

If f and g are implicit equations of two curves C_1 and C_2, the intersection multiplicity between C_1 and C_2 at P_0 is given by $i_{P_0}(C_1, C_2) := i_{P_0}(f, g)$.

Note that the intersection multiplicity is additive with respect of the product of functions:

$$i_{P_0}(f, g_1 g_2) = i_{P_0}(f, g_1) + i_{P_0}(f, g_2).$$

When f is an implicit equation of a branch C and $\phi(t)$ a primitive parametrization, then the intersection multiplicity is also given by

$$i_{P_0}(f, g) = \mathrm{ord}_t(g \circ \phi) \tag{1.3}$$

(see [56]). Note that when C_1 and C_2 are two regular curves, we have that

$$i_{P_0}(C_1, C_2) \geq 1.$$

When the previous intersection multiplicity is exactly one, we say that C_1 and C_2 are *transverse*. Otherwise, we say that they are *tangent*.

1.3 Resolution of Singularities

The resolution of singularities is a central problem is analytic/algebraic geometry, the reader is referred to [4] for more a detailed introduction. We are going to describe the resolution of singularities of germs of plane curves (see [14, 56]).

We consider a local system of coordinates (x, y) of the surface M_0 at a point $P_0 \in M_0$, defined in an open subset $U \subset M_0$, that is, the point P_0 corresponds with $(x = 0, y = 0)$. We consider the next subvariety

$$Bl_{U,P_0} := \{((x, y), [\alpha_1 : \alpha_2]) \in U \times \mathbb{P}^1(\mathbb{C}) : x\alpha_2 = y\alpha_1\} \subset U \times \mathbb{P}^1(\mathbb{C}).$$

The first projection defines is a birational map $\sigma : Bl_{U,P_0} \longrightarrow U$. We say that σ is the *blow-up* of U with *center* P_0. In particular, we have that $\sigma^{-1}(P)$ is just a point if $P \neq P_0$ and $\sigma^{-1}(P_0) \cong \mathbb{P}^1(\mathbb{C})$. The restriction map $\sigma : Bl_{U,P_0} \setminus \sigma^{-1}(P_0) \longrightarrow U \setminus \{P_0\}$ is an isomorphism.

Thus, following the approach in [14], we can "patch" Bl_{U,P_0} at M_0 along $U \setminus \{P_0\}$ via σ, obtaining a new complex regular surface M_1. This surface M_1 is also called the *blow-up* of M_0 at P_0. In other words, M_1 is obtained by changing the set U by Bl_{U,P_0}.

We emphasize that the map σ extends to the whole surface M_1. As a simplification, we also denote by $\sigma : M_1 \longrightarrow M_0$ this extension map of $\sigma : Bl_{U,P_0} \longrightarrow U$. The term blow-up will be indistinguishably used to refer to the surface M_1 or the projection map σ.

The curve $E^1 = \sigma^{-1}(P_0) = \mathbb{P}^1(\mathbb{C})$ is named the *exceptional divisor* of the blow-up σ.

Let us describe Bl_{U,P_0} in terms of an atlas. Recall that the atlas of $\mathbb{P}^1(\mathbb{C})$ is defined by two charts $\tilde{V}_1, \tilde{V}_2 \subset \mathbb{P}^1(\mathbb{C})$ homeomorphic to two copies V_1 and V_2 of $\mathbb{C}$. The charts $\tilde{V}_1$ and $\tilde{V}_2$ correspond to the sets $\{[\alpha_1 : \alpha_2] \in \mathbb{P}^1(\mathbb{C}) : \alpha_i \neq 0\}$ with $i = 1, 2$.

This atlas allows to define two open charts $\tilde{U}_1, \tilde{U}_2$ of Bl_{U,P_0}, which are identified with two open sets of U_1 and U_2 in $\mathbb{C}^2$. These charts are given by the maps

$$U_1 \longrightarrow \tilde{U}_1 \subset Bl_{U,P_0},$$
$$(x_1, y_1) \mapsto ((x_1, x_1 y_1), [1 : y_1])$$

$$U_2 \longrightarrow \tilde{U}_2 \subset Bl_{U,P_0},$$
$$(x_2, y_2) \mapsto ((x_2 y_2, y_2), [x_2 : 1])$$

The transition map between $U_1 \setminus \{y_1 = 0\}$ and $U_2 \setminus \{x_2 = 0\}$ is given by the identification $x_2 = 1/y_1$ and $y_2 = x_1 y_1$. Sometimes, we will just write $(x, y) = (x_1, x_1 y_1)$ or $(x, y) = (x_2 y_2, y_2)$ to denote the previous coordinate systems in the charts U_1, U_2.

In these charts, we have that σ is defined by $\sigma(x_1, y_1) = (x_1, x_1 y_1)$ and $\sigma(x_2, y_2) = (x_2 y_2, y_2)$ with $(x_i, y_i) \in U_i$ for $i = 1, 2$. Additionally, the exceptional divisor E^1 is given by the implicit equation $x_1 = 0$ in all the points of $E^1 \cap U_1$. Similarly, $y_2 = 0$ is a local implicit equation of E^1 in all the points of $E^1 \cap U_2$.

Now we consider finite compositions of blow-ups starting a germ of surface (M_0, P_0). That is, we consider compositions

$$\pi : (M_N, E^N) \xrightarrow{\sigma_N} (M_{N-1}, E^{N-1}) \xrightarrow{\sigma_{N-1}} (M_{N-2}, E^{N-2}) \xrightarrow{\sigma_{N-2}} \ldots \xrightarrow{\sigma_1} (M_0, P_0).$$

Here the morphism $\sigma_1 : (M_1, E^1) \longrightarrow (M_0, P_0)$ is the blow-up of (M_0, P_0) with center P_0, the morphism $\sigma_2 : (M_2, E^2) \longrightarrow (M_1, E^1)$ is the blow-up of the germ space (M_1, E^1) with center a point $P_1 \in E^1$. We put $E^2 = (\sigma_1 \circ \sigma_2)^{-1}(P_0)$. Inductively, the morphism $\sigma_i : (M_i, E^i) \longrightarrow (M_{i-1}, E^{i-1})$ is the blow-up of (M_{i-1}, E^{i-1}) with center a point $P_{i-1} \in E^{i-1}$ and $E^i = (\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_i)^{-1}(P_0)$, for $i = 2, 3, \ldots, N$. The curve E^N is called the exceptional divisor of the sequence π. The irreducible components of E^N are isomorphic to $\mathbb{P}^1(\mathbb{C})$, they cut two by two in a transverse way and a point belongs to at most to two of them (normal crossings).

Given $f = 0$ a curve C, we define its *total transform* by π as the curve $\pi^*(f) = f \circ \pi = 0$. We define the *strict transform* of C by π as the union of the irreducible components of the total transform of C different from the components of exceptional divisor. A local implicit equation of the strict transform is obtained locally by dividing $\pi^*(f)$ by local implicit equations of the exceptional divisor as many times as possible.

Consider an intermediate exceptional divisor E^i with $i = 1, \ldots, N$. We have that E^i is the union of i irreducible components E^i_j, with $j = 1, \ldots, i$. In an inductive way the curve E^i_j is the strict transform of E^{i-1}_j by σ_i for $j < i$ and $E^i_i = \sigma_i^{-1}(P_{i-1})$. A point $P \in E^i$ is said to be a *corner point* if P is in the intersection point of $E^i_\ell \cap E^i_j$ for two different indices $1 \leq \ell, j \leq i$, otherwise we say that P is a *free point*.

Remark 1.3.1. The intersection of two irreducible components of E^i is at most one point. In fact, given $j < i$, then E^{i+1}_{i+1} intersects E^{i+1}_j if and only if $P_i \in E^i_j$. An irreducible component of the exceptional divisor intersects at most two other irreducible components.

Given a curve C defined by $f = 0$ and $\pi : (M_N, E^N) \longrightarrow (M_0, P_0)$ a sequence of blow-ups starting at P_0, we say that C *passes through* $P \in E^N$ if the strict transform of f at P is not a unit.

More precisely, we define an *infinitely near point* of C as a point such that there exists a sequence of blow-ups as before.

As mentioned before, the exceptional divisor of any sequence of blow-ups defines a curve with normal crossings. More explicitly, given a curve (C, P_0) in (M_0, P_0), we say that it is a *curve with normal crossings* at P_0, if in some local coordinates (x, y) and for $\epsilon = 0$ or $\epsilon = 1$, then $xy^{\varepsilon} = 0$ is an implicit equation of C.

The theorem of resolution of singularities of plane curves is stated as follows (see for instance [56]).

Theorem 1.3.2 (Resolution of singularities). *Consider (C, P_0) a curve at the point $P_0 \in M_0$. There exists a finite sequence of blow-ups $\pi : (M_N, E^N) \longrightarrow (M_0, P_0)$, with $N \geq 1$, satisfying that the total transform of C has normal crossings at any point of E^N.*

The sequence π can be taken minimal, in the sense that for any other sequence of blow-ups π' satisfying the previous conditions, then $\pi' = \rho \circ \pi$ with ρ the identity map or a sequence of blow-ups.

Note that, under the conditions of the previous theorem, the strict transform of the curve C by π is non singular.

A sequence of blow-ups as in Theorem 1.3.2 is said to be a *resolution of singularities* of C. If π is the minimal one, then we will say that π is the *minimal resolution of singularities* of the curve C.

Remark 1.3.3. In order to obtain the minimal resolution of singularities it is enough to blow-up successively at points where the total transforms has no normal crossings. These points are necessary in the strict transform of C.

Proposition 1.3.4. *Consider C a curve in (M_0, P_0), let us fix local coordinates (x, y) in P_0 and let T_C be the tangent cone of C in this coordinates. Let $\pi : (M_1, E^1) \longrightarrow (M_0, P_0)$ be the blow-up with center P_0. Then we have that*

$$C' \cap E^1 = (T_C)' \cap E^1,$$

where C' and $(T_C)'$ denote respectively the strict transforms by π of C and T_C.

Proof. Consider a local system (x, y) of coordinates in (M_0, P_0). Take $f = 0$ an implicit equation of C, then we can write

$$f(x, y) = \sum_{i \geq n = \nu_{P_0}(f)} f_i(x, y),$$

where $f_i \in \mathbb{C}[x, y]$ is an homogeneous polynomial of degree i and $f_n = 0$ defines the tangent cone T_C of C, with $n \geq 1$.

Now, we perform the blow-up $\sigma : (M_1, E^1) \longrightarrow (M_0, P_0)$ with center P_0. We consider the chart U_1 of (M_1, E^1) with coordinate system $(x, y) = (x_1, x_1 y_1)$. The second chart is treated in a similar way. In the coordinates (x_1, y_1) we see that the total transform F of f is given by

$$F = f \circ \sigma = x_1^n \sum_{i \geq n = \nu_{P_0}(f)} x_1^{i-n} f_i(1, y_1) = x_1^n \Big(f_n(1, y_1) + x_1 \sum_{i \geq n+1} x_1^{i-n-1} f_i(1, y_1) \Big).$$

The strict transform of f is given by $x_1^{-n} F$ in the chart U_1, and the the strict transform of the tangent T_C is given by $f_n(1, y_1) = 0$. Noting that $U_1 \cap E^1$ is $x_1 = 0$, from the above formula we conclude that

$$C' \cap E^1 \cap U_1 = (T_C)' \cap E^1 \cap U_1,$$

as desired. □

Example 1.3.5. Let us describe the minimal resolution of singularities of the curve $(C, \mathbf{0})$ in $(\mathbb{C}^2, \mathbf{0})$ defined by the implicit equation $f = y^2 - x^3$. First, we consider the blow-up $\sigma_1 : (M_1, E^1) \to (\mathbb{C}^2, \mathbf{0})$ with center $\mathbf{0}$. We are going to find the points where the total transform has no normal crossings. These points must belong to the intersection of the strict transform C_1 of C and the exceptional divisor $E^1 = \sigma_1^{-1}(\mathbf{0})$. The only point P_1 in $C_1 \cap \sigma_1^{-1}(\mathbf{0})$ is in the strict transform of the tangent cone of $y = 0$ of C. It is precisely the origin of the first chart U_1 of M_1. In this chart we have coordinates (x_1, y_1) such that $(x, y) = (x_1, x_1 y_1)$. The total transform of f is $f_1 = x_1^2(y_1^2 - x_1)$. It has no normal crossings at P_1 because the parabola $y_1^2 = x_1$ is tangent to the line $x_1 = 0$. Then we have to perform a new blow-up with center P_1.

Note that this parabola is the strict transform C_1 of C and its tangent cone is defined by $x_1 = 0$. Let $\sigma_2 : (M_2, E^2) \to (M_1, E^1)$ be the blow-up with center P_1. By the same argument as above, the next point $P_2 \in E^2$ without normal crossings, if it exists, it is given by the strict transform of $y_1^2 = x_1$. Then P_2 belongs to the strict transform of the tangent cone $x_1 = 0$. Thus, it is precisely the origin of the second chart U_2 of the blow-up σ_2, given in coordinates by $(x_1, y_1) = (x_2 y_2, y_2)$. The exceptional divisor E^2 is locally given at P_2 by the equation $x_2 y_2 = 0$, the total transform of C is given $f_2 = x_2^2 y_2^3(y_2 - x_2)$. We have three different lines at the point P_2 and this object has no normal crossings. Note that the strict transform of C at P_2 is given by the line $y_2 = x_2$.

Now, let us perform the blow-up $\sigma_3 : (M_3, E^3) \to (M_2, E^2)$ with center P_2. We can read the point $P_3 \in E^3$ of the strict transform of $y_2 = x_2$ in the first chart $(x_2, y_2) = (x_3, x_3 y_3)$ as $x_3 = 0$ and $y_3 = 1$. In this point we have normal crossings for the total transform $f_3 = x_3^6 y_3^3(y_3 - 1)$. Note that in coordinates $\tilde{x}_3 = x_3$ and $\tilde{y}_3 = y_3 - 1$ centered at P_3, the total transform is written as $f_3 = (\tilde{y}_3 + 1)^3 \tilde{x}_3^6 \tilde{y}_3$. Hence, it is a unit times a monomial and the property of having normal crossings is satisfied.

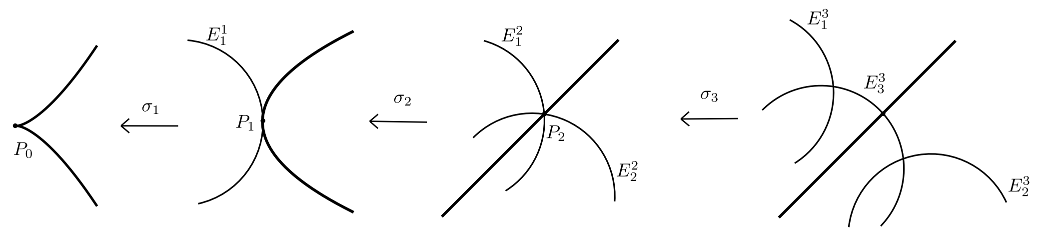

Figure 1.2: *Schematic drawing with the resolution of singularities of $y^2 - x^3$ step by step. The thick curves represent different strict transforms of C by the different blow-ups.*

We end this section showing how to perform the previous computations from a parametrization.

Assume that C is a branch in (M_0, P_0) defined by a primitive parametrization $\phi(t) = (t^n, \sum_{i \ge n} a_i t^i) = (x(t), y(t))$. We take a blow-up $\sigma_1 : (M_1, E^1) \to (M_0, P_0)$, and we consider the two charts U_1 and U_1' of M_1, defined respectively by the systems of coordinates $(x, y) = (x_1, x_1 y_1)$ and $(x_1' y_1', y_1')$. The reader can check that C does not pass through any point of $E^1 \cap U_1'$.

The strict transform C_1 of C by σ_1 in the chart U_1 is defined by the parametrization $(x_1(t), y_1(t))$, where $x_1(t) = x(t) = t^n$ and $y_1(t) = y(t)/x(t) = \sum_{i \ge n} a_i t^{i-n}$. We see that the curve passes through the point $P_1 = (x_1 = 0, y_1 = a_n)$. If we take the coordinates $\tilde{x}_1 = x_1$ and $\tilde{y}_1 = y_1 + a_n$, then we read the point P_1 as $(\tilde{x}_1 = 0, \tilde{y}_1 = 0)$, and the parametrization of C_1 becomes $(\tilde{x}_1(t), \tilde{y}_1(t)) = (t^n, \sum_{i > n} a_i t^{i-n})$. Denote by k the minimum index, such that $a_i \neq 0$ in the previous parametrization of C_1

Now let us see what happens if we iterate the process. Consider the blow-up $\sigma_2 : (M_2, E^2) \to (M_1, E^1)$ with center P_1. Again there are two charts to consider, U_2 and U_2' defined by the coordinates $(\tilde{x}_1, \tilde{y}_1) = (x_2, x_2 y_2)$ and $(\tilde{x}_1, \tilde{y}_1) = (x'_2 y'_2, y'_2)$. There are two cases: first, that $k \geq 2n$. Second, that $k < 2n$. In the first one, we are in the same situation as above. For this reason, let us focus on the second case. If $k < 2n$, we can check that the curve C does not pass through any point of $E^2 \cap U_2$. Thus, we need to compute the strict transform C_2 of C by $\sigma_1 \circ \sigma_2$ in the coordinates (x'_2, y'_2). We see that C_2 has as parametrization

$$(x'_2(t), y'_2(t)) = (\tilde{x}_1(t)/\tilde{y}_1(t), \tilde{y}_1(t)) = (u(t)t^{n-k}, \tilde{y}_1(t)),$$

where $u(t) = t^k/\tilde{y}_1(t)$ is an unit in $\mathbb{C}\{t\}$. We can write the previous parametrization as a primitive parametrization as follows: we have that $\tilde{y}_1(t) = t^k/u(t)$, if we take a change of variables $t' = t/u^{1/k}(t)$, with $u^{1/k}$ a k-root of the series $u(t)$. Then, we notice that $y'_2(t') = (t')^k$. Giving a primitive parametrization.

We see that C passes through the point $P = (x'_2 = 0, y'_2 = 0)$. Notice that the point P is the corner point of the exceptional divisor E^2. Furthermore, it follows that if $n = 1$, that is, if the branch is regular, then this second case is impossible to happen. We summarize this last observation in the next remark.

Remark 1.3.6. Assume that C is a branch in (M_0, P_0). They are equivalent:

- C is regular.
- for any sequence of blow ups $\pi : (M_N, E^N) \longrightarrow (M_0, P_0)$, the curve C never passes through any of the corner points of the exceptional divisor E^N.

Following the evolution of the parametrizations as indicated, we can show the following result (see for instance [56] Theorem 3.5.5).

Theorem 1.3.7. *Let C be a branch and $\tilde{C}$ the strict transform of C by a blow-up $\sigma : (M_1, E^1) \longrightarrow (M_0, P_0)$. Denote by $(\beta_0, \beta_1, \ldots, \beta_g)$ the characteristic exponents of C. Then the characteristic exponents of C_1 are:*

- $(\beta_0, \beta_1 - \beta_0, \beta_2 - \beta_0, \ldots, \beta_g - \beta_0)$*, if* $\beta_1 > 2\beta_0$.
- $(\beta_1 - \beta_0, \beta_0, \beta_2 - \beta_1 + \beta_0, \ldots, \beta_g - \beta_1 + \beta_0)$*, if* $\beta_1 < 2\beta_0$ *and* $(\beta_1 - \beta_0) \nmid \beta_0$.
- $(\beta_1 - \beta_0, \beta_2 - \beta_1 + \beta_0, \ldots, \beta_g - \beta_1 + \beta_0)$*, if* $(\beta_1 - \beta_0) \mid \beta_0$.

In the case of a branch with a single Puiseux pair (n, m), the previous theorem is stated as follows:

Theorem 1.3.8. *Let C be a branch and C_1 the strict transform of C by a blow-up $\sigma : (M_1, E^1) \longrightarrow (M_0, P_0)$. Assume that (n, m) are the characteristic exponents of C. There are three possible cases:*

- $(n, m - n)$ *are the characteristic exponents of* C_1*, if* $m > 2n$.
- $(m - n, n)$ *are the characteristic exponents of* C_1*, if* $m < 2n$ *and* $(m - n) \nmid n$.
- C_1 *is regular, if* $(n, m) = (n, n + 1)$*. Moreover, the intersection multiplicity* $i_{P_1}(C_1, E^1)$ *at* $P_1 = C_1 \cap E^1$ *is given by* $i_{P_1}(C_1, E^1) = n$.

1.4 Topological Invariants

Consider two regular complex analytic surfaces (S_1, P_1) and (S_2, P_2). O. Zariski in a series of papers [57, 58, 59] studied, among other problems, when two reduced plane curves (C_1, P_1) in (S_1, P_1) and (C_2, P_2) in (S_2, P_2) are *topologically equivalent*. This is the so-called Zariski's equisingularity theory. In other words, when there exists an homeomorphism $\Psi : U_1 \longrightarrow U_2$ such that $\Psi(C_1 \cap U_1) = C_2 \cap U_2$, where U_1 and U_2 are open neighborhoods of P_1 and P_2 in S_1

and S_2 respectively, taken such that two representatives of the germs C_1 and C_2 are well defined. Note that being topologically equivalent defines an equivalence relation.

The notion of being topologically equivalent induces the one of *topological invariant*. A property is a topological invariant if it remains equal for all the elements in same class under the topological equivalence. A topological invariant is a *complete topological invariant* if it determines the topological class.

If we ask the map Ψ to be a biholomorphism between U_1 and U_2, we obtain the notion of two curves being *analytically equivalent*. We can define in a similar way when a property is an *analytic invariant* or a *complete analytic invariant*.

Zariski found out that the class of a curve under the topological equivalence only depends on the resolution of singularities, in the following sense. Consider (C, P_0) a singular branch, and $\pi : (M_N, E^N) \longrightarrow (M_0, P_0)$ the minimal resolution of singularities of C. As always, we put $E^N = \cup_{i=1}^N E_i^N$ the decomposition into irreducible components of the exceptional divisor $E^N = \pi^{-1}(P_0)$, and denote P_{i-1} the center of each blow-up σ_i. We define the *dual graph* of the resolution as a labelled graph of with N vertices v_i for $i = 1, \dots, N$, where v_i represents the curve E_i^N. Additionally, for $i \neq j$, the vertices v_i and v_j are connected by an edge if and only if $E_i^N \cap E_j^N \neq \emptyset$. Note that the label of each of the vertices is determined by the order of the components of the exceptional divisor, as defined in the previous section. Finally, for any vertex v_i in the graph, we add an arrow for each irreducible component of the strict transform of C by π that passes through one of the points of E_i^N.

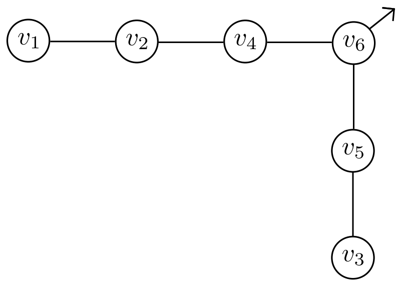

Figure 1.3: *Dual graph of the curve* $y^5 - x^{13}$.

Theorem 1.4.1 (Zariski's Equisingularity)**.** *Two branches* (C_1, P_1) *and* (C_2, P_2) *are topologically equivalent if and only if their minimal resolutions of singularities produce equivalent dual graphs.*

The Zariski's Equisingularity Theorem states that the equivalence class of a dual graph is a complete topological invariant of the curve. During the last decades many complete topological invariants have appeared and their relations have been studied, see for instance [24, 56]. Nonetheless, we are going to describe the ones we will use in this work. Furthermore, we will only consider the irreducible case.

Another complete topological invariant is the *semigroup of the branch* C, denoted by Γ_C. It is defined as the set of all possible intersection multiplicities of C with any other curve. More precisely, consider $\phi(t)$ a primitive parametrization of C, then

$$\Gamma_C := \{\nu_C(h) : h \in \mathbb{C}\{x, y\}\}; \ \nu_C(h) := \mathrm{ord}_t(h \circ \phi)\}$$

We recall that by Equation (1.3), $\nu_C(h)$ can be seen as the intersection multiplicity of f and h, for $f = 0$ an implicit equation of C.

Note that Γ_C is endowed with a semigroup structure, hence its name, because of the additivity of the intersection multiplicity with respect to the product of functions. The concept of semigroup of a branch can be extended to the non irreducible case, see [20].

In [56] is given a method to compute a minimal set of generators of the semigroup $(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g)$, in the sense that any element of the semigroup can be written as a non negative integer combination of $(\overline{\beta}_0, \ldots, \overline{\beta}_g)$. In other words, given $\gamma \in \Gamma_C$, there exist $\alpha_i \in \mathbb{Z}_{\geq 0}$ for $i = 0, \ldots, g$ such that

$$\gamma = \sum_{i=0}^{g} \alpha_i \overline{\beta}_i.$$

Moreover, any $\overline{\beta}_i$ cannot be written as a combination with non negative integer coefficients of the other generators.

Let us describe how to compute the *minimal set of generators* of Γ_C from the characteristic exponents $(\beta_0, \beta_1, \ldots, \beta_g)$ of C. We start by putting $\overline{\beta}_0 = \beta_0$ and $\overline{\beta}_1 = \beta_1$. Now for $i = 2, \ldots, g$, we have that

$$\overline{\beta}_i = \frac{e_{i-2}}{e_{i-1}} \overline{\beta}_{i-1} + \beta_i - \beta_{i-1}. \tag{1.4}$$

We recall that the numbers e_i are defined as $e_0 = \beta_0$ and $e_j = gcd(\beta_j, e_{j-1})$ for $j = 1, \ldots, g$. The previous formula allows us to obtain the generators of the semigroup from the characteristic exponents and vice-versa. In fact, we have the following theorem, see for instance [56].

Theorem 1.4.2. *Given an irreducible plane curve, its set of characteristic exponents, its semigroup or its dual graph are equivalent complete topological invariants.*

By Equation (1.4), since $gcd(\beta_0, \ldots, \beta_g) = e_g = 1$, then we have $gcd(\overline{\beta}_0, \overline{\beta}_1, \ldots \overline{\beta}_g) = 1$. Hence, there exists a minimum element $c_\Gamma \in \Gamma_C$ such that, for any natural number $k \geq c_\Gamma$, then we have that $k \in \Gamma_C$. The element c_Γ is called the *conductor* of the semigroup. In [60] p. 13 it is shown that the conductor satisfies the following formula:

$$c_\Gamma = \sum_{i=1}^{g} (e_{i-1} - e_i)(\beta_i - 1) = \overline{\beta}_g e_{g-1} - \beta_g - \beta_0 + 1. \tag{1.5}$$

The Equation (1.5) only holds for semigroups of branches. Note that to simplify the notation, we write c_Γ instead of c_{Γ_C}.

When computing Saito bases in Chapter 8, we use the following property associated to the conductor of the semigroup.

Lemma 1.4.3. *([56] Lemma 11.6.1.) Consider the map induced by a primitive parametrization $\phi^\# : \mathbb{C}\{x, y\} \longrightarrow \mathbb{C}\{t\}$, then we have that the ideal $(t^{c_\Gamma}) \subset \mathrm{Im}(\phi^\#)$.*

Finally, note that if the branch C has genus one and characteristic exponents (n, m), then the semigroup is $\Gamma_C = \langle n, m \rangle$. Conversely, if $\Gamma_C = \langle n, m \rangle$ then the characteristic exponents of C are (n, m). In this cuspidal case, we have that the conductor of Γ_C, according to Equation (1.5), is $c_\Gamma = (n-1)(m-1)$.

Example 1.4.4. The conductor c_Γ of the semigroup Γ_C of a branch C does not determine Γ_C. In other words, c_Γ is topological invariant which is not complete. To see this observation, we can consider the semigroups $\Gamma_1 = \langle 3, 4 \rangle$ and $\Gamma_2 = \langle 2, 7 \rangle$. They are the semigroups of the branches defined by the parametrizations (t^3, t^4) and (t^2, t^7) respectively. Moreover, we see that the conductor in both cases takes the value 6.

Remark 1.4.5. The conductor has a geometrical interpretation. Assume that C is a curve with implicit equation $f = 0$, and denote by $\mathbb{S}^3_\epsilon \subset \mathbb{C}^2$ a 3-sphere centered at the origin with radius $\epsilon > 0$ small enough. Then we can define the locally trivial fibration (called the Milnor fibration)

$g : \mathbb{S}^3_\epsilon \setminus \{f = 0\} \to \mathbb{S}^1$ given by the map $g(p) = f(p)/||f(p)||$. In [45], J. Milnor showed that the fibers of this map have the homotopy type of the joint union μ_f of 2-spheres, where μ_f is the Milnor number of f. In the irreducible case, we have that μ_f is the conductor of the semigroup of C. In general, we have that the Milnor number is the complex dimension of the $\mathbb{C}$-vector space $\mathbb{C}\{x, y\}/(f_x, f_y)$.

1.5 The Analytic Classification

The problem of the analytical classification of equisingular curves was solved by A. Hefez, M.E. Hernandes and M.E.R. Hernandes in a series of papers [34, 35, 36]. In this section we overview their result for the particular case of branches.

1.5.1 Differentials and Differential Values

Here we recall some concepts related to the module of differentials. For a more detailed introduction see [22]. Consider, as always, $(M_0, \mathcal{O}_{M_0})$ a complex analytic regular surface. We denote by $\Omega^q_{M_0}$ the $\mathcal{O}_{M_0}$-module of holomorphic q forms of M_0, with $q = 1, 2$. Given $P_0 \in M_0$ a point and (x, y) a local system of coordinates of M_0 at P_0, we have that $\Omega^1_{M_0,P_0}$ and $\Omega^2_{M_0,P_0}$ are generated as $\mathcal{O}_{M_0,P_0}$-modules by dx and dy; and $dx \wedge dy$ respectively.

Take (C, P_0) a branch in (M_0, P_0) with primitive parametrization $\phi(t) = (x(t), y(t))$. We refer to the module of differentials of C as Ω^1_{C,P_0}, which is the $\mathbb{C}\{x(t), y(t)\}$-submodule of $\Omega^1_{\mathbb{C},0}$ generated by $x'(t)dt$ and $y'(t)dt$. See [31] for other approximations.

Remark 1.5.1. By Remark 1.2.4, we have a natural epimorphism from $\Omega^1_{M_0,P_0}$ to Ω^1_{C,P_0} defined by $\omega \mapsto \phi^*\omega$, where $\phi^*\omega$ is the pull-back of the 1-form ω by the parametrization ϕ. We recall that if we write $\omega = Adx + Bdy$, then the pull-back of ω is

$$\phi^*(\omega) = \Big(A(\phi(t))x'(t) + B(\phi(t))y'(t)\Big)dt.$$

We define the *differential value* of ω by $\nu_C(\omega) := \mathrm{ord}_t(\alpha(t)) + 1$, with $\phi^*(\omega) = \alpha(t)dt$. Then the *set of differential values* of C is

$$\Lambda_C := \{\nu_C(\omega) : \omega \in \Omega^1_{M_0,P_0}\}.$$

Note that

$$\Lambda_C = \{\mathrm{ord}_t(\alpha(t)) + 1 : \alpha(t)dt \in \Omega^1_{C,P_0}\}.$$

The set of differential values satisfies the following properties:

1. For any function $h \in \mathcal{O}_{M_0,P_0}$ satisfying that $h(P_0) = 0$, we have that $\nu_C(h) = \nu_C(dh)$ where dh is the differential of h. This implies that $\Gamma_C \setminus \{0\} \subset \Lambda_C$.
2. Given $h \in \mathcal{O}_{M_0,P_0}$ and $\omega \in \Omega^1_{M_0,P_0}$, we have that $\nu_C(h\omega) = \nu_C(h) + \nu_C(\omega)$. Thus, we conclude that for any $\gamma \in \Gamma_C$ and $\lambda \in \Lambda_C$, it is satisfied that $\lambda + \gamma \in \Lambda_C$.
3. The set of differential values Λ_C is an analytic invariant of C. More generally, the differential value of a 1-form does not depend on the analytic system of coordinates, see [34].

Since the semigroup Γ_C has a conductor c_Γ, the first property means that Λ_C is determined by Γ_C and a finite set $\Lambda_C \setminus \Gamma_C$. Moreover, we have that any element of $\Lambda_C \setminus \Gamma_C$ is bounded by c_Γ.

The second property of Λ_C implies that Λ_C is a Γ_C-semimodule. For this reason, from now on, we will refer to the set of differential values of C as *the semimodule of differential values* of C. In Chapter 3, we give more details on the theory of semimodules.

1.5.2 The Normal Form Parametrization Theorem

In [60], O. Zariksi shows that a singular branch (C, P_0) satisfies that $\Lambda_C = \Gamma_C \setminus \{0\}$ if and only if C is quasi-homogeneous. We recall that C is a quasi-homogeneous branch if in some coordinates (x, y) in (M_0, P_0), then $y^n - x^m$ is an implicit equation of C, for a Puiseux pair (n, m) with $2 \leq n$.

If $\Lambda_C \neq \Gamma_C \setminus \{0\}$, we can consider the number $\lambda_Z = \min(\Lambda_C \setminus \Gamma_C) - n$, where n is the multiplicity of C at P_0. The number λ_Z is the *Zariski's invariant* introduced in [60]. Since Λ_C is an analytic invariant, the Zariski's invariant is also an analytic invariant.

Zariski's idea was to use the information of the semimodule of differential values to find an analytic change of coordinates which allows to compute a parametrization of the curve as simple as possible. With this idea in mind, C. Delorme in [21] computed what he called ultra short parametrizations of a branch. These ultra short parametrizations are primitive parametrizations of the shape $\phi(t) = (t^n, y(t))$, where $y(t)$ is a polynomial with as many zero coefficients as the author could determine. However the computation is restricted to the case of cusps. It was in [34] where the authors could find a parametrization as simple as possible, in the following sense:

Theorem 1.5.2 ([34] Theorem 2.1)**.** *Let (C, P_0) be a branch whose semigroup is $\Gamma_C = \langle n, \overline{\beta}_1, \dots, \overline{\beta}_g \rangle$. Then there exists a system of local coordinates in (M_0, P_0) such that C has a normal form parametrization defined as follows: if $\Lambda_C \setminus \Gamma_C = \emptyset$, then we put $(t^n, t^{\overline{\beta}_1})$. Otherwise, if we have that $\Lambda_C \setminus \Gamma_C \neq \emptyset$, we have that the normal form parametrization is*

$$\left(t^n, t^{\overline{\beta}_1} + t^{\lambda_Z} + \sum_{i > \lambda_Z, i \notin \Lambda_C - n} a_i t^i \right), \tag{1.6}$$

where $\lambda_Z = \min(\Lambda_C \setminus \Gamma_C) - n$ is the Zariski's invariant of C. The curve (C, P_0) is analytically equivalent to another branch (C', P_0) if and only if there exists $r \in \mathbb{C}^*$ with $r^{\lambda_Z - \overline{\beta}_1} = 1$ and $a_i = r^{i - \overline{\beta}_1} a'_i$ for every coefficient a'_i of a normal form parametrization of C'.

For a regular branch, its normal form parametrization is $(t, 0)$.

The previous theorem shows the relevance of the study semimodules of differential values and its gaps. In Chapter 3, we give some results, in the cuspidal case, that points towards an effective way on the determination of the gaps of a semimodule.

The idea of these normal forms or even the ultra short parametrizations is the following one: we can take a primitive parametrization $\phi(t) = (t^n, y(t))$ of a branch C and write $y(t) = \sum a_i t^i$. We follow an iterative argument from the terms of smaller degree in $y(t)$ to the greater ones. We determine the smallest i such that we can find a change of coordinates that sends ϕ to a new parametrization $\phi^1(t) = (t^n, y^1(t))$, satisfying that $y^1(t)$ and $y(t)$ coincide up order $i - 1$, and with $a_i^1 = 0$. Then we restart the process with $\phi^1(t)$ as many times as needed. After a finite number of steps, we obtain a parametrization $\phi^k(t) = (t^n, y^k(t))$. If k is big enough we can guarantee that we can find a last parametrization $\phi^{k+1}(t) = (t^n, y^{k+1}(t))$ defined as before, with the extra condition that $y^{k+1}(t)$ is a polynomial in $\mathbb{C}[t]$.

It is worth remarking that P. Fortuny Ayuso in [23] gives the previous changes of coordinate in terms of the 1-forms that give place to the semimodule of differential values. Even though we are not going to compute any normal form parametrization, these results show the importance of the semimodule of differential values to study the analytic classification of germs of plane curves.

Our main goal is to connect the semimodule of differential values with other analytic invariants of a branch. All the main results of this work points towards that direction. Specially those in Chapters 8 and 9, when computing Saito bases for a cusp C, or when determining some roots of the Bernstein-Sato polynomial of C.

1.6 Cuspidal Sequences

We end this chapter with a brief description of the resolution of singularities of a cusp. We follow the notations from Section 1.3. Consider $\pi : (M_N, E^N) \longrightarrow (M_0, P_0)$ a sequence of blow-ups of length N, where as before we write $\pi = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_N$ as a composition of N blow-ups σ_i with center P_{i-1} for $i = 1, 2, \ldots, N$.

We say that π is a *bamboo* if the center P_j of the blow-up σ_{j+1} belongs to E_j^j for $j = 1, \ldots, N-1$. The minimal resolution of singularities of a branch is always a bamboo. A bamboo is said to be a *cuspidal sequence* if $P_1, P_2, \ldots, P_\ell$ are free points for $1 \leq \ell < N$ and $P_{\ell+1}, P_{\ell+2}, \ldots, P_{N-1}$ are corner points. The index ℓ is the *index of freeness* of π. The last irreducible component $D := E_N^N$ is called the *cuspidal divisor* of π.

Given a singular branch C, we say that C is a *cusp* if its minimal resolution of singularities is a cuspidal sequence. Moreover, take a cuspidal sequence π with cuspidal divisor D. We say that a branch C is a *D-cusp* if the strict transform $\tilde{C}$ of C by π passes through a free point $P \in D$, and the curves D and $\tilde{C}$ define a curve with normal crossings at P. We denote by $Cusp(D)$ the set of all D-cusps.

Remark 1.6.1. Note that, if for some $i < N-1$ the center point P_i of a cuspidal sequence π is a corner point, by definition of bamboo, we have that $P_i \in E_i^i \cap E_j^i$ for an index $j < i$. Therefore by Remark 1.3.1, the next center corner point P_{i+1} satisfies that: either $P_{i+1} \in E_{i+1}^{i+1} \cap E_i^{i+1}$ or $E_{i+1}^{i+1} \cap E_j^{i+1}$. It follows that if π is a cuspidal sequence of length $N > 1$, then the sequence $\rho_2 = \sigma_2 \circ \sigma_3 \circ \ldots \circ \sigma_N$ is also cuspidal.

We define the Puiseux pair (n, m) of a cuspidal sequence π in an inductive way as follows. If $N = 1$, then we put $(n, m) = (1, 1)$. Otherwise, if $N > 1$, we take ρ_2 with Puiseux pair (n_1, m_1). Denote by ℓ_1 the index of freeness of ρ_2. Now if $\ell = 1$, we put $(n, m) = (m_1, n_1 + m_1)$. If $\ell > 1$, then we put $(n, m) = (n_1, m_1 + n_1)$, in this case, we have that $\ell_1 = \ell - 1$.

Notice that by this construction, we obtain that the index of freeness of the sequence π is $\ell = \lfloor m/n \rfloor$. Moreover, we always have that pair (n, m) constructed as above satisfies that $gcd(n, m) = 1$.

Lemma 1.6.2. *Assume that π is a cuspidal sequence with Puiseux pair (n, m), with cuspidal divisor D. If C is a singular D-cusp, then the characteristic exponents of C are exactly (n, m).*

Proof. The result follows by a recursively application of Theorem 1.3.8 and its proof, knowing that the strict transform is at the end a regular branch. □

The previous lemma explains why the pair (n, m) is called the Puiseux pair of a cuspidal sequence π. It corresponds exactly with the Puiseux pair of a branch whose minimal resolution of singularities is π. Nonetheless, we are extending the definition to include the case where $n = 1$.

Now, fix π a cuspidal sequence with Puiseux pair (n, m), index of freeness ℓ and C a D-cusp. We say that a regular branch (Y, P_0) has *maximal contact* with π or with C, if $P_1, P_2, \ldots, P_\ell$ are infinite near points of Y. By Remark 1.3.6, the corner point $P_{\ell+1}$ is never an infinitely near point of the regular branch Y.

Remark 1.6.3. After a big enough number of blow-ups, we can obtain a maximal contact branch in a natural way. Assume that the index of freeness ℓ of π satisfies that $\ell < N-1$, and take the sequence of blow-ups $\rho_{\ell+1} = \sigma_{\ell+1} \circ \sigma_{\ell+2} \circ \ldots \circ \sigma_N$ with index of freeness ℓ'. Then the curve E_ℓ^ℓ has maximal contact with $\rho_{\ell+1}$.

Indeed, the points $P_{\ell+1}, P_{\ell+2}, \ldots, P_{\ell+\ell'}$ are free for the sequence $\rho_{\ell+1}$, but they are corner points for π. We only need to show that $\{P_{\ell+j}\} = E_{\ell+j}^{\ell+j} \cap E_{\ell}^{\ell+j}$, for $j = 1, \ldots, \ell'$. Since π is a bamboo, by definition we have that $P_{\ell+j} \in E_{\ell+j}^{\ell+j}$.

Additionally, $P_{\ell+j}$ is a corner point for π, then $P_{\ell+j} \in E_k^{\ell+j}$ for $k < \ell + j$. If $k > \ell$, we have that $P_{\ell+j}$ is a corner point for $\rho_{\ell+1}$, which is a contradiction. Moreover, we cannot have $k < \ell$, because $E_{j+\ell}^{j+\ell} \cap E_i^{j+\ell} = \emptyset$ for $i < \ell$, otherwise we will have that $P_{j+\ell-1} \in E_i^{\ell+j-1}$ contradicting the fact that π is a bamboo. Therefore, $k = \ell$ concluding the result.

We can characterize a maximal contact branch with respect to a D-cusp, in terms of intersection multiplicities.

Lemma 1.6.4. *Let C be a D-cusp with Puiseux pair (n, m). A regular branch Y has maximal contact with respect to C if and only if*

$$i_{P_0}(C, Y) = m.$$

To prove the last lemma, we use a weaker version of the well known Noether's formula.

Lemma 1.6.5 ([14] Lemma 3.3.4)**.** *Assume that C_1 and C_2 are two curves in (M_0, P_0), and $\sigma : (M_1, E^1) \longrightarrow (M_0, P_0)$ is a blow-up with center P_0. Denote by $S_1, S_2 \subset E^1$, the points where C_1 and C_2 pass through respectively; and by $\tilde{C}_1$ and $\tilde{C}_2$ the strict transforms of C_1 and C_2 by σ. Then*

$$i_{P_0}(C_1, C_2) = \nu_{P_0}(C_1)\nu_{P_0}(C_2) + \sum_{P \in S_1 \cap S_2} i_P(\tilde{C}_1, \tilde{C}_2).$$

Proof of Lemma 1.6.4. Assume that Y has maximal contact with C, then C and Y share $P_1, \ldots, P_\ell$ as infinitely near points. We recall that $\ell = \lfloor m/n \rfloor$. Now, we denote by C_i and Y_i the strict transform of C and Y by $\pi_i = \sigma_1 \circ \sigma_2 \circ \ldots \sigma_i$, with $1 \leq i \leq N$.

By an iterative use of Lemma 1.6.5, and by definition of maximal contact branch, we have that:

$$i_{P_0}(C, Y) = \nu_{P_0}(C)\nu_{P_0}(Y) + \nu_{P_1}(C_1)\nu_{P_1}(Y_1) + \ldots \nu_{P_\ell}(C_\ell)\nu_{P_\ell}(Y_\ell) + i_{P_{\ell+1}}(C_{\ell+1}, Y_{\ell+1}). \qquad (1.7)$$

We note the following:

- $\nu_{P_i}(Y_i) = 1$ for $1 \leq i \leq \ell$, because Y_i is a regular branch.
- The characteristic exponents of C_i are $(n, m - ni)$ for $1 \leq i < \ell$, and the ones of C_ℓ are $(m - n\ell, n)$.
- $i_{P_{\ell+1}}(C_{\ell+1}, Y_{\ell+1}) = 0$, because $P_{\ell+1}$ is not an infinitely near point of Y.

Then, Equation (1.7) becomes the following:

$$i_{P_0}(C, Y) = n\ell + (m - n\ell) = m.$$

Now, for the converse result, the previous observations show that if the regular branch Y has intersection multiplicity m with C, then they must share the points $P_1, P_2, \ldots, P_\ell$ as infinitely near points. Otherwise, by applying Lemma 1.6.5 we would end up with a lower intersection multiplicity. □

We can always find a branch with maximal contact. Assume that

$$\phi(t) = (x(t), y(t)) = (t^n, \sum_{i \geq n} a_i t^i)$$

is a primitive parametrization of C. Then we can consider the change of coordinates $x_1 = x$ and $y_1 = y - \sum_{i\geq n}^{m-1} x^{i/n}$. Note that by definition of the characteristic exponents, we always have that i/n is an integer number. We see that $\nu_C(y_1) = m$, thus the branch defined by $y_1 = 0$ has maximal contact with respect to C.

Note that if (x, y) is a system of local coordinates in (M_0, P_0) such that $y = 0$ has maximal contact with respect to C, then we can write a primitive parametrization of the form

$$\phi(t) = (t^n, a_m t^m + h.o.t.), \text{ with } a_m \neq 0. \tag{1.8}$$

Equivalently, by Equation (1.1), we have that C has an implicit equation f given by:

$$f = y^n + bx^m + \sum_{ni+mj>nm} z_{ij} x^i y^j, \text{ with } b \neq 0. \tag{1.9}$$

A system of coordinates (x, y) as above, is said to be a *adapted system of coordinates* with respect to C. More generally, if (x, y) is a system of coordinates, such that $y = 0$ has maximal contact with respect to a cuspidal sequence π, we say the (x, y) is an adapted system of coordinates with respect to π.

Cuspidal sequences are determined by their Puiseux pairs and by a branch with maximal contact. In fact, we have the following result.

Proposition 1.6.6. *Take $1 \leq n \leq m$ and let (Y, P_0) be a regular curve. There exists a unique cuspidal sequence $\pi : (M_N, E^N) \to (M_0, P_0)$ such that Y has maximal contact with π and (n, m) is the Puiseux pair of π.*

Proof. If $n = m = 1$, the only possibility is that $N = 1$ and then π consists in the blow-up of P_0. Let us proceed by induction on $n + m$ and assume that $n + m > 2$. We necessarily have that $N \geq 2$, let σ_1 be the first blow-up with center P_0 and P_1 the infinitely near point of Y in E_1^1, we denote by Y_1 the strict transform of Y by σ_1 at P_1.

Assume first that $2n \leq m$. We apply induction to Y_1 with respect to the pair n', m' where $n' = n$, $m' = m - n$; and we obtain a cuspidal sequence π' over (M_1, P_1) of length N' with the required properties. We construct π of length $N = N' + 1$ by taking σ_j with center the point P'_{j-2}, for $j = 2, 3, \ldots, N' + 1$.

In the case that $n \leq m < 2n$, we consider the branch $Y_1' = E_1^1$ at P_1, we apply induction to Y_1' with respect to the pair n', m' where $n' = m - n$, $m' = n$ and we obtain a cuspidal sequence π' over (M_1, P_1) of length N'. We construct π of length $N = N' + 1$ as before.

The uniqueness of π follows by an inductive argument invoking the uniqueness after one blow-up. □

By Theorem 1.4.1, all the elements in $Cusp(D)$ are topologically equivalent. Besides, any cusp is analytically equivalent to a D-cusp.

Proposition 1.6.7. *Consider a cuspidal sequence π with Puiseux pair (n, m) and cuspidal divisor D. Let C be a branch in (M_0, P_0) with characteristic exponents (n, m). There is an D-cusp analytically equivalent to C.*

Proof. Choose a local coordinate system (x, y) adapted to π and let $f \in \mathbb{C}\{x, y\}$ be an implicit equation of C.

If $n = 1$, the branch C is nonsingular. Then, there is an automorphism $\phi : (M_0, P_0) \to (M_0, P_0)$ such that $\phi^{\#}(f) = y$. The result follows in this case since $y = 0$ is a D-cusp.

Assume that $2 \leq n < m$. As we showed before, we can find a non singular branch Y defined by $g = 0$ having maximal contact with C, that is, with the property that

$$i_{P_0}(Y, C) = m.$$

Take an automorphism $\phi : (M_0, P_0) \to (M_0, P_0)$ such that $\phi^{\#}(g) = y$. We have that $\phi^{\#}(f)$ is an D-cusp, where $\phi^{\#}$ is the associated map ϕ between local rings $\mathcal{O}_{M_0,P_0} \to \mathcal{O}_{M_0,P_0}$. □

According to the above result, the analytic moduli of the family of branches equisingular to the irreducible cusp $y^n - x^m = 0$ is faithfully represented by the analytic moduli of the family Cusps(D). In other words, every branch of genus 1 is analytically equivalent to a cusp.

2

Totally Dicritical Foliations

This chapter is devoted to study a family of two dimensional foliations called totally D-dicritical, where D is a cuspidal divisor. In Section 2.1, we present the concept of foliation. In Section 2.2, we recall the theorem of resolutions of singularities for foliations in dimension two. In Section 2.3, we define the notion of divisorial value along an irreducible component of the exceptional divisor of a sequence of blow-ups. We particularize to the case of cuspidal sequences, as presented in Chapter 1. Finally, in Section 2.4, we characterize the totally D-dicritical foliations. Briefly speaking, they satisfy that D is a dicritical divisor without singularities in its free points. This notion can be understood by saying that the foliation is transverse to the cuspidal divisor. Our main references in this chapter are [11, 12].

We recall that $(M_0, \mathcal{O}_{M_0})$ is a regular complex analytic surface.

2.1 Basic Notions

As in Chapter 1, we denote by $\Omega^1_{M_0}$ and $\Omega^2_{M_0}$ the sheaves of differential 1-forms and 2-forms respectively. Given $\omega \in \Omega^1_{M_0}(U)$, we denote by $Sing(\omega)$ the *singular locus* of ω, that is, the set of points of U such that ω takes the value zero. A point $P \in U$ is a *singular* point of ω, if P belongs to the singular locus of ω, otherwise we say that P is regular.

A *foliation* $\mathcal{F}$ is defined as a *local data* $\mathcal{S} = \{(U_i, \omega_i)\}_{i \in I}$ with the following properties:

1. The set $\{U_i\}_{i\in I}$ is an open cover of M_0.
2. For every $i \in I$, the element ω_i is a differential 1-form of $\Omega_{M_0}(U_i)$.
3. For every $i, j \in I$, $i \neq j$, there exists a unit $h_{ij} \in \mathcal{O}_{M_0}(U_i \cap U_j)$ such that

$$\omega_i \mid_{U_i \cap U_j} = h_{ij} \omega_i \mid_{U_i \cap U_j},$$

   where $\omega_i \mid_{U_i \cap U_j}$ denotes the restriction of ω_i to the open set $U_i \cap U_j$.
4. The singular locus $Sing(\omega_i) \subset U_i$ has codimension two for every $i \in I$.

In dimension two, a 1-form ω always satisfies the integrability condition $\omega \wedge d\omega = 0$. Including the integrability condition to the 1-forms in the previous definition gives place to the notion of codimension one foliation, for any dimension.

We can reinterpret the fourth condition about the singular locus as follows: assume that ω is one of the 1-forms defining conditions of a foliation $\mathcal{F}$, and take (x, y) a local system of coordinates in M_0 where ω is defined. Then we can write $\omega = Adx + Bdy$. Saying that the singular locus of ω has codimension 2 is equivalent to saying that $gcd(A, B) = 1$.

Note that a foliation $\mathcal{F}$ in (M_0, P_0) is defined, in a small enough neighborhood U of P_0, by a holomorphic 1-form ω, such that the singular locus of ω is either the empty set or $\{P_0\}$. For this reason, given a 1-form ω whose coefficients are coprime, we also say that ω defines a foliation in (M_0, P_0). Besides, when talking about foliations in (M_0, P_0), we will refer indistinctly to the geometrical object or a 1-form defining it.

More generally, given a non null 1-form $\omega \in \Omega^1_{M_0,P_0}$, that we write in some coordinates as $\omega = Adx + Bdy$, when we refer to the foliation defined by ω, we mean the foliation defined by $\omega/gcd(A, B)$.

As in the case of curves, we can define the multiplicity of a foliation. More precisely, given $\omega \in \Omega^1_{M_0,P_0}$, we can write ω in a system of coordinates as $\omega = Adx + Bdy$. Then the *multiplicity* of ω at P_0 is $\nu_{P_0}(\omega) := \min\{\nu_{P_0}(A), \nu_{P_0}(B)\}$. If a foliation $\mathcal{F}$ is defined by a 1-form ω, then the multiplicity of $\mathcal{F}$ at P_0 is $\nu_{P_0}(\mathcal{F}) := \nu_{P_0}(\omega)$. It can be checked that the multiplicity at P_0 does not depend on the coordinates.

Consider $\mathcal{F}$ a foliation defined by $\omega \in \Omega_{M_0,P_0}$ and let (C, P_0) be a curve defined by an implicit equation $f \in \mathcal{O}_{M_0,P_0}$. We say that C is an *invariant curve* for ω, or for $\mathcal{F}$, if $df \wedge \omega = f\eta$, where $\eta \in \Omega^2_{M_0,P_0}$. When C is a branch, we say that C is an *invariant branch* for ω or for $\mathcal{F}$.

Lemma 2.1.1 ([11] Lemma 3.4)**.** *Consider a local system of coordinates (x, y) in (M_0, P_0). Let C be a branch with implicit equation $f \in \mathbb{C}\{x, y\}$ and primitive parametrization $\phi(t) = (x(t), y(t))$. Then, given $\omega \in \Omega^1_{M_0,P_0}$, it is equivalent to say that C is an invariant branch for ω than saying that $\phi^*(\omega) = 0$. In other words, that the differential value of ω is $\nu_C(\omega) = \infty$.*

Proof. Since $f(\phi(t)) = 0$, then

$$\frac{d}{dt} f(\phi(t)) = f_x(\phi(t))x'(t) + f_y(\phi(t))y'(t) = 0,$$

with f_x and f_y the partial derivatives of f with respect x and y respectively. Assume without loss of generality that $x(t) \neq 0$. Since the curve is defined at (M_0, P_0), it implies that $x'(t) \neq 0$. Thus we have that

$$f_x(\phi(t)) = -\frac{f_y(\phi(t))y'(t)}{x'(t)}. \tag{2.1}$$

Now write $\omega = Adx + Bdy$, and consider $\omega \wedge df = (Af_y - Bf_x)dx \wedge dy$. We have that C is invariant by ω if and only if, along the points of C, it is satisfied that

$$A(\phi(t))f_y(\phi(t)) - B(\phi(t))f_x(\phi(t)) = 0.$$

By Equation (2.1), this condition is equivalent to

$$\frac{f_y(\phi(t))}{x'(t)}(A(\phi(t))x'(t) + B(\phi(t))y'(t)) = 0. \tag{2.2}$$

Note that $f_y(\phi(t)) \neq 0$, since otherwise, f will divide f_y which is not possible. □

According to Camacho-Sad theorem, a foliation $\mathcal{F}$ in (M_0, P_0) always has at least an invariant curve (C, P_0), see [10].

Consider $\mathcal{F}$ a regular foliation in (M_0, P_0) defined by a non singular 1-form $\omega \in \Omega^1_{M_0,P_0}$ at P_0. Let C be a regular branch with $g = 0$ an implicit equation. We say that C is *transverse* to $\mathcal{F}$ if $\omega \wedge dg = vdx \wedge dy$, with $v \in \mathcal{O}_{M_0,P_0}$ a unit. If v is not a unit, then we say that C is *tangent*. Notice that being invariant is a particular case of being tangent.

2.2 Resolution of Singularities of Plane Foliations

As in the case of curves, thanks to the results due to A. Seidenberg [51], we can find a resolution of singularities of a foliation in (M_0, P_0). It is worth noting that, up to this moment, in the general dimension case, it is unknown if a foliation admits a resolution of singularities. However, in the two dimensional case, there always exists a resolution, and the way of finding one recalls the case of plane curves. Even though, there are differences on the behaviour between the cases of foliations and curves.

As in Section 1.3, consider $\pi : (M_N, E^N) \rightarrow (M_0, P_0)$ a sequence of blow-ups starting at P_0, with $\pi = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_N$. Given $\omega \in \Omega^1_{M_0,P_0}$, then $\pi^*(\omega)$ is the *total transform* of ω. Assume that ω defines a foliation $\mathcal{F}$ in (M_0, P_0). We define the strict transform of ω by π in an inductive way. We do it for $\sigma_1 : (M_1, E^1) \rightarrow (M_0, P_0)$, and it is done recursively for π.

We take $\sigma_1^*(\omega)$ the total transform of ω by σ_1, then we have an atlas in (M_1, P_1) defined by the two charts U_1, U_2, as in Section 1.3. In each chart, the 1-form $\sigma_1^*(\omega)$ may not define a foliation, this is because the singular locus may not have codimension 2 in each chart. Thus, we need to remove the common factors of the coefficients of $\sigma_1^*(\omega)$ in each of the charts. In this way, we obtain a foliation $\widetilde{\mathcal{F}}$ which is the *strict transform* of $\mathcal{F}$ by σ_1.

To clarify the concepts, let us make some computations. Consider $\omega = Adx + Bdy$ with $gcd(A, B) = 1$ and let $\sigma_1 : (M_1, E^1) \rightarrow (M_0, P_0)$ be the blow-up at P_0. Take U_1 the chart of (M_1, E^1) defined by the coordinate system $(x, y) = (x_1, x_1 y_1)$. Then, we have that

$$\begin{aligned} \sigma_1^*(\omega) &= A(x_1, x_1y_1)dx_1 + B(x_1, x_1y_1)(x_1dy_1 + y_1dx_1) = \\ &= (A(x_1, x_1y_1) + y_1B(x_1, x_1y_1))dx_1 + x_1B(x_1, x_1y_1)dy_1 \\ &= A_1(x_1, y_1)dx_1 + B_1(x_1, y_1)dy_1. \end{aligned}$$

The strict transform of ω by σ_1 is $x_1^{-k}\sigma_1^*(\omega)$, with $k > 0$ such that $gcd(A_1, B_1) = x_1^k$. It is satisfied that $k = \nu_{P_0}(\omega) + \epsilon$, where $\epsilon \in \{0, 1\}$. If $\epsilon = 0$, we have that E_1^1 is an invariant curve of the strict transform of ω by σ_1. In this case, we say that E_1^1 is a *non dicritical divisor*. Otherwise, if $\epsilon = 1$, then E_1^1 is not invariant by the strict transform of ω by σ_1, and we say that E_1^1 is a *dicritical divisor*.

Now, assume that we have defined the strict transform of ω by $\pi_{N-1} = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_{N-1}$, and denote it by ω'. Then the strict transform of ω by π is the strict transform of ω' by σ_N.

Similarly, given a foliation $\mathcal{F}$ defined by ω, the strict transform of $\mathcal{F}$ by π is the foliation defined by the strict transform of ω by π.

Remark 2.2.1. Consider $\mathcal{F}$ a foliation in (M_0, P_0), and denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ by a single blow-up $\sigma : (M_1, E_1^1) \rightarrow (M_0, P_0)$. By classical results of differential equations, if P_0 is a regular point of $\mathcal{F}$, then there exists a unique regular invariant curve Y passing through P_0.

Similarly, given a point $P \in E_1^1$, if $\mathcal{F}'$ is regular at P. There are two cases: first, E_1^1 is a non dicritical divisor, that is, E_1^1 is invariant by $\mathcal{F}'$. Otherwise, E_1^1 is a dicritical divisor of $\mathcal{F}'$. In this second case, there must be a different branch Y from E_1^1, invariant by $\mathcal{F}'$. In general, given any point P of a dicritical divisor E_1^1, there must be at least one invariant branch of $\mathcal{F}'$ different from E_1^1, passing through P.

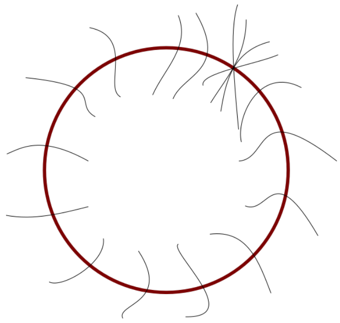

Figure 2.1: *Schematic drawing of a dicritical divisor, the red circle is a real representation of* $E_1^1 = \mathbb{P}^1(\mathbb{C})$. *At any point* $P \in E_1^1$, *there is at least one invariant curve of the foliation passing through* P.

Lemma 2.2.2. *Consider* $\mathcal{F}$ *a foliation in* (M_0, P_0) *and a branch* C. *Denote by* π *a sequence of blow-ups starting at* P_0. *Denote by* $\mathcal{F}'$ *and* C' *the strict transforms by* π *of* $\mathcal{F}$ *and* C. *Then the branch* C *is invariant by* $\mathcal{F}$ *if and only if* C' *is invariant by* $\mathcal{F}'$.

Proof. By a recursive argument, it is only needed to verify the statement for the case when $\pi = \sigma_1$ is a blow-up. Consider the coordinate systems (x, y) and (x_1, y_1) in (M_0, P_0) and in an open set of (M_1, E^1) respectively, such that $(x, y) = (x_1, x_1 y_1)$. We are assuming without loss of generality that C' passes through a point of the chosen chart of (M_1, E^1).

Denote by $f = 0$ a reduced irreducible implicit equation of C and by $f' = 0$ its strict transform. Then we have that $f' = x_1^{-k}(f \circ \pi)$ and $\omega' = x_1^{-k'}\pi^*\omega$, with $k, k' \geq 0$. Therefore

$$\omega' \wedge df' = x_1^{-k-k'}\pi^*(\omega) \wedge \left(-k\frac{f \circ \pi}{x_1}dx_1 + \pi^*(df)\right).$$

We see that f' divides $\omega' \wedge df'$ if and only if f divides $\omega \wedge df$. □

We remark that finding a resolution of singularities of a foliation resembles the case of plane curves. Before, we were interested in finding a curve with normal crossings, here this role will be played by simple points.

Let $\omega \in \Omega^1_{M_0,P_0}$ define a singular foliation $\mathcal{F}$ in (M_0, P_0) at P_0. We write $\omega = Adx + Bdy$ and we consider the matrix

$$J(\omega) = \begin{pmatrix} -\frac{\partial B}{\partial x}(\mathbf{0}) & -\frac{\partial B}{\partial y}(\mathbf{0}) \\ \frac{\partial A}{\partial x}(\mathbf{0}) & \frac{\partial A}{\partial y}(\mathbf{0}) \end{pmatrix}.$$

Assume that the matrix $J(\omega)$ is neither null nor nilpotent. Denote by $\lambda, \mu \in \mathbb{C}$ the two eigenvalues of $J(\omega)$, such that $\mu \neq 0$. If the ratio λ/μ is not a positive rational number, then we say that P_0 is a *simple singularity of the foliation* $\mathcal{F}$. Besides, if $\lambda = 0$, we say that P_0 is a *saddle node* singularity, otherwise, we say that P_0 is an *hyperbolic singularity*. If P_0 is a singularity, but it is not a simple singularity, then we say that P_0 is a non reduced singularity.

Two important remarks must be made about simple singularities of foliations that reminds the case of curves:

1. A simple singularity is stable by blow-up, that is, if we consider $\sigma_1 : (M_1, E^1) \longrightarrow (M_0, P_0)$ the blow-up at P_0, we have that all the singularities in E^1 of the strict transform of $\mathcal{F}$ by σ are simple.
2. By Briot-Bouquet theorem, see [11], if P_0 is an hyperbolic singularity of $\mathcal{F}$, then ω has exactly two invariant regular branches C_1, C_2, such their tangent vectors are eigenvectors of the matrix $J(\omega)$. If P_0 is a saddle node singularity, then we have a similar result. The only difference is that the branch associated to the eigenspace of zero eigenvalue may be

non convergent. In other words, we can have an element $f \in \mathbb{C}[[x,y]] \setminus \mathbb{C}\{x,y\}$, such that $\omega \wedge df = f\eta$ with η a formal 2-form.

Simple singularities are the only ones allowed in a resolution of singularities of a foliation, in the following sense: consider a normal crossings curve $E^0 \subset M_0$. We say that the point P_0 is a *simple point* of $(M_0, E^0, \mathcal{F})$ of a foliation $\mathcal{F}$ in (M_0, P_0), if one of two following conditions is satisfied:

a) P_0 is a simple singularity of $\mathcal{F}$, there exists an irreducible component of E^0 through P_0, and all the irreducible components of E^0 are invariant by $\mathcal{F}$.
b) The point P_0 is a regular point $\mathcal{F}$ and $\mathcal{F}$ has normal crossings with E^0. That is to say, if L is the only invariant curve of $\mathcal{F}$ through P_0, then $E^0 \cup L$ is a normal crossings divisor.

Theorem 2.2.3 (Resolution of Singularities [51])**.** *Consider a germ of foliation $\mathcal{F}$ in (M_0, P_0). There exists a finite sequence of blow-ups $\pi : (M_N, E^N) \to (M_0, P_0)$, such that the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ by π satisfies that every point $P \in E^N$ is a simple point of $(M_N, E^N, \widetilde{F})$.*

We say that π as above is a resolution of singularities of $\mathcal{F}$.

2.3 Divisorial Value

The goal of this section and the next one is to characterize the differential 1-forms that appear when computing the semimodule of differential values. To do so, we introduce in this section the notion of divisorial value, we use it when studying the Newton Polygon of a 1-form. Most of the results presented can be found in Section 3 of [12].

We fix $\pi : (M_N, E^N) \to (M_0, P_0)$ a sequence of blow-ups, with the usual notations established in Section 1.3. Consider a holomorphic function h in (M_N, E^N) defined globally in $E := E_N^N \subset E^N$. The *divisorial value $\nu_E(h)$ of h* is obtained as follows. Take a point $P \in E$ and choose a reduced local equation $u = 0$ of the germ (E, P), then we define the divisorial value of f as

$$\nu_E(h) := \max\{a \in \mathbb{Z} : \ u^{-a}h \in \mathcal{O}_{M_N,P}\}.$$

Remark 2.3.1. Notice that $\nu_E(h)$ does not depend on the point $P \in E$ chosen. In fact, assume that $u = 0$ is an implicit equation of E defined in an open set $U \subset E$. Given a different point $Q \in U$, since $u = 0$ is an implicit equation of E at Q, the divisorial value $\nu_E(h)$ is the same at the points Q and P.

Finally, for any pair of points $P, Q \in E$, we can take a sequence of open sets $U_1, U_2, \ldots, U_k \subset E$, where U_1 and U_k are open neighbourhoods of P and Q respectively, with $U_j \cap U_{j+1} \neq \emptyset$ for $j = 1, 2, \ldots, k-1$. We denote by $u_i = 0$ an implicit equation of E defined along all the points U_i, for $i \in \{1, \ldots, k\}$. Since the divisorial value must be the same at $U_j \cap U_{j+1}$ we conclude the desired result.

We are using the fact that two implicit equations $h' = 0$ and $h = 0$ of a curve C differ by an unit, this implies that the definition of divisorial value is independent of the chosen local equation.

Consider one of the center points P_j of the sequence of blow-ups π, with $j \in \{0, 1, \ldots, N-1\}$ and a germ of holomorphic function $h \in \mathcal{O}_{M_j,P_j}$. Then $\rho_j^* h$ is a germ of function in (M_N, E), that is, $\rho_j^* h$ is globally defined in E, where $\rho_j = \sigma_j \circ \sigma_{j+1} \circ \ldots \circ \sigma_N$. We extend the definition *divisorial value* with respect to E to h by putting $\nu_E(h) := \nu_E(\rho_j^* h)$.

Proposition 2.3.2. *Take (x, y) a local system of adapted coordinates with respect to a cuspidal sequence $\pi = \pi_y^{n,m}$, and denote by D the cuspidal divisor of $\pi_y^{n,m}$. Consider a germ $h \in \mathcal{O}_{M_0,P_0}$ that we write as*

$$h = \sum_{\alpha,\beta} h_{\alpha\beta} x^\alpha y^\beta, \quad h_{\alpha\beta} \in \mathbb{C}.$$

Then $\nu_D(h) = \min\{n\alpha + m\beta;\ h_{\alpha\beta} \neq 0\}$.

Proof. If $n = m = 1$, π is just a single blow. In this case, we see that the divisorial value of a function h is coincides with its multiplicity at P_0, as desired. Let us work by induction on $n + m$ and assume that $n + m \geq 2$. We remark that $\nu_D(h) = \nu_D(\sigma_1^* h)$. Consider the first intermediate sequence $\rho_2 = \sigma_2 \circ \ldots \circ \sigma_N$, with adapted coordinates (x_1, y_1). Denote by ℓ the index of freeness of $\pi_y^{n,m}$. There are two options: if $\ell = 1$, then the coordinate system is given by $(x, y) = (x_1 y_1, y_1)$. Otherwise, if $\ell \geq 2$, then $(x, y) = (x_1, x_1 y_1)$. Therefore, we have that

$$\sigma_1^* h = \sum_{\alpha,\beta} h_{\alpha\beta} x_1^{\alpha+\beta} y_1^\beta \text{ if } \ell \geq 2;$$
$$\sigma_1^* h = \sum_{\alpha,\beta} h_{\alpha\beta} x_1^\beta y_1^{\alpha+\beta} \text{ if } \ell = 1.$$

Furthermore, if $\ell = 1$ the Puiseux pair of ρ_2 is given by $(n, m - n)$. On the contrary, if $\ell \geq 2$, then the Puiseux pair of ρ_2 is $(n - m, n)$. We conclude by applying the induction hypothesis to the previous expressions. □

Remark 2.3.3. The computations from the proof of Proposition 2.3.2 show that in the case when $\pi_y^{n,m}$ is a single blow-up, then $\nu_D(h) = \nu_{P_0}(h)$. Furthermore, in this case every system of coordinates is adapted with respect to $\pi_y^{1,1}$.

2.3.1 Divisorial Value of a Differential Form

From now fix $\pi = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_N$ a sequence of blow-ups, where as always $\sigma_i : (M_i, E^i) \to (M_{i-1}, E^{i-1})$ is a blow-up with center $P_i \in E^{i-1}$, for $i = 1, 2, \ldots, N$ and $E^0 = \{P_0\}$. We denote by $\pi_j = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_j$ and $\rho_j = \sigma_j \circ \sigma_{j+1} \circ \ldots \circ \sigma_N$ the intermediate sequences, for $j = 1, 2, \ldots, N$. Finally, we fix the divisors

$$H_0 = (xy = 0) \subset M_0, \quad H_j = \pi_j^{-1}(H_0) \subset M_j,$$

such that H_j is locally given at P_j by $x_j y_j = 0$ for $0 \leq j \leq N - 1$. We also consider $H_N = H = \pi^{-1}(H_0) \subset M_N$. Each H_j is, at every point, a curve with normal crossings in (M_j, E^j), containing E^j. We also write $E = E_N^N$.

Take a point $Q \in E_j$, not necessarily equal to P_j, in particular we consider also the case $j = N$. Select a local system of coordinates (u, v) in chart of (M_j, E^j) such that $(u = 0) \subset H_j \subset (uv = 0)$, then we have that either $H_j = (u = 0)$ or $H_j = (uv = 0)$ locally at Q. The $\mathcal{O}_{M_j,Q}$-module $\Omega^1_{M_j,Q}[\log H_j]$ of germs of H_j-*logarithmic 1-forms* is the rank two free $\mathcal{O}_{M_j,Q}$-module generated by

$$du/u, dv \quad \text{if} \quad H_j = (u = 0),$$
$$du/u, dv/v \quad \text{if} \quad H_j = (uv = 0).$$

In Chapter 8 we retake the notion of logarithmic forms in a more general setting. For the moment, note that $\Omega^1_{M_j,Q} \subset \Omega^1_{M_j,Q}[\log H_j]$. Indeed, a differential 1-form $\omega = Adu + Bdv$ may be written as

$$\omega = uA\frac{du}{u} + Bdv = uA\frac{du}{u} + vB\frac{dv}{v}.$$

In a similar way, we define the $\mathcal{O}_{M_j,Q}$-module $\Omega^2_{M_j,Q}[\log H_j]$ of H_j-*logarithmic 2-forms* as the rank one free $\mathcal{O}_{M_j,Q}$-module generated by

$$
\begin{aligned}
(du/u)\wedge dv \quad &\text{if} \quad H_j=(u=0),\\
(du/u)\wedge dv/v \quad &\text{if} \quad H_j=(uv=0).
\end{aligned}
$$

Again, we have that $\Omega^2_{M_j,Q}\subset\Omega^2_{M_j,Q}[\log H_j]$.

Now, let us consider a q-form $\omega\in\Omega^q_{M_N}[\log H]$, $q=1,2$, defined in E. Select a point $Q\in E$ and a local reduced equation $u=0$ of E at Q. We define the *divisorial value* $\nu_E(\omega)$ by

$$\nu_E(\omega)=\max\{\ell\in\mathbb{Z};\ u^{-\ell}\omega\in\Omega^q_{M_N,Q}[\log H]\}.$$

As in Remark 2.3.1 we can show that the divisorial value of a q-form is well defined, that is, it does not depend neither on the chosen equation of E nor on the point Q.

Remark 2.3.4. Let $\omega\in\Omega^q_{M_N}[\log E]$ be a q-form globally defined on E as before. Since E is one of the irreducible components of H, we have that

$$\Omega^q_{M_N}[\log E]\subset\Omega^q_{M_N}[\log H].$$

Let us choose a reduced local equation $u=0$ of E at a point $Q\in E$ as before. A direct computation shows that

$$\nu_E(\omega)=\max\{\ell\in\mathbb{Z};\ u^{-\ell}\omega\in\Omega^q_{M_N,Q}[\log E]\}. \tag{2.3}$$

This remark shows that the divisorial value of a q-form $\omega\in\Omega^q_{M_N}[\log E]$ is independent of the choice of the adapted coordinate system that defines H_0.

Definition 2.3.5. *For any $\omega\in\Omega^q_{M_j,P_j}$, the* divisorial value $\nu_E(\omega)$ *is defined by $\nu_E(\omega)=\nu_E(\rho_j^*\omega)$.*

Remark 2.3.6. The notion of divisorial value of a 1-forms can be defined without using logarithmic 1-forms with respect to a curve. We could have defined it as the maximum number of times that the implicit equation of E divides $\pi_j^*\omega$ up to obtaining a holomorphic 1-form. However, with this definition, the divisorial value is no longer determined by just the monomials of the 1-form, as we show later. There is an indeterminacy depending on if the divisor E is dicritical or not for the 1-form ω. By introducing the logarithmic 1-forms, this problem disappears.

Proposition 2.3.7. *Consider a differential 1-form $\omega=Adx+Bdy\in\Omega^1_{M_0,P_0}$, we can write ω as*

$$\omega=xA(dx/x)+yB(dy/y)\in\Omega^1_{M_0,P_0}[\log H_0].$$

Then, we have that $\nu_E(\omega)=\min\{\nu_E(xA),\nu_E(yB)\}$.

Proof. Write $g=xA$ and $h=yB$, that is, $\omega=g(dx/x)+h(dy/y)$. We proceed by induction on N. If $N=1$, we have that $E=(x_1=0)$ where $(x,y)=(x_1,x_1y_1)$ and

$$\sigma_1^*\omega=(\sigma_1^*g+\sigma_1^*h)(dx_1/x_1)+\sigma_1^*(h)(dy_1/y_1).$$

Then $\nu_E(\omega)=\min\{\nu_E(\sigma_1^*g+\sigma_1^*h),\nu_E(\sigma_1^*h)\}=\min\{\nu_E(g),\nu_E(h)\}$ and as desired. If $N\geq 2$, then

$$\nu_E(\omega)=\nu_E(\pi^*\omega)=\nu_E(\rho_2^*(\sigma_1^*\omega))=\nu_E(\sigma_1^*\omega).$$

By induction hypothesis, we have

$$\nu_E(\sigma_1^*\omega)=\min\{\nu_E(\sigma_1^*g+\sigma_1^*h),\nu_E(\sigma_1^*h)\}=\min\{\nu_E(g),\nu_E(h)\}$$

and we are finished. □

Corollary 2.3.8. *Given $f \in \mathcal{O}_{M_0,P_0}$ with $f(P_0) = 0$ and $\omega = df$, then $\nu_E(\omega) = \nu_E(f)$.*

Proof. If we write $f = \sum_{\alpha,\beta\geq 1} h_{\alpha\beta} x^\alpha y^\beta$, then we have that

$$x f_x = \sum_{\alpha,\beta} \alpha h_{\alpha\beta} x^\alpha y^\beta; \quad y f_y = \sum_{\alpha,\beta} \beta h_{\alpha\beta} x^\alpha y^\beta.$$

By Proposition 2.3.7, we have that $\nu_E(df) = \min\{\nu_E(x f_x), \nu_E(y f_y)\}$. Now, by Proposition 2.3.2, we obtain that

$$\min\{\nu_E(x f_x), \nu_E(y f_y)\} = \min\{n\alpha + m\beta : \alpha h_{\alpha\beta} \neq 0 \text{ or } \beta h_{\alpha\beta} \neq 0\}.$$

Since $f(P_0) = 0$, the condition $\alpha h_{\alpha\beta} \neq 0$ or $\beta h_{\alpha\beta} \neq 0$, is equivalent to $h_{\alpha\beta} \neq 0$. Concluding the desired result. □

Similarly, we show that

Proposition 2.3.9. *Consider a differential 2-form $\eta = g dx \wedge dy \in \Omega^2_{M_0,P_0}$, that we can write as*

$$\omega = xyg(dx/x) \wedge (dy/y) \in \Omega^2_{M_0,P_0}[\log H_0].$$

Then, we have that $\nu_E(\eta) = \nu_E(xyg)$.

Proof. As in the proof of Proposition 2.3.7, we use an inductive argument on the length N of the sequence of blow-ups π. If $N = 1$, we take coordinates $(x, y) = (x_1, x_1 y_1)$, such that E is defined by $x_1 = 0$. Additionally, we see that

$$\sigma_1^* \eta = \sigma_1^*(xyg) dx_1/x_1 \wedge dy_1/y_1.$$

Showing the desired result.

Finally, if $N \geq 2$, we have that

$$\nu_E(\eta) = \nu_E(\pi^* \eta) = \nu_E(\rho_2^*(\sigma_1^* \eta)) = \nu_E(\sigma_1^* \eta).$$

By induction hypothesis, we have that

$$\nu_E(\sigma_1^* \eta) = \nu_E(\sigma_1^*(xyg)) = \nu_E(xyg).$$

□

Corollary 2.3.10. *Consider $\omega, \omega' \in \Omega^1_{M_0,P_0}$, then we have that*

$$\nu_E(\omega) + \nu_E(\omega') \leq \nu_E(\omega \wedge \omega').$$

Proof. Let us write

$$\omega = xA(dx/x) + yB(dy/y); \quad \omega' = xA'(dx/x) + yB'(dy/y),$$

then we have that

$$\omega \wedge \omega' = (xAyB' - xA'yB) dx/x \wedge dy/y.$$

By Propositions 2.3.7 and 2.3.9, we get that

$$\begin{aligned} \nu_E(\omega \wedge \omega') &= \nu_E(xAyB' - xA'yB) \geq \min\{\nu_E(xA) + \nu_E(yB'), \nu_E(xA') + \nu_E(yB)\} \\ &\geq \min\{\nu_E(xA), \nu_E(yB)\} + \min\{\nu_E(xA'), \nu_E(yB')\} = \nu_E(\omega) + \nu_E(\omega'). \end{aligned}$$

□

Remark 2.3.11. Propositions 2.3.7 and 2.3.9 show that we can compute divisorial values of logarithmic forms in terms of their coefficients. In the cuspidal case, by Proposition 2.3.2, we have that the divisorial value of a function can be seen as a monomial value. Therefore, we can extend this monomial valuation to consider 1-forms and 2-forms.

We are not only interested on the divisorial values, but also on the terms of the coefficients that they determine. They are what we call initial parts and we procced to define them.

2.3.2 Weighted Initial Parts

From now on, we assume that (x, y) is a local system of coordinates with respect to the cuspidal sequence $\pi = \pi_y^{n,m}$, with cuspidal divisor $D = E$. Consider a non zero germ $h \in \mathcal{O}_{M_0,P_0}$ that we write as $h = \sum_{\alpha,\beta} h_{\alpha\beta} x^\alpha y^\beta$. Suppose that $q = \nu_D(h)$. We define the *weighted initial part* $\mathrm{In}_{n,m;x,y}(h)$ by

$$\mathrm{In}_{n,m;x,y}(h) = \sum_{n\alpha+m\beta=q} h_{\alpha\beta} x^\alpha y^\beta.$$

We can write

$$h = \mathrm{In}_{n,m;x,y}(h) + \tilde{h}, \quad \nu_D(\tilde{h}) > q.$$

This definition extends to logarithmic differential 1-forms $\omega \in \Omega^1_{M_0,P_0}[\log(H_0)]$ as follows. If $q = \nu_D(\omega)$, we write

$$\omega = f(dx/x) + g(dy/y); \quad f = \sum_{\alpha\geq 1,\beta\geq 0} f_{\alpha\beta} x^\alpha y^\beta, \quad g = \sum_{\alpha\geq 0,\beta\geq 1} g_{\alpha\beta} x^\alpha y^\beta.$$

We define

$$\mathrm{In}_{n,m;x,y}(\omega) = \sum_{n\alpha+m\beta=q} x^\alpha y^\beta (f_{\alpha\beta} dx/x + g_{\alpha\beta} dy/y).$$

As before, we have $\omega = \mathrm{In}_{n,m;x,y}(\omega) + \tilde{\omega}$, with $\nu_D(\tilde{\omega}) > q$. When there is no confusion on the Puiseux pair (n, m) and the coordinate system, we just write $\mathrm{In}(-)$ instead of $\mathrm{In}_{n,m;x,y}(-)$.

Remark 2.3.12. Note that the definition of initial part can be made in terms of graduated rings and modules to be free of coordinates. Nonetheless, this "coordinate-based" definition is enough for our purposes.

Next proposition shows the behaviour of the initial part under blow-up.

Proposition 2.3.13. *Assume that* $N > 1$, *take* $\omega \in \Omega^1_{M_0,P_0}[\log(H_0)]$. *If* $W = \mathrm{In}_{n,m;x,y}(\omega)$, *then* $\sigma_1^*(W) = \mathrm{In}_{n_1,m_1;x_1,y_1}(\sigma_1^*\omega)$, *where* (x_1, y_1) *is a local system of adapted coordinates in a chart of* (M_1, E^1), *with respect to the intermediate sequence of blow-ups* ρ_2.

Proof. Put $q = \nu_D(\omega)$ and denote by ℓ the index of freeness of π. There are two cases: either $\ell = 1$ or $\ell \geq 2$. If $\ell \geq 2$, then we consider the Puiseux pair of ρ_2 is given by $(n_1, m_1) = (n, m - n)$ and the coordinate system is defined by $(x, y) = (x_1, x_1 y_1)$. Now we write

$$\omega = \Big(\sum_{\alpha,\beta\geq 0} a_{\alpha+1\beta} x^\alpha y^\beta \Big) dx + \Big(\sum_{\alpha,\beta\geq 0} b_{\alpha\beta+1} x^\alpha y^\beta \Big) dy,$$

and we have that in the local system (x_1, y_1),

$$\sigma_1^*(\omega) = \Big(\sum_{\alpha,\beta\geq 0} (a_{\alpha+1\beta} + b_{\alpha+1\beta}) x^{\alpha+\beta} y^\beta \Big) dx + \Big(\sum_{\alpha,\beta\geq 0} b_{\alpha\beta+1} x^{\alpha+\beta} y^\beta \Big) dy.$$

Note that $n\alpha + m\beta = q$ is equivalent to $n(\alpha+\beta) + (m-n)\beta = q$. Hence, just by doing a standard computation, we show that $\sigma_1^*(W) = \mathrm{In}_{n_1,m_1;x_1,y_1}(\sigma_1^*\omega)$.

If $\ell = 1$, we do the same, but in this case $(x, y) = (x_1 y_1, y_1)$ and $(n_1, m_1) = (m - n, n)$. □

Now, consider a 1-form η whose initial part is given by

$$\mathrm{In}_{n,m;x,y}(\eta) = x^a y^b \left\{ \mu \frac{dx}{x} + \zeta \frac{dy}{y} \right\}.$$

We say that η is *resonant* if and only if $n\mu + m\zeta = 0$.

The next proposition shows the relationship between the divisorial value of a function (resp. 1-form) with its intersection multiplicity (resp. differential values) with a cusp, as explained in Chapter 1.

Proposition 2.3.14. *Consider C a D-cusp in (M_0, P_0) with Puiseux pair (n, m), for any function $g \in \mathcal{O}_{M_0,P_0}$ we have the following:*

a) $\nu_C(g) \geq \nu_D(g)$.
b) *If* $\nu_C(g) > \nu_D(g)$, *then* $\nu_D(g) \geq nm$.

Similarly, for any 1-form $\omega \in \Omega^1_{M_0,P_0}$, we have that

a') $\nu_C(\omega) \geq \nu_D(\omega)$.
b') *If* $\nu_C(\omega) > \nu_D(\omega)$ *and* $\nu_D(\omega) < nm$, *then ω is resonant.*

The proof is based on the following remark.

Remark 2.3.15. Given $(a, b), (a', b') \in \mathbb{Z}_{\geq 0}$, such that $na + mb = na' + mb' = c < nm$, then we have that $(a, b) = (a', b')$.

Proof. Since (x, y) is an adapted system of coordinates with respect to $\pi_y^{n,m}$, then we can take a primitive parametrization of C of the form $(t^n, v(t)t^m)$, with $v(t) \in \mathbb{C}\{t\}$ a unit, see Equation (1.8). Take A, B two different monomials, Statements a) and a') follow by noting that:

1. $\nu_C(x^a y^b) = \nu_D(x^a y^b) = na + mb$.
2. $\nu_C(A + B) \geq \min\{\nu_C(A), \nu_C(B)\}$.
3. By Proposition 2.3.2, we have that $\nu_D(A + B) = \min\{\nu_D(A), \nu_D(B)\}$.

Now, for Statement b), assume that $\nu_D(g) < nm$, then by Remark 2.3.15, we can write $\mathrm{In}_{n,m;x,y}(g) = A$, with A a monomial. We see that $\nu_C(g) = \nu_C(A) = \nu_D(A)$. This is because of Statement a) and items 1.- 3.

Finally, for Statement b'), consider a 1-form ω whose divisorial value is smaller than nm, then we have that

$$\mathrm{In}_{n,m;x,y}(\omega) = \eta = x^a y^b \left\{ \mu \frac{dx}{x} + \zeta \frac{dy}{y} \right\}.$$

Just a mere computation shows that $\nu_C(\eta) > \nu_D(\eta) = \nu_D(\omega)$ if and only if η is resonant. As in Statement b), we show that if η is non resonant, then $\nu_C(\omega) = \nu_C(\eta) = \nu_D(\eta) = \nu_D(\omega)$. □

In Proposition 2.3.14 Statement a'), we can give an interpretation on whether we have an inequality instead of an equality.

Proposition 2.3.16. *Let C be a D-cusp in (M_0, P_0). Consider $\phi(t) = (t^n, at^m + h.o.t) = (t^n, v(t)t^m)$ a primitive parametrization of C, and take $\omega \in \Omega^1_{M_0,P_0}$. We have that the curve $\tilde{C}$ defined by the primitive parametrization $\tilde{\phi}(t) = (t^n, at^m)$ is invariant by the 1-form $\eta = \mathrm{In}_{n,m;x,y}(\omega)$ if and only if $\nu_C(\omega) > \nu_D(\omega)$.*

Proof. Denote by $q_0 = \nu_D(\omega)$ and write

$$\omega = \eta + \sum_{q>q_0} \omega_q,$$

in such a way $\mathrm{In}_{n,m;x,y}(\omega_q) = \omega_q$ and $\nu_D(\omega_q) = q$ for all $q > q_0$. We notice that, for any 1-form θ such that $\mathrm{In}_{n,m;x,y}(\theta) = \theta$, we have that $\tilde{\phi}^*(\theta) = \alpha t^{q-1}dt$, with $\alpha \in \mathbb{C}$ and $q = \nu_D(\theta)$. Besides, we also see that $\phi^*(\theta) = t^{q-1}(\alpha + h.o.t.)dt$. This applies to the terms ω_q and η. If we write $\tilde{\phi}^*(\eta) = \mu t^{q_0-1}dt$, then we have that $\tilde{C}$ is invariant by η if and only $\mu = 0$. The condition $\mu = 0$ is equivalent to $\nu_C(\omega) > \nu_D(\omega)$. □

2.4 Basic and Pre-basic 1-Forms

This section is devoted to characterize the 1-forms $\omega \in \Omega^1_{M_0,P_0}$ whose total transform $\pi^*\omega$ defines a foliation that is transverse to D, and which has normal crossings with E^N at every point of D.

2.4.1 Reduced Divisorial Value and Basic 1-Forms

Let us consider a non null differential 1-form $\omega \in \Omega^1_{M_0,P_0}$. Let $V_\omega = x^a y^b$ be the monomial defined by the property that $\omega = V_\omega \eta$, where $\eta \in \Omega^1_{M_0,P_0}[\log(H_0)]$ is a logarithmic form that cannot be divided by any nonconstant monomial. We define the *reduced divisorial value* $\mathrm{rdv}_D(\omega)$ to be $\mathrm{rdv}_D(\omega) = \nu_D(\eta)$.

Definition 2.4.1. *We say that $\omega \in \Omega^1_{M_0,P_0}$ is a basic 1-form if and only if its reduced divisorial value satisfies that $\mathrm{rdv}_D(\omega) < nm$.*

Given ω a basic 1-form, then the initial part of ω can be written as

$$\mathrm{In}_{n,m;x,y}(\omega) = x^a y^b W, \quad \text{where} \quad W = \mathrm{In}_{n,m;x,y}(\eta).$$

If ω is a basic 1-form with $\omega = V_\omega \eta$. By Remark 2.3.15, there is exactly one pair $(c, d) \in \mathbb{Z}^2_{\geq 0}$ such that $cn + dm = \nu_D(\eta) < nm$, then we have that

$$W = x^c y^d \left\{ \mu \frac{dx}{x} + \zeta \frac{dy}{y} \right\}.$$

Now we show that being basic is preserved by blow-up.

Proposition 2.4.2. *Assume that $N \geq 2$ and take $\omega \in \Omega^1_{M_0,P_0}$. If ω is a basic 1-form, then $\sigma_1^*\omega$ is also a basic 1-form.*

Proof. Write $\omega = V_\omega \eta$ as before and denote $q = \mathrm{rdv}_D(\omega) = \nu_D(\eta) < nm$. Recall that $\nu_D(\eta) = \nu_D(\sigma_1^*\eta)$. Since monomials are well behaved under the point center blow-up σ_1, it is enough to show that there are $c, d \geq 0$ such that $\sigma_1^*\eta = x_1^c y_1^d \eta'$, with $\nu_D(\eta') < n_1 m_1$, where (n_1, m_1) is the Puiseux pair of ρ_2 and (x_1, y_1) a system of adapted coordinates with respect to ρ_2. Write

$$\eta = \sum_{\alpha,\beta} x^\alpha y^\beta \eta_{\alpha\beta}, \text{ where } \eta_{\alpha\beta} = \mu_{\alpha\beta}\frac{dx}{x} + \zeta_{\alpha\beta}\frac{dy}{y}, \text{ and } (\mu_{\alpha\beta}, \zeta_{\alpha\beta}) \in \mathbb{C}^2.$$

Recall that $q = \min\{n\alpha + m\beta;\ \eta_{\alpha\beta} \neq 0\}$. Put $r = \min\{\alpha + \beta;\ \eta_{\alpha\beta} \neq 0\}$, that is, $r = \nu_{P_0}(\eta) + 1$. We have two cases: $\ell = 1$ and $\ell \geq 2$, where ℓ is the index of freeness of $\pi_y^{n,m}$.

Assume first that $\ell \geq 2$ and hence $2n \leq m$. In this situation, we have that $(x, y) = (x_1, x_1 y_1)$, $n_1 = n$, $m_1 = m - n \geq n$ and

$$\sigma_1^*(\eta) = x_1^r \eta', \text{ where } \eta' = \sum_{\alpha,\beta} x_1^{\alpha+\beta-r} y_1^{\beta} \eta'_{\alpha\beta}, \text{ with } \eta'_{\alpha\beta} = \left(\mu_{\alpha\beta} + \zeta_{\alpha\beta}\right) \frac{dx_1}{x_1} + \zeta_{\alpha\beta} \frac{dy_1}{y_1}.$$

We only need to check that $\nu_D(\eta') < n_1 m_1$. Note that $\eta'_{\alpha\beta} \neq 0$ if and only if $\eta_{\alpha\beta} \neq 0$. Hence, we have that

$$\begin{aligned} \nu_D(\eta') &= \min\{n_1(\alpha + \beta - r) + m_1 \beta;\ \eta_{\alpha\beta} \neq 0\} = \\ &= \min\{n(\alpha + \beta - r) + (m - n)\beta;\ \eta_{\alpha\beta} \neq 0\} = \\ &= \min\{n\alpha + m\beta - nr;\ \eta_{\alpha\beta} \neq 0\} = q - nr. \end{aligned}$$

We have to verify that $q - nr < n_1 m_1$, where $n_1 m_1 = n(m - n) = nm - n^2$. If $r \geq n$, the result follows since by hypothesis we have that $q < nm$. Assume that $r < n$. There are $\tilde{\alpha}, \tilde{\beta}$ with $\eta_{\tilde{\alpha}\tilde{\beta}} \neq 0$ such that $\tilde{\alpha} + \tilde{\beta} = r$. Then

$$\begin{aligned} q - nr &\leq n\tilde{\alpha} + m\tilde{\beta} - nr = n(\tilde{\alpha} + \tilde{\beta}) + (m - n)\tilde{\beta} - nr = \\ &= (m - n)\tilde{\beta} < (m - n)n, \end{aligned}$$

since $\tilde{\beta} \leq r < n$.

Assume now that $\ell = 1$ and thus $n < m < 2n$. We have that $(x, y) = (y_1, x_1 y_1)$, $n_1 = m - n < n$, $m_1 = n$ and

$$\sigma_1^*(\eta) = y_1^r \eta'', \text{ where } \eta'' = \sum_{\alpha,\beta} x_1^{\beta} y_1^{\alpha+\beta-r} \eta''_{\alpha\beta}, \text{ with } \eta''_{\alpha\beta} = \zeta_{\alpha\beta} \frac{dx_1}{x_1} + \left(\mu_{\alpha\beta} + \zeta_{\alpha\beta}\right) \frac{dy_1}{y_1}.$$

Again, we have to verify that $\nu_D(\eta') < n_1 m_1$. As before, we have that $\eta''_{\alpha\beta} \neq 0$ if and only if $\eta_{\alpha\beta} \neq 0$. Hence

$$\begin{aligned} \nu_D(\eta'') &= \min\{n_1 \beta + m_1(\alpha + \beta - r);\ \eta_{\alpha\beta} \neq 0\} = \\ &= \min\{(m - n)\beta + n(\alpha + \beta - r);\ \eta_{\alpha\beta} \neq 0\} = \\ &= \min\{m\beta + n\alpha - nr;\ \eta_{\alpha\beta} \neq 0\} = q - nr. \end{aligned}$$

We show that $q - nr < n_1 m_1$ exactly as before. □

We have the next result that follows directly from the computations in the proof of Proposition 2.4.2:

Corollary 2.4.3. *Assume that $N \geq 2$. A basic differential 1-form $\omega \in \Omega^1_{M_0,P_0}$ is resonant if and only if $\sigma_1^* \omega$ is resonant.*

2.4.2 Pre-Basic 1-Forms

Let us introduce a slightly more general class of 1-forms that we call *pre-basic forms*. Consider the pair of coprime positive integers (n, m), with $1 \leq n \leq m$. There are unique $b, d \in \mathbb{Z}_{\geq 0}$ such that $dn - bm = 1$ with the property that $0 \leq b < n$ and $0 < d \leq m$. We call (b, d) the co-pair of (n, m).

Definition 2.4.4. *We define the region $R^{n,m}$ by $R^{n,m} = H_-^{n,m} \cap H_+^{n,m}$, where*

$$\begin{aligned} H_-^{n,m} &= \{(\alpha, \beta) \in \mathbb{R}^2;\ (n - b)\alpha + (m - d)\beta \geq 0\}, \\ H_+^{n,m} &= \{(\alpha, \beta) \in \mathbb{R}^2;\ b\alpha + d\beta \geq 0\}, \end{aligned}$$

and (b, d) is the co-pair of (n, m).

Remark 2.4.5. If $n = m = 1$, the co-pair of $(1,1)$ is $(b,d) = (0,1)$. Then

$$H_-^{1,1} = \{(\alpha,\beta);\ \alpha \geq 0\}, \quad H_+^{1,1} = \{(\alpha,\beta);\ \beta \geq 0\}.$$

Thus, we have that $R^{1,1}$ is the quadrant $R^{1,1} = \mathbb{R}^2_{\geq 0}$.

Remark 2.4.6. The slopes $-(n-b)/(m-d)$ and $-b/d$ satisfy that

$$-(n-b)/(m-d) < -n/m < -b/d.$$

Indeed, we have $-n/m < -b/d$ if and only if $-dn < -mb = -dn + 1$. On the other hand

$$-(n-b)/(m-d) < -n/m \quad \Leftrightarrow \quad m(n-b) > n(m-d) \Leftrightarrow bm < dn = bm + 1.$$

We conclude that $R^{n,m}$ is a positively convex region of $\mathbb{R}^2$ such that $(0,0)$ is its only vertex and we have that

$$R^{n,m} \cap \{(\alpha,\beta) \in \mathbb{R}^2;\ n\alpha + m\beta = 0\} = \{(0,0)\}.$$

Given a point $(a,b) \in \mathbb{R}^2_{\geq 0}$, we define $R^{n,m}(a,b)$ by $R^{n,m}(a,b) = R^{n,m} + (a,b)$.

Fix (x,y) a local system of coordinates at (M_0, P_0). Given a 1-form $\omega \in \Omega^1_{M_0,P_0}$, we write it as $\omega = Adx + Bdy$. The *Newton cloud* of ω is defined by

$$\mathcal{NC}_{x,y}(\omega) = \mathcal{NC}_{x,y}(xA) \cup \mathcal{NC}_{x,y}(yB),$$

and the *Newton polygon* of ω is given by

$$\mathcal{NP}_{x,y}(\omega) := \text{ convex hull of } \Big(\bigcup_{(i,j)\in\mathcal{NC}_{x,y}(\omega)} ((i,j) + (\mathbb{R}_{\geq 0})^2) \Big).$$

Definition 2.4.7. *We say that $\omega \in \Omega^1_{M_0,P_0}$ is a* pre-basic 1-form *if and only if there is a point $(a,b) \in \mathcal{NC}_{x,y}(\omega)$ such that $\mathcal{NC}_{x,y}(\omega) \subset R^{n,m}(a,b)$.*

If ω is pre-basic, we have that

$$\mathcal{NC}_{x,y}(\omega) \cap \{(\alpha,\beta) \in \mathbb{R}^2;\ n\alpha + m\beta = \nu_D(\omega)\} = \{(a,b)\}.$$

Thus, similar to the case of basic 1-forms, the initial part W of ω has the form

$$W = x^a y^b \left\{ \mu_{ab} \frac{dx}{x} + \zeta_{ab} \frac{dy}{y} \right\}. \tag{2.4}$$

Example 2.4.8. We consider the pair $(5,8)$, whose co-pair is $(5,3)$. The 1-form

$$\omega = x^3 dy + x^2 y dx - 7x^5 dx + 11 y^5 dy$$

is pre-basic, as we can see in the following figure.

Next lemma shows the behaviour of the co-pairs under blow-up.

Lemma 2.4.9. *Let us consider a pair (n,m) with $1 \leq n < m$ and n,m are without common factor. Let (b,d) be the co-pair of (n,m). We have that*

- *If $m \geq 2n$ and we put $(n_1,m_1) = (n, m-n)$, then the co-pair of (n_1,m_1) is $(b_1,d_1) = (b, d-b)$.*
- *If $m < 2n$ and we put $(n_1,m_1) = (m-n, n)$, then the co-pair of (n_1,m_1) is $(b_1,d_1) = (m-n-d+b, n-b)$.*

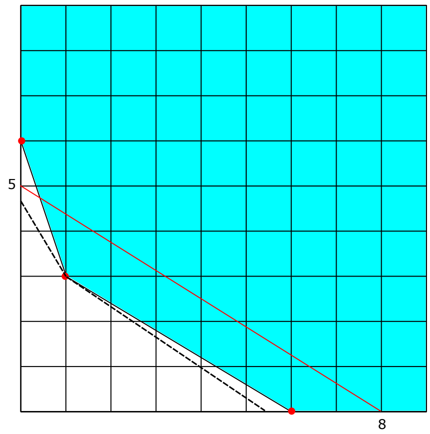

Figure 2.2: *Newton polygon of ω, the dashed lines mark the borders of the region $R^{5,8}(1,3)$.*

Moreover, we have that $\Psi(R^{n,m}) = R^{n_1,m_1}$, where Ψ is the linear automorphism of $\mathbb{R}^2$ given by $\Psi(\alpha,\beta) = (\alpha+\beta,\beta)$, if $m \geq 2n$, and $\Psi(\alpha,\beta) = (\beta,\alpha+\beta)$, if $m < 2n$.

Proof. Let us show the first two statements. If $m \geq 2n$, we have that

$$d_1n_1 - b_1m_1 = (d-b)n - b(m-n) = 1.$$

Moreover, since $0 \leq b_1 = b < n_1 = n$ we conclude that (b_1,d_1) is the co-pair of (n_1,m_1), in view of Remark 2.3.15. If $m < 2n$, we have

$$d_1n_1 - b_1m_1 = (n-b)(m-n) - (m-n-d+b)n = 1.$$

We know that $0 \leq b < n$, hence $0 < d_1 = n-b \leq m_1 = n$ and by Remark 2.3.15, we deduce that (b_1,d_1) is the co-pair of (n_1,m_1).

Now consider $(\alpha,\beta) \in \mathbb{R}^2$ and put $(\alpha_1,\beta_1) = \Psi(\alpha,\beta)$.

Case $m \geq 2n$. In order to prove that $\Psi(R^{n,m}) = R^{n_1,m_1}$ it is enough to see that

$$(\alpha,\beta) \in H_-^{n,m} \Leftrightarrow (\alpha_1,\beta_1) \in H_-^{n_1,m_1} \text{ and } (\alpha,\beta) \in H_+^{n,m} \Leftrightarrow (\alpha_1,\beta_1) \in H_+^{n_1,m_1}.$$

We verify these properties as follows:

$$\begin{aligned}
(\alpha_1,\beta_1) \in H_-^{n_1,m_1} \quad &\Leftrightarrow \quad (n_1-b_1)\alpha_1 + (m_1-d_1)\beta_1 \geq 0 \Leftrightarrow \\
&\Leftrightarrow \quad (n-b)(\alpha+\beta) + (m-n-d+b)\beta \geq 0 \Leftrightarrow \\
&\Leftrightarrow \quad (n-b)\alpha + (m-d)\beta \geq 0 \Leftrightarrow (\alpha,\beta) \in H_-^{n,m}. \\
(\alpha_1,\beta_1) \in H_+^{n_1,m_1} \quad &\Leftrightarrow \quad b_1\alpha_1 + d_1\beta_1 \geq 0 \Leftrightarrow \\
&\Leftrightarrow \quad b(\alpha+\beta) + (d-b)\beta \geq 0 \Leftrightarrow \\
&\Leftrightarrow \quad b\alpha + d\beta \geq 0 \Leftrightarrow (\alpha,\beta) \in H_+^{n,m}.
\end{aligned}$$

Case $m < 2n$. In this case, we have that

$$(\alpha,\beta) \in H_+^{n,m} \quad \Leftrightarrow \quad (\alpha_1,\beta_1) \in H_-^{n_1,m_1} \tag{2.5}$$

$$(\alpha,\beta) \in H_-^{n,m} \quad \Leftrightarrow \quad (\alpha_1,\beta_1) \in H_+^{n_1,m_1}. \tag{2.6}$$

and this also implies that $\Psi(R^{n,m}) = R^{n_1,m_1}$. We verify the properties in Equations (2.5) and (2.6) as follows:

$$
\begin{aligned}
(\alpha_1,\beta_1)\in H_-^{n_1,m_1} &\Leftrightarrow (n_1-b_1)\alpha_1+(m_1-d_1)\beta_1\geq 0\Leftrightarrow\\
&\Leftrightarrow (m-n-(m-n-d+b))\beta+(n-n+b)(\alpha+\beta)\geq 0\Leftrightarrow\\
&\Leftrightarrow d\beta+b\alpha\geq 0\Leftrightarrow(\alpha,\beta)\in H_+^{n,m}.\\
(\alpha_1,\beta_1)\in H_+^{n_1,m_1} &\Leftrightarrow b_1\alpha_1+d_1\beta_1\geq 0\Leftrightarrow\\
&\Leftrightarrow (m-n-d+b)\beta+(n-b)(\alpha+\beta)\geq 0\Leftrightarrow\\
&\Leftrightarrow (m-d)\beta+(n-b)\alpha\geq 0\Leftrightarrow(\alpha,\beta)\in H_-^{n,m}.
\end{aligned}
$$

The proof is ended. □

Now we show the stability of being pre-basic under blow-up.

Proposition 2.4.10. *Assume that $N\geq 2$. For any $\omega\in\Omega^1_{M_0,P_0}$, we have that*

1. *ω is pre-basic if and only if $\sigma_1^*\omega$ is pre-basic.*
2. *ω is pre-basic and resonant if and only if $\sigma_1^*\omega$ is pre-basic and resonant.*

Proof. We consider two cases as in the statement of Lemma 2.4.9, the case $m\geq 2n$ and $m<2n$ and we define the linear automorphism Ψ accordingly to these cases, as well as the Puiseux's pair (n_1,m_1). A monomial by monomial computation shows that

$$\mathcal{NC}_{x,y}(\sigma_1^*\omega)=\Psi(\mathcal{NC}_{x,y}(\omega)). \tag{2.7}$$

In view of Lemma 2.4.9, we have that

$$\Psi(R^{n,m}(a,b))=R^{n_1,m_1}(\Psi(a,b)). \tag{2.8}$$

Statement *1* is now a direct consequence of Equations (2.7) and (2.8). Statement *2* follows from Statement *1* and Corollary 2.4.3. □

We end this section studying simple properties about basic and pre-basic 1-forms.

Proposition 2.4.11. *Take a differential 1-form $\omega\in\Omega^1_{M_0,P_0}$. We have*

1. *If $N=1$, then ω is pre-basic if and only if it is basic.*
2. *If ω is basic then it is pre-basic.*
3. *If ω is basic and resonant then it is pre-basic and resonant.*

Proof. If $N=1$, we have $n=m=1$ and $R^{1,1}(a,b)=(\mathbb{R})^2_{\geq 0}+(a,b)$. Then being basic is the same property of being pre-basic: the Newton Polygon has a single vertex.

Assume now that ω is basic. In view of the stability result of basic 1-forms by blow-up given in Proposition 2.4.2, we have that $\tilde\omega$ is basic, where $\tilde\omega$ is the pull-back of ω in the last center of blow-up P_{N-1} of the cuspidal sequence. By Statement 1, we have that $\tilde\omega$ is pre-basic. Now we apply Proposition 2.4.10 to conclude that ω is pre-basic.

Statement 3 is easily deduced from Statement 2. □

2.5 Totally dicritical Forms

Consider a 1-form $\omega\in\Omega^1_{M_N}$ defined around the divisor cuspidal divisor D of $\pi_y^{n,m}$. Recall that we have a normal crossings divisor H such that $H\supset D$, coming from our choice of adapted coordinates, although if $n\geq 2$ the divisor H around D is intrinsically defined and it coincides with E^N. We say that ω is *totally D-dicritical with respect to H* if for any point $P \in D$ there are local coordinates (u, v) such that $D = (u = 0)$, $H \subset (uv = 0)$ and ω has the form

$$\omega = u^a v^b dv,$$

where $b = 0$ when $H = (u = 0)$. Note that ω defines a non-singular foliation around D, this foliation has normal crossings with H and D is transverse to the leaves.

The property of being totally D-dicritical can be read in terms of having a resonant pre-basic 1-form.

Proposition 2.5.1. *For any $\omega \in \Omega^1_{M_0,P_0}$, the following properties are equivalent:*

1. *The 1-form ω is pre-basic and resonant.*
2. *$\pi^*\omega$ is totally D-dicritical with respect to H.*

The proof of Proposition 2.5.1 requires the use of a simple version of the Frobenius Theorem, see [11] Theorem 2.4, and a technical lemma:

Theorem 2.5.2 (Frobenius Codimension 1)**.** *Let $(M, \mathcal{O}_M)$ be an analytic space of dimension $r \geq 2$ regular at a point $P \in M$. Consider $\mathcal{F}$ a regular foliation of codimension one in (M, P) defined by a 1-form $\omega \in \Omega^1_{M,P}$. Then there are functions $u, g \in \mathcal{O}_{M,P}$, such that u is a unit, P is a regular point of dg and $\omega = u dg$.*

We will only use this theorem when the ambient space is two dimensional.

Lemma 2.5.3. *Assume that $\omega \in \Omega^1_{M_0,P_0}$ satisfies that $\pi^*\omega$ is totally D-dicritical with respect to H. Consider P_{N-1} the last center point of π and (x_{N-1}, y_{N-1}) a coordinate system at P_{N-1}, such that H_{N-1} is locally defined by the implicit equation $x_{N-1}y_{N-1} = 0$. Denote by ω_{N-1} the strict transform of ω by π_{N-1} at P_{N-1}. Then both $x_{N-1} = 0$ and $y_{N-1} = 0$ are invariant curves by ω_{N-1}.*

Proof. Denote by ω_N the strict transform of ω in (M_N, E^N). After the last blow-up we obtain the coordinate system (x_N, y_N) in a chart of (M_N, E^N) defined by $(x_{N-1}, y_{N_1}) = (x_N, x_N y_N)$. Since $\pi^*\omega$ is totally D-dicritical, then, at the point $P = (x_N = 0, y_N = 0)$ there are coordinates (u, v) such that we can write locally ω_N as $\omega_N = dv$. Moreover, in this coordinate system the divisor H is defined at P by $uv = 0$, where $u = 0$ is an implicit equation of the cuspidal divisor D. Note that H is also defined at P by $x_N y_N = 0$ with $x_N = 0$ defining D and $y_N = 0$ the strict transform of $y_{N-1} = 0$ by σ_N. Thus, $v = 0$ and $y_N = 0$ are implicit equations of the same curve. Therefore, $y_N = 0$ is invariant by ω_N. By Lemma 2.2.2 that is equivalent to say that $y_{N-1} = 0$ is invariant by ω_{N-1}.

By taking the coordinate system defined by $(x_{N-1}, y_{N-1}) = (x'_N y'_N, y'_N)$, we show in a similar way that $x_{N-1} = 0$ is invariant by ω_{N-1}. □

Proof Proposition 2.5.1. In view of the stability of the property "pre-basic and resonant" under the successive blow-ups in the sequence π, see Proposition 2.4.10, it is enough to consider the case when $N = 1$. In this case we have a single blow-up.

Part 1: Statement *1* implies Statement *2*

Assume that ω is pre-basic and resonant and let us see that $\pi^*\omega$ is totally D-dicritical. By definition of being pre-basic and resonant, we have that

$$\omega = h(x, y)x^a y^b \left[\left\{ \frac{dx}{x} - \frac{dy}{y} \right\} + \sum_{\alpha+\beta\geq 1} x^\alpha y^\beta \left\{ \mu_{\alpha\beta} \frac{dx}{x} + \zeta_{\alpha\beta} \frac{dy}{y} \right\} \right], \quad a, b \geq 1,$$

where $h(0,0) \neq 0$. Consider the local system of coordinates (x_1, y_1) defined locally in (M_1, E^1), and given by $(x, y) = (x_1, x_1 y_1)$. The case $(x, y) = (x_1 y_1, y_1)$ is treated in a similar way. We show first that the at point $P = (x_1 = 0, y_1 = 0)$ there are coordinates (u, v) such that $\pi^*\omega = u^\alpha v^\beta dv$. Note that H is defined by $x_1 y_1 = 0$. In this system of coordinates we have that

$$\pi^*\omega = h(x_1, x_1 y_1) x_1^{a+b} y_1^b \left[-\frac{dy_1}{y_1} + \sum_{\alpha+\beta \geq 1} x_1^{\alpha+\beta} y_1^\beta \left\{ (\mu_{\alpha\beta} + \zeta_{\alpha\beta}) \frac{dx_1}{x_1} + \zeta_{\alpha\beta} \frac{dy_1}{y_1} \right\} \right].$$

Since $\alpha + \beta \geq 1$ and $b \geq 1$, then we see that the 1-form $\eta = h^{-1} x_1^{-a-b} y_1^{b-1} \pi^*\omega$ is holomorphic and can be written as

$$\eta = -dy_1 + y_1 A(x_1, y_1) dx_1 + x_1 B(x_1, y_1) dy_1. \tag{2.9}$$

By Theorem 2.5.2 there exist a unit u and a function g, which can be assumed that $g(0,0) = 0$, such that $\eta = u dg$. Note that $\eta \wedge dy_1 = y_1 A(x_1, y_1) dx_1 \wedge dy_1$. This implies that y_1 divides g. Since g is regular at the point P and $g(0,0) = 0$, we conclude that $g = y_1 v$, where v is another unit, in particular $x_1 = 0$ is transverse to $g = 0$. Thus if we take the coordinate system (x_1, g), then

$$\pi^*\omega = u' x_1^{a+b} g^{b-1} dg; \quad \text{where } u' \text{ is an appropriate unit.}$$

Finally, taking the coordinate system (h, g) with $h = x_1 (u')^{1/(a+b)}$ we obtain the desired result. Now consider the point $Q = (x_1 = 0, y_1 = q)$ with $q \neq 0$. The divisor H at Q is defined by $x_1 = 0$. Consider the local coordinates $x_2 = x_1$ and $y_2 = y_1 + q$. By Equation (2.9), we have that

$$\eta = -dy_2 + A(x_2, y_2) dx_2 + x_2 B(x_2, y_2) dy_2.$$

Now, we have that terms of multiplicity 0 at Q of η are $-dy_2 + \mu dx_2$, where μ may not be 0. Again, by Theorem 2.5.2, we write $\eta = u_2 dg_2$ with the same properties as before. As in the previous case, $x_2 = 0$ is transverse to $g_2 = 0$, because $\eta \wedge dx_2 = (-1 - x_2 B) dx_2 \wedge dy_2$. Hence we can take the coordinate system (x_2, g_2), obtaining that

$$\pi^*\omega = u_2' x_2^{a+b} dg_2; \quad \text{where } u_2' \text{ is an appropriate unit.}$$

Note that y_1 is a unit when seeing as an element of $\mathbb{C}\{x_2, g_2\}$. We conclude as before, ending the first part of the proof.

Part 2: Statement 2 implies Statement *1*

Now assume that $\pi^*\omega$ is totally D-dicritical, we are going to show that ω is pre-basic and resonant. Suppose that ω is not pre-basic, and write $\omega = x^c y^d \eta$, with η a holomorphic 1-form whose coefficients share no common factor. Put $W = \mathrm{In}_{1,1;x,y}(\eta)$. There are two ways for ω not being pre-basic: first, if the Newton cloud $\mathcal{NC}_{x,y}(W)$ is more than one point. Second, if $\mathcal{NC}_{x,y}(W) = (a, b)$, but $\mathcal{NC}_{x,y}(\omega) \not\subseteq R^{1,1}(a, b)$. Assume the first situation, and write

$$W = \sum_{k=0}^{j} x^{a-k} y^{b+k} \left\{ \mu_k \frac{dx}{x} + \zeta_k \frac{dy}{y} \right\},$$

with $j \geq 1$. Note that the multiplicity of η at P_0 is $a + b - 1$. As before, consider the local coordinate system given by $(x, y) = (x_1, x_1 y_1)$ defined in a open neighbourhood of (M_1, E^1) . If the strict transform η' of η by π is given by $x_1^{-a-b+1} \pi^*\eta$, then the cuspidal divisor D, given by $x_1 = 0$ is invariant by η'. Hence η' defines a foliation which is non transverse to D. Recall that a totally D-dicritical 1-form defines a foliation which is transverse to D, and also that η' and $\pi^*\omega$

define the same foliation. Therefore D must not be invariant by η'. It follows that the divisor D is a dicritical component of the foliation defined by η', that is to say, the strict transform is given by $\eta' = x_1^{-a-b}\pi^*\eta$. Since all the terms of multiplicity $a+b-1$ in η are those of W, then we have that

$$\theta' = x_1^{-a-b}\pi^*\theta, \quad \text{where } \theta = \eta - W, \tag{2.10}$$

is also holomorphic.

We see that

$$\pi^* W = \sum_{k=0}^{j} x_1^{a+b} y_1^{b+k} \left\{ (\mu_k + \zeta_k)\frac{dx_1}{x_1} + \zeta_k \frac{dy_1}{y_1} \right\}.$$

After dividing by x_1^{a+b}, we obtain

$$W' = x_1^{-a-b}\pi^* W = \sum_{k=0}^{j} y_1^{b+k} \left\{ (\mu_k + \zeta_k)\frac{dx_1}{x_1} + \zeta_k \frac{dy_1}{y_1} \right\},$$

note that, since η' is holomorphic we must have that $\mu_k + \zeta_k = 0$ for all k, that is

$$W' = \sum_{k=0}^{j} \zeta_k y_1^{b+k-1} dy_1.$$

Now, consider the polynomial $p(T) = \sum_{k=0}^{j} \zeta_k T^k$. Since $j \geq 1$, we have that $p(T)$ is non constant. Denote by $\gamma \in \mathbb{C}$ any root of $p(T)$. We put new coordinates $x_1 = x_2$ and $y_1 = y_2 + \gamma$. In this system of coordinates, we have that the multiplicity of W' at the point $P = (x_2 = 0, y_2 = 0)$ is at least 1. Moreover, we notice two facts about the 1-form θ' from (2.10):

1. dx_1 may appear with a non zero constant coefficient in the expression of θ'.
2. θ' does not have non zero terms of the shape $y_1^s dy_1$, with $s \geq 0$. Indeed, if θ' has a non zero term such as $y_1^s dy_1$ is because there is a a term of the shape $x_1^{a+b} y_1^s dy_1$ in $\pi^*\theta$. This last term $x_1^{a+b} y_1^s dy_1$ must come from a non zero term of θ written as $x^{a+b-s-1} y^s dy$. The multiplicity of $x^{a+b-s-1} y^s dy$ at P_0 is $a+b-1$, which contradicts the fact that all the terms of θ have multiplicity at least $a+b$.

Thus, there are two cases: either η' has multiplicity at least one at P or it has multiplicity zero. If $\nu_P(\eta') \geq 1$, then P is a singular point of η', hence $\pi^*\omega$ in not totally D-dicritical leading to a contradiction. Recall that in the definition of totally D-dicriticalness, we have a foliation defined locally by regular 1-forms. If $\nu_P(\eta') = 0$, since there are non zero terms of the shape $y_1^s dy_1$ in θ'. Then $dx_1 = dx_2$ is the unique term, up to constant multiplication, of multiplicity 0 of η' . Therefore, η' is tangent with $x_2 = 0$, which is a local equation of D, again contradicting the totally D-dicriticalness property. We conclude that $\mathcal{NC}_{x,y}(W)$ cannot be more than one point.

Now, assume that

$$W = x^a y^b \left\{ \mu \frac{dx}{x} + \zeta \frac{dy}{y} \right\}.$$

We are going to show that $\mathcal{NC}_{x,y}(\eta) \subset R^{1,1}(a,b)$. This is the same as showing that ω is pre-basic, because η and ω are related by the multiplication of a monomial. As before, we have that D must be a dicritical component of the foliation defined by η'. Hence $\mu + \zeta = 0$. Since $W \neq 0$, this implies that η, or equivalently ω, is resonant. We write

$$\eta = \mu x^a y^b \left\{ \frac{dx}{x} - \frac{dy}{y} \right\} + \sum_{\alpha+\beta > a+b} x^\alpha y^\beta \left\{ \mu_{\alpha\beta}\frac{dx}{x} + \zeta_{\alpha\beta}\frac{dy}{y} \right\} = W + \theta.$$

Where $\mu_{0\beta} = \zeta_{\alpha 0} = 0$ in order to have η holomorphic. If we show that, in the previous expression, all the non zero terms of the summation θ satisfy that that $a = b = 1$ and $\alpha, \beta \geq 1$, then $\mathcal{NC}_{x,y}(\eta) \subset R^{1,1}(1,1)$.

Assume first that $b > 1$. Since y does not divide η, then there exists $\alpha \geq 1$, such that: either $\mu_{\alpha 0} \neq 0$ or $\zeta_{\alpha 1} \neq 0$.

If there is $\alpha \geq 1$ with $\mu_{\alpha 0} \neq 0$, then we can write

$$\eta \wedge dy = \sum_{j=1} \mu_{j0} x^{j-1} dx \wedge dy + yH,$$

where H is a holomorphic 2-form. We have that $\eta \wedge dy - yH$ is non zero and non divisible by y. Therefore, $y = 0$ is non invariant by η. However by Lemma 2.5.3, we have the opposite result, leading to a contradiction. This also shows that $\beta \geq 1$ for all non zero coefficients in the summation θ.

If there exists $\alpha \geq 1$, such that $\zeta_{\alpha 1} \neq 0$. Then we can consider the coordinate system $(x, y) = (x_1, x_1 y_1)$ induced by π. We have that

$$\eta' = x_1^{-a-b} \pi^* \eta = y_1^b \left\{ -\frac{dy_1}{y_1} \right\} + \sum_{\alpha+\beta > a+b} x_1^{\alpha+\beta-a-b} y_1^{\beta} \left\{ (\mu_{\alpha\beta} + \zeta_{\alpha\beta}) \frac{dx_1}{x_1} + \zeta_{\alpha\beta} \frac{dy_1}{y_1} \right\}.$$

By assumption we have that there is a coefficient $\zeta_{\alpha 1} \neq 0$, thus y_1 does not divide η'. This last observation, combined by the extra assumption that $b > 1$, implies that η' defines a singular foliation at the point $(x_1 = 0, y_1 = 0)$. This contradicts that $\pi^* \omega$ is totally D-dicritical. Therefore, we have shown that $b = 1$ and $\beta \geq 1$. In a similar way we show that $a = 1$ and $\alpha \geq 1$. □

Remark 2.5.4. If $n \geq 2$ the axes $x_{N-1} y_{N-1} = 0$ around P_{N-1} coincide with the germ of E^{N-1} at P_{N-1}. In this situation, the property of being basic and resonant does not depend on the chosen adapted coordinate system.

Definition 2.5.5. *Given a resonant pre-basic* 1*-form* ω*, we say that a branch* (C, P_0) *in* (M_0, P_0) *is a* ω-cusp *if and only if it is invariant by* ω *and the strict transform of* (C, P_0) *by* π *cuts* D *at a free point.*

Let us note that each free point of D defines a ω-cusp and conversely, in view of the fact that $\pi^* \omega$ is totally D-dicritical with respect to H.

3

Cuspidal Semimodules

In Chapter 1 we saw that a germ of a plane curve (C, P_0) in the analytic space (M_0, P_0) defines a set of differential values Λ_C and a semigroup Γ_C. As mentioned there, Λ_C has the structure of a Γ_C-semimodule. We proceed to study these semimodules, specially when the semigroup we take is the one a of a cusp. Most of the results that we present here can be found in [12, 13].

3.1 Basis of a Semimodule

Take $\Gamma \subset \mathbb{Z}_{\geq 0}$ an additive numerical semigroup, that is, Γ is a monoid generated by $\langle \overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g \rangle$ with $gcd(\overline{\beta}_0, \ldots, \overline{\beta}_g) = 1$, see [6]. A set $\Lambda \subset \mathbb{Z}_{\geq 0}$ is a Γ-*semimodule*, if $\gamma + \lambda \in \Lambda$ for all $\gamma \in \Gamma$ and $\lambda \in \Lambda$.

Definition 3.1.1. *A nonempty finite increasing sequence of non negative integer numbers $\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ is a basis of the semimodule Λ if for any $0 \leq j \leq s$ we have that $\lambda_j \notin \Gamma(\mathcal{B}_{j-1})$, where $\Gamma(\mathcal{B}_{j-1}) = (\lambda_{-1} + \Gamma) \cup (\lambda_0 + \Gamma) \cup \cdots \cup (\lambda_{j-1} + \Gamma)$.*

If $\Lambda = \Gamma(\mathcal{B})$, we have a chain of semimodules

$$\lambda_{-1} + \Gamma = \Lambda_{-1} \subset \Lambda_0 \subset \cdots \subset \Lambda_s = \Lambda, \tag{3.1}$$

where $\Lambda_j = \Gamma(\mathcal{B}_j)$. We call *decomposition sequence of* Λ to this chain of semimodules. Let us note that

$$\lambda_{-1} = \min \Lambda \ \text{ and } \ \lambda_j = \min(\Lambda \setminus \Lambda_{j-1}), \quad 0 \leq j \leq s. \tag{3.2}$$

These definitions are justified by next Proposition 3.1.2

Proposition 3.1.2. *Given a semimodule Λ, there is a unique basis $\mathcal{B}$ such that $\Lambda = \Gamma(\mathcal{B})$.*

Proof. We start with $\lambda_{-1} = \min \Lambda$. Note that $\Gamma(\lambda_{-1}) \subset \Lambda$. If $\Gamma(\lambda_{-1}) = \Lambda$, we stop and we put $s = -1$. If $\Gamma(\lambda_{-1}) \neq \Lambda$, we put $\lambda_0 = \min(\Lambda \setminus \Gamma(\lambda_{-1}))$. Note that $\Gamma(\lambda_{-1}, \lambda_0) \subset \Lambda$. We continue in this way and after finitely many steps we obtain that $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$.

Let us show the uniqueness of $\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$. Assume that $\Lambda = \Gamma(\mathcal{B}')$, for another Γ-basis $\mathcal{B}' = (\lambda'_{-1}, \lambda'_0, \ldots, \lambda'_{s'})$. By definition of basis, we have that $\lambda_{-1} = \lambda'_k + \gamma$ for some $-1 \leq k \leq s'$ and $\gamma \in \Gamma$. Since $\lambda_{-1} = \min \Lambda$ and $\lambda'_k \in \Lambda$, we see that $\gamma = 0$. Besides $\lambda'_{-1} \leq \lambda'_k$. Thus $\lambda_{-1} = \lambda'_k$. Moreover, since $\mathcal{B}'$ is an increasing sequence, we have $k = -1$. Now assume that $\lambda_j = \lambda'_j$ for any $0 \leq j \leq i-1$. Again, by definition of basis, we have that $\lambda_i = \lambda'_k + \gamma$ for some $-1 \leq k \leq s'$ and $\gamma \in \Gamma$. If $k < i$, then we know that $\lambda_k = \lambda'_k$ and it would imply that $\lambda_i \in \lambda_k + \Gamma$, in contradiction with the definition of basis of the semimodule, thus $k \geq i$. In view of Equation (3.2) we have that $\lambda_i = \min(\Lambda \setminus \Gamma(\mathcal{B}_{k-1})) = \min\left(\Lambda \setminus \Gamma(\mathcal{B}'_{k-1})\right)$. Hence $\gamma = 0$, noting that $\lambda'_i \leq \lambda'_k$, we conclude that $\lambda_i = \lambda'_k = \lambda'_i$. □

We say that the basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \dots, \lambda_s)$ of $\Lambda = \Gamma(\mathcal{B})$ has *length* s. Moreover, the element λ_i is called the *i-element of the basis* $\mathcal{B}$, for $-1 \leq i \leq s$. When $\lambda_{-1} = 0$, we say that Λ is a *normalized semimodule*.

Denote by $n = \min(\Gamma \setminus \{0\})$, given the basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_s)$, we have that $\lambda_i \neq \lambda_j \mod n$. Hence, the length s is bounded by $n - 2$.

In Chapter 1, we introduced the notion of conductor for the semigroup of a branch. More generally, we define the *conductor* of a Γ-semimodule Λ as:

$$c_\Lambda = \min\{k \in \mathbb{Z}_{\geq 0} : \mathbb{Z}_{\geq k} \subset \Lambda\},$$

see [3]. We notice the following: if λ_{-1} is the minimum element in Λ, then we have $c_\Lambda \leq c_\Gamma + \lambda_{-1}$. More generally, given $\Lambda_i \subset \Lambda_{i+1}$ two consecutive semimodules of the decomposition sequence of Λ, then we have that $c_{\Lambda_{i+1}} \leq c_{\Lambda_i}$.

We say that a numerical semigroup Γ is *cuspidal* if it is generated by two positive coprime integers (n, m) with $2 \leq n < m$. A Γ-semimodule Λ is *cuspidal* when Γ is cuspidal. From now on, we fix a cuspidal semigroup Γ, and we denote $n < m$ its generators. By Equation (1.5), we have that the conductor of a cuspidal semigroup Γ is

$$c_\Gamma = (n-1)(m-1).$$

As explained in Section 1.4, the semigroup of a cusp is a cuspidal semigroup. Moreover, the semimodule of differential values of a cusp is a cuspidal semimodule.

3.2 Axes, Limits and Critical Values

From now on, we assume that Λ is a cuspidal Γ-semimodule, where Γ is generated by the two coprime integers $2 \leq n < m$. We denote by $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_s)$ the basis of Λ. We introduce the following structural values associated to Λ.

For $1 \leq i \leq s+1$, we define the *axes* u_i^n, u_i^m, u_i and $\tilde{u}_i$ of Λ as follows:

- $u_i^n = \min\{\lambda_{i-1} + n\ell \in \Lambda_{i-2};\ \ell \geq 1\}$. We write $u_i^n = \lambda_{i-1} + n\ell_i^n$.
- $u_i^m = \min\{\lambda_{i-1} + m\ell \in \Lambda_{i-2};\ \ell \geq 1\}$. Similarly, we put $u_i^m = \lambda_{i-1} + m\ell_i^m$.
- $u_i = \min\{u_i^n, u_i^m\}$ and $\tilde{u}_i = \max\{u_i^n, u_i^m\}$.

The numbers ℓ_i^n and ℓ_i^m are called the *limits* of Λ.

Example 3.2.1. Consider the cuspidal semigroup $\Gamma = \langle 7, 15\rangle$ and let Λ be the Γ-semimodule defined by the basis $\mathcal{B} = (7, 15, 27, 46)$. We are going to compute the axes to show how the computations work.

We have that $u_1^n = \min\{15 + 7\ell \in (7+\Gamma)\} = 22 = 15 + 7 \cdot 1 = 7 + 15$. Similarly, we have that $u_1^m = \min\{15 + 15\ell \in (7+\Gamma)\} = 105 = 15 + 15 \cdot 6 = 7 + 7 \cdot 14$. Hence, we have that $u_1 = u_1^n = 22$ and $\tilde{u}_1 = u_1^m = 105$.

Next, we see that

$$u_2^n = \min\{27 + 7\ell \in (7+\Gamma) \cup (15+\Gamma)\} = 90 = 27 + 7 \cdot 9 = 15 + 15 \cdot 15.$$

$$u_2^m = \min\{27 + 15\ell \in (7+\Gamma) \cup (15+\Gamma)\} = 42 = 27 + 15 \cdot 1 = 7 + 7 \cdot 5.$$

Thus $u_2 = u_2^m = 42$ and $\tilde{u}_2 = u_2^n = 90$. Finally

$$u_3^n = \min\{46 + 7\ell \in (7+\Gamma) \cup (15+\Gamma) \cup (27+\Gamma)\} = 60 = 46 + 7 \cdot 2 = 15 + 15 \cdot 3.$$

$$u_3^m = \min\{46 + 15\ell \in (7+\Gamma) \cup (15+\Gamma) \cup (27+\Gamma)\} = 76 = 46 + 15 \cdot 2 = 27 + 7 \cdot 7.$$

Therefore, $u_3 = u_3^n = 60$ and $\tilde{u}_3 = u_3^m = 76$. The reader can list all the elements of Λ and check that the conductor of Λ is $c_\Lambda = 55$.

Remark 3.2.2. If we consider the semimodule $\Lambda' = \Lambda - \lambda$, the new basis and the axes are shifted by λ and we obtain the same limits as for Λ. This is particularly interesting when $\lambda = \lambda_{-1}$ and hence Λ' is a normalized semimodule.

Remark 3.2.3. Let us note that $1 \leq \ell_i^m < n$ and that $1 \leq \ell_i^n < m$. To see this we can suppose that Λ is normalized and thus $c_{\Lambda_j} \leq c_\Gamma = (n-1)(m-1)$ for any $j = -1, 0, 1, \dots, s$. Assume that $\ell_i^m \geq n$, we have

$$\lambda_{i-1} + m(\ell_i^m - 1) \geq (n-1)m \geq c_\Gamma \geq c_{\Lambda_{i-2}}.$$

Then $\lambda_{i-1} + m(\ell_i^m - 1) \in \Lambda_{i-2}$ in contradiction with the minimality of ℓ_i^m. A similar argument proves that $\ell_i^n < m$.

Remark 3.2.4. Notice that $u_i^n \neq u_i^m$ for each index $1 \leq i \leq s+1$. Indeed, if $u_i^n = u_i^m$, then $n\ell_i^n = m\ell_i^m$; given that n and m are coprime, then $mk = \ell_i^n$, for a positive integer k and hence $\ell_i^n \geq m$ which is a contradiction, by the previous remark.

Lemma 3.2.5. *Let Λ be a cuspidal semimodule of length s. Take $1 \leq i \leq s+1$. If $\lambda_{i-1} + na + mb \in \Lambda_{i-2}$, where $a, b \in \mathbb{Z}_{\geq 0}$, then either $a \geq \ell_i^n$ or $b \geq \ell_i^m$.*

Proof. By definition, we have that:

$$\lambda_{i-1} + na + mb = \lambda_k + nc + md, \quad k < i-1,$$

where c, d are non negative integers. We proceed by induction on $\alpha = ac + bd \geq 0$. If $\alpha = 0$, then $ac = bd = 0$. This implies that $ab = 0$, otherwise, $ab \neq 0$ and hence $c = d = 0$, that is

$$\lambda_{i-1} + na + mb = \lambda_k,$$

which is a contradiction because $\lambda_k < \lambda_{i-1}$. Now if $a = 0$, we end with the minimality of ℓ_i^m and, similarly, if $b = 0$, we end by the minimality of ℓ_i^n.

Assume that $\alpha > 0$. Then $ac \neq 0$ or $bd \neq 0$. If $ac \neq 0$, let us put $a' = a - 1 \geq 0$ and $c' = c - 1 \geq 0$. We have that

$$\lambda_{i-1} + na' + mb = \lambda_r + nc' + md.$$

We conclude by applying an inductive argument. We apply a similar argument if $bd \neq 0$. □

We define inductively the *critical values* t_i^n, t_i^m, t_i, and $\tilde{t}_i$, for $-1 \leq i \leq s+1$ by putting $t_{-1} = \lambda_{-1} = n$ and $t_0 = \lambda_0 = m$ and

$$\left.\begin{array}{rclrcl} t_i^n &=& t_{i-1} + n\ell_i^n, & t_i^m &=& t_{i-1} + m\ell_i^m \\ t_i &=& \min\{t_i^n, t_i^m\}, & \tilde{t}_i &=& \max\{t_i^n, t_i^m\} \end{array}\right\} \quad 1 \leq i \leq s+1.$$

Noting that $n\ell_i^n = u_i^n - \lambda_{i-1}$ and $m\ell_i^m = u_i^m - \lambda_{i-1}$, we have that:

$$\begin{array}{rclrcl} t_i^n &=& t_{i-1} + u_i^n - \lambda_{i-1}, & t_i^m &=& t_{i-1} + u_i^m - \lambda_{i-1}, \\ t_i &=& t_{i-1} + u_i - \lambda_{i-1}, & \tilde{t}_i &=& t_{i-1} + \tilde{u}_i - \lambda_{i-1}. \end{array}$$

Definition 3.2.6. *We say that the cuspidal semimodule Λ is* increasing *if we have $\lambda_i > u_i$, for any $1 \leq i \leq s$, see [2].*

The previous definition is motivated by the results in [21], where C. Delorme shows that the semimodule of differential values of a cusp is increasing, with the (-1)-element of the basis equal to n and the 0-element equal to m. Reciprocally, in [2], the authors show that given an increasing semimodule Λ with (-1)-element and 0-element of the basis n and m respectively, then there exists a cusp whose semimodule of differential values is Λ. That's way we mainly consider cuspidal semimodules. Besides, it also explains why we use the convention of starting the basis of the semimodule at index -1, because when considering the semimodule of differential values of a cusp, the first two elements of the basis are given by the semigroup. We want to remark that we could extend the notion of increasing semimodules for semimodules which are non cuspidal, however, the semimodules of differential values of non cuspidal branches, they are not increasing.

We finally note that if Λ is increasing, then each Λ_i is also increasing, for $1 \leq i \leq s$.

Example 3.2.7. We continue with the computations from Example 3.2.1. We have have that $\Gamma = \langle 7, 15 \rangle$ and the basis of the semimodule Λ is $\mathcal{B} = (7, 15, 27, 46)$. We showed that

$$\begin{aligned} u_1 &= u_1^n = 22; & \tilde{u}_1 &= u_1^m = 105 \\ u_2 &= u_2^m = 42; & \tilde{u}_2 &= u_2^n = 90 \\ u_3 &= u_3^n = 60; & \tilde{u}_3 &= u_3^m = 76 \end{aligned}$$

and hence we obtain

$$\begin{aligned} t_1 &= t_1^n = 22; & \tilde{t}_1 &= t_1^m = 105 \\ t_2 &= t_2^m = 37; & \tilde{t}_2 &= t_2^n = 85 \\ t_3 &= t_3^n = 51; & \tilde{t}_3 &= t_3^m = 67. \end{aligned}$$

Finally, we notice that $\lambda_1 = 27 > 22 = u_1$ and $\lambda_2 = 46 > 42 = u_2$. Thus, Λ is an increasing Γ-semimodule.

Lemma 3.2.8. *Let Λ be an increasing cuspidal semimodule. For any index $1 \leq i \leq s$, we have that $\lambda_i - \lambda_j > t_i - t_j$, for $-1 \leq j < i$.*

Proof. By a telescopic argument, it is enough to prove the following statements:

- $\lambda_r - \lambda_{r-1} > t_r - t_{r-1}$, for $1 \leq r \leq s$.
- $\lambda_0 - \lambda_{-1} \geq t_0 - t_{-1}$.

The second statement is straightforward, because $t_{-1} = \lambda_{-1}$ and $t_0 = \lambda_0$. Let us prove that $\lambda_r - \lambda_{r-1} > t_r - t_{r-1}$, for $1 \leq r \leq s$.

The inequality $\lambda_r - \lambda_{r-1} > t_r - t_{r-1}$ is equivalent to

$$t_r = t_{r-1} + u_r - \lambda_{r-1} > t_r + u_r - \lambda_r$$

and this is equivalent to say that $\lambda_r > u_r$. The result follows by recalling that Λ is increasing. □

Corollary 3.2.9. *Let Λ be an increasing cuspidal semimodule. For any $1 \leq i \leq s$, we have that*

$$u_{i+1}^n > t_{i+1}^n \quad \text{and} \quad u_{i+1}^m > t_{i+1}^m.$$

Proof. Recalling that $t_{i+1}^n = u_{i+1}^n - (\lambda_i - t_i)$, it is enough to prove that $\lambda_i - t_i > 0$. In view of Lemma 3.2.8 and putting $j = -1$, we have that $\lambda_i - t_i > \lambda_{-1} - t_{-1} = 0$. □

3.3 Circular Intervals

The circular intervals we describe here are useful for understanding the distribution of the elements of an increasing cuspidal semimodule. We introduced the notion of circular interval in [12], it is an alternative approach to the one in [2] where the authors use the notion of "lean sets", or the "mancala games" in [41] . Recall that we have fixed a pair of integer numbers (n, m) with $2 \leq n < m$ and $\gcd(n, m) = 1$. The motivation comes from the following example.

Example 3.3.1. Consider the cuspidal semigroup $\Gamma = \langle 5, 11\rangle$ and the cuspidal semimodule $\Lambda = \Gamma(5, 11, 18)$.

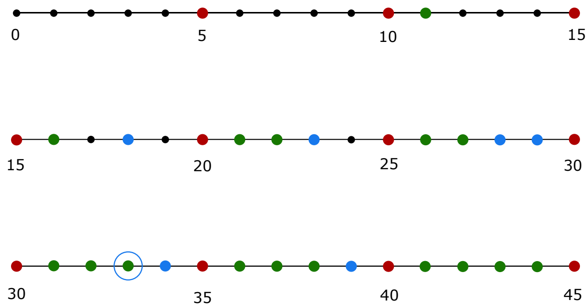

Figure 3.1: *First elements of Λ. The black dots represent the elements that do not belong to Λ. The red circles are the multiples of 5 belonging to Λ. The green circles are the rest of elements belonging to $\Gamma(5, 11)$. The blue circles are the elements in $(18 + \Gamma) \setminus \Gamma(5, 11)$. Finally, the element 33 is marked, since it can be checked that corresponds with the axis u_2.*

We notice that in the previous figure we can split the representation of the semimodule by boxes of size $n = 5$. Each one of them is almost identical to the next one. Additionally, each time a new multiple of $m = 11$ appears, we represent it by a green circle that is at distance one from the previous one. This is because $11 \equiv 1 \bmod 5$. The idea behind circular intervals is to obtain a similar representation of theses boxes when m may not be congruent with 1 modulo n.

We are going to consider the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ as a "clock" with "n-hours" as we explain below.

Let $\varepsilon : \mathbb{R} \longrightarrow \mathbb{S}^1$ be the map given by

$$\varepsilon(t) = \exp\left(-\frac{2\pi t\sqrt{-1}}{n}\right).$$

We see the n-roots of unit as $\mathbb{S}^1_n := \varepsilon(\mathbb{Z})$. Note that there is a bijection

$$c : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{S}^1_n$$

given by $c(k + n\mathbb{Z}) = \varepsilon(k)$. More than that, the bijection c is an isomorphism of abelian groups, where $\mathbb{S}^1_n \subset \mathbb{C}$ has the induced multiplicative structure coming from the complex numbers $\mathbb{C}$. Note that

$$c((k + n\mathbb{Z}) + (k' + n\mathbb{Z})) = \varepsilon(k)\varepsilon(k').$$

In particular $c(k + 1 + n\mathbb{Z}) = \varepsilon(k)\varepsilon(1)$.

Notation 3.3.2. In order to visualize in a better way the arithmetic of the abelian multiplicative group $\mathbb{S}^1_n$, we introduce the following notations:

$$\varepsilon(k) = k_\varepsilon, \quad \varepsilon(k)\varepsilon(k') = k_\varepsilon + k'_\varepsilon.$$

Note that there is no confusion possible with the addition in $\mathbb{C}$. For instance, we have $(-1)_\varepsilon = (n-1)_\varepsilon$, $(k+1)_\varepsilon = k_\varepsilon + 1_\varepsilon$ and $(k-1)_\varepsilon = k_\varepsilon - 1_\varepsilon = k_\varepsilon + (n-1)_\varepsilon$.

Let us consider two points $P, Q \in \mathbb{S}^1_n$. There are $\alpha \in \mathbb{Z}$ and an integer number β with $0 \le \beta \le n-1$ such that $P = \varepsilon(\alpha)$ and $Q = \varepsilon(\alpha+\beta)$. This number β, with $0 \le \beta \le n-1$, does not depend on the chosen α such that $P = \varepsilon(\alpha)$ and we call it the *separation* $S(P,Q)$ *from* P *to* Q, that is, if $P = \varepsilon(\alpha)$, we have that $Q = \varepsilon(\alpha + S(P,Q))$. Observe that $S(P,P) = 0$ and that

$$S(P,Q) + S(Q,P) = n, \quad \text{if } Q \neq P.$$

Additionally, if $Q \neq Q'$ then $S(P,Q) \neq S(P,Q')$ and $S(Q,P) \neq S(Q',P)$.

We define the *circular interval* $< P, Q >$ to be

$$< P, Q >= \{\varepsilon(\alpha + k);\ k = 0, 1, \ldots, S(P,Q)\} \subset \mathbb{S}^1_n.$$

Note that if $P \neq Q$, we have that

$$< P, Q > \cup < Q, P >= \mathbb{S}^1_n, \quad < P, Q > \cap < Q, P >= \{P, Q\}.$$

Remark 3.3.3. Given three points $P, Q, R \in \mathbb{S}^1_n$ such that

$$R \in < P, Q >,$$

We have that $S(P,Q) = S(P,R) + S(R,Q) \le n-1$.

To simplify the notation, we consider the total order in $\mathbb{S}^1_n$, defined by

$$0_\varepsilon < 1_\varepsilon < 2_\varepsilon < \cdots < (n-1)_\varepsilon.$$

With this ordering, we see that given two elements $z, z' \in \mathbb{S}^1_n$, then we have the following possibilities:

$$< z, z' >= \begin{cases} \{z\} & \text{if } z = z'. \\ \{z, z+1_\varepsilon, z+2_\varepsilon, \ldots, z'-1_\varepsilon, z'\} & \text{if } z < z'. \\ \{z, z+1_\varepsilon, z+2_\varepsilon, \ldots, (n-1)_\varepsilon, 0_\varepsilon, 1_\varepsilon, \ldots, z'-1_\varepsilon, z'\} & \text{if } z > z'. \end{cases}$$

Consider a list $B = (z_{-1}, z_0, z_1, \ldots, z_s)$ of two by two distinct points $z_j \in \mathbb{S}^1_n$, with $s \ge 0$. For any index $0 \le i \le s$, we define the *i-left bound* $b^\ell_i(B)$ *and the i-right bound* $b^r_i(B)$ *of* B to be integer numbers such that

$$-1 \le b^\ell_i(B), b^r_i(B) \le i-1$$

and, moreover, the following holds:

1. If $k = b^\ell_i(B)$, then $S(z_k, z_i) \le S(z_q, z_i)$, for any $-1 \le q \le i-1$.
2. If $\tilde{k} = b^r_i(B)$, then $S(z_i, z_{\tilde{k}}) \le S(z_i, z_q)$, for any $-1 \le q \le i-1$.

By the properties of the separation between two points in $\mathbb{S}^1_n$, we have that $S(z_k, z_i) < S(z_q, z_i)$ if $q \neq k$ and $S(z_i, z_{\tilde{k}}) < S(z_i, z_q)$ if $\tilde{k} \neq q$.

Remark 3.3.4. Denote $k = b^\ell_i(B)$ and $\tilde{k} = b^r_i(B)$. The bounds are the integer numbers $k, \tilde{k}$ with $-1 \le k, \tilde{k} \le i-1$ defined by the two following properties:

1. $z_i \in < z_k, z_{\tilde{k}} >$.
2. If $z_j \in < z_k, z_{\tilde{k}} >$ with $-1 \le j \le i$, then $j \in \{i, k, \tilde{k}\}$.

Taking into account that $S(P,Q) + S(Q,P) = n$ for $Q \neq P$, then $b^\ell_i(B) = b^r_i(B)$ if and only if $i = 0$.

3.4 Circular Intervals in a Cuspidal Semimodule

Let us recall that Γ is generated by pair (n, m), with $2 \leq n < m$ and n, m are without common factors.

We consider the quotient map $\rho : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$, which we also denote by $\rho(k) = \bar{k}$. Since $gcd(n, m) = 1$, the class $\bar{m}$ is a unit in $\mathbb{Z}/n\mathbb{Z}$, thus we have a ring isomorphism

$$\xi : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}, \quad \xi(\bar{k}) = \bar{k}/\bar{m}.$$

Let $\zeta : \mathbb{Z} \longrightarrow \mathbb{S}^1_n$ be the map defined by $\zeta(k) = (c \circ \xi \circ \rho)(k)$. Notice that $\zeta(k + an) = \zeta(k)$ and that $\zeta(mk) = \varepsilon(k) = k_\varepsilon$.

Consider the intervals $I_q = \{nq, nq + 1, \ldots, nq + n - 1\} \subset \mathbb{Z}$, $q \in \mathbb{Z}$. For a set $S \subset \mathbb{Z}$, we define the *q-level set* $R_q(S)$ by

$$R_q(S) = \zeta(S \cap I_q) \subset \mathbb{S}^1_n.$$

Example 3.4.1. Consider the cuspidal semigroup $\Gamma = \langle 7, 17 \rangle$ and the increasing Γ-semimodule Λ generated by the basis $\mathcal{B} = (7, 17, 26)$. We have that $\Lambda \cap I_6 = \{42, 43, 45, 47, 48\}$. Now we apply the map ζ to $\Lambda \cap I_6$. First, we compute the residue modulo of the elements, obtaining $A = \{\bar{0}, \bar{1}, \bar{3}, \bar{5}, \bar{6}\}$.

Now, we apply ξ to A, which corresponds with the division by $\overline{17}$. Note that $3 \equiv 17 \bmod 7$, and $\bar{3}^{-1} = \bar{5}$. Thus the image of A by ξ is $B = \{\bar{0}, \bar{5}, \bar{1}, \bar{4}, \bar{2}\}$. Finally, we send the elements of B by c to the 7 roots of the unity in $\mathbb{C}$. We see that $R_6(\Lambda)$ has two circular intervals, $\{0_\varepsilon, 1_\varepsilon, 2_\varepsilon\}$ and $\{4_\varepsilon, 5_\varepsilon\}$.

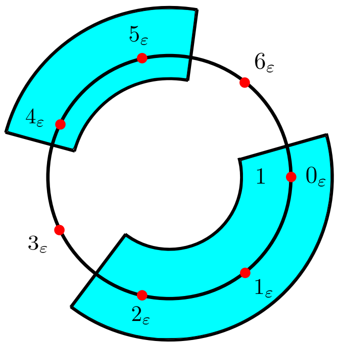

Figure 3.2: *Level set $R_6(\Lambda)$ from Example 3.4.1, the blue sectors represent the two circular intervals of the level set.*

Remark 3.4.2. If $S \subset \mathbb{Z}$ satisfies the property that $n + S \cap I_{q-1} \subset S \cap I_q$, we have that $R_{q-1}(S) \subset R_q(S)$. This is the case of cuspidal semimodules.

Before continuing, let us remark the following thing: as it was stated, the construction of circular intervals requires working with cuspidal semigroups, since we are using explicitly the condition that $gcd(n, m) = 1$. One of the problems when generalizing all the results for more complicated semigroups, such as the ones of curves which are not cusps, it is that we do not know how to order the elements of $\mathbb{Z}/n\mathbb{Z}$ in a proper way.

Now let us consider a cuspidal semimodule Λ of length $s \geq 0$ with basis

$$\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s).$$

We see the basis $\mathcal{B}$ in $\mathbb{S}^1_n$ as $B = \zeta(\mathcal{B}) = (z_{-1}, z_0, z_1, \ldots, z_s)$, where we have that $z_j = \zeta(\lambda_j)$, for $j = -1, 0, 1, \ldots, s$.

Note that $z_i \neq z_j$, if $i \neq j$; indeed, saying that $z_i = z_j$ means that $\lambda_i - \lambda_j \in n\mathbb{Z}$, that is not possible in view of the definition of basis.

Take an index $1 \leq i \leq s+1$. We define the *tops* q_i^n and q_i^m of Λ by the property that $u_i^n \in I_{q_i^n}$ and $u_i^m \in I_{q_i^m}$. We also define the *tops* q_i and $\tilde{q}_i$ to be such that $u_i \in I_{q_i}$ and $\tilde{u}_i \in I_{\tilde{q}_i}$. Recall that

$$\{u_i^n, u_i^m\} = \{u_i, \tilde{u}_i\}.$$

As a consequence, we have that $\{q_i^n, q_i^m\} = \{q_i, \tilde{q}_i\}$. Note that $q_i \leq \tilde{q}_i$, since $u_i < \tilde{u}_i$.

We also need to consider the integers v_i that indicate the first levels $R_{v_i}(\Lambda)$ such that $z_i \in R_{v_i}(\Lambda)$. In other words, each v_i is defined by the property that $\lambda_i \in I_{v_i}$, for $i = -1, 0, 1, \ldots, s$.

The following statements concern the properties of being circular intervals for the levels of Λ and some derived properties of the conductor.

Lemma 3.4.3. *Consider $\mu \in I_v$, denote $r = \zeta(\mu)$ and let q be such that $q \geq v$. For any $p \in R_q(\mu + \Gamma)$ we have that $< r, p > \subset R_q(\mu + \Gamma)$. In particular, the set $R_q(\mu + \Gamma)$ is a circular interval.*

Proof. The second statement is straightforward, since the union of circular intervals with a common point is a circular interval. To prove the first statement, we proceed by induction on the number ℓ of elements in $< r, p >$. If $\ell \leq 2$, there is nothing to prove since $< r, p > \subset \{r, p\} \subset R_q(\mu + \Lambda)$. Assume that $\ell > 2$; in particular we have that $r \neq p$. Consider the point $\tilde{p} = p - 1_\varepsilon$. We have that $< r, p > = < r, \tilde{p} > \cup \{p\}$ and the length of $< r, \tilde{p} >$ is $\ell - 1$. Then, it is enough to show that $\tilde{p} \in R_q(\mu + \Gamma)$. Take an element $\mu + na + mb \in I_q \cap (\mu + \Gamma)$ such that $\zeta(\mu + na + mb) = p$. Noting that $r \neq p$, we have that $b \geq 1$. There is $q' \leq q$ such that $\mu + na + m(b-1) \in I_{q'}$ and hence

$$\mu + n(a + q - q') + m(b-1) \in I_q \cap (\mu + \Gamma).$$

We have that $\zeta(\mu + n(a + q - q') + m(b-1)) = \tilde{p}$ and thus $\tilde{p} \in R_q(\mu + \Gamma)$. □

Remark 3.4.4. For any $\mu \in \mathbb{Z}_{\geq 0}$, we have

$$\#R_q(\mu + \Gamma) \leq \#R_{q-1}(\mu + \Gamma) + 1.$$

Indeed, this is equivalent to show that $\#\rho((\mu + \Gamma) \cap I_q) \leq \#\rho((\mu + \Gamma) \cap I_{q-1}) + 1$. Assume that $\bar{p}_1, \bar{p}_2 \in \rho((\mu + \Gamma) \cap I_q) \setminus \rho((\mu + \Gamma) \cap I_{q-1})$. Notice that given $p = \mu + na + mb \in I_q$ with $a > 0$, then $p - n \in I_{q-1}$, thus we can take representatives $p_1, p_2 \in (\mu + \Gamma) \cap I_q$ of $\bar{p}_1$ and $\bar{p}_2$ of the form $p_1 = \mu + mb_1$, $p_2 = \mu + mb_2$. If $p_1 \neq p_2$, we have that $|p_1 - p_2| \geq m > n$ and this is not possible.

Proposition 3.4.5. *Assume that Λ is normalized (that is $\lambda_{-1} = 0$) and that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v_s$. We have:*

1. *$< 0_\varepsilon, z_s - 1_\varepsilon > \subset R_q(\Lambda_{s-1})$, for $q \geq q_{s+1}^n - 1$.*
2. *$< z_s, (n-1)_\varepsilon > \subset R_q(\Lambda)$, for $q \geq q_{s+1}^m - 1$.*

In particular, we have that $R_q(\Lambda) = \mathbb{S}_n^1$, for any $q \geq \tilde{q}_{s+1} - 1$. That is, $c_\Lambda \leq n(\tilde{q}_{s+1} - 1)$.

Proof. Statement 1: By Remark 3.4.2 it is enough to show that we have $< 0_\varepsilon, z_s - 1_\varepsilon > \subset R_{q_{s+1}^n - 1}(\Lambda_{s-1})$. Since $u_{s+1}^n = \lambda_s + n\ell_{s+1}^n \in \Lambda_{s-1}$, there is an index $k \leq s-1$ such that $\lambda_s + n\ell_{s+1}^n = \lambda_k + na + mb$. By the minimality of ℓ_{s+1}^n, we have that $a = 0$ and hence $\lambda_s + n\ell_{s+1}^n = \lambda_k + mb$.

Assume that the next statements are true:

a) If $z_k > z_s$, then $< 0_\varepsilon, z_s > \subset R_{q_{s+1}^n}(\lambda_k + \Gamma)$.
b) If $z_k < z_s$, then $< z_k, z_s > \subset R_{q_{s+1}^n}(\lambda_k + \Gamma)$ and $< 0_\varepsilon, z_k > \subset R_{q_{s+1}^n - 1}(\Lambda_{s-1})$.

If $z_k > z_s$, by the minimality of ℓ^n_{s+1}, we have that $z_s \notin R_{q^n_{s+1}-1}(\lambda_k + \Gamma)$; now, in view of Remark 3.4.4 and noting that $< 0_\varepsilon, z_s >=< 0_\varepsilon, z_s - 1_\varepsilon > \cup \{z_s\}$, we obtain that

$$< 0_\varepsilon, z_s - 1_\varepsilon > \subset R_{q^n_{s+1}-1}(\lambda_k + \Gamma) \subset R_{q^n_{s+1}-1}(\Lambda_{s-1}).$$

If $z_k < z_s$, we obtain as above that $< z_k, z_s - 1_\varepsilon > \subset R_{q^n_{s+1}-1}(\lambda_k + \Gamma)$, then

$$< 0_\varepsilon, z_s - 1_\varepsilon >=< 0_\varepsilon, z_k > \cup < z_k, z_s - 1_\varepsilon > \subset R_{q^n_{s+1}-1}(\Lambda_{s-1}).$$

If remains to prove a) and b).

Proof of a): We can apply Lemma 3.4.3 to have that $< z_k, z_s > \subset R_{q^n_{s+1}}(\lambda_k + \Gamma)$. We end by noting that $< 0_\varepsilon, z_s > \subset < z_k, z_s >$.

Proof of b): Again by Lemma 3.4.3 we have that $< z_k, z_s > \subset R_{q^n_{s+1}}(\lambda_k + \Gamma)$. On the other hand, we know that $R_{q^n_{s+1}-1}(\Lambda_{s-1})$ is a circular interval since $q^n_{s+1} - 1 \geq v_s$ and it contains 0_ε and z_k. Moreover $z_s \notin R_{q^n_{s+1}-1}(\Lambda_{s-1})$ and $z_s > z_k$, then the circular interval $R_{q^n_{s+1}-1}(\Lambda_{s-1})$ contains $< 0_\varepsilon, z_k >$.

Statement 2: It is enough to show that $< z_s, (n-1)_\varepsilon > \subset R_{q^m_{s+1}-1}(\Lambda)$. By an argument as before, there is an index $k \leq s - 1$ such that $u^m_{s+1} = \lambda_s + m\ell^m_{s+1} = \lambda_k + na$. Take $z_k \neq z_s$ as above. By Lemma 3.4.3, we have that $< z_s, z_k > \subset R_{q^m_{s+1}}(\lambda_s + \Gamma)$. Let us see that $z_k \notin R_{q^m_{s+1}-1}(\lambda_s + \Gamma)$. We proceed by contradiction, assume that $z_k \in R_{q^m_{s+1}-1}(\lambda_s + \Gamma)$. This implies that $\lambda_k + n(a-1) \in \lambda_s + \Gamma$ and hence there are non negative integer numbers α, β such that

$$u^m_{s+1} > \lambda_s + n\alpha + m\beta = \lambda_k + n(a-1).$$

If $a - 1 \leq \alpha$, we have that $\lambda_k = \lambda_s + n(\alpha - a + 1) + m\beta$ and this contradicts the fact that $\lambda_k < \lambda_s$; hence $a - 1 > \alpha$ and we have

$$\lambda_s + m\beta = \lambda_k + n(a - 1 - \alpha).$$

Since $a - 1 - \alpha < a$, we have that $\beta < \ell^m_{s+1}$. This contradicts the minimality of the limit ℓ^m_{s+1}.

Since $z_k \notin R_{q^m_{s+1}-1}(\lambda_s + \Gamma)$, we can apply Remark 3.4.4 which tells us that

$$< z_s, z_k > \setminus \{z_k\} \subset R_{q^m_{s+1}-1}(\lambda_s + \Gamma) \subset R_{q^m_{s+1}-1}(\Lambda).$$

Note also that $z_k \in R_{q^m_{s+1}-1}(\Lambda)$. Then we have that $< z_s, z_k > \subset R_{q^m_{s+1}-1}(\Lambda)$. As before we have to consider two cases: if $z_s > z_k$, then $< z_s, (n-1)_\varepsilon > \subset < z_s, z_k > \subset R_{q^m_{s+1}-1}(\Lambda)$. Otherwise, if $z_s < z_k$, we recall that $z_s \notin R_v(\Lambda_{s-1})$; since $R_v(\Lambda_{s-1})$ is a circular interval containing z_k and 0_ε, but not containing z_s, we have that

$$< z_k, (n-1)_\varepsilon > \subset R_v(\Lambda_{s-1}) \subset R_{q^m_{s+1}-1}(\Lambda_{s-1}) \subset R_{q^m_{s+1}-1}(\Lambda).$$

We conclude that $< z_s, (n-1)_\varepsilon >=< z_s, z_k > \cup < z_k, (n-1)_\varepsilon > \subset R_{q^m_{s+1}-1}(\Lambda)$. □

Proposition 3.4.6. *Assume that Λ is normalized and increasing. Then $R_q(\Lambda)$ is a circular interval for any $q \geq q_{s+1}$.*

Proof. Let us proceed by induction on the length s of Λ. If $s = -1$, we have $\Lambda = \Lambda_{-1} = \Gamma$ and the result follows by Lemma 3.4.3 applied to $\mu = 0$. Let us suppose that $s \geq 0$ and assume by induction that the result is true for Λ_{s-1}. We have that $\Lambda = \Lambda_{s-1} \cup (\lambda_s + \Gamma)$. This implies that

$$R_q(\Lambda) = R_q(\Lambda_{s-1}) \cup R_q(\lambda_s + \Gamma), \quad q \geq 0.$$

By induction hypothesis, we know that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq q_s$. Moreover, by the increasing property, we have that

$$u_{s+1} > \lambda_s > u_s \geq \lambda_{s-1}.$$

In particular, we have that $q_{s+1} \geq q_s$ and $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq q_{s+1}$. On the other hand, by Lemma 3.4.3, we know that $R_q(\lambda_s + \Gamma)$ is a circular interval for any $q \geq v_s$. Since $q_{s+1} \geq v_s$, we have that $R_q(\lambda_s + \Gamma)$ is a circular interval for any $q \geq q_{s+1}$. Thus, both

$$R_q(\Lambda_{s-1}) \text{ and } R_q(\lambda_s + \Gamma)$$

are circular intervals for $q \geq q_{s+1}$. We need to show that their union is also a circular interval. It is enough to show that they are not disjoint, because the union of two non disjoint circular intervals is again a circular interval. We have two cases:

If $u_{s+1} = u^n_{s+1}$, then as in the proof of Proposition 3.5.9, we have that $u_{s+1} = \lambda_{s+1} + n\ell^n_{s+1} = \lambda_k + mb$ for $k < s$, hence we have that $z_s \in R_{q_{s+1}}(\lambda_k + \Gamma) \subset R_{q_{s+1}}(\Lambda_{s-1})$ obtaining the desired result. If $u_{s+1} = u^m_{s+1}$, similarly as the previous case we have that $u_{s+1} = \lambda_{s+1} + m\ell^m_{s+1} = \lambda_k + na$ for $k < s$, and we have that $z_k \in R_{q_{s+1}}(\lambda_s + \Gamma)$.

□

Corollary 3.4.7. *Assume that Λ is increasing. Then $\tilde{u}_{s+1} \geq c_\Lambda + n$, where c_Λ is the conductor of Λ.*

Proof. We assume without loss of generality that Λ is normalized, because we notice that for the semimodule $\Lambda' = \lambda' + \Lambda$, we have that its conductor and axes are shifted by λ' with respect the ones of Λ.

First, let us show that $R_q(\Lambda_{s-1})$ is a circular interval for $q \geq v_s$.

If $s = 0$, we have that $\Lambda_{s-1} = \Lambda_{-1} = \Gamma$, we apply Lemma 3.4.3 by taking $\mu = 0$. Assume now that $s \geq 1$. By Proposition 3.4.6, we know that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq q_s$. Moreover, we have that $\lambda_s > u_s$ since Λ is an increasing semimodule. This implies that $v_s \geq q_s$, hence we get that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v_s$, as desired.

We end the proof as follows. By Proposition 3.4.5, we have that $R_q(\Lambda) = \mathbb{S}^1_n$, for any $q \geq \tilde{q}_{s+1} - 1$. This implies that for any $k \geq n\tilde{q}_{s+1} - n$, we have that $k \in \Lambda$, and hence $k \geq c_\Lambda$. We conclude by that, by definition of the tops, we have that $\tilde{u}_{s+1} \geq n\tilde{q}_{s+1}$. □

Remark 3.4.8. Notice that Proposition 3.4.5 and 3.4.6 are also true for increasing cuspidal semimodules such that λ_{-1} is a multiple nk of n. Indeed, in this case, we obtain the desired statements by applying the propositions to $\Lambda - nk$.

In [3] is given an explicit formula for the conductor of a semimodule. Nonetheless, for our purposes, the previous bound in terms of the axis $\tilde{u}_{s+1}$ is enough.

3.5 Distribution of the Elements of the Basis

Along this section, we consider a cuspidal semimodule Λ of length $s \geq 0$ with basis $\mathcal{B}$, that we read in $\mathbb{S}^1_n$ as $B = \zeta(\mathcal{B})$ as in the previous section. We are going to describe a pattern for the distribution of the points z_i in

$$B = (z_{-1}, z_0, z_1, \ldots, z_s)$$

by computing the bounds $b^\ell_i(B)$ and $b^r_i(B)$ of B in terms of the axes u^n_{i+1} and u^m_{i+1}.

Lemma 3.5.1. *Take $0 \leq i \leq s$. There are unique integer numbers k^n_i and k^m_i such that:*

1. $-1 \le k_i^n, k_i^m \le i-1$.
2. *There is* $b_{i+1} \ge 0$ *such that* $u_{i+1}^n = \lambda_i + n\ell_{i+1}^n = \lambda_{k_i^n} + mb_{i+1}$.
3. *There is* $a_{i+1} \ge 0$ *such that* $u_{i+1}^m = \lambda_i + m\ell_{i+1}^m = \lambda_{k_i^m} + na_{i+1}$.

Proof. The existence of k_i^n and k_i^m comes from the definition of axes and limits. Let us show their uniqueness. Assume that there is another $k \ne k_i^n$ with $-1 \le k \le i-1$ and a natural number b such that

$$u_{i+1}^n = \lambda_i + n\ell_{i+1}^n = \lambda_{k_i^n} + mb_{i+1} = \lambda_k + mb.$$

Then either $\lambda_k \in (\lambda_{k_i^n} + \Gamma)$ or $\lambda_{k_i^n} \in (\lambda_k + \Gamma)$ in contradiction with definition of basis. The uniqueness of k_i^m is shown in the same way. □

The numbers b_{i+1} and a_{i+1} are the *colimits* of Λ.

Notation 3.5.2. We denote the indexes k_i and $\tilde{k}_i$ of $\tilde{u}_{i+1}$ by

$$k_i = \begin{cases} k_i^n, & \text{if } u_{i+1} = u_{i+1}^n, \\ k_i^m, & \text{if } u_{i+1} = u_{i+1}^m. \end{cases} \qquad \tilde{k}_i = \begin{cases} k_i^n, & \text{if } \tilde{u}_{i+1} = u_{i+1}^n, \\ k_i^m, & \text{if } \tilde{u}_{i+1} = u_{i+1}^m. \end{cases}$$

Remark 3.5.3. Note that $1 \le b_{i+1} < n$. To see this, it is enough to consider when Λ is normalized. Indeed, if $b_{i+1} \ge n$, we have that

$$\lambda_i + n(\ell_{i+1}^n - 1) = \lambda_{k_i^n} + m(b_{i+1}) - n \ge (m-1)n \ge c_\Gamma \ge c_{\Lambda_{i-1}}.$$

Thus $\lambda_i + n(\ell_{i+1}^n - 1) \in \Lambda_{i-1}$ in contradiction with the minimality of ℓ_{i+1}^n. Now, as a consequence, we have that the separation $S(z_{k_i^n}, z_i)$ is given by $S(z_{k_i^n}, z_i) = b_{i+1}$. Recalling that $1 \le \ell_{i+1}^m < n$, see Remark 3.2.3, we have that the separation $S(z_i, z_{k_i^m})$ is given by $S(z_i, z_{k_i^m}) = \ell_{i+1}^m$.

We will show that $k_i^n = b_i^\ell(B)$ and $k_i^m = b_i^r(B)$, for this reason we will use the same terminology of i-left bound, i-right bound or bound, established for the indexes b_i, b_i^n, b_i^m and $\tilde{b}_i$, to the indexes k_i, k_i^n, k_i^m and $\tilde{k}_i$. Before showing this relationship, we make a simple example.

Example 3.5.4. Take the semimodule $\Lambda = \Gamma \setminus \{0\}$. The basis of Λ is $\mathcal{B} = (n, m)$. Note that $\lambda_{-1} = n$ and $\lambda_0 = m$; thus, we have $\Lambda_{-1} = n + \Gamma$ and $\Lambda_0 = \Lambda = \Gamma \setminus \{0\}$.

The limit ℓ_1^n is the smallest positive integer such that

$$m + n\ell_1^n = \lambda_0 + n\ell_1^n \in \Lambda_{-1} = n + \Gamma.$$

After solving the equation $m + n\ell_1^n = n + mb_1$, we obtain that $\ell_1^n = 1 = b_1$. Moreover, we have $u_1^n = n + m = t_1^n$.

In the same way, in order to compute ℓ_1^m, we solve $m + m\ell_1^m = n + na_1$, obtaining $\ell_1^m = n-1$ and $a_1 = m-1$. Therefore, $u_1^m = nm = t_1^m$.

We conclude that $u_1 = u_1^n = n + m$, $\tilde{u}_1 = u_1^m = nm$, $t_1 = t_1^n$ and $\tilde{t}_1 = t_1^m$. As expected, we have that $k_0^n = k_0^m = -1$, that are the 0-bounds of the list

$$B = (0_\varepsilon, 1_\varepsilon) = (z_{-1}, z_0),$$

(note that $\zeta(m) = 1_\varepsilon$).

Any cuspidal semimodule Λ with basis $(n, m, \dots)$ has the same first axes, first critical values, first limits, first colimits and 0-bounds as the ones computed above, since their computation depends only on $\Lambda_0 = \Gamma \setminus \{0\}$. As we said before, the semimodule of differential values of any cusp satisfies this property.

Lemma 3.5.5. *Consider $0 \leq i \leq s$ and take integer numbers $-1 \leq k, k' \leq i-1$, with $k \neq k'$. Assume that we have the following equalities:*

$$\lambda_i + ne = \lambda_k + mb; \qquad \lambda_i + ne' = \lambda_{k'} + mb', \tag{3.3}$$

where $e, e' \in \mathbb{Z}$ and $0 \leq b, b' < n$. Then we have that $e < e'$ if and only if $b < b'$.

Proof. Equations (3.3) lead us to:

$$\begin{aligned} \lambda_k &= \lambda_{k'} + n(e - e') + m(b' - b), \\ \lambda_{k'} &= \lambda_k + n(e' - e) + m(b - b'). \end{aligned}$$

Note that $\lambda_k \notin \lambda_{k'} + \Gamma$ and $\lambda_{k'} \notin \lambda_k + \Gamma$, since λ_k and $\lambda_{k'}$ are different elements of the basis of Λ. We conclude that $b < b'$ if and only if $e < e'$. □

Proposition 3.5.6. *Consider $0 \leq i \leq s$ and take integer numbers $-1 \leq k, k' \leq i-1$, with $k \neq k'$. We have*

1. *Assume that $\lambda_i + ne = \lambda_k + mb$, $\lambda_i + ne' = \lambda_{k'} + mb'$, where $e, e' \in \mathbb{Z}$ and $0 \leq b, b' < n$. Then $e < e' \Leftrightarrow \lambda_i + ne < \lambda_i + ne' \Leftrightarrow S(z_k, z_i) < S(z_{k'}, z_i)$. In particular, taking $k = k_i^n$, we have $S(z_{k_i^n}, z_i) < S(z_{k'}, z_i)$.*
2. *Assume that $\lambda_i + mf = \lambda_k + na$, $\lambda_i + mf' = \lambda_{k'} + na'$ where $a, a' \in \mathbb{Z}$ and $0 \leq f, f' < n$. Then $f < f' \Leftrightarrow \lambda_i + mf < \lambda_i + mf' \Leftrightarrow S(z_i, z_k) < S(z_i, z_{k'})$. In particular, taking $k = k_i^m$, we have $S(z_i, z_{k_i^m}) < S(z_i, z_{k'})$.*

Proof. Notice that $S(z_i, z_k) = f$ and $S(z_i, z_{k'}) = f'$, this proves the second statement. For the first statement, we apply Lemma 3.5.5, by noting that $S(z_k, z_i) = b$ and $S(z_{k'}, z_i) = b'$. □

Corollary 3.5.7. *We have that $k_i^n = b_i^\ell(B)$ and $k_i^m = b_i^r(B)$, for $0 \leq i \leq s$.*

Remark 3.5.8. Take an integer number $\lambda \in \mathbb{Z}$. Then $\mathcal{B}' = \lambda + \mathcal{B}$ is the basis of $\Lambda' = \lambda + \Lambda$ and $B' = \zeta(\mathcal{B}') = B + \lambda_\varepsilon$. Thus, the bounds of B' are the same ones as the bounds of B. Anyway, the axes for Λ' are the ones of Λ shifted by λ, this implies also that bounds, limits and colimits coincide for both semimodules.

For the particular case when the semimodule Λ is increasing, we can give a more accurate description of the bounds, as shown in next proposition:

Proposition 3.5.9. *Assume that Λ is increasing and take $1 \leq i \leq s$. We have*

1. *If $u_i = u_i^n$, then $k_i^n = i - 1$ and $k_i^m = k_{i-1}^m$.*
2. *If $u_i = u_i^m$, then $k_i^n = k_{i-1}^n$ and $k_i^m = i - 1$.*

Proof. In view of Remark 3.5.8, it is enough to consider the normalized case $\lambda_{-1} = 0$. Let us do the proof of Statement *1*; the proof of Statement *2* is similar and we do not explicit it. Thus, we take the assumption that $u_i = u_i^n$.

First, let us suppose that $i = 1$. By considering the bounds in the list (z_{-1}, z_0, z_1), we deduce that $k_0^n = k_0^m = -1$ and either $z_1 \in < z_{-1}, z_0 >$, or $z_1 \in < z_0, z_{-1} >$. Let us show that we actually have that $z_1 \in < z_0, z_{-1} >$, which implies that $k_1^n = 0$ and $k_1^m = -1$ as desired.

Since $\Lambda_{-1} = \Gamma$, we have that $R_q(\Lambda_{-1})$ is a circular interval for $q \geq 0$, due to Lemma 3.4.3. Recall that $u_1^n = u_1 \in I_{q_1}$. Noting that $z_{-1} = 0_\varepsilon$ and applying Proposition 3.4.5 we have that

$$< z_{-1}, z_0 - 1_\varepsilon > \subset R_q(\Lambda_{-1}), \quad q \geq q_1^n - 1 = q_1 - 1.$$

On the other hand, we have that $z_0 \in R_q(\Lambda_0)$, for any $q \geq q_1$ since $\lambda_0 < u_1$ and hence $v_0 \leq q_1$. Thus, we have $< z_{-1}, z_0 > \subset R_{q_1}(\Lambda_0)$. Note that $\lambda_1 > u_1$, since Λ is increasing; this implies that $z_1 \notin R_{q_1}(\Lambda_0)$, hence we necessarily have that $z_1 \notin < z_{-1}, z_0 >$, therefore $z_1 \in < z_0, z_{-1} >$.

Now, assume that $i > 1$. Our first step is to show that $z_i \in < z_{k^n_{i-1}}, z_{k^m_{i-1}} >$. By Proposition 3.4.6, we have that $R_q(\Lambda_{i-2})$ is a circular interval for $q \geq q_{i-1}$. Since $z_{k^n_{i-1}}$ and $z_{k^m_{i-1}}$ belong to $R_{q_{i-1}}(\Lambda_{i-2})$ we have that

$$\text{Either } < z_{k^n_{i-1}}, z_{k^m_{i-1}} > \subset R_{q_{i-1}}(\Lambda_{i-2}), \text{ or } < z_{k^m_{i-1}}, z_{k^n_{i-1}} > \subset R_{q_{i-1}}(\Lambda_{i-2}).$$

Noting that and $z_{i-1} \notin R_{q_{i-1}}(\Lambda_{i-2})$ and $z_{i-1} \in < z_{k^n_{i-1}}, z_{k^m_{i-1}} >$, we conclude that

$$< z_{k^m_{i-1}}, z_{k^n_{i-1}} > \subset R_{q_{i-1}}(\Lambda_{i-2}).$$

Noting also that $z_i \notin R_{q_{i-1}}(\Lambda_{i-2})$, we obtain that $z_i \in < z_{k^n_{i-1}}, z_{k^m_{i-1}} >$, as desired.

Thus, we have $z_i, z_{i-1} \in < z_{k^n_{i-1}}, z_{k^m_{i-1}} >$ and hence there are two possibilities: either $z_i \in < z_{k^n_{i-1}}, z_{i-1} >$, or $z_i \in < z_{i-1}, z_{k^m_{i-1}} >$. If we show that $z_i \in < z_{i-1}, z_{k^m_{i-1}} >$ holds, then the proof is over. By Proposition 3.4.5 we have that

$$< 0_\varepsilon, z_{i-1} - 1_\varepsilon > \subset R_{q_i-1}(\Lambda_{i-2}) \subset R_{q_i-1}(\Lambda_{i-1}).$$

Since $z_{i-1} \in R_{q_i}(\Lambda_{i-1})$, we have that $< 0_\varepsilon, z_{i-1} > \subset R_{q_i}(\Lambda_{i-1})$. If $z_{k^n_{i-1}} \notin < 0_\varepsilon, z_{i-1} >$, taking into account that $0_\varepsilon = z_{-1}$, then we would have that $S(0_\varepsilon, z_{i-1}) < S(z_{k^n_{i-1}}, z_{i-1})$. This contradicts the fact that $k^n_{i-1} = b^\ell_{i-1}(B)$. Hence, we have that $z_{k^n_{i-1}} \in < 0_\varepsilon, z_{i-1} >$ and

$$< z_{k^n_{i-1}}, z_{i-1} > \subset R_{q_i}(\Lambda_{i-1}).$$

Since $z_i \notin R_{q_i}(\Lambda_{i-1})$, we obtain that $z_i \in < z_{i-1}, z_{k^m_{i-1}} >$ as desired. □

Remark 3.5.10. Note that Proposition 3.5.9 implies the following statements:

1. If $k^n_i = i - 1$, then $u_i = u^n_i$.
2. If $k^m_i = i - 1$, then $u_i = u^m_i$.

Indeed, we have that $i - 1 \in \{k^n_i, k^m_i\}$; if $k^n_i = i - 1$, then necessarily $k^m_i \neq i - 1$ (note that $i \geq 1$) and we are in the situation of the first statement of Proposition 3.5.9. Similar argument when $k^m_i = i - 1$.

3.6 Relations between Parameters

Let Λ be an increasing cuspidal semimodule with basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ and let us denote

$$B = \zeta(\mathcal{B}) = (z_{-1}, z_0, z_1, \ldots, z_s).$$

In this section we describe inductive features of axes, limits and co-limits of Λ.

Lemma 3.6.1. *Take $1 \leq k < i \leq s + 1$. We have:*

1. *The axes and the critical values u_i, t_i satisfy that $u_i > u_k$ and $t_i > t_k$.*
2. *The axes and the critical values $\tilde{u}_i, \tilde{t}_i$ satisfy that $\tilde{u}_i < \tilde{u}_k$ and $\tilde{t}_i < \tilde{t}_k$.*

Proof. It is enough to consider the case $k = i - 1$.

Let us prove Statement *1*. By definition of the axes, we have that $u_i > \lambda_{i-1}$. Since the semimodule is increasing, we have that $\lambda_{i-1} > u_{i-1}$. We get that $u_i > u_{i-1}$. Moreover, by definition of critical value, we see that

$$t_i = t_{i-1} + (u_i - \lambda_{i-1}) > t_{i-1}.$$

This ends the proof of Statement *1*.

Let us prove Statement 2. We do it for the case that $\tilde{u}_i = u_i^n = \lambda_{i-1} + n\ell_i^n$, the proof for the case $\tilde{u}_i = u_i^m$ runs in a similar way. By Proposition 3.5.9, there are two cases: either $k_{i-1}^n = i-2$ or $k_{i-1}^m = i-2$. We shall see that $\tilde{u}_i < \tilde{u}_{i-1}$ and that $\tilde{t}_i < \tilde{t}_{i-1}$ simultaneously in each of the cases above.

Case $k_{i-1}^n = i-2$. By Remark 3.5.10, we see that $u_{i-1} = u_{i-1}^n$ and $\tilde{u}_{i-1} = u_{i-1}^m$. Hence we can write:

$$\tilde{u}_i = u_i^n = \lambda_{i-1} + n\ell_i^n = \lambda_{i-2} + mb_i \tag{3.4}$$

$$\tilde{u}_{i-1} = u_{i-1}^m = \lambda_{i-2} + m\ell_{i-1}^m = \lambda_k + na_{i-1} \quad \text{with } k < i-2. \tag{3.5}$$

In order to see that $\tilde{u}_i < \tilde{u}_{i-1}$, we need to show that $b_i < \ell_{i-1}^m$. To do this, we are going to exclude the possibility $b_i \geq \ell_{i-1}^m$:

- If $\ell_{i-1}^m = b_i$, we deduce that $\tilde{u}_{i-1} = \tilde{u}_i$ from equations (3.4) and (3.5). Hence, we have

$$\tilde{u}_i = \lambda_{i-1} + n\ell_i^n = \lambda_k + na_{i-1} = \tilde{u}_{i-1}, \quad k < i-2.$$

  Then $\lambda_{i-1} \in \lambda_k + \Gamma$ or $\lambda_k \in \lambda_{i-1} + \Gamma$, contradicting the fact that $\mathcal{B}$ is a basis.
- If $\ell_{i-1}^m < b_i$, by equation (3.4) and by Corollary 3.4.7, we have that:

$$\begin{aligned} \tilde{u}_i - n = \lambda_{i-1} + n(\ell_i^n - 1) &= \lambda_{i-2} + mb_i - n \geq \lambda_{i-2} + m\ell_{i-1}^m + m - n \\ &= \tilde{u}_{i-1} + m - n \geq c_{\Lambda_{i-2}} + m > c_{\Lambda_{i-2}}. \end{aligned}$$

  We get that $\lambda_{i-1} + n(\ell_i^n - 1) > c_{\Lambda_{i-2}}$ and thus $\lambda_{i-1} + n(\ell_i^n - 1) \in \Lambda_{i-2}$, contradicting the minimality of ℓ_i^n.

It follows that $b_i < \ell_{i-1}^m$, concluding that $\tilde{u}_i < \tilde{u}_{i-1}$, in the case $k_{i-1}^n = i-2$.

Let us see now that $\tilde{t}_i < \tilde{t}_{i-1}$ in this case $k_{i-1}^n = i-2$. From equations (3.4), (3.5), using the fact that $\tilde{u}_i < \tilde{u}_{i-1}$ and the property of increasing semimodule, we have that:

$$\lambda_{i-2} + m\ell_{i-1}^m = \tilde{u}_{i-1} > \tilde{u}_i = \lambda_{i-1} + n\ell_i^n > u_{i-1} + n\ell_i^n.$$

Consequently, $m\ell_{i-1}^m > u_{i-1} - \lambda_{i-2} + n\ell_i^n$ and

$$\tilde{t}_{i-1} = t_{i-2} + m\ell_{i-1}^m > t_{i-2} + u_{i-1} - \lambda_{i-2} + n\ell_i^n = t_{i-1} + n\ell_i^n = \tilde{t}_i.$$

This ends the proof that $\tilde{t}_i < \tilde{t}_{i-1}$ in this case.

Case $k_{i-1}^m = i-2$. Note that $\tilde{u}_{i-1} = u_{i-1}^n$ and $k_{i-1}^n = k_{i-2}^n$ in view of Remark 3.5.10 and Proposition 3.5.9 . Thus, we can write

$$\tilde{u}_i = u_i^n = \lambda_{i-1} + n\ell_i^n = \lambda_k + mb_i, \text{ with } k = k_{i-1}^n = k_{i-2}^n < i-2, \tag{3.6}$$

$$\tilde{u}_{i-1} = u_{i-1}^n = \lambda_{i-2} + n\ell_{i-1}^n = \lambda_k + mb_{i-1}, \text{ with } k = k_{i-2}^n < i-2. \tag{3.7}$$

Let us proceed in a similar way as before to show that $b_{i-1} > b_i$:

- Assume that $b_{i-1} = b_i$. Then $\rho(\lambda_{i-1}) = \rho(\lambda_{i-2})$, absurd.

- Assume that $b_{i-1} < b_i$. Then, we have that

$$\begin{aligned} \tilde{u}_i - n = \lambda_{i-1} + n(\ell_i^n - 1) &= \lambda_k + mb_i - n \\ &> \lambda_k + (b_i - 1)m = \tilde{u}_{i-1} + (b_i - b_{i-1} - 1)m \\ &\geq \tilde{u}_{i-1} \geq n + c_{\Lambda_{i-2}}. \end{aligned}$$

  Then $\lambda_{i-1} + n(\ell_i^n - 1) \in \Lambda_{i-2}$, in contradiction with the minimality of ℓ_i^n.

We conclude that $b_{i-1} > b_i$ and thus $\tilde{u}_{i-1} > \tilde{u}_i$.

Let us see now that $\tilde{t}_i < \tilde{t}_{i-1}$ in this case $k_{i-1}^m = i-2$. From equations (3.6), (3.7), using the fact that $\tilde{u}_i < \tilde{u}_{i-1}$ and the property of increasing semimodule, we have that:

$$\lambda_{i-2} + n\ell_{i-1}^n = \tilde{u}_{i-1} > \tilde{u}_i = \lambda_{i-1} + n\ell_i^n > u_{i-1} + n\ell_i^n.$$

Consequently, $n\ell_{i-1}^n > u_{i-1} - \lambda_{i-2} + n\ell_i^n$ and

$$\tilde{t}_{i-1} = t_{i-2} + n\ell_{i-1}^n > t_{i-2} + u_{i-1} - \lambda_{i-2} + n\ell_i^n = t_{i-1} + n\ell_i^n = \tilde{t}_i.$$

This ends the proof. □

Corollary 3.6.2. *Let Λ be a cuspidal increasing semimodule with basis*

$$\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$$

such that $\lambda_{-1} = n$ and $\lambda_0 = m$. We have that $\tilde{t}_1 = \tilde{u}_1 = nm$ and the following holds

$$t_{i+1}^n < \tilde{t}_1 = nm \quad \text{and} \quad t_{i+1}^m < \tilde{t}_1 = nm,$$

for any $1 \leq i \leq s$. Similarly, for the axes.

Proof. It is enough to recall that $\tilde{t}_1 = nm$ in view of Example 3.5.4. □

We finish this chapter with two technical results required in Chapter 8.

Proposition 3.6.3. *Consider $1 \leq i \leq s$. We have*

1. *If $k_i^n = i-1$, then $\ell_{i+1}^n + a_{i+1} = a_i$ and $\ell_{i+1}^m + b_{i+1} = \ell_i^m$.*
2. *If $k_i^m = i-1$, then $\ell_{i+1}^n + a_{i+1} = \ell_i^n$ and $\ell_{i+1}^m + b_{i+1} = b_i$.*

Proof. Notice that shifting the semimodule any integer number does not change the value of the limits and the colimits. Therefore, we can assume without loss of generality that Λ is normalized and thus $\lambda_{-1} = 0$.

Let us prove Statement *1*. By hypothesis, we have that $k_i^n = i-1$. In view of Remark 3.5.10 and Proposition 3.5.9, we also have that $k_i^m = k_{i-1}^m$. Let us write:

$$u_{i+1}^n = \lambda_i + n\ell_{i+1}^n = \lambda_{i-1} + mb_{i+1}, \tag{3.8}$$

$$u_{i+1}^m = \lambda_i + m\ell_{i+1}^m = \lambda_{k_i^m} + na_{i+1}, \tag{3.9}$$

$$u_i^m = \lambda_{i-1} + m\ell_i^m = \lambda_{k_{i-1}^m} + na_i = \lambda_{k_i^m} + na_i. \tag{3.10}$$

From equations (3.8) and (3.9) we obtain that

$$n\ell_{i+1}^n + na_{i+1} + \lambda_{k_i^m} = mb_{i+1} + m\ell_{i+1}^m + \lambda_{i-1}. \tag{3.11}$$

By equation (3.10) we can substitute $\lambda_{k_i^m} = \lambda_{i-1} + m\ell_i^m - na_i$ in equation (3.11) to obtain

$$n(\ell_{i+1}^n + a_{i+1} - a_i) = m(\ell_{i+1}^m + b_{i+1} - \ell_i^m). \tag{3.12}$$

Since n and m have no common factor, we have that n divides $\ell^m_{i+1} + b_{i+1} - \ell^m_i$.

Let us see that $\ell^m_{i+1} + b_{i+1} - \ell^m_i = 0$ and hence $\ell^m_{i+1} + b_{i+1} = \ell^m_i$ as desired. If $\ell^m_{i+1} + b_{i+1} - \ell^m_i \neq 0$ we are in one of the following three cases

$$a)\ |\ell^m_{i+1} + b_{i+1} - \ell^m_i| \geq 2n, \quad b)\ \ell^m_{i+1} + b_{i+1} - \ell^m_i = -n, \quad c)\ \ell^m_{i+1} + b_{i+1} - \ell^m_i = n.$$

Let us see that each of these cases leads to a contradiction.

Assume first that we are in case a). Noting that $\ell^m_{i+1}, b_{i+1}, \ell^m_i \geq 1$, there is at least one of them that is strictly bigger than n. Let us consider the three possibilities:

- If $\ell^m_{i+1} > n$, we have that $m\ell^m_{i+1} > nm$ and then $\lambda_i + m\ell^m_{i+1} > nm$. This implies that $\lambda_i + m(\ell^m_{i+1} - 1) > (n-1)m \geq c_\Gamma \geq c_{\Lambda_{i-1}}$. Then, we have that $\lambda_i + m(\ell^m_{i+1} - 1) \in \Lambda_{i-1}$, contradicting the minimality of ℓ^m_{i+1}. Recall that the conductor of the semigroup is $c_\Gamma = (n-1)(m-1)$.
- If $\ell^m_i > n$, we do the same argument as before.
- If $b_{i+1} > n$, we have that $\lambda_i + n\ell^n_{i+1} = \lambda_{i-1} + mb_{i+1} > nm$ and then

  $$\lambda_i + n(\ell^n_{i+1} - 1) > (m-1)n \geq c_\Gamma \geq c_{\Lambda_{i-1}}.$$

  Then $\lambda_i + n(\ell^n_{i+1} - 1) \in \Lambda_{i-1}$ and this contradicts the minimality of ℓ^n_{i+1}.

Assume that we are in case b), that is $\ell^m_{i+1} + b_{i+1} - \ell^m_i = -n$. this implies that $\ell^m_i > n$ and we do the same argument as before to obtain a contradiction.

Assume that we are in case c), that is $\ell^m_{i+1} + b_{i+1} - \ell^m_i = n$. We have that $\ell^m_{i+1} + b_{i+1} > n$. By Remark 3.5.3 we see that the separation $S(z_{i-1}, z_i)$ is given by $S(z_{i-1}, z_i) = b_{i+1}$ (recall that $k^n_i = i-1$) and that the separation $S(z_i, z_{k^m_i})$ is given by $S(z_i, z_{k^m_i}) = \ell^m_{i+1}$. Noting that $z_i \in < z_{i-1}, z_{k^m_i} >$ and $z_{i-1} \neq z_{k^m_i}$, we conclude that

$$n > S(z_{i-1}, z_{k^m_i}) = S(z_{i-1}, z_i) + S(z_i, z_{k^m_i}) = b_{i+1} + \ell^m_{i+1}.$$

This contradicts $b_{i+1} + \ell^m_{i+1} > n$. The proof that $\ell^m_{i+1} + b_{i+1} = \ell^m_i$ is ended. Moreover, since $\ell^m_{i+1} + b_{i+1} - \ell^m_i = 0$, by equation (3.12), we conclude that $\ell^n_{i+1} + a_{i+1} = a_i$, as desired.

The proof of Statement 2 runs in a similar way to the above arguments. □

Next corollary will be useful in our computation of Saito bases in Chapter 8:

Corollary 3.6.4. *Consider* $2 \leq j+1 < q \leq s+1$. *Then*

$$\ell^m_{j+1} - (\ell^m_{j+2} + \ell^m_{j+3} + \cdots + \ell^m_q) = b_q > 0,$$

under the assumption that $\tilde{t}_{j+1} = t^m_{j+1}$ *and* $\tilde{t}_\ell = t^n_\ell$, *for* $j+2 \leq \ell \leq q-1$. *In a symmetric way, we have that*

$$\ell^n_{j+1} - (\ell^n_{j+2} + \ell^n_{j+3} + \cdots + \ell^n_q) = a_q > 0,$$

under the assumption that $\tilde{t}_{j+1} = t^n_{j+1}$ *and* $\tilde{t}_\ell = t^m_\ell$, *for* $j+2 \leq \ell \leq q-1$.

Proof. We prove the first assertion, the second one is similar. Let us consider the difference

$$\ell^m_{j+1} - \ell^m_{j+2}.$$

Since $\tilde{t}_{j+1} = t^m_{j+1}$, by Proposition 3.5.9, we have that $k^n_{j+1} = j$. By Proposition 3.6.3, we conclude that

$$\ell^m_{j+1} - \ell^m_{j+2} = b_{j+2}.$$

Now, let us study the difference $b_{j+2} - \ell^m_{j+3}$. Since $t_{j+2} = t^m_{j+2}$, we have that $k^m_{j+2} = j + 1$. By Proposition 3.6.3 we conclude that

$$b_{j+2} - \ell^m_{j+3} = b_{j+3}.$$

Following in this way, we conclude that

$$\ell^m_{j+1} - (\ell^m_{j+2} + \ell^m_{j+3} + \cdots + \ell^m_q) = b_q > 0,$$

as desired. □

4

Standard Bases

In order to compute the semimodule of differential values of branches, we need to compute standard bases. This notion is our main computational tool in this work. Standard bases were firstly introduced by H. Hironaka in [38] when he was studying the resolution of singularities of algebraic varieties in characteristic 0. The concept of standard basis is quite similar, in its definition, to the one of Gröbner basis. In this chapter, we basically give the notations and definitions needed to understand this concept, additionally, we give the algorithms that compute a standard basis.

We treat the notion of standard basis in the cases of ideals, algebras and modules. As a remark, in the case of cusps, the computations are easier. For this reason the reader can skip the sections about of algebras and submodules. Nonetheless, we add them to the text for completeness, this way the reader may find the required techniques to study semimodules of differential values for branches which are not cusps.

We use [31] as our main reference, there it is only treated the formal case. Since we are only interested in the holomorphic case, we added minor modifications to the formal case.

4.1 Standard Bases of an Ideal

In this section we recall the notion of standard basis for an ideal. We follow [31] along the first three sections.

Let p be a positive integer. Fix a *monomial order* $\preceq$ of $(\mathbb{Z}_{\geq 0})^p$, that is, $\preceq$ is a total order satisfying that $\mathbf{0} \preceq s$ for all $s \in (\mathbb{Z}_{\geq 0})^p$, and if $s_1 \preceq s_2$, then $s_1 + s \preceq s_2 + s$ for all $s, s_1, s_2 \in (\mathbb{Z}_{\geq 0})^p$. Additionally, we ask $\preceq$ to satisfy the *finiteness property*. In other words, for all $s_0 \in (\mathbb{Z}_{\geq 0})^p$, the set $\{s \preceq s_0 : s \in (\mathbb{Z}_{\geq 0})^p\}$ is finite.

Example 4.1.1. The following two monomial orders are the main ones that we are going to use along this work.

- The natural order in $\mathbb{Z}_{\geq 0}$.
- Consider a pair of positive integers (n, m) such that $2 \leq n < m$. Then given $(a, b), (c, d) \in (\mathbb{Z}_{\geq 0})^2$, we say that $(a, b) < (c, d)$ if and only if either $na+mb < nc+md$ or $na+mb = nc+md$ with $a < c$. We call this order the *weighted order* with respect (n, m).

Given $s, t \in (\mathbb{Z}_{\geq 0})^p$, we say that s *divides* t, or that t is *divisible* by s, if $t - s \in (\mathbb{Z}_{\geq 0})^p$, we denote it by $s \mid t$. Consider a set $A \subset (\mathbb{Z}_{\geq 0})^p$, a subset $D \subset A$ is said to be a *set of divisors* of A if for all $s \in A$ there exists $t \in D$ such that $t \mid s$. The set D is a *minimal set of divisors* if $t \mid t'$ implies that $t = t'$, for all $t, t' \in D$.

Remark 4.1.2. Any non empty set $A \subset (\mathbb{Z}_{\geq 0})^p$ has always a finite minimal set of divisors, see [31] p.3 Theorem 1.

If $g \neq 0$, then we define the *leading power* of g with respect to $\preceq$ as

$$lp(g) := \min(\mathcal{NC}_{x_1,\ldots,x_n}(g)),$$

we recall that $\mathcal{NC}_{x_1,\ldots,x_n}(g)$ is the Newton cloud of g, as defined in Chapter 1. Besides, we set $lp(0) = (\infty, \infty, \ldots, \infty) \succ \alpha$ for any $\alpha \in (\mathbb{Z}_{\geq 0})^p$. The *leading term* of g is $lt(g) := a_\beta x^\beta$ where $\beta = lp(g)$ if $g \neq 0$ and $lt(0) := 0$.

Definition 4.1.3. *Given an ideal $I \subset \mathbb{C}\{x_1, \ldots, x_p\}$ and a subset $B \subset I$. We say that B is a standard basis of I if B generates I as an ideal, and for any $h \in I$, there exists $b \in B$ such that $lp(b) \mid lp(h)$. We say that B is a minimal standard basis, if for all $b \in B$, then $B \setminus \{b\}$ is not a standard basis.*

We are going to explain how to compute a minimal standard basis. First, we introduce some terminology. Given a set $B \subset \mathbb{C}\{x_1, \ldots, x_p\}$ and two elements $g, r \in \mathbb{C}\{x_1, \ldots, x_p\}$, we say that r is a *reduction* modulo B of g, if there exist $a \in \mathbb{C}$, $\alpha \in (\mathbb{Z}_{\geq 0})^p$, and $b \in B$, such that

$$r = g - a x^\alpha b,$$

with either $r = 0$ or $lp(g) \prec lp(r)$. If there exists a reduction r of g modulo B, we say that g is reducible modulo B.

We denote by r_∞ a *final reduction* as the Krull limit of a sequence of reductions starting of g until obtaining either the zero element or an element non reducible by B any longer. More precisely, we consider a sequence

$$r_0 = g \longrightarrow r_1 \longrightarrow r_2 \longrightarrow \ldots \longrightarrow r_k \longrightarrow \ldots$$

where r_i is a reduction of r_{i-1} modulo B. An element $r' \in \mathbb{C}\{x_1, \ldots, x_p\}$ is called a *partial reduction* of g modulo B if there exists a finite sequence of reductions, as above, starting at g that gives r'. In the definition of final (resp. partial) reduction of g, we are including the case where g is non reducible modulo B, in that case g will be its own final (resp. partial) reduction.

Remark 4.1.4. Artin's Approximation Theorem, see [5], states that given a formal solution of a system of holomorphic equations there exists a convergent solution as close as desired to the formal one in the Krull topology. Thus, when B is finite, saying that a final reduction of g modulo B is zero implies that g belongs to the ideal generated by B. Additionally, the reciprocal result is also true, that is, if $g \in (B)$, then g is reducible modulo B. In particular, 0 would be a final reduction of g modulo B.

Take non zero functions $g_1, g_2 \in \mathbb{C}\{x_1, \ldots, x_p\}$ with $lt(g_i) = a_i x^{\alpha_i}$ for $i = 1, 2$, the *minimal S-process* of g_1, g_2 is

$$S_{min}(g_1, g_2) := \frac{lcm(x^{\alpha_1}, x^{\alpha_2})}{x^{\alpha_1}} g_1 - \frac{a_1}{a_2} \frac{lcm(x^{\alpha_1}, x^{\alpha_2})}{x^{\alpha_2}} g_2,$$

where $lcm(x^{\alpha_1}, x^{\alpha_2})$ is the usual least common multiple.

We recall now an algorithm to compute a standard basis.

Büchberger's Algorithm

INPUT: $(g_1, \ldots, g_j) = I \subset \mathbb{C}\{x_1, \ldots, x_p\}$ ideal and a monomial order $\preceq$.
OUTPUT: B standard basis of I.
START:

Put $B = \{g_1, \dots, g_j\}$.

loop {

for all distinct pairs of elements $h_1, h_2 \in B$:

Compute $s = S_{\min}(h_1, h_2)$ and r_∞ a final reduction of s modulo B.

if $r_\infty \neq 0$ **then**:

Add r_∞ to B.

if all the final reductions computed are 0 **then**:

Return.

} **end loop**

Consider $I \subset \mathbb{C}\{x_1, \dots, x_p\}$ an ideal and the $\mathbb{C}$-vector space $Q = \mathbb{C}\{x_1, \dots, x_p\}/I$, when the complex dimension of Q is finite we have that:

$$dim_{\mathbb{C}} Q = \#((\mathbb{Z}_{\geq 0})^p \setminus lp(I)).$$

If $p = 2$ we can rewrite the previous formula. Assume that $B = \{g_1, \dots, g_j\}$ is a minimal standard basis of an ideal $I \subset \mathbb{C}\{x, y\}$. Put $lp(g_i) = (a_i, b_i)$ for $i = 1, \dots, j$, and suppose that they are ordered such that:

$$0 \leq a_1 < a_2 < \dots < a_j; \quad b_1 > b_2 > \dots > b_j \geq 0.$$

Proposition 4.1.5 ([31] p.32 Lemma 1)**.** *With the notations as above, the dimension of $Q = \mathbb{C}\{x, y\}/I$ as $\mathbb{C}$-vector space is finite if and only if $a_1 = b_j = 0$. Besides, in the finite case, we have that*

$$dim_{\mathbb{C}} Q = \sum_{i=2}^{j} b_{i-1}(a_i - a_{i-1}).$$

Remark 4.1.6. The set of leading powers $\{lp(g_i)\}_{i=1,\dots,j}$, with $\{g_1, \dots, g_j\}$ a minimal standard basis of $I \subset \mathbb{C}\{x_1, \dots, x_p\}$, does not depend on the minimal standard basis chosen. In particular, we also have that two minimal standard bases of an ideal I have the same number of elements. Moreover, there always exists a finite standard basis of an ideal $I \subset \mathbb{C}\{x_1, \dots, x_n\}$.

4.2 Standard Bases of a Subalgebra

As we said before, we are interested in computing minimal standard bases of submodules, in order to do that, we need first to compute them for the case of subalgebras.

The purpose of using this theory is to study the case where our subalgebra is $\mathbb{C}\{x(t), y(t)\}$, being $(x(t), y(t))$ a primitive parametrization of a curve.

Consider a set $G \subset \mathbb{C}\{x_1, \dots, x_p\}$, and define G^α a *G-product* as

$$G^\alpha = \prod_{\substack{i \in I \\ I \text{ finite}}} g_i^{\alpha_i}, \quad g_i \in G \text{ for all } i \in I, \text{ where } \alpha = (\alpha_1, \dots, \alpha_{\#I}).$$

The G-products play a similar role as the monomials in the previous section. For simplicity, we are going to assume that $G = \{g_1, \dots, g_\ell\}$ is always a finite set, in this case, we omit the index I in the product, and we use the usual notation

$$G^\alpha = \prod_{i=1}^{\ell} g^{\alpha_i}.$$

We also consider $T = \mathbb{C}\{f_1, \ldots, f_\ell\} \subset \mathbb{C}\{x_1, \ldots, x_p\}$ a $\mathbb{C}$-subalgebra, where f_i is a non zero element of the ideal $(x_1, x_2, \ldots, x_p)$ for $1 \leq i \leq \ell$.

The following definitions are adaptations of the ones given in the previous section, but for the case of subalgebras.

Given $g, r \in \mathbb{C}\{x_1, \ldots, x_p\}$, we say that r is a *reduction* of g modulo G and that g is reducible modulo G, if there exist a G-product G^α and $a \in \mathbb{C}$, such that

$$r = g - aG^\alpha$$

with either $r = 0$ or $lp(r) \succ lp(g)$. As before, we can consider a sequence of the form:

$$g = r_0 \longrightarrow r_1 \longrightarrow r_2 \longrightarrow \ldots \longrightarrow r_k \longrightarrow \ldots$$

where r_i is a reduction of r_{i-1} modulo G for $i \geq 1$. If $G \subset T$, again, because of the finiteness property of the monomial order, we have that the sequence converges to an element r_∞. In fact, it is satisfied that $r_\infty \in T$, that is, it is a convergent power series. Indeed, there are two cases: either $r_\infty = 0$ or $r_\infty \neq 0$. If $r_\infty = 0$, then is clear that $r_\infty \in T$. Otherwise, if $r_\infty \neq 0$, then by the finiteness property of the monomial order, we have that the sequence that defines r_∞ must be finite. Therefore, the element r_∞ is the finite sum of holomorphic power series, and thus, it is also convergent.

We say that the limit r_∞ is a *final reduction* of g modulo G, if either $r_\infty = 0$ or $lp(r_\infty) \neq lp(G^\alpha)$ for every G-product. As before, we refer to any of the intermediate elements of the sequence as *partial reductions* of g modulo G.

We say that G is a standard basis of algebras if

$$\langle lp(G) \rangle = lp(\mathbb{C}\{G\}).$$

The set G is a *standard basis* of T, if G is a standard basis of algebras and T is generated by G. We say that G is *minimal* if for any $g \in G$, $lp(g) \notin \langle lp(G \setminus \{g\}) \rangle$.

Up this point, everything is similar to what was defined in the previous section. However, the notion of minimal S-process must be changed.

Consider $G = \{f_1, \ldots, f_\ell\} \subset \mathbb{C}\{x_1, \ldots, x_p\}$, for any $(\alpha_1, \ldots, \alpha_\ell) = \alpha \in (\mathbb{Z}_{\geq 0})^\ell$, we denote by $pr_i(\alpha) := \alpha_i$ with $1 \leq i \leq p$. We consider the system of diophantine equations $\mathcal{S}$:

$$\begin{array}{rcl} \sum_{i=1}^{\ell} \alpha_i pr_1(lp(f_i)) & = & \sum_{i=1}^{\ell} \beta_i pr_1(lp(f_i)) \\ \vdots & & \vdots \\ \sum_{i=1}^{\ell} \alpha_i pr_p(lp(f_i)) & = & \sum_{i=1}^{\ell} \beta_i pr_p(lp(f_i)) \end{array}$$

The set of solutions of $\mathcal{S}$ has a minimal set of divisors $D(f_1, \ldots, f_\ell) \subset (\mathbb{Z}_{\geq 0})^{2\ell}$. We will denote any element of $D(f_1, \ldots, f_\ell)$ as (α, β), with $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^\ell$.

We define the *minimal S-process* $S_{min}(G, \alpha, \beta)$ of G as a sum of two G-products

$$S_{min}(G, \alpha, \beta) := G^\alpha + bG^\beta,$$

where (α, β) is an element of $D(g_1, \ldots, g_\ell)$ and $b \in \mathbb{C}^*$, such that

$$S_{min}(G, \alpha, \beta) = 0 \quad \text{or} \quad lp(S_{min}(G, \alpha, \beta)) \succ lp(G^\alpha) = lp(G^\beta).$$

It is possible that T does not have a finite standard basis, see [31], however in the case it exists, we can compute one in a similar way as in the case of ideals:

Büchberger's Algorithm

INPUT: $T = \mathbb{C}\{f_1, \ldots, f_\ell\}$ a $\mathbb{C}\{x_1, \ldots, x_p\}$-subalgebra and a monomial order $\leq$ in $(\mathbb{Z}_{\geq 0})^p$.
OUTPUT: G standard basis of T.
START:

Put $G = \{f_1, \ldots, f_\ell\}$.

loop {

Compute any S-minimal process $s = S_{\min}(G, \alpha, \beta)$ of G and r_∞ a final reduction of s modulo G.

if $r_\infty \neq 0$ **then**:

Add r_∞ to G.

if all the final reductions computed are $r_\infty = 0$ **then**:

Return.

} **end loop**

4.3 Standard Bases of a Submodule

Here we describe what is a standard basis for a submodule. We put as before $T = \mathbb{C}\{f_1, \ldots, f_\ell\} \subset \mathbb{C}\{x_1, \ldots, x_p\}$ a $\mathbb{C}\{x_1, \ldots, x_p\}$-subalgebra. We only deal with a particular type of submodules of $\mathbb{C}\{x_1, \ldots, x_p\}$. We set $\hat{T} = \mathbb{C}[[f_1, \ldots, f_\ell]] \subset \mathbb{C}[[x_1, \ldots, x_p]]$ and let $\hat{M}$ be a complete $\hat{T}$ submodule of $\mathbb{C}[[x_1, \ldots, x_p]]$. We consider the T submodule $M \subset \mathbb{C}\{x_1, \ldots, x_p\}$ as the set of convergent elements of $\hat{M}$.

As before, we say that $H \subset M$ is a *standard basis* of M if H generates M as T-module and

$$\langle lp(H) \rangle = \langle lp(M) \rangle.$$

We say that H is a *minimal standard basis* if for any $m \in H$ we have that $lp(m) \notin \langle lp(H \setminus \{m\}) \rangle$.

Assume that G is a standard basis of T. Take $H \subset \mathbb{C}\{x_1, \ldots, x_p\}$ and $h, r \in \mathbb{C}\{x_1, \ldots, x_p\}$. We say that r is a *reduction* of g modulo (H, G), or that that g is reducible modulo (H, G), if there are a G−product G^α, $m \in H$ and $a \in \mathbb{C}$ with:

$$r = g - aG^\alpha m$$

satisfying that: either $r = 0$ or $lp(r) \succ lp(g)$. We are going to assume that G and H are finite sets, in fact, we can assume that $G = \{f_1, \ldots, f_\ell\}$. We consider, again, a sequence of reductions:

$$g = r_0 \to r_1 \to r_2 \to \ldots \to r_q \to \ldots$$

which converges to an element $r_\infty \in M$, because of the finiteness property of the order and the completeness of M. If $r_\infty = 0$ or r_∞ cannot be reduced more, we say that r_∞ is a *final reduction* of g modulo (H, G). The intermediate elements of the sequence are called *partial reduction* of g modulo (H, G).

Given m_1, m_2 two different elements of H, we consider the system of diophantine equations $\mathcal{S}$:

$$\begin{aligned} pr_1(m_1) + \sum_{i=1}^{\ell} \alpha_i pr_1(lp(f_i)) &= pr_1(m_2) + \sum_{i=1}^{\ell} \beta_i pr_1(lp(f_i)) \\ \vdots \qquad & \qquad \vdots \\ pr_p(m_1) + \sum_{i=1}^{\ell} \alpha_i pr_p(lp(f_i)) &= pr_p(m_2) + \sum_{i=1}^{\ell} \beta_i pr_p(lp(f_i)), \end{aligned}$$

We denote by $D(m_1, m_2; f_1, \ldots, f_\ell) \subset (\mathbb{Z}_{\geq 0})^{2\ell}$ the minimal set of divisors associated to the set of solutions of $\mathcal{S}$. Again any of its elements is denoted by (α, β) with $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^{\ell}$.

Given a $m_1, m_2 \in H$, we define the *minimal S-process* $S_{min}(m_1, m_2, \alpha^1, \alpha^2)$ as:

$$S_{min}(m_1, m_2, \alpha^1, \alpha^2) := G^{\alpha^1} m_1 + a G^{\alpha^2} m_2, \text{ with } G^{\alpha^q} = \prod_{i=1}^{\ell} g_i^{\alpha_i^q}, q = 1, 2.$$

where $(\alpha^1, \alpha^2) \in D(m_1, m_2; g_1, \ldots, g_j)$ and $a \in \mathbb{C}^*$ such that

$$lp(S_{min}(m_1, m_2, \alpha^1, \alpha^2)) \succ lp(G^{\alpha^1} m_1) = lp(G^{\alpha^2} m_2).$$

We now give the corresponding algorithm that computes a minimal standard basis of M.

Büchberger's Algorithm

INPUT: $T = \mathbb{C}\{G\}$ a $\mathbb{C}\{x_1, \ldots, x_p\}$-subalgebra with G a standard basis, M a T-submodule of $\mathbb{C}\{x_1, \ldots, x_p\}$ generated by $B = \{m_1, \ldots, m_q\}$ and a monomial order $\preceq$ in $\mathbb{C}\{x_1, \ldots, x_p\}$.

OUTPUT: H standard basis of M.

START:

Put $H = B = \{m_1, \ldots, m_q\}$.

loop {

Compute any S-minimal process $s = S_{\min}(h_1, h_2, \alpha^1, \alpha^2)$ with $h_1, h_2 \in M$. Compute r_∞ a final reduction of s modulo (H, G).

if $r_\infty \neq 0$ **then**:

Add r_∞ to H.

if all the final reductions computed are $r_\infty = 0$ **then**:

Return.

} **end loop**

4.4 Formal vs Convergent

We end this chapter with a brief explanation on the differences between the formal and the convergent cases. Since we find those differences quite small, we chose the previous presentation instead of restarting the theory from scratch, proofs included. Besides, this work did not pretend to be a reference on standard basis, rather on its applications when applied to plane curves. Nonetheless, just to make things clear we put this small note.

We first remark that we demanded our monomial order to satisfy the finiteness property. This condition is required in both the formal and convergent cases when computing standard bases for algebras and modules. However, for ideals, in the formal case it is not, see [31]. Next example shows that modifications are needed when passing from the formal case to the convergent one.

Example 4.4.1. In $(\mathbb{Z}_{\geq 0})^2$ we consider the lexicographic monomial order, that is, $(a,b) < (c,d)$ if either $b < d$ or $b = d$ and $a < c$. This monomial order does not have the finiteness property since $(k,0) < (0,1)$ for all $k \in \mathbb{Z}_{\geq 0}$.

We take the following family:

$$B = \{f_n = x^n - x^{n+1} + n!y^n : n \geq 1\},$$

and the function $r_1 = -x$. Note that $lt(f_n) = x^n$ for all n. We have that $r_2 = r_1 + f_1 = -x^2 + y$ is a reduction of r_1 modulo B. Similarly, $r_3 = r_2 + f_2 = -x^3 + y + 2y^2$ is a reduction of r_2 modulo B. We see that we can continue this process indefinitely. After an infinite number of reductions, we obtain a final reduction $r_\infty = \sum_{n\geq 1} n!y^n$ modulo B, which is not convergent.

Example 4.4.1 shows that when considering any monomial order the final reduction of a function r modulo a set of functions B may not be an element of the ring $\mathbb{C}\{x_1, \ldots, x_p\}$. Recall that having a ring closed under final reductions is necessary when applying the Büchberger's algorithm.

If we impose the finiteness property in the monomial order, as we did, then the number of reductions used to compute a non zero final reduction is always finite. In other words, if we start with convergent elements then a final reduction is always written as a polynomial combination of convergent functions, which is again convergent.

We notice an extra thing, Example 4.4.1 is constructed by using an infinite set B. When applying Büchberger's algorithm for ideals, we only deal with sets of finite cardinal. We do not know if only imposing the set B to have finite cardinal would solve the convergence problem.

For the rest of theory, it is the same as the one we can found in [31]. We only restrict ourselves to work over the complex numbers and hence we can apply Artin's Approximation Theorem. We recall that we needed it, because saying in our context that a final reduction of a function is 0, implies that the function can be written as a formal combination of the elements our standard basis. By means of that theorem, we can assure the existence of a convergent one, and state that the original element is generated by the standard basis.

5

Delorme's Decompositions

We need to find the semimodule of differential values of a cusp C. This could be done by means of Büchberger's algorithm of modules from Chapter 4. Nonetheless, due to the works of C. Delorme [21], we can improve the previous method in the cuspidal case. The goal of this chapter is to explain in detail, and to generalize, the results of Delorme. To do so, we use the combinatorics introduced in Chapter 3. In relationship with the results from [9, 48], in the last section we show how to do all the computations with an implicit equation. Note that Büchberger's algorithm is thought to be used in principle with a primitive parametrization. These last results are used in Chapter 9, when discussing about the Bernstein-Sato polynomial.

5.1 Standard Bases for the Module of Differentials

We first start with an explanation on how standard bases can allow us to compute the semimodule of differential values. Before that, we have to give the relationship between standard bases and the semigroup of a branch. We are going to use usual order in $\mathbb{Z}_{\geq 0}$ as a monomial order.

Fix (C, P_0) a branch, with $\phi(t) = (x(t), y(t))$ a primitive parametrization. By Remark 1.2.4, we have that the $\mathbb{C}\{t\}$-subalgebra $\mathbb{C}\{x(t), y(t)\}$ is isomorphic to the local ring of C.

Assume that $G = \{h_0, h_1 \ldots, h_g\} \subset \mathbb{C}\{x(t), y(t)\}$ is a minimal standard basis of $\mathbb{C}\{x(t), y(t)\}$. Hence, given $g(t) \in \mathbb{C}\{x(t), y(t)\}$, we can write

$$\operatorname{ord}_t(g) = \sum_{h \in G} \gamma_h \operatorname{ord}_t(h), \quad \gamma_h \in \mathbb{Z}_{\geq 0}.$$

Recall that the parametrization ϕ induces a surjective map $\mathbb{C}\{x, y\} \to \mathbb{C}\{x(t), y(t)\}$. Thus we can write $g(t) = r \circ \phi(t)$ for some $r \in \mathbb{C}\{x, y\}$, that is, $\nu_C(r) = \sum_{h \in G} \gamma_h \operatorname{ord}_t(h)$. By the minimality of the standard basis, we have that the set $\{\operatorname{ord}_t(h) : h \in G\}$ is the minimal set of generators of the semigroup Γ_C of the curve C, as defined in Chapter 1. In other words, if the the elements of G are properly ordered, we have that $\operatorname{ord}_t(h_i) = \overline{\beta}_i$, for $i = 0, 1, \ldots, g$.

For example, assume that C is a cusp with Puiseux pair (n, m) and (x, y) is a system of adapted coordinates with respect to C. By Equation (1.8), we can write $\phi(t) = (t^n, v(t)t^m)$, where $v(t)$ is a unit in $\mathbb{C}\{t\}$. In this situation we have that $\Gamma_C = \langle n, m \rangle$. Then $\{x(t), y(t)\}$ is a minimal standard basis of $\mathbb{C}\{x(t), y(t)\}$.

Now return to the general case, where the branch C may no longer be a cusp, and take the module Ω^1_{C,P_0} of differentials of C defined in Section 1.5. Note that Ω^1_{C,P_0} can be seen as a $\mathbb{C}\{t\}$-module. Let $H = (\psi_{-1}, \psi_0, \ldots, \psi_s)$ be a minimal standard basis of the module of differentials Ω^1_{C,P_0}. By definition of standard basis, given an element $\psi \in \Omega^1_{C,P_0}$, we can write

$$\mathrm{ord}_t(\psi) = \mathrm{ord}_t(\psi_i) + \sum_{j=0}^{g} \alpha_j \overline{\beta}_j, \quad \text{for some } i \in \{-1, 0, \ldots, s\} \text{ and } \alpha_j \geq 0 \text{ for all } j.$$

By Remark 1.5.1, we have a surjection $\Omega^1_{M_0,P_0} \to \Omega_{C,P_0}$ given by the pull-back map ϕ^*. Recalling the definition of differential value $\nu_C(\omega)$ of a 1-form ω, introduced in Section 1.5, we have that $\mathcal{B} = (\mathrm{ord}_t(\psi_{-1}) + 1, \mathrm{ord}_t(\psi_0) + 1, \ldots, \mathrm{ord}_t(\psi_s) + 1)$ is the basis of the semimodule of differential values of C, assuming that the values are properly ordered. Note that for any $i \neq j$, we do not have $\mathrm{ord}_t(\psi_i) \notin \mathrm{ord}_t(\psi_j) + \Gamma_C$, otherwise, ψ_i is reducible modulo $(\{\psi_j\}, G)$. Hence, in order to find the basis of the semimodule of differentials of C, we only need to compute a minimal standard basis of Ω^1_{C,P_0}.

Furthermore, we can take a sequence of 1-forms $\mathcal{S} = (\omega_{-1}, \omega_0, \ldots, \omega_s)$ in $\Omega^1_{M_0,P_0}$, such that $\phi^*(\omega_i) = \psi_i$ for $i = -1, 0, \ldots, s$. As an abuse of notation, we say that $\mathcal{S}$ is a minimal standard basis of the module of differentials of C.

Note that a sequence like $\mathcal{S}$ can be defined by the property $\nu_C(\omega_i) = \lambda_i$ for $i = -1, 0, \ldots, s$, where λ_i is the i-element of the basis of the semimodule of differential values of C, see Section 3.1. In fact, from now on, we only use this notion of minimal standard basis of the module of differentials of C and never the original one. Moreover, we just write standard basis, instead of standard basis of an object, when the reference of the object is well understood. Since, in the last section of this chapter, we are going to compute standard bases of some ideal in $\mathbb{C}\{x, y\}$, we thought that it would be appropriate to give a brief explanation on why a series of two variables holomorphic 1-forms are called a standard basis of a curve. This treatment can be also found in [32].

5.2 Structure of the Semimodule of Differential Values

The structure of the semimodule of differential values of an irreducible curve C was described by Delorme in [21]. In this section, we use a different approach to describe the semimodule of differential values of a cusp. This approach will be useful when constructing a Saito basis of C in Chapter 8.

Fix C a cusp with Puiseux pair (n, m) and $n \geq 2$. Let $\Lambda_C = \Gamma_C(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ be the semimodule of differential values of C with

$$\Lambda_{-1} \subset \Lambda_0 \subset \ldots \subset \Lambda_s = \Lambda_C,$$

its decomposition sequence (see (3.1)). We select a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ of the cusp C. From now on, we fix a system of adapted coordinates (x, y) with respect to C, and D the cuspidal divisor of C.

Furthermore, as the coordinate system and the Puiseux pair are fixed, we are going to denote the initial parts as In, instead of $\mathrm{In}_{n,m;x,y}$, as defined in Chapter 2.

We begin by giving some results about the 1-forms ω_{-1} and ω_0.

Lemma 5.2.1. *We have that $\lambda_{-1} = n$ and $\lambda_0 = m$.*

Proof. Since (x, y) is system of adapted coordinates with respect to C, we can take a parametrization of C given by $(t^n, v(t)t^m)$, where $v(t)$ is a unit. Recall that $\nu_C(Adx + Bdy)$ is the order in t of the expression

$$nt^n A(t^n, v(t)t^m) + v(t)t^m B(t^n, v(t)t^m)\{m + tv'(t)/v(t)\}. \tag{5.1}$$

We see that this order is $\geq n$ and that $\nu_C(dx) = n$. Hence $n = \lambda_{-1}$. Moreover, the terms in Equation (5.1) of degree smaller than m come only from the first part $nt^n A(t^n, t^m v(t))$ of the sum and they are values in Γ_C. Since $m = \nu_C(dy)$, we conclude that $\lambda_0 = m$. □

Remark 5.2.2. The 1-forms ω_{-1}, ω_0 can be written as

$$\begin{aligned} \omega_{-1} &= h_{-1}dx + g_{-1}dy, \text{ with } h_{-1}(0,0) \neq 0; \\ \omega_0 &= h_0 dx + g_0 dy, \text{ with } g_0(0,0) \neq 0, \ \nu_C(h_0 dx) > m. \end{aligned}$$

Thus, we can write any differential 1-form ω in a unique way as $\omega = A\omega_{-1} + B\omega_0$ and we have that $\{\omega_{-1}, \omega_0\}$ is a basis of $\Omega^1_{M_0,P_0}$.

5.2.1 The Zariski's Invariant

In this subsection, we deal with properties of the 1-forms that give place to λ_1 as their differential value. In other words, we study the properties about the 1-form ω_1. In the next subsection we treat the behaviour of the 1-forms ω_i with $i = 2, \ldots, s$. This is done in terms of divisorial values. We cite the work of O. Gómez-Martínez [28] that essentially contains several of the results in this subsection.

Recall that $\lambda_1 = \min(\Lambda_C \setminus \Lambda_0)$. Since $\lambda_{-1} = n$ and $\lambda_0 = m$, we have that $\Lambda_0 = (n + \Gamma_C) \cup (m + \Gamma_C) = \Gamma_C \setminus \{0\}$ and we get that $\lambda_1 = \min(\Lambda_C \setminus \Gamma_C)$. In particular, we have that $\lambda_1 - n$ is the Zariski's invariant, see Section 1.5.2.

When studying the 1-forms associated to a minimal standard basis, we do it in terms of their initial parts. For this reason we introduce the concept of reachability that we proceed to define.

Given two 1-forms $\omega, \eta \in \Omega^1_{M_0,P_0}$, we say that η is *reachable* by ω if there exist a monomial $x^a y^b$ and a non zero constant μ, such that

$$\mathrm{In}(\eta) = \mu x^a y^b \,\mathrm{In}(\omega).$$

In the situation above, we get that $\nu_D(\eta - \mu x^a y^b \omega) > \nu_D(\eta)$.

Remark 5.2.3. Let ω be a 1-form with $\mathrm{In}(\omega) = nxdy - mydx$. If ω' is a resonant 1-form, then ω' is reachable by ω.

Many of the following proofs are based in computational techniques. Thus, in order to simplify the notations, we introduce the next convention.

Notation 5.2.4. Take two 1-forms $\omega, \omega' \in \Omega^1_{M_0,P_0}$, with the same differential value $\nu_C(\omega) = \nu_C(\omega') < \infty$. We will denote by μ^+ the unique non zero constant with the property that $\eta_1 = \omega + \mu^+\omega'$ satisfies that $\nu_C(\eta_1) > \nu_C(\omega)$. The constant μ^+ is called the *tuning constant*. We want to remark that in some occasions, we will write recursively $\eta_2 = \eta_1 + \mu^+\omega''$ for $\omega'' \in \Omega^1_{M_0,P_0}$ with $\nu_C(\omega'') = \nu_C(\eta_1)$. In this situation, the new tuning constant μ^+ constant may be different to the one we use when constructing η_1, but both of them will be denoted by μ^+ without any other indication.

Next result characterizes the 1-forms with differential value equal to λ_1.

Proposition 5.2.5. *We have the following properties:*

1. *If $s = 0$, then $\infty = \sup\{\nu_C(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_D(\omega) = n + m\}$.*
2. *If $s \geq 1$, then $\lambda_1 = \sup\{\nu_C(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_D(\omega) = n + m\} < \infty$.*

Proof. Assume that $s = 0$ and hence $\Lambda_C = \Lambda_0 = \Gamma_C \setminus \{0\}$. Let us consider the 1-form $\eta = x\omega_0 + \mu^+ y\omega_{-1}$, for the tuning constant μ^+. We have that $\nu_C(\eta) > n + m = \nu_D(\eta)$. Moreover, since $s = 0$, we have that $\nu_C(\eta) \in \Gamma_C$ and then there is a monomial function g such that

$$\nu_D(dg) = \nu_C(dg) = \nu_C(\eta) > n + m.$$

Consider the 1-form, such that $\eta^1 = \eta + \mu^+ dg$, we have that $\nu_D(\eta^1) = \nu_D(\eta) = n + m$ and $\nu_C(\eta^1) > \nu_C(\eta)$. We repeat the argument with η^1 and in this way we obtain 1-forms η^k with $\nu_D(\eta^k) = n + m$ and $\nu_C(\eta^k) \geq n + m + 1 + k$. This proves the first statement.

Assume now that $s \geq 1$. Let us first show that

$$\lambda_1 \leq \sup\{\nu_C(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_D(\omega) = n + m\}.$$

We put again $\eta = x\omega_0 + \mu^+ y\omega_{-1}$. If $\nu_C(\eta) \notin \Gamma_C$, we have that $\nu_C(\eta) \geq \lambda_1$ since λ_1 is the minimum of the differential values not in Γ_C. Assume that $\nu_C(\eta) \in \Gamma_C$ and hence

$$\nu_C(\eta) = na + mb > n + m.$$

Taking the function $g = x^a y^b$, we can consider $\eta_1 = \eta + \mu^+ dg$. Note that $\nu_D(\eta_1) = n + m$, since $\nu_D(dg) = na + mb > n + m$. We restart with η_1 instead of η, noting that $\nu_C(\eta) < \nu_C(\eta_1)$. Repeating finitely many times this procedure, we obtain a new 1-form $\tilde{\eta} = \eta - d\tilde{g}$ such that $\nu_D(\tilde{\eta}) = n + m$ and either $\nu_C(\tilde{\eta}) \geq c_\Gamma = (n-1)(m-1)$ or $\nu_C(\tilde{\eta}) \notin \Gamma_C$, in both cases we have that $\nu_C(\tilde{\eta}) \geq \lambda_1$.

It remains to show that $\lambda_1 \geq \sup\{\nu_C(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_D(\omega) = n + m\}$. Consider the 1-form ω_1 from $\mathcal{S}$, we recall that $\nu_C(\omega_1) = \lambda_1$. Let us show that it is not possible to have $\tilde{\omega}$ such that $\nu_D(\tilde{\omega}) = n + m$ and $\nu_C(\tilde{\omega}) > \nu_C(\omega_1)$. In this situation, both ω_1 and $\tilde{\omega}$ are basic and resonant, see Proposition 2.3.14. By Remark 5.2.3, the 1-form ω_1 is reachable by $\tilde{\omega}$ and thus there is a constant μ and $a, b \geq 0$ such that the 1-form $\omega_1^1 = \omega_1 - \mu x^a y^b \tilde{\omega}$ satisfies that

$$\nu_D(\omega_1^1) > \nu_D(\omega_1),$$

and we have that $\nu_C(\omega_1^1) = \nu_C(\omega_1) = \lambda_1$. We restart with the pair $\omega_1^1, \tilde{\omega}$. In this way, we obtain an infinite sequence of 1-forms $\omega_1, \omega_1^1, \omega_1^2, \ldots$ with strictly increasing divisorial values. Up to a finite number of steps, we find an index k such that $\nu_D(\omega_1^k) > \lambda_1 = \nu_C(\omega_1^k)$. This contradicts the fact $\nu_C(\omega_1^k) \geq \nu_D(\omega_1^k)$. □

Corollary 5.2.6. *Any 1-form $\omega \in \Omega^1_{M_0,P_0}$ such that $\nu_D(\omega) = n + m$ and $\nu_C(\omega) \notin \Gamma_C$ satisfies that $\nu_C(\omega) = \lambda_1$.*

Proof. In view of the previous result, we have that $\nu_C(\omega) \leq \lambda_1$. Since $\nu_C(\omega) \notin \Gamma_C$, we also have that $\nu_C(\omega) \geq \lambda_1$. □

Corollary 5.2.7. *Any 1-form $\omega \in \Omega^1_{M_0,P_0}$ such that $\nu_C(\omega) = \lambda_1$ satisfies that $\nu_D(\omega) = n + m$.*

Proof. In virtue of Proposition 5.2.5, there exists a 1-form η such that $\nu_C(\eta) = \lambda_1$ and $\nu_D(\eta) = n+m$. Assume that

$$\nu_D(\omega) > n + m$$

in order to obtain a contradiction. Since $\lambda_1 \notin \Gamma_C$, both ω and η are basic and resonant. By Remark 5.2.3, the 1-form ω is reachable by η. Then there is a function g with $\nu_C(g) > 0$ such that

$$\nu_D(\omega - g\eta) > \nu_D(\omega).$$

Put $\omega^1 = \omega - g\eta$. Since $\nu_C(g\eta) > \lambda_1$, we have that $\nu_C(\omega^1) = \lambda_1$. We restart with the pair ω^1, ω. After finitely many repetitions we find ω^k with $\nu_C(\omega^k) = \lambda_1$ and $\nu_D(\omega^k) > \lambda_1$, this is a contradiction. □

Note that Corollary 5.2.7 applies to ω_1, that is, $\nu_D(\omega_1) = n + m$. The following two lemmas are necessary steps in order to prove an inductive version of Proposition 5.2.5 valid for all indices $i = 1, 2, \ldots, s$:

Lemma 5.2.8. *Assume that $s \geq 1$ and consider an integer number $k = na + mb + \lambda_1 \in \lambda_1 + \Gamma_C$. The following statements are equivalent:*

1. $k \notin \Gamma_C$.
2. $\nu_D(\omega) \leq \nu_D(x^a y^b \omega_1)$ *for any* $\omega \in \Omega^1_{M_0,P_0}$ *such that* $\nu_C(\omega) = k$.

Proof. Note that $k = \nu_C(x^a y^b \omega_1) > n(a+1) + m(b+1) = \nu_D(x^a y^b \omega_1)$.

Assume that $k \in \Gamma_C$, then $k = na' + mb' > \nu_D(x^a y^b \omega_1)$. Taking $\omega = d(x^{a'} y^{b'})$, we have $\nu_C(\omega) = \nu_D(\omega) = k > \nu_D(x^a y^b \omega_1)$.

Now assume that $k \notin \Gamma_C$. Let us reason by contradiction assuming that there is ω with $\nu_C(\omega) = k$ and $\nu_D(\omega) > \nu_D(x^a y^b \omega_1)$. We have that ω is basic and resonant since $\nu_C(\omega) \notin \Gamma_C$. Then ω is reachable by ω_1, that is, there are $a', b' \geq 0$ and a constant μ such that $\nu_D(x^{a'} y^{b'} \omega_1) = \nu_D(\omega)$ and

$$\nu_D(\omega - \mu x^{a'} y^{b'} \omega_1) > \nu_D(\omega) > \nu_D(x^a y^b \omega_1).$$

Since $na' + mb' > na + mb$, we have that $\nu_C(x^{a'} y^{b'} \omega_1) > k$ and hence $\nu_C(\omega^1) = k$, where $\omega^1 = \omega - \mu x^{a'} y^{b'} \omega_1$. Repeating the procedure with the pair ω^1, ω_1, we obtain a sequence

$$\omega, \omega^1, \omega^2, \ldots$$

with strictly increasing divisorial value and such that $\nu_C(\omega^j) = k$ for any j. This is a contradiction. □

Next lemma describes the divisorial value $\nu_D(\omega)$ of the 1-forms ω whose differential values are not in the semigroup Γ_C of C.

Lemma 5.2.9. *Let $\omega \in \Omega^1_{M_0,P_0}$ be a 1-form such that $\nu_C(\omega) = \lambda \notin \Gamma_C$. There are unique $a, b \geq 0$ such that $\nu_D(\omega) = \nu_D(x^a y^b \omega_1)$. Moreover, we have that $\lambda \geq na + mb + \lambda_1$.*

Proof. Note that ω is basic and resonant and thus, by Remark 5.2.3, the existence and uniqueness of a, b is assured. Moreover, if $\lambda < na + mb + \lambda_1$, we can find a constant μ such that

$$\nu_D(\omega - \mu x^a y^b \omega_1) > \nu_D(x^a y^b \omega_1)$$

and $\nu_C(\omega - \mu x^a y^b \omega_1) = \lambda$. Put $\omega^1 = \omega - \mu x^a y^b \omega_1$, we have that $\nu_C(\omega^1) = \lambda \notin \Gamma_C$. As before, we get that

$$\nu_D(\omega^1) = \nu_D(x^{a_1} y^{b_1} \omega_1), \quad \text{with} \quad na_1 + mb_1 > na + mb$$

and thus $\lambda < na_1 + mb_1 + \lambda_1$. We repeat the process with the pair ω^1, ω_1, in order to have a sequence $\omega, \omega^1, \omega^2, \ldots$ with strictly increasing divisorial values and such that $\nu_C(\omega^j) = \lambda$ for any j. This is a contradiction. □

5.2.2 General Case

In the previous subsection, we have shown that $\nu_D(\omega_1) = t_1 = n + m$, where t_1 is the critical value introduces in Section 3.2. Here we prove the 1-forms ω with differential value $\nu_C(\omega) = \lambda_i$ have divisorial value $\nu_D(\omega)$ equal to the critical value t_i.

Theorem 5.2.10. *For each $1 \leq i \leq s$ we have the following statements*

1. *$\lambda_i = \sup\{\nu_C(\omega) : \nu_D(\omega) = t_i\}$.*
2. *If $\nu_C(\omega) = \lambda_i$, then $\nu_D(\omega) = t_i$.*
3. *For each 1-form ω with $\nu_C(\omega) \notin \Lambda_{i-1}$, there is a unique pair $a, b \geq 0$ such that $\nu_D(\omega) = \nu_D(x^a y^b \omega_i)$. Moreover, we have that $\nu_C(\omega) \geq \lambda_i + na + mb$.*
4. *We have that $\lambda_i > u_i$.*
5. *Let $k = \lambda_i + na + mb$, then $k \notin \Lambda_{i-1}$, if and only if for all ω such that $\nu_C(\omega) = k$, we have that $\nu_D(\omega) \leq \nu_D(x^a y^b \omega_i)$.*

In particular, the semimodules Λ_i are increasing, for $i = 1, 2, \ldots, s$.

Proof. Assume that $i = 1$ and then $t_1 = n + m = u_1$. We have

- Statement *1* is proven in Proposition 5.2.5.
- Statement *2* is proven in Corollary 5.2.7.
- Statement *3* is proven in Lemma 5.2.9.
- Statement *4* follows from the fact that $\lambda_1 > n + m = \nu_D(\omega_1) = u_1$.
- Statement *5* is proven in Lemma 5.2.8.

Now, consider $i \geq 2$ and assume by the induction hypothesis that the Statements *1-5* are true for indices ℓ with $1 \leq \ell \leq i - 1$.

As in many proofs in Chapter 3, we have two cases: either $u_i = u_i^n$ or $u_i = u_i^m$ (we follow the notations established there). Assume that $u_i = u_i^n = \lambda_{i-1} + n\ell_i^n$. The computations in the case $u_i = u_i^m = \lambda_{i-1} + m\ell_i^m$ are similar ones.

The proof is founded in three claims as follows:

• *Claim 1: There is a 1-form η with $\nu_D(\eta) = t_i$, whose initial part is proportional, by a constant, to the initial part of $x^{\ell_i^n}\omega_{i-1}$ and such that either $\nu_C(\eta) \geq c_\Gamma$ or $\nu_C(\eta) \notin \Lambda_{i-1}$.*

• *Claim 2: Any 1-form ω with $\nu_C(\omega) \notin \Lambda_{i-1}$ is reachable by $x^{\ell_i^n}\omega_{i-1}$.*

• *Claim 3: Let η be a 1-form such that $\nu_D(\eta) = t_i$ whose initial part is proportional to the initial part of $x^{\ell_i^n}\omega_{i-1}$ and such that either $\nu_C(\eta) \geq c_\Gamma$ or $\nu_C(\eta) \notin \Lambda_{i-1}$. Then $\nu_C(\eta) = \lambda_i$.*

We recall to the reader that the notion "initial part" refers to the concept of weighted initial part defined in Section 2.3.

Proof of Claim 1: Recall that $t_i = \nu_D(\omega_{i-1}) + u_i - \lambda_{i-1} = \nu_D(\omega_{i-1}) + n\ell_i^n$. Let us start with $\eta_1 = x^{\ell_i^n}\omega_{i-1}$. We have that

$$\nu_D(\eta_1) = n\ell_i^n + \nu_D(\omega_{i-1}) = t_i, \quad \nu_C(\eta_1) = n\ell_i^n + \lambda_{i-1} = u_i \in \Lambda_{i-2}.$$

By Statement *5* applied to $\nu_C(\eta_1) \in \Lambda_{i-2}$, there is η_1' with $\nu_C(\eta_1') = \nu_C(\eta_1)$ and $\nu_D(\eta_1') > \nu_D(\eta_1)$. Since $\nu_C(\eta_1') = \nu_C(\eta_1)$, then we can write

$$\tilde{\eta} = \eta_1 + \mu^+\eta_1' \quad \text{with} \quad \nu_C(\tilde{\eta}) > \nu_C(\eta_1) = u_i.$$

Recall that μ^+ denotes the tuning constant. Since $\nu_D(\eta_1') > \nu_D(\eta_1)$, we have that $\nu_D(\tilde{\eta}) = \nu_D(\eta_1) = t_i$ and the initial part of $\tilde{\eta}$ is the same one as the initial part of $\eta_1 = x^{\ell_i^n}\omega_{i-1}$. If $\nu_C(\tilde{\eta}) \geq c_\Gamma$ or $\nu_C(\tilde{\eta}) \notin \Lambda_{i-1}$, then $\eta = \tilde{\eta}$ is the 1-form we were looking for. Assume that $\nu_C(\tilde{\eta}) \in \Lambda_{i-1}$. Let us write

$$\nu_C(\tilde{\eta}) = na + mb + \lambda_\ell, \quad \ell \leq i - 1.$$

Let us see that $\nu_D(\tilde{\eta}) < \nu_D(x^a y^b \omega_\ell)$; this is equivalent to verify that $t_i - t_\ell < na + mb$. Since $\nu_C(\tilde{\eta}) > u_i$, in view of Lemma 3.2.8 we have

$$na + mb > u_i - \lambda_\ell = n\ell_i^n + \lambda_{i-1} - \lambda_\ell \geq n\ell_i^n + t_{i-1} - t_\ell = t_i - t_\ell.$$

On the other hand, we have that $\nu_C(\tilde{\eta}) = \nu_C(x^a y^b \omega_\ell)$. Thus, writing $\tilde{\eta}_1 = \tilde{\eta} + \mu^+ x^a y^b \omega_\ell$, we have that $\nu_C(\tilde{\eta}_1) > \nu_C(\tilde{\eta})$ and $\nu_D(\tilde{\eta}_1) = \nu_D(\tilde{\eta})$. Moreover, the initial part of $\tilde{\eta}_1$ is the same one as the initial part of $x^{\ell_i^n}\omega_{i-1}$.

If $\nu_C(\tilde{\eta}_1) \in \Lambda_{i-1}$, we repeat the procedure starting with $\tilde{\eta}_1$, to obtain $\tilde{\eta}_2$ such that $\nu_C(\tilde{\eta}_2) > \nu_C(\tilde{\eta}_1)$ and $\nu_D(\tilde{\eta}_2) = t_i$. After finitely many repetitions, we get a 1-form η such that $\nu_D(\eta) = t_i$, whose initial part is the same one as the one of $x^{\ell_i^n}\omega_{i-1}$ and either $\nu_C(\eta) \geq c_\Gamma$ or $\nu_C(\eta) \notin \Lambda_{i-1}$. This proves Claim 1.

Proof of Claim 2: Take ω such that $\lambda = \nu_C(\omega) \notin \Lambda_{i-1}$. Note that $\lambda \notin \Lambda_{i-2}$. By Statement 3, we have that ω is reachable by ω_{i-1}. Thus, there are $a, b \geq 0$ and a constant μ such that

$$\nu_D(\omega - \mu x^a y^b \omega_{i-1}) > \nu_D(\omega) = \nu_D(x^a y^b \omega_{i-1}) = na + mb + t_{i-1}.$$

and moreover, we have that $\lambda = \nu_C(\omega) > na + mb + \lambda_{i-1} = k$ (note that $\lambda \neq k$ since $\lambda \notin \Lambda_{i-1}$).

Consider the 1-form $\omega' = \omega - \mu x^a y^b \omega_{i-1}$. We know that

$$\nu_C(\omega') = k, \quad \nu_D(\omega') > \nu_D(x^a y^b \omega_{i-1}).$$

By Statement 5, we conclude that $k \in \Lambda_{i-2}$. Therefore, by applying Lemma 3.2.5, we have that either $a \geq \ell_i^n$ or $b \geq \ell_i^m$. Let us show that we necessarily have that $a \geq \ell_i^n$, this would imply that ω' is reachable by $x^{\ell_i^n}\omega_{i-1}$. Assume that $b \geq \ell_i^m$. By Statement 4, we know that Λ_{i-1} is an increasing semimodule. By Corollary 3.4.7, we know that $k = \tilde{u}_i \geq c_{\Lambda_{i-1}} + n$. Since $\nu_C(\omega) = \lambda > k$, we have a contradiction with the assumption that $\lambda \notin \Lambda_{i-1}$. Therefore $a \geq \ell_i^n$. This ends the proof of Claim 2.

Proof of Claim 3: The conditions which satisfy $\nu_C(\eta)$ imply that $\nu_C(\eta) \geq \lambda_i$. Assume that $\lambda = \nu_C(\eta) > \lambda_i$. Recalling that $\nu_C(\omega_i) = \lambda_i \notin \Lambda_{i-1}$ and that the initial part of η is proportional to the initial part of $x^{\ell_i^n}\omega_{i-1}$, we can apply Claim 2 and we get that ω_i is reachable by η. Then there are $a, b \geq 0$ and a constant μ such that $\nu_D(\omega_i - \mu x^a y^b \eta) > \nu_D(\omega_i)$. Put $\omega_i^1 = \omega_i - \mu x^a y^b \eta$. We have that $\nu_C(\omega_i^1) = \lambda_i$ since $\nu_C(\mu x^a y^b \eta) \geq \lambda > \lambda_i$. In this way we produce an infinite list of 1-forms with strictly increasing divisorial value

$$\omega_i = \omega_i^0, \omega_i^1, \omega_i^2, \ldots$$

such that $\nu_C(\omega_i^j) = \lambda_i$, for any $j \geq 0$. Therefore, we exists an index j we such that $\nu_D(\omega_i^j) \geq c_\Gamma$ and then $\lambda_i \geq \nu_D(\omega_i^j) \geq c_\Gamma$ and this is a contradiction. So we necessarily have that $\nu_C(\eta) = \lambda_i$. This ends the proof of Claim 3.

Proof of Statements 1 and 2: In view of Claim 1 and Claim 3, there is a 1-form η with $\nu_D(\eta) = t_i$ such that $\nu_C(\eta) = \lambda_i$ and whose initial part is proportional to the initial part of $x^{\ell_i^n}\omega_{i-1}$. In order to prove Statement 1, it remains to prove that if $\nu_D(\omega) = t_i$ then $\nu_C(\omega) \leq \lambda_i$. Assume that $\lambda = \nu_C(\omega) > \lambda_i = \nu_C(\eta)$. The 1-form ω is basic and resonant and it has the same divisorial value as η. Hence there is a constant $\mu \neq 0$ such that the 1-form $\eta^1 = \eta - \mu\omega$ verifies that

$$\nu_D(\eta^1) > t_i = \nu_D(\eta) = \nu_D(\omega).$$

The 1-form η^1 satisfies that $\nu_C(\eta^1) = \lambda_i \notin \Lambda_{i-1}$. By Claim 2, there are $a, b \geq 0$ and a constant μ' such that

$$\nu_D(\eta^2) > \nu_D(\eta^1), \quad \text{with} \quad \eta^2 = \eta^1 - \mu' x^a y^b \eta.$$

We have that $\nu_C(\eta^2) = \lambda_i$ and $\nu_D(\eta^2) > \nu_D(\eta^1)$. Repeating this procedure, we have a list of 1-forms $\eta^1, \eta^2, \ldots$ with strictly increasing divisorial value such that $\nu_C(\eta^j) = \lambda_i$ for any j. We find a contradiction just by considering one of such η^j with $\nu_D(\eta^j) \geq c_\Gamma$. This ends the proof of Statement *1*.

Let us prove Statement 2. Choose ω with $\nu_C(\omega) = \lambda_i$. By Claim 2, we have that ω is reachable by η and hence $\nu_D(\omega) \geq t_i$. Assume by contradiction that $\nu_D(\omega) > t_i$. There is a constant μ and $a, b \geq 0$ with $a + b \geq 1$ such that

$$\nu_D(\omega^1) > \nu_D(\omega), \quad \text{where} \quad \omega^1 = \omega - \mu x^a y^b \eta.$$

Since $\nu_C(\mu x^a y^b \eta) = na + mb + \lambda_i > \lambda_i$, we have that $\nu_C(\omega^1) = \lambda_i$. Repeating the argument, we get a sequence of 1-forms $\omega^0 = \omega, \omega^1, \ldots$ with strictly increasing divisorial value such that $\nu_C(\omega^j) = \lambda_i$ for any j. This is a contradiction.

Proof of Statement 3: By Claim 2, we have that ω_i is reachable by $x^{\ell_i^n} \omega_{i-1}$. By Statement *2* (already proved) we have that $\nu_D(\omega_i) = t_i$. Hence the initial part of ω_i is proportional to the initial part of $x^{\ell_i^n} \omega_{i-1}$. Consider a 1-form ω with $\nu_C(\omega) \notin \Lambda_{i-1}$. By Claim 2 the 1-form ω is reachable by $x^{\ell_i^n} \omega_{i-1}$ and hence it is reachable by ω_i. Then, there are $a, b \geq 0$ such that

$$\nu_D(x^a y^b \omega_i) = na + mb + t_i = \nu_D(\omega).$$

Since $\nu_C(\omega) \notin \Lambda_{i-1}$, we have that $nm > \nu_C(\omega) > \nu_D(\omega) > na + mb$, this implies the uniqueness of a, b. Let us show that $\nu_C(\omega) \geq na + mb + \lambda_i$. Assume by contradiction that $\nu_C(\omega) < an + bm + \lambda_i$. Consider $\omega^1 = \omega - \mu x^a y^b \omega_i$ such that $\nu_D(\omega^1) > \nu_D(\omega)$. In view of the hypothesis about $\nu_C(\omega)$, we have that $\nu_C(\omega^1) = \nu_C(\omega)$. Moreover, if $\nu_D(\omega^1) = \nu_D(x^{a_1} y^{b_1} \omega_i)$ we also have that $\nu_C(\omega) < na_1 + mb_1 + \lambda_i$. The situation repeats and we obtain an infinite sequence of 1-forms $\omega^0 = \omega, \omega^1, \omega^2, \ldots$ with strictly increasing divisorial values, such that $\nu_C(\omega^j) = \nu_C(\omega)$ for any $j \geq 0$. This is a contradiction.

Proof of Statement 4: Noting that $\nu_D(x^{\ell_i^n} \omega_{i-1}) = t_i$, by Statement *1* we have $\lambda_i \geq \nu_C(x^{\ell_i^n} \omega_{i-1}) = n\ell_i^n + \lambda_{i-1} = u_i$. On the other hand, since $\lambda_i \notin \Lambda_{i-1}$, we have that $\lambda_i \neq u_i$ and hence $\lambda_i > u_i$.

Proof of Statement 5: Consider $k = \lambda_i + na + mb$. Assume first that $k \notin \Lambda_{i-1}$. Let ω be such that $\nu_C(\omega) = k$. We have to prove that

$$\nu_D(\omega) \leq \nu_D(x^a y^b \omega_i) = na + mb + t_i.$$

In view of Statement *3*, we know that ω is reachable by ω_i. Hence there are $a', b' \geq 0$ and a constant μ such that $\nu_D(\omega - \mu x^{a'} y^{b'} \omega_i) > \nu_D(\omega)$. Hence

$$\nu_D(\omega) = \nu_D(x^{a'} y^{b'} \omega_i) = na' + mb' + t_i.$$

Assume by contradiction that $\nu_D(\omega) > \nu_D(x^a y^b \omega_i) = na + mb + t_i$. This implies that $na' + mb' > na + mb$ and thus

$$\nu_C(x^{a'} y^{b'} \omega_i) = na' + mb' + \lambda_i > k = na + mb + \lambda_i = \nu_C(x^a y^b \omega_i) = \nu_C(\omega).$$

Put $\omega^1 = \omega - \mu x^{a'} y^{b'} \omega_i$. We have that $\nu_C(\omega^1) = k$. Repeating the argument with ω^1, we obtain an infinite list of 1-forms $\omega^0 = \omega, \omega^1, \omega^2, \ldots$ with increasing divisorial values and such that $\nu_C(\omega^j) = k \notin \Lambda_{i-1}$. This is a contradiction.

Assume now that $k \in \Lambda_{i-1}$. There is an index $\ell \leq i - 1$ such that

$$k = na + mb + \lambda_i = na' + mb' + \lambda_\ell.$$

By Lemma 3.2.8, we have that $\lambda_i - \lambda_\ell > t_i - t_\ell$ and hence $na + mb + t_i < na' + mb' + t_\ell$. The 1-form $x^{a'} y^{b'} \omega_\ell$ satisfies that $k = \nu_C(x^{a'} y^{b'} \omega_\ell)$ and

$$\nu_D(x^{a'} y^{b'} \omega_\ell) = na' + mb' + t_\ell > na + mb + t_i = \nu_D(x^a y^b \omega_i).$$

This ends the proof. □

As a consequence of the previous theorem and its proof, we have the following result about the initial parts of the elements of a minimal standard basis.

Corollary 5.2.11. *For each $1 \leq i \leq s$, the 1-forms ω_i are basic and resonant. In particular, for the adapted coordinate system (x, y), the initial parts can be written as*

$$\mathrm{In}(\omega_i) = \mu_i x^{c_i} y^{d_i} \left(m \frac{dx}{x} - n \frac{dy}{y} \right), \quad nc_i + md_i = \nu_D(\omega_i) = t_i < nm.$$

Proof. Applying Lemma 3.2.8, since Λ_C is increasing, we have that

$$\nu_D(\omega_i) = t_i < \lambda_i = \nu_C(\omega_i) < nm.$$

The statement follows from these inequalities, and Proposition 2.3.14. □

Remark 5.2.12. From the previous corollary and Proposition 2.5.1, we have that the 1-forms ω_i, with $i \geq 1$, are totally D-dicritical.

5.2.3 Delorme's Algorithm

We end this section presenting an algorithm that computes minimal standard bases of the module of differential of a cusp. This method is based in the previous Theorem 5.2.10.

Following the ideas and the notation from Chapter 4, given $\eta \in \Omega^1_{M_0,P_0}$, $\mathcal{S} = \{\theta_1, \ldots, \theta_k\} \subset \Omega^1_{M_0,P_0}$ and $G = \{g_1, \ldots, g_{k'}\} \subset \mathcal{O}_{M_0,P_0}$. We say that η' is reduction of η modulo $(\mathcal{S}, G)$, or just $\mathcal{S}$ if there is no confusion with G, if $\eta' = \eta + \mu^+ G^\alpha \theta_i$, where μ^+ is the tuning constant, $\nu_C(G^\alpha \theta_i) = \nu_C(\eta) < \infty$ and G^α is a G-product.

Similarly, if there is a sequence

$$\eta = \eta_0 \to \eta_1 \to \eta_2 \to \ldots \to \eta_q \to \ldots$$

such that η_i is reduction modulo $(\mathcal{S}, G)$ of η_{i-1}, for $i \geq 1$, we say that any η_i is a partial reduction of η. Moreover, if we denote by η_∞ the limit of that sequence, we say that η_∞ is a final reduction of η, that is, η_∞ is no longer reducible modulo $(\mathcal{S}, G)$.

Delorme's Algorithm

INPUT: A cusp C and $G = \{f_{-1}, f_0\}$ with $\nu_C(f_{-1}) = n$ and $\nu_C(f_0) = m$.
OUTPUT: $\mathcal{S}$ minimal standard basis of the cusp C.
START:

Put $\mathcal{S}_0 = \{\omega_{-1} = df_{-1}, \omega_0 = df_0\}$, $i = 1$, $\mathcal{B}_0 = (\lambda_{-1} = n, \lambda_0 = m)$ and $\Lambda_{-1} = (n + \Gamma)$.

loop {

Compute $u_i = \min\{(\lambda_{i-1} + \Gamma) \cap \Lambda_{i-2}\}$ and put $\Lambda_{i-1} = \Lambda_{i-2} \cup (\lambda_{i-1} + \Gamma)$.

Find $k < i - 1$, h, h' G–products such that $\nu_C(h\omega_{i-1}) = \nu_C(h'\omega_k) = u_i$.

Put $\eta = h\omega_{i-1} + \mu^+ h'\omega_k$.

Compute η_∞ a final reduction of η modulo $\mathcal{S}_{i-1}$.

if $\nu_C(\eta_\infty) = \infty$ **then**:

Put $\mathcal{S} = \mathcal{S}_{i-1}$ and Return.

otherwise:

Put $\omega_i = \eta_\infty$ and $\mathcal{S}_i = \mathcal{S}_{i-1} \cup \{\omega_i\}$.

Put $\lambda_i = \nu_C(\omega_i)$, $\mathcal{B}_i = (n, m, \lambda_1, \ldots, \lambda_i)$ and $i = i + 1$.

} **end loop**

Recall that in a system of adapted coordinates (x, y) with respect to the cusp C, we can put $f_{-1} = x$ and $f_0 = y$. The previous algorithm works because of the following ideas: first, by means of Lemma 3.2.8, we have that $\nu_D(\eta) = t_i$, and that no partial reduction of η has a different divisorial value. Second, by Theorem 5.2.10 we have that the i-element of the basis of the module of differential values λ_i, in case it exists, it is greater than the axis u_i. Moreover, we also have that $\lambda_i = \sup\{\nu_C(\omega) : \nu_D(\omega) = t_i\} = \min(\Lambda_C \setminus \Lambda_{i-1})$. Thus, if η has a non trivial final reduction η_∞ modulo $\mathcal{S}_{i-1} = \{\omega_{-1}, \omega_0, \ldots, \omega_{i-1}\}$, then it must be that $\nu_C(\eta_\infty) = \lambda_i$. Otherwise, it implies that the i-element of the basis does not exist.

Notice that the condition on finding a trivial final reduction can be simplified by just demanding to compute a partial reduction η_r modulo $\mathcal{S}_{i-1}$, such that $\nu_C(\eta_r) \geq c_{\Lambda_{i-1}}$. This implies that η_r is again reducible modulo $\mathcal{S}_{i-1}$.

This algorithm is an adapted version of the one presented in Chapter 4 when computing standard bases of modules. The Büchberger's algorithm for modules can be used with any branch, however, as mentioned, the algorithm stated above is specific for cusps. The main problem on giving a generalization of the main results in this work is to adapt the Büchberger's algortithm to more general branches.

We now give an example of application of the previous algorithm.

Example 5.2.13. We consider the cusp C defined by the primitive parametrization

$$\phi(t) = (t^5, t^{11} + t^{12} + 7t^{13}) = (x(t), y(t)).$$

We notice that the Puiseux pair of C is $(5, 11)$ and that (x, y) is an adapted system of coordinates with respect to C, see Section 1.6. We are going to compute a minimal standard basis of C.

We start by putting $\omega_{-1} = dx$ and $\omega_0 = dy$. Note that $\phi^* dx = 5t^4 dt$ and $\phi^* dy = (11t^{10} + 12t^{11} + 91t^{12})dt$. It follows that $\nu_C(dx) = 5 = \lambda_{-1}$ and $\nu_C(dy) = 11 = \lambda_0$, as we already knew.

This example will also show how the results from Chapter 3 can be used to simplify the computations. We need to find u_1. As we saw in Example 3.5.4, we have that $u_1 = n + m = 16$. Moreover, we recall that the 0-left and right bounds are $k_0^n = -1 = k_0^m$, see Example 3.5.4. We need to compute the smallest positive solutions of these equations.

$$\begin{aligned} u_1^n &= \lambda_0 + n\ell_1^n = \lambda_{-1} + mb_1 = 11 + n\ell_1^n = 5 + mb_1 \\ u_1^m &= \lambda_0 + m\ell_1^m = \lambda_{-1} + na_1 = 11 + m\ell_1^m = 5 + na_1 \end{aligned}$$

We deduce that $\ell_1^n = 1 = b_1$, $\ell_1^m = 4$ and $a_1 = 10$. Therefore, we obtain $16 = u_1 = u_1^n < u_1^m = \tilde{u}_1 = 55$. By Proposition 3.5.9, the new bounds are $k_1^n = 0$ and $k_1^m = -1$.

Hence, we have that $\nu_C(x\omega_0) = \nu_C(y\omega_{-1}) = 16$. By taking the pull-back by ϕ, we obtain that

$$\begin{aligned} \phi^*(x\omega_0) &= (11t^{15} + 12t^{16} + 91t^{17})dt \\ \phi^*(y\omega_{-1}) &= (5t^{15} + 5t^{16} + 35t^{17})dt. \end{aligned}$$

If we put $\eta = x\omega_0 - 11/5y\omega_{-1}$, we have that $\phi^*\eta = (t^{16} + 14t^{17})dt$. Thus, $\nu_C(\eta) = 17$. We note that $17 \notin (5 + \Gamma_C) \cup (11 + \Gamma_C)$. We add $\omega_1 = x\omega_0 - 11/5y\omega_{-1}$ to our candidate set of minimal standard basis, and $\lambda_1 = 17$ to the basis of the semimodule of differential values. Now, we repeat the process, we start by computing the axis u_2.

$$\begin{aligned} u_2^n &= \lambda_1 + n\ell_2^n = \lambda_0 + mb_2 = 17 + n\ell_2^n = 11 + mb_2 \\ u_2^m &= \lambda_1 + m\ell_2^m = \lambda_{-1} + na_2 = 17 + m\ell_2^m = 5 + na_2. \end{aligned}$$

We obtain that $\ell_2^n = 1 = b_2$, $\ell_2^m = 3$ and $a_2 = 9$. This implies that $22 = u_2 = u_2^n < u_2^m = \tilde{u}_2 = 50$. Again, by Proposition 3.5.9, we have that $k_2^n = 1$ and $k_2^m = -1$.

The previous computation shows that $\nu_C(x\omega_1) = \nu_C(y\omega_0) = 22$.

$$\begin{aligned} \phi^*(x\omega_1) &= (t^{21} + 14t^{22})dt \\ \phi^*(y\omega_0) &= (11t^{21} + 23t^{22} + 180t^{23} + 175t^{24} + 637t^{25})dt. \end{aligned}$$

By putting $\eta = x\omega_1 - 1/11y\omega_0$, we see that $\phi^*\eta = (131/11t^{22} + h.o.t.)dt$, that is, $\nu_C(\eta) = 23$. As before, we notice that $23 \notin (5 + \Gamma_C) \cup (11 + \Gamma_C) \cup (17 + \Gamma_C)$. Hence we put $\omega_2 = \eta$ and $\lambda_2 = 23$. A new iteration shows that

$$\begin{aligned} u_3^n &= 23 + 5\ell_3^n = 17 + mb_3 \longrightarrow u_3^n &= 28 \\ u_3^m &= 23 + 11\ell_2^3 = 5 + 5a_2 \longrightarrow u_3^m &= 45. \end{aligned}$$

Then $\nu_C(x\omega_2) = \nu_C(y\omega_1) = 28$. We check that $\omega_3 = x\omega_2 - 131/11y\omega_1$ satisfies that $\nu_C(\omega_3) = 29$ which does not belong to $(5 + \Gamma_C) \cup (11 + \Gamma_C) \cup (17 + \Gamma_C) \cup (23 + \Gamma_C)$.

We could check that the algorithm stops by computing a new 1-form with the previous procedure whose final reduction gives infinite differential value. However, we also recall that the length of the basis of a semimodule is bounded by $n - 2$. Since we have computed up to the 3-element of the basis of Λ_C, we can conclude that the algorithm has finished and $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \omega_2, \omega_3)$ is a minimal standard basis of C, and $\mathcal{B} = (5, 11, 17, 23, 29)$ is the basis of the semimodule of differential values Λ_C.

5.3 Delorme's Decomposition

In this section we prove the following result:

Theorem 5.3.1. *Consider indices $0 \leq j \leq i \leq s$ and $* \in \{n, m\}$. Take ω a 1-form such that $\nu_D(\omega) = t^*_{i+1}$ and $\nu_C(\omega) > u^*_{i+1}$. Then, there is a decomposition of the 1-form ω given by*

$$\omega = \sum_{\ell=-1}^{j} f_\ell^{ij} \omega_\ell, \tag{5.2}$$

such that the following properties hold. Let v^*_{ij} be defined by $v^*_{ij} = \nu_C(f_j^{ij}\omega_j)$. Then we obtain:

1. *$v^*_{ij} = \min\{\nu_C(f_\ell^{ij}\omega_\ell);\ -1 \leq \ell < j\}$.*
2. *$v^*_{ij} = \lambda_j + t^*_{i+1} - t_j$, in particular, if $j = i$ we have that $v^*_{ii} = \lambda_i + t^*_{i+1} - t_i = u^*_{i+1}$.*
3. *If $j < i$, we have that $\nu_C(f_\ell^{ij}\omega_\ell) = v^*_{ij}$, for $\ell = k_j$ and $\nu_C(f_\ell^{ij}\omega_\ell) > v^*_{ij}$, for any $\ell \neq k_j$ and $-1 \leq \ell < j$.*
4. *If $j = i$, we have that $\nu_C(f_\ell^{ii}\omega_\ell) = v^*_{ii}$, for $\ell = k_j^*$ and $\nu_C(f_\ell^{ii}\omega_\ell) > v^*_{ii}$, for any $\ell \neq k_j^*$ and $-1 \leq \ell < j$.*

We recall that k_i makes reference to the bounds introduced in Section 3.5.

A decomposition of a 1-form ω as the one in Theorem 5.3.1 is called a *Delorme's decomposition* of ω.

Remark 5.3.2. Notice that Theorem 5.3.1 can be used to write ω_{i+1} in terms of $\omega_{-1}, \omega_0, \ldots, \omega_i$ for $0 \leq i \leq s-1$. Indeed, let us choose $* \in \{n, m\}$ such that $u^*_{i+1} = u_{i+1}$, and hence $t^*_{i+1} = t_{i+1}$. We have that $\nu_D(\omega_{i+1}) = t_{i+1}$ and $\nu_C(\omega_{i+1}) = \lambda_{i+1} > u_{i+1}$. By Theorem 5.3.1, if we fix j with $0 \leq j \leq i$ we have an expression

$$\omega_{i+1} = f^{ij}_j \omega_j + f^{ij}_{j-1}\omega_{j-1} + \cdots f^{ij}_0 \omega_0 + f^{ij}_{-1}\omega_{-1}, \tag{5.3}$$

such that $\lambda_j + t_{i+1} - t_j = \nu_C(f^{ij}_j \omega_j) = \nu_C(f^{ij}_{k_j}\omega_{k_j}) < \nu_C(f^{ij}_\ell \omega_\ell)$, for any $\ell \neq k_j$, with $-1 \leq \ell \leq j-1$.

Theorem 5.3.1 was proved by C. Delorme in [21] for the case where $j = i < s$ and $\omega = \omega_i$ is an element of a minimal standard basis of C. In [12], we prove a first generalization of the result of Delorme that included the case where ω was a 1-form with divisorial value t_{s+1} and C invariant, that is, $\nu_C(\omega) = \infty$. The version stated here can be found in [13].

The proof of Theorem 5.3.1 requires the following Lemma.

Lemma 5.3.3. *Consider $1 \leq i \leq s$ and $* \in \{n, m\}$. Given a 1-form η with $\nu_C(\eta) > u^*_{i+1}$ and $\nu_D(\eta) > t^*_{i+1}$, we have that:*

1. *If $\nu_D(\eta) < nm$, there is a 1-form α such that:*
   (a) $\nu_D(\eta - \alpha) > \nu_D(\eta)$.
   (b) There is a decomposition $\alpha = \sum_{\ell=-1}^{i} g_\ell \omega_\ell$, where $\nu_C(g_\ell \omega_\ell) > u^*_{i+1}$ and $\nu_D(g_\ell\omega_\ell) > t^*_{i+1}$, for any $-1 \leq \ell \leq i$.
2. *If $\nu_D(\eta) \geq nm$, there is a decomposition $\eta = \sum_{\ell=-1}^{i} g_\ell \omega_\ell$ where each term satisfies that $\nu_C(g_\ell\omega_\ell) > u^*_{i+1}$.*

Proof. Let us prove first statement (2). By Remark 5.2.2, we have that $\{\omega_{-1}, \omega_0\}$ is a basis of $\Omega^1_{M_0,P_0}$. Therefore, we can write η in a unique way

$$\eta = g_{-1}\omega_{-1} + g_0\omega_0. \tag{5.4}$$

Moreover, since the initial parts are $\mathrm{In}(\omega_{-1}) = \lambda dx$ and $\mathrm{In}(\omega_0) = \mu dy$, then we have that

$$\nu_D(\eta) = \min\{\nu_D(g_{-1}\omega_{-1}),\ \nu_D(g_0\omega_0)\}$$

Noting that $\nu_D(\omega) \geq nm$, we have that $\nu_D(g_{-1}\omega_{-1}) \geq nm$ and $\nu_D(g_0\omega_0) \geq nm$. By Lemma 3.6.1, we have that $\tilde{u}_{i+1} < \tilde{u}_1$, besides $u^*_{i+1} \leq \tilde{u}_{i+1}$, and hence

$$u^*_{i+1} \leq \tilde{u}_{i+1} < \tilde{u}_1 = nm.$$

We conclude that $\nu_C(g_\ell\omega_\ell) \geq \nu_D(g_\ell\omega_\ell) \geq nm > u^*_{i+1}$, for $\ell = -1, 0$. Then the decomposition in equation (5.4) satisfies the required properties.

Let us prove now statement (1). By Remark 2.3.15, the Newton cloud of initial part of η is a single point:

$$\mathrm{In}(\eta) = x^a y^b \left(\alpha_{-1}\frac{dx}{x} + \alpha_0 \frac{dy}{y}\right).$$

where $\nu_D(\eta) = na + mb$. There are two possibilities: either η is resonant or not. If η is not resonant, by Proposition 2.3.14, we have that $\nu_C(\eta) = \nu_D(\eta) = na + mb$. Since $\mathrm{In}(\omega_{-1}) = \lambda dx$ and $\mathrm{In}(\omega_0) = \mu dy$, we have that $\{\mathrm{In}(\omega_{-1}), \mathrm{In}(\omega_0)\}$ is a basis of $\Omega^1_{M_0,P_0}$. Thus, we can consider

$$\alpha = \mathrm{In}(\eta) = h_{-1}\,\mathrm{In}(\omega_{-1}) + h_0\,\mathrm{In}(\omega_0), \quad h_{-1} = \mu_{-1}x^{a-1}y^b,\ h_0 = \mu_0 x^a y^{b-1}. \tag{5.5}$$

For μ_{-1} and μ_0 appropriate non zero constants. We have that $\nu_D(\eta-\alpha)>\nu_D(\eta)$. Moreover we also have that $\nu_D(\alpha)=\nu_D(\eta)>u^*_{i+1}$. Since

$$\nu_D(\alpha)=\min\{\nu_D(g_{-1}\omega_{-1}),\nu_D(g_0\omega_0)\},$$

we conclude that $\nu_C(g_\ell\omega_\ell)\geq\nu_D(g_\ell\omega_\ell)\geq\nu_D(\alpha)>u^*_{i+1}$, for $i=-1,0$. Moreover, in view of Corollary 3.2.9 we have that $u^*_{i+1}>t^*_{i+1}$, hence we also get that $\nu_D(g_\ell\omega_\ell)>t^*_{i+1}$, for $\ell=-1,0$. Thus, the expression in Equation (5.5) satisfies the desired properties.

Now, let us assume that η is resonant. Up to multiply η by a non-null scalar, we have that

$$\operatorname{In}(\eta)=x^ay^b\left(m\frac{dx}{x}-n\frac{dy}{y}\right),\qquad \nu_D(\eta)=na+mb>t^*_{i+1}.$$

Let us define the index $k:=\max\{\ell\leq i:\eta \text{ is reachable by } \omega_\ell\}$. Since η is resonant, then $k\geq 1$. By definition of k, there exists a monomial μx^cy^d such that $\nu_D(\mu x^cy^d\omega_k)=\nu_D(\eta)$ and

$$\nu_D(\eta')>\nu_D(\eta),\qquad \text{where } \eta'=\eta-\mu x^cy^d\omega_k.$$

The desired decomposition will be given just by the expression $\alpha=\mu x^cy^d\omega_k$. Since $\nu_D(\alpha)=\nu_D(\eta)>t^*_{i+1}$, we only need to verify that $\nu_C(x^cy^d\omega_k)>u^*_{i+1}$.

First, let us assume that $k=i$ and hence $\alpha=\mu x^cy^d\omega_i$. Write

$$\nu_D(\alpha)=nc+md+t_i=\nu_D(\eta)>t^*_{i+1}.$$

Recalling that $t^*_{i+1}=t_i+u^*_{i+1}-\lambda_i$, we obtain that $nc+md+\lambda_i>u^*_{i+1}$. Hence, we conclude by noting that

$$\nu_C(x^cy^d\omega_i)=nc+md+\lambda_i>u^*_{i+1}.$$

Now, let us consider the case when $1\leq k\leq i-1$. Assume by contradiction that $\nu_C(x^cy^d\omega_k)\leq u^*_{i+1}<\nu_C(\eta)$. Taking into account that $\eta'=\eta-\mu x^cy^d\omega_k$, we see the following:

$$\nu_D(\eta')>\nu_D(x^cy^d\omega_k)=nc+md+t_k;$$
$$\nu_C(\eta')=\nu_C(x^cy^d\omega_k)=nc+md+\lambda_k.$$

By Statement 5 in Theorem 5.2.10, we have that $nc+md+\lambda_k\in\Lambda_{k-1}$. In view of Lemma 3.2.5, this implies that either $c\geq\ell^n_{k+1}$ or $d\geq\ell^m_{k+1}$. There are four possibilities:

$$u_{k+1}=\lambda_k+n\ell^n_{k+1} \text{ and } c\geq\ell^n_{k+1};\qquad u_{k+1}=\lambda_k+n\ell^n_{k+1} \text{ and } d\geq\ell^m_{k+1};$$
$$u_{k+1}=\lambda_k+m\ell^m_{k+1} \text{ and } c\geq\ell^n_{k+1};\qquad u_{k+1}=\lambda_k+m\ell^m_{k+1} \text{ and } d\geq\ell^m_{k+1}.$$

The cases from the first line behave in a similar way as those in the second one, therefore, we will only show what happens in the first two cases.

Case $u_{k+1}=u^n_{k+1}=\lambda_k+n\ell^n_{k+1}$ and $c\geq\ell^n_{k+1}$. In this case we have that η is reachable by $x^{\ell^n_{k+1}}\omega_k$. If we show that $x^{\ell^n_{k+1}}\omega_k$ is reachable by ω_{k+1}, we contradict the maximality of k, as desired. By Corollary 5.2.11, the 1-forms ω_{k+1} and ω_k are resonant and it is enough to show that

$$\nu_D(x^{\ell^n_{k+1}}\omega_k)=\nu_D(\omega_{k+1}).$$

We have that $\nu_D(\omega_{k+1})=t_{k+1}$ and $\nu_D(x^{\ell^n_{k+1}}\omega_k)=t_k+n\ell^n_{k+1}$. Let us see that $t_{k+1}=t_k+n\ell^n_{k+1}$ in our case. We have that $t^n_{k+1}=t_k+n\ell^n_{k+1}$. Moreover, the fact that $u_{k+1}=u^n_{k+1}$ implies that $t_{k+1}=t^n_{k+1}$, as desired.

Case $u_{k+1}=u^n_{k+1}=\lambda_k+n\ell^n_{k+1}$ and $d\geq\ell^m_{k+1}$. By Lemma 3.6.1, we see that

$$nc+md+\lambda_k\geq\lambda_k+m\ell^m_{k+1}=\tilde{u}_{k+1}\geq\tilde{u}_i>u^*_{i+1}.$$

This ends the proof. □

Proof of Theorem 5.3.1. Let us take ω being such that $\nu_D(\omega) = t^*_{i+1}$ and $\nu_C(\omega) > u^*_{i+1}$ as in the statement. We will consider three cases:

$$\text{a) } i = 0; \quad \text{b) } i > 0,\ j = i; \quad \text{c) } i > 0,\ 0 \leq j < i.$$

Case a): $i = 0$. Since $\{\omega_{-1}, \omega_0\}$ is a basis of $\Omega^1_{M_0,P_0}$, the 1-form ω can be written as

$$\omega = f^{00}_{-1}\omega_{-1} + f^{00}_0\omega_0. \tag{5.6}$$

Looking at the computations in Example 3.5.4, we see that $t^*_1 = u^*_1 \leq nm$ and $k^*_0 = -1$. Therefore, we need to prove that $\nu_C(f^{00}_{-1}\omega_{-1}) = \nu_C(f^{00}_0\omega_0) = u^*_1$.

Recall that, up to constant, we have that $\mathrm{In}(\omega_{-1}) = dx$ and $\mathrm{In}(\omega_0) = dy$. Then, one of the following cases occurs:

(i) $\nu_D(f^{00}_0\omega_0) = t^*_1$ and $\nu_D(f^{00}_{-1}\omega_{-1}) \geq t^*_1$.
(ii) $\nu_D(f^{00}_{-1}\omega_{-1}) = t^*_1$ and $\nu_D(f^{00}_0\omega_0) \geq t^*_1$.

Assume that we are in case (i). Since $\nu_D(f^{00}_0) + \nu_D(\omega_0) \leq nm$, we have that

$$\nu_D(f^{00}_0) < nm.$$

This implies, by Proposition 2.3.14, that $\nu_D(f^{00}_0) = \nu_C(f^{00}_0)$. Therefore, we can write

$$\nu_C(f^{00}_0\omega_0) = \nu_C(f^{00}_0) + \nu_C(\omega_0) = \nu_D(f^{00}_0) + \nu_D(\omega_0) = \nu_D(f^{00}_0\omega_0) = t^*_1 = u^*_1.$$

Moreover, since $\nu_D(f^{00}_{-1}\omega_{-1}) \geq t^*_1 = u^*_1$, we have that

$$\nu_C(f^{00}_{-1}\omega_{-1}) \geq \nu_D(f^{00}_{-1}\omega_{-1}) \geq t^*_1 = u^*_1.$$

Noting that $\nu_C(f^{00}_{-1}\omega_{-1} + f^{00}_0\omega_0) > u^*_1$ and that $\nu_C(f^{00}_0\omega_0) = u^*_1$, we conclude that $\nu_C(f^{00}_{-1}\omega_{-1}) = \nu_C(f^{00}_0\omega_0) = u^*_1$.

We do a similar argument in the case that $\nu_D(f^{00}_{-1}\omega_{-1}) = t^*_1$.

Case b)*: $i > 0$ and $j = i$*. We do the proof in the case $* = n$, the case $* = m$ runs in a similar way. Note that:

$$\nu_D(\omega) = t^n_{i+1} < nm, \quad \nu_D(\omega) = t^n_{i+1} < u^n_{i+1} < \nu_C(\omega),$$

in view of Corollary 3.6.2 and Corollary 3.2.9. We deduce that the 1-form ω is resonant. Since ω_i is also resonant and $t^n_{i+1} = t_i + n\ell^n_{i+1}$, we have that

$$\nu_D(\omega) = \nu_D(x^{\ell^n_{i+1}}\omega_i).$$

We deduce that there is a non-null scalar $\mu \neq 0$ such that

$$\mathrm{In}(\omega) = \mu\,\mathrm{In}(x^{\ell^n_{i+1}}\omega_i) = \mu x^{\ell^n_{i+1}}\,\mathrm{In}(\omega_i).$$

Thus, the 1-form $\eta_1 = \omega - \mu x^{\ell^n_{i+1}}\omega_i$ satisfies the following two properties:

$$\nu_D(\eta_1) > t^n_{i+1}, \quad \nu_C(\eta_1) = \nu_C(x^{\ell^n_{i+1}}\omega_i) = u^n_{i+1}.$$

The second one comes from the fact that $\nu_C(\omega) > u^n_{i+1} = \lambda_i + n\ell^n_{i+1}$. Take the bound $k = k^n_i$ and the colimit $b = b_{i+1}$. We recall that

$$u^n_{i+1} = \lambda_i + n\ell^n_{i+1} = \lambda_k + mb.$$

Hence, the 1-form $y^b\omega_k$ satisfies that $\nu_C(y^b\omega_k) = u^n_{i+1}$. On the other hand, the divisorial value $\nu_D(y^b\omega_k)$ is given by

$$\nu_D(y^b\omega_k) = mb + t_k.$$

Let us show that $\nu_D(y^b\omega_k) > t^n_{i+1} = \nu_D(\omega)$. We have

$$t^n_{i+1} < mb + t_k \Leftrightarrow t_i + n\ell^n_{i+1} < t_k + mb \Leftrightarrow$$
$$t_i - t_k < mb - n\ell^n_{i+1} = mb - n\ell^n_{i+1} + u^n_{i+1} - u^n_{i+1} \Leftrightarrow$$
$$t_i - t_k < (u^n_{i+1} - n\ell^n_{i+1}) - (u^n_{i+1} - mb) = \lambda_i - \lambda_k.$$

We conclude, since $t_i - t_k < \lambda_i - \lambda_k$ in view of Lemma 3.2.8.

Take $\eta_2 = \eta_1 - \mu^+ y^b\omega_k$ such that $\nu_C(\eta_2) > u^n_{i+1}$. Note that $\nu_D(\eta_2) > t^n_{i+1}$. Applying Lemma 5.3.3, we get a decomposition

$$\eta_2 = \omega - \mu x^{\ell^n_{i+1}}\omega_i - \mu_2 y^b\omega_k = \sum_{\ell=-1}^{i} h_\ell\omega_\ell, \quad \nu_C(h_\ell\omega_\ell) > u^n_{i+1},\ \nu_D(h_\ell\omega_\ell) > t^n_{i+1},$$

having the desired properties.

Case c): $i > 0, 0 \le j < i$. Let us reason by inverse induction on j, recalling that the case $j = i$ has already been proven. By induction hypothesis, we can decompose ω as:

$$\omega = \textstyle\sum_{\ell=-1}^{j+1} f_\ell^{ij+1}\omega_\ell, \tag{5.7}$$

where $v^*_{ij+1} = \nu_C(f^{ij+1}_{j+1}\omega_{j+1}) = \min\{\nu_C(f^{ij+1}_\ell\omega_\ell); -1 \le \ell < j+1\}$. Notice that in the Case b), we have proven the case where $j + 1 = i$. In view of Remark 5.3.2, we can apply Case b) to ω_{j+1} to obtain a decomposition:

$$\omega_{j+1} = \textstyle\sum_{\ell=-1}^{j} f_\ell^{jj}\omega_\ell, \tag{5.8}$$

where $u_{j+1} = \nu_C(f^{jj}_j\omega_j) = \min\{\nu_C(f^{jj}_\ell\omega_\ell);\ \ell < j\}$, and the minimum is only reached at the bound $k = k_{j+1}$. If we substitute the expression of ω_{j+1} given in (5.8) into the expression of ω given in (5.7), we obtain

$$\omega = \textstyle\sum_{\ell=-1}^{j}(f^{ij+1}_\ell + f^{ij+1}_{j+1}f^{jj}_\ell)\omega_\ell. \tag{5.9}$$

Let us show that equation (5.9) gives the desired decomposition. In order to do this, we only have to show that

i) $\nu_C((f^{ij+1}_j + f^{jj}_j f^{ij+1}_{j+1})\omega_j) = \nu_C((f^{ij+1}_k + f^{jj}_k f^{ij+1}_{j+1})\omega_k) = v^*_{ij}$.
ii) $\nu_C((f^{ij+1}_\ell + f^{jj}_\ell f^{ij+1}_{j+1})\omega_\ell) > v^*_{ij}$ for $\ell \ne j, k$.

Recall that $v^*_{ij+1} = \lambda_{j+1} + t^*_{i+1} - t_{j+1}$ and $v^*_{ij} = \lambda_j + t^*_{i+1} - t_j$. Hence, by Lemma 3.2.8, we have that $v^*_{ij} < v^*_{ij+1}$. Moreover, by the properties of the decomposition given in equation (5.7), we get that:

$$\nu_C(f^{ij+1}_{j+1}) = v^*_{ij+1} - \lambda_{j+1}; \tag{5.10}$$
$$\nu_C(f^{ij+1}_\ell\omega_\ell) \ge v^*_{ij+1} > v^*_{ij}, \text{ for } \ell < j+1. \tag{5.11}$$

Using the expression given in (5.10) and the properties of the decomposition given in (5.8), it follows that:

$$\begin{aligned}\nu_C(f^{ij+1}_{j+1}f^{jj}_\ell\omega_\ell) =&\nu_C(f^{ij+1}_{j+1}) + \nu_C(f^{jj}_\ell\omega_\ell) =\\ =&v^*_{ij+1} - \lambda_{j+1} + \nu_C(f^{jj}_\ell\omega_\ell) \ge\\ \ge&v^*_{ij+1} - \lambda_{j+1} + u_{j+1},\end{aligned}$$

where the last inequality is an equality just for $\ell = j, k$. Now, taking into account that $u_{j+1} = \lambda_j + t_{j+1} - t_j$ and that $v^*_{ij+1} = t^*_{i+1} + \lambda_{j+1} - t_{j+1}$, we obtain that

$$v^*_{ij+1} - \lambda_{j+1} + u_{j+1} = \lambda_j + t^*_{i+1} - t_j = v^*_{ij}.$$

Finally, since $\nu_C(f_\ell^{ij+1}\omega_\ell) > v^*_{ij}$ for $\ell < j+1$, by expression (5.11), we get that

$$\nu_C((f_\ell^{ij+1} + f_\ell^{jj} f_{j+1}^{ij+1})\omega_\ell) \geq v^*_{ij},$$

where, again, we have an equality just for $\ell = j, k$. □

From the previous result we can deduce that a 1-form with divisorial value equal to t^n_{i+1} (resp. t^m_{i+1}) and differential value greater than u^n_{i+1} (resp. u^m_{i+1}) is basic and resonant.

Corollary 5.3.4. *Consider $1 \leq i \leq s$ and let ω be a 1-form such that $\nu_D(\omega) = t^*_{i+1}$ and $\nu_C(\omega) > u^*_{i+1}$ with $* \in \{n, m\}$. For any decomposition*

$$\omega = \textstyle\sum_{\ell=-1}^{j} f_\ell^{ij}\omega_\ell, \qquad 1 \leq j \leq i.$$

satisfying the stated properties in Theorem 5.3.1, then we have that $\mathrm{In}(\omega) = \mathrm{In}(f_j^{ij}\omega_j)$. In particular, ω is basic and resonant.

Proof. We only need to show that $\nu_D(f_j^{ij}\omega_j) < \nu_D(f_\ell^{ij}\omega_\ell)$ for $\ell < j$. We know that $\nu_C(f_j^{ij}\omega_j) \leq \nu_C(f_\ell^{ij}\omega_\ell)$ for $\ell < j$. Besides, $\nu_D(\omega) = t^*_{i+1} < nm$ and hence ω is basic because $i \geq 1$. Therefore, we have that $nm > \nu_D(f_j^{ij})$, and consequently, the divisorial value and the differential value coincide $\nu_D(f_j^{ij}) = \nu_C(f_j^{ij})$. Furthermore:

$$\nu_C(f_\ell^{ij}\omega_\ell) = \nu_C(f_\ell^{ij}) + \lambda_\ell \geq \nu_C(f_j^{ij}\omega_j) = \nu_C(f_j^{ij}) + \lambda_j.$$

By Lemma 3.2.8, we have that $\lambda_j - \lambda_\ell > t_j - t_\ell$, thus

$$\nu_C(f_\ell^{ij}) + t_\ell > \nu_C(f_j^{ij}\omega_j) = \nu_C(f_j^{ij}) + t_j.$$

If $\nu_C(f_\ell^{ij}) > nm$, then $\nu_D(f_\ell^{ij}) \geq nm$, see Proposition 2.3.14. Hence, we have that

$$\nu_D(f_\ell^{ij}\omega_\ell) > t^*_{i+1} = \nu_D(f_j^{ij}\omega_j).$$

Indeed, if $\nu_C(f_\ell^{ij}) \leq nm$, we get that $\nu_C(f_\ell^{ij}) = \nu_D(f_\ell^{ij})$. With this, we conclude that

$$\nu_D(f_\ell^{ij}\omega_\ell) = \nu_D(f_\ell^{ij}) + t_\ell = \nu_C(f_\ell^{ij}) + t_\ell > \nu_C(f_j^{ij}) + t_j = \nu_D(f_j^{ij}) + t_j = \nu_D(f_j^{ij}\omega_j).$$

Finally, we see that ω is resonant, because $\nu_C(\omega) > \nu_D(\omega)$ and Proposition 2.3.14. □

Now we describe the initial parts of ω_1 and $\tilde{\omega}_1$.

Proposition 5.3.5. *Let ω be a 1-form such that $\nu_D(\omega) = t^*_1$ and $\nu_C(\omega) > u^*_1$. Let us write (in a unique way)*

$$\omega = f_{-1}\omega_{-1} + f_0\omega_0.$$

Then, we have that

1. *If $t^*_1 = t_1$, we have that $\mathrm{In}(\omega) = \mu(mydx - nxdy)$, where $\mu \neq 0$.*
2. *If $t^*_1 = \tilde{t}_1$, we have that $\mathrm{In}(\omega) = \mu(\mathrm{In}(df))$, where $f = 0$ is a reduced equation of the cusp C.*

In particular, we have that $\mathrm{In}(\omega) = \mathrm{In}(f_{-1}\omega_{-1}) + \mathrm{In}(f_0\omega_0)$.

Proof. If $t_1^* = t_1$, since $t_1 = n + m$, we see that $\mathrm{In}(\omega)$ can be written as

$$\mathrm{In}(\omega) = \mu_{-1} y dx - \mu_0 x dy.$$

Moreover, we have that $t_1 = u_1 = n + m$ and hence $\nu_C(\omega) > \nu_D(\omega)$. The result follows by Proposition 2.3.14, which asserts that ω is a resonant 1-form.

If $t_1^* = \tilde{t}_1 = nm$, we also have that $\tilde{u}_1 = nm$. The initial part $\mathrm{In}(\omega)$ has the form

$$\mathrm{In}(\omega) = \mu_{-1} x^{m-1} dx + \mu_0 y^{n-1} dy.$$

If the initial part of ω is not a multiple of $\mathrm{In}(f)$, we get that $\nu_C(\omega) = nm$, which is a contradiction with the hypothesis. □

5.4 Standard Bases from an Implicit Equation

This section is devoted to explain how to use Delorme's algorithm without using a primitive parametrization. The techniques from this section will be used when computing roots of the Bernstein-Sato polynomial.

In this section we approach two questions. First, computing a differential value using an implicit equation. Second, computing the tuning constants μ^+ needed in several steps of Delorme's algorithm. In fact, we can find a solution of both problems in [31] for the more general case of the namely complete intersection curves. However, the approach used to find tuning constants by means of resultants implies some computational problems. For this reason we include our version here.

The first question has a well known solution. More precisely, denote by $\mathcal{X}_{M_0,P_0}$ the $\mathcal{O}_{M_0,P_0}$-module of germs of vector field in (M_0, P_0). Take C a branch defined by the implicit equation $f = 0$. Consider $\omega = Adx + Bdy$ a 1-form, we denote by $X_\omega \in \mathcal{X}_{M_0,P_0}$ to the vector field

$$X_\omega := B\frac{\partial}{\partial x} - A\frac{\partial}{\partial y}.$$

Notice that the definition of X_ω depends on the chosen coordinate system. The following lemma is a weaker version of the Proposition B.1 in [18]

Lemma 5.4.1 ([18] Proposition B.1)**.** *Let C be a branch (not necessarily a cusp) defined by the implicit equation $f = 0$. Then we have that for any 1-form $\omega \in \Omega^1_{M_0,P_0}$:*

$$\nu_C(\omega) = i_{P_0}(X_\omega(f), f) - c_\Gamma + 1. \tag{5.12}$$

In virtue of Proposition 4.1.5, we can obtain $i_{P_0}(X_\omega(f), f)$ by computing a minimal standard basis of the ideal $(X_\omega(f), f)$. This is done by means of an implicit equation, hence we can compute the differential value $\nu_C(\omega)$ without using directly a parametrization.

Now assume that C is a cusp with Puiseux pair (n, m). Suppose that (x, y) is a local system of coordinates adapted to C and take an implicit equation $f = 0$ of C as in Equation (1.9), that is,

$$f = \mu x^m + y^n + \sum_{\substack{\alpha,\beta \geq 0 \\ n\alpha + m\beta > nm}} z_{\alpha\beta} x^\alpha y^\beta, \quad \text{with } \mu \neq 0 \text{ and } z_{\alpha\beta} \in \mathbb{C}.$$

With this setting the computation of the tuning constants μ^+ relies on finding a minimal standard basis of the ideal $(X_\omega(f), f)$. We want to remark that our procedure for computing tuning constants only works in the cuspidal case. As in Example 4.1.1, we consider the weighted order with respect to (n, m), where we recall that $(a, b) \prec (c, d)$ if and only if either $na + mb < nc + md$ or $na + mb = nc + md$ and $a < c$.

Proposition 5.4.2. *Consider a vector field $X \in \mathcal{X}_{M_0,P_0}$ and the ideal $I = (X(f), f)$, with the assumption that $f \nmid X(f)$.*

1. *Suppose that $lp(X(f)) \neq (0, n-1)$. Denote by h a final reduction of $X(f)$ modulo $\{f\}$. Then:*
   a) *If $lp(h) = (a, 0)$, then $\{f, h\}$ is a minimal standard basis of I.*
   b) *If $lp(h) = (a, b)$ with $b > 0$, then $\{f, h, S_{min}(f, h)\}$ is a minimal standard basis of I, where $S_{min}(f, h)$ is a minimal S-process of f and h, where $lp(S_{min}(f, h) = (a + m, 0)$.*
2. *Suppose that $lp(X(f)) = (0, n-1)$, and let g be a final reduction of f modulo $\{X(f)\}$. Then $\{X(f), g\}$ is a minimal standard basis of I, where $lp(g) = (m, 0)$.*

Notice that in the case *1.b*), we are saying that $S_{min}(f, h)$ is not reducible modulo $\{f, h\}$.

Proof. The proofs of the three statements are pretty similar; for this reason, we omit the one for Statement *2*. Assume, as in Statement *1*, that $lp(X(f)) \neq (0, n-1)$. We recall that we write f as in Equation (1.9).

$$f = \mu x^m + y^n + \sum_{\substack{\alpha,\beta \geq 0 \\ n\alpha + m\beta > nm}} z_{\alpha\beta} x^\alpha y^\beta, \quad \text{with } \mu \neq 0 \text{ and } z_{\alpha\beta} \in \mathbb{C}.$$

Thus, the condition $lp(X(f)) \neq (0, n-1)$ is equivalent to $lp(X(f)) \neq lp(f_y)$. It follows that the term $\alpha \frac{\partial}{\partial y}$ with $\alpha \in \mathbb{C}$ of the vector field X is zero. From the equation of f, we deduce that there are no monomials of the form xy^k with $k < n$. Therefore, $lp(X(f)) \neq (0, c)$ with $c < n$.

To compute a minimal standard basis of the ideal I, we apply Büchberger's algorithm. First, we find a generator system $B = \{f_1, f_2\}$ of I, such that neither f_1 is reducible by f_2, nor vice versa. Next, we will compute the minimal S-process $S_{min}(f_1, f_2)$ and a final reduction modulo $\{f_1, f_2\}$. Finally, we iterate the process as many times as needed. In fact, we show that either $\{f_1, f_2\}$ is the desired standard basis, or the algorithm finds one after the first iteration.

Since f does not divide $X(f)$, we know that $I \neq (f)$, meaning that we need at least two generators for the minimal standard basis. Because $lp(X(f)) \neq (0, c)$ with $c < n$ and $lp(f) = (0, n)$, if f is reducible modulo $\{X(f)\}$, then $lp(X(f)) = (0, n)$, implying that $X(f)$ is reducible modulo $\{f\}$. Therefore, we set $f_1 = f$ and $f_2 = h$, where h is a final reduction of $X(f)$ modulo $\{f\}$.

Let the leading power of h be $lp(h) = (a, b)$ with $b < n$. Otherwise, $h - \mu' x^a y^{n-b} f$ would give us a new reduction of h for an appropriate constant μ', contradicting the fact that h is a final reduction of $X(f)$ modulo $\{f\}$.

We have to check that for the case $b = 0$, the algorithm has found a standard basis, and hence minimal, since neither f nor h can be omitted. On the contrary, if $b \neq 0$, we have to compute an extra element. In both cases, we can write

$$S_1 = S_{min}(f, h) = x^a f - \mu_1 y^{n-b} h,$$

where μ_1 is the unique constant such that $lp(S_1) \succ lp(x^a f) = lp(y^{n-b} h)$.

We claim that the leading term of S_1 is $lt(S_1) = \mu x^{a+m}$, where μ is the constant appearing in the implicit equation f. This is consequence of the following two facts: first, the Newton cloud of h has no points of the form (c, d) with $nc + md = na + mb$ and $d \geq n$. Otherwise, it would follow that $c < a$ and that $lp(h) \neq (a, b)$. Second, the leading term of $x^a f - lt(x^a f)$ is μx^{a+m}. Since $lt(S_1) = \mu x^{a+m}$, then S_1 does not admit any reduction modulo $\{f, h\}$, unless $b = 0$.

Case $b = 0$: for any partial reduction r of S_1, we have that

$$lp(r) = (c, d) \succ lp(S_1) = (a + m, 0) \succ (a, n) = lp(x^a f).$$

We have the following:

- If $d \geq n$, then r can be reduced modulo $\{f\}$.
- If $c \geq a$, then r can be reduced modulo $\{h\}$.
- If $c < a$ and $d < n$, then we have that $nc + md < na + mn$, in contradiction with the assumption that $(c, d) \succ (a, n)$.

In conclusion, if $b = 0$, then r is reducible modulo $\{f, h\}$. Since r is any partial reduction of S_1 modulo $\{f, h\}$, it follows that 0 is a final reduction of S_1 modulo $\{f, h\}$. Thus the Büchberger's algorithm stops and $\{f, h\}$ is a minimal standard basis of I. This ends the proof of Statement *1.a*).

Case $b \neq 0$: As we said before, since $lt(S_1) = \mu x^{a+m}$, we have that in this case S_1 is its own final reduction modulo $\{f, h\}$, in particular we have that $lp(S_1) = (a+m, 0)$ as desired. Therefore, by the Büchberger's algorithm, it is necessary to add S_1 to our candidate of standard basis $\{f, h\}$. We only need to verify that the algorithm stops here. In other words, we have to compute all new possible minimal S-process and see that they have 0 as a final reduction modulo $\{f, h, S_1\}$.

There are only two new minimal S-process to consider:

$$\begin{aligned} S_2 = S_{min}(f, S_1) &= x^{a+m} f - \mu_2 y^n S_1 \\ S_3 = S_{min}(h, S_1) &= x^m h - \mu_3 y^b S_1. \end{aligned}$$

Here μ_2 and μ_3 are the unique constants such that $lp(S_2) \succ lp(x^{a+m} f) = (a + m, n)$ and $lp(S_3) \succ (x^m h) = (a + m, b)$. In order to show that their final reductions modulo $\{f, h, S_1\}$ are zero, it suffices to show that any pair $(c, d) \succ (a + m, b)$ is divisible by $(0, n)$, (a, b) or $(a + m, 0)$, that is, the leading powers of f, h and S_1.

We have that $nc + md \geq n(a + m) + mb$, hence

- If $0 \leq c \leq a$, then $d \geq n + b > n$ and $(0, n)$ divides (c, d).
- If $a < c < m + a$, then $d > b$ and (a, b) divides (c, d).
- If $c \geq a + m$, then $(a + m, 0)$ divides (c, d).

Hence the final reductions of S_2 and S_3 modulo $\{f, h, S_1\}$ are 0, ending the proof of Statement *1.b*). □

The previous result can be used to particularize Lemma 5.4.1.

Proposition 5.4.3. *Take a 1-form* $\omega \in \Omega^1_{M_0,P_0}$ *such that* $f \nmid X_\omega(f)$. *Put* $h \in \mathcal{O}_{M_0,P_0}$ *a final reduction of* $X_\omega(f)$ *modulo* $\{f\}$, *with leading power* $lp(h) = (a, b)$. *Then we have that*

$$\nu_C(\omega) = n(a + 1) + m(b + 1) - nm.$$

Proof. By Lemma 5.4.1 we have that

$$\nu_C(\omega) = i_{P_0}(X_\omega(f), f) - c_\Gamma + 1.$$

By Proposition 5.4.2, we can determine the set of leading powers of a minimal standard basis of the ideal $(X_\omega(f), f)$. By direct application of Proposition 4.1.5, we find that

$$i_{P_0}(X_\omega(f), f) = na + mb.$$

The last equality combined with fact that $c_\Gamma = (n - 1)(m - 1)$ give us the desired result.

□

Remark 5.4.4. Proposition 5.4.3 solves the problem of computing the tuning constants μ^+ in the case we are interested in. Suppose that $\eta_1, \eta_2 \in \Omega^1_{M_0,P_0}$ are two 1-forms with $\nu_C(\eta_1) = \nu_C(\eta_2) < \infty$. Let us see how to compute the tuning constant of the 1-form $\eta_1 + \mu^+\eta_2$. First, consider h_1 and h_2 final reductions of $X_{\eta_1}(f)$ and $X_{\eta_2}(f)$ modulo $\{f\}$ respectively. Put $lp(h_1) = (a, b)$ and $lp(h_2) = (c, d)$ the leading powers, where $b, d < n$. By Proposition 5.4.3, we have that

$$n(a + 1) + m(b + 1) - nm = \nu_C(\eta_1) = \nu_C(\eta_2) = n(c + 1) + m(d + 1) - nm.$$

The conditions $0 \leq b, d < n$ imply that $(a, b) = (c, d)$. The previous equalities can be translated to leading terms as

$$lt(h_1) = \mu_1 x^a y^b; \quad lt(h_2) = \mu_2 x^a y^b.$$

Here μ_1 and μ_2 are non zero constants. If we put $\mu^+ = -\mu_1/\mu_2$ as the tuning constant, we observe the following: the leading power $lp(h_1 + \mu^+ h_2) > lp(h_1), lp(h_2)$. Since $h_1 + \mu^+ h_2$ is, at least, a partial reduction of $X_{\eta_1+\mu^+\eta_2}(f)$, by Propositions 5.4.3, we verify that $\nu_C(\eta_1+\mu^+\eta_2) > \nu_C(\eta_1), \nu_C(\eta)$.

Since we know how to compute differential values and tuning constants, we can apply Delorme's algorithm without a parametrization.

To end this chapter, we show how a minimal standard basis of the module of differentials is related with a minimal standard basis of the extended jacobian ideal of C, when considering the weighted monomial order with respect (n, m). We recall that the extended jacobian ideal of f is $\mathcal{J}(f) = (f, f_x, f_y)$, where f_x, f_y are the partial derivatives of f with respect x and y respectively.

In [9], it is given a description of minimal standard basis of the extended jacobian ideal, when f is generic. Besides, according to [48], if we have a minimal standard basis of $\mathcal{J}(f)$, then we can obtain the semimodule of differential values of C. Moreover, in [16], standard bases of (f) are used to find a similar result to Theorem 1.5.2 when dealing with the implicit equation of a cusp.

Theorem 5.4.5. *Assume that C is a cusp with a Puiseux pair (n, m) and let $f \in \mathcal{O}_{M_0,P_0}$ be an implicit equation of C. Suppose also, that the local system of coordinates (x, y) is adapted with respect C. Denote by $(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ the basis of the semimodule of differential values Λ_C of C and take $(\omega_{-1}, \omega_0, \ldots, \omega_s)$ a minimal standard basis of the module of differentials of C. For $i = -1, 0, 1, \ldots, s$, let $h_i \in \mathcal{O}_{M_0,P_0}$ be a final reduction of $X_{\omega_i}(f)$ modulo $\{f\}$, then*

$$B = \{h_{-1}, h_0, \ldots, h_s\}$$

is a minimal standard basis of $\mathcal{J}(f)$ with respect the weighted order with respect (n, m).

Proof. Put

$$f = \mu x^m + y^n + \sum_{\substack{\alpha,\beta \geq 0 \\ n\alpha + m\beta > nm}} z_{\alpha\beta} x^\alpha y^\beta, \quad \text{with } \mu \neq 0 \text{ and } z_{\alpha\beta} \in \mathbb{C}.$$

It is enough to show the following three statements:

1. B is a generator system of the ideal $\mathcal{J}(f)$.
2. Given $g \in \mathcal{J}(f)$, there exists at least one element $b \in B$, such that $lp(b)$ divides $lp(g)$.
3. Given $h_i, h_j \in B$ with $h_i \neq h_j$, then the leading powers satisfy that $lp(h_i) \nmid lp(h_j)$.

Statement 1: We need to show that the ideal generated by B coincides with $\mathcal{J}(f)$. Given $\omega \in \Omega^1_{M_0,P_0}$, by definition we have that any final reduction of $X_\omega(f)$ is an element of $\mathcal{J}(f)$. Therefore, we only need to show that f_x, f_y and f belong to the ideal (B).

By Remark 5.2.2, the pair $\{\omega_{-1},\omega_0\}$ is a basis of the $\mathcal{O}_{M_0,P_0}$-module $\Omega^1_{M_0,P_0}$. Hence, we can write the 1-form dx as

$$dx = g_{-1}\omega_{-1} + g_0\omega_0 \text{ with } g_{-1}, g_0 \in \mathcal{O}_{M_0,P_0}. \tag{5.13}$$

Besides, we also have that $\nu_C(\omega_{-1}) = n$ and $\nu_C(\omega_0) = m$, thus by Equation (5.12), we see that $i_{P_0}(X_{\omega_{-1}}(f), f) = nm - m$ and $i_{P_0}(X_{\omega_0}(f), f) = nm - n$. Therefore, by Proposition 5.4.3, the leading powers of h_{-1} and h_0 are $lp(h_{-1}) = (0, n-1)$ and $lp(h_0) = (m-1, 0)$. Since both leading powers are smaller than $(0,n)$, that is, $(0,n-1), (m-1,0) \prec (0,n)$, then the only possibility is that $X_{\omega_{-1}}(f)$ and $X_{\omega_0}(f)$ are not reducible modulo $\{f\}$. In particular, $lp(X_{\omega_{-1}}(f)) = lp(h_{-1}) = (0, n-1)$ and $lp(X_{\omega_0}(f)) = lp(h_0) = (m-1, 0)$. These last equalities imply that $X_{\omega_{-1}}(f) = h_{-1}$ and $X_{\omega_0}(f) = h_0$. Thus, by Equation (5.13)

$$-f_y = X_{dx}(f) = g_{-1}h_{-1} + g_0 h_0.$$

This shows that $f_y \in (B)$. In the same way, we find that $f_x \in (B)$. We only need to prove that $f \in (B)$. Since, we already know that $f_x, f_y \in (B)$, it is equivalent to show that

$$f - \left(\frac{1}{m}xf_x + \frac{1}{n}yf_y\right) = \sum_{\substack{\alpha,\beta\geq 0\\ n\alpha+m\beta>nm}} \frac{nm-n\alpha-m\beta}{nm} z_{\alpha\beta}x^\alpha y^\beta \in (B).$$

We are going to define a sequence of functions f_ℓ, g_ℓ for $\ell \geq 1$ satisfying the following two conditions:

- $f_\ell = f - g_\ell$, with $g_\ell \in (B)$.
- $g_\ell = X_{\eta_\ell}(f) - pf$, for some $\eta_\ell \in \Omega^1_{M_0,P_0}$ and $p \in \mathcal{O}_{M_0,P_0}$.

The functions f_ℓ and g_ℓ will show that $f \in (B)$. We construct them in an iterative way. We start with $g_1 = \left(\frac{1}{m}xf_x + \frac{1}{n}yf_y\right)$ and $\eta_1 = \frac{1}{m}xdy - \frac{1}{n}ydx$, that is, $f_1 = f - g_1$.

Now assume that we have constructed f_ℓ and g_ℓ as desired. Then there are three cases:

a) $f_\ell = 0$.
b) $lp(f_\ell)$ is not divisible by the leading power of any element of B.
c) $lp(f_\ell)$ is divisible by the leading power of some element of B.

Case a): the proof of the statement is over since we can write $f = g_\ell \in (B)$.

Case b): write $lp(f_\ell) = (\alpha, \beta)$, which by assumption is not divisible by any leading power of any element of B. In particular, (α,β) is not divisible by $lp(h_{-1}) = (0, n-1)$. Hence, (α,β) is not divisible by $(0,n)$. This means that $-f_\ell$ is a final reduction of $X_{\eta_\ell}(f)$ modulo $\{f\}$. Therefore, by Proposition 5.4.3, we obtain that $\nu_C(\eta_\ell) = \lambda = n(\alpha+1) + m(\beta+1) - nm$. Since λ is a differential value, we have that $\lambda = \lambda_i + np + mq$ for some $-1 \leq i \leq s$ and $p, q \geq 0$.

Set $(c_i, d_i) = lp(h_i)$, this means that $\lambda_i = n(c_i+1) + m(d_i+1) - nm$. Thus, we find that $n\alpha + m\beta = n(c_i + p) + m(d_i + q)$. Therefore,

$$\alpha = c_i + p - km; \quad \beta = d_i + q + kn; \quad k \in \mathbb{Z}.$$

If $k > 0$, then $\beta > n$ and (α,β) is divisible by $(0, n-1) = lp(h_{-1})$ which is a contradiction. If $k < 0$, then (α,β) is divisible by $(m-1, 0) = lp(h_0)$ again a contradiction. Finally, if $k = 0$, then $(\alpha,\beta) = (c_i + p, d_i + q)$ and (α,β) is divisible by (c_i, d_i) which is another contradiction. Thus, Case b) cannot happen.

Case c): in this case we construct the functions $f_{\ell+1}$ and $g_{\ell+1}$. Again, set $(\alpha,\beta) = lp(f_\ell)$ and consider $(c_i, d_i) = lp(h_i)$, such that (α,β) is divisible by (c_i, d_i). Then, we write

$$\eta_{\ell+1} = \eta_\ell + \mu x^{\alpha-c_i} y^{\beta-d_i}\omega_i,$$

where μ is the unique constant such that the function

$$g_{\ell+1} = g_\ell + \mu x^{\alpha-c_i} y^{\beta-d_i} h_i,$$

satisfies that $lp(f - g_{\ell+1}) > lp(f - g_\ell)$.

By hypothesis on g_ℓ, we can write $g_\ell = X_{\eta_\ell}(f) - p'f$ with $p' \in \mathcal{O}_{M_0,P_0}$. Additionally, h_i is a final reduction of $X_{\omega_i}(f)$ modulo $\{f\}$, we have that

$$\mu x^{\alpha-c_i} y^{\beta-d_i} h_i = X_{\mu x^{\alpha-c_i} y^{\beta-d_i} \omega_i}(f) - p''f, \quad p'' \in \mathcal{O}_{M_0,P_0}.$$

Thus, if we write $p = p' + p''$, we get that $g_{\ell+1} = X_{\eta_{\ell+1}}(f) - pf$. Finally, we put $f_{\ell+1} = f - g_{\ell+1}$ and we restart the process.

This entire procedure shows that a final reduction of f_1 modulo B is 0, implying that $f_1 \in (B)$, in particular, we find that $f \in (B)$. This concludes the proof of Statement 1.

Statement 2: Take $g \in \mathcal{J}(f)$ such that $lp(g) = (\alpha, \beta)$ is not divisible by any leading power of any element of B. By Statement 1, we can write

$$g = \sum_{i=-1}^{s} g_i h_i,$$

for some functions g_i with $i = -1, 0, \ldots, s$. We consider the 1-form $\omega = \sum_{i=-1}^{s} g_i \omega_i$. As in Statement 1, the assumption of non divisibility implies, in particular, that (α, β) is not divisible by $(0, n-1) = lp(h_{-1})$. Hence, $lp(g)$ is not divisible by $(0, n)$ and g is a final reduction of $X_\omega(f)$.

By Proposition 5.4.3, $\nu_C(\omega) = \lambda = n(\alpha+1) + m(\beta+1) - nm$. Repeating exactly the same arguments that in Case b) from Statement 1, we find a contradiction with the assumption that (α, β) is not divisible by any leading power of the elements of B.

Statement 3: Consider $h_i, h_j \in B$, with $i \neq j$. Assume that $(c_j, d_j) = lp(h_j)$ is divisible by $lp(h_i) = (c_i, d_i)$. Since h_i, h_j are, respectively, final reductions of $X_{\omega_i}(f)$ and $X_{\omega_j}(f)$ modulo $\{f\}$, then by Proposition 5.4.3, we have that

$$\begin{aligned} \nu_C(\omega_i) &= \lambda_i = n(c_i+1) + m(d_i+1) - nm \\ \nu_C(\omega_j) &= \lambda_j = n(c_j+1) + m(d_j+1) - nm \end{aligned}$$

Since $(c_i, d_i) \mid (c_j, d_j)$, we have that $c_j - c_i \geq 0$ and $d_j - d_i \geq 0$. Therefore, $\lambda_j - \lambda_i = n(c_j - c_i) + m(d_i - d_j) \in \Gamma_C$. This contradicts the fact that λ_i and λ_j are two different elements of the basis of Λ_C. □

We give an example of application of the previous theorem.

Example 5.4.6. As in Example 5.2.13, we take the cusp C defined by the primitive parametrization

$$\phi(t) = (t^5, t^{11} + t^{12} + 7t^{13}) = (x(t), y(t)).$$

We showed that $\mathcal{B} = (5, 11, 17, 23, 29)$ is the basis of the semimodule Λ_C of differential values. We are going to check this by using an implicit equation. Take a minimal standard basis $(\omega_{-1}, \omega_0, \ldots, \omega_s)$ of the module of differentials of C. As in Theorem 5.4.5, we denote by h_i a final reduction of $X_{\omega_i}(f)$ modulo $\{f\}$, for $i = -1, 0, \ldots, s$, where f stands for an implicit equation of C.

By Proposition 5.4.3, in order to compute the basis of the semimodule of differential values of C, we only need to know the pairs $lp(h_i)$. Since the set $\{h_{-1}, h_0, \ldots, h_s\}$ is a minimal standard basis of the extended jacobian ideal $\mathcal{J}(f)$ of C, we do not need to compute the 1-forms to obtain the basis of semimodule. More precisely, it is enough to obtain any minimal standard basis of $\mathcal{J}(f)$, recall that by Remark 4.1.6, the set of leading powers does not depend on the minimal standard basis chosen.

Since we do not want to compute the basis of Λ_C with a parametrization, we need to obtain an implicit equation of C. We can compute one by means of Equation (1.1). An easier option, in terms of complexity, is to compute the resultant of the polynomials $(x - x(t))$ and $(y - y(t))$ with respect to the variable t, as explained in Section 1.2. This time we relay on SageMath to do the computations, included those about the computation of the basis (see [54]). We proceed to give, and explain, the commands used.

We first define the polynomial ring in three variables x, y, t with coefficients in the algebraic closure of $\mathbb{Q}$. We recommend not to use the field of complex numbers, since they have a numerical precision and this could induce some errors on the computations.

```
R.<x,y,t> = PolynomialRing(QQbar, 3)
```

We define the parametrization of the branch.

```
u = t^5
v = t^11 + t^12 + 7*t^13
```

We compute an implicit equation of the branch by using the resultant method mentioned above.

```
f = (x - u).resultant(y - v, t)
```

We obtain as a result:

```
16807*x^13 + 211*x^12 + x^11 + 1470*x^10*y + 5*x^9*y + 40*x^7*y^2
+ 35*x^5*y^3 - y^5
```

We can check that

```
f(u,v,t)=0.
```

Now, we consider a new polynomial ring in two variables x, y with the desired monomomial order given by the pair $(5, 11)$.

```
M.<x,y>= PolynomialRing(QQbar,2,order=TermOrder('negwdeglex',(5,11)))
```

Since we have changed the ring, we have to redefine `f` to be an element in `M`. This is done just by rewriting:

```
f = 16807*x^13 + 211*x^12 + x^11 + 1470*x^10*y + 5*x^9*y + 40*x^7*y^2 + 35*x^5*y^3
- y^5
```

Next, we define our ideal:

```
fx = f.derivative(x)
fy = f.derivative(y)
I = ideal(fx, fy, f)
```

We compute a standard basis of `I`.

```
B = I.groebner_basis()
```

By Proposition 5.4.3, the basis of the semimodule of differential values is given by:

```
conductor = (5-1)*(11-1)
results = [b.lt().degree() - conductor + 1 for b in B]
```

We check that the array `results` returns the desired basis of the semimodule of differential values Λ_C.

6

Standard Systems

In the cuspidal case, we can enlarge the concept of minimal standard basis of the module of differentials. This way we can include extra 1-forms whose role will be prominent when describing Saito bases in Chapter 8. Up to this point we have mainly focus on the critical values t_i; in contrast, standard systems include also information coming from the critical values $\tilde{t}_i$ for $1 \leq i \leq s+1$.

This chapter is devoted to give the definition of standard systems and to show their existence.

Fix C a cusp with Puiseux pair (n, m) and cuspidal divisor D. We say that a sequence of 1-forms $\mathcal{E} = (\omega_{-1}, \omega_0, \dots, \omega_s, \omega_{s+1})$ is an *extended standard basis* of C, if $\mathcal{S} = (\omega_{-1}, \omega_0, \dots, \omega_s)$ is a minimal standard basis of C and ω_{s+1} satisfies the following two conditions:

1. $\nu_D(\omega_{s+1}) = t_{s+1}$.
2. $\nu_C(\omega_{s+1}) = \infty$, that is, C is invariant by ω_{s+1} (see Lemma 2.1.1).

Definition 6.1. *A standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ for the cusp C is the data of an extended standard basis $\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \dots, \omega_s, \omega_{s+1})$ and a family $\tilde{\mathcal{E}} = (\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_s, \tilde{\omega}_{s+1})$ of 1-forms satisfying that*

$$\nu_D(\tilde{\omega}_j) = \tilde{t}_j, \quad \nu_C(\tilde{\omega}_j) = \infty, \quad 1 \leq j \leq s+1.$$

We say that a standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ for C is a special standard system if there are expressions $\tilde{\omega}_j = h_j \omega_{s+1} + f_j \tilde{\omega}_{s+1}$, where $h_j, f_j \in \mathcal{O}_{M_0, P_0}$ for any $1 \leq j \leq s$.

The inclusion of the notion of special standard system is just a formalism. Indeed, by the already mentioned Theorem 8.2, all standard systems are special.

Note that the 1-forms ω_i and $\tilde{\omega}_i$, except ω_{-1}, ω_0 and $\tilde{\omega}_1$ are basic and resonant, see Corollary 3.6.2 and Proposition 2.3.14. Thus, by Proposition 2.5.1, they are totally D-dicritical.

Remark 6.2. We introduce the notion of extended standard basis separated from the one of standard system because the 1-forms in $\tilde{\mathcal{E}}$ do not share some of the properties that have the 1-forms in $\mathcal{E}$. We will explain this in more detail in Chapter 7.

In the next propositions, we saw the existence of standard systems. The proof of both propositions is very similar, however, we prefer this split presentation.

Proposition 6.3. *Let C be a cusp and Λ_C its semimodule of differential values, with basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_s)$. Assume that $s \geq 1$. There are two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ having C as an invariant curve such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$.*

Proof. Let us select a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ of the cusp C. As we have already done, let us denote by $*$ a chosen element $* \in \{n, m\}$. We have to find $\omega^*_{s+1} \in \Omega^1_{M_0,P_0}[C]$ such that $\nu_D(\omega^*_{s+1}) = t^*_{s+1}$.

We do the detailed proof for $* = n$. The case $* = m$ runs in a similar way. Then, we have to find $\omega^n_{s+1} \in \Omega^1_{M_0,P_0}[C]$ such that $\nu_D(\omega^n_{s+1}) = t^n_{s+1}$.

Let us recall that $u^n_{s+1} = \lambda_s + n\ell = \lambda_k + mb$, where we denote $\ell = \ell^n_{s+1}$, $k = k^n_s$ and $b = b_{s+1}$, the limits, bounds and colimits, see Lemma 3.5.1. Recall that $k < s$. Consider the 1-forms

$$\eta_0 = x^\ell \omega_s, \quad \eta_1 = y^b \omega_k.$$

Note that $\nu_D(\eta_0) = t^n_{s+1} = t_s + n\ell$ and $\nu_D(\eta_1) > t^n_{s+1}$. Since we have

$$\begin{aligned} \nu_D(\eta_1) = mb + t_k > \nu_D(\eta_0) &= t_s + n\ell \Leftrightarrow \\ t_s - t_k < mb - n\ell &= (u^n_{s+1} - \lambda_k) - (u^n_{s+1} - \lambda_s) = \lambda_s - \lambda_k \end{aligned}$$

and $\lambda_s - \lambda_k > t_s - t_k$ follows by Lemma 3.2.8. Moreover, the differential values coincide

$$\nu_C(\eta_0) = \nu_C(\eta_1) = u^n_{s+1}.$$

Thus, we write $\theta_1 = \eta_0 + \mu^+\eta_1$, where μ^+ is the tuning constant. We get that

$$\nu_D(\theta_1) = t^n_{s+1}, \quad \nu_C(\theta_1) > \nu_C(\eta_0) = \nu_C(\eta_1) = u^n_{s+1}.$$

We consider three cases:

a) $\nu_C(\theta_1) = \infty$. Then we end by taking $\omega^n_{s+1} = \theta_1$.
b) $\nu_C(\theta_1) \geq nm$.
c) $\nu_C(\theta_1) < nm$.

Case b): let $\phi(t)$ be a primitive parametrization of C. We have that $\phi^*(\theta_1) = \psi(t)dt$, with $\mathrm{ord}_t(\psi(t)) \geq nm - 1 > c_\Gamma$. In view of Lemma 1.4.3, there is a function $h(x, y)$ such that $\phi^*(dh) = \psi(t)dt$. If we take $\omega^n_{s+1} = \theta_1 - dh$, we have that $\nu_C(\omega^n_{s+1}) = \infty$. In order to finish, we have to see that $\nu_D(dh) > t^n_{s+1}$. Since $t^n_{s+1} < \tilde{t}_1 = nm$ (see Lemma 3.6.1), we just need to see that $\nu_D(dh) \geq nm$. By Proposition 2.3.14, if $\nu_D(dh) < nm$, we obtain that $\nu_C(dh) = \nu_D(dh)$, in contradiction with the fact that $\nu_C(dh) \geq nm$. Thus $\nu_D(dh)$ is at least nm as desired.

Case c): write $\nu_C(\theta_1) = \lambda_i + n\alpha + m\beta > u^n_{s+1}$, for a certain index $-1 \leq i \leq s$. Consider the 1-form η_2 given by

$$\eta_2 = x^\alpha y^\beta \omega_i, \quad \nu_D(\eta_2) = t_i + n\alpha + m\beta.$$

Let us see that $\nu_D(\eta_2) > t^n_{s+1} = t_s + n\ell = \nu_D(\theta_1)$. Assume first that $i = s$, we know that $u^n_{s+1} = \lambda_s + n\ell < \nu_C(\eta_2) = \lambda_s + n\alpha + m\beta$, hence $n\alpha + m\beta > n\ell$ and $t_s + n\alpha + m\beta > t_s + n\ell$ as desired.

Assume now that $i < s$. We have

$$\begin{aligned} \nu_C(\eta_2) = \lambda_i + n\alpha + m\beta > u^n_{s+1} = \lambda_s + n\ell \Rightarrow \\ \Rightarrow n\alpha + m\beta - n\ell > \lambda_s - \lambda_i > t_s - t_i \Rightarrow \\ \Rightarrow t_i + n\alpha + m\beta > t_s + n\ell. \end{aligned}$$

Consequently, $\nu_D(\eta_2) > \nu_D(\theta_1)$.

On the other hand, we have that $\nu_C(\eta_2) = \nu_C(\theta_1)$. Hence, we can write $\theta_2 = \theta_1 + \mu^+\eta_2$, we obtain that

$$\nu_D(\theta_2) = \nu_D(\theta_1) = t^n_{s+1}, \quad \nu_C(\theta_2) > \nu_C(\theta_1).$$

We restart the procedure with θ_2, since the differential value is strictly increasing, in a finite number of steps we arrive to case b) or to case a), this ends the proof. □

Now, we proof the existence of the 1-forms $\tilde{\omega}_i$ for $i \leq s$. The proof runs in a very similar way.

Proposition 6.4. *Let C be a cusp and Λ_C its semimodule of differential values, with basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$. Assume that $s \geq 1$. For any $i = 1, \ldots, s$, there is 1-form $\tilde{\omega}_i$ having C as an invariant curve, such that $\nu_D(\tilde{\omega}_i) = \tilde{t}_i$.*

Proof. If $i = 1$, we can take $\tilde{\omega}_1 = df$, where $f = 0$ is an implicit equation of C. Indeed, it is clear that C is invariant by df. Moreover, by Example 3.5.4, we know that $\nu_D(df) = nm = \tilde{t}_1$.

Assume that $i \geq 2$. Let us select a minimal standard basis $\mathcal{S} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ of the cusp C. In fact, we only need the elements of a minimal standard basis up to index $i - 1$. Without loss of generality we are going to assume that $\tilde{t}_i = t_i^n$, hence $\tilde{u}_i = u_i^n$.

Let us recall that $u_i^n = \lambda_{i-1} + n\ell = \lambda_k + mb$, where we denote $\ell = \ell_i^n$, $k = k_{i-1}^n$ and $b = b_i$ the limits, bounds and colimits, as defined in Chapter 3. Recall that $k < i - 1$. Consider the 1-forms

$$\eta_0 = x^\ell \omega_{i-1}, \quad \eta_1 = y^b \omega_k.$$

Note that $\nu_D(\eta_0) = t_i^n = t_{i-1} + n\ell$ and $\nu_D(\eta_1) > t_i^n$. Since we have

$$\begin{aligned} \nu_D(\eta_1) = mb + t_k > \nu_D(\eta_0) &= t_{i-1} + n\ell \Leftrightarrow \\ t_{i-1} - t_k < mb - n\ell &= (u_i^n - \lambda_k) - (u_i^n - \lambda_{i-1}) = \lambda_{i-1} - \lambda_k \end{aligned}$$

and $\lambda_{i-1} - \lambda_k > t_{i-1} - t_k$ follows by Lemma 3.2.8. Moreover, the differential values coincide

$$\nu_C(\eta_0) = \nu_C(\eta_1) = u_i^n.$$

Thus, we write $\theta_1 = \eta_0 + \mu^+ \eta_1$, where μ^+ is the tuning constant. We get that

$$\nu_D(\theta_1) = t_i^n, \quad \nu_C(\theta_1) > \nu_C(\eta_0) = \nu_C(\eta_1) = u_i^n.$$

We consider three cases:

a) $\nu_C(\theta_1) = \infty$. Then we end by taking $\tilde{\omega}_i = \theta_1$.
b) $\nu_C(\theta_1) \geq nm$.
c) $\nu_C(\theta_1) < nm$.

Case b): let $\phi(t)$ be a primitive parametrization of C. We have that $\phi^*(\theta_1) = \psi(t)dt$, with $\operatorname{ord}_t(\psi(t)) \geq nm - 1 > c_\Gamma$. In view of Lemma 1.4.3, there is a function $h(x, y)$ such that $\phi^*(dh) = \psi(t)dt$. If we take $\tilde{\omega}_i = \theta_1 - dh$, we have that $\nu_C(\tilde{\omega}_i) = \infty$. In order to finish, we have to see that $\nu_D(dh) > t_i^n$. Since $t_i^n < \tilde{t}_1 = nm$ (see Lemma 3.6.1), if we only need to see that $\nu_D(dh) \geq nm$. By Proposition 2.3.14, if $\nu_D(dh) < nm$, we obtain that $\nu_C(dh) = \nu_D(dh)$, in contradiction with the fact that $\nu_C(dh) \geq nm$. Thus $\nu_D(dh)$ is at least nm as desired.

Case c): we have that $\nu_C(\theta_1) > \tilde{u}_i$, thus by Corollary 3.4.7, we have that $\nu_C(\theta_1)$ is greater than the conductor of the semimodule Λ_{i-1} appearing in the decomposition sequence of Λ_C (see Chapter 3). Therefore, $\nu_C(\theta_1) \in \Lambda_{i-1}$ and we can write $\nu_C(\theta_1) = \lambda_j + n\alpha + m\beta > u_i^n$, for a certain index $-1 \leq j \leq i - 1$. Consider the 1-form η_2 given by

$$\eta_2 = x^\alpha y^\beta \omega_j, \quad \nu_D(\eta_2) = t_j + n\alpha + m\beta.$$

Let us see that $\nu_D(\eta_2) > t_i^n = t_{i-1} + n\ell = \nu_D(\theta_1)$. Assume first that $j = i - 1$, we know that $u_i^n = \lambda_{i-1} + n\ell < \nu_C(\eta_2) = \lambda_{i-1} + n\alpha + m\beta$, hence $n\alpha + m\beta > n\ell$ and $t_{i-1} + n\alpha + m\beta > t_{i-1} + n\ell$ as desired.

Assume now that $j < i-1$. We have

$$\begin{aligned} \nu_C(\eta_2) = \lambda_j + n\alpha + m\beta > u_i^n = \lambda_{i-1} + n\ell \Rightarrow \\ \Rightarrow n\alpha + m\beta - n\ell > \lambda_{i-1} - \lambda_j > t_{i-1} - t_j \Rightarrow \\ \Rightarrow t_j + n\alpha + m\beta > t_{i-1} + n\ell. \end{aligned}$$

Therefore, $\nu_D(\eta_2) > \nu_D(\theta_1)$.

On the other hand, we have that $\nu_C(\eta_2) = \nu_C(\theta_1)$. Hence, we can write $\theta_2 = \theta_1 + \mu^+\eta_2$, we obtain that

$$\nu_D(\theta_2) = \nu_D(\theta_1) = t_i^n, \quad \nu_C(\theta_2) > \nu_C(\theta_1).$$

We re-start the procedure with θ_2, since the differential value is strictly increasing, in a finite number of steps we arrive to case b) or to case a), this ends the proof. □

7

Analytic Semiroots

In this chapter we introduce the concept of analytic semiroot of a cusp C. The goal is to find cusps whose analytic type "approximate" the one of C. This resembles the case of approximate roots with a similar property with topological classes instead (see [1]).

Consider C a cusp with Puiseux pair (n, m) and cuspidal divisor D. Suppose that $\omega \in \Omega^1_{M_0,P_0}$ defines a totally D-dicritical foliation. Then we have that each non corner point P of D defines a branch C_P^ω invariant by ω whose strict transform, by the minimal resolution of singularities of C, is non singular and transverse to D at P. The curve C_P^ω has the same resolution of singularities as C. We recall that C_P^ω is called an ω-cusp through P, see Definition 2.5.5.

When ω is an element of an extended standard basis, we obtain the desired notion of analytic semiroot.

Definition 7.1. *Consider C a cusp and let $\mathcal{E} = (\omega_{-1}, \omega_0, \ldots, \omega_s, \omega_{s+1})$ be an extended standard basis of C. Denote by D the cuspidal divisor of C. For any index $1 \le i \le s+1$ and any non corner point $P \in D$, we say that the ω_i-cusp $C_P^{\omega_i}$ is an weak analytic $\mathcal{E}$-semiroot of index i of C. When P is the infinitely near point of C, we just say that $C_P^{\omega_i}$ is the analytic $\mathcal{E}$-semiroot of index i of C.*

The next result shows the relationship between the semimodule of differential values of a cusp and its analytic semiroots.

Theorem 7.2 (Theorem 8.8 [12])**.** *Consider C a cups with $\mathcal{E} = (\omega_{-1}, \omega_0, \ldots, \omega_{s+1})$ an extended standard basis. For any $1 \le i \le s+1$ and $\gamma = C_P^{\omega_i}$ a weak analytic $\mathcal{E}$-semiroot of C, we have that*

$$\mathcal{E}_i = (\omega_{-1}, \omega_0, \ldots, \omega_i)$$

is an extended standard basis of γ and the semimodule of differential values is $\Lambda_\gamma = \Lambda_{i-1}$. Moreover, we have the equality of differential values

$$\nu_C(\omega_k) = \nu_\gamma(\omega_k), \quad \text{for} \quad -1 \le k \le i-1.$$

Recalling the definition of increasing semimodule satisfied by Λ_C ($\lambda_i > u_i$), we have that the previous theorem is a consequence of the next more general technical results.

Proposition 7.3. *Consider an index $1 \le i \le s+1$ and a element $* \in \{n, m\}$, excluding the case where $i = 1$ and $* = m$. Let ω be a 1-form such that $\nu_C(\omega) > u_i^*$ and $\nu_D(\omega) = t_i^*$. Take $\gamma = C_P^\omega$ an ω-cusp. Then we have that $\nu_C(\omega_\ell) = \nu_\gamma(\omega_\ell) = \lambda_\ell$ for $\ell = -1, \ldots, i-1$. Moreover, λ_ℓ is precisely the ℓ-element of the basis of Λ_C.*

Recall that these 1-forms are totally D-dicritical, since they are basic and resonant.

Proposition 7.4. *Consider an index $1 \leq i \leq s+1$. Let ω be a 1-form such that $\nu_C(\omega) > u_i$ and $\nu_D(\omega) = t_i$. Take $\gamma = C_P^{\omega}$ an ω-cusp. Then $\Lambda_\gamma = \Lambda_{i-1}$.*

Before giving the proofs of both propositions, let us note that Proposition 7.3 implies the following.

Corollary 7.5. $\Lambda_{C_P^i} \supset \Lambda_{i-1}$.

Moreover, Proposition 7.3 gives a version of Theorem 7.2 for the case of standard systems. In particular, for the 1-forms $\tilde{\omega}_i$, with $\nu_D(\tilde{\omega}_i)$ equal to the critical value $\tilde{t}_i$ and with C invariant, see Definition 6.1.

Theorem 7.6. *Consider $\tilde{\gamma} = C_P^{\tilde{\omega}_i}$ an $\tilde{\omega}_i$-cusp with $2 \leq i \leq s+1$. Then we have the equality of differential values*

$$\nu_C(\omega_\ell) = \nu_{\tilde{\gamma}}(\omega_\ell), \quad \text{for} \quad -1 \leq \ell \leq i-1.$$

Moreover, we have the inclusion $\Lambda_{i-1} \subset \Lambda_{\tilde{\gamma}}$.

Proof of Proposition 7.3. Note that for any basic non resonant 1-form η, by Proposition 2.3.14, we have that

$$\nu_C(\eta) = \nu_D(\eta) = \nu_\gamma(\eta).$$

This is particularly true for the case of exact 1-forms. Notice that if $\eta = dg$ with $g(0,0) = 0$, then we know that $\nu_D(\eta) = \nu_D(g)$. Hence, for any germ of function $g \in \mathcal{O}_{M_0,P_0}$, we have that

$$\min\{\nu_C(dg), nm\} = \min\{\nu_\gamma(dg), nm\}. \tag{7.1}$$

Since (x,y) is not only a system of adapted coordinates with respect to C, but also with respect to γ, then

$$\nu_C(dx) = \nu_\gamma(dx) = n = \lambda_{-1}, \quad \nu_C(dy) = \nu_\gamma(dy) = m = \lambda_0.$$

The previous equalities imply that the statement of the theorem is true for $\ell = -1, 0$. Let us assume that theorem is true for any $\ell = -1, 0, 1, \ldots, j$, with $0 \leq j \leq i-2$. We apply Theorem 5.3.1 to obtain a decomposition

$$\omega = \sum_{\ell=-1}^{j+1} f_\ell \omega_\ell$$

such that $\nu_C(f_\ell \omega_\ell) \geq v_{ij+1}$, where $v_{ij+1} \leq v_{ii} = u_{i+1} < nm$ (Corollary 3.6.2) and there is a single $k \leq j$ such that $\nu_C(f_{j+1}\omega_{j+1}) = \nu_C(f_k \omega_k) = v_{ij+1}$. We deduce that

$$\nu_C\left(\sum_{\ell=-1}^{j} f_\ell \omega_\ell\right) = \nu_C(f_k \omega_k) = v_{ij+1}.$$

On the other hand, by induction hypothesis and noting that $v_{ij+1} < nm$, we have that

$$\min\{\nu_C(f_\ell \omega_\ell), nm\} = \min\{\nu_\gamma(f_\ell \omega_\ell), nm\}, \quad \ell = -1, 0, 1, \ldots, j.$$

In particular, we have that

$$\nu_\gamma(f_k \omega_k) = v_{ij+1}, \quad \nu_\gamma\left(\sum_{\ell=-1}^{j} f_\ell \omega_\ell\right) = v_{ij+1}.$$

Since $\nu_\gamma(\omega) = \infty$, we have that $\nu_\gamma(f_{j+1}\omega_{j+1}) = v_{ij+1}$. Hence we have

$$v_{ij+1} = \nu_C(f_{j+1}\omega_{j+1}) = \nu_\gamma(f_{j+1}\omega_{j+1}).$$

Noting that $\nu_C(f_{j+1}) = \nu_\gamma(f_{j+1})$, we conclude that $\nu_C(\omega_{j+1}) = \nu_\gamma(\omega_{j+1})$. This shows that $\nu_C(\omega_\ell) = \nu_\gamma(\omega_\ell) = \lambda_\ell$ for $\ell = -1, 0, 1, \ldots, i-1$.

Let us inductively prove that λ_ℓ is the ℓ-element of the basis of Λ_γ, with $\ell = -1, 0, 1, \ldots, i-1$. The result is clear for $\ell = -1, 0$. If $\ell = 1$, we notice that the critical value t_1 is the for both Λ_C and Λ_γ since they share the (-1)-element and the 0-element of their bases. Then, by Theorem 5.2.10, we have that

$$\lambda_1 \leq \sup\{\nu_\gamma(\eta) : \nu_D(\eta) = t_1\} = \min(\Lambda_\gamma \setminus \Lambda_0) \leq \lambda_1.$$

Hence all the previous inequalities are equalities and we have that λ_1 is the 1-element of the basis of Λ_γ.

Now assume that the result holds for $1 \leq \ell \leq i-2$. Let us show that the $\lambda_{\ell+1}$ is the $(\ell+1)$-element of the basis of Λ_γ. As before, we notice that $\nu_\gamma(\omega_{\ell+1}) = \lambda_{\ell+1}$ and $\nu_D(\omega_{\ell+1}) = t_{\ell+1}$, since the elements of the bases of Λ_C and Λ_γ are the same up to index ℓ. We have that the critical value $t_{\ell+1}$ is the same for both semimodules. Thus

$$\lambda_{\ell+1} \leq \sup\{\nu_\gamma(\eta) : \nu_D(\eta) = t_{\ell+1}\} = \min(\Lambda_\gamma \setminus \Lambda_\ell) \leq \lambda_{\ell+1}.$$

We conclude the desired result. □

Proof of Proposition 7.4. Denote by $\mathcal{B}' = (\lambda'_{-1}, \lambda'_0, \ldots, \lambda'_{s'})$ the basis of Λ_γ. In virtue of Theorem 7.3 we have that $\lambda_\ell = \lambda'_\ell$ for $\ell = -1, 0, \ldots, i-1$. We only must show that $s' = i-1$. If $s' > i-1$, by definition there exists a 1-form ω' such that $\nu_\gamma(\omega') = \lambda'_i < \infty$. By Theorem 5.2.10, we see that $\nu_D(\omega') = t_i = \nu_D(\omega_i)$ since the elements of the bases of Λ_C and Λ_γ are common up to index $i-1$. Therefore,

$$\lambda'_i = \max\{\nu_\gamma(\eta);\ \nu_D(\eta) = t_i\}.$$

But we know that $\nu_\gamma(\omega) = \infty > \lambda'_i$, this is the desired contradiction. □

Now, we give several examples of different phenomena related to the previous results. First, we show that the inclusion in Corollary 7.5 may be strict.

Example 7.7. We consider the 1-form $\omega = 7x^2dy - 18xydx - 14/9ydy$. Notice that ω is basic and resonant with respect to the pair $(7, 18)$. We take the ω-cusp C given by the primitive parametrization

$$\phi(t) = (t^7, t^{18} + t^{22} + 10/9t^{26} + 319/243t^{30} + 1178/729t^{34} + h.o.t.).$$

As a remark, we can compute $\phi(t)$, by considering the parametrization $\varphi(t) = (t^7, t^{18} + \sum_{j=19}^{\infty} a_j t^j)$. Imposing the condition $\varphi^*\omega = 0$, we find the values of the coefficients a_j.

The reader can use the techniques from Chapter 5 to see that the basis of the semimodule of differential values of C is $(7, 18, 29)$. We also have that $\tilde{u}_2 = u_2^m = 119 = 29 + 5 \cdot 18 = 7 + 16 \cdot 7$ and $\tilde{t}_2 = 115$.

Now, we take the 1-form $\eta = 7xy^5dy - 18y^6dx - 4x^{16}dx$. We have that $\nu_D(\eta) = \tilde{t}_2$ and $\nu_C(\eta) = 123 > 119$. We consider the η-cusp C_1 defined by the primitive parametrization $\phi_1(t) = (t^7, 2t^{18} + 1/32t^{22} - 5/4096t^{26} + h.o.t.)$. The basis of the semimodule of differential values of C_1 is $(7, 18, 29, 40, \ldots)$. We do not compute all the elements of the basis, since we would need a more detailed parametrization. With the truncated parametrization that we have, we can check that $\nu_{C_1}(\omega_2) = 40$, where $\omega_2 = 7x^2dy - 18xydx - 64ydy$. This shows that the semimodules of differential values of C and C_1 are different.

The next example shows that in general the analytic semiroots are not analytically equivalent.

Example 7.8. Consider the curve C given by the primitive parametrization $\phi(t) = (t^7, t^{17} + t^{30} + t^{33} + t^{36})$ with $\Gamma_C = \langle 7, 17\rangle$ and semimodule of differential values $\Lambda_C = \Gamma_C(7, 17, 37, 57)$. A minimal standard basis of C is given by the 1-forms $\omega_{-1} = dx$, $\omega_0 = dy$, $\omega_1 = 7xdy - 17ydx$ and

$$\omega_2 = 3757x^2ydx - 1547x^3dy - 4624y^2dx + 1904xydy + 1183y^2dy.$$

All the ω_2-cusps are defined by the primitive parametrization

$$\varphi_a(t) = (t^7, at^{17} + a^3t^{30} + a^4t^{33} + \cdots)$$

with $a \in \mathbb{C}^*$. If we consider a new parameter $t = a^{-2/13}u$ and the we make the analytic change of variables $x_1 = a^{14/13}x$, $y_2 = a^{21/13}y$, we obtain that the family of cusps of ω_2 are the curves $C_a^{\omega_2}$ given by the parametrizations

$$\phi_a(u) = (u^7, u^{17} + u^{30} + a^{7/13}u^{33} + \cdots).$$

From the results above, we have that $\Lambda_{C_a^2} = \Lambda_1 = \Gamma_C(7, 17, 37)$ for all $a \in \mathbb{C}^*$. Since $33 \notin \Lambda_1 - 7$, by Theorem 1.5.2, two curves $C_{a_1}^2$ and $C_{a_2}^2$ are not, in general, analytically equivalent for $a_1, a_2 \in \mathbb{C}^*$. Recall that as it was mentioned in Section 1.5, finding normal form parametrizations, used to determine if two curves are analytically equivalent, is done by modifying the terms of smaller degree in the parametrization, and then iterating for bigger degree terms. Therefore, the displayed coefficients of the parametrizations $\phi_a(u)$ correspond with the ones of their normal form parametrizations.

Finally, we give an example showing the underlying problems to generalize the concept of analytic semiroot to more general families of branches.

Example 7.9. Consider the branch C defined by the primitive parametrization $\phi(t) = (t^{10}, t^{15}+t^{18})$. Notice that the characteristic exponents of C, as defined in Chapter 1, are $(10, 15, 18)$. Thus, by Equation (1.4), we have $\Gamma_C = \langle 10, 15, 33\rangle$. In fact, we see that for $z = y^2 - x^3$, we have that $\nu_C(z) = 33$.

By means of Büchberger's algorithm presented in Section 4.3, we can compute a minimal standard basis for C. Because of the difficulty of the computations we omit all of them. Moreover, for our purposes we only give a truncated minimal standard basis. We take the following 1-forms: $\omega_{-1} = dx$, $\omega_0 = dy$, $\omega_1 = 2xdy - 3ydx$, $\omega_2 = dz$, $\omega_3 = -11y\omega_1 + xdz$, $\omega_4 = -11x^2\omega_1 + ydz$. They give us the differential values $\lambda_{-1} = 10$, $\lambda_0 = 15$, $\lambda_1 = 28$, $\lambda_2 = 33$, $\lambda_3 = 46$, $\lambda_4 = 51$, and the basis of the semimodule of differential values of C is $(10, 15, 28, 33, 46, 51, \ldots)$, as we said, we do not need the whole basis.

Our interest now focus in both ω_3 and ω_4. If we denote by $\pi : (M_N, E^N) \to (M_0, P_0)$ a minimal resolution of singularities of C, notice that the divisor E_N^N is dicritical for both ω_3 and ω_4. More precisely, both 1-forms have infinite families of curves topologically equivalent to C. We consider the branches C_3 and C_4, defined by the primitive parametrizations $\phi_3(t)$ and $\phi_4(t)$ respectively, as follows:

$$\begin{aligned}\phi_3(t) &= (t^{10}, t^{15} + t^{18} - 1/2t^{21} + 1/2t^{24} - 5/8t^{27} + 7/8t^{30} + h.o.t.)\\ \phi_4(t) &= (t^{10}, t^{15} + t^{18} + 17/10t^{21} + 84/25t^{24} + 7163/1000t^{27} + h.o.t.),\end{aligned}$$

such that C_3 is invariant by ω_3 and C_4 is invariant by ω_4. We can check that the basis of the semimodule of differential values of C_3 is $(10, 15, 28, 33, 51, \ldots)$, with $\nu_{C_3}(\omega_4) = 51$. In the case of C_4, we have that the basis is $(10, 15, 28, 33, 46, \ldots)$, where $\nu_{C_4}(\omega_3) = 46$. In other words, ω_4 is a 1-form that belongs to a minimal standard basis of C_3, and ω_3 belongs to a minimal standard basis of C_4. Therefore, if we would like to extend our notion of analytic semiroots to non cuspidal branches, we have that C_3 and C_4 are analytic semiroots of C. However, we also see that C_3 is an analytic semiroot of C_4, and vice-versa, C_4 is an analytic semiroot of C_3. Nonetheless, $46 \in \Lambda_{C_4}$, but $46 \notin \Lambda_{C_3}$. Concluding that we can not generalize Theorem 7.4 to branches with a more complicated semigroup.

8

Saito Bases and Other Analytic Invariants

Let (C, P_0) be a plane curve in (M_0, P_0). In Chapter 2 we introduced informally the module of *logarithmic 1-forms* along C. We now precise the definition. Consider a meromorphic 1-form η in (M_0, P_0), that is, in any coordinate system (x, y), we can write $\eta = Adx + Bdy$, where A, B are meromorphic functions. Take $f = 0$ an implicit equation of C.

We say that η is a *logarithmic 1-form* along C if $f\eta$ and $\eta \wedge df$ are both holomorphic. We denote by $\Omega^1_{M_0,P_0}[\log C]$ the module of logarithmic 1-forms along C.

By changing the curve by any hypersurface in any regular ambient space, and the 1-forms for q-forms we extend the notion to the one of logarithmic q-forms along a hypersurface. However, as always, we are only going to work with our two dimensional case, thus we do not need this notion with such generality.

K. Saito introduced in [49] the notion of the logarithmic forms. Its relevance comes from the study of the Gauss-Manin connection, which appears when dealing with the monodromy map around a singular point of a hypersurface, see [8, 47].

Saito showed that $\Omega^1_{M_0,P_0}[\log C]$ is a free $\mathcal{O}_{M_0,P_0}$-module of rank two. Denote by $\Omega^1_{M_0,P_0}[C]$ the $\mathcal{O}_{M_0,P_0}$-submodule of $\Omega^1_{M_0,M_0}$ given by the 1-forms ω such that C is invariant by ω. We have that $\Omega^1_{M_0,P_0}[\log C]$ is isomorphic to $\Omega^1_{M_0,P_0}[C]$ as $\mathcal{O}_{M_0,P_0}$-modules, this isomorphism is given by the multiplication by f. Hence $\Omega^1_{M_0,P_0}[C]$ is also free module of rank two. Any basis $\{\eta_1, \eta_2\}$ of $\Omega^1_{M_0,P_0}[C]$ is called a *Saito basis* for C. A part from this, he also gave a characterization of the elements of a Saito basis. The result is the following.

Lemma 8.1 (Saito's Criterion [49])**.** *Let C' be a curve defined by the implicit equation $g = 0$. Given $\eta_1, \eta_2 \in \Omega^1_{M_0,P_0}[C']$, then $\{\eta_1, \eta_2\}$ is a Saito basis for C' if and only if*

$$\eta_1 \wedge \eta_2 = ugdx \wedge dy,$$

where $u \in \mathcal{O}_{M_0,P_0}$ is a unit, and (x, y) is the chosen coordinate system.

For the rest of the chapter, we assume that C is a cusp with Puiseux pair (n, m). As before, we denote by D the cuspidal divisor associated to the minimal resolution of singularities of C and (x, y) is a system of adapted coordinates with respect to C. Our main goal in this chapter is to prove the following result:

Theorem 8.2. *Denote by Λ_C the semimodule of differential values for the cusp C, with length $s \geq 0$. Let t_{s+1} and $\tilde{t}_{s+1}$ be the last critical values of Λ_C. Then, there are two 1-forms $\omega_{s+1}, \tilde{\omega}_{s+1}$ having C as an* *invariant curve and such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Moreover, for any pair of 1-forms as above, the set $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis for C.*

Note that the existence of the 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ is given by Propositions 6.3 and 6.4. Hence, we have to prove that $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ is a Saito basis for C. We prove it in several steps:

1. We prove Theorem 8.2 in the case $s = 0$.
2. We show that $\tilde{\mathcal{E}} \cup \{\omega_{s+1}\}$ generates the $\mathcal{O}_{M_0,P_0}$-module $\Omega^1_{M_0,P_0}[C]$, for any standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ for C that includes ω_{s+1} and $\tilde{\omega}_{s+1}$.
3. We show that any pair of 1-forms $\omega_{s+1}, \tilde{\omega}_{s+1}$ having C as invariant curve and such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$ are included in at least one special standard system $(\mathcal{E}, \tilde{\mathcal{E}})$. This result is proved in Proposition 8.3.1.
4. We conclude as follows. We start with $\{\omega_{s+1}, \tilde{\omega}_{s+1}\}$ and we consider a special standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ containing them. By Statement 3, any 1-form ω in the Saito module $\Omega^1_{M_0,P_0}[C]$ is a combination
$$\omega = h\omega_{s+1} + \textstyle\sum_{\ell=-1}^{s+1} f_\ell \tilde{\omega}_\ell.$$
Since $(\mathcal{E}, \tilde{\mathcal{E}})$ is a special standard system, each 1-form $\tilde{\omega}_\ell$ is a combination of $\omega_{s+1}, \tilde{\omega}_{s+1}$, for any $\ell = -1, 0, 1, \ldots, s$. In this way, we find a writing $\omega = f\omega_{s+1} + g\tilde{\omega}_{s+1}$, as desired.

The proof of Statement 2 relies on having proved Theorem 8.2 when $s = 0$, which corresponds to a quasi-homogeneous cusp. We consider this situation in next section.

8.1 The Quasi-Homogeneous Case

The statement of Theorem 8.2 when $s = 0$ is well know, see for instance [49]. Let us show it, for the sake of completeness. We recall that, by Theorem 1.5.2, the cusp C is quasi-homogeneous, and thus, C is analytically equivalent to the curve $f = 0$, where $f = y^n + \mu x^m$, for any $\mu \in \mathbb{C}^*$. In fact, we could assume $\mu = -1$. We can take

$$\omega_1 = nxdy - mydx, \quad \tilde{\omega}_1 = df = \mu m x^{m-1}dx + ny^{n-1}dy.$$

By Lemma 8.1, we have that $\{\omega_1, \tilde{\omega}_1\}$ is a Saito basis for C. Note that $\nu_D(\omega_1) = t_1 = n + m$ and $\nu_D(\tilde{\omega}_1) = \tilde{t}_1 = nm$.

Take now $\omega, \tilde{\omega}$ in $\Omega^1_{M_0,P_0}[C]$ being such that

$$\nu_D(\omega) = t_1 = n + m, \quad \nu_D(\tilde{\omega}) = \tilde{t}_1 = nm.$$

Let us see that $\{\omega, \tilde{\omega}\}$ is a Saito basis for C. Write

$$\omega = A\omega_1 + B\tilde{\omega}_1, \quad \tilde{\omega} = \tilde{A}\omega_1 + \tilde{B}\tilde{\omega}_1.$$

Since $n + m < nm$, we see that A is a unit. It is also obvious that $\tilde{A}$ is not a unit. If we show that $\tilde{B}$ is a unit, the determinant $A\tilde{B} - B\tilde{A}$ is a unit and hence $\{\omega, \tilde{\omega}\}$ is a Saito basis. Note that

$$\nu_D(\tilde{A}\omega_1) \neq nm.$$

Indeed, if $\nu_D(\tilde{A}\omega_1) = \nu_D(\tilde{A}) + n + m = nm$, we conclude that

$$\nu_D(\tilde{A}) = nm - n - m = c_\Gamma - 1 \in \Gamma_C.$$

This is a contradiction, since $c_\Gamma - 1 \in \Gamma_C$. Recall that by definition of conductor, c_Γ is the smallest element in Γ_C such that $k \in \Gamma_C$, for any $k \geq c_\Gamma$. We also recall that the divisorial value with respect to D always belongs to Γ_C. Then, we have that

$$\nu_D(\tilde{\omega}) = nm = \nu_D(\tilde{B}\tilde{\omega}_1) = \nu_D(\tilde{B}) + nm.$$

This implies that $\tilde{B}$ is a unit, as desired.

8.2 Generators of the Saito Module

Let us consider a standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ of C given by

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1}), \quad \tilde{\mathcal{E}} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}).$$

In next Proposition 8.2.5 we describe a generator system of the Saito module $\Omega^1_{M_0,P_0}[C]$.

Our arguments run by first by writing initial forms as a combination of those of our candidate of generator system. This would allow us to conclude in virtue of Artin's Approximation Theorem. This is done by working in an ordering the initial parts in terms of their divisorial values. To do this, we just need the concept of "partial standard system".

Consider an index $0 \leq j \leq s$. A *j-partial standard system associated to the extended standard basis* $\mathcal{E}$ is a pair $(\mathcal{E}, \tilde{\mathcal{E}}^j)$, where $\tilde{\mathcal{E}}^j$ is a list

$$\tilde{\mathcal{E}}^j = (\tilde{\omega}_{j+1}, \tilde{\omega}_{j+2}, \ldots, \tilde{\omega}_{s+1}),$$

such that $\nu_D(\tilde{\omega}_\ell) = \tilde{t}_\ell$ and $\tilde{\omega}_\ell \in \Omega^1_{M_0,P_0}[C]$, for $j+1 \leq \ell \leq s+1$.

We start by a lemma concerning the structure of the critical values of an increasing cuspidal semimodule:

Lemma 8.2.1. *Let Λ be an increasing cuspidal Γ-semimodule of length $s \geq 1$, where $\Gamma = \langle n, m\rangle$. Assume that the basis $\mathcal{B} = (\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ satisfies that $\lambda_{-1} = n$ and $\lambda_0 = m$. Consider the set*

$$T = \{t_{s+1}, \tilde{t}_2, \tilde{t}_3, \ldots, \tilde{t}_{s+1}\},$$

where $t_j, \tilde{t}_j$ are the critical values of Λ corresponding to the index j. Then, there are two nonnegative integer numbers $p, q \in \mathbb{Z}_{\geq} 0$ such that

$$\{np + n + m, mq + n + m\} \subset T.$$

Moreover, we have that $p < m - 2$ and $q \leq n - 2$.

We remark that, as we saw in Chapter 3, we have that $s \leq n - 2$. Thus, the assumption $s \geq 1$ implies that $n \geq 3$.

Proof. We start by noting that by Corollary 3.6.2, we have that any element in T is strictly smaller than $\tilde{t}_1 = nm$. This gives us the desired bounds on the indexes p and q.

Now, by definition of critical values, we know that one of the following mutually excluding properties holds:

(I) $\tilde{t}_2 = t_1 + n\ell_2^n = n + m + n\ell_2^n$
(II) $\tilde{t}_2 = t_1 + m\ell_2^m = n + m + m\ell_2^m$

Let us do the proof in the case (I), the case (II) has a similar proof. We can take $p = \ell_2^n$ and hence $\tilde{t}_2 = n + m + np \in T$. Then, it is enough to find an element of T of the form $n + m + mq$.

Assume first that $s = 1$. Then $t_{s+1} = t_2 = t_1 + m\ell_2^m = n + m + m\ell_2^m$. Taking $q = \ell_2^m$, we have that $t_{s+1} = n + m + mq \in T$, ending the proof when $s = 1$.

Assume now that $s > 1$. There are two cases:

a) For any $2 \leq i \leq s$, we have that $t_{i+1} - t_i = m\ell_{i+1}^m$.
b) There is an index i (that we take to be the minimum one) with $2 \leq i \leq s$ such that $t_{i+1} - t_i = n\ell_{i+1}^n$.

Assume we are in case a). Recall that $t_2 = t_1 + m\ell_2^m$, since $\tilde{t}_2 = t_1 + n\ell_2^n$. By a telescopic computation, we see that $t_{s+1} \in T$ may be written as

$$t_{s+1} = t_1 + m\left(\sum_{\ell=2}^{s+1} \ell_\ell^m\right) = n + m + mq.$$

Assume we are in case b). For any $2 \leq j \leq i$, we have that $t_j = t_{j-1} + m\ell_j^m$. By a telescopic computation, we obtain that $t_i = t_1 + mp_i$. The element $\tilde{t}_{i+1} \in T$ is given by $\tilde{t}_{i+1} = t_i + m\ell_{i+1}^m$ and hence we have that

$$\tilde{t}_{i+1} = t_1 + m(p_i + \ell_{i+1}^m),$$

as desired. This ends the proof. □

Remark 8.2.2. As a consequence of Lemma 8.2.1, we have the following property. Assume that Λ_C is the semimodule of differential values of a cusp C and $(\mathcal{E}, \tilde{\mathcal{E}})$ is a standard system, where

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1}), \quad \tilde{\mathcal{E}} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}).$$

Consider the set $\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_2, \ldots, \tilde{\omega}_s, \tilde{\omega}_{s+1}\}$. Assuming that (x, y) is a system of adapted coordinates with respect to the cusp C, there are two 1-forms $\eta_1, \eta_2 \in \mathcal{T}$ such that

$$\mathrm{In}(\eta_1) = \mu_1 x^p(mydx - nxdy), \quad \mathrm{In}(\eta_2) = \mu_2 y^q(mydx - nxdy),$$

where $\mu_1 \neq 0 \neq \mu_2$ and $0 \leq p < m - 2$, $0 \leq q \leq n - 2$.

Next lemma is the key argument for finding our generator system of Saito's module. It will be also important in order to find the Saito bases we are looking for.

Lemma 8.2.3. *Let us consider a standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ and a 1-form $\omega \in \Omega^1_{M_0,P_0}[C]$. Assume that (x, y) is a system of adapted coordinates with respect to C. Then, the initial form $\mathrm{In}(\omega)$ is a combination, with quasi-homogeneous coefficients, of the initial forms*

$$\mathrm{In}(\tilde{\omega}_1), \ldots, \mathrm{In}(\tilde{\omega}_{s+1}), \mathrm{In}(\omega_{s+1}).$$

Proof. Assume that $\phi(t) = (t^n, a_m t^m + h.o.t)$ is a primitive parametrization of C, with $a_m \neq 0$. Let us denote by $W = \mathrm{In}(\omega)$. By Proposition 2.3.16, the 1-form W has a quasi-homogeneous curve C_1 as an invariant curve, where $\phi_1(t) = (t^n, a_m t^m)$ is a primitive parametrization of C. Equivalently, C_1 is defined by the implicit equation $y^n + \mu x^m = \mathrm{In}(f)$, for $f = 0$ an implicit equation of C, where again $\mu \neq 0$. Let us invoke the result of Theorem 8.2 for the case of length zero established in subsection 8.1. In this case we consider the two 1-forms

$$W_1 = nxdy - mydx, \quad \tilde{W}_1 = ny^{n-1}dy + \mu m x^{m-1}dx,$$

that give a Saito basis $\{W_1, \tilde{W}_1\}$ of C_1. This gives a decomposition

$$W = HW_1 + \tilde{G}_1\tilde{W}_1,$$

where we can take $H, \tilde{G}_1$ to be quasi-homogeneous with respect to the weights (n, m). By Statement 2 of Proposition 5.3.5 and up to multiply $\tilde{\omega}_1$ by a constant, we have that

$$\mathrm{In}(\tilde{\omega}_1) = \tilde{W}_1.$$

Now, we are going to show the existence of a decomposition

$$HW_1 = G_{s+1}W_{s+1} + \sum_{\ell=2}^{s+1} \tilde{G}_\ell \tilde{W}_\ell, \quad \text{where } \tilde{W}_\ell = \mathrm{In}(\tilde{\omega}_\ell),\ W_{s+1} = \mathrm{In}(\omega_{s+1}), \tag{8.1}$$

with all the coefficients G_{s+1} and $\tilde{G}_\ell$ being quasi-homogeneous.

Let $\delta = \nu_D(HW_1)$. Since H is a quasi-homogeneous polynomial, we can write

$$HW_1 = \sum_{\alpha n + \beta m = \delta} W_{\alpha\beta}, \quad W_{\alpha\beta} = \mu_{\alpha\beta} x^\alpha y^\beta \left(n\frac{dy}{y} - m\frac{dx}{x} \right), \quad \alpha, \beta \geq 1.$$

Now, it is enough to show that each of the 1-forms $W_{\alpha\beta}$ is reachable by one of the 1-forms in the set

$$\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_2, \ldots, \tilde{\omega}_{s+1}\}.$$

We consider two cases:

a) There is a differential 1-form $W_{\alpha\beta} \neq 0$ such that $\alpha \geq m$ or $\beta \geq n$.
b) For any $W_{\alpha\beta} \neq 0$ we have that $\alpha < m$ and $\beta < n$.

Assume we are in case a). By a straightforward verification, we see that all the terms $W_{\alpha\beta} \neq 0$ satisfy the condition that either $\alpha \geq m$ or $\beta \geq n$. Note that the indexes of the 1-forms $W_{\alpha\beta}$ and $W_{\alpha'\beta'}$ are related by $\alpha = \alpha' + m\ell$ and $\beta = \beta' + n\ell$, for some $\ell \in \mathbb{Z}$. In view of Lemma 8.2.1 and Remark 8.2.2, we see that each $W_{\alpha\beta} \neq 0$ is reachable by an element of $\mathcal{T}$.

Assume now that we are in case b). Then, following the same argument as before, we have that there is only one 1-form $W_{\alpha\beta} \neq 0$ and hence, we have

$$HW_1 = \mu_{\alpha\beta} x^{\alpha-1} y^{\beta-1}(mydx - nxdy), \quad 1 \leq \alpha < m,\ 1 \leq \beta < n.$$

Moreover, we have that $\tilde{G}_1\tilde{W}_1 = 0$. Indeed, we know that

$$\tilde{G}_1\tilde{W}_1 = \tilde{G}_1(ny^{n-1}dy + \mu m x^{m-1}dx)$$

and, if we write, as before,

$$\tilde{G}_1\tilde{W}_1 = \sum_{\alpha' n + \beta' m = \delta} \tilde{W}_{\alpha'\beta'}, \quad \tilde{W}_{\alpha'\beta'} = x^{\alpha'} y^{\beta'} \left(\mu_{\alpha'\beta'}\frac{dy}{y} \xi_{\alpha'\beta'}\frac{dx}{x} \right), \quad \alpha, \beta \geq 1.$$

Then, the terms $\tilde{W}_{\alpha'\beta'}$ fit in the description of a) and with the same divisorial value as $W_{\alpha\beta}$, this contradicts the already proven fact that if one term satisfies condition a), then the rest of the terms must also satisfy condition a). We conclude that

$$\mathrm{In}(\omega) = W = HW_1 = \mu_{\alpha\beta} x^{\alpha-1} y^{\beta-1}(mydx - nxdy) = \mu_{\alpha\beta} x^{\alpha-1} y^{\beta-1} W_1.$$

Note that ω is then reachable by ω_1. Let q be the maximum index $1 \leq q \leq s+1$ such that ω is reachable by ω_q. Note that the case $q = s+1$ is precisely a case covered by the statement we aim to prove. Thus, we assume that $1 \leq q \leq s$. Write

$$\eta = \omega - \mu'' x^a y^b \omega_q, \quad \nu_D(\eta) > \nu_D(\omega).$$

We have that $\nu_C(\eta) = \nu_C(x^a y^b \omega_q)$. We can invoke Statement 4 in Theorem 5.2.10 to obtain that $\nu_C(\eta) \in \Lambda_{q-1}$, that is

$$\lambda_q + na + mb \in \Lambda_{q-1}.$$

Where we recall that Λ_{q-1} is an element of the decomposition sequence of Λ_C. By Lemma 3.2.5, we have that either $a \geq \ell^n_{q+1}$ or $b \geq \ell^m_{q+1}$. Assume that $a \geq \ell^n_{q+1}$. If $u_{q+1} = u^n_{q+1}$, then ω is reachable by ω_{q+1}, contradicting the maximality of q, if $u_{q+1} = u^m_{q+1}$, we obtain that ω is reachable by $\tilde{\omega}_{q+1}$, as required. We recall that the reachability is obtained by the fact that the initial parts of ω_{q+1} and $\tilde{\omega}_{q+1}$ are proportional to the ones of $x^{\ell^n_{q+1}}\omega_q$ and $y^{\ell^m_{q+1}}\omega_q$, see Theorem 5.3.1. Same arguments for the case that $b \geq \ell^m_{q+1}$. This ends the proof. □

Remark 8.2.4. Let $(\mathcal{E}, \tilde{\mathcal{E}}^j)$, with $\tilde{\mathcal{E}}^j = (\tilde{\omega}_{j+1}, \tilde{\omega}_{j+2}, \dots, \tilde{\omega}_{s+1})$, be a j-partial standard system, with $j \geq 1$ and take a 1-form $\omega \in \Omega^1_{M_0,P_0}[C]$ such that $\nu_D(\omega) < \tilde{t}_j$. By the same arguments as in the preceding lemma, noting that $\tilde{t}_j < \tilde{t}_{j-1} < \cdots < \tilde{t}_1$, we see that there is a combination

$$\mathrm{In}(\omega) = G_{s+1}W_{s+1} + \sum_{\ell=j+1}^{s+1} \tilde{G}_\ell \tilde{W}_\ell, \quad \text{where } \tilde{W}_\ell = \mathrm{In}(\tilde{\omega}_\ell),\ W_{s+1} = \mathrm{In}(\omega_{s+1}), \tag{8.2}$$

all the coefficients being quasi-homogeneous of the corresponding degree.

Proposition 8.2.5. *The set $\mathcal{T} = \{\omega_{s+1}, \tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_{s+1}\}$ is a generating system of the Saito $\mathcal{O}_{M_0,P_0}$-module $\Omega^1_{M_0,P_0}[C]$.*

Proof. Take $\omega \in \Omega^1_{M_0,P_0}[C]$, we know the existence of a decomposition

$$\mathrm{In}(\omega) = G_{s+1}W_{s+1} + \sum_{\ell=1}^{s+1} \tilde{G}_\ell \tilde{W}_\ell, \quad \text{where } \tilde{W}_\ell = \mathrm{In}(\tilde{\omega}_\ell),\ W_{s+1} = \mathrm{In}(\omega_{s+1}),$$

with all the coefficients G_{s+1} and $\tilde{G}_\ell$ being quasi-homogeneous. We re-start the procedure of Lemma 8.2.3 with

$$\omega' = \omega - \left(G_{s+1}\omega_{s+1} + \sum_{\ell=1}^{s+1} \tilde{G}_\ell \tilde{\omega}_\ell \right).$$

In this way, we obtain a formal expression $\omega = \hat{g}_{s+1}\omega_{s+1} + \sum_{\ell=1}^{s+1} \widehat{\tilde{g}}_\ell \tilde{\omega}_\ell$. By a direct application of Artin's Approximation Theorem [5], we obtain the desired convergent expression

$$\omega = g_{s+1}\omega_{s+1} + \textstyle\sum_{\ell=1}^{s+1} \tilde{g}_\ell \tilde{\omega}_\ell.$$

□

8.3 Existence of Special Standard Systems

We recall that in Definition 6.1, we introduce the notion of special standard system. This subsection is devoted to provide a proof of the following result

Proposition 8.3.1. *Assume that the length s of the semimodule Λ_C of differential values of the cusp C is $s \geq 1$. Take two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ in $\Omega^1_{M_0,P_0}[C]$ such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Then, there is a special standard system $(\mathcal{E}, \tilde{\mathcal{E}})$ for C containing $\omega_{s+1}, \tilde{\omega}_{s+1}$ in the sense that*

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \dots, \omega_s, \omega_{s+1}), \quad \tilde{\mathcal{E}} = (\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_s, \tilde{\omega}_{s+1}).$$

The proof of the above proposition follows directly from next result

Proposition 8.3.2. *Assume that the length s of the semimodule Λ_C of differential values of the cusp C is $s \geq 1$. Take two 1-forms ω_{s+1} and $\tilde{\omega}_{s+1}$ in $\Omega^1_{M_0,P_0}[C]$ such that $\nu_D(\omega_{s+1}) = t_{s+1}$ and $\nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. For any index $1 \leq j \leq s$ there are functions $f_j, \tilde{f}_j$ such that*

$$\nu_D(f_j\omega_{s+1} + \tilde{f}_j\tilde{\omega}_{s+1}) = \tilde{t}_j.$$

Along the whole proof, we consider an extended standard basis

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s, \omega_{s+1})$$

ending at ω_{s+1}. The proof of Proposition 8.3.2 is quite long. In order to make clear the arguments, we do it in two steps:

- *Step 1: case $j = s$.* That is, we find $\tilde{\omega}_s \in \Omega^1_{M_0,P_0}[C]$ such that $\nu_D(\tilde{\omega}_s) = \tilde{t}_s$.
- *Step 2: The general case.*

Even though in *Step 2* we assume taking a in index $j < s$, in fact, the proof also holds to the case of *Step 1*. Thus, the reader can skip the *Step 1*.

8.3.1 First Case

This subsection is devoted to the proof of Proposition 8.3.2 when $j = s$. We are going to prove that there is a combination

$$\tilde{\omega}_s = \tilde{f}_s\tilde{\omega}_{s+1} + f_s\omega_{s+1}$$

such that $\nu_D(\tilde{\omega}_s) = \tilde{t}_s$.

There are two possible cases: $t_{s+1} = t_s + n\ell^n_{s+1}$ and $t_{s+1} = t_s + m\ell^m_{s+1}$. Both cases run in a similar way. We assume from now on that $t_{s+1} = t_s + n\ell^n_{s+1}$ and hence we have $\tilde{t}_{s+1} = t_s + m\ell^m_{s+1}$. With the notations as in Chapter 5, let us write Delorme's decompositions of $\tilde{\omega}_{s+1}$ and ω_{s+1} as follows

$$\tilde{\omega}_{s+1} = \tilde{\mu}_1 y^{\ell^m_{s+1}}\omega_s + \tilde{\mu}_2 x^{a_{s+1}}\omega_{k^m_s} + \tilde{\eta}, \quad \tilde{\eta} = \sum_{\ell=-1}^{s} \tilde{h}_\ell\omega_\ell, \tag{8.3}$$

$$\omega_{s+1} = \mu_1 x^{\ell^n_{s+1}}\omega_s + \mu_2 y^{b_{s+1}}\omega_{k^n_s} + \eta, \quad \eta = \sum_{\ell=-1}^{s} h_\ell\omega_\ell \tag{8.4}$$

where we have the following properties:

1. $\text{In}(\omega_{s+1}) = \mu_1 \text{In}(x^{\ell^n_{s+1}}\omega_s)$. Recall that $t_{s+1} = t_s + n\ell^n_{s+1}$.
2. $\text{In}(\tilde{\omega}_{s+1}) = \tilde{\mu}_1 \text{In}(y^{\ell^m_{s+1}}\omega_s)$. Recall that $\tilde{t}_{s+1} = t_s + m\ell^m_{s+1}$.
3. $\nu_C(\mu_1 x^{\ell^n_{s+1}}\omega_s + \mu_2 y^{b_{s+1}}\omega_{k^n_s}) > \nu_C(\mu_1 x^{\ell^n_{s+1}}\omega_s) = \nu_C(\mu_2 y^{b_{s+1}}\omega_{k^n_s}) = u^n_{s+1} = u_{s+1}$. Recall that $u^n_{s+1} = \lambda_s + n\ell^n_{s+1} = \lambda_{k^n_s} + mb_{s+1}$.
4. $\nu_C(\tilde{\mu}_1 y^{\ell^m_{s+1}}\omega_s + \tilde{\mu}_2 x^{a_{s+1}}\omega_{k^m_s}) > \nu_C(\mu_1 y^{\ell^m_{s+1}}\omega_s) = \nu_C(\mu_2 x^{a_{s+1}}\omega_{k^m_s}) = u^m_{s+1} = \tilde{u}_{s+1}$. Recall that $u^m_{s+1} = \lambda_s + m\ell^m_{s+1} = \lambda_{k^m_s} + na_{s+1}$.
5. For any $-1 \leq \ell \leq s$, we have that $\nu_C(h_\ell\omega_\ell) > u^n_{s+1}$ and $\nu_C(\tilde{h}_\ell\omega_\ell) > u^m_{s+1}$.

Let us consider the 1-form $\theta_0 \in \Omega^1_{M_0,P_0}[C]$ defined by

$$\theta_{\mathbf{0}} = \mu_1 x^{\ell^n_{s+1}}\tilde{\omega}_{s+1} - \tilde{\mu}_1 y^{\ell^m_{s+1}}\omega_{s+1} = \xi + \zeta_{\mathbf{0}},$$

where $\xi = \tilde{\mu}_3 x^{\ell^n_{s+1}+a_{s+1}}\omega_{k^m_s} - \mu_3 y^{\ell^m_{s+1}+b_{s+1}}\omega_{k^n_s}$, with $\tilde{\mu}_3 = \mu_1\tilde{\mu}_2$, $\mu_3 = \tilde{\mu}_1\mu_2$ and such that $\zeta_{\mathbf{0}} = \sum_{\ell=-1}^{s} g^{\mathbf{0}}_\ell\omega_\ell$. In virtue of Equations (8.3) and (8.4), we obtain that $\zeta_{\mathbf{0}} = \mu_1 x^{\ell^n_{s+1}}\tilde{\eta} - \tilde{\mu}_1 y^{\ell^m_{s+1}}\eta$, therefore

$$g^{\mathbf{0}}_\ell = \mu_1 x^{\ell^n_{s+1}}\tilde{h}_\ell - \tilde{\mu}_1 y^{\ell^m_{s+1}}h_\ell, \quad \text{for } \ell = -1, 0, \ldots, s. \tag{8.5}$$

In a more general way, given a pair of functions $\tilde{f}, f \in \mathcal{O}_{M_0,P_0}$, we write

$$\theta_{\tilde{f},f} = \theta_{\mathbf{0}} + \tilde{f}\tilde{\omega}_{s+1} + f\omega_{s+1} = \xi + \zeta_{\tilde{f},f} \in \Omega^1_{M_0,P_0}[C],$$

where $\zeta_{\tilde{f},f} = \zeta_{\mathbf{0}} + \tilde{f}\tilde{\omega}_{s+1} + f\omega_{s+1}$. We also write $\zeta_{\tilde{f},f} = \sum_{\ell=-1}^{s} g_\ell^{\tilde{f},f}\omega_\ell$. Let us note that $\theta_{\mathbf{0}} = \theta_{0,0}$, $\zeta_{\mathbf{0}} = \zeta_{0,0}$ and $g_\ell^{\mathbf{0}} = g_\ell^{0,0}$, for $-1 \le \ell \le s$.

In order to prove the desired result, we are going to show the existence of a pair $\tilde{f}, f$ such that $\nu_D(\theta_{\tilde{f},f}) = \tilde{t}_s$.

We have two options: $u_s = u_s^n$ and $u_s = u_s^m$. Both cases run in a similar way. So, we fix the case that $u_s = u_s^n$. Hence, we have $t_s = t_s^n$, $\tilde{u}_s = u_s^m$ and $\tilde{t}_s = t_s^m$. By Proposition 3.5.9, we know that $k_s^n = s-1$ and $k_s^m = k_{s-1}^m$. Let us check that the divisorial value of ξ is $\tilde{t}_s$.

Lemma 8.3.3. $\nu_D(\xi) = \tilde{t}_s$.

Proof. By Proposition 3.6.3, the colimits a_{s+1} and b_{s+1} satisfy that $b_{s+1} + \ell_{s+1}^m = \ell_s^m$ and $a_{s+1} + \ell_{s+1}^n = a_s$. Hence, we have that

$$\xi = -\mu_3 y^{\ell_s^m}\omega_{k_s^n} + \tilde{\mu}_3 x^{a_s}\omega_{k_s^m} = -\mu_3 y^{\ell_s^m}\omega_{s-1} + \tilde{\mu}_3 x^{a_s}\omega_{k_{s-1}^m}.$$

Let us show that $\nu_D(\xi) = \tilde{t}_s$. Note that $\nu_D(y^{\ell_s^m}\omega_{s-1}) = m\ell_s^m + t_{s-1} = t_s^m = \tilde{t}_s$. Thus, it is enough to show that $\nu_D(x^{a_s}\omega_{k_{s-1}^m}) > \tilde{t}_s = t_s^m$. We have $\nu_D(x^{a_s}\omega_{k_{s-1}^m}) = na_s + t_{k_{s-1}^m}$. Since $u_s^m = m\ell_s^m + \lambda_{s-1} = na_s + \lambda_{k_{s-1}^m}$, then, by Lemma 3.2.8, it follows that

$$\begin{aligned} na_s - m\ell_s^m &= \lambda_{s-1} - \lambda_{k_{s-1}^m} > t_{s-1} - t_{k_{s-1}^m} \Rightarrow \\ &\Rightarrow na_s + t_{k_{s-1}^m} > \tilde{t}_s = t_{s-1} + m\ell_s^m. \end{aligned}$$

We conclude that $\nu_D(\xi) = \tilde{t}_s$. □

The problem is reduced to finding $\tilde{f}, f$ such that $\nu_D(\zeta_{f,\tilde{f}}) > \tilde{t}_s$. We proceed to verify this.

We say that a pair of functions $\tilde{f}, f$ is a *good pair* if and only if we have that $\nu_C(g_\ell^{\tilde{f},f}\omega_\ell) > \tilde{u}_s$, for any $\ell = -1, 0, \ldots, s$.

We end the proof as a direct consequence of the following lemmas:

Lemma 8.3.4. *The pair $\tilde{f} = 0, f = 0$ is a good pair.*

Lemma 8.3.5. *If $\tilde{f}, f$ is a good pair, then $\nu_D(g_\ell^{\tilde{f},f}\omega_\ell) > \tilde{t}_s$, for $-1 \le \ell \le s-1$ and $\nu_D(g_s^{\tilde{f},f}\omega_s) \ne \tilde{t}_s$.*

Corollary 8.3.6. *Assume that $\tilde{f}, f$ is a good pair. Then, we have that either $\nu_D(\theta_{\tilde{f},f}) = \tilde{t}_s$ or $\nu_D(\theta_{\tilde{f},f}) = \nu_D(g_s^{\tilde{f},f}\omega_s) < \tilde{t}_s$.*

Lemma 8.3.7. *If $\tilde{f}, f$ is a good pair and $\nu_D(\theta_{\tilde{f},f}) < \tilde{t}_s$, then there is another good pair $\tilde{f}_1, f_1$ such that $\nu_D(g_s^{\tilde{f}_1,f_1}\omega_s) > \nu_D(g_s^{\tilde{f},f}\omega_s)$.*

Indeed, by Lemma 8.3.4, there is at least one good pair, by Lemma 8.3.5 and Lemma 8.3.3 we obtain Corollary 8.3.6. Now, we apply repeatedly Lemma 8.3.7 to get that $\nu_D(g_s^{\tilde{f},f}\omega_s) \ge \tilde{t}_s$, hence, in view of Lemmas 8.3.3 and 8.3.4, we get that $\nu_D(g_s^{\tilde{f},f}\omega_s) > \tilde{t}_s$ and $\nu_D(\theta_{\tilde{f},f}) = \tilde{t}_s$ as desired.

The rest of this subsection is devoted to proving the above three Lemmas 8.3.4, 8.3.5 and 8.3.7.

Proof of Lemma 8.3.4. We have to prove that

$$\nu_C(g_\ell^{\mathbf{0}}\omega_\ell) > \tilde{u}_s, \text{ for any } \ell = -1, 0, \ldots, s.$$

By Equation (8.5) we have that $g_\ell^{\mathbf{0}} = \mu_1 x^{\ell_{s+1}^n}\tilde{h}_\ell - \tilde{\mu}_1 y^{\ell_{s+1}^m} h_\ell$, for any $\ell = -1, 0, \ldots, s$. Now, it is enough to show that

$$\nu_C(x^{\ell_{s+1}^n}\tilde{h}_\ell\omega_\ell) > \tilde{u}_s \text{ and } \nu_C(y^{\ell_{s+1}^m} h_\ell\omega_\ell) > \tilde{u}_s.$$

We have that

$$\begin{aligned}
\nu_C(x^{\ell_{s+1}^n}\tilde{h}_\ell\omega_\ell) &= n\ell_{s+1}^n + \nu_C(\tilde{h}_\ell\omega_\ell) > n\ell_{s+1}^n + u_{s+1}^m = n\ell_{s+1}^n + \tilde{u}_{s+1}\\
&= n\ell_{s+1}^n + na_{s+1} + \lambda_{k_s^m} = n(\ell_{s+1}^n + a_{s+1}) + \lambda_{k_{s-1}^m}\\
&= na_s + \lambda_{k_{s-1}^m} = u_s^m = \tilde{u}_s\\
\nu_C(y^{\ell_{s+1}^m} h_\ell\omega_\ell) &= m\ell_{s+1}^m + \nu_C(h_\ell\omega_\ell) > m\ell_{s+1}^m + u_{s+1}^n = m\ell_{s+1}^m + u_{s+1}\\
&= m\ell_{s+1}^m + mb_{s+1} + \lambda_{k_s^n} = m(\ell_{s+1}^m + b_{s+1}) + \lambda_{s-1}\\
&= m\ell_s^m + \lambda_{s-1} = u_s^m = \tilde{u}_s.
\end{aligned}$$

Recall that the equalities $\ell_{s+1}^n + a_{s+1} = a_s$ and $\ell_{s+1}^m + b_{s+1} = \ell_s^m$ comes from Proposition 3.6.3. This ends the proof of Lemma 8.3.4. □

Proof of Lemma 8.3.5. Along the proof of this lemma, we just write $g_\ell^{\tilde{f},f} = g_\ell$, in order to simplify the notation.

Let us first show that $\nu_D(g_\ell\omega_\ell) > \tilde{t}_s$, for any $-1 \le \ell \le s-1$. Recall that $\nu_C(g_\ell\omega_\ell) > \tilde{u}_s$ and write

$$\nu_C(g_\ell\omega_\ell) = \nu_C(g_\ell) + \lambda_\ell > \tilde{u}_s = u_s^m = \lambda_{s-1} + m\ell_s^m.$$

Noting that $\lambda_{s-1} - \lambda_\ell \ge t_{s-1} - t_\ell$, in view of Lemma 3.2.8, we have that

$$\nu_C(g_\ell) + \lambda_{s-1} > \lambda_{s-1} + t_{s-1} - t_\ell + m\ell_s^m$$

and thus we have $\nu_C(g_\ell) + t_\ell > t_{s-1} + m\ell_s^m = t_s^m = \tilde{t}_s$.

There are two cases: if $\nu_D(g_\ell) < nm$, then $\nu_C(g_\ell) = \nu_D(g_\ell)$ (Proposition 2.3.14). Second, $\nu_D(g_\ell) \ge nm$. Noting that $\tilde{t}_s \le nm$, see Corollary 3.6.2, we conclude in both cases that

$$\nu_D(g_\ell\omega_\ell) = \nu_D(g_\ell) + t_\ell > \tilde{t}_s,$$

as desired.

Let us show that $\nu_D(g_s\omega_s) \neq \tilde{t}_s$. Assume by contradiction that $\nu_D(g_s\omega_s) = \tilde{t}_s$. Recalling that $t_s = t_s^n, \tilde{t}_s = t_s^m, t_s^n = t_{s-1} + n\ell_s^n$ and $t_s^m = t_{s-1} + m\ell_s^m$, we have

$$\begin{aligned}
\nu_D(g_s\omega_s) = \tilde{t}_s &\Rightarrow \nu_D(g_s) + t_s = \tilde{t}_s \Rightarrow \nu_D(g_s) + t_s^n = t_s^m \Rightarrow\\
&\Rightarrow \nu_D(g_s) + t_{s-1} + n\ell_s^n = t_{s-1} + m\ell_s^m \Rightarrow\\
&\Rightarrow m\ell_s^m = \nu_D(g_s) + n\ell_s^n.
\end{aligned}$$

This implies that $m\ell_s^m \in \Gamma_C$ is written in two different ways as a combination of n, m with nonnegative integer coefficients. This is not possible, since $m\ell_s^m < nm$, in view of Remark 3.2.3. The proof of Lemma 8.3.5 is ended. □

Proof of Lemma 8.3.7. Assume that $\tilde{f}, f$ is a good pair with $\nu_D(\theta_{\tilde{f},f}) < \tilde{t}_s$. Let us find another good pair $\tilde{f}_1, f_1$ such that $\nu_D(g_s^{\tilde{f}_1,f_1}\omega_s) > \nu_D(g_s^{\tilde{f},f}\omega_s)$.

Since $\nu_D(\xi) = \tilde{t}_s$ (Lemma 8.3.3), $\theta_{\tilde{f},f} = \xi + \zeta_{\tilde{f},f}$ and $\nu_D(\theta_{\tilde{f},f}) < \tilde{t}_s$, we know that $\mathrm{In}(\theta_{\tilde{f},f}) = \mathrm{In}(\zeta_{\tilde{f},f})$. In particular $\nu_D(\zeta_{\tilde{f},f}) = \nu_D(\theta_{\tilde{f},f})$. Applying Lemma 8.3.5, we get that

$$\mathrm{In}(\theta_{\tilde{f},f}) = \mathrm{In}(\zeta_{\tilde{f},f}) = \mathrm{In}(g_s^{\tilde{f},f}\omega_s) = \mathrm{In}(g_s^{\tilde{f},f})\,\mathrm{In}(\omega_s).$$

Noting that $\nu_D(\theta_{\tilde{f},f}) < \tilde{t}_s \leq nm$, we have that $\nu_D(g_s^{\tilde{f},f}) < nm$. Hence, for certain $a, b \geq 0$ and $\mu' \neq 0$, we can write $\mathrm{In}(g_s^{\tilde{f},f}) = \mu' x^a y^b$. Now we consider the decomposition

$$\theta_{\tilde{f},f} = \mu' x^a y^b \omega_s + \eta', \quad \nu_D(\eta') > \nu_D(x^a y^b \omega_s).$$

As $\nu_C(\theta_{\tilde{f},f}) = \infty$, we have that $\nu_C(\eta') = \nu_C(\mu' x^a y^b \omega_s) = na + mb + \lambda_s$. Let us apply Theorem 5.2.10, Statement 5, to the integer number $k = na + mb + \lambda_s$. Since the 1-form η' satisfies both $\nu_C(\eta') = k$ and $\nu_D(\eta') > \nu_D(x^a y^b \omega_s)$, we conclude that $k \in \Lambda_{s-1}$. By Lemma 3.2.5, we know that one of the following properties holds:

$$a \geq \ell_{s+1}^n \quad \text{or} \quad b \geq \ell_{s+1}^m.$$

Let us show that $\theta_{\tilde{f},f}$ is reachable by ω_{s+1} or from $\tilde{\omega}_{s+1}$. Assume that $a \geq \ell_{s+1}^n$, then we have that

$$\nu_D(\theta_{\tilde{f},f}) \quad = \quad na + mb + t_s = n\ell_{s+1}^n + t_s + n(a - \ell_{s+1}^n) + mb = t_{s+1}^n + n(a - \ell_{s+1}^n) + mb.$$

Noting that $t_{s+1} = t_{s+1}^n$, we have that $\theta_{\tilde{f},f}$ and $x^{a-\ell_{s+1}^n} y^b \omega_{s+1}$ have the same initial parts (up to a constant) and thus $\theta_{\tilde{f},f}$ is reachable by ω_{s+1}. In the same way, if we assume that $b \geq \ell_{s+1}^m$, we have

$$\nu_D(\theta_{\tilde{f},f}) \quad = \quad na + mb + t_s = m\ell_{s+1}^m + t_s + m(b - \ell_{s+1}^m) + na = t_{s+1}^m + m(b - \ell_{s+1}^m) + na.$$

We conclude as above that $\theta_{\tilde{f},f}$ is reachable by $\tilde{\omega}_{s+1}$.

Assume now that $a \geq \ell_{s+1}^n$ and hence $\theta_{\tilde{f},f}$ is reachable by ω_{s+1}, the case $b \geq \ell_{s+1}^m$ is treated in a similar way. There is a constant $\mu_3 \neq 0$ such that

$$\nu_D(\theta_{\tilde{f},f} - \mu_3 x^{a-\ell_{s+1}^n} y^b \omega_{s+1}) > \nu_D(\theta_{\tilde{f},f}).$$

Let us put $\tilde{f}_1 = \tilde{f}$ and $f_1 = f - \mu_3 x^{a-\ell_{s+1}^n} y^b$. Note that

$$\theta_{\tilde{f}_1,f_1} = \theta_{\tilde{f},f} - \mu_3 x^{a-\ell_{s+1}^n} y^b \omega_{s+1}$$

and hence $\nu_D(\theta_{\tilde{f}_1,f_1}) > \nu_D(\theta_{\tilde{f},f})$.

Let us verify that $\tilde{f}_1, f_1$ is a good pair. Using the decomposition of ω_{s+1} in Equation (8.4), we write

$$x^{a-\ell_{s+1}^n} y^b \omega_{s+1} = \textstyle\sum_{\ell=-1}^{s} g'_\ell \omega_\ell,$$

We note that

$$\zeta_{\tilde{f}_1,f} = \zeta_{\tilde{f},f} - \mu_3 x^{a-\ell_{s+1}^n} y^b \omega_{s+1}.$$

Since $\tilde{f}, f$ is a good pair, we have that $\nu_C(g_\ell^{\tilde{f},f}\omega_\ell) > \tilde{u}_s$. Thus $\tilde{f}_1, f_1$ is a good pair if $\nu_C(g'_\ell \omega_\ell) > \tilde{u}_s$, for $\ell = -1, 0, \ldots, s$. Let us show that this is true. Since the terms $g'_\ell \omega_\ell$, for $-1 \leq \ell \leq s$, come from the decomposition of ω_{s+1} times a monomial, as a consequence of Theorem 5.3.1 we see that

$$\nu_C(g'_s \omega_s) \leq \nu_C(g'_\ell \omega_\ell), \quad \text{for } -1 \leq \ell \leq s.$$

Hence, it is enough to show that $\nu_C(g'_s \omega_s) > \tilde{u}_s$. Notice that

$$\mathrm{In}(\zeta_{\tilde{f},f}) = \mathrm{In}(g_s^{\tilde{f},f}\omega_s) = \mu_3\,\mathrm{In}(x^{a-\ell_{s+1}^n} y^b \omega_{s+1}) = \mu_3\,\mathrm{In}(g'_s \omega_s),$$

where the last equality comes from Corollary 5.3.4. Thus, we have

$$\nu_D(g_s^{\tilde f,f}\omega_s) = \nu_D(g_s'\omega_s) < \tilde t_s \le nm.$$

Therefore, $\nu_D(g_s^{\tilde f,f}) = \nu_D(g_s') < nm$. This implies that

$$\nu_D(g_s^{\tilde f,f}) = \nu_C(g_s^{\tilde f,f}) = \nu_C(g_s') = \nu_D(g_s').$$

Since $\tilde f, f$ is a good pair, we conclude that $\nu_C(g_s'\omega_s) = \nu_C(g_s^{\tilde f,f}\omega_s) > \tilde u_s$. If $b \ge \ell^m_{s+1}$, then $\theta_{\tilde f,f}$ is reachable by $\tilde\omega_{s+1}$ and we proceed in a similar way. This ends the proof of Lemma 8.3.7. □

8.3.2 Induction Step

This subsection is devoted to the proof of Proposition 8.3.2 when $1 \le j < s$, assuming that the result is true for $j+1, j+2, \ldots, s$. We are going to prove that there is a combination

$$\tilde\omega_j = \tilde f_j \tilde\omega_{s+1} + f_j \omega_{s+1}$$

such that $\nu_D(\tilde\omega_j) = \tilde t_j$, under the assumption that for any $j+1 \le \ell \le s$ there is a combination $\tilde\omega_\ell = \tilde f_\ell \tilde\omega_{s+1} + f_\ell \omega_{s+1}$, such that $\nu_D(\tilde\omega_\ell) = \tilde t_\ell$.

The proof is very similar to the case $j = s$. Recall that $\nu_D(\tilde\omega_{j+1}) = \tilde t_{j+1}$. There are two options, either $\tilde t_{j+1} = t^n_{j+1}$ or $\tilde t_{j+1} = t^m_{j+1}$. In both cases, the proof runs in a similar way. We fix from now on the option $\tilde t_{j+1} = t^m_{j+1}$.

Let us define the number $q \in \{j+2, \ldots, s+1\}$ as follows

$$q = \begin{cases} s+1, \text{ if } \tilde t_\ell = t^n_\ell, \text{ for } \ell = j+2, j+3, \ldots, s+1, \\ \min\{\ell;\ \tilde t_\ell = t^m_\ell,\ j+2 \le \ell \le s+1\}, \text{ otherwise}, \end{cases}$$

and define the 1-form $\widehat\omega_q$ as follows:

$$\widehat\omega_q = \begin{cases} \omega_{s+1}, & \text{if } \tilde t_\ell = t^n_\ell, \text{ for } \ell = j+2, j+3, \ldots, s+1, \\ \tilde\omega_q, & \text{otherwise.} \end{cases}$$

Let us note that $\nu_D(\widehat\omega_q) = t^m_q$ in both cases.

Now, we proceed as follows:

1. First, we find a combination θ_0 of $\tilde\omega_{j+1}$ and $\widehat\omega_q$ such that $\nu_D(\theta_0) \le \tilde t_j$. Note that θ_0 should be a combination of $\tilde\omega_{s+1}$ and ω_{s+1}, in view of the induction hypothesis.
2. Next, we find a 1-form $\tilde\omega_j - \theta_0$ that is a combination of

$$\tilde\omega_{j+1}, \tilde\omega_{j+2}, \ldots, \tilde\omega_{s+1}, \omega_{s+1},$$

in such a way that $\nu_D(\tilde\omega_j) = \tilde t_j$.

Consider Delorme's decompositions of $\tilde\omega_{j+1}$ and $\widehat\omega_q$ as introduced in Theorem 5.3.1, that we write as follows

$$\tilde\omega_{j+1} = \tilde\mu_1 y^{\ell^m_{j+1}}\omega_j + \tilde\mu_2 x^{a_{j+1}}\omega_{k^m_j} + \tilde\eta, \quad \tilde\eta = \sum_{\ell=-1}^{j} \tilde h_\ell \omega_\ell, \tag{8.6}$$

$$\widehat\omega_q = M\omega_j + N\omega_{k^n_j} + \eta, \quad \eta = \sum_{\ell=-1}^{j} h_\ell \omega_\ell, \tag{8.7}$$

where M, N are monomials in such a way that we have the following properties:

1. $\mathrm{In}(\tilde{\omega}_{j+1}) = \tilde{\mu}_1 \mathrm{In}(y^{\ell^m_{j+1}} \omega_j) = \tilde{\mu}_1 y^{\ell^m_{j+1}} \mathrm{In}(\omega_j)$. Recall that $\tilde{t}_{j+1} = t_j + m\ell^m_{j+1}$.
2. $\nu_C(\tilde{\mu}_1 y^{\ell^m_{j+1}} \omega_j + \tilde{\mu}_2 x^{a_{j+1}} \omega_{k^m_j}) > \nu_C(y^{\ell^m_{j+1}} \omega_j) = \nu_C(x^{a_{j+1}} \omega_{k^m_j}) = u^m_{j+1} = \tilde{u}_{j+1}$. Recall that $u^m_{j+1} = \lambda_j + m\ell^m_{j+1} = \lambda_{k^m_j} + na_{j+1}$.
3. $\nu_C(\tilde{h}_\ell \omega_\ell) > \tilde{u}_{j+1} = u^m_{j+1}$, for $\ell = -1, 0, 1, \ldots, j$.
4. $\mathrm{In}(\widehat{\omega}_q) = \mathrm{In}(M\omega_j) = M\,\mathrm{In}(\omega_j)$.
5. $\nu_C(M\omega_j + N\omega_{k^n_j}) > \nu_C(M\omega_j) = \nu_C(N\omega_{k^n_j}) = \lambda_j + t^m_q - t_j = v^m_{q-1,j}$.
6. $\nu_C(h_\ell \omega_\ell) > \lambda_j + t^m_q - t_j = v^m_{q-1,j}$, for $\ell = -1, 0, 1, \ldots, j$.

Let us compute the monomials M and N. We have that

$$t^m_q = \nu_D(\widehat{\omega}_q) = \nu_D(M) + \nu_D(\omega_j) \Rightarrow \nu_D(M) = t^m_q - t_j.$$

By a telescopic argument, we obtain

$$\begin{aligned}
t^m_q - t_j &= t^m_q - t_{j+1} + (t_{j+1} - t_j) \\
&= t^m_q - t_{j+1} + n\ell^n_{j+1} \\
&= t^m_q - t_{j+2} + (t_{j+2} - t_{j+1}) + n\ell^n_{j+1} \\
&= t^m_q - t_{j+2} + m\ell^m_{j+2} + n\ell^n_{j+1} \\
&= t^m_q - t_{j+3} + (t_{j+3} - t_{j+2}) + m\ell^m_{j+2} + n\ell^n_{j+1} \\
&= t^m_q - t_{j+3} + m(\ell^m_{j+3} + \ell^m_{j+2}) + n\ell^n_{j+1} \\
\cdots\ & \cdots \\
&= t^m_q - t_{q-1} + m(\ell^m_{q-1} + \cdots + \ell^m_{j+3} + \ell^m_{j+2}) + n\ell^n_{j+1} \\
&= m(\ell^m_q + \ell^m_{q-1} + \cdots + \ell^m_{j+3} + \ell^m_{j+2}) + n\ell^n_{j+1}.
\end{aligned}$$

This implies that $M = \mu_1 x^a y^b$, where

$$a = \ell^n_{j+1}, \quad b = \ell^m_q + \ell^m_{q-1} + \cdots + \ell^m_{j+3} + \ell^m_{j+2}, \quad t^m_q - t_j = na + mb. \tag{8.8}$$

Let us compute now the monomial N. We know that

$$\nu_C(N\omega_{k^n_j}) = \nu_D(N) + \lambda_{k^n_j} = \nu_C(M\omega_j) = \lambda_j + na + mb.$$

Then, we have that

$$\nu_D(N) = \lambda_j - \lambda_{k^n_j} + na + mb.$$

Recalling that $u^n_{j+1} = \lambda_j + n\ell^n_{j+1} = \lambda_{k^n_j} + mb_{j+1}$, we obtain that

$$\begin{aligned}
\nu_D(N) &= \lambda_j - \lambda_{k^n_j} + na + mb = \\
&= mb_{j+1} - n\ell^n_{j+1} + na + mb = m(b_{j+1} + b).
\end{aligned}$$

This implies that $N = \mu_2 y^{b_{j+1}+b}$.

Let us note that $b < \ell^m_{j+1}$, in view of Corollary 3.6.4. In a more precise way, we have that $\ell^m_{j+1} - b = b_q$. Now, we consider the 1-form θ_0 given by

$$\theta_{\mathbf{0}} = \mu_1 x^a \tilde{\omega}_{j+1} - \tilde{\mu}_1 y^{b_q} \widehat{\omega}_q = \mu_1 x^{\ell^n_{j+1}} \tilde{\omega}_{j+1} - \tilde{\mu}_1 y^{b_q} \widehat{\omega}_q.$$

We write $\theta_{\mathbf{0}} = \xi + \zeta_{\mathbf{0}}$, where

$$\begin{aligned}
\xi &= \mu_1 \tilde{\mu}_2 x^{a + a_{j+1}} \omega_{k^m_j} - \tilde{\mu}_1 \mu_2 y^{b_q + b + b_{j+1}} \omega_{k^n_j} \\
&= \mu_1 \tilde{\mu}_2 x^{\ell^n_{j+1} + a_{j+1}} \omega_{k^m_j} - \tilde{\mu}_1 \mu_2 y^{\ell^m_{j+1} + b_{j+1}} \omega_{k^n_j}
\end{aligned}$$

and $\zeta_{\mathbf{0}} = \sum_{\ell=-1}^{j} g_\ell^{\mathbf{0}}\omega_\ell$, with

$$g_\ell^{\mathbf{0}}\omega_\ell = (\mu_1 x^{\ell_{j+1}^n}\tilde{h}_\ell - \tilde{\mu}_1 y^{b_q} h_\ell)\omega_\ell, \quad \text{for } \ell = -1, 0, \ldots, j. \tag{8.9}$$

In a more general way, given a list of functions $\widetilde{\mathbf{f}}, f$ in $\mathcal{O}_{M_0,P_0}$, where

$$\widetilde{\mathbf{f}} = (\tilde{f}_{j+1}, \tilde{f}_{j+2}, \ldots, \tilde{f}_{s+1}),$$

we write

$$\theta_{\widetilde{\mathbf{f}},f} = \theta_{\mathbf{0}} + \textstyle\sum_{\ell=j+1}^{s+1} \tilde{f}_\ell\tilde{\omega}_\ell + f\omega_{s+1} = \xi + \zeta_{\widetilde{\mathbf{f}},f} \in \Omega^1_{M_0,P_0}[C],$$

where $\zeta_{\widetilde{\mathbf{f}},f} = \zeta_{\mathbf{0}} + \sum_{\ell=j+1}^{s+1} \tilde{f}_\ell\tilde{\omega}_\ell + f\omega_{s+1}$. We also write $\zeta_{\widetilde{\mathbf{f}},f} = \sum_{\ell=-1}^{s} g_\ell^{\widetilde{\mathbf{f}},f}\omega_\ell$. Let us note that $\theta_{\mathbf{0}} = \theta_{\mathbf{0},0}$, $\zeta_{\mathbf{0}} = \zeta_{\mathbf{0},0}$ and $g_\ell^{\mathbf{0}} = g_\ell^{\mathbf{0},0}$, for $-1 \le \ell \le s$.

In order to prove the desired result, we are going to show the existence of a list $\widetilde{\mathbf{f}}, f$ such that $\nu_D(\theta_{\widetilde{\mathbf{f}},f}) = \tilde{t}_j$.

We have two options: $u_j = u_j^n$ and $u_j = u_j^m$. Both cases run in a similar way. We fix the case that $u_j = u_j^n$. Hence, we have $t_j = t_j^n$, $\tilde{u}_j = u_j^m$ and $\tilde{t}_j = t_j^m$. By Proposition 3.5.9, we know that $k_j^n = j-1$ and $k_j^m = k_{j-1}^m$. Let us see that

Lemma 8.3.8. $\nu_D(\xi) = \tilde{t}_j$.

Proof of Lemma 8.3.8. By Proposition 3.6.3, the colimits a_{j+1} and b_{j+1} satisfy that $b_{j+1} + \ell_{j+1}^m = \ell_j^m$ and $a_{j+1} + \ell_{j+1}^n = a_j$. Hence,we have that

$$\begin{aligned} \xi &= \mu_1\tilde{\mu}_2 x^{\ell_{j+1}^n + a_{j+1}}\omega_{k_j^m} - \tilde{\mu}_1\mu_2 y^{\ell_{j+1}^m + b_{j+1}}\omega_{k_j^n} \\ &= \mu_1\tilde{\mu}_2 x^{a_j}\omega_{k_{j-1}^m} - \tilde{\mu}_1\mu_2 y^{\ell_j^m}\omega_{j-1}. \end{aligned}$$

Note that $\nu_D(y^{\ell_j^m}\omega_{j-1}) = m\ell_j^m + t_{j-1} = t_j^m = \tilde{t}_j$. Thus, it is enough to show that $\nu_D(x^{a_j}\omega_{k_{j-1}^m}) > \tilde{t}_j = t_j^m$. We have $\nu_D(x^{a_j}\omega_{k_{j-1}^m}) = na_j + t_{k_{j-1}^m}$. Since $u_j^m = m\ell_j^m + \lambda_{j-1} = na_j + \lambda_{k_{j-1}^m}$, then

$$\begin{aligned} na_j - m\ell_j^m &= \lambda_{j-1} - \lambda_{k_{j-1}^m} > t_{j-1} - t_{k_{j-1}^m} \Rightarrow \\ &\Rightarrow na_j + t_{k_{j-1}^m} > \tilde{t}_j = t_{j-1} + m\ell_j^m, \end{aligned}$$

by Lemma 3.2.8. We conclude that $\nu_D(\xi) = \tilde{t}_j$. □

Now, the problem is reduced to finding a list $(\widetilde{\mathbf{f}}, f)$ such that $\nu_D(\zeta_{\widetilde{\mathbf{f}},f}) > \tilde{t}_j$. We proceed to verify this. We say that a list of functions $(\widetilde{\mathbf{f}}, f)$ is a *good list* if and only if we have that $\nu_C(g_\ell^{\widetilde{\mathbf{f}},f}\omega_\ell) > \tilde{u}_j$, for any $\ell = -1, 0, \ldots, j$.

We end the proof as a direct consequence of the following lemmas:

Lemma 8.3.9. *The list* $(\widetilde{\mathbf{f}}, f) = (\mathbf{0}, 0)$ *is a good list.*

Lemma 8.3.10. *If* $(\widetilde{\mathbf{f}}, f)$ *is a good list, then* $\nu_D(g_\ell^{\widetilde{\mathbf{f}},f}\omega_\ell) > \tilde{t}_j$, *for* $-1 \le \ell \le j-1$ *and* $\nu_D(g_j^{\widetilde{\mathbf{f}},f}\omega_j) \ne \tilde{t}_j$.

Corollary 8.3.11. *Assume that* $(\widetilde{\mathbf{f}}, f)$ *is a good list. Then, either we have that* $\nu_D(\theta_{\widetilde{\mathbf{f}},f}) = \tilde{t}_j$ *or* $\nu_D(\theta_{\widetilde{\mathbf{f}},f}) = \nu_D(g_j^{\widetilde{\mathbf{f}},f}\omega_j) < \tilde{t}_j$.

Lemma 8.3.12. *If* $(\widetilde{\mathbf{f}}, f)$ *is a good list and* $\nu_D(\theta_{\widetilde{\mathbf{f}},f}) < \tilde{t}_j$, *then there is another good list* $(\widetilde{\mathbf{f}}^1, f^1)$ *such that* $\nu_D(g_j^{\widetilde{\mathbf{f}}^1,f^1}\omega_j) > \nu_D(g_j^{\widetilde{\mathbf{f}},f}\omega_j)$.

Indeed, by Lemma 8.3.9, there is at least one good list, by Lemma 8.3.10 and Lemma 8.3.8 we obtain Corollary 8.3.11. Now, we apply repeatedly Lemma 8.3.12 to get that $\nu_D(g_j^{\tilde{\mathfrak{k}},f}\omega_j) \geq \tilde{t}_j$, hence, in view of Lemmas 8.3.8 and 8.3.9, we get that $\nu_D(g_j^{\tilde{\mathfrak{k}},f}\omega_j) > \tilde{t}_j$ and $\nu_D(\theta_{\tilde{\mathfrak{k}},f}) = \tilde{t}_j$ as desired.

The rest of this subsection is devoted to proving the above Lemmas 8.3.9, 8.3.10 and 8.3.12.

Proof of Lemma 8.3.9. We need to show that

$$\nu_C(g_\ell^{\mathbf{0}}\omega_\ell) > \tilde{u}_j, \text{ for any } \ell = -1, 0, \ldots, j.$$

By Equation (8.9), we have $g_\ell^{\mathbf{0}}\omega_\ell = (\mu_1 x^{\ell_{j+1}^n}\tilde{h}_\ell - \tilde{\mu}_1 y^{b_q} h_\ell)\omega_\ell$. Now, it is enough to show that

$$\nu_C(x^{\ell_{j+1}^n}\tilde{h}_\ell\omega_\ell) > \tilde{u}_j \text{ and } \nu_C(y^{b_q}h_\ell\omega_\ell) > \tilde{u}_j.$$

We have

$$\begin{aligned}
\nu_C(x^{\ell_{j+1}^n}\tilde{h}_\ell\omega_\ell) &= n\ell_{j+1}^n + \nu_C(\tilde{h}_\ell\omega_\ell) > n\ell_{j+1}^n + u_{j+1}^m = n\ell_{j+1}^n + \tilde{u}_{j+1} \\
&= n\ell_{j+1}^n + na_{j+1} + \lambda_{k_j^m} = n(\ell_{j+1}^n + a_{j+1}) + \lambda_{k_{j-1}^m} \\
&= na_j + \lambda_{k_{j-1}^m} = u_j^m = \tilde{u}_j.
\end{aligned}$$

Let us consider now $\nu_C(y^{b_q}h_\ell\omega_\ell)$. We have that

$$\nu_C(y^{b_q}h_\ell\omega_\ell) > mb_q + \lambda_j + t_q^m - t_j.$$

Let us show that $mb_q + \lambda_j + t_q^m - t_j = \tilde{u}_j$. Recall that $\tilde{u}_j = u_j^m = \lambda_{j-1} + m\ell_j^m$. Thus, we have to prove that

$$mb_q + \lambda_j + t_q^m - t_j - \lambda_{j-1} - m\ell_j^m = 0.$$

Note that $k_j^n = j - 1$ and then $\lambda_j - \lambda_{j-1} = -n\ell_{j+1}^n + mb_{j+1}$. Then we have to verify that

$$mb_q - n\ell_{j+1}^n + mb_{j+1} + t_q^m - t_j - m\ell_j^m = 0.$$

Recalling that, by Equation (8.8), $t_q^m - t_j = na + mb = n\ell_{j+1}^n + mb$ and that $b_q = \ell_{j+1}^m - b$, we have to verify that

$$m(\ell_{j+1}^m - b) - n\ell_{j+1}^n + n\ell_{j+1}^n + mb + mb_{j+1} - m\ell_j^m = 0.$$

We have to see that $b_{j+1} + \ell_{j+1}^m = \ell_j^m$, and this follows from Proposition 3.6.3. □

Proof of Lemma 8.3.10. Along the proof of this lemma, we just write $g_\ell^{\tilde{\mathfrak{k}},f} = g_\ell$, in order to simplify the notation.

Let us first show that $\nu_D(g_\ell\omega_\ell) > \tilde{t}_j$, for any $-1 \leq \ell \leq j-1$. Recall that $\nu_C(g_\ell\omega_\ell) > \tilde{u}_j$ and write

$$\nu_C(g_\ell\omega_\ell) = \nu_C(g_\ell) + \lambda_\ell > \tilde{u}_j = u_j^m = \lambda_{j-1} + m\ell_j^m.$$

Noting that $\lambda_{j-1} - \lambda_\ell \geq t_{j-1} - t_\ell$, in view of Lemma 3.2.8, we have that

$$\nu_C(g_\ell) + \lambda_{j-1} > \lambda_{j-1} + t_{j-1} - t_\ell + m\ell_j^m$$

and thus we have $\nu_C(g_\ell) + t_\ell > t_{j-1} + m\ell_j^m = t_j^m = \tilde{t}_j$.

There are two cases: if $\nu_D(g_\ell) < nm$, then $\nu_C(g_\ell) = \nu_D(g_\ell)$ (Proposition 2.3.14). Second, $\nu_D(g_\ell) \geq nm$. Noting that $\tilde{t}_j \leq nm$, see Corollary 3.6.2, we conclude in both cases that

$$\nu_D(g_\ell\omega_\ell) = \nu_D(g_\ell) + t_\ell > \tilde{t}_j,$$

as desired.

Let us show that $\nu_D(g_j\omega_j) \neq \tilde{t}_j$. Assume by contradiction that $\nu_D(g_j\omega_j) = \tilde{t}_j$. Recalling that $t_j = t_j^n$, $\tilde{t}_j = t_j^m$, $t_j^n = t_{j-1} + n\ell_j^n$ and $t_j^m = t_{j-1} + m\ell_j^m$, we have

$$\begin{aligned} \nu_D(g_j\omega_j) = \tilde{t}_j &\Rightarrow \nu_D(g_j) + t_j = \tilde{t}_j \Rightarrow \nu_D(g_j) + t_j^n = t_j^m \Rightarrow \\ &\Rightarrow \nu_D(g_j) + t_{j-1} + n\ell_j^n = t_{j-1} + m\ell_j^m \Rightarrow \\ &\Rightarrow m\ell_j^m = \nu_D(g_j) + n\ell_j^n. \end{aligned}$$

This implies that $m\ell_j^m \in \Gamma_C$ is written in two different ways as a combination of n, m with non negative integer coefficients. This is not possible, since $m\ell_j^m < nm$, in view of Remark 3.2.3. □

Proof of Lemma 8.3.12. Assume that $\widetilde{\mathbf{f}}, f$ is a good list with $\nu_D(\theta_{\widetilde{\mathbf{f}},f}) < \tilde{t}_j$. Let us find another good list $\widetilde{\mathbf{f}}^1, f^1$ such that

$$\nu_D(g_j^{\widetilde{\mathbf{f}}^1,f^1}\omega_j) > \nu_D(g_j^{\widetilde{\mathbf{f}},f}\omega_j).$$

Let us note that $\nu_D(\theta_{\widetilde{\mathbf{f}},f}) = \nu_D(g_j^{\widetilde{\mathbf{f}},f}\omega_j) < \tilde{t}_j$ and, more precisely, we have that

$$W = \mathrm{In}(g_j^{\widetilde{\mathbf{f}},f}\omega_j) = \mathrm{In}(\theta_{\widetilde{\mathbf{f}},f}).$$

In view of Remark 8.2.4, there is a decomposition

$$W = G_{s+1}W_{s+1} + \sum_{\ell=j+1}^{s+1} \tilde{G}_\ell \tilde{W}_\ell,$$

where the coefficients are quasi-homogeneous. Moreover, all the forms $W, W_{s+1}, \tilde{W}_\ell$, for $j+1 \leq \ell \leq s+1$ are resonant with divisorial value $< nm$. We conclude that all those forms are given by the product of monomial and 1-form

$$m\frac{dx}{x} - n\frac{dy}{y}.$$

Since $\theta_{\widetilde{\mathbf{f}},f}$ is resonant, we can assume without loss of generality that all the coefficients $G_{s+1}, \tilde{G}_{j+1}, \tilde{G}_{j+2}, \ldots, \tilde{G}_{s+1}$ are zero except exactly one of them. Note that $\nu_D(W) < \tilde{t}_j \leq nm$ implies that the Newton cloud of W is a single point. Thus, we have that

$$W = G_{s+1}W_{s+1} \text{ or there is } \ell_0 \text{ such that } W = \tilde{G}_{\ell_0}\tilde{W}_{\ell_0}.$$

Let us write $S = W_{s+1}$ in the first case and $S = \tilde{W}_{\ell_0}$ in the second one. Then we have that $W = GS$, where $G = G_{s+1}$ in the first case and $G = \tilde{G}_{\ell_0}$ in the second one.

Now we define the list $(\widetilde{\mathbf{f}}^1, f^1)$ by

$$(\tilde{f}_{j+1}^1, \tilde{f}_{j+1}^1, \ldots, \tilde{f}_{s+1}^1, f^1) = (\widetilde{\mathbf{f}}, f) - (\tilde{G}_{j+1}, \tilde{G}_{j+2}, \ldots, \tilde{G}_{s+1}, G_{s+1}).$$

By construction we obtain that $\nu_D(g_j^{\widetilde{\mathbf{f}}^1,f^1}\omega_j) > \nu_D(g_j^{\widetilde{\mathbf{f}},f}\omega_j)$.

We have just to verify that $(\widetilde{\mathbf{f}}^1, f^1)$ is a good list. We do it in the case that $S = W_{s+1}$, the other cases run in a similar way. Note that

$$\zeta_{\widetilde{\mathbf{f}}^1,f^1} = \zeta_{\widetilde{\mathbf{f}},f} - G_{s+1}\omega_{s+1} = \sum_{\ell=-1}^{j} g_\ell^{\widetilde{\mathbf{f}}^1,f^1}\omega_\ell.$$

Let us write ω_{s+1} using Delorme's decomposition: $\omega_{s+1} = \sum_{\ell=-1}^{j} c_\ell\omega_\ell$, where we know that

1. $\mathrm{In}(\omega_{s+1}) = \mathrm{In}(c_j\omega_j)$.
2. $\nu_C(c_j\omega_j) \le \nu_C(c_\ell\omega_\ell)$, for $\ell = -1, 0, 1, \ldots, j$.

Note that $g_\ell^{\widetilde{\mathbf{f}}^1, f^1} = g_\ell^{\widetilde{\mathbf{f}}, f} - G_{s+1}c_\ell$, for $\ell = -1, 0, 1, \ldots, j$. Then, in order to show that we have a good list, it is enough to show that $\nu_C(G_{s+1}c_j\omega_j) > \tilde{u}_j$. We now verify this inequality.

We know that $\nu_D(g_j\omega_j) = \nu_D(G_{s+1}c_j\omega_j) < nm$, since they share initial part. Noting that the divisorial values are under nm, we have that

$$\nu_D(g_j) = \nu_C(g_j), \quad \nu_D(G_{s+1}c_j) = \nu_C(G_{s+1}c_j).$$

We conclude that $\nu_C(G_{s+1}c_j\omega_j) = \nu_C(g_j\omega_j) > \tilde{u}_j$, as desired. □

8.4 New Discrete Analytic Invariants

Consider C a plane curve and $\pi : (M_N, E^N) \to (M_0, P_0)$ a sequence of blow ups starting at P_0. Fix $E \subset E^N$ any of the irreducible components of E^N.

Lemma 8.4.1. *For any Saito basis ω, ω' of C, we have that*

$$\nu_E(\omega) + \nu_E(\omega') \le \nu_E(xyf), \quad j = 1, 2, \ldots, N,$$

where $f = 0$ is an implicit equation of C.

Proof. Since ω, ω' is a Saito basis, by Saito's Criterion (Lemma 8.1, we have that

$$\omega \wedge \omega' = uf\,dx \wedge dy = uxyf\left(\frac{dx}{x} \wedge \frac{dy}{y}\right),$$

where u is a unit. The property follows from Corollary 2.3.10, that states that

$$\nu_E(\omega) + \nu_E(\omega') \le \nu_E(\omega \wedge \omega').$$

□

Thanks to the previous lemma we can define *the pair* $(\mathfrak{s}_E(C), \widetilde{\mathfrak{s}}_E(C))$ *of Saito multiplicities at* E by

$$\mathfrak{s}_E(C) = \min\{\nu_E(\omega);\ \omega \text{ belongs to a Saito basis for } C\}. \tag{8.10}$$
$$\widetilde{\mathfrak{s}}_E(C) = \max\{\nu_E(\omega);\ \omega \text{ belongs to a Saito basis for } C\}. \tag{8.11}$$

Note that $\mathfrak{s}_E(C)$ is equal to the minimal divisorial value of the elements of any Saito basis, whereas $\widetilde{\mathfrak{s}}_E(C)$ does not follow directly from a given Saito basis.

Remark 8.4.2. In [26], the author introduces an invariant related with $(\mathfrak{s}_{E^1}(C), \widetilde{\mathfrak{s}}_{E^1}(C))$, where E^1 is the exceptional divisor appearing after the blow-up of P_0 in (M_0, P_0). More precisely, the author defined in a similar way the pair of Saito multiplicities, but just considering multiplicities at P_0, instead of divisorial values. The relationship between both objects comes from observing that

$$\nu_{E^1}(\omega) = \nu_{P_0}(\omega) + 1.$$

Moreover, note that the invariance under local biholomorphism of divisorial values shows that the pair of Saito multiplicities is an analytic invariant of the curve.

With the idea of finding relationships among different analytic invariants of plane curves. We would like to know if the pairs of Saito multiplicities may be deduced from the knowledge of the semimodule of differential values, at least, when C is a cusp. The answer is positive when considering the last pair $(\mathfrak{s}_D(C), \widetilde{\mathfrak{s}}_D(C))$ and D the cuspidal divisor of the minimal resolution of C as we show in next result.

On the other hand, we present an example of two D-cusps having the same semimodule of differential values such that the pair of Saito multiplicities at E^1 do not coincide (see Examples 8.4.4 and 8.4.5).

Theorem 8.4.3. *Take $\pi_y^{n,m}$ a cuspidal sequence with $n \geq 2$, D its cuspidal divisor. Let C be a D-cusp, then $(\mathfrak{s}_D(C), \widetilde{\mathfrak{s}}_D(C)) = (t_{s+1}, \tilde{t}_{s+1})$, where t_{s+1} and $\tilde{t}_{s+1}$ are the last critical values of the semimodule of differential values of C.*

Proof. By Theorem 8.2, we have that there are two 1-forms $\omega_{s+1}, \widetilde{\omega}_{s+1} \in \Omega^1_{M_0,P_0}[C]$ defining a Saito basis for C and such that

$$\nu_D(\omega_{s+1}) = t_{s+1} < \tilde{t}_{s+1} = \nu_D(\tilde{\omega}_{s+1}).$$

This proves that $\mathfrak{s}_D(C) = t_{s+1}$ and $\tilde{t}_{s+1} \leq \widetilde{\mathfrak{s}}_D(C)$. Now, let ω, ω' be another Saito basis, with $\nu_D(\omega) = t_{s+1}$ and $\nu_D(\omega') \geq \nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}$. Let us write

$$\omega = h\omega_{s+1} + \tilde{h}\tilde{\omega}_{s+1}, \quad \omega' = g\omega_{s+1} + \tilde{g}\tilde{\omega}_{s+1},$$

where $\delta = h\tilde{g} - g\tilde{h}$ is a unit in $\mathcal{O}_{M_0,P_0}$. Therefore, the divisorial values verify that $\nu_D(h) = 0$ and $\nu_D(g) > 0$, hence h is a unit and g is not a unit. Since δ is a unit, we have that $\tilde{g}$ is a unit. If $\nu_D(\omega') > \tilde{t}_{s+1} = \nu_D(\tilde{\omega}_{s+1})$, we necessarily have that

$$\nu_D(g\omega_{s+1}) = \nu_D(\tilde{g}\tilde{\omega}_{s+1}) = \nu_D(\tilde{\omega}_{s+1}) = \tilde{t}_{s+1}.$$

Let us see that this is not possible. Assume that $t_{s+1} = t_s + n\ell^n_{s+1}$ and hence $\tilde{t}_{s+1} = t_s + m\ell^m_{s+1}$ (the case $t_{s+1} = t_s + m\ell^m_{s+1}$ runs in a similar way). We have

$$\nu_D(g) + t_{s+1} = \tilde{t}_{s+1} \Rightarrow \nu_D(g) + n\ell^n_{s+1} = m\ell^m_{s+1}.$$

Noting that $\nu_D(g) \in \Gamma_C$, we obtain two different ways of writing $m\ell^m_{s+1} < nm$ as a combination of n, m with non-negative integer coefficients. This is a contradiction. □

We are going now to present the example of two cusps C_1 and C_2 corresponding to the Puiseux pair $(7, 36)$, such that the (common) semimodule of differential values has a basis $\mathcal{B} = (7, 36, 123)$ and such that the Saito pairs of multiplicities with respect to the first divisor E^1 are different for C_1 and C_2.

Example 8.4.4. Consider the cusp C_1 invariant by the 1-form

$$\omega = 36x^3(7xdy - 36ydx) - 560y^3dy,$$

with a parametrization $\phi_1(t) = (t^7, t^{36} + t^{116} + \frac{28}{9}t^{196} + h.o.t.)$. The basis of the semimodule of differential values of C_1 is $(7, 36, 123)$, with a minimal standard basis given by

$$\mathcal{S} = (\omega_{-1} = dx, \omega_0 = dy, \omega_1 = 7xdy - 36ydx).$$

We have $u^n_2 = \lambda_1 + n\ell^n_2 = \lambda_0 + mb_2$, that is $123 + 7\ell^n_2 = 36 + 36b_2$, we obtain that

$$\ell^n_2 = b_2 = 3, \quad u^n_2 = 144.$$

Similarly, we found out that

$$u_2^m = 231 = 123 + 36\ell_2^m = 7 + 7a_2, \quad \ell_2^m = 3, \quad a_2 = 32.$$

Hence $u_2 = u_2^n$ and $\tilde{u}_2 = u_2^m$. Moreover, we have

$$t_2 = t_2^n = t_1 + n\ell_2^n = 43 + 7 \cdot 3 = 64, \quad \tilde{t}_2 = t_2^m = t_1 + m\ell_2^m = 43 + 36 \cdot 3 = 151.$$

We see that $\nu_D(\omega) = t_2 = 64$. Hence we can take $\omega_2 = \omega$ to obtain an extended standard basis of C_1 and also, we consider $\omega_2 = \omega$ as one of the generators of a Saito basis for C_1. Notice that $\nu_{E^1}(\omega) = 4$, since $\nu_{P_0}(\omega) = 3$. We can take $\tilde{\omega}_2$ to be a 1-form with divisorial order $\nu_D(\tilde{\omega}_2) = \tilde{t}_2 = 151$ and C_1 being invariant by $\tilde{\omega}_2$. By Delorme's decomposition in Theorem 5.3.1, we can write $\tilde{\omega}_2$ as

$$\tilde{\omega}_2 = y^3\omega_1 + \mu^+ x^{32}dx + \eta_2; \qquad \eta_2 = f_{-1}dx + f_0 dy + f_1(7xdy - 36ydx),$$

where μ^+ is the tunning constant, such that $\nu_{C_1}(f_\ell\omega_\ell) > \tilde{u}_2 = 231$, for $\ell = -1, 0, 1$.

Let us compute $\nu_{E^1}(\tilde{\omega}_2)$. Assume that we have $\nu_{E^1}(f_\ell\omega_\ell) > 5$, for $\ell = -1, 0, 1$, then we obtain that $\nu_{E^1}(\tilde{\omega}_2) = 5$. In view of Lemma 8.4.1, we know that

$$\mathfrak{s}_{E^1}(C_1) + \widetilde{\mathfrak{s}}_{E^1}(C_1) \leq \nu_{E^1}(xyf) = 7 + 2 = 9,$$

Thus, we have $(\mathfrak{s}_{E^1}(C_1), \widetilde{\mathfrak{s}}_{E^1}(C_1)) = (4, 5)$ since the Saito basis $\omega, \tilde{\omega}_2$ gives the maximal pair $(4, 5)$.

It remains to show that $\nu_{E^1}(f_\ell\omega_\ell) > 5$, for $\ell = -1, 0, 1$. We consider two situations; $\nu_D(f_\ell) \geq nm$ and $\nu_D(f_\ell) < nm$. In the first situation we have that

$$\nu_{P_0}(f_\ell) \geq n = 7.$$

In the case that $\nu_D(f_\ell) < nm$ we have that

$$\nu_D(f_\ell) = \nu_{C_1}(f_\ell) > 231 - \lambda_\ell.$$

Moreover, looking at the monomials in the expression of f_ℓ, we have that

$$\nu_D(f_\ell) \leq m\nu_{P_0}(f_\ell) = 36\nu_{P_0}(f_\ell).$$

Thus we have:

$$\nu_{E^1}(f_\ell\omega_\ell) = \begin{cases} \nu_{P_0}(f_{-1}) + 1 \geq \frac{\nu_D(f_{-1})}{36} + 1 & > & \frac{231-\lambda_{-1}}{36} + 1 = \frac{260}{36} \geq 5; & \ell = -1. \\ \nu_{P_0}(f_0) + 1 \geq \frac{\nu_D(f_0)}{36} + 1 & > & \frac{231-\lambda_0}{36} + 1 = \frac{231}{36} \geq 5; & \ell = 0. \\ \nu_{P_0}(f_1) + 2 \geq \frac{\nu_D(f_1)}{36} + 2 & > & \frac{231-\lambda_1}{36} + 2 = \frac{180}{36} = 5; & \ell = 1. \end{cases}$$

Example 8.4.5. Take the cusp C_2 with Puiseux pair $(7, 36)$ invariant by the 1-form

$$\omega' = 36x^3(7xdy - 36ydx) - 560y^3dy + y(7xdy - 36ydx).$$

and defined by a parametrization as follows

$$\phi_2(t) = (t^7, t^{36} + t^{116} - \tfrac{4}{171}t^{131} + \tfrac{1}{1782}t^{146} - \tfrac{1}{72900}t^{161} + h.o.t.).$$

The basis of the semimodule of differential values is $(7, 36, 123)$. We can take

$$\mathcal{S} = (\omega_{-1} = dx, \omega_0 = dy, \omega_1 = 7xdy - 36ydx).$$

as minimal standard basis for C_2 (thus, it is the same one as for C_1). We repeat the arguments done before for C_1. We can take $\omega_2' = \omega'$ as one of the generators of a Saito basis for C_2, with

$\nu_D(\omega') = t_2$. Again, we obtain a partial standard system $(\omega_{-1}, \omega_0, \omega_1, \omega_2' = \omega', \tilde{\omega}_2')$, where $\tilde{\omega}_2'$ can be written as

$$\tilde{\omega}_2' = y^3\omega_1 + \mu' x^{32} dx + \eta_2'; \qquad \eta_2' = \sum_{\ell=-1}^{1} f_\ell' \omega_\ell,$$

with μ' being an appropriate constant and $\nu_{C_2}(f_\ell'\omega_\ell) > 231$. Thus, again we found that $\nu_{E^1}(f_\ell'\omega_\ell) > 5$. We have that $\nu_{E^1}(\tilde{\omega}_2') = 5$.

Now, we have that

$$(\nu_{E^1}(\omega'), \nu_{E^1}(\tilde{\omega}_2')) = (3, 5).$$

This implies that $\mathfrak{s}_{E^1}(C_2) = 3 < 4 = \mathfrak{s}_{E^1}(C_1)$. Hence the Saito pairs of multiplicities at E^1 for C_1 and C_2 are different.

Moreover, the pair $(3, 5)$ is not maximal yet: the 1-form $\eta = \tilde{\omega}_2' - y^2\omega_2'$ satisfies that $\{\eta, \omega_2'\}$ is a Saito basis and $\nu_{E^1}(\eta) = 6$. Hence the Saito's pair of multiplicities at E^1 for the cusp C_2 is equal to $(\mathfrak{s}_{E^1}(C_2), \widetilde{\mathfrak{s}}_{E^1}(C_2)) = (3, 6)$.

We end this chapter remarking that our Theorem 8.2 and the method used to construct the previous examples were used in [27] to show the following. Consider $n = n_1 e_1$ and $m = m_1 e_1$, with n_1, m_1 coprime positive integers. For any $2 \leq k \leq \lfloor n/2 \rfloor + 1$ there exists a curve C with the same topological type as the one defined by the implicit equation $y^n - x^m = 0$ such that $s_{E^1}(C) = k$. In fact, 2 and $\lfloor n/2 \rfloor + 1$ are the minimum and the maximum values that the number $s_{E^1}(C)$ can take, showing that all the possibilities for $s_{E^1}(C)$ are achieved.

9

Roots of the Bernstein-Sato Polynomial

In this chapter, we prove that some roots of Bernstein-Sato polynomial of a cusp C are determined by the semimodule of differential values of C.

Let us start giving the definition of Bernstein-Sato polynomial in a general context. Consider the ring of non-commutative power series $A = \mathbb{C}\{x_1, \ldots, x_p, \partial_1, \ldots, \partial_p\}$ in $2p$ variables, and define $\mathcal{D}$ to be the quotient of A by the commutators $[x_i, x_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$, where δ_{ij} is the Kronecker's delta. The ring $\mathcal{D}$ is the set of differential operators in p variables, whose action on $\mathbb{C}\{x_1, \ldots, x_p\}$ is defined by the partial derivative ∂_i with respect to x_i.

We take $\mathcal{D}[\rho]$ the ring of polynomials in the variable ρ and coefficients in $\mathcal{D}$. Given any function $g \in \mathbb{C}\{x_1, \ldots, x_p\}$, we can extend the action of $\mathcal{D}$ to functions of the form g^ρ, just by putting $\partial_i \cdot g^\rho = \rho g^{\rho-1}\partial_i g$.

According to [8], there exist non zero $P \in \mathcal{D}[\rho]$ and $B(\rho) \in \mathbb{C}[\rho]$ such that

$$P(\rho) \cdot g^{\rho+1} = B(\rho) g^\rho.$$

Then the ideal of $\mathbb{C}[\rho]$ of all $B(\rho)$ for which there is an operator $P \in \mathcal{D}[\rho]$ satisfying the last condition is non zero. Since it is a principal ideal, it admits a monic generator denoted by $b(\rho)$ which is called the *Bernstein-Sato polynomial* of g. The name is due to I.N. Bernstein and M. Sato who discovered, in the algebraic case, the existence of such a polynomial independently in [7] and [50].

From the works of M. Kashiwara and B. Malgrange, we know that all roots of the Bernstein-Sato polynomial are negative rational numbers (see [39, 43]). The Bernstein-Sato polynomial is an analytic invariant of the hypersurface $g = 0$ (see [55]). The relevance of this polynomial in the singularity theory comes from the fact that the roots of the Bernstein-Sato polynomial of a hypersurface H with isolated singularity determine the eigenvalues of the monodromy of the Milnor fiber of H (see [44]).

In this chapter we prove the following results:

Theorem 9.1. *Let C be a cusp with semigroup $\Gamma_C = \langle n, m\rangle$ and semimodule of differential values Λ_C. Assume that $\lambda_1 = \min(\Lambda_C \setminus \Gamma_C)$ exists. Then for any element $\lambda \in (\lambda_1 + \Gamma_C) \setminus \Gamma_C \subset \Lambda_C$, the rational number $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C.*

Theorem 9.2. *Let C be a cusp with semigroup $\Gamma_C = \langle n, m\rangle$ and semimodule of differential values Λ_C. Assume that $n \leq 4$. Then for any element $\lambda \in \Lambda_C \setminus \Gamma_C$, the rational number $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C.*

We conjectured that Theorem 9.2 holds for any n, that is:

Conjecture 9.3. *Let C be a cusp with semigroup $\Gamma_C = \langle n, m\rangle$ and semimodule of differential values Λ_C. Then for any element $\lambda \in \Lambda_C \setminus \Gamma_C$, the rational number $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C.*

P. Cassou-Noguès in [17] gives algebraic conditions to assure that a rational number is a root of the Bernstein-Sato polynomial of a cusp C. These algebraic conditions are given in terms of the coefficients of an implicit equation of the cusp, which is written in a particular kind of coordinates.

The idea of the proofs of both theorems is the following one: we take an implicit equation of C and we find algebraic conditions on its coefficients such that given natural number is a differential value. This is done by using the techniques from Chapter 5. Afterwards, we compare the computed conditions with those in [17].

9.1 Cuspidal Sets and Systems of Nice Coordinates

In this section, we introduce three sets $P, M \subset (\mathbb{Z}_{\geq 0})^2$ and $J \subset \mathbb{Z}_{\geq 0}$ which are related with the semigroup of a cusp, and also with roots of the Bernstein-Sato polynomial. Moreover, the results in [17] require to write an implicit equation of the cusp C in a particular system of coordinates with respect to C. These systems of coordinates will be also introduced in this section.

Fix (n, m) a pair with $gcd(n, m)$ and $2 \leq n$. We define the *cuspidal sets* P, J and M as:

$$\begin{aligned} P &:= \{(p_1, p_2) \in (\mathbb{Z}_{\geq 0})^2 : 0 \leq p_1 < m-1, 0 \leq p_2 < n-1 \text{ and } np_1 + mp_2 > nm\}, \\ J &:= \{j = p_{1,j}n + p_{2,j}m - nm : (p_{1,j}, p_{2,j}) \in P\}, \\ M &:= \{(m - p_1 - 1, n - p_2 - 1) : (p_1, p_2) \in P\}. \end{aligned}$$

Note that the previous sets are empty if $n = 2$. These cuspidal sets appear in a natural way when studying cusps (see [15, 17, 46]). Given $j \in J$, we write $(p_{1,j}, p_{2,j})$ to refer to the unique element in P such that $j = p_{1,j}n + p_{2,j}m - nm$.

Remark 9.1.1. If we consider the weighted order $\preceq$ with weights (n, m), then we have that $j < j'$ if and only if $(p_{1,j}, p_{2,j}) \prec (p_{1,j'}, p_{2,j'})$.

Remark 9.1.2. The cuspidal sets are empty if and only if $n = 2$ or $(n, m) = (3, 4), (3, 5)$. Moreover, if they are non empty, then we have that $(1, 1) \in M$ since $(m-2, n-2) \in P$. We will use that $(1, 1) \in M$ in some of the proofs ahead.

Next lemma shows the relationship between the elements in J and the elements in the semigroup of C. We will see later that the elements in $\Lambda_C \setminus \Gamma_C$ are described in terms of the set J, where Γ_C is the semigroup of the cusp and Λ_C the semimodule of differential values, see Lemma 9.2.1.

Lemma 9.1.3. *We have that*

$$J = \{\ell \in \mathbb{Z}_{\geq 0} : \ell + n, \ell + m \notin \Gamma_C\}.$$

As a consequence, for any $\ell \in J$, the element $\ell + n + m$ does not belong to the semigroup Γ_C.

Proof. The proof of the equality $J = \{\ell \in \mathbb{N} : \ell + n, \ell + m \notin \Gamma_C\}$ is given in [46], Lemma 1.4.

Now, if $\ell + n + m \in \Gamma_C$, this implies that $\ell + n + m = na + mb$, with at least one of the coefficients a, b different from 0. Hence, either $\ell + n \in \Gamma_C$ or $\ell + m \in \Gamma_C$. □

Next lemma describes the relationship between the semigroup Γ_C and the elements in the set M.

Lemma 9.1.4. *Consider a cuspidal semigroup $\Gamma_C = \langle n, m\rangle$ with $n \geq 3$. We have that*

1. *if $\lambda \notin \Gamma_C$ with $\lambda > n + m$, then there exists $j \in J$ such that $\lambda = j + n + m$.*
2. *If $\lambda + na + mb \notin \Gamma_C$ with $\lambda > n + m$ and $a, b \geq 0$, then we have that the element $(a + 1, b + 1)$ belongs to M.*

Proof. Statement 1 is a consequence of Lemma 9.1.3, because we have that $\lambda - n - m, \lambda - n, \lambda - m \notin \Gamma_C$.

For Statement 2. Write $\lambda = j + n + m$ with $j = np_{1,j} + mp_{2,j} - nm$ and $\lambda + na + mb = j' + n + m$ with $j' = np_{1,j'} + mp_{2,j'} - nm$, where $(p_{1,j}, p_{2,j}), (p_{1,j'}, p_{2,j'}) \in P$. Since $\lambda + na + mb \notin \Gamma_C$, we have that $\lambda + na + mb < c_\Gamma = (n - 1)(m - 1)$. Therefore,

$$na + mb \leq nm - n - m - \lambda < nm.$$

Thus, in virtue of Remark 2.3.15, we obtain

$$\begin{aligned} a + 1 &= p_{1,j'} - p_{1,j} + 1 = m - (m - p_{1,j'} + p_{1,j} - 2) - 1 \\ b + 1 &= p_{2,j'} - p_{2,j} + 1 = n - (n - p_{2,j'} + p_{2,j} - 2) - 1. \end{aligned}$$

We need to check that $(m - p_{1,j'} + p_{1,j} - 2, n - p_{2,j'} + p_{2,j} - 2) \in P$. By definition of P, we have to show that:

a) $0 \leq m - p_{1,j'} + p_{1,j} - 2 \leq m - 2$.
b) $0 \leq n - p_{2,j'} + p_{2,j} - 2 \leq n - 2$.
c) $n(m - p_{1,j'} + p_{1,j} - 2) + m(n - p_{2,j'} + p_{2,j} - 2) > nm$.

Let us show a) and b): recall that $na + mb \leq nm - n - m - \lambda$. Moreover $\lambda > n + m$ and we get that

$$na + mb < nm - 2n - 2m. \tag{9.1}$$

Consequently, we obtain that

$$0 \leq a \leq m - 3; \quad 0 \leq b \leq n - 3.$$

These last inequalities combined with the fact that

$$a = p_{1,j'} - p_{1,j}; \quad b = p_{2,j'} - p_{2,j},$$

give us the desired result.

Now, let us show c): we have that

$$n(m - p_{1,j'} + p_{1,j} - 2) + m(n - p_{2,j'} + p_{2,j} - 2) = 2nm - na - mb - 2n - 2m.$$

By Equation (9.1), we have that $na+mb < nm-2n-2m$, thus we conclude $2nm-na-mb-2n-2m > nm$. □

As mentioned before, the results in [17] require to write the implicit equation of the cusp C in a particular system of coordinates that we are going to introduce now.

In [60], Zariski proved the existence of a system of coordinates (x, y) in (M_0, P_0) such that C has an implicit equation given by

$$f = x^m + y^n + \sum_{j \in J} z_j x^{p_{1,j}} y^{p_{2,j}}; \quad z_j \in \mathbb{C}, \tag{9.2}$$

We will call this system of coordinates (x, y) a *nice system of coordinates* of C. An implicit equation as the one of the Equation (9.2) will be called a *nice equation* of C. Note that a system of nice coordinates is an adapted system of coordinates with respect to C.

Now we show how a nice equation allows us to compute the roots of the Bernstein-Sato polynomial of a cusp.

For any $j \in J$, we define the rational numbers

$$\alpha_j = \frac{np_{1,j} + mp_{2,j} + n + m}{nm}; \quad \beta_j = \alpha_j - 1 = \frac{j + n + m}{nm}.$$

By Lemma 9.1.4, given $\lambda \in \Lambda \setminus \Gamma_C$, then λ/nm corresponds with β_j for some $j \in J$.

According to [17], we have that either $-\alpha_j$ or $-\beta_j$ is a root of the Bernstein polynomial of C. Moreover, for any $(a, b) \in M$, there is a complex function $I_0((a, b), f)(\rho)$, such that its residue at $\rho = -\beta_j$ is given by

$$Res_f(a,b)(\beta_j) = \frac{\Gamma(\beta_j)^{-1}}{nm} \sum_{\substack{\ell \in J,\ \delta_\ell \in \mathbb{N} \\ \sum \delta_\ell \ell = k}} (-1)^{\sum \delta_\ell} \Gamma\left(\frac{\sum \delta_\ell p_{1,\ell} + a}{m}\right) \Gamma\left(\frac{\sum \delta_\ell p_{2,\ell} + b}{n}\right) \prod \frac{z_\ell^{\delta_\ell}}{\delta_\ell !}, \tag{9.3}$$

where $k = \beta_j nm - na - mb$ and $\Gamma(-)$ is the Euler's Gamma function. Furthermore, next result characterizes when the values α_j and β_j are roots of the Bernstein-Sato polynomial of C.

Theorem 9.4 ([17])**.** *Assume that C is defined by the nice equation*

$$f = x^m + y^n + \sum_{j \in J} z_j x^{p_{1,j}} y^{p_{2,j}}.$$

and consider $b(\rho)$ its Bernstein-Sato polynomial. Then $-\beta_j$ is a root of $b(\rho)$ if and only if there exists $(a, b) \in M$, such that $Res_f(a, b)(\beta_j) \neq 0$. Otherwise, if $-\beta_j$ is not a root of $b(\rho)$, then $-\alpha_j$ is.

We want to emphasize that some of the roots of the Bernstein-Sato polynomial of a cusp only depend on the topological class. However, the ones of the form β_j are not topological invariants. In fact, we can consider the most easy example: we take the quasi-homogeneous curve $f = x^m + y^n$. Then all the possible residues are zero and no β_j is a root of the Bernstein-Sato polynomial of $f = 0$. In contrast, in the next example we find a cusp with a root of the shape β_j.

Example 9.1.5. Consider the cusp C defined by the nice equation

$$f = x^{11} + y^5 + x^9 y + 3x^7 y^2.$$

We notice that the Puiseux pair of C is $(5, 11)$. The cuspidal sets are

$$\begin{aligned} P &= \{(9,1),(7,2),(5,3),(8,2),(6,3),(9,2),(7,3),(8,3),(9,3)\}, \\ J &= \{1,2,3,7,8,12,13,18,23\}, \\ M &= \{(1,1),(2,1),(3,1),(1,2),(4,1),(2,2),(5,1),(3,2),(1,3)\}. \end{aligned}$$

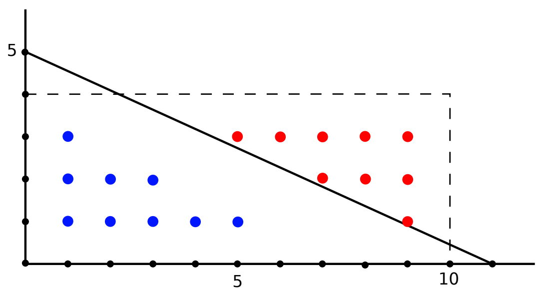

Figure 9.1: *The blue points represent the elements of M, the red ones the elements in P.*

We get that $z_1 = 1$ and $z_2 = 3$.

Let us compute the residue $Res_f(2,1)(\beta_8)$, where $\beta_8 = 23/55$. We see that $k = 23-5\cdot 2-11\cdot 11 = 2$. The sequences of non negative integers $(\delta_\ell)_{\ell\in J}$, such that $\sum_{\ell\in J}\delta_\ell \ell = 2$:

- $\delta_1 = 2$ and $\delta_\ell = 0$ if $\ell \neq 1$.
- $\delta_2 = 1$ and $\delta_\ell = 0$ if $\ell \neq 2$.

Now we apply Equation (9.3) to compute the residue $Res_f(2,1)(23/55)$. Before that, recall that the Euler Gamma function satisfies that $\Gamma(\rho+1) = \rho\Gamma(\rho)$.

$$\begin{aligned} Res_f(2,1)(23/55) &= \frac{\Gamma(23/55)^{-1}}{55}\left(\Gamma\left(\tfrac{2\cdot 9+2}{11}\right)\Gamma\left(\tfrac{2\cdot 1+1}{5}\right)\cdot\frac{1^2}{2!}-\right. \\ &- \left.\Gamma\left(\tfrac{7\cdot 1+2}{11}\right)\Gamma\left(\tfrac{2\cdot 1+1}{5}\right)\cdot 3\right) = \\ &= \frac{\Gamma(23/55)^{-1}}{55}\Gamma\left(\tfrac{9}{11}\right)\Gamma\left(\tfrac{3}{5}\right)\left(\frac{9}{22}-3\right) = \\ &= \frac{\Gamma(23/55)^{-1}}{55}\Gamma\left(\tfrac{9}{11}\right)\Gamma\left(\tfrac{3}{5}\right)\left(-\frac{57}{22}\right). \end{aligned}$$

In fact, we are not interested in developing the product involving the Gamma functions. It is enough to see that they are non-zero, hence we can conclude that the residue $Res_f(2,1)(23/55)$ is non zero. By Theorem 9.4, we conclude that $-23/55$ is a root of the Bernstein-Sato polynomial of C. Using the techniques exposed in Section 5.4, we can check that $23 \notin \Gamma_C$ is the 2-element of the basis of semimodule of differential values of C. This is in correspondence with what we expected from Conjecture 9.3.

9.2 Roots of the Bernstein-Sato Polynomial and Zariski's Invariant

Fix (C, P_0) a cusp with Puiseux pair (n, m) and $n \geq 3$. Consider the associated cuspidal sets P, J, M. The cuspidal divisor of C will be denoted by D. The goal of this section is to prove Theorem 9.1.

Next lemma relates the 1-element of the basis of the semimodule of differential values of C with the coefficients of a nice equation of C. Moreover, these conditions are also related with the residues introduced in Equation (9.3).

In the proofs of the results of this chapter, we are going to consider the weighted monomial order $\preceq$ with weights (n, m), as in Example 4.1.1.

Lemma 9.2.1. *Let C be a cusp with Puiseux pair (n, m). Consider f a nice equation of C, as in Equation (9.2). The following statements are equivalent:*

1. *$\lambda_1 = j_1 + n + m$ with $j_1 \in J$ is the 1-element of the basis of Λ_C.*
2. *$z_\ell = 0$ for $\ell < j_1$ and $z_{j_1} \neq 0$, where the z_j denotes the coefficient of the nice equation f.*
3. *$Res_f(1,1)((\ell+n+m)/nm) = 0$ for $\ell < j_1$ and $Res_f(1,1)((j_1+n+m)/nm) \neq 0$.*

Before proving the Lemma 9.2.1, we remark that the equivalence of the Statements *1* and *2* is well known (see [15]). However, we include here a proof using an approach similar to the one that we will use later on the proof of Theorem 9.2.

Notation 9.2.2. Given $r, g \in \mathbb{C}\{x, y\}$, when we say that r is a reduction of g, we mean that r is a reduction of g modulo $\{f\}$. We do similarly with final and partial reductions.

Proof Lemma 9.2.1. We prove first the equivalence between the two firsts statements, and later the equivalence between the two last ones.

Part 1: Statement *1* is equivalent to Statement *2*.

Assume that the 1-element of the basis of the semimodule Λ_C of differential values of C is $\lambda_1 = j_1 + na + mb$, with $j_1 \in J$. We are going to compute the first elements of a minimal standard basis of the module of differentials of C with Delorme's algorithm.

Since (x, y) is local system of nice coordinates, we can put $\omega_{-1} = dx$ and $\omega_0 = dy$. Then the axis u_1 is

$$u_1 = \min(\Lambda_0 \cap (\lambda_{-1} + \Gamma_C)) = n + m = \nu_C(x\omega_0) = \nu_C(y\omega_{-1}),$$

where Λ_i denotes the i^{th} element of the decomposition sequence of Λ_C, with $i = -1, 0, \ldots, s$.

Let us consider the 1-form $\theta = x\omega_0 + \mu^+ y\omega_{-1}$ and compute the tuning constant μ^+. As explained in Remark 5.4.4, in order to find μ^+, it is enough to compute final reductions of $X_{x\omega_0}(f)$ and $X_{y\omega_{-1}}(f)$. Starting by $X_{x\omega_0}(f)$, we find:

$$r_0 = X_{x\omega_0}(f) = x\tfrac{\partial}{\partial x}(f) = xf_x = mx^m + \sum_{\ell \in J} p_{1,\ell} z_\ell x^{p_{1,\ell}} y^{p_{2,\ell}}. \tag{9.4}$$

The leading power $lp(r_0) = (m, 0)$ is not divisible by $(0, n) = lp(f)$. Thus, r_0 is its own final reduction. For $X_{y\omega_{-1}}(f)$, we have:

$$X_{y\omega_{-1}}(f) = -y\tfrac{\partial}{\partial y}(f) = -yf_y = -ny^n - \sum_{\ell \in J} p_{2,\ell} z_\ell x^{p_{1,\ell}} y^{p_{2,\ell}}.$$

Here, $lp(X_{y\omega_{-1}}(f)) = (0, n)$. We can take the reduction

$$r_{-1} = X_{y\omega_{-1}}(f) + nf = nx^m + \sum_{\ell \in J} (n - p_{2,\ell}) z_\ell x^{p_{1,\ell}} y^{p_{2,\ell}}. \tag{9.5}$$

Since $lp(r_{-1}) = (m, 0)$ is not divisible by $(0, n)$, we can put r_{-1} as final reduction of $X_{y\omega_{-1}}(f)$. By Proposition 4.1.5, we find:

$$i_{P_0}(X_{y\omega_{-1}}(f), f) = nm = i_{P_0}(X_{x\omega_0}(f), f)$$

From Equations (9.4) and (9.5), the leading terms are $lt(r_{-1}) = nx^m$ and $lt(r_0) = mx^m$. Therefore, the tuning constant μ^+ is $-m/n$. For convenience, instead of θ, we take the 1-form

$$\eta = n\theta = nx\omega_0 - my\omega_{-1} = nxdy - mydx.$$

We define r_1 to be the following partial reduction of $X_\eta(f)$:

$$\begin{aligned} r_1 := nr_0 - mr_{-1} &= \sum_{\ell \in J} (np_{1,\ell} + mp_{2,\ell} - nm) z_\ell x^{p_{1,\ell}} y^{p_{2,\ell}} = \\ &= \sum_{\ell \in J} \ell z_\ell x^{p_{1,\ell}} y^{p_{2,\ell}}. \end{aligned} \tag{9.6}$$

The leading power of r_1 is an element $(p_{1,j}, p_{2,j}) \in P$. By definition of the cuspidal set P, we have that $0 \leq p_{2,j} < n - 1$. Thus $(p_{1,j}, p_{2,j})$ is not divisible by $(0, n)$. Therefore r_1 is a final reduction of $X_\eta(f)$. Moreover, by Propositions 4.1.5 and 5.4.3

$$\begin{aligned} i_{P_0}(X_\eta(f), f) &= np_{1,j} + mp_{2,j}, \\ \nu_C(\eta) &= np_{1,j} + mp_{2,j} - (n-1)(m-1) + 1 = j + n + m. \end{aligned}$$

By Lemma 9.1.3, we have that $j + n + m \notin \Gamma_C = \Lambda_0 \cup \{0\}$. We also note that the divisorial value of η is $\nu_D(\eta) = n + m = t_1$. Since λ_1 is the 1-element of the basis, we have that

$$\lambda_1 = \min(\Lambda_C \setminus \Lambda_0),$$

see Section 3.1. Furthermore, by Theorem 5.2.10, it is also satisfied that

$$\lambda_1 = \sup\{\nu_C(\omega) : \nu_D(\omega) = t_1\}.$$

We conclude that $\lambda_1 = j + n + m$, that is, $j = j_1$. In fact, we have shown that $\nu_C(\eta) = \lambda_1$. This implies that η is its own final reduction modulo $\{\omega_{-1}, \omega_0\}$, where r_1 is a final reduction of $X_\eta(f)$.

By Equation (9.6) and Remark 9.1.1, stating that the leading power of r_1 is (p_{1,j_1}, p_{2,j_1}) is equivalent to saying that $z_\ell = 0$ for $\ell < j_1$ and $z_{j_1} \neq 0$. This shows that Statement *1* is equivalent to Statement *2*.

Part 2: Statement 2 is equivalent to Statement 3.

Fix $j_1 \in J$. Let us compute $Res_f(1,1)((j + n + m)/nm)$, for $j \in J$ with $j \leq j_1$. Since $J \neq \emptyset$, then the set M is not empty and then we have that $(1,1) \in M$ by Remark 9.1.2. We can apply Equation (9.3). Notice that $\beta_j = (j + n + m)/nm$.

Start by taking $\ell_1 = \min(J)$. By Equation (9.3), we have to find sequences $(\delta_\ell)_{\ell \in J}$ of non negative integer numbers, such that $\sum_{\ell \in J} \delta_\ell \ell = \ell_1$. Since $\ell_1 = \min(J)$, the only possible sequence is the one defined by $\delta_{\ell_1} = 1$ and $\delta_\ell = 0$ for $\ell \neq \ell_1$. Therefore, by Equation (9.3):

$$Res_f(1,1)((\ell_1 + n + m)/nm) = 0 \Leftrightarrow z_{\ell_1} = 0.$$

This proves that Statement *2* is equivalent to Statement *3* if $\ell_1 = j_1$.

Now assume that $\ell_1 < j_1$, and proceed in an inductive way. Take $k \in J$ such that $k < j_1$. Suppose that $Res_f(1,1)((\ell + n + m)/nm) = 0$ for all $\ell \leq k < j_1$ is equivalent to $z_\ell = 0$ for all $\ell \leq k$. Denote by $\ell_k = \min\{\ell \in J : \ell > k\}$, let us show that, if $z_\ell = 0$ for $\ell \leq k$, then

$$Res_f(1,1)((\ell_k + n + m)/nm) = 0 \Leftrightarrow z_{\ell_k} = 0.$$

Applying this argument inductively will prove the equivalence between Statements *2* and *3*. We compute the sequences of non negative integer numbers $(\delta_\ell)_{\ell \in J}$ such that $\sum_{\ell \in J} \delta_\ell \ell = \ell_k$. There are two kind of possible sequences: first, the one given by $\delta_{\ell_k} = 1$ and $\delta_\ell = 0$ if $\ell \neq \ell_k$. Second, all the non zero δ_ℓ satisfies that $\ell < \ell_k$. Since $z_\ell = 0$ for $\ell < \ell_k$, we have, again by Equation (9.3), that

$$Res_f(1,1)((\ell_k + n + m)/nm) = 0 \Leftrightarrow z_{\ell_k} = 0,$$

as desired. □

As a consequence of the previous lemma we have the next result showing that several residues are non zero. In fact, next lemma and Theorem 9.4 prove Theorem 9.1.

Lemma 9.2.3. *Let C be a cusp with Puiseux pair (n, m) and assume that the 1-element λ_1 of the basis of Λ_C is given by*

$$\lambda_1 = j_1 + n + m, \textit{ with } j_1 \in J.$$

Then for any $\lambda = \lambda_1 + na + mb \in \Lambda_C \setminus \Lambda_0 = \Lambda_C \setminus \Gamma_C$, we have that $Res_f(a+1, b+1)(\lambda/nm) \neq 0$.

Proof. By Lemma 9.1.4, we can write $\lambda_1 = j_1 + n + m$ with $j_1 \in J$ and $j_1 = np_{1,j_1} + mp_{2,j_1} - nm$. Moreover, by Lemma 9.2.1, we have that if f is a nice equation of C as in Equation (9.2), then $z_\ell = 0$ for $\ell < j_1$ and $z_{j_1} \neq 0$.

Now take $\lambda = \lambda_1 + na + mb \in \Lambda_C \setminus \Lambda_0$ that we write as $\lambda = q + n + m$ with $q = np_{1,q} + mp_{2,q} - nm$ and $q \in J$. Note that we have the equalities: $p_{1,q} = p_{1,j_1} + a$ and $p_{2,q} = p_{2,j_1} + b$, where (p_{1,j_1}, p_{2,j_1}) is the element in P associated to $j_1 \in J$.

As before, we can express:

$$\frac{q+n+m}{nm} = \frac{np_{1,q} + mp_{2,q} + n + m}{nm} - 1 = \beta_q.$$

By Lemma 9.1.4, we know that $(a+1, b+1) \in M$. Observe that $j_1 = \beta_q nm - n(a+1) - m(b+1)$. Again, since $z_\ell = 0$ for $\ell < j_1$, we only have to consider a single non zero sequence $(\delta_\ell)_{\ell \in J}$: $\delta_{j_1} = 1$ and $\delta_\ell = 0$ for $\ell \neq j_1$, because the other sequences have zero contribution to the computation of the residue $Res_f(a+1, b+1)(\lambda/nm)$, when applying Equation (9.3). We conclude that

$$Res_f(a+1, b+1)(\lambda/nm) = 0 \Leftrightarrow z_{j_1} = 0.$$

However, this contradicts the fact that $z_{j_1} \neq 0$, ending the proof. □

9.3 Cusps with Multiplicity up to 4

As in the previous section, we fix C a cusp with Puiseux pair (n, m), semigroup Γ_C, semimodule of differential values Λ_C and cuspidal divisor D. In this section we prove Theorem 9.2. Hence, we impose the extra condition that $n \leq 4$.

We recall that the length s of the basis $(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ of the semimodule of differential values of C is bounded above by $n-2$. Additionally, we have that $\lambda_{-1} = n$ and $\lambda_0 = m$. Therefore, Theorem 9.2 is trivial if $n = 2$. In that case, we have that $\Lambda_C \setminus \Gamma_C = \emptyset$ and there is nothing to prove. By the same argument if $n = 3$, we have that either $\Lambda_C \setminus \Gamma_C = \emptyset$ or $\Lambda_C \setminus \Gamma_C = (\lambda_1 + \Gamma_C) \setminus \Gamma_C$. Thus by Theorem 9.1, Theorem 9.2 is also true when $n = 3$. We are left to show that it also holds when $n = 4$. The rest of the section is devoted to show that the theorem holds under the assumption $n = 4$. Consider $f \in \mathbb{C}\{x, y\}$ a nice equation of C as in Equation (9.2):

$$f = x^m + y^4 + \sum_{j \in J} z_j x^{p_{1,j}} y^{p_{2,j}}; \qquad z_j \in \mathbb{C}.$$

We proceed in a similar way as in the previous section. We are going to find all possible Γ_C-semimodules Λ such that Λ can be the semimodule of differential values of a cusp C with multiplicity four. Later, we are going to find the conditions on the coefficients of an implicit equation of C imposed by the restriction of having Λ as the semimodule of differential values of C. Finally, we will see that the computed conditions imply that certain residues, described in Equation (9.3), are non zero.

Since $gcd(4, m) = 1$, it follows that $m = 4\alpha + \epsilon$ with $\alpha \geq 1$ and $\epsilon \in \{1, 3\}$. Before studying all possible semimodules of differential values, we give the following remark about cuspidal sets and nice equations.

Remark 9.3.1. Take $n = 4$ and $m = 4\alpha + \epsilon$ with $\alpha \geq 2$ and $\epsilon \in \{1, 3\}$. For $0 \leq \beta \leq \alpha - 2$ and $0 \leq \beta' \leq 2\alpha - 2$ we have that

$$\begin{aligned} \epsilon + 4\beta &= 4(3\alpha + \epsilon + \beta) + (4\alpha + \epsilon)1 - nm, \\ 2\epsilon + 4\beta' &= 4(2\alpha + \epsilon + \beta') + (4\alpha + \epsilon)2 - nm. \end{aligned}$$

Thus, the cuspidal sets J and P are

$$J = \{\epsilon + 4\beta,\ 2\epsilon + 4\beta' :\ 0 \leq \beta \leq \alpha - 2,\ 0 \leq \beta' \leq 2\alpha - 2\},$$

and

$$P = \{(3\alpha + \epsilon + \beta, 1), (2\alpha + \epsilon + \beta', 2) :\ 0 \leq \beta \leq \alpha - 2,\ 0 \leq \beta' \leq 2\alpha - 2\},$$

with the natural correspondence between J and P.

Therefore , we can write the nice equation f of C as:

$$f = x^{4\alpha+\epsilon} + y^4 + \sum_{\beta=0}^{\alpha-2} z_{\epsilon+4\beta} x^{3\alpha+\epsilon+\beta} y + \sum_{\beta'=0}^{2\alpha-2} z_{2\epsilon+4\beta'} x^{2\alpha+\epsilon+\beta'} y^2. \tag{9.7}$$

We can particularize Remark 9.1.1. More precisely, for any $\beta \geq 0$, we have that

$$(3\alpha + \epsilon + \beta, 1) \prec (2\alpha + \epsilon + \beta, 2) \prec (3\alpha + \epsilon + \beta + 1, 1).$$

Lemma 9.3.2. *Denote by $\mathcal{B}$ the basis of the semimodule Λ_C. Then one of the following statements must be satisfied:*

1. $\mathcal{B} = (4, 4\alpha + \epsilon)$.
2. $\mathcal{B} = (4, 4\alpha + \epsilon, \lambda_1)$ *with* $\lambda_1 > u_1 = n + m = 4(\alpha + 1) + \epsilon$ *and* $\lambda_1 \notin \Gamma_C$.
3. $\mathcal{B} = (4, 4\alpha + \epsilon, \lambda_1, \lambda_2)$ *with*

$$\begin{aligned} \lambda_1 &= n + m + \epsilon + 4q = 4(\alpha + 1) + 2\epsilon + 4q \quad \textit{with} \quad 0 \leq q \leq \alpha - 2. \\ \lambda_2 &= 8\alpha + 3\epsilon + 4q' \quad \textit{with} \quad 0 \leq q' \leq q. \end{aligned}$$

If $\alpha = 1$, then Case 3 is not possible.

Lemma 9.3.2 was proven in [33]. There, the authors use their normal form theorem from [34] to show the result. For completeness, we provide a proof using only combinatorial techniques.

Proof. Put $\lambda_{-1} = n = 4$ and $\lambda_0 = m = 4\alpha + \epsilon$. We recall that, as explained in Section 3.2, the only conditions that an increasing sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ must satisfy in order to be the basis of a semimodule of differential values are the following: $\lambda_{-1} = n$, $\lambda_0 = m$, $\lambda_i > u_i$ for $i = 1, \ldots, s$ and $\lambda_i \notin \lambda_j + \Gamma_C$, for $i \neq j$ and $i, j = -1, 0, 1, \ldots, s$. The three options above verify these conditions. Let us show that there are no other possibilities.

The number of the elements of the basis is at most 4 because $s \leq 2$. Additionally, we see that all possible semimodules of differentials values with $s = 0, 1$ are covered. Case 1. corresponds to a quasi-homogeneous curve and hence $s = 0$. Of $s = 1$, we only need that the 1-element of the basis λ_1 is greater than u_1. Therefore, we must show that if the length of the basis s is 2, the basis corresponds to Case 3.

Assume that $\alpha \geq 2$ and $s = 2$ later we will show that it is not possible to have $\alpha = 1$ and $s = 2$. There are two possibilities for the 1-element of the basis λ_1:

a) $\lambda_1 \equiv 3m \mod 4$.
b) $\lambda_1 \equiv 2m \mod 4$.

Case a). If $\lambda_1 \equiv 3m \mod 4$, let us see that the 2-element of the basis cannot exists. To do that, let us compute the axis u_2 and see that for any $\lambda > u_2$, we have that $\lambda \in \Lambda_1$. Recalling the results from Chapter 3, we have by Example 3.5.4 and Proposition 3.5.9 that the bounds are $k_1^n = 0$ and $k_1^m = -1$. Thus, the axis u_2 can only take the following two values:

i) $u_2 = u_2^m = \lambda_1 + m\ell_m^2 = n + na_2$.

ii) $u_2 = u_2^n = \lambda_1 + n\ell_n^2 = m + mb_2$.

i). If $u_2 = \lambda_1 + m\ell_m^2 = n(a_2+1)$, since $n = 4$ and $\lambda_1 \equiv 3m \mod 4$, then, $\ell_m^2 = 1$. Hence, for any $\lambda > u_2$ we find that $\lambda > u_2 = \lambda_1 + m > 2m$. Assume that we write $\lambda \equiv \delta m \mod 4$ with $\delta \in \{0,1,2,3\}$. If $\delta = 0,1,2$, we could write $\lambda = \delta m + 4c$ for $c \geq 0$ and $\lambda_2 \in \Gamma_C$. If $\delta = 3$, then $\lambda = \lambda_1 + 4c'$ with $c' \geq 0$ and $\lambda \in \lambda_1 + \Gamma_C$.

ii). If $u_2 = \lambda_1 + n\ell_n^2 = m(b_2+1)$, we have that $b_2 \geq 2$ since we are assuming that $\lambda_1 \equiv 3m \mod n$. Thus we obtain $\lambda > u_2 \geq 3m$. As before, we could write $\lambda = \delta m + 4c$ with $c \geq 0$ and $\delta \in \{0,1,2,3\}$. This means that $\lambda \in \Gamma_C$. Therefore, the case $u_2 = \lambda_1 + n\ell_n^2 = m(b_2+1)$ implies the non-existence of the 2-element of the basis.

In conclusion the assumption $\lambda_1 \equiv 3m \mod 4$ implies that the 2-element of the basis cannot exist.

Case b). Assume that $\lambda_1 \equiv 2m \mod 4$. First, we note that $\lambda_1 < 2m$, otherwise, we could write $\lambda_1 = 2m + 4\ell$ for $\ell \geq 0$ implying that $\lambda_1 \in \Gamma_C$. This would contradict the fact that $\lambda_1 \in \Lambda_C \setminus \Gamma_C$. Moreover, we have the extra condition $\lambda_1 > u_1 = n + m = 4(\alpha+1) + \epsilon$. In other words

$$4(\alpha+1) + \epsilon < \lambda_1 < 2m = 4(2\alpha) + 2\epsilon.$$

Taking into account that we are assuming that $\lambda_1 \equiv 2m \mod 4$, the last two inequalities are equivalent to

$$\lambda_1 = 4(\alpha+1) + 2\epsilon + 4q, \quad \text{with} \quad 0 \leq q \leq \alpha - 2.$$

Hence λ_1 must be as stated in Case 3. We only have to determine the possibilities for the 2-element of the basis. First, we find the value of the axis u_2, since the bounds are $k_1^n = 0$ and $k_{-1}^m = -1$, we have the to compute the smallest value associated to the minimal solutions of the following equations

$$\lambda_1 + n\ell_2^n = m(b_2+1); \quad \lambda_1 + m\ell_2^m = n(a_2+1).$$

We can check that $u_2 = \lambda_1 + 4(\alpha - q - 1) = 2m = 8\alpha + 2\epsilon$. Assume that $\lambda > u_2$ and that $\lambda \notin \Lambda_1$, as the 2-element of the basis must satisfy.

The conditions $\lambda > u_2 = 2m > \lambda_1$ and $\lambda \notin \Lambda_1$ imply that $\lambda \equiv 3m \mod 4$. Otherwise, we could write $\lambda = \lambda_k + 4c$ for $k = \{-1,0,1\}$ and $c \geq 0$, recall that $\lambda_{-1} = n$, $\lambda_0 = m$ and the assumtion $\lambda_1 \equiv 2m \mod 4$. Arguing as before, we find that

$$u_2 < \lambda < \lambda_1 + m = 8\alpha + 4 + 4q + 3\epsilon < 3m.$$

From the previous inequalities and $\lambda \equiv 3m \mod 4$ we get

$$\lambda = 8\alpha + 3\epsilon + 4q' \quad \text{with} \quad 0 \leq q' \leq q.$$

This shows that the basis is as in Case 3.

Finally, we notice that if $\alpha = 1$, it is not possible to have a cusp C such that its semimodules of differential values has basis of length 2. Indeed, assume that $\epsilon = 1$, then we have that $\Gamma_C = \langle 4,5\rangle$. The conductor of the semigroup is $c_\Gamma = (4-1)(5-1) = 12$. Then the axis is $u_1 = 9$. This implies that if the 1-element of the basis λ_1 exists, then it must be $\lambda_1 = 11 > 9$ (note that $10 = 2\cdot 5 \in \Gamma_C$). Thus, we compute the new axis and we have that $u_2 = 15 > c_\Gamma$. Therefore given $\lambda > u_2$, we see that $\lambda \in \Gamma_C$. We conclude that the semimodule of differential values of a cusp with Puiseux pair $(4,5)$ cannot have a basis of length 2. If $\epsilon = 3$ we proceed in a similar way. □

Remark 9.3.3. The proof of Lemma 9.3.2 shows that in case of having a basis as in Case 3, then for any $\lambda \notin \Lambda_1$ with $\lambda > u_2$, we have that

$$\lambda = 8\alpha + 3\epsilon + 4\beta \quad \text{with} \quad 0 \leq \beta \leq q.$$

Now, consider the 2-element of the basis $\lambda_2 = 8\alpha + 3\epsilon + 4q'$ with $0 \leq \beta \leq q'$. Given $\lambda \in \Lambda_2 \setminus \Lambda_1 = \Lambda_C \setminus \Lambda_1$, then $\lambda = 8\alpha + 3\epsilon + 4\beta$ with $q' \leq \beta \leq q$. This is equivalent to saying that $\lambda = \lambda_2 + na$ with $0 \leq a \leq q - q'$.

By Lemma 9.3.2, there are three cases to consider in the proof of Theorem 9.2. In the first one, where the length of the basis is $s = 0$, there is nothing to prove because $\Lambda_C \setminus \Gamma_C = \emptyset$. In the second one, with $s = 1$, since $\Lambda_C \setminus \Gamma_C = (\lambda_1 + \Gamma_C) \setminus \Gamma_C$, the theorem holds by Theorem 9.1. Hence we only need to prove the third case with $s = 2$. In that case, again by Theorem 9.1, we only need to show that for any element $\lambda \in \Lambda_C \setminus \Lambda_1$, then $-\lambda/nm$ is a root of the Bernstein-Sato polynomial. Note the elements $\Lambda_C \setminus \Lambda_1$ are described in the previous remark. The rest of the chapter is devoted to show that given $\lambda \in \Lambda_C \setminus \Lambda_1$, then $-\lambda/nm$ is a root.

As in the previous section, we split the study of the relationships between differential values, coefficients of a nice equation and residues in several technical lemmas.

Lemma 9.3.4. *Let C be a cusp and assume that the basis of the semimodule of differential values is $(n, m, \lambda_1, \lambda_2)$ with*

$$n = 4, \quad m = 4\alpha + \epsilon, \quad \lambda_1 = 4(\alpha + 1) + 2\epsilon + 4q,$$

where $\alpha \geq 2$, $\epsilon \in \{1, 3\}$ and $0 \leq q \leq \alpha - 2$. They following statements are equivalent:

1. *$\lambda_2 = 8\alpha + 3\epsilon + 4q'$ with $0 \leq q' \leq q$ is the 2-element of the basis.*
2. *$z_{2\epsilon+4\beta'} = 0$ for $q \leq \beta' < q + q'$ and $z_{2\epsilon+4(q+q')} \neq 0$, if $q' < q$. Or*

$$2(4\alpha + \epsilon)z_{2\epsilon+8q} - (3\alpha + \epsilon + q)z^2_{\epsilon+4q} \neq 0,$$

if $q' = q$.

Before giving the proof, we recall that the term "reduction" means "reduction modulo $\{f\}$" when referring to functions.

Proof. Assume that the 2-element of the basis of Λ_C is $\lambda_2 = 8\alpha + 3\epsilon + 4q'$ with $0 \leq q' \leq q$. Essentially, the proof follows a similar reasoning as in Lemma 9.2.1 Part 1. We will apply Delorme's algorithm to compute a minimal standard basis of C. In this way, we will obtain a 1-form ω_2 whose differential value is λ_2. During the process, we will derive the desired algebraic conditions given in Statement 2.

In order to simplify the computations along all the proof, we ignore the terms with leading power greater than $(3\alpha + \epsilon - 1 + q, 2)$, that is, given $g \in \mathbb{C}\{x, y\}$ and assume that $g = \sum a_{jk}x^j y^k$. When we write

$$g = g_1 + h.o.t.,$$

we mean that $g_1 = \sum b_{jk}x^j y^k$ where $b_{jk} = 0$ for $(j, k) > (3\alpha + \epsilon - 1 + q, 2)$ and $b_{jk} = a_{jk}$ otherwise. We introduce the previous convention for the following reason: consider a 1-form η with divisorial value $\nu_D(\eta) = t_2$. By Theorem 5.2.10, we know that $\nu_C(\eta) \leq \lambda_2$. Thus, by Propositions 5.4.2 and 5.4.3, we have that

$$i_{P_0}(X_\eta(f), f) \leq \lambda_2 + nm - n - m = 4(3\alpha + q' + \epsilon - 1) + (4\alpha + \epsilon)2.$$

Therefore, if r is a partial or final reduction of $X_\eta(f)$, then r satisfies the following property:

(⋆) The leading power of r is at most $(3\alpha + q' + \epsilon - 1, 2)$.

Since $q' \leq q$, we are not concerned with the behaviour of the monomials with leading power greater than $(3\alpha + q + \epsilon - 1, 2)$.

We start computing a 1-form that could later be identified as ω_2. Since $\lambda_1 = 4(\alpha+1)+2\epsilon+4q = m + n + \epsilon + 4q$ with $0 \leq q \leq \alpha - 2$, by Lemma 9.2.1, we have that $z_\ell = 0$ for $\ell < \epsilon + 4q$ and $z_{\epsilon+4q} \neq 0$. Hence, we can write Equation (9.7) as

$$f = x^{4\alpha+\epsilon} + y^4 + \sum_{\beta=q}^{\alpha-2} z_{\epsilon+4\beta} x^{3\alpha+\epsilon+\beta} y + \sum_{\beta'=q}^{2q} z_{2\epsilon+4\beta'} x^{2\alpha+\epsilon+\beta'} y^2. \tag{9.8}$$

We now apply Delorme's algorithm. Since (x, y) is a system of nice coordinates with respect to C, we can take $\omega_{-1} = dx$ and $\omega_0 = dy$ and then we have $\nu_C(\omega_{-1}) = n$ and $\nu_C(\omega_0) = m$. Similarly as in proof of Lemma 9.2.1, we find that the 1-form $\omega_1 = nxdy - mydx$ satisfies that the function r_1 given by

$$\begin{aligned} r_1 = X_{\omega_1}(f) - nmf = nxf_x + myf_y - nmf = \sum_{j\in J} jz_j x^{p_{1,j}} y^{p_{2,j}} \\ = \sum_{\beta=q}^{\alpha-2} (\epsilon + 4\beta) z_{\epsilon+4\beta} x^{3\alpha+\epsilon+\beta} y + \sum_{\beta'=q}^{2q} (2\epsilon + 4\beta') z_{2\epsilon+4\beta'} x^{2\alpha+\epsilon+\beta'} y^2, \end{aligned} \tag{9.9}$$

is a final reduction of $X_{\omega_1}(f)$. Moreover, the leading power of r_1 is $lp(r_1) = (3\alpha + \epsilon + q, 1)$. By Proposition 5.4.3, the differential value of ω_1 is $\nu_C(\omega_1) = \lambda_1$.

We now compute the candidate for ω_2. As in the proof of Lemma 9.3.2, the axis u_2 is $u_2 = \lambda_1 + 4(\alpha - q - 1) = 2m$. In particular, we have:

$$u_2 = \nu_C(x^{\alpha-q-1}\omega_1) = \nu_C(ydy).$$

This implies that $t_2 = t_1 + 4(\alpha - q - 1) = 4(\alpha - q) + m < 2m$. Therefore, by Delorme's algorithm we need to compute the tuning constant μ^+ for the 1-form $\theta = x^{\alpha-q-1}\omega_1 + \mu^+ ydy$. Later, we will compute a final reduction of θ modulo $\{\omega_{-1}, \omega_0, \omega_1\}$.

Computation of μ^+: We must compute final reductions of $X_{x^{\alpha-q-1}\omega_1}(f)$ and $X_{ydy}(f)$. For the first one, by Equation (9.9), we have that

$$\begin{aligned} r_2 = x^{\alpha-q-1} r_1 = x^{\alpha-q-1}(X_{\omega_1}(f) - nmf) \\ = \sum_{\beta=q}^{\alpha-2} (\epsilon + 4\beta) z_{\epsilon+4\beta} x^{4\alpha+\epsilon+\beta-q-1} y + \\ + \sum_{\beta'=q}^{2q} (2\epsilon + 4\beta') z_{2\epsilon+4\beta'} x^{3\alpha+\epsilon+\beta'-q-1} y^2, \end{aligned} \tag{9.10}$$

which is indeed a final reduction of $X_{x^{\alpha-q-1}\omega_1}(f)$ since its leading power $(4\alpha + \epsilon - 1, 1)$ is not divisible by $(0, 4)$. Similarly, for $X_{ydy}(f)$, we have that

$$\begin{aligned} r_0 = X_{ydy}(f) = y\frac{\partial}{\partial x} \\ = (4\alpha + \epsilon) x^{4\alpha+\epsilon-1} y + (3\alpha + \epsilon + q) z_{\epsilon+4q} x^{3\alpha+\epsilon+q-1} y^2 + h.o.t. \end{aligned} \tag{9.11}$$

Again, the leading power $lp(r_0) = (4\alpha + \epsilon - 1, 1)$ is not divisible by $(0, 4)$. Therefore, by Equations (9.10) and (9.11), the tuning constant is $\mu^+ = -(\epsilon + 4q) z_{\epsilon+4q}/(4\alpha + \epsilon)$. Equivalently, we can write:

$$\omega = (4\alpha + \epsilon)\theta = (4\alpha + \epsilon) x^{\alpha-q-1}\omega_1 - (\epsilon + 4q) z_{\epsilon+4q} ydy. \tag{9.12}$$

This way we have that $\nu_C(\omega) > \nu_C(x^{\alpha-q-1}\omega_1) = \nu_C(ydy)$. Notice that $\nu_D(\omega) = t_2 = 4(\alpha - q) + m$, because $\nu_D(x^{\alpha-q-1}\omega_1) = t_2 = 4(\alpha - q) + m < 2m = \nu_D(ydy)$.

Computation of a final reduction of ω: Note that it is the same than computing a final reduction of θ. For this purpose, we should recursively construct 1-forms ω^i of the form

$$\omega^i = \omega^{i-1} + \mu^+ h_i \omega_k,$$

where $k \in \{-1, 0, 1\}$, $i \geq 0$, h_i is a monomial, and $\omega^0 = \omega$. This process continues until we obtain a 1-form whose differential value is λ_2.

To shorten the proof we do it in a single step, we take all the reductions modulo $\{\omega_{-1}, \omega_0, \omega_1\}$ simultaneously. We compute a partial reduction X of $X_\omega(f)$. Using Equations (9.10) and (9.11), we define X as

$$\begin{aligned} X &= (4\alpha + \epsilon)r_2 - ((\epsilon + 4q)z_{\epsilon+4q})r_0 = \\ &= \sum_{\beta=q+1}^{2q} (4\alpha + \epsilon)(\epsilon + 4\beta)z_{\epsilon+4\beta}x^{4\alpha+\epsilon+\beta-q-1}y + \\ &+ \sum_{\beta'=q}^{2q} (4\alpha + \epsilon)(2\epsilon + 4\beta')z_{2\epsilon+4\beta'}x^{3\alpha+\epsilon+\beta'-q-1}y^2 - \\ &- (\epsilon + 4q)(3\alpha + \epsilon + q)z^2_{\epsilon+4q}x^{3\alpha+\epsilon+q-1}y^2 + h.o.t., \end{aligned} \tag{9.13}$$

since r_0 and r_2 are final reductions of $X_{ydy}(f)$ and $X_{x^{\alpha-q-1}\omega_1}(f)$ respectively. Comparing Equation (9.12) with Equation (9.13), we verify that X is a partial reduction of $X_\omega(f)$.

Since $\nu_D(\omega) = t_2$, then by the property ($\star$), we have that $lp(X) \leq (3\alpha + q' + \epsilon - 1, 2)$. In particular, this implies that $X \neq 0$. Moreover, by Equation (9.13), we observe that $lp(X) \geq (3\alpha + \epsilon - 1, 2)$.

The function X encodes the necessary information to compute the desired final reduction of ω. Write

$$X = \sum_{j,k \geq 0} a_{jk} x^j y^k,$$

and assume that there exists a minimum index ℓ with $q \leq \ell \leq q + q'$, satisfying that $a_{3\alpha+\epsilon+\ell-q-1,2} \neq 0$. Note that by Equation (9.13), having $a_{3\alpha+\epsilon+\beta'-q-1,2} \neq 0$ is the same as having $z_{2\epsilon+4\beta'} \neq 0$, where $q \leq \beta' < 2q$. In other words, we are assuming that $z_{2\epsilon+4\beta'} = 0$ for $q \leq \beta' < \ell$ and that $z_{2\epsilon+4\ell} \neq 0$.

We take the 1-form ω' given by

$$\omega' = \omega - \sum_{\beta=q+1}^{\ell} (\epsilon + 4\beta)z_{\epsilon+4\beta}x^{\beta-q}ydy, \tag{9.14}$$

which is going to be a final reduction of ω modulo $\{\omega_{-1}, \omega_0, \omega_1\}$. Notice that for any $\beta > q$, we have that

$$\nu_D(x^{\beta-q}ydy) > \nu_D(ydy) = 2m > t_2 = 4(\alpha - q) + m = 8\alpha + \epsilon - 4q.$$

Thus, $\nu_D(\omega') = t_2$. Next, observe that:

$$X_{x^{\beta-q}ydy}(f) = (4\alpha + \epsilon)x^{4\alpha+\epsilon+\beta-q-1}y + h.o.t.,$$

where we see that $X_{x^{\beta-q}ydy}(f)$ is non reducible modulo $\{f\}$. Since X is a partial reduction of

$X_\omega(f)$, we can define a partial reduction X' of $X_{\omega'}(f)$, given by the expression:

$$\begin{aligned} X' = X - \sum_{\beta=q+1}^{\ell} (\epsilon+4\beta) z_{\epsilon+4\beta} X_{x^{\beta-q} y dy}(f) = \\ = \sum_{\beta=\ell+1}^{2q} (4\alpha+\epsilon)(\epsilon+4\beta) z_{\epsilon+4\beta} x^{4\alpha+\epsilon+\beta-q-1} y + \\ + \sum_{\beta'=\ell}^{2q} (4\alpha+\epsilon)(2\epsilon+4\beta') z_{2\epsilon+4\beta'} x^{3\alpha+\epsilon+\beta'-q-1} y^2 - \\ - (\epsilon+4q)(3\alpha+\epsilon+q) z_{\epsilon+4q}^2 x^{3\alpha+\epsilon+q-1} y^2 + h.o.t. \end{aligned} \tag{9.15}$$

Since $z_{2\epsilon+4\ell} \neq 0$, we have that the leading power of X' is $lp(X') = (3\alpha+\epsilon+\ell-q-1, 2)$, note that the first summation starts at the index $\ell+1$ and $\epsilon \leq 3$. Thus, we have that $lp(X')$ is not divisible by $lp(f) = (0,4)$. Hence, X' is a final reduction of $X_{\omega'}(f)$. By Proposition 5.4.3, we have that

$$\nu_C(\omega') = 8\alpha + 3\epsilon + 4(\ell - q) = \lambda.$$

We note the following: first, $\lambda \leq \lambda_2$, with the equality achieved if and only if $\ell = q + q'$. Second, $\lambda \notin \Lambda_1$. Thus, for λ_2 to be the minimum element in $\Lambda_C \setminus \Lambda_1$, we need $\ell = q + q'$.

Finally, if ℓ does not exist, meaning $a_{3\alpha+\epsilon+\beta'-1,2} = 0$ for $q \leq \beta' \leq q+q'$, we can construct X' as before by setting $\ell = q + q'$ in the expression of ω' in Equation (9.14). However, this time, by Equation (9.15), X' may not be a final reduction of $X_{\omega'}(f)$. Nonetheless, we see that its leading power is greater than $(3\alpha+\epsilon+q'-1, 2)$. But, as we saw before, that is not possible, since $\nu_D(\omega') = t_2$ and property ($\star$).

Therefore, ℓ exists and it takes the value $\ell = q + q'$. In this situation, we put $\omega_2 = \omega'$ and X' is a final reduction of $X_{\omega_2}(f)$.

Conclusion: As mentioned before, by Equation (9.15), having $a_{3\alpha+\epsilon+\beta'-q-1,2} = 0$ for $q \leq \beta' < q + q'$ is equivalent to having $z_{2\epsilon+4\beta'} = 0$. Additionally, the condition $a_{3\alpha+\epsilon+q'-1,2} \neq 0$ is the same as $z_{2\epsilon+4(q+q')} \neq 0$ if $q' < q$, and to

$$\begin{aligned} (4\alpha+\epsilon)(2\epsilon+8q) z_{2\epsilon+8q} - (\epsilon+4q)(3\alpha+\epsilon+q) z_{\epsilon+4q}^2 &\neq 0 \Leftrightarrow \\ 2(4\alpha+\epsilon) z_{2\epsilon+8q} - (3\alpha+\epsilon+q) z_{\epsilon+4q}^2 &\neq 0, \end{aligned}$$

if $q' = q$, as desired. Finally, all the previous computations also show that if we assume Statement 2, then $\nu_C(\omega_2) = 8\alpha + 3\epsilon + 4q'$. □

Next lemma shows the relationship between differential values and residues.

Lemma 9.3.5. *Let C be a cusp and assume that the basis of the semimodule of differential values is $(n, m, \lambda_1, \lambda_2)$ with*

$$n = 4, \quad m = 4\alpha + \epsilon, \quad \lambda_1 = 4(\alpha+1) + 2\epsilon + 4q,$$

where $\alpha \geq 2$, $\epsilon \in \{1, 3\}$ and $0 \leq q \leq \alpha - 2$. They following statements are equivalent:

1. *$\lambda_2 = 8\alpha + 3\epsilon + 4q'$ with $0 \leq q' \leq q$ is the 2-element of the basis.*
2. *$Res_f(\alpha - q, 1)((8\alpha + 3\epsilon + 4\gamma)/nm) = 0$ for all non negative integers $\gamma < q'$ and $Res_f(\alpha - q, 1)((8\alpha + 3\epsilon + 4q')/nm) \neq 0$.*

Remark 9.3.6. Note that $(\alpha-\beta,1)$ belongs to the cuspidal set M for $0\le\beta\le\alpha-2$, and in particular, $(\alpha-q,1)\in M$, showing that we are in position to apply Equation (9.3). To verify this, note that

$$(\alpha-\beta,1)=(m-1,n-1)-(p_1,p_2),$$

where $(p_1,p_2)=(3\alpha+\epsilon+\beta-1,2)$ is an element of the cuspidal set P, as shown in Remark 9.3.1.

Proof. Assume that $\lambda_2=8\alpha+3\epsilon+4q'$ with $0\le q'\le q$. We need to compute the residues $Res_f(\alpha-q,1)((8\alpha+3\epsilon+4\gamma)/nm)$ for $0\le\gamma\le q'$. Let us consider two cases: $\gamma<q'$ and $\gamma=q'$.

If $0\le\gamma<q'$, we observe that:

$$k=8\alpha+3\epsilon+4\gamma-4(\alpha-q)-(4\alpha+\epsilon)1=2\epsilon+4(q+\gamma)\in J.$$

Hence, in virtue of Equation (9.3), we need to find the sequences of non negative integer numbers $(\delta_\ell)_{\ell\in J}$, such that:

$$\sum_{\ell\in J}\ell\delta_\ell=2\epsilon+4(q+\gamma)=k.$$

Note that by Remark 9.3.1 the elements in J are of the form $\epsilon+4\beta$ and $2\epsilon+4\beta'$, where $0\le\beta\le\alpha-2$ and $0\le\beta'\le2\alpha-2$. Moreover, as shown in the proof of Lemma 9.3.4, we can take a nice equation f of C, given by:

$$f=x^{4\alpha+\epsilon}+y^4+\sum_{\beta=q}^{\alpha-2}z_{\epsilon+4\beta}x^{3\alpha+\epsilon+\beta}y+\sum_{\beta'=q}^{2\alpha-2}z_{2\epsilon+4\beta'}x^{2\alpha+\epsilon+\beta'}y^2.$$

Noting that both summations begin at index $q>\gamma$, the previous two observations lead to the fact that $\delta_{2\epsilon+4(q+\gamma)}=1$ and $\delta_\ell=0$, for $\ell\ne2\epsilon+4(q+\gamma)$, is the single sequence of $(\delta_\ell)_{\ell\in J}$ that is relevant in the computation of $Res_f(\alpha-q,1)((8\alpha+3\epsilon+4\gamma)/nm)$. More precisely, by Equation (9.3), it can be checked that any other sequence will give a zero contribution to the value of the residue. Then, again by Equation (9.3), we conclude that

$$Res_f(\alpha-q,1)((8\alpha+3\epsilon+4\gamma)/nm)\ne0\Leftrightarrow z_{2\epsilon+4(\gamma+q)}\ne0. \tag{9.16}$$

Consider the case $\gamma=q'$. Again we have to consider sequences $(\delta_\ell)_{\ell\in J}$ such that

$$\sum_{\ell\in J}\ell\delta_\ell=2\epsilon+4(q+q').$$

There are two possibilities: $q'<q$ or $q'=q$. Assume first $q'<q$. Following the same argument as before, all the sequences except at most one have zero contribution to the computation of the residue. The only possible relevant sequence is $\delta_{2\epsilon+4(q+q')}=1$ and $\delta_\ell=0$ if $\ell\ne2\epsilon+4(q+q')$. By Equation 9.3

$$Res_f(\alpha-q,1)((8\alpha+3\epsilon+4q')/nm)\ne0\Leftrightarrow z_{2\epsilon+4(q+q')}\ne0. \tag{9.17}$$

Finally, if $q'=q$, then there are two relevant sequences: first, $\delta_{2\epsilon+8q}=1$ and $\delta_\ell=0$ if $\ell\ne2\epsilon+8q$. Second, $\delta_{\epsilon+4q}=2$ and $\delta_\ell=0$ if $\ell\ne\epsilon+4q$. Hence, by Equation (9.3) we obtain the following:

$$\begin{aligned}Res_f(\alpha-q,1)\left(\tfrac{8\alpha+3\epsilon+4q}{nm}\right) &= \frac{\Gamma\left(\frac{8\alpha+3\epsilon+4q}{nm}\right)^{-1}}{nm}\left[\frac{z_{\epsilon+4q}^2}{2}\Gamma\left(\tfrac{2(3\alpha+\epsilon+q)+(\alpha-q)}{m}\right)\Gamma\left(\tfrac{3}{4}\right)-\right.\\ &- \left. z_{2\epsilon+8q}\Gamma\left(\tfrac{(2\alpha+\epsilon+2q)+(\alpha-q)}{m}\right)\Gamma\left(\tfrac{3}{4}\right)\right].\end{aligned}$$

Using the fact that the Euler's Gamma function satisfies that $\Gamma(\rho+1)=\rho\Gamma(\rho)$, we can extract a common factor in the previous equation. This leads to:

$$Res_f(\alpha-q,1)\left(\tfrac{8\alpha+3\epsilon+4q}{nm}\right)\ne0\Leftrightarrow2(4\alpha+\epsilon)z_{2\epsilon+q}-(3\alpha+\epsilon+q)z_{\epsilon+4q}^2\ne0. \tag{9.18}$$

By Equations (9.16)-(9.18), we have to show that Statement *1* is equivalent to

- $z_{2\epsilon+4(\gamma+q)} = 0$ for $0 \leq \gamma < q'$.
- if $q' < q$, then $z_{2\epsilon+4(q'+q)} \neq 0$.
- if $q' = q$, then $2(4\alpha+\epsilon)z_{2\epsilon+q} - (3\alpha+\epsilon+q)z^2_{\epsilon+4q} \neq 0$

Thus, we see that Statement *1* and *2* are equivalent in virtue of Lemma 9.3.4. □

As a consequence of the previous lemmas we obtain the following.

Lemma 9.3.7. *Let C be a cusp and assume that the basis of the semimodule of differential values is $(n, m, \lambda_1, \lambda_2)$ with*

$$n = 4, \quad m = 4\alpha + \epsilon, \quad \lambda_1 = 4(\alpha+1) + 2\epsilon + 4q,$$

where $\alpha \geq 2$, $\epsilon \in \{1,3\}$ and $0 \leq q \leq \alpha - 2$. If $\lambda_2 = 8\alpha + 3\epsilon + 4q'$ with $0 \leq q' \leq q$ is the 2-element of the basis, then for any $\lambda_2 + na + mb \notin \Lambda_1$, we have that

$$Res_f(\alpha - q + a, b+1)((\lambda_2 + na + mb)/nm) \neq 0.$$

Before giving the proof of the lemma, let us explain why this gives the proof of Theorem 9.2. As mentioned before, we were left to show that given $\lambda \in \Lambda \setminus \Gamma_C$, then $-\lambda/nm$ is a root of the Bernstein-Sato polynomial of C. By Theorem 9.4 it is sufficient to show that a certain residue is non-zero, which is given by Lemma 9.3.7.

Proof. Fix f a nice equation of C as in Equation (9.7). We need to show that for a given $\lambda = \lambda_2 + na + mb \notin \Lambda_1$ with $a, b \geq 0$, then

$$Res_f(\alpha - q + a, b+1)(\lambda/nm) \neq 0.$$

By Remark 9.3.3, since $\lambda \notin \Lambda_1$, we have that $b = 0$ and $0 \leq a \leq q - q'$. If $a = 0$, we have already shown that $Res_f(\alpha - q, 1)(\lambda_2/nm) \neq 0$, see Lemma 9.3.5. Thus, assume that $1 \leq a \leq q - q'$, notice that we are assuming that $q' < q$. By Remark 9.3.6, we have $(\alpha - q + a, 1) \in M$.

We can write λ explicitly as $\lambda = 8\alpha + 3\epsilon + 4q' + 4a$. Subtracting $n(\alpha - q + a) + m$, we find:

$$\lambda - n(\alpha - q + a) - m = 2\epsilon + 4(q + q').$$

In other words, we have to compute sequences $(\delta_\ell)_{\ell \in J}$ such that $\sum \ell\delta_\ell = 2\epsilon + 4(q+q')$. Since $q' < q$, there is only one relevant sequence to the computation of the desired residue: $\delta_{2\epsilon+4(q+q')} = 1$ and the rest $\delta_\ell = 0$ for $\ell \neq 2\epsilon + 4(q+q')$. Any other sequence gives zero contribution to the residue. Thus,

$$Res_f(\alpha - q + a, 1)(\lambda/nm) \neq 0 \Leftrightarrow z_{2\epsilon+4(q+q')} \neq 0$$

and we have shown in Lemma 9.3.4 that $z_{2\epsilon+4(q+q')} \neq 0$. □

We end this Chapter with a brief discussion about the Conjecture in 9.3 higher multiplicities. When considering cases where $n \geq 5$, several problems arise. The most important one is that it does not seem possible to apply all the previous techniques in an inductive way. However, following similar steps, we could try to give a proof for each particular value of n. If we want to give a proof for the next case $n = 5$, we would need to determine all possible semimodules, as in Lemma 9.3.4. Nonetheless, the complexity of this point is pretty high when compared with the multiplicity four case. Not only that, but we also need to deal with some partitions of natural numbers, as shown in Equation (9.3), to find residues. Some description about these partitions may be needed.

There is a last problem not at all mention on this text. The choice of the element $(a, b) \in M$ when computing $Res_f(a, b)(\beta_j)$. Note that Theorem 9.4 only demands to have a non zero residue for a particular element $(a,b) \in M$. Along all the proofs we did not explain how we found the appropriate value to take. The idea was the following: we have an element $\lambda \in \Lambda_C \setminus \Gamma_C$, which corresponds with the differential value $\nu_C(\omega)$ of a 1-form ω. We look at the divisorial value $\nu_D(\omega) = na + mb$. Then we check that $(a,b) \in M$ is our desired candidate. When computing examples, this idea does not give the expected results anymore when $n = 5$. Nonetheless, Conjecture 9.3 still holds in those cases.

This conjecture was initially motivated by the results in [17] where the author gives an stratification of the topological class associated to the Puiseux pair $(6,7)$ in terms of the roots of the Bernstein-Sato polynomial. Let us remark that in [17] there is no mention to differential values. However, we can check, with the methods exposed in this thesis, that every branch with Puiseux pair $(6,7)$ satisfies Conjecture 9.3. Besides, more particular examples can be checked with the method `checkRoot` in `Singular` (see [19]).

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Foliaciones y valores diferenciales en la clasificación analítica de cúspides planas

David Senovilla Sanz

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