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RESEARCH

Justifying Linearization for Nonlinear Boundary Homogenization on a Grill-Type Winkler Foundation

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Abstract
We address a nonlinear boundary homogenization problem associated to the deformations of a block of an elastic material with small *reaction regions* periodically distributed along a plane. We assume a nonlinear Winkler-Robin law which implies that a strong reaction takes place in these reaction regions. Outside, on the plane, the surface is traction-free while the rest of the surface is clamped to an absolutely rigid profile. When dealing with *critical sizes* of the reaction regions, we show that, asymptotically, they behave as stuck regions, the homogenized boundary condition being a linear one with a new reaction term which contains a *capacity matrix* depending on the macroscopic variable. This matrix is defined through the solution of a parametric family of microscopic problems, the macroscopic variable being its parameter. Among others, to show the convergence of the solutions, we develop techniques that extend those both for nonlinear scalar problems and linear vector problems in the literature. We also address the *extreme cases*.

Keywords Nonlinear Winkler foundations · Boundary homogenization · Elasticity operator · Capacity matrix · Critical relations

Mathematics Subject Classification 35B27 · 35J65 · 35J57 · 35B25 · 74B05 · 74Q20

1 Introduction

Rapidly alternating Winkler-Robin type boundary conditions appear naturally in foundation models for many Civil Engineering constructions, where strong reactions concentrated in small regions ("the springs") may occur. These reactions may be represented by a nonlinear monotonic vector function of the displacements. A homogenization process is required to save numerical computations, and also, often, a linearization process is performed in order

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to simplify the model. However, the linearization may be wrong or it may not be justified (cf. [20] to compare). In this paper, we justify both the averaged boundary conditions and the linearization process for certain relations between the period, sizes of the reaction regions and reaction coefficients.

As a matter of fact, boundary homogenization for linear elasticity systems has been widely considered in the literature, let us mention [4, 12, 15–17, 19, 22, 23, 31, 32, 36, 39–41] and references therein. Winkler-Steklov type boundary conditions are addressed in [15, 23] while linear Winkler-Robin boundary conditions with large parameters are in [16, 17, 19, 21]. Among the previously mentioned papers there exist only a few addressing critical sizes of the heterogeneities or *reaction regions* on the boundary giving rise to the so-called *strange terms* in the homogenized boundary conditions (cf. [4, 17, 31] for grill type foundations) and none of them address nonlinear boundary conditions as we do it here.

In this paper, we study the asymptotic behavior of the deformations of a non-homogeneous elastic body which has very large surface reaction terms concentrated in small regions (*the reaction regions*) periodically placed along a plane part of the surface. The reaction terms are represented by a nonlinear vector function which depends on the points where the reaction regions are placed and on the displacement vector, and also it contains a very large *reaction parameter* $\beta(\varepsilon)$.

We assume that the elastic material fills the domain Ω of the upper half space $\mathbb{R}^{3+}$, and a part Σ of its surface lies on the plane $\{x_3 = 0\}$ and contains small regions T^ε of "size" r_ε, at a distance $O(\varepsilon)$ between them (cf. Fig. 1), ε measures the period of the structure. The boundary conditions are nonlinear, of Winkler-Robin type, on T^ε. Outside, the surface Σ is free of charges while the rest of the surface $\partial\Omega \setminus \overline{\Sigma}$ is assumed to be fixed. Here ε and r_ε are two small parameters $r_\varepsilon \ll \varepsilon \ll 1$ while $\beta(\varepsilon)$ tends to $+\infty$ as $\varepsilon \to 0$. Let us introduce the limits that relate the three parameters:

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon^2} = r_0, \tag{1.1}$$

and

$$\lim_{\varepsilon\to 0} r_\varepsilon \beta(\varepsilon) = +\infty. \tag{1.2}$$

It should be noted that the products $r_\varepsilon \varepsilon^{-2}$, $r_\varepsilon \beta(\varepsilon)$ as well as $r_\varepsilon^2 \varepsilon^{-2}\beta(\varepsilon)$ have proven to play an important role in the description of the limits of the linear models [16, 17] and the scalar models in [18, 46]. For nonlinear Winkler-Robin boundary conditions we mention the recent results in [20] for large sizes of the reaction regions and *critical reactions*, namely $r_\varepsilon \gg \varepsilon^2$ and $r_\varepsilon^2 \varepsilon^{-2}\beta(\varepsilon) = O(1)$ which implies $\beta(\varepsilon) \ll (r_\varepsilon)^{-1}$, and henceforth, a very different situation from that in this paper, cf. (1.2) and (1.1). In [20] we proved that an averaged surface reaction term of the same type of the original one is asymptotically imposed. However, in the case here considered, where we have a very large reaction $\beta(\varepsilon) \gg (r_\varepsilon)^{-1}$, cf. (1.2), and $r_\varepsilon = O(\varepsilon^2)$, cf. (1.1), the study of the nonlinear model here considered requires a different treatment and a thorough analysis due to the strong contrast induced by the large reaction. For $r_0 > 0$, the case $\beta(\varepsilon) = O(r_\varepsilon)^{-1}$ somehow makes a threshold where the homogenized model changes from linear to nonlinear, this process being the object of actual development. In this paper, we show that the nature of the homogenized boundary conditions are different from those in the original problem. Let us explain this in further detail.

The homogenization problem being (2.11), we provide a general framework for nonlinear reaction terms $\beta(\varepsilon)M(x, u^\varepsilon)$ to show the convergence of solutions u^ε, as $\varepsilon \to 0$ (cf. (2.7)-(2.10) and Remark 2), towards the solution u^0 of a homogenized problem. In the case where $r_0 > 0$ the homogenized boundary condition is a linear Winkler-Robin one. Further specifying, it contains the so-called *the strange term*

$$r_0 \mathcal{C}(\hat{x}) u^0,$$

cf. (3.10), and links stresses and displacements, the elastic coefficients of this "averaged spring" being given by a matrix, $\mathcal{C}(\hat{x})$, the so-called *capacity matrix* which depends on the macroscopic variable, but which is defined through the solution of a family of microscopic problems in the unbounded half-space, with the macroscopic variable $\hat{x} \in \Sigma$ as its parameter (cf. (3.5) and (3.1)). The limit problem coincides with that obtained for the case of alternating boundary conditions of Dirichlet and Neumann type in [4, 31] and also in the case where the reaction term is linear, cf. (7.5), [16, 17]. The study in [4, 31] dealt with a homogeneous and isotropic media which means that the capacity matrix is constant, and also, it deals with different boundary conditions. In our case, asymptotically, at the microscopic level the media is homogeneous, but it depends on the position of the reaction regions. It behaves as if the material were stuck on these reaction regions. Due to this strong interaction between macroscopic and microscopic scale, the periodicity becomes necessary to obtain the strange term (cf. Remark 2). This case $r_0 > 0$ is referred to as a *critical size*, since the behavior of the solutions asymptotically becomes somehow "extreme" when $r_0 = 0$ or $r_0 = +\infty$, being a Neumann condition or a Dirichlet one asymptotically imposed over the whole region Σ.

It should be emphasized that the nonlinear Winkler-Robin law here used has appeared in the literature in very different models for porous media (cf. [11]) or rough walls (cf. [20, 50]) but always with sizes of heterogeneities of the same order of magnitude as the period. Also note that this law might be replaced by others arising in many contact problems (cf. e.g., [2, 25, 44]) with the suitable modifications in the treatment and the homogenized boundary conditions, cf. also Remark 4. See, e.g., [4, 43, 50] and references therein for the homogenization of Signorini type conditions outside critical size ranges. Also, we refer to [37, 38] for the introduction of other capacity matrices in very different problems of elasticity.

Let us mention that, in homogenization problems, strange terms in the partial differential equation or on boundary/transmission conditions, as well as related terminology, were introduced in the literature many years ago, cf. [8, 33, 48] for the Laplacian, [1] for Stokes equations, [31] for the elasticity operator, [24] for semilinear boundary conditions in perforated media, and [5, 6, 9, 10] for different settings involving non periodic media; the list not being exhaustive.

Finally, we describe the structure of the paper. Section 2 contains the description of the geometry, the setting of the homogenization problem and some a priori estimates which are necessary for our asymptotic analysis. Section 3 contains the main results (cf. Theorems 3.1-3.3) including the homogenized problems and the parametric family of local problems necessary to describe the strange term when $r_0 > 0$, the parameter of this family being the macroscopic variable. It also contains the correct setting of these problems in the suitable Sobolev spaces. The preliminary results for the proofs are in Sect. 4. In Sect. 5, we obtain homogenized and local problems using matched asymptotic expansions. Sections 6 and 7 are devoted to the proof of the convergence stated in Theorems 3.1-3.3, where the emphasis is placed on the nonlinear term $\beta(\varepsilon) M(x, u^\varepsilon)$ (cf. Theorems 6.1 and 6.2), avoiding the repetition of the linear case but also preserving self-containment of the paper. Also we note that we avoid using variational inequalities as used to be the case when dealing with semilinear boundary conditions (cf. [18, 20]). In particular, it should be emphasized that the technique applies to scalar problems and linear problems for the elasticity system (see Remark 3), but new convergence measures and comparison results have been developed for the treatment of this nonlinear model (cf. [16–19, 46] to compare). Finally, note that stronger convergence results follow when $r_0 = 0$ and $r_0 = +\infty$, which, for completeness, are addressed in Sect. 7.

2 The Setting of the Problem

Let Ω be an open bounded domain of $\mathbb{R}^3$ situated in the upper half-space $\mathbb{R}^{3+} = \{x \in \mathbb{R}^3 : x_3 > 0\}$, with a Lipschitz boundary $\partial\Omega$. Let $\partial\Omega$ be $\partial\Omega = \overline{\Gamma}_\Omega \cup \overline{\Sigma}$ where $\Sigma \neq \emptyset$ is the part of the boundary in contact with the plane $\{x_3 = 0\}$ and Γ_Ω the rest. Let T denote an open bounded domain of the plane $\{x_3 = 0\}$ with a Lipschitz boundary. Without any restriction, we can assume that both Σ and T contain the origin of coordinates.

Let ε be a small parameter, i.e., $\varepsilon \ll 1$. Let r_ε be an order function such that $r_\varepsilon \ll \varepsilon$. For $\mathrm{k} = (k_1, k_2) \in \mathbb{Z}^2$, we denote by $\widetilde{x}^\varepsilon_\mathrm{k}$ the point of the plane $\{x_3 = 0\}$ with coordinates $\widetilde{x}^\varepsilon_\mathrm{k} = (k_1\varepsilon, k_2\varepsilon, 0)$, and by $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$ the homothetic domain of T of ratio r_ε after translation to the point $\widetilde{x}^\varepsilon_\mathrm{k}$:

$$T^\varepsilon_{\widetilde{x}_\mathrm{k}} = \widetilde{x}^\varepsilon_\mathrm{k} + r_\varepsilon T\,.$$

If there is no ambiguity, we shall write $\widetilde{x}_\mathrm{k}$ instead of $\widetilde{x}^\varepsilon_\mathrm{k}$, and T^ε instead of $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$ while $\widetilde{x}^\varepsilon_\mathrm{k}$ is referred to as the center of $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$.

Let $\mathcal{J}^\varepsilon$ denote $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 : T^\varepsilon_{\widetilde{x}_\mathrm{k}} \subset \Sigma\}$. Let also N_ε denote the number of elements of $\mathcal{J}^\varepsilon$:

$$N_\varepsilon \approxeq \frac{|\Sigma|}{\varepsilon^2} = O(\varepsilon^{-2}). \tag{2.1}$$

For brevity, we denote by $\bigcup T^\varepsilon$ the union of all the T^ε contained in Σ, namely,

$$\bigcup T^\varepsilon \equiv \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} T^\varepsilon_{\widetilde{x}_\mathrm{k}}.$$

Also, in what follows $x = (x_1, x_2, x_3)$ denotes the usual coordinates of the Cartesian coordinate system, while by $\hat{x} = (x_1, x_2)$ we refer to the two first components of $x \in \mathbb{R}^3$. In addition, for the sake of completeness, we introduce the so-called *microscopic variable*

$$y = \frac{x - \widetilde{x}_\mathrm{k}}{r_\varepsilon}, \tag{2.2}$$

which transforms each region $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$ into the unit region T, $\mathrm{k} \in \mathcal{J}^\varepsilon$ (cf. Fig. 1).

The geometrical configuration in the plane is analogous to that in [34, 46, 48] for scalar problems and that in [4, 17, 23, 31] for the elasticity system; cf. also further references in these papers.

For $i, j, k, l = 1, 2, 3$, we denote by $a_{ijkl}(x)$ the elastic coefficients of the material, which are assumed to be globally Lipschitz functions defined in $\overline{\Omega}$ and satisfy the standard symmetry and coercivity properties (cf., e.g., [42])

$$a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x), \quad i, j, k, l = 1, 2, 3, \quad \forall x \in \overline{\Omega}, \tag{2.3}$$

and

$$\exists \alpha_1 > 0 : a_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha_1 \xi_{ij}\xi_{ij}, \ \forall \text{matrix } \xi : \xi_{ij} = \xi_{ji},\ i, j = 1, 2, 3,\ \forall x \in \overline{\Omega}. \tag{2.4}$$

For a given displacement vector $u(x) = (u_1(x), u_2(x), u_3(x))$ we use the standard notations for stress and strain tensors, $\sigma(u) = (\sigma_{ij}(u))_{i,j=1,2,3}$ and $e(u) = (e_{ij}(u))_{i,j=1,2,3}$, related by Hooke's law

$$\sigma_{ij}(u) = a_{ijkl}(x)e_{kl}(u), \tag{2.5}$$

where

$$e_{kl}(u) = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big). \tag{2.6}$$

Above, and in what follows, we use the convention of summation over repeated indexes.

As for the reaction terms, let us introduce a vector function

$$M(\hat{x}, u) = (M_1(\hat{x}, u), M_2(\hat{x}, u), M_3(\hat{x}, u)),$$

with components $M_i(\hat{x}, u) \equiv M_i(x_1, x_2, u_1, u_2, u_3)$, $M_i \in C(\overline{\Sigma} \times \mathbb{R}^3)$, $i = 1, 2, 3$, satisfying

$$M_i(\hat{x}, 0) = 0, \quad \forall \hat{x} \in \overline{\Sigma}, \quad i = 1, 2, 3, \tag{2.7}$$

the strong monotonicity condition in the u variable

$$(M_i(\hat{x}, u) - M_i(\hat{x}, v))(u_i - v_i) \geq K(u_i - v_i)^2 \equiv K|u - v|^2, \quad \forall \hat{x} \in \overline{\Sigma},\ u, v \in \mathbb{R}^3, \tag{2.8}$$

and *the globally Lipschitz condition in x and u variables*

$$|M_i(\hat{x}, u) - M_i(\hat{x}', v)| \leq L_i(|\hat{x} - \hat{x}'| + |u - v|), \quad \forall \hat{x}, \hat{x}' \in \overline{\Sigma},\ u, v \in \mathbb{R}^3, \tag{2.9}$$

$i = 1, 2, 3$, for certain positive constants K, L_1, L_2 and L_3. The last condition being a particular case when $\delta = 0$ of the most general one

$$|M_i(\hat{x}, u) - M_i(\hat{x}', v)| \leq L_i(|\hat{x} - \hat{x}'| + |u - v| + |u - v|^{1+\delta}), \quad \forall \hat{x}, \hat{x}' \in \overline{\Sigma},\ u, v \in \mathbb{R}^3, \tag{2.10}$$

and $\delta \in [0, 2]$. Note that (2.9) holds with further smoothness of M, cf. (7.4); see Remarks 2 and 3 in this connection.

For $f = (f_1, f_2, f_3) \in (L^2(\Omega))^3$ let us consider the problem

$$\begin{cases} -\dfrac{\partial \sigma_{ij}^{\varepsilon}}{\partial x_j} = f_i & \text{in } \Omega, \\ u^{\varepsilon} = 0 & \text{on } \Gamma_{\Omega}, \\ \sigma_{i3}^{\varepsilon} = 0 & \text{on } \Sigma \setminus \overline{\bigcup T^{\varepsilon}}, \\ \sigma_{i3}^{\varepsilon} - \beta(\varepsilon) M_i(x, u^{\varepsilon}) = 0 & \text{on } \bigcup T^{\varepsilon}, \end{cases} \qquad i = 1, 2, 3. \tag{2.11}$$

Above, $u^{\varepsilon} = (u_1^{\varepsilon}, u_2^{\varepsilon}, u_3^{\varepsilon})$ denotes the displacement vector, and on account of (2.5), (2.6) we have denoted by

$$\sigma_{ij}^{\varepsilon} \equiv \sigma_{ij}(u^{\varepsilon}) = a_{ijkl} e_{kl}(u^{\varepsilon}).$$

The *Robin/Winkler coefficient of reaction* (*reaction parameter*, in short) $\beta(\varepsilon)$ arising in the equations on T^{ε} is a positive parameter that converges towards $+\infty$: $\beta(\varepsilon) \gg r_{\varepsilon}^{-1}$, cf. (1.2).

The problem (2.11) may represent a model associated to the interaction of a block of an elastic material with the soil via a series of small "springs" with a nonlinear Winkler law $\sigma_{i3}^{\varepsilon} = -\beta(\varepsilon) M_i(x, u^{\varepsilon})$. Outside these springs, *the reaction regions* $\bigcup T^{\varepsilon}$, the surface is free of forces, cf., [17] for linear-Winkler springs, [3] for scalar models in the framework of variational inequalities and [18] for a nonlinear scalar model.

For fixed $\varepsilon > 0$, the weak formulation of problem (2.11) reads: find $u^{\varepsilon} \in \mathbf{V}$, satisfying

$$\int_{\Omega} \sigma_{ij}(u^{\varepsilon}) e_{ij}(v)\, dx + \beta(\varepsilon) \int_{\bigcup T^{\varepsilon}} M_i(\hat{x}, u^{\varepsilon}) v_i\, d\hat{x} = \int_{\Omega} f_i v_i\, dx, \ \forall v \in \mathbf{V}, \tag{2.12}$$

where $\mathbf{V}$ denotes the space completion of $\{\phi \in (C^1(\overline{\Omega}))^3 : \phi = 0 \text{ on } \Gamma_{\Omega}\}$ with the norm generated by the scalar product

$$(u, v)_{\mathbf{V}} = \int_{\Omega} e_{ij}(u) e_{ij}(v)\, dx\,. \tag{2.13}$$

On account of (2.3) and (2.4), the first integral on the left hand side of (2.12) defines a bilinear, symmetric continuous and coercive form on $\mathbf{V} \subset (L^2(\Omega))^3$. As for the second integral, on account of (2.8), we can write

$$\beta(\varepsilon) \int_{\bigcup T^{\varepsilon}} (u_i^{\varepsilon})^2\, d\hat{x} \leq C\beta(\varepsilon) \int_{\bigcup T^{\varepsilon}} M_i(\hat{x}, u^{\varepsilon}) u_i^{\varepsilon}\, d\hat{x}. \tag{2.14}$$

Here and in what follows C denotes a positive constant independent of ε.

The existence and uniqueness of solution of (2.11) is obtained from properties (2.7), (2.8) and (2.10), that allows to write the problem in terms of a variational inequality for a coercive and monotonic hemicontinuous operator on $\mathbf{V}$ (cf., e.g., Theorems 8.2-8.4 in Sections II.8.2 and II.8.3 of [30] for a general abstract framework and [20] for applications to this model). So that we can state the following result

Theorem 2.1 *For fixed* $\varepsilon > 0$, *there is a unique solution* $u^{\varepsilon} \in \mathbf{V}$ *of the integral identity* (2.12). *In addition the sequence* $\{u^{\varepsilon}\}_{\varepsilon}$ *satisfies*

$$\|u^{\varepsilon}\|_{\mathbf{V}} \leq C \quad \textit{and} \quad \beta(\varepsilon) \int_{\bigcup T^{\varepsilon}} (u_i^{\varepsilon})^2\, d\hat{x} \leq C. \tag{2.15}$$

As a consequence of the uniform bound (2.15), for any sequence, we can extract a subsequence, still denoted by ε, such that

$$u^{\varepsilon} \longrightarrow u^0 \text{ in } (H^1(\Omega))^3 - weak, \text{ as } \varepsilon \to 0, \tag{2.16}$$

for some $u^0 \in \mathbf{V}$.

The aim of this work is to identify u^0 with the unique solution of a homogenized problem which depends on $r_0 = 0$, $r_0 = +\infty$ or $r_0 > 0$ in (1.1).

3 Homogenized Problems and Main Results

In order to make the reading of the paper easier, in this section, we state the three homogenized problems which depend on the different relations between the parameters, cf. (1.1)

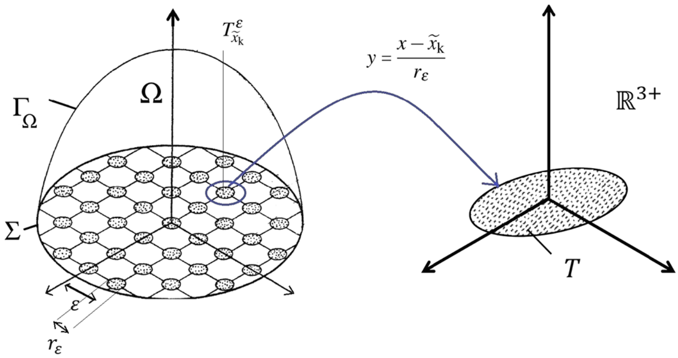

Fig. 1 The geometrical configurations for macroscopic and microscopic variables

and (1.2). We also introduce the parametric family of microscopic problems, the so-called *local problems* (3.1), that allow us to describe the strange term in the homogenized problem. Finally, we state the main results of the paper cf. Theorems 3.1-3.3. We obtain microscopic and homogenized problems in Sect. 5, by using the technique of matched asymptotic expansions, while in Sects. 6-7 we use these local problems to show the convergence.

3.1 The Parametric Family of Local Problems

Let us introduce the $\hat{x}$*-dependent local problems*

$$\begin{cases} -\dfrac{\partial \sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}})}{\partial y_j} = 0 & \text{in } \mathbb{R}^{3+}, \\ \sigma^{\hat{x}}_{i3,y}(W^{l,\hat{x}}) = 0 & \text{on } \{y_3 = 0\} \setminus \overline{T}, \\ W^{l,\hat{x}}(y) = \mathrm{e}^l & \text{on } T, \\ W^{l,\hat{x}}(y) \longrightarrow 0 & \text{as } |y| \to \infty,\ y_3 > 0, \end{cases} \qquad i = 1, 2, 3, \tag{3.1}$$

where $y = (y_1, y_2, y_3)$ denotes an auxiliary variable in $\mathbb{R}^3$, e^l stands for the unit vector in the y_l-direction, while $l = 1, 2, 3$, and the upper index $\hat{x}$ is a parameter which refers to the elastic homogeneous media with *freezing coefficients at* $\hat{x}$. Namely,

$$\sigma^{\hat{x}}_{ij,y}(V) = a_{ijkl}(\hat{x}) e_{kl,y}(V), \tag{3.2}$$

where the lower index y denotes the variable of derivation.

The variable y is referred to as *the local variable*; its connection with the macroscopic variable x is given by the change (2.2). Hence, since we need to distinguish between differentiation in x and y, in what follows, lower indexes x or y in the components of the stress and strain tensors mean the variable for derivation.

It should be emphasized that, in (3.1), the macroscopic variable $\hat{x} \in \Sigma$, becomes a parameter arising in the stress tensor (3.2), and we have a parametric family of local problems (3.1) whose solutions satisfy the equilibrium equations for a homogeneous media filling the half-space $\mathbb{R}^{3+}$. The proof of the existence and uniqueness of the solution of (3.1) in suitable functional spaces follows the scheme in [31] and [16] for an isotropic media and [17] for the anisotropic media. For the sake of completeness, we outline here below the main results.

Let $\mathcal{D}(\overline{\mathbb{R}^{3+}})$ be the space of functions that are restrictions to $\overline{\mathbb{R}^{3+}}$ of the elements of $\mathcal{D}(\mathbb{R}^3)$, and let $\mathcal{D}_T(\overline{\mathbb{R}^{3+}})$ be the space of functions of $\mathcal{D}(\overline{\mathbb{R}^{3+}})$ such that they vanish in a neighborhood of $\overline{T}$. Let us define the functional spaces $\mathcal{W}$ and $\mathcal{W}_0$ as the completion of $(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$ and $(\mathcal{D}_T(\overline{\mathbb{R}^{3+}}))^3$ respectively, with the norm

$$\Big(\sum_{i,j=1}^{3}\Big\|e_{ij,y}(U)\Big\|^2_{L^2(\mathbb{R}^{3+})}\Big)^{1/2}. \tag{3.3}$$

Due to Korn's inequality in bounded Lipschitz domains, the continuous embedding of $\mathcal{W}_0$ into $(H^1_{loc}(\mathbb{R}^{3+}))^3$ holds, and the elements U of $\mathcal{W}_0$ have null traces on T and satisfy $e_{ij,y}(U)\in L^2(\mathbb{R}^{3+})$, $i,j=1,2,3$. We refer to, e.g., [26, 27, 37] for further definitions and properties of spaces $\mathcal{W}$ and $\mathcal{W}_0$ involving (3.3).

For each $l=1,2,3$, we take a function $\Psi^l\in(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$ such that $\Psi^l=\mathrm{e}^l$ in a neighborhood of $\overline{T}$. Then, there is a unique solution $W^{l,\hat{x}}\in\Psi^l+\mathcal{W}_0$ satisfying

$$\int_{\mathbb{R}^{3+}}\sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}})e_{ij,y}(V)dy=0\quad\forall V\in\mathcal{W}_0. \tag{3.4}$$

This is a weak formulation of problem $(3.1)_1$-$(3.1)_3$ and the representation of the solution (4.3) (cf. also (4.6)) provides its precise behavior at infinity in $(3.1)_3$.

Also, $\sigma_{i3}(W^{l,\hat{x}})\big|_{y_3=0}$ is a distribution having a compact support contained in $\overline{T}$ and belongs to $H^{-1/2}(T)$. Hence, it makes sense to define the matrix $\mathcal{C}=(\mathcal{C}_{il})_{i,l=1,2,3}$ as

$$\mathcal{C}_{il}(\hat{x})=-\big\langle\sigma^{\hat{x}}_{i3,y}(W^{l,\hat{x}}),1\big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}, \tag{3.5}$$

which is referred to us the *capacity matrix.*

Applying the Green formula in (3.1), for any $V\in(\mathcal{D}(\overline{\mathbb{R}^{3+}}))^3$, we can write

$$\int_{\mathbb{R}^{3+}}\sigma^{\hat{x}}_{pj,y}(W^{l,\hat{x}})e_{pj,y}(V)\,dy=\big\langle-\sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}),V_i\big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}, \tag{3.6}$$

and, by density, it holds for any $V\in\mathcal{W}$. Consequently, taking $V=W^{p,\hat{x}}$ in (3.6) we have the representation for the components of the matrix $\mathcal{C}$:

$$\int_{\mathbb{R}^{3+}}\sigma^{\hat{x}}_{pj,y}(W^{l,\hat{x}})e_{pj,y}(W^{i,\hat{x}})\,dy=-\big\langle\sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}),\mathrm{e}^i_p\big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}=\mathcal{C}_{il}(\hat{x}), \tag{3.7}$$

which proves useful in the following sections.

3.2 The Homogenized Problems

For the critical size $r_0>0$, we introduce the homogenized problem

$$-\frac{\partial\sigma_{ij,x}(u^0)}{\partial x_j}=f_i\quad\text{in }\Omega,\qquad i=1,2,3, \tag{3.8}$$

$$u^0_i=0\quad\text{on }\Gamma_\Omega,\qquad i=1,2,3, \tag{3.9}$$

$$\sigma_{i3,x}(u^0)-r_0\mathcal{C}_{ij}(\hat{x})u^0_j=0\quad\text{on }\Sigma,\qquad i=1,2,3, \tag{3.10}$$

with the strange term $r_0\mathcal{C}_{ij}(\hat{x})u_j^0$ containing the capacity matrix $\mathcal{C}(\hat{x})$ defined by (3.5) or, equivalently, by (3.7). Therefore, the correct setting of problem (3.8)-(3.10) is linked to that of the corresponding $\hat{x}$-dependent family of microscopic problems (3.1) as well as to the properties of their solutions. Note that (3.10) is a linear Winkler-Robin boundary condition on Σ which does not depend on the function M.

Indeed, the variational formulation of problem (3.8), (3.9), (3.10), reads:

find $u^0 \in \mathbf{V}$ satisfying

$$\int\limits_{\Omega} \sigma_{ij,x}(u^0)e_{ij,x}(v)\,dx + r_0 \int\limits_{\Sigma} \mathcal{C}_{ij}(\hat{x})u_i^0 v_j\,d\hat{x} = \int\limits_{\Omega} f_i v_i\,dx, \quad \forall v \in \mathbf{V}. \tag{3.11}$$

The properties of the matrix $\mathcal{C}$ in (3.5), cf. Proposition 4.2, along with the Poincaré and Korn's inequalities on $\mathbf{V}$, imply the existence and uniqueness of solution of (3.11).

For the extreme cases where $r_0 = 0$ the homogenized problem is the mixed boundary value problem defined by (3.8), (3.9) and

$$\sigma_{i3,x}(u^0) = 0 \quad \text{on} \quad \Sigma, \qquad i = 1, 2, 3. \tag{3.12}$$

Its variational formulation is

$$\int\limits_{\Omega} \sigma_{ij,x}(u^0)e_{ij,x}(v)\,dx = \int\limits_{\Omega} f_i v_i\,dx, \quad \forall v \in \mathbf{V}, \tag{3.13}$$

whose existence and uniqueness of solution in $\mathbf{V}$ is well known on account of the Poincaré and Korn's inequalities on $\mathbf{V}$.

Similarly, for the extreme case where $r_0 = +\infty$ the homogenized problem is the Dirichlet problem defined by (3.8) and

$$u^0 = 0 \quad \text{on} \quad \partial\Omega, \tag{3.14}$$

which has a variational formulation

$$\int\limits_{\Omega} \sigma_{ij,x}(u^0)e_{ij,x}(v)\,dx = \int\limits_{\Omega} f_i v_i\,dx, \quad \forall v \in (H_0^1(\Omega))^3, \tag{3.15}$$

and a unique solution in $(H_0^1(\Omega))^3$. Note that the variational formulations (3.11), (3.13) and (3.15) are written in terms of bilinear, continuous, and coercive forms on a couple of Hilbert spaces with a compact and dense embedding $\mathbf{V} \subset \mathbf{H}$ and therefore classical in the literature (cf., e.g., [42, 47] as regards the elasticity framework for these formulations).

Theorem 3.1 *Let $r_0 \geq 0$ in (1.1) and let M satisfy (2.7)-(2.9). Then, the solution u^ε of (2.12) converges in $(H^1(\Omega))^3$-weak, as $\varepsilon \to 0$, towards the solution u^0 of (3.11).*

Theorem 3.2 *For $r_0 = 0$ in (1.1) and M satisfying (2.7), (2.8) and (2.10), the solution u^ε of (2.12) converges in $(H^1(\Omega))^3$, as $\varepsilon \to 0$, towards the solution u^0 of (3.13).*

Theorem 3.3 *For $r_0 = +\infty$ in (1.1) and M satisfying (2.7), (2.8) and (2.10), the solution u^ε of (2.12) converges in $(H_0^1(\Omega))^3$, as $\varepsilon \to 0$, towards the solution u^0 of (3.15).*

4 Preliminary Results

For the sake of completeness, we state here the necessary results for the proofs of Theorems 3.1-3.3 in Sects. 6-7.

Boundedness and dependence on the parameter $\hat{x} \in \Sigma$*:* $r_0 \geq 0$.

The following results deal with the correct setting of the homogenized and local problems as well as the continuous dependence of $W^{l,\hat{x}}$ and $\mathcal{C}(\hat{x})$ on the macroscopic variable $\hat{x} \in \overline{\Sigma}$. In their statements C denotes a positive constant independent of $\hat{x}$.

Proposition 4.1 *For* $l = 1, 2, 3$, *the solution* $W^{l,\hat{x}}$ *of* (3.6) *depends continuously on* $\hat{x} \in \overline{\Sigma}$ *in the topology of* $\mathcal{W}$. *In addition, for* $l, p, i, j = 1, 2, 3$, *the functions*

$$\widehat{x} \mapsto \int_{\mathbb{R}^{3+}} e_{ij,y}(W^{l,\hat{x}}) e_{ij,y}(W^{p,\hat{x}}) dy \qquad \text{and} \qquad \widehat{x} \mapsto \langle \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}), \mathrm{e}^i_p \rangle_{H^{-1/2}(T)\times H^{1/2}(T)}$$

also depend continuously on $\hat{x} \in \overline{\Sigma}$, *and the bounds*

$$\sum_{i,j=1}^{3} \left\| e_{ij,y}(W^{l,\hat{x}}) \right\|_{L^2(\mathbb{R}^{3+})} \leq C, \qquad \left| \langle \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}), \mathrm{e}^i_p \rangle_{H^{-1/2}(T)\times H^{1/2}(T)} \right| \leq C \tag{4.1}$$

and

$$\left\| \sigma^{\hat{x}}_{p3,y}(W^{l,\hat{x}}) \right\|_{H^{-1/2}(T)} \leq C \tag{4.2}$$

hod true $\forall \hat{x} \in \overline{\Sigma}$.

Proposition 4.2 *For each fixed* $\hat{x} \in \Sigma$, $\mathcal{C}(\hat{x})$ *defined by* (3.5) *is a symmetric and positive definite matrix. In addition, its entries depend continuously on* $\hat{x} \in \overline{\Sigma}$.

The proofs of Propositions 4.1 and 4.2 can be found in Sect. 7 of [17].

Related to solutions of $\hat{x}$-dependent problems (3.1) we state the following results:

Theorem 4.3 *For each* $\hat{x} \in \overline{\Sigma}$ *and* $l = 1, 2, 3$, *the solution* $W^{l,\hat{x}} \in \Psi^l + \mathcal{W}_0$ *of problem* (3.4) *can be represented in terms of the Green matrix-function* $G^{\hat{x}}_{ij}$ *as follows*

$$W^{l,\hat{x}}_i(y_1, y_2, y_3) = \langle \sigma^{l,\hat{x}}_j, G^{\hat{x}}_{ij}(y_1 - \cdot, y_2 - \cdot, y_3) \rangle_{H^{-1/2}(T)\times H^{1/2}(T)}, \tag{4.3}$$

where $\sigma^{l,\hat{x}}$ *is defined by*

$$\sigma^{l,\hat{x}} = (\sigma^{l,\hat{x}}_1, \sigma^{l,\hat{x}}_2, \sigma^{l,\hat{x}}_3) := (\sigma^{\hat{x}}_{13,y}(W^{l,\hat{x}}), \sigma^{\hat{x}}_{23,y}(W^{l,\hat{x}}), \sigma^{\hat{x}}_{33,y}(W^{l,\hat{x}})),$$

and $G^{\hat{x}} = (G^{\hat{x}}_{ij})_{i,j=1,2,3}$ *is a symmetric tensor which depends on the elastic moduli* $a_{ijkp}(\hat{x})$ *of the media and admits the representation*

$$G^{\hat{x}}(y) = |y|^{-1}\, \Phi^{\hat{x}}(\omega) \quad \text{for} \quad y \in \mathbb{R}^{3+} \quad \text{and} \quad \omega \in \overline{S^2_+} := \{y \in \mathbb{R}^{3+} : |y| = 1\}, \tag{4.4}$$

where $\Phi^{\hat{x}}$ *is a symmetric matrix whose elements are smooth functions on the unit semi-sphere in* $\mathbb{R}^{3+}$, $\overline{S^2_+}$. *In addition,* $\Phi^{\hat{x}}$ *depends continuously on the parameter* $\hat{x} \in \overline{\Sigma}$, *in such a* *way that for* $i, j = 1, 2, 3$,

$$\max_{\hat{x}\in\overline{\Sigma},\, \omega\in\overline{S^2_+}} |\Phi^{\hat{x}}_{ij}(\omega)| \le C \quad \textit{and} \quad \max_{\hat{x}\in\overline{\Sigma},\, \omega\in\overline{S^2_+}} \left|\nabla_\omega \Phi^{\hat{x}}_{ij}(\omega)\right| \le C, \tag{4.5}$$

where ∇_ω *is the gradient-operator on the sphere, and* C *a certain constant.*

Corollary 4.4 *There is a positive constant* C *independent of* $\hat{x}$, *such that for* $y \in \mathbb{R}^{3+}$ *and* $|y| > 2R_T$, *while* T *falls into the disc of radius* R_T, *we have*

$$|W^{l,\hat{x}}_i(y)| \le C\frac{1}{|y|} \quad \textit{and} \quad \left|\frac{\partial W^{l,\hat{x}}_i}{\partial y_p}(y)\right| \le C\frac{1}{|y|^2}, \quad i, p = 1, 2, 3. \tag{4.6}$$

The proofs of Theorem 4.3 and Corollary 4.4 have been performed in [17].

Since, at the microscopic level the medium is homogeneous, in the case of an original macroscopic isotropic media, $G^{\hat{x}}$ in (4.4) has an explicit representation in terms of homogeneous functions of y, and of the Lamé constants $\lambda(\hat{x})$, $\mu(\hat{x})$. For ease of reading, we introduce here the definition of $G^{\hat{x}}$, the so-called Green's tensor for the equilibrium equations of a semi-infinite isotropic and homogeneous medium, see Section I.8 of [29]. Indeed, when (2.5) reads

$$\sigma_{ij,x}(u) = \lambda(x)\delta_{ij}e_{kk,x}(u) + 2\mu(x)e_{ij,x}(u),$$

the components of the stress tensor arising in (3.1), cf. (3.2), are

$$\sigma^{\hat{x}}_{ij,y}(U) = \lambda(\hat{x})\delta_{ij}e_{kk,y}(U) + 2\mu(\hat{x})e_{ij,y}(U),$$

and the components of $G^{\hat{x}}$ in terms of Young's modulus and Poisson's ratio, $E(\hat{x})$ and $\varsigma(\hat{x})$, are given by:

$$G^{\hat{x}}_{11}(\xi) = g(\hat{x})\left(\frac{2(1-\varsigma(\hat{x}))\rho + \xi_3}{\rho(\rho+\xi_3)} + \frac{\xi_1^2(2\rho(\varsigma(\hat{x})\rho + \xi_3) + \xi_3^2)}{\rho^3(\rho+\xi_3)^2}\right),$$

$$G^{\hat{x}}_{22}(\xi) = g(\hat{x})\left(\frac{2(1-\varsigma(\hat{x}))\rho + \xi_3}{\rho(\rho+\xi_3)} + \frac{\xi_2^2(2\rho(\varsigma(\hat{x})\rho + \xi_3) + \xi_3^2)}{\rho^3(\rho+\xi_3)^2}\right),$$

$$G^{\hat{x}}_{12}(\xi) = g(\hat{x})\left(\frac{\xi_1\xi_2(2\rho(\varsigma(\hat{x})\rho + \xi_3) + \xi_3^2)}{\rho^3(\rho+\xi_3)^2}\right),\quad G^{\hat{x}}_{13}(\xi) = g(\hat{x})\left(\frac{\xi_1\xi_3}{\rho^3} - \frac{(1-2\varsigma(\hat{x}))\xi_1}{\rho(\rho+\xi_3)}\right),$$

$$G^{\hat{x}}_{23}(\xi) = g(\hat{x})\left(\frac{\xi_2\xi_3}{\rho^3} - \frac{(1-2\varsigma(\hat{x}))\xi_2}{\rho(\rho+\xi_3)}\right),\quad G^{\hat{x}}_{33}(\xi) = g(\hat{x})\left(\frac{\xi_3^2}{\rho^3} + \frac{2(1-\varsigma(\hat{x}))}{\rho}\right)$$

where

$$g(\hat{x}) = \frac{1+\varsigma(\hat{x})}{2\pi E(\hat{x})}, \quad \rho = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^{3+}, \quad G^{\hat{x}}_{ij} = G^{\hat{x}}_{ji},$$

while we recall

$$\lambda(\hat{x}) = \frac{E(\hat{x})\varsigma(\hat{x})}{(1+\varsigma(\hat{x}))(1-2\varsigma(\hat{x}))} \quad \text{and} \quad \mu(\hat{x}) = \frac{E(\hat{x})}{2(1+\varsigma(\hat{x}))}.$$

See [31] in connection with these formulas and the proof of Theorem 4.3 and Corollary 4.4 in the case of a homogeneous and isotropic original media.

For the anisotropic media, cf. (3.2), we note that $\Phi^{\hat{x}}$ in (4.4) is an analog of the Kelvin tensor in the whole space $\mathbb{R}^3$ constructed from the analogues to Boussinesq and Cerruti tensors realizing actions of concentrated forces on the edge half-plane. Since the half-space is a cone with generator the semi-sphere, the representation (4.3) is supported by general results in [35]. We note that explicit formulas for the Green matrix-function and accompanying tensors are known in the case of isotropy (cf. the above formulas) and, for their existence and main properties in anisotropic media, we refer to Sects. 2 and 5 of [35] where more general boundary value problems for elliptic systems in conical domains are considered. See also [17] for further related references and proofs of the structure (4.4) and (4.5).

The boundary layer functions.

Below, using the solutions of (3.1), we introduce some functions which prove to be essential to define the test functions for obtaining the convergence of the solution of (2.12) towards that of the homogenized problem (3.11), when $r_0 \geq 0$.

First, let us consider $\varphi \in C^\infty[0,1]$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0, 1/8]$ and $\mathrm{Supp}(\varphi) \subset [0, 1/4]$. We define the function

$$\varphi^\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \bigcup\limits_{\mathrm{k}\in\mathcal{J}^\varepsilon} \overline{B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \dfrac{\varepsilon}{8}\right)}, \\ \varphi\Big(\dfrac{|x-\widetilde{x}_\mathrm{k}| - r_\varepsilon}{\varepsilon}\Big) & \text{for } x \in \mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}, \quad \mathrm{k} \in \mathcal{J}^\varepsilon \\ 0 & \text{for } x \in \Omega \setminus \bigcup\limits_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \dfrac{\varepsilon}{4}\right). \end{cases} \tag{4.7}$$

where $\mathcal{J}^\varepsilon = \{\mathrm{k} \in \mathbb{Z}^2 : T^\varepsilon_{\widetilde{x}_\mathrm{k}} \subset \Sigma\}$, $B^+(\widetilde{x}_\mathrm{k}, r) = B(\widetilde{x}_\mathrm{k}, r) \cap \{x_3 > 0\}$ is the half-ball of radius r centered at the point $\widetilde{x}_\mathrm{k}$, and $\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}$ the half-annulus:

$$\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}} = B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right) \setminus \overline{B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{8}\right)}. \tag{4.8}$$

For $l = 1, 2, 3$, and $\mathrm{k} \in \mathcal{J}^\varepsilon$, we construct the functions $\widetilde{W}^{l,\mathrm{k},\varepsilon}(x)$ and $\widetilde{W}^{l,\varepsilon}(x)$ using the solutions $W^{l,\widetilde{x}_\mathrm{k}}$ of the local problems (3.1), as follows:

$$\widetilde{W}^{l,\mathrm{k},\varepsilon}(x) = \mathrm{e}^l - W^{l,\widetilde{x}_\mathrm{k}}\left(\frac{x-\widetilde{x}_\mathrm{k}}{r_\varepsilon}\right)\varphi^\varepsilon(x) \quad \text{for} \quad x \in B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right). \tag{4.9}$$

Now, we extend it by e^l in $\Omega \setminus \bigcup_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right)$ and, finally, we set

$$\widetilde{W}^{l,\varepsilon}(x) = \begin{cases} \widetilde{W}^{l,\mathrm{k},\varepsilon}(x) & \text{for } x \in B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \frac{\varepsilon}{4}\right), \quad \mathrm{k} \in \mathcal{J}^\varepsilon, \\ \mathrm{e}^l & \text{for } x \in \Omega \setminus \bigcup\limits_{\mathrm{k}\in\mathcal{J}^\varepsilon} B^+\left(\widetilde{x}_\mathrm{k}, r_\varepsilon + \dfrac{\varepsilon}{4}\right). \end{cases} \tag{4.10}$$

Proposition 4.5 *For $x \in \mathcal{C}^{\varepsilon,+}_{\widetilde{x}_\mathrm{k}}$, and ε sufficiently small, the inequalities*

$$\left|\frac{\partial \varphi^\varepsilon}{\partial x_j}(x)\right| \leq C\frac{1}{\varepsilon}, \quad j = 1, 2, 3, \tag{4.11}$$

$$\left|W_i^{l,\widetilde{x}_\mathrm{k}}\Big(\frac{x-\widetilde{x}_\mathrm{k}}{r_\varepsilon}\Big)\right| \leq C\frac{r_\varepsilon}{\varepsilon}, \quad \left|\frac{\partial W_i^{l,\widetilde{x}_\mathrm{k}}}{\partial x_j}\Big(\frac{x-\widetilde{x}_\mathrm{k}}{r_\varepsilon}\Big)\right| \leq C\frac{r_\varepsilon}{\varepsilon^2}, \quad i, j, l = 1, 2, 3, \tag{4.12}$$

and

$$\left|\widetilde{W}_i^{l,\mathrm{k},\varepsilon}(x)\right| \le C\frac{r_\varepsilon}{\varepsilon}, \quad \left|\frac{\partial \widetilde{W}_i^{l,\mathrm{k},\varepsilon}}{\partial x_j}(x)\right| \le C\frac{r_\varepsilon}{\varepsilon^2}, \quad i,j,l=1,2,3, \tag{4.13}$$

are satisfied, C being a constant independent of $\widetilde{x}_\mathrm{k}$ and ε.

In addition, for $l, p = 1,2,3$, we have

$$\|\widetilde{W}^{l,\varepsilon}\|_{(H^1(\Omega))^3} \le C \quad \text{and} \quad \widetilde{W}^{l,\varepsilon} \xrightarrow{\varepsilon\to 0} \mathrm{e}^l \quad \text{weakly in } (H^1(\Omega))^3 \tag{4.14}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega_1} \sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})e_{ij,x}(\widetilde{W}^{p,\varepsilon})\Phi dx &= r_0\int_{\Sigma_1}\Phi(\hat{x})\int_{\mathbb{R}^{3+}}\sigma^{\hat{x}}_{ij,y}(W^{l,\hat{x}})e_{ij,y}(W^{p,\hat{x}})dy\,d\hat{x}\\ &= r_0\int_{\Sigma_1}\mathcal{C}_{pl}(\hat{x})\Phi(\hat{x})\,d\hat{x},\end{aligned} \tag{4.15}$$

where $\Phi \in C(\overline{\Omega}_1)$ and Ω_1 is any open domain, $\Omega_1 \subseteq \Omega$, with $\overline{\Sigma}_1 := \partial\Omega_1 \cap \overline{\Sigma} \neq \emptyset$.

See Propositions 3 and 4 of [17] for the proof of Proposition 4.5.

The case where $r_0 = +\infty$.

We state a necessary result to prove the convergence when $r_0 = +\infty$ whose proof has been performed in Sect. 5 of [20].

Theorem 4.6 *Let us assume $r_0 = +\infty$ in (1.1) and properties (2.7), (2.8) and (2.10) for M. For $\phi \in (C^1(\overline{\Omega}))^3$ and $v \in (H^1(\Omega))^3$,*

$$\left|\frac{\varepsilon^2}{r_\varepsilon^2}\int_{\bigcup T^\varepsilon} M_i(x,\phi)v_i\,d\hat{x} - |T|\int_\Sigma M_i(x,\phi)v_i\,d\hat{x}\right| \le C\Big(\varepsilon^{1/2} + \frac{\varepsilon}{r_\varepsilon^{1/2}}\Big)\|v\|_{(H^1(\Omega))^3}, \tag{4.16}$$

where C is a constant independent of ε and v.

5 Matched Asymptotic Expansions

Throughout this section we provide the main steps to obtain, by means of matched asymptotic expansions, the homogenized problems introduced in Sect. 3.

Taking (2.15) into account we consider the asymptotic expansions for the displacement vector u^ε:

Assume the outer expansion

$$u^\varepsilon(x) = u^0(x) + r_\varepsilon u^1(x)\cdots, \quad \text{in } \Omega\cap\{x_3 > d\} \quad \forall d > 0, \tag{5.1}$$

which in fact, is supposed to be valid for x "far" from regions the $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$, namely, at a distance $\rho \gg r_\varepsilon$ from the centers $\widetilde{x}_\mathrm{k}$. Also, we assume the inner expansion in a neighborhood of each reaction region $T^\varepsilon_{\widetilde{x}_\mathrm{k}}$

$$u^\varepsilon(y) = V^0(y) + r_\varepsilon V^1(y)\cdots \quad \text{for } y \in \overline{\mathbb{R}^{3+}}, \tag{5.2}$$

where variable y is given by (2.2) in a neighborhood of each center $\widetilde{x}_{\mathrm{k}}$, $\mathrm{k} \in \mathcal{J}^{\varepsilon}$, while by $r_{\varepsilon} u^1$, $r_{\varepsilon} V^1$ and dots we denote regular terms in the asymptotic series containing higher order functions of ε that we are not using in our asymptotic analysis.

By matching the inner and outer expansions for u^{ε}, at the first order, we have

$$\lim_{|y|\to\infty} V^0(y) = \lim_{x\to\widetilde{x}_{\mathrm{k}}} u^0(x). \tag{5.3}$$

By replacing (5.1) in (2.11) we obtain the equations for u^0: (3.8), (3.9) and another boundary condition on Σ to be determined again by matching outer and inner expansions. To do it, let us determine $V^0(y)$ in the inner expansion (5.2). We first obtain formal asymptotics expansions for $M_i(x, u^{\varepsilon}(x))$, while $i = 1, 2, 3$, in a neighborhood of each region $T^{\varepsilon}_{\widetilde{x}_k}$. Under the assumption of regularity for the function M_i, cf. (7.4), and of (5.2), we can write

$$M_i(x, u^{\varepsilon}(x)) = M_i(r_{\varepsilon} y + \widetilde{x}_{\mathrm{k}}, u^{\varepsilon}(y)) = M_i(\widetilde{x}_{\mathrm{k}}, V^0(y)) + r_{\varepsilon} M_i^1(y) + \cdots, \tag{5.4}$$

for a certain regular function M_i^1.

Taking derivatives in (2.11) with respect to y, cf. (2.2), we replace (5.2) and (5.4) in (2.11), and take into account the continuity of the elastic coefficients $a_{ijkl}(x)$ and (5.3). Then, we obtain that V^0 satisfies

$$\begin{cases} -\dfrac{\partial \sigma^{\mathrm{k}}_{ij,y}(V^0)}{\partial y_j} = 0 & \text{in } \mathbb{R}^{3+}, \\ \sigma^{\mathrm{k}}_{i3,y}(V^0) = 0 & \text{on } \{y_3 = 0\} \setminus \overline{T}, \\ \sigma^{\mathrm{k}}_{i3,y}(V^0) - r_{\varepsilon}\beta(\varepsilon) M_i(\widetilde{x}_{\mathrm{k}}, V^0) = 0 & \text{on } T, \\ V^0(y) \longrightarrow u^0(\widetilde{x}_{\mathrm{k}}) & \text{as } |y| \to \infty,\ y_3 > 0. \end{cases} \quad i = 1, 2, 3. \tag{5.5}$$

Above, and in what follows, for simplicity, we write the superscript k in the strain tensor (3.2) when $\hat{x} \equiv \widetilde{x}_{\mathrm{k}}$, namely,

$$\sigma^{\mathrm{k}}_{ij,y}(V) = a_{ijkl}(\widetilde{x}_{\mathrm{k}}) e_{kl,y}(V). \tag{5.6}$$

Since $r_{\varepsilon}\beta(\varepsilon) \to +\infty$, then we get $M_i(\widetilde{x}_{\mathrm{k}}, V^0) \approx 0$ for $i = 1, 2, 3$, and consequently $M_i(\widetilde{x}_{\mathrm{k}}, V^0) V_i^0 \approx 0$, but properties (2.7) and (2.8) ensure that $(V_i^0)^2 \approx 0$.

Following the idea for linear problems in boundary homogenization (cf. [16, 17, 31]), we assume the approach to V^0 as follows

$$V^0(y) \approx u_l^0(\widetilde{x}_{\mathrm{k}})(\mathrm{e}^l - W^l(y)) \equiv \sum_{l=1}^{3} u_l^0(\widetilde{x}_{\mathrm{k}})(\mathrm{e}^l - W^l(y)), \tag{5.7}$$

where, for $l = 1, 2, 3$, $W^l(y) \equiv W^{l,\widetilde{x}_{\mathrm{k}}}(y)$ is the solution of (cf. (5.5))

$$\begin{cases} -\dfrac{\partial \sigma^{\mathrm{k}}_{ij,y}(W^{l,\widetilde{x}_{\mathrm{k}}})}{\partial y_j} = 0 \ \text{ in } \mathbb{R}^{3+}, \\ \sigma^{\mathrm{k}}_{i3,y}(W^{l,\widetilde{x}_{\mathrm{k}}}) = 0 \quad \text{on } \{y_3 = 0\} \setminus \overline{T}, \\ W^{l,\widetilde{x}_{\mathrm{k}}}(y) = \mathrm{e}^l \quad \text{on } T, \\ W^{l,\widetilde{x}_{\mathrm{k}}}(y) \longrightarrow 0 \quad \text{as } |y| \to \infty,\ y_3 > 0, \end{cases} \quad i = 1, 2, 3. \tag{5.8}$$

Now, we note that writing $\widetilde{x}_{\rm k}=\hat{x}$, (5.8) is (3.1).

In order to obtain the boundary condition on Σ for u^0, we perform an integration by parts over the equilibrium equations in *coin-like domains*, neglecting the stresses across the lateral surface (cf., e.g., Sect. 3 in [14] for the technique). We define one of these domains as follows. Let us consider Σ_1 an open domain contained in Σ such that $\partial\Sigma_1$ does not touch any region $T^{\varepsilon}_{\widetilde{x}_{\rm k}}$. Let $\delta(\varepsilon)$ be positive, $r_\varepsilon \ll \delta(\varepsilon)\ll 1$. We consider the coin-like domain

$$\Omega^{\delta(\varepsilon)}_{\Sigma_1}=\Omega\cap(\Sigma_1\times(0,\delta(\varepsilon))). \tag{5.9}$$

Let $\Gamma_{\delta(\varepsilon)}$ denote the lateral boundary of $\Omega^{\delta(\varepsilon)}_{\Sigma_1}$ in such a way that

$$\partial\Omega^{\delta(\varepsilon)}_{\Sigma_1}=\overline{\Gamma_{\delta(\varepsilon)}}\cup\overline{\Sigma_1^{\delta(\varepsilon)}}\cup\overline{\Sigma_1} \tag{5.10}$$

where $\Sigma_1^{\delta(\varepsilon)}$ denotes the set $\{x:(x_1,x_2,0)\in\Sigma_1,\ x_3=\delta(\varepsilon)\}$. On $\Sigma_1^{\delta(\varepsilon)}$, we are "far" from the reaction regions $T^{\varepsilon}_{\widetilde{x}_{\rm k}}$ and (5.1) hold. "Near" each region T^ε, we need to use the inner expansion (5.2), which in terms of the macroscopic variable reads $u^\varepsilon(x)=V^0\big((x-\widetilde{x}_{\rm k})r_\varepsilon^{-1}\big)+\cdots$. In particular, on each reaction region $T^{\varepsilon}_{\widetilde{x}_{\rm k}}$ we have (cf. (2.2), (5.2) and (5.6))

$$\sigma_{i3,x}(u^\varepsilon)=\sigma_{i3,x}(V^0(y))\approx a_{i3kh}(\widetilde{x}_{\rm k})\frac{1}{r_\varepsilon}e_{kh,y}(V^0(y))+\cdots=\frac{1}{r_\varepsilon}\sigma^{\rm k}_{i3,y}(V^0(y))+\cdots \tag{5.11}$$

Now, considering (2.11), we apply the Green formula over $\Omega^{\delta(\varepsilon)}_{\Sigma_1}$ (cf. (5.9) and (5.10)) to obtain

$$\int\limits_{\Sigma_1\cap\bigcup T^{\varepsilon}_{\widetilde{x}_{\rm k}}}\sigma_{i3,x}(u^\varepsilon)d\hat{x}=\int\limits_{\Omega^{\delta(\varepsilon)}_{\Sigma_1}} f_i dx+\int\limits_{\Gamma_{\delta(\varepsilon)}}\sigma_{ij,x}(u^\varepsilon)n_j\,d\Gamma_\delta+\int\limits_{\Sigma_1^{\delta(\varepsilon)}}\sigma_{i3,x}(u^\varepsilon)d\hat{x}\,, \tag{5.12}$$

where n stands for the unit outer normal to $\Omega^{\delta(\varepsilon)}_{\Sigma_1}$ along $\Gamma_{\delta(\varepsilon)}$. Then, using (5.1), (5.2), (5.11), (5.7), performing the change of variable (2.2), considering that $\sigma^{\rm k}_{i3,y}(W^l)\big|_{\{y_3=0\}}$ is a distribution of compact support on $\overline{T}$, and taking limits in (5.12), as $\varepsilon\to 0$, lead us to

$$\begin{aligned}\int\limits_{\Sigma_1}\sigma_{i3,x}(u^0)d\hat{x}&:=\lim_{\varepsilon\to0} r_\varepsilon\sum_{\widetilde{x}_k\in\Sigma_1}u^0_l(\widetilde{x}_{\rm k})\big\langle\sigma^{\rm k}_{i3,y}({\rm e}^l-W^l)\,,\,1\big\rangle_{H^{-1/2}(T)\times H^{1/2}(T)}\\&=\lim_{\varepsilon\to0}\sum_{\widetilde{x}_k\in\Sigma_1}r_\varepsilon\mathcal{C}_{il}(\widetilde{x}_{\rm k})u^0_l(\widetilde{x}_{\rm k}),\end{aligned} \tag{5.13}$$

where, $W^l\equiv W^{l,\widetilde{x}_{\rm k}}$ is the solution of (5.8) and $\mathcal{C}_{il}$ are the components of a matrix $\mathcal{C}\big(\widetilde{x}_{\rm k}\big)$ defined by (3.5). Under the hypothesis of smoothness of $u^0(\hat{x})$ and that of $\mathcal{C}(\hat{x})$ (cf. Proposition 4.2), and under the assumption that $r_0>0$ in (1.1) we obtain that (5.13) reads

$$\int\limits_{\Sigma_1}\sigma_{i3,x}(u^0)d\hat{x}=r_0\int\limits_{\Sigma_1}\mathcal{C}_{il}\big(\hat{x}\big)u^0_l(\hat{x})\,d\hat{x}. \tag{5.14}$$

Obviously, when $r_0=0$, the limit (5.13) gives

$$\int\limits_{\Sigma_1}\sigma_{i3,x}(u^0)d\hat{x}=0. \tag{5.15}$$

Formulas (5.13) and (5.14) furnish the asymptotic behavior of the solution u^ε when $r_0 \geq 0$. The reasoning above must be slightly modified in the case where $r_0 = +\infty$ as follows: we multiply both sides of the equality in (5.12) by $\varepsilon^2 r_\varepsilon^{-1}$. Since $\varepsilon^2 r_\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$, the reasoning in (5.13)-(5.14) provides

$$0 = \lim_{\varepsilon\to 0} \sum_{\widetilde{x}_k \in \Sigma_1} \varepsilon^2 \int_{T^\varepsilon_{\widetilde{x}_k}} \sigma^{\mathrm{k}}_{i3,y}(V^0(y))d\hat{y} = \int_{\Sigma_1} \mathcal{C}_{il}(\hat{x})u^0_l(\hat{x})\,d\hat{x}. \tag{5.16}$$

Gathering all the possible values for r_0, cf. (1.1), on account of the somewhat arbitrary choice of $\Sigma_1 \subset \Sigma$, from (5.14)-(5.16) we obtain the following boundary conditions on Σ to be added to (3.8) and (3.9) in order to determine the first terms of the asymptotic expansions (5.1), namely u^0.

If $r_0 > 0$, then we have (5.14). $\sigma_{i3,x}(u^0) - r_0 \mathcal{C}^e_{il}(x, u^0)u^0_l = 0$ on Σ. This gives that u^0 is the solution of problem (3.8), (3.9), (3.10).

If or $r_0 = 0$, then we have (5.15). This gives that u^0 is the solution of problem (3.8), (3.9), (3.12).

If $r_0 = +\infty$, then, $u^0 = 0$ on Σ. This gives that u^0 is the solution of problem (3.8), (3.9), (3.14). Here, we have used (5.16) and hence that $\mathcal{C}_{il}u^0_l = 0$ on Σ, for $i = 1, 2, 3$. Then, the positive definite property for the matrix $\mathcal{C}_{il}$ (cf. Proposition 4.2) gives $(u^0_l)^2 = 0$ as above-announced.

6 The Convergence when $r_0 \geq 0$

In this section, we prove Theorem 3.1. In order to do this, we first show two results of convergence which are essential for the proofs. The fist one deals with precise estimates for the solutions of (2.11) in spaces of traces which allows to pass to the limit in sums of products of duality (for *surface terms*, cf. (6.18)). The second result deals with the construction of a sequence $\widetilde{u}^\varepsilon$ which somehow behaves as u^ε allowing us to pass to the limit but controlling the discrepancies of $\widetilde{u}^\varepsilon - u^\varepsilon$. As a matter of fact, Theorem 6.1 extends the result of Proposition 5.2 in [18] (for scalar problems) to solutions of vector problems using less restrictions. See Remark 1 in connection with Theorem 3.2.

Theorem 6.1 *Let $r_0 \geq 0$ and M satisfy (2.7)-(2.9). Then, the solution u^ε of (2.12) verifies*

$$\sum_{i=1}^3 \left(\sum_{\widetilde{x}_\mathrm{k}} \left\| u^\varepsilon_i \right\|^2_{H^{1/2}(T^\varepsilon_{\widetilde{x}_\mathrm{k}})} \right)^{1/2} \leq C\Big(r^2_\varepsilon + \frac{1}{\beta(\varepsilon)^2}\Big)^{1/2} \leq C r_\varepsilon, \tag{6.1}$$

with a certain constant C independent of ε.

Proof From the definition of the $H^{1/2}$-norm, for each component u^ε_i, while $i = 1, 2, 3$, we write the left hand side of (6.1) as follows

$$\mathfrak{L}_\varepsilon := \sum_{\widetilde{x}_\mathrm{k}} \left\| u^\varepsilon_i \right\|^2_{L^2(T^\varepsilon_{\widetilde{x}_\mathrm{k}})} + \sum_{\widetilde{x}_\mathrm{k}} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \int_{T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \frac{|u^\varepsilon_i(\hat{x}) - u^\varepsilon_i(\hat{x}')|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x} d\hat{x}'. \tag{6.2}$$

For the first summation we have, cf. (2.15)

$$\sum_{\widetilde{x}_\mathrm{k}} \left\| u^\varepsilon_i \right\|^2_{L^2(T^\varepsilon_{\widetilde{x}_\mathrm{k}})} = \frac{1}{\beta(\varepsilon)} \int_{\bigcup T^\varepsilon_{\widetilde{x}_\mathrm{k}}} \beta(\varepsilon)(u^\varepsilon_i)^2 d\hat{x} \leq C \frac{1}{\beta(\varepsilon)}. \tag{6.3}$$

Let us analyze the second summation in (6.2). Using (2.8) we deduce the inequality

$$\widetilde{K}(u_i - v_i)^2 \leq (M_i(x,u) - M_i(x,v))^2 \quad \forall x \in \overline{\Sigma},\ u, v \in \mathbb{R}^3, \tag{6.4}$$

for some $\widetilde{K} > 0$. Indeed, let us consider a constant $\hat{K} > (2K)^{-1}$, with K the constant arising in (2.8), we write

$$\begin{aligned}\hat{K}K(u_i - v_i)^2 &\leq \hat{K}(M_i(x,u) - M_i(x,v))(u_i - v_i)\\ &\leq \frac{\hat{K}^2(M_i(x,u) - M_i(x,v))^2}{2} + \frac{(u_i - v_i)^2}{2}\end{aligned}$$

and therefore (6.4) holds. This along with (2.9), leads us to the following chain of inequalities

$$\sum_{\widetilde{x}_k} \int_{T^\varepsilon_{\widetilde{x}_k}} \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{|u_i^\varepsilon(\hat{x}) - u_i^\varepsilon(\hat{x}')|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x}d\hat{x}'$$

$$\leq \widetilde{C} \sum_{\widetilde{x}_k} \int_{T^\varepsilon_{\widetilde{x}_k}} \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{|M_i(\hat{x}, u^\varepsilon(\hat{x})) - M_i(\hat{x}, u^\varepsilon(\hat{x}'))|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x}d\hat{x}'$$

$$\leq 2\widetilde{C} \int_{\bigcup T^\varepsilon_{\widetilde{x}_k}} \int_{\bigcup T^\varepsilon_{\widetilde{x}_k}} \frac{|M_i(\hat{x}, u^\varepsilon(\hat{x})) - M_i(\hat{x}', u^\varepsilon(\hat{x}'))|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x}d\hat{x}' + \widehat{C} \sum_{\widetilde{x}_k} \int_{T^\varepsilon_{\widetilde{x}_k}} \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{1}{|\hat{x} - \hat{x}'|} d\hat{x}d\hat{x}'$$

$$\leq 2\widetilde{C} \left\| M_i(\cdot, u^\varepsilon(\cdot)) \right\|^2_{H^{1/2}(\bigcup T^\varepsilon_{\widetilde{x}_k})} + \widehat{C} \sum_{\widetilde{x}_k} \int_{T^\varepsilon_{\widetilde{x}_k}} \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{1}{|\hat{x} - \hat{x}'|} d\hat{x}d\hat{x}' := 2\widetilde{C}\mathfrak{L}^{\mathrm{I}}_\varepsilon + \widehat{C}\mathfrak{L}^{\mathrm{II}}_\varepsilon,$$

where $\widetilde{C}$ and $\widehat{C}$ denote two constants independent of ε. Let us obtain suitable bounds for $\mathfrak{L}^{\mathrm{I}}_\varepsilon$ and $\mathfrak{L}^{\mathrm{II}}_\varepsilon$.

For the first term, we have

$$\left(\mathfrak{L}^{\mathrm{I}}_\varepsilon\right)^{1/2} \leq \frac{1}{\beta(\varepsilon)} \left\| \beta(\varepsilon)\chi_{\bigcup T^\varepsilon} M_i(\cdot, u^\varepsilon(\cdot)) \right\|_{H^{1/2}(\Sigma)} = \frac{1}{\beta(\varepsilon)} \left\| \sigma_{i3}(u^\varepsilon) \right\|_{H^{-1/2}(\Sigma)} \leq C\frac{1}{\beta(\varepsilon)}, \tag{6.5}$$

where $\chi_{\bigcup T^\varepsilon}$ stands for the characteristic function of the set $\bigcup_{k\in\mathcal{J}^\varepsilon} T^\varepsilon_{\widetilde{x}_k}$, and we have used the equation on Σ in (2.12), cf. (2.11), the trace embedding theorem and (2.15).

As for the other term, on each $T^\varepsilon_{\widetilde{x}_k}$, we consider the function defined as

$$U^{\varepsilon,k}(\hat{x}') = \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{1}{|\hat{x} - \hat{x}'|} d\hat{x},$$

and apply the result in Section I.6.1 of [49] (cf. also Lemma 5 in Section I.2 of [28]) for integrals of potential type to obtain

$$|U^{\varepsilon,k}(\hat{x}')| \leq C|T^\varepsilon_{\widetilde{x}_k}|^{1/3}(r_\varepsilon)^{1/3} \quad \forall \hat{x}' \in T^\varepsilon_{\widetilde{x}_k}. \tag{6.6}$$

Taking into account the volume of each $T^\varepsilon_{\widetilde{x}_k}$, (2.1) and $r_0 > 0$ (1.1), we get

$$\mathfrak{L}^{\mathrm{II}}_\varepsilon := \sum_{\widetilde{x}_k} \int_{T^\varepsilon_{\widetilde{x}_k}} \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{1}{|\hat{x} - \hat{x}'|} d\hat{x}d\hat{x}' \leq C(r_\varepsilon)^3\varepsilon^{-2} \leq C(r_\varepsilon)^2. \tag{6.7}$$

It should be noted that, for the proofs above, we have used the fact $\chi_{\bigcup T^\varepsilon} M_i(\cdot, u^\varepsilon(\cdot)) \in H^{1/2}(\Sigma)$. To see this, we can follow the same reasoning above. Indeed, without any restriction we can proceed with $M_i(\cdot, u^\varepsilon(\cdot))$ and, from (2.7) and (2.9), we have

$$\int_\Sigma |M_i(\hat{x}, u^\varepsilon(\hat{x}))|^2\, d\hat{x} + \int_\Sigma \int_\Sigma \frac{|M_i(\hat{x}, u^\varepsilon(\hat{x})) - M_i(\hat{x}', u^\varepsilon(\hat{x}'))|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x} d\hat{x}'$$
$$\leq C \left(\int_\Sigma |u^\varepsilon(\hat{x})|^2\, d\hat{x} + \int_\Sigma \int_\Sigma \frac{|u^\varepsilon(\hat{x}) - u^\varepsilon(\hat{x}')|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x} d\hat{x}' + \int_\Sigma \int_\Sigma \frac{1}{|\hat{x} - \hat{x}'|} d\hat{x} d\hat{x}' \right),$$

for a certain constant C. Since $u^\varepsilon \in (H^{1/2}(\Sigma))^3$ and the last integral is finite (cf. e.g. Lemma 5 in Section I.2 of [28]), all the integrals above are finite and this implies $M_i(\cdot, u^\varepsilon(\cdot)) \in H^{1/2}(\Sigma)$.

Finally, using (6.3), (6.5) and (6.7) in (6.2), we obtain the first estimate in (6.1), while the second estimate holds due to (1.2). Consequently, the theorem is proved. □

Theorem 6.2 *For any $u^0 \in \mathbf{V}$ which is the weak limit in $(H^1(\Omega))^3$ of a subsequence u^ε, cf.* (2.16), *we construct a sequence $\widetilde{u}^\varepsilon \in \mathbf{V}$, such that*

$$\widetilde{u}^\varepsilon = 0 \text{ on } \bigcup T^\varepsilon, \qquad \widetilde{u}^\varepsilon \xrightarrow{\varepsilon \to 0} u^0 \text{ weakly in } (H^1(\Omega))^3, \tag{6.8}$$

and, for any $\phi \in (C^1(\overline{\Omega}))^3$ with $\phi = 0$ on Γ_Ω the following convergence occurs:

$$\lim_{\varepsilon \to 0} \int_\Omega \sigma_{ij,x}(\widetilde{u}^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx = r_0 \int_\Sigma \mathcal{C}_{ij}(\hat{x}) u_i^0 \phi_j d\hat{x}. \tag{6.9}$$

In addition, under the assumptions (2.7)-(2.9), *we have*

$$\lim_{\varepsilon \to 0} \int_\Omega \sigma_{ij,x}(\widetilde{u}^\varepsilon - u^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx = 0. \tag{6.10}$$

Proof The existence of a sequence $\widetilde{u}^\varepsilon$ satisfying (6.8) and (6.9) has been shown in Theorem 7 of [17]: it follows for any converging sequence u^ε not necessarily solution of (2.11) (cf. also [4, 45].) For completeness, we sketch here the main steps of the construction of such a function and refer to Sects. 6 and 7 of [17] for further details. Then, using this function $\widetilde{u}^\varepsilon$ we provide a proof of (6.10) which is specific for the solution of (2.11) and for the nonlinear reaction term M, and requires Theorem 6.1.

The construction of $\widetilde{u}^\varepsilon$ satisfying (6.8)-(6.9).

Since $\sigma_{ij,x}(\widetilde{W}^{l,\varepsilon})$ takes values different from zero only in a neighborhood of Σ, and $u^\varepsilon = 0$ on Γ_Ω, there is no loss of generality for the proof to assume that the domain Ω is a polyhedron and the boundary Γ_Ω can be written as a finite union of plane faces. For each fixed $h > 0$, we consider a regular triangulation $\{\triangle_{h_q}\}_{q=1}^{\mathcal{M}_h}$ of the domain Ω composed of tetrahedrons of diameter h (see, e.g., [7])

$$\overline{\Omega} = \bigcup_{q=1}^{\mathcal{M}_h} \overline{\triangle}_{h_q}.$$

Let u^h denote the orthogonal projection of the element $u \in H^1(\Omega)$, with $u = 0$ on Γ_Ω, on the subspace $\mathcal{Y}^h$ of the continuous functions over $\overline{\Omega}$ which are affine functions on each tetrahedron $\triangle_{h_q}$ and take the value 0 on Γ_Ω. We set

$$\widetilde{u}^{\varepsilon h} = u_l^{\varepsilon h} \widetilde{W}^{l,\varepsilon}. \tag{6.11}$$

On account of the convergence of the coefficients of the polynomials on each $\triangle_{h_q}$ (that is, the restriction of $u^{\varepsilon h}$ to each $\triangle_{h_q}$), $q = 1, 2, \dots, \mathcal{M}_h$, and the convergence (4.14) for $W^{l,\varepsilon}$, as $\varepsilon \to 0$, we get

$$\widetilde{u}^{\varepsilon h} \xrightarrow{\varepsilon\to 0} u^{0h} \text{ weakly in } (H^1(\Omega))^3. \tag{6.12}$$

Also, using interpolation operators for smooth functions (see, e.g., Ch. 4 in [7]) and a density argument, the following convergence holds:

$$u^{0h} \to u^0 \text{ in } H^1(\Omega), \qquad \text{as } h \to 0. \tag{6.13}$$

In addition, in the integral on the left hand side in (6.9) we can replace $\widetilde{u}^\varepsilon$ by $\widetilde{u}^{\varepsilon h}$, namely,

$$\int_\Omega \sigma_{ij,x}(\widetilde{u}^{\varepsilon h}) e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l \, dx \tag{6.14}$$

and take limits first for $\varepsilon \to 0$, to get

$$r_0 \int_\Sigma \mathcal{C}_{ij}(\hat{x}) u_i^{0h} \phi_j d\hat{x},$$

and then for $h \to 0$, to get the integral on the right hand side of (6.9).

Finally, in the above framework (6.11)-(6.14), we apply a result on convergence for double indexed subsequences, see, e.g., Corollary 1.18 in Section I.2 of [3], to extract a sequence $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that $\widetilde{u}^\varepsilon := u^{\varepsilon h(\varepsilon)}$ satisfies the desired properties (6.8) and (6.9).

The proof of (6.10).

Let us denote by d^ε the difference $d^\varepsilon := u^\varepsilon - \widetilde{u}^\varepsilon$ which satisfies

$$d^\varepsilon \xrightarrow{\varepsilon\to 0} 0 \text{ weakly in } (H^1(\Omega))^3. \tag{6.15}$$

Noting that

$$e_{ij,x}(\phi_l d^\varepsilon) = e_{ij,x}(d^\varepsilon)\phi_l + \frac{1}{2}\Big(\frac{\partial \phi_l}{\partial x_j} d_i^\varepsilon + \frac{\partial \phi_l}{\partial x_i} d_j^\varepsilon\Big), \tag{6.16}$$

and using (4.14) and (6.15), for the integral in (6.10) we can write

$$\mathbf{I}_\varepsilon := \int_\Omega \sigma_{ij,x}(d^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l \, dx = \int_\Omega \sigma_{ij,x}(d^\varepsilon \phi_l) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) dx + o_\varepsilon(1),$$

where, here and in what follows $o_\varepsilon(1)$ denotes a certain function satisfying $o_\varepsilon(1) \to 0$, as $\varepsilon \to 0$.

Hence, using definitions (4.9) and (4.10), we show that

$$\mathbf{I}_\varepsilon = \sum_{\widetilde{x}_k} \int_{B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{4})} \sigma_{ij,x}(d^\varepsilon \phi_l) e_{ij,x}(\tau_x W^{l,\widetilde{x}_k} \varphi^\varepsilon) dx + o_\varepsilon(1)$$

$$= \sum_{\widetilde{x}_k} \int_{B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{4})} \sigma_{ij,x}(d^\varepsilon \phi_l \varphi^\varepsilon) e_{ij,x}(\tau_x W^{l,\widetilde{x}_k}) dx + o_\varepsilon(1),$$

where τ_x denotes the change $y \mapsto x$, cf. (2.2), and in the last term $o_\varepsilon(1)$ we have gathered the terms of the integrals in the half-annuli $\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}$, cf. (4.7)-(4.8), which are sums of

$$\sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} \sigma_{ij,x}(d^\varepsilon \phi_l) \tau_x W_p^{l,\widetilde{x}_k} \frac{\partial \varphi^\varepsilon}{\partial x_q} dx \text{ and } \sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} a_{ijkl} d_p^\varepsilon \phi_l e_{ij}(\tau_x W_p^{l,\widetilde{x}_k}) \frac{\partial \varphi^\varepsilon}{\partial x_q} dx,$$

for $i, j, p, q, l, k = 1, 2, 3$. Let us show that indeed, all these terms vanish in the limit as $\varepsilon \to 0$.

Using estimates (4.11), (4.12), (4.13) and (2.1), we prove that the first sums above are bounded by

$$C\varepsilon^{-1} \|d^\varepsilon\|_{\mathbf{V}} \sum_{p=1}^{3} \Big(\sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} |\tau_x W_p^{l,\widetilde{x}_k}|^2 dx \Big)^{1/2} \le C\varepsilon^{1/2}.$$

For the second sums, we use the same estimates above-written and get the bound

$$C\varepsilon^{-1} \sum_{p=1}^{3} \Big(\int_{\bigcup \mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} (d_p^\varepsilon)^2 dx \Big)^{\frac{1}{2}} \cdot \sum_{i,j=1}^{3} \Big(\sum_{\widetilde{x}_k} \int_{\mathcal{C}^{\varepsilon,+}_{\widetilde{x}_k}} \big| e_{ij,x}(\tau_x W_p^{l,\widetilde{x}_k}) \big|^2 dx \Big)^{\frac{1}{2}}$$
$$\le C \sum_{p=1}^{3} \Big(\varepsilon^{-1} \int_{0<x_3<\varepsilon} (d_p^\varepsilon)^2 dx \Big)^{\frac{1}{2}}$$

Now, on account of the convergence (6.15) and the following convergence result

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{0<x_3<\varepsilon} (d_p^\varepsilon)^2 dx = 0, \quad p = 1, 2, 3, \tag{6.17}$$

(see, e.g., Lemma 2.4 in [33]) we also get that the second sums above mentioned converge towards zero.

All together, along with (2.5), (3.2) and (2.2), give

$$\mathbf{I}_\varepsilon = \sum_{\widetilde{x}_k} \int_{B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{4})} \sigma_{ij,x}(\tau_x W^{l,\widetilde{x}_k}) e_{ij,x}(d^\varepsilon \phi_l \varphi^\varepsilon) dx + o_\varepsilon(1)$$
$$= r_\varepsilon \sum_{\widetilde{x}_k} \int_{B^+(0, 1 + \frac{\varepsilon}{4 r_\varepsilon})} \sigma^{\widetilde{x}_k}_{ij,y}(W^{l,\widetilde{x}_k}) e_{ij,y}(d^\varepsilon \phi_l \varphi^\varepsilon) dy + o_\varepsilon(1),$$

where, to obtain the last $o_\varepsilon(1)$, we have used the globally Lipschitz property of the elastic coefficients, estimates (4.1) and (4.11) and convergences (6.15) and (6.17).

Hence, since $\varphi^\varepsilon = 0$ on $\partial B^+(\widetilde{x}_k, r_\varepsilon + \frac{\varepsilon}{4}) \cap \{x_3 > 0\}$, $\varphi^\varepsilon = 1$ on $T^\varepsilon_{\widetilde{x}_k}$, and $\widetilde{u}^\varepsilon = 0$ on $T^\varepsilon_{\widetilde{x}_k}$ and $W^{l,\widetilde{x}_k}$ satisfies (5.8), by applying the Green formula we get

$$|\mathbf{I}_\varepsilon| = r_\varepsilon \Big| \sum_{\widetilde{x}_k} \langle \sigma^{\widetilde{x}_k}_{i3,y}(W^{l,\widetilde{x}_k}),\ \tau_y(d^\varepsilon_i \phi_l) \rangle_{H^{-1/2}(T)\times H^{1/2}(T)} \Big| + o_\varepsilon(1)$$
$$\leq r_\varepsilon \sum_{\widetilde{x}_k} \big\| \sigma^{\widetilde{x}_k}_{i3,y}(W^{l,\widetilde{x}_k}) \big\|_{H^{-1/2}(T)} \big\| \tau_y(u^\varepsilon_i \phi_l) \big\|_{H^{1/2}(T)} + o_\varepsilon(1). \tag{6.18}$$

Using (4.2), the smoothness of ϕ, (6.6) for $T^\varepsilon_{\widetilde{x}_k} \equiv T$ and the definition of the $H^{1/2}$-morm,

$$\big\| \tau_y u^\varepsilon_i \big\|^2_{H^{1/2}(T)} = \int_T |\tau_y u^\varepsilon_i|^2 d\hat{y} + \int_T \int_T \frac{|u^\varepsilon_i(\hat{y}) - u^\varepsilon_i(\hat{y}')|^2}{|\hat{y} - \hat{y}'|^3} d\hat{y} d\hat{y}', \tag{6.19}$$

we derive the inequality

$$|\mathbf{I}_\varepsilon| \leq C r_\varepsilon \varepsilon^{-1} \sum_{i=1}^3 \Big(\sum_{\widetilde{x}_k} \big\| \tau_y u^\varepsilon_i \big\|^2_{H^{1/2}(T)} \Big)^{1/2} + o_\varepsilon(1).$$

Then, performing the change $y \mapsto x$ in the integrals in (6.19), cf. (2.2), we write

$$\big\| \tau_y u^\varepsilon_i \big\|^2_{H^{1/2}(T)} \leq \frac{1}{r_\varepsilon^2} \big\| u^\varepsilon \big\|^2_{L^2(T^\varepsilon_{\widetilde{x}_k})} + \frac{1}{r_\varepsilon} \int_{T^\varepsilon_{\widetilde{x}_k}} \int_{T^\varepsilon_{\widetilde{x}_k}} \frac{|u^\varepsilon_i(\hat{x}) - u^\varepsilon_i(\hat{x}')|^2}{|\hat{x} - \hat{x}'|^3} d\hat{x} d\hat{x}'$$
$$\leq \frac{1}{r_\varepsilon^2} \big\| u^\varepsilon \big\|^2_{H^{1/2}(T^\varepsilon_{\widetilde{x}_k})}.$$

Now, using this and the result in Theorem 6.1, we obtain

$$|\mathbf{I}_\varepsilon| \leq C \varepsilon^{-1} \sum_{i=1}^3 \Big(\sum_{\widetilde{x}_k} \big\| u^\varepsilon_i \big\|^2_{H^{1/2}(T^\varepsilon_{\widetilde{x}_k})} \Big)^{1/2} + o_\varepsilon(1) \leq C \frac{1}{\varepsilon} r_\varepsilon + o_\varepsilon(1).$$

Since $r_0 > 0$ in (1.1), we have that $\lim_{\varepsilon\to 0} \mathbf{I}_\varepsilon = 0$, and (6.10) holds. Thus, the theorem is proved. □

6.1 Proof of Theorem 3.1

Let us show that the limit u^0 in (2.16) is the solution of the homogenized problem (3.11)

In order to do it, for $\phi \in (C^1(\overline{\Omega}))^3$, $\phi = 0$ on Γ_Ω, and $\widetilde{W}^{l,\varepsilon}$ defined by (4.10), we consider the vector function $\phi_l(x)\widetilde{W}^{l,\varepsilon}(x)$. On account of (4.14), we have that $\phi_l \widetilde{W}^{l,\varepsilon} \to \phi$ weakly in $(H^1(\Omega))^3$, as $\varepsilon \to 0$.

Then, we take the *test function* $v = \phi_l \widetilde{W}^{l,\varepsilon}$ in (2.12). Since it vanishes on $\bigcup T^\varepsilon$, we have

$$\int_\Omega \sigma_{ij,x}(u^\varepsilon) e_{ij,x}(\phi_l \widetilde{W}^{l,\varepsilon})\, dx = \int_\Omega f_i \phi_l \widetilde{W}^{l,\varepsilon}_i\, dx. \tag{6.20}$$

Considering (6.16), where we replace d^ε by $\widetilde{W}^{l,\varepsilon}$, (4.14) and (2.16), the limit passage as $\varepsilon \to 0$ in (6.20) gives

$$\int_\Omega \sigma_{ij,x}(u^0) e_{ij,x}(\phi)\, dx - \int_\Omega f_i \phi_i\, dx = -\lim_{\varepsilon\to 0} \int_\Omega \sigma_{ij,x}(u^\varepsilon) e_{ij,x}(\widetilde{W}^{l,\varepsilon}) \phi_l\, dx. \tag{6.21}$$

Now, replacing in this last integral u^ε by the functions $\widetilde{u}^\varepsilon$ constructed in Theorem 6.2, and using (6.9) and (6.10) we obtain the chain of equalities

$$\int_\Omega \sigma_{ij,x}(u^0)e_{ij,x}(\phi)\,dx - \int_\Omega f_i\phi_i\,dx = -\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij,x}(\widetilde{u}^\varepsilon)e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l\,dx$$
$$= r_0\int_\Sigma \mathcal{C}_{ij}(\hat{x})u_i^0\phi_j\,d\hat{x}. \tag{6.22}$$

Indeed, to show (6.22), we write the left hand side of (6.21) as

$$\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij,x}(u^\varepsilon)e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l\,dx = \lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij,x}(u^\varepsilon-\widetilde{u}^\varepsilon)e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l\,dx$$
$$+\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij,x}(\widetilde{u}^\varepsilon)e_{ij,x}(\widetilde{W}^{l,\varepsilon})\phi_l\,dx$$

and apply (6.10) to get the first equality in (6.22), and then (6.9) gives the last equality in (6.22).

Consequently, we deduce that u^0 satisfies

$$\int_\Omega \sigma_{ij,x}(u^0)e_{ij,x}(\phi)\,dx + r_0\int_\Sigma \mathcal{C}_{ij}(\hat{x})u_i^0\phi_j\,d\hat{x} = \int_\Omega f_i\phi_i\,dx \quad \forall\phi\in(C^1(\overline{\Omega}))^3,\ \phi\big|_{\Gamma_\Omega}=0.$$

By a density argument, the above equality holds for all $\phi\in\mathbf{V}$, and we conclude that u^0 is the unique solution of (3.11). Therefore, we have proved Theorem 3.1. □

Remark 1 Note that the proof of Theorem 3.1 holds when $r_0=0$; that is, the term containing the integral on Σ in (6.21) vanishes. Thus, for $r_0=0$, u^0 is the unique solution of (3.13). Also we observe that, since the convergence (4.14) holds in the strong topology of $(H^1(\Omega))^3$ (cf. Proposition 3 in [17] when $r_\varepsilon\varepsilon^{-2}\to 0$), taking limits directly in (6.20), we can avoid using Theorems 6.1 and 6.2 for $r_0=0$.

7 The Improved Convergence: $r_0=0$ or $r_0=+\infty$

In this section we show the strong convergence in $(H^1(\Omega))^3$ of the solutions of (2.11) when the sizes of the reaction regions are very small ($r_0=0$) or very large ($r_0=+\infty$): that is, Theorems 3.2 and 3.3.

Proof of Theorem 3.2 Repeating the proof of Theorem 3.1 for $r_0=0$ (see Remark 1) implies that the weak limit of u^ε, cf. (2.16), is given by the solution u^0 of (3.13). Let us show that the convergence takes place in $(H^1(\Omega))^3$. To do it, we use the lower semi-continuity of the norm for the weak topology (cf., e.g., [4] and references therein).

Indeed, we can write the chain of inequalities and equalities, respectively,

$$\begin{aligned}\int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)dx &\le \liminf_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)dx\\ &\le \liminf_{\varepsilon\to 0}\left(\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)\,dx+\beta(\varepsilon)\int_{\bigcup T^\varepsilon} M_i(\hat{x},u^\varepsilon)u_i^\varepsilon\,d\hat{x}\right)\\ &= \lim_{\varepsilon\to 0}\int_\Omega f_i u_i^\varepsilon\,dx=\int_\Omega f_i u_i^0\,dx=\int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)\,dx,\end{aligned} \tag{7.1}$$

where we have used the fact that

$$\int_\Omega \sigma_{ij}(u)e_{ij}(u)dx,$$

defines a norm in $\mathbf{V}$ equivalent to that generated by (2.13) (cf. (2.5) and (2.4)), (2.7), (2.8), (2.12) for $v=u^\varepsilon$, and (3.13). Consequently, the inequalities above convert into equalities resulting in

$$\lim_{\varepsilon\to 0}\int_\Omega \sigma_{ij}(u^\varepsilon)e_{ij}(u^\varepsilon)\,dx=\int_\Omega \sigma_{ij}(u^0)e_{ij}(u^0)\,dx.$$

Therefore, the convergence of u^ε in $(H^1(\Omega))^3$ holds true, and the result in Theorem 3.2 is proved. □

Proof of Theorem 3.3 First, let us show that the limit u^0 arising in (2.16) vanishes on Σ.

In order to do it, we use (4.16) for $v=u^\varepsilon$. This allows us to write

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{r_\varepsilon^2}\int_{\bigcup T^\varepsilon} M_i(\hat{x},\phi)u_i^\varepsilon\,d\hat{x}=|T|\int_\Sigma M_i(\hat{x},\phi)u_i^0\,d\hat{x}\quad \forall\phi\in(C^1(\overline{\Omega}))^3,\ \phi=0 \text{ on } \Gamma_\Omega. \tag{7.2}$$

Using Cauchy-Schwarz inequality, the smoothness of ϕ and (2.1), for the integral on the left hand side of (7.2) we also have

$$\frac{\varepsilon^2}{r_\varepsilon^2}\left|\int_{\bigcup T^\varepsilon} M_i(\hat{x},\phi)u_i^\varepsilon\,d\hat{x}\right|\le\frac{\varepsilon^2}{r_\varepsilon^2}\,\frac{r_\varepsilon}{\varepsilon}\,(\beta(\varepsilon))^{-1/2}\left(\beta(\varepsilon)\int_{\bigcup T^\varepsilon}(u_i^\varepsilon)^2\,d\hat{x}\right)^{1/2},$$

that converge towards zero because of (2.15), (1.1) for $r_0=+\infty$ and (1.2). Therefore,

$$\int_\Sigma M_i(\hat{x},\phi)u_i^0\,d\hat{x}=0,\quad \forall\phi\in(C^1(\overline{\Omega}))^3,\quad \phi=0 \text{ on } \Gamma_\Omega.$$

Let us use (2.10), the fact that $\delta\in[0,2]$ and Hölder's inequality, to get

$$\begin{aligned}\left|\int_\Sigma (M_i(\hat{x},\phi)-M_i(\hat{x},v))u_i^0\,d\hat{x}\right| &\le C\sum_{i,j=1}^3\int_\Sigma\left(|\phi_j-v_j|+|\phi_i-v_j|^{1+\delta}\right)|u_i^0|\,d\hat{x}\\ &\le C\sum_{i,j=1}^3\left(\|u_i^0\|_{L^2(\Sigma)}\|\phi_j-v_j\|_{L^2(\Sigma)}+\|\phi_j-v_j\|^{1+\delta}_{L^{(1+\delta)4/3}(\Sigma)}\|u_i^0\|_{L^4(\Sigma)}\right)\ \forall v\in\mathbf{V}.\end{aligned} \tag{7.3}$$

Thus, because of the continuous embedding $H^1(\Omega) \subset L^4(\Sigma)$, taking limits as ϕ tends to v in $(H^1(\Omega))^3$ gives

$$\int_\Sigma M_i(\hat{x}, v) u_i^0 \, d\hat{x} = 0, \quad \forall v \in \mathbf{V}.$$

Consequently, on account of (2.8), we can write

$$\int_\Sigma (u_i^0)^2 \, d\hat{x} \leq C \int_\Sigma M_i(\hat{x}, u_i^0) u_i^0 \, d\hat{x} = 0 \Longrightarrow u^0 = 0 \text{ on } \Sigma.$$

Finally, since $u^0 \in (H^1(\Omega))^3$ and vanishes on Γ_Ω we claim that $u^0 \in (H_0^1(\Omega))^3$.

Next, let us show that u^0 is the unique solution of (3.13). This follows taking $v \in (H_0^1(\Omega))^3$ in (2.12), and passing to the limit as $\varepsilon \to 0$: we see that u^0 satisfies (3.13).

Now, the proof of the strong convergence in $(H^1(\Omega))^3$ follows taking into account the equality (3.15) for $v = u^0$ and rewriting the chain of equalities and inequalities (7.1) with minor modifications.

This ends the proof of Theorem 3.3. □

Remark 2 It should be noted that conditions (2.7)-(2.10) provide a general framework for the setting of the homogenization problem as well as for the convergence of solutions for many limit values of the products $r_\varepsilon \varepsilon^{-2}$, $r_\varepsilon \beta(\varepsilon)$ as well as $r_\varepsilon^2 \varepsilon^{-2} \beta(\varepsilon)$, even for scalar problems. Let us mention that because of the surface integral in (2.12), $\delta \in [0, 2]$ in (2.10) allows the correct setting of the problem (2.11) (cf. (7.3)) but different less restrictive hypothesis and functions could be accepted in this respect (cf. Remark 3 in [14] and references therein). However, for certain limit values of these products, e.g., for $r_0 > 0$, restrictions on δ could be required to show convergence, like $\delta = 0$ in (2.10) which becomes important to obtain (6.1). As matter of fact, it allows the function $M_i(\cdot, u^\varepsilon(\cdot))$ to be in $H^{1/2}(\Sigma)$, however proofs may hold under other hypothesis that ensure this condition.

Note that (2.9) holds when we have a greater smoothness of M such as:

$$M_i \in C^1(\overline{\Sigma} \times \mathbb{R}) \text{ and } \left|\frac{\partial M_i}{\partial x_j}(x, u)\right|, \left|\frac{\partial M_i}{\partial u_j}(x, u)\right| \leq D_{ij}, \quad \forall (x, u) \in \overline{\Sigma} \times \mathbb{R}^3, \tag{7.4}$$

$i, j = 1, 2, 3$, for certain positive constants D_{ij}. Also, in practice M_i may depends only on the displacement in the i-th direction $M_i(x, u) \equiv M_i(x, u_i)$, or even be independent of x, satisfying further smoothness conditions (7.4) (cf., e.g., Sect. 3.3.1 in [44]). All this simplifies computations.

As regards the strong monotonicity condition (2.8) with $K > 0$, it may also be weakened taking $K = 0$ depending on the above mentioned limits; this is the case, in the framework of the present paper, when $r_0 = 0$.

As regards the periodicity condition, as outlined in Sect. 1, it becomes necessary when $r_0 > 0$ due, for instance, to the dependence of the strange term on the geometry of T, while a certain non periodical distribution of the reaction regions can be allowed in the extreme cases (cf. [20] and [21]).

Remark 3 In the linear case, the functions M_i read: $M_i(\hat{x}, u) \equiv M_{ij}(\hat{x}) u_j$, where $(M_{ij})_{i,j=1,2,3}$ is symmetric and positive definite 3×3-matrix:

$$\exists \alpha_2 > 0 : M_{ij}(\hat{x}) \xi_i \xi_j \geq \alpha_2 \xi_i^2, \quad \forall \xi \in \mathbb{R}^3, \quad \forall \hat{x} \in \overline{\Sigma}, \tag{7.5}$$

with components M_{ij} globally Lipschitz functions on $\overline{\Sigma}$. As a consequence, in this paper we extend the results for the linear boundary conditions considered in [16, 17] to a nonlinear framework.

Also, it should be emphasized that the homogenized problems here appearing coincide with those obtained for the homogenization of linear Winkler-Robin boundary conditions, see [16, 17], and, in the case of homogeneous and isotropic media, they also coincide with those obtained for rapidly alternating boundary conditions of Dirichlet and Neumann type in [4, 31]. This is due to the large reaction occurred on the reaction regions T^ε which likely stuck "asymptotically" these regions to the plane.

Remark 4 Notice that this is the first work in the literature that addresses the homogenization of nonlinear Winkler-Robin conditions for the ratio of the reaction regions $r_\varepsilon = O(\varepsilon^2)$. In particular, the very large reaction, cf. (1.2), allows a justification of the linearization process, which was not guessed before this study (cf. [20] to compare with different ranges of parameters). Conditions (2.7)-(2.10) for the nonlinear function M provide a general framework for mathematical justifications, but the linearization process may hold without such hypothesis, cf. [14] for different functions in a scalar problem. Also note that the nonlinear law in (2.11) might be replaced by others arising in contact problems (cf., e.g., [25, 44]), with the suitable modifications in the treatment and the homogenized boundary conditions: see [4, 43, 50] and references therein for the homogenization of Signorini-type conditions along planes outside critical size ranges; see [13] and references therein for a scalar model in perforated media with nonlinear restrictions. Among others, the technique developed in this paper for the elasticity system could be applied to different functions arising in [14] and restrictions in [13], both of which consider scalar problems.

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