universe

MDPI

Article

The Quantum Electromagnetic Field in the Weyl–Wigner Representation

Emilio Santos

Departamento de Física, Universidad de Cantabria, 39005 Santander, Spain; emilio.santos@unican.es

Abstract: The quantum electromagnetic (EM) field is formulated in the Weyl–Wigner representation (WW), which is equivalent to the standard Hilbert space one (HS). In principle, it is possible to interpret within WW all experiments involving the EM field interacting with macroscopic bodies, the latter treated classically. In the WW formalism, the essential difference between classical electrodynamics and the quantum theory of the EM field is just the assumption that there is a random EM field-filling space, i.e., the existence of a zero-point field with a Gaussian distribution for the field amplitudes. I analyze a typical optical test of a Bell inequality. The model admits an interpretation compatible with local realism, modulo a number of assumptions assumed plausible.

Keywords: quantized electromagnetic field; Wigner representation; tests of Bell inequalities; Weyl transform

1. Introduction

1.1. The Standard Quantization Method for Fields

Quantum field theory started with the electromagnetic field (EM) quantization by Dirac in 1927. As is well known, the procedure consists of expanding the field in plane waves, or more generally, normal modes, writing every mode in terms of two complex conjugated time-dependent amplitudes (c-numbers) $\left\{a_j(t), a_j^*(t)\right\}$, where j labels a mode, and then promoting the amplitudes to be operators in a Hilbert space [1].

In order to define the commutation properties of the operators, it is convenient to introduce two auxiliary quantities $\{x_j(t), p_j(t)\}$ as follows:

$$x_j(t) \equiv \frac{c}{\sqrt{2}\omega_j}\left(a_j(t) + a_j^*(t)\right), p_j(t) \equiv \frac{i\hbar\omega_j}{\sqrt{2}c}\left(a_j(t) - a_j^*(t)\right), \quad (1)$$

where i is the imaginary unit. Then, the (Maxwell) evolution of every field amplitude $a_j(t)$ corresponds to the time change of the quantities $(x_j(t), p_j(t))$ as if they were the coordinate and momentum of a harmonic oscillator with unit mass and proper frequency ω_j. Therefore, the EM field may be treated formally as a collection of harmonic oscillators. (For the sake of clarity, I write operators in a Hilbert space with a 'hat', e.g., $\hat{a}_j, \hat{a}_j^\dagger$, and numerical (c-number) amplitudes without a 'hat', e.g., a_j, a_j^*). Then, the field is quantized via the said oscillators, as in elementary quantum mechanics, promoting $\{x_j(t), p_j(t)\}$ to be operators fulfilling the standard commutation rules. Hence, field quantization is completed with the following commutation rules at equal times:

$$[\hat{a}_j, \hat{a}_k] = \left[\hat{a}_j^\dagger, \hat{a}_k^\dagger\right] = 0, \left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}, \quad (2)$$

δ_{jk} being Kronecker delta. The operators $\hat{a}_j$ $\left(\hat{a}_j^\dagger\right)$ are usually named annihilation (creation) operators of photons.

Dirac's procedure is the standard quantization method for fields. It rests on the mathematical theory of Hilbert spaces. In this article, I will study an alternative formalism

Citation: Santos, E. The Quantum Electromagnetic Field in the Weyl–Wigner Representation. *Universe* **2024**, *10*, 452. https://doi.org/10.3390/universe10120452

Academic Editor: Ralf Hofmann

Received: 24 October 2024
Revised: 4 December 2024
Accepted: 6 December 2024
Published: 9 December 2024

for the quantum EM field deriving from the work undertaken by H. Weyl [2] and E. P. Wigner [3] on (non-relativistic) quantum mechanics of particles (QM in the following).

1.2. Wigner Representation in Quantum Mechanics

In 1927, Weyl proposed a quantization method for systems of particles via a transform that converts classical (c-number) coordinates and momenta into operators in a Hilbert space [2]. The idea may be put as follows: If we have a classical function in phase space, $F(\mathbf{x}, \mathbf{p})$, involving the position and momentum of a particle, we may obtain a quantum counterpart, $Q(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}})$, involving operators in a Hilbert space via the following symbolic integral:

$$Q(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}) = \int d\mathbf{x}\delta(\mathbf{x} - \hat{\mathbf{x}}) \int d\mathbf{p}\delta(\mathbf{p} - \hat{\mathbf{p}})F(\mathbf{x}, \mathbf{p}),$$

where δ is Dirac delta. The integrals extend over the whole 3D space. The delta function of an operator is not well defined, but we may give a meaning to the symbolic equation substituting an integral representation for the "deltas", that is

$$Q(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}) = \frac{1}{4\pi^2} \int d\boldsymbol{\lambda} \int d\boldsymbol{\mu} \int d\mathbf{x} \exp[i\boldsymbol{\lambda} \cdot (\mathbf{x} - \hat{\mathbf{x}})] \int d\mathbf{p} \exp[i\boldsymbol{\mu} \cdot (\mathbf{p} - \hat{\mathbf{p}})]F(\mathbf{x}, \mathbf{p}),$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are vector variables. This equation presents a problem, namely, that the function Q may change if the order of the $\mathbf{x}$ and $\mathbf{p}$ integrals is reversed, due to the fact that the operators $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ do not commute. Hence, there is an ambiguity in the ordering of the operators and Weyl chose the following symmetrical order, that is

$$Q(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}) = \frac{1}{4\pi^2} \int d\boldsymbol{\lambda} \int d\boldsymbol{\mu} \int d\mathbf{x} \int d\mathbf{p} \exp[i\boldsymbol{\lambda} \cdot (\mathbf{x} - \hat{\mathbf{x}}) + i\boldsymbol{\mu} \cdot (\mathbf{p} - \hat{\mathbf{p}})]F(\mathbf{x}, \mathbf{p}), \tag{3}$$

where $Q(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}})$ is called the Weyl transform of $F(\mathbf{x}, \mathbf{p})$. The generalization to a system of particles is straightforward. The resulting equations from Equation (3) do not agree with those of QM in general, whence the Weyl quantization method is not too useful.

In 1932, Wigner [3] proposed a method to achieve the reverse of quantization via introducing a formalism with classical appearance for quantum mechanics. He proposed to obtain a function $W_\psi(\mathbf{r}, \mathbf{p}, t)$ in the phase space of the particle from the quantum wavefunction $\psi(\mathbf{r}, t)$ of a state as follows:

$$W_\psi(\mathbf{r}, \mathbf{p}.t) = \int d\mathbf{u}\psi^*(\mathbf{r} + \mathbf{u}, t)\psi(\mathbf{r} - \mathbf{u}, t) \exp(2i\mathbf{u} \cdot \mathbf{p}), \tag{4}$$

where $W_\psi(\mathbf{r}, \mathbf{p})$ is named the Wigner function of the state $\psi(\mathbf{r})$. The evolution equation, in the form of a time derivative of $W_\psi(\mathbf{r}, \mathbf{p}.t)$, may be obtained taking the Schrödinger equation into account but I omit the (cumbersome) expression [4,5]. It is also possible to obtain the Wigner function when the state is given by a density operator in the abstract Hilbert space, rather than a wavefunction as in Equation (4). The Wigner function of a state $\hat{M}$ is obtained in this case as follows:

$$\begin{aligned} W_{\hat{M}} \quad &= \quad T_W[\hat{M}] \equiv \frac{1}{4\pi^2} \int d\boldsymbol{\lambda} \int d\boldsymbol{\mu} \exp[-i\boldsymbol{\lambda} \cdot \mathbf{x} - i\boldsymbol{\mu} \cdot \mathbf{p}] \\ &\quad \times Tr\{\hat{M} \exp[i\boldsymbol{\lambda} \cdot (\mathbf{x} - \hat{\boldsymbol{x}}) + i\boldsymbol{\mu} \cdot (\mathbf{p} - \hat{\boldsymbol{p}})]\}, \end{aligned} \tag{5}$$

where $Tr\{\}$ means the trace operation. The Wigner transform Equation (5) is actually the inverse of the Weyl transform when the operator $\hat{M}$ may be written as a function of the canonical operators $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$.

The Wigner transform (or inverse Weyl transform) Equation (5) may be applied not only to density operators representing states but also to observables, with the result that the whole QM may be formulated in terms of phase space functions. That formalism is usually named the *Wigner representation of QM* [4,5]. In particular, the Schrödinger equation of a system of particles becomes a diferential equation for the Wigner function $W_\psi\left(\left\{\mathbf{r}_j, \mathbf{p}_j\right\}, t\right)$ and the expectation value of the observable $\hat{\text{M}}$ in the state ψ becomes an integral

$$\langle\psi|\hat{M}|\psi\rangle = \int W_M\left(\left\{\mathbf{r}_j, \mathbf{p}_j\right\}\right) W\psi\left(\left\{\mathbf{r}_j, \mathbf{p}_j\right\}, t\right) \prod_j d\mathbf{r}_j d\mathbf{p}_j.$$

The Wigner representation is equivalent to the standard Hilbert space formulation of QM and it is useful for some purposes. However, it does not solve the problems of interpretation because it does not admit a realistic interpretation. In fact, the Wigner functions of states are positive definite but for a slight fraction of pure quantum states, that is, when the wavefunction is Gaussian [6]. Therefore, they cannot be interpreted as probability distributions in phase space. Here, I name as realistic the interpretation of a formalism fitting in the celebrated EPR article [7].

The Wigner representation of QM may be extended to the electromagnetic field. In fact, the inverse of the Weyl transform may be written, taking the change of variables Equation (1) into account, as follows:

$$\begin{aligned} W_{\hat{M}} &= T_W\left[\hat{M}\right] \equiv \frac{1}{(2\pi^2)^n} \prod_{j=1}^{n} \int_{-\infty}^{\infty} d\lambda_j \int_{-\infty}^{\infty} d\mu_j \exp\left[-2i\lambda_j \text{Re}a_j - 2i\mu_j \text{Im}a_j\right] \\ &\times Tr\left\{\hat{M} \exp\left[i\lambda_j\left(\hat{a}_j + \hat{a}_j^\dagger\right) + \mu_j\left(\hat{a}_j - \hat{a}_j^\dagger\right)\right]\right\}, \end{aligned} \tag{6}$$

where $T_W\left[\hat{M}\right]$ stands for the (inverse) Weyl transform of the operator $\hat{M}$. The result obtained, $W_{\hat{M}}\left(\left\{a_j, a_j^*\right\}\right)$, is a function of (c-number) field amplitudes. Here, the operator $\hat{M}$ may be either an observable or the density operator of a state. That is, the transform Equation (6) obtains the passage from the standard (HS) quantum formalism to an alternative Weyl–Wigner (WW) representation for the same theory in terms of the amplitudes $\left\{a_j, a_j^*\right\}$ of the EM field.

Unlike standard QM, the WW representation of the quantized EM field *does suggest a realistic picture for the field* as argued below in this article. In Sections 2–4, I will be concerned mainly with the formal aspects of WW, leaving further discussion about interpretation for the last Section 5. The WW representation of the quantized EM field has been studied elsewhere [8–11]. In this article, I will present further elaboration and applications of the formalism. A less formal, semi-quantitative, realistic interpretation of the quantized EM field has been provided elsewhere [12] for many phenomena currently assumed to be purely quantum.

In Section 2, I shall derive the most relevant properties of the WW formalism obtained from HS via the inverse Weyl transform Equation (6). After that, I will discuss two matters related to the WW formalism, namely, whether it may be seen as a new method for field quantization in Section 3, and the usefulness of the WW formalism for the interpretation of some experiments in Section 4. Several aspects of interpretation will be discussed in Section 5.

2. From Hilbert Space to Weyl–Wigner Representation of Fields

Here, I will study the EM field in the Coulomb gauge because it is more appropriate than the Lorentz gauge for the applications considered in Section 4, although the latter is more common and useful in (relativistic) quantum electrodynamics. The Maxwell theory is compatible with special relativity, which is explicitly exhibited in the covariant Lorentz gauge. However, it may be formulated in the Coulomb gauge, in terms of a vector potential $\mathbf{A}(\mathbf{x},t)$ plus a scalar potential $\phi(\mathbf{x},t)$, the latter taking account of the instantaneous electrostatic interaction between charges (but this does not mean that the theory violates relativistic invariance) [13]. Then, the quantity expanded in plane waves is the vector potential. The effect of the electrostatic interaction is straightforward and will be ignored in the following.

In free space, the expansion may be written

$$\mathbf{A}(\mathbf{x},t) = \sum_{l} a_l \boldsymbol{\varepsilon}_l \exp(i\mathbf{k}_l \cdot \boldsymbol{x} - i\omega_l t) + \text{complex } conjugate, \tag{7}$$

where $\mathbf{k}_l$ is the wavevector and $\boldsymbol{\varepsilon}_l$ the polarization vector of a mode with frequency $\omega_l = c|\mathbf{k}_l|$. As stated above, the standard quantization of the field consists of introducing a Hilbert space and promoting the classical amplitudes to be operators $\left\{\hat{a}_j, \hat{a}_j^\dagger\right\}$ acting on that space. As a difference with Equation (1), here and in the following, I will write explicitly the time dependence so that the operators $\left\{\hat{a}_j, \hat{a}_j^\dagger\right\}$ are time-independent and they fulfil the commutation rules Equation (2).

In the following, I summarize the most relevant properties of WW.

Transform of products of field amplitudes. Equation (6) allows deriving the product of amplitudes in WW for any HS product of creation and annihilation operators. If we have a product of WW amplitudes like $a_j^m a_j^{*n}$, the HS counterpart is

$$a_j^m a_j^{*n} \to \left(\hat{a}_j^m \hat{a}_j^{\dagger n}\right)_{sym}, \tag{8}$$

where *sym* stands for symmetric and it means writing a sum of the $m+n$ operator products in all possible orderings and then dividing by the number of terms that is $(m+n)!/(m!n!)$. The reverse passage from HS to WW requires a prior rewriting the HS expression as a sum of products of operators in symmetrical order, taking the commutation rules into account.

States. In WW, the states are defined via the (inverse) Weyl transform of the HS states. Thus, any state in WW, corresponding to a given state in HS, is a function of the amplitudes $\left\{a_j, a_j^*\right\}$, named the "Wigner function" of the state.

The "vacuum" state. In HS, the ground state of the free field (called the vacuum state) is represented by a vector $\mid 0\rangle$ defined by

$$\hat{a}_j \mid 0\rangle = 0 \Rightarrow \langle 0 \mid \hat{a}_j^\dagger = 0, \text{ for all radiation modes}, \tag{9}$$

0 being here the null vector in the HS. Alternatively, it may be represented by the density operator

$$\hat{\rho} = \mid 0\rangle\langle 0 \mid. \tag{10}$$

If this operator is inserted in place of $\hat{M}$ in Equation (6), we obtain after some algebra the Wigner function of the vacuum state, that is [9,10],

$$W_0(\{a_j\}) = \prod_j \frac{2}{\pi} \exp\left(-2|a_j|^2\right), \tag{11}$$

which is normalized for the integration with respect to $\prod_j d\text{Re}a_j d\text{Im}a_j$. Hence, the mean square average value of each amplitude is

$$\left\langle |a_j|^2 \right\rangle = \frac{1}{2}. \tag{12}$$

If Equation (11) was interpreted as a probability distribution, then Equation (12) would be the variance of $|a_j|$. The field represented by Equation (11) will be labeled the "zeropoint field" (ZPF).

Taking the last Equation (19) (see below) into account, Equation (11) leads to the following distribution of the energy amongst the radiation modes:

$$W_E(\{E_j\}) = \prod_j \frac{2}{\hbar\omega_j} \exp\left(-\frac{2E_j}{\hbar\omega_j}\right), \tag{13}$$

where E_j is the energy of mode j and the normalization is appropriate for integration with respect to $\prod_j dE_j$. I point out that Equation (13) leads to the following spectral density (energy of the field per unit volume and unit frequency interval)

$$\rho(\omega) = \frac{1}{2}\hbar\omega^3, \tag{14}$$

which possesses the important property of being Lorentz invariant. Indeed, the relativistic (Lorentz) invariance fixes the spectral density Equation (14) modulo the scale factor $\hbar$ that should be identified with the Planck constant in order to agree with the predictions of QM.

Excited pure states of the field correspond in HS to vectors which may be obtained by application of the creation operators of photons $\left\{\hat{a}_j^\dagger\right\}$ on the vacuum state. For instance, $\hat{a}_j\dagger \mid 0\rangle$ is the vector of HS representing the state of a single photon of kind j, the vector $\hat{a}_j^\dagger \hat{a}_k^\dagger \mid 0\rangle$ represents a two-photon state, etc. The set of all these states is named the Fock space. It is usual to admit that any linear combinations of state-vectors belonging to the Fock space is also a possible pure state. Hence, any vector in HS obtained by the action of a function of the creation operators on the vacuum state leads to a pure state $\mid \psi\rangle$, that is

$$\mid \psi\rangle = \hat{f} \mid 0\rangle, \hat{f} \equiv c_0 + \sum_j \sum_n c_{jn}\hat{a}_j^{\dagger n}, \langle\psi \mid \psi\rangle = 1, \tag{15}$$

where j labels a radiation mode and n is a natural number, c_0 and c_{jn} being complex numbers, with the constraint that the vector $\mid \psi\rangle$ is normalized. Mixed states, which might be represented by density operators, correspond to the probability distributions of pure states.

For all states defined by Equation (15) in the HS formalism, we may obtain the corresponding expression in WW via performing the (inverse) Weyl transform, in a similar way as we made for the vacuum state going from Equation (10) to Equation (11). That is, the set of states in WW will consist of functions of the amplitudes with the form

$$\Psi\left(\left\{a_j, a_j^*\right\}, t\right) = T_W\left[\hat{f} \mid 0\rangle\langle 0 \mid \hat{f}^\dagger\right]. \tag{16}$$

Working these transforms is straightforward although lengthy in most cases. For the Fock states $\left\{\hat{a}_j^{\dagger n} \mid 0\rangle\right\}$, the results are well known but will not be reproduced here.

I shall point out that the definition of states in HS is controversial. In fact, the *question whether a given state of the set Equation* (15) *is physical* cannot be answered within the nude HS formalism. It would require the existence of a "preparation procedure" able, in principle, to manufacture the said state in the laboratory. For some states belonging to the set Equation (15), that preparation seems not possible because some of the state properties are contradictory. For instance, the state $\hat{a}_j^\dagger \mid 0\rangle$ is not localized but *extended* over the whole space (or over a large normalization volume) if $\hat{a}_j^\dagger$ is associated to a plane wave with definite wavevector $\mathbf{k}_j$. But, at the same time, it should represent a (*localized?)* particle, i.e., one "photon". Thus, many authors assume that only a fraction of the set Equation (15) are physical states, e.g., those in the form of (localized) wavepackets.

Observables. Equation (6) allows obtaining the WW counterparts of the observables in the HS formalism. They would be functions of the amplitudes $\left\{a_j, a_j^*\right\}$.

Evolution. It is remarkable that the evolution of the *quantized free EM field* within the WW formalism is just *the classical (Maxwell) evolution*. In fact, the evolution of a field in the WW formalism is given by the Moyal equation [4,5,10]

$$\begin{aligned}\frac{\partial W_{\hat{M}}}{\partial t} &= 2\{\sin\left[\frac{1}{4}\left(\frac{\partial}{\partial \mathrm{Re}a_j'}\frac{\partial}{\partial \mathrm{Im}a_j''} - \frac{\partial}{\partial \mathrm{Im}a_j'}\frac{\partial}{\partial \mathrm{Re}a_j''}\right)\right] \\ &\quad \times W_{\hat{M}}\left\{a_j', a_j^{*\prime}, t\right\} H\left(a_j'', a_j^{*\prime\prime}\right)\}_{a_j},\end{aligned} \tag{17}$$

where $\{\}_{a_j}$ means making $a'_j = a''_j = a_j$ and $a^{*\prime}_j = a^{*\prime\prime}_j = a^*_j$ after performing the derivatives. The density operator $\hat{M}$ is the representative of a state in HS and $W_{\hat{M}}$ its Wigner function. For the free electromagnetic field, the Hamiltonian $H\left(a_j, a^*_j\right)$ is quadratic in the amplitudes (see Equation (19) below) whence $\sin x$ reduces to x in Equation (17). This means that the Moyal equation becomes

$$\frac{\partial W_{\hat{M}}}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{Re} a'_j}\frac{\partial}{\partial \mathrm{Im} a''_j} - \frac{\partial}{\partial \mathrm{Im} a'_j}\frac{\partial}{\partial \mathrm{Re} a''_j}\right) \times W_{\hat{M}}\left\{a'_j, a^{*\prime}_j, t\right\} H\left(a''_j, a^{*\prime\prime}_j\right)\}_{a_j}, \tag{18}$$

where the right side is a Poisson bracket, taking Equation (1) into account, and Equation (18) is the Liouville equation of classical evolution. For details see, e.g., Ref. [10].

Interactions. Studying, within WW, the evolution when the EM field is not free requires the inclusion of the interactions with other fields or with macroscopic bodies. The interaction with other fields should be made also within a WW representation for the other fields. See comments on this possibility at the end of Section 3 below. In particular, we cannot study QED within the WW approach, because this would require the Dirac electron-positron field in a formalism compatible with WW of the EM field. This difficulty dramatically reduces the possibilities of our approach.

In sharp contrast, it is easy to define the interactions of the EM field with macroscopic bodies, which allows the study of interesting experiments. A relevant example is worked in Section 4. I propose that the interactions of the quantized EM field, in the WW representation, with macroscopic bodies are given by classical Maxwell–Lorentz electrodynamics. The assumption is plausible because both macroscopic bodies and the free EM field are governed by classical laws, that is Equation (18).

Hamiltonian. The free field Hamiltonians in the HS and WW formalisms might be defined as follows

$$\hat{H}_{HS} = \hbar\sum_j \omega_j(\hat{a}^\dagger_j \hat{a}_j + \frac{1}{2}) = \frac{1}{2}\hbar\sum_j \omega_j(\hat{a}^\dagger_j \hat{a}_j + \hat{a}_j \hat{a}^\dagger_j), H_{WW} = \hbar\sum_j \omega_j |a_j|^2, \tag{19}$$

respectively, where I have taken Equation (8) into account. However, with this definition the energy of the vacuum state $\mid 0\rangle$ is not zero (in fact the vacuum becomes the ZPF with distribution Equation (11)), whence it is standard practice to redefine the Hamiltonian by putting the annihilation operators on the right, which is named the 'normal ordering rule'. Thus, the HS and WW Hamiltonians become, respectively,

$$\hat{H}^{norm}_{HS} = \hbar\sum_j \omega_j \hat{a}^\dagger_j \hat{a}_j, H^{norm}_{WW} = \hbar\sum_j \omega_j\left(|a_j|^2 - \frac{1}{2}\right). \tag{20}$$

Actually, the normal order produces the same effect on the field energy as subtracting from the Hamiltonian Equation (19) its vacuum expectation value, that is

$$\hat{H}^{subtract}_{HS} = \hat{H}_{HS} - \langle 0 \mid \hat{H}_{HS} \mid 0\rangle = \hbar\sum_j \omega_j \hat{a}^\dagger_j \hat{a}_j, \tag{21}$$

also giving nil vacuum energy.

When there are interactions with other quantized fields, the Hamiltonian cannot be defined prior to obtaining formalisms similar to WW for those fields, which at the moment are unavailable. For the case when the quantized EM field interacts with macroscopic bodies, I propose that the interaction Hamiltonian is given by classical electrodynamics.

Vacuum energy expectation value. Let us calculate the expectation value of the energy within WW for a general state W_ϕ. In order to agree with HS predictions, we shall use the normally ordered Hamiltonian Equation (20). For a state $\mid \phi\rangle$ of the field, the calculation, firstly in HS then in WW, may be written (see Equation (12))

$$\begin{aligned} \langle E \rangle &= \left\langle \phi \middle| \sum_l \hbar\omega_l \hat{a}_l^\dagger \hat{a}_l \middle| \phi \right\rangle \\ &= \sum_l \hbar\omega_l \int W_\phi(\{a_l\}) \left(|a_l|^2 - \frac{1}{2} \right) d\text{Re}a_l d\text{Im}a_l \\ &= \sum_l \hbar\omega_l \int |a_l|^2 (W_\phi(\{a_l\}) - W_0(\{a_l\})) d\text{Re}a_l d\text{Im}a_l, \end{aligned} \quad (22)$$

where $W_\phi(\{a_l\})$ is the state (Wigner function) that in WW represents the HS state $\mid \phi\rangle$ and W_0 is the vacuum Wigner function Equation (11). The result is that the *normal ordering rule of HS corresponds to removing the vacuum energy* that would appear according to the initial definition Equation (19). Thus, the vacuum Wigner function Equation (11) plays a strange role in the WW representation. On the one hand, the vacuum Equation (11) does not imply the absence of any field, whence the vacuum field should interact with charges. On the other hand, the vacuum is devoid of energy according to Equation (22), then appearing as an inert stuff without any physical relevance, except for its fluctuations. This paradoxical condition will be studied in more detail in later sections.

The expectation value of the observable $\hat{O}$ in the state $\hat{\rho}$ reads $Tr(\hat{\rho}\hat{O})$, or in particular, $\langle \psi \mid \hat{O} \mid \psi\rangle$, in the HS formalism. The counterpart in the WW formalism leads to the integral of the product of two functions of the amplitudes, that is

$$Tr(\hat{\rho}\hat{O}) = \int W_{\hat{\rho}} \left\{ a_j, a_j^* \right\} W_{\hat{O}} \left\{ a_j, a_j^* \right\} \prod_j d\text{Re}a_j d\text{Im}a_j, \quad (23)$$

where $W_{\hat{\rho}}$ and $W_{\hat{O}}$ are the counterparts of a density operator, $\hat{\rho}$, and a quantum observable $\hat{O}$. A particular case of Equation (23) is the vacuum expectation value when $W_{\hat{\rho}}$ becomes W_0.

In summary, Equation (23) proves that *the predictions of both formalisms, HS and WW, are identical for the same states and observables. This implies that both correspond to the same physical theory for the quantized EM field, although the mathematical formalisms are quite different.*

3. The WW Formalism as a New Field Quantization Method

In the previous section, I have shown that WW is a new formalism for the quantized EM field, equivalent to the (HS) standard one. The equivalence means that we may represent states and observables in either formalism, passing from WW to HS via the Weyl transform (not written explicitly in this paper, see, e.g., [9]) or from HS to WW via the inverse transform Equation (6). The point is that expectation values may be obtained in either HS or WW giving the same numerical values in both cases.

In this section, I propose to take WW as a new quantization method alternative to the (HS) standard one. That is, I will skip the passage through HS in order to obtain WW, as made in the previous section. In fact, I will quantize directly the classical (Maxwell) EM theory via a set of rules shown below. I will use that method for the EM field interacting with macroscopic bodies, these characterized by their charges or their bulk electric and magnetic properties. This includes most of the devices used in quantum optics, like mirrors, polarizers, beam splitters, etc. The motion of the macroscopic bodies is assumed non-relativistic.

The new quantization method of the EM field may be stated by the following rules:

1. The formalism rests on spacetime functions as in the classical Maxwell theory. In particular, the field is defined in the Coulomb gauge by a vector potential **A(r**,t), Equation (7), and a scalar potential $\phi(\mathbf{r}, t)$, whence we might derive the electric **E(r**,t) and the magnetic **B(r**,t) fields. The vector potential may be expanded in normal modes and written in terms of amplitudes $\left\{ a_j, a_j^* \right\}$ if necessary.

2. The evolution of the field is governed by classical Maxwell–Lorentz electrodynamics and the motion of macroscopic bodies by the standard laws of mechanics, that is Newton's second law or Hamiltonian dynamics.

3. The *essential difference* between classical electrodynamics and the quantum theory of the EM field in WW *is just the assumption that there is a random EM field-filling space*, i.e., the ZPF with the probability distribution Equation (11). Actually, that distribution might be obtained without any reference to the HS formalism. In fact, Equations (11) and (14) may be derived from the following two conditions put on the ZPF distribution, except for a scaling parameter. That is maximal information (Shanon) entropy compatible with relativistic (Lorentz) invariance of the spectrum. The scaling parameter that appears should be identified with the Planck constant ħ.

4. The Hamiltonian of the free field is given as a function of the amplitudes by

$$H = \hbar \sum_j \omega_j |a_j|^2 - \langle H \rangle_{vac} = \hbar \sum_j \omega_j |a_j|^2 - \frac{1}{2}\hbar \sum_j \omega_j, \tag{24}$$

where $\langle H \rangle_{vac}$ is the part coming from the ZPF which must be subtracted, see Equation (12), so that the energy of the ground state is zero. The Hamiltonian including interactions with macroscopic bodies should be defined by Equation (24) plus the appropriate interaction terms, according to classical electrodynamics.

I point out that the assumption of a random radiation filling space (the ZPF) in *rule 3* and Equation (24) of *rule 4* look inconsistent. Indeed, the ZPF is assumed to act on electric charges, but itself is devoid of energy. Of course, Equation (24) is related to the "normal ordering rule", which in standard (HS) quantum theory looks artificial but not inconsistent. The label of inconsistency may be overthrown by taking into account that our approach has a limited domain, namely, the EM field plus macroscopic bodies. We may assume that in a wider domain that included all fields (i.e., those of the standard model of high-energy physics) and interactions, the total energy of the ZPF would be plausibly nil. Indeed, we may assume that the (negative) contributions of Fermi fields should balance the (positive) contributions of Bose fields. In the meantime, when a theory similar to WW does not exist for all fields, it is appropriate to remove the ZPF contribution of the EM field in calculations.

5. The set of states are functions $\Psi\left(\left\{a_j, a_j^*\right\}\right)$ of the amplitudes (or functions of the electric **E** and magnetic **B** fields). However, I will not fix the set of states as given in Equation (16) of Section 2, but I will define the possible states as those probability distributions (given by positive definite functions) of the EM field. I point out that this definition of states might break the full agreement between WW and HS derived in Section 2. However, I believe that both formulations could not be discriminated empirically.

6. In principle, all functions of the amplitudes, $\mathrm{F}\left(\left\{a_j, a_j^*\right\}\right)$ (or functions of the electric **E** and magnetic **B** fields) may be observables.

7. The expectation value of the observable $F\left(\left\{a_j, a_j^*\right\}\right)$ in the state $\Psi\left(\left\{a_j, a_j^*\right\}\right)$ may be obtained via the integral

$$\int F\left\{a_j, a_j^*\right\} \Psi\left\{a_j, a_j^*\right\} \prod_j d\mathrm{Re}a_j d\mathrm{Im}a_j. \tag{25}$$

I shall point out that in Sections 2 and 3, I have ignored all effects of the scalar potential $\phi(\mathbf{r},\mathrm{t})$, that appear when we work in the Coulomb gauge, deriving from the instantaneous electrostatic interactions. We must also take into account that potential in the interpretation of experiments, which would be straightforward.

A most relevant question is whether a quantization formalism similar to the WW one studied here may be applied to other quantum fields. My answer is that I do not know, but certainly the search for this possibility is worthwhile. I believe that the generalization of WW to Bose fields is likely possible because the essential properties needed are fulfilled. In fact, the fields are represented by ordinary scalar or vector functions of the coordinates;

they may be expanded in normal modes (in particular, plane waves) and the amplitudes may be quantized via promoting the amplitudes of the modes to be operators fulfilling the standard commutation rules Equation (2). Of course, I exclude the gravitational field, which being nonlinear cannot be quantized that way. In sharp contrast, the standard quantization of Fermi fields involves field amplitudes fulfilling anticommutation rules (e.g., the Dirac electron-positron field). Furthermore, in that case, the amplitudes are not simple numerical (scalar or vector) functions of the coordinates, but spinors with several components. Therefore, finding a quantization method similar to WW will be substantially more difficult if possible at all.

4. Entangled Photons Experiments

In this section, I will study a relevant experiment within the WW quantization formalism, that is, an optical test of the Bell inequality [14] involving parametric down conversion. The experiment embraces the EM field interacting with macroscopic bodies. Therefore, the interactions will be treated as in the classical Maxwell–Lorentz electrodynamics, as stated in the previous section. I will study an experiment similar to the one reported in Refs. [15–17], but in order to simplify the argument, I shall consider maximally entangled photons, although non-maximally entangled photons were used in the quoted test.

In experiments with "entangled photon pairs" produced via spontaneous parametric down-conversion (SPDC), a crystal having nonlinear electric susceptibility is pumped by a laser with frequency ω_P. In the opposite side of the crystal, a rainbow with several colors appears. Amongst the radiation emitted, two narrow beams, named "signal" and "idler", are selected via apertures and lens systems. For two appropriate (conjugated) directions, these beams are strongly correlated, which in the HS formalism is labeled entanglement. Indeed, in HS, the two beams are interpreted as consisting of a flow of entangled photon pairs, one photon in every pair going to the "signal" beam and the partner photon to the "idler" beam. The energy of one photon from the laser is $\hbar\omega_P$ and the SPDC process is currently interpreted by saying that one of the laser photons is split by the coupling with the crystal, giving rise to two photons with energies $\hbar\omega_s$ and $\hbar\omega_i$, fulfilling $\omega_s + \omega_i = \omega_P$, this equality interpreted as energy conservation. In contrast, the photon momenta are not conserved because a part of the momentum is absorbed by the crystal. This is the common view, but I stress that in the WW representation, there are no photons, just (continuous) wave fields.

SPDC is actually a process that may be produced at the macroscopic level, when it can be interpreted classically [18]. In the process, two macroscopic light signals with wavevectors $\mathbf{k}_s$ and $\mathbf{k}_i$, and frequencies ω_s and ω_i, respectively, plus a strong laser beam with frequency $\omega_P = \omega_s + \omega_i$ are sent to a nonlinear crystal. For appropriate $\mathbf{k}_s$ and $\mathbf{k}_i$, two beams appear on the opposite side of the crystal with the same wavevectors and frequencies as the incoming ones, but greater intensities, which may be represented as follows:

$$\begin{aligned} incoming &: \quad a_s \exp(i\mathbf{k}_s \cdot \boldsymbol{r} - i\omega_s t), a_i \exp(i\mathbf{k}_i \cdot \boldsymbol{r} - i\omega_i t), \\ outgoing &: \quad [a_s + D a_i^* \exp(i\zeta)] \exp(i\mathbf{k}_s \cdot \boldsymbol{r} - i\omega_s t), \\ & \quad\;\; [a_i + D a_s^* \exp(i\xi)] \exp(i\mathbf{k}_i \cdot \boldsymbol{r} - i\omega_i t), \end{aligned} \tag{26}$$

where D is a small coupling parameter, i.e., $0 < D << 1$, ζ and ξ are phases of the outgoing beams, those in the incoming beams being included in the definition of the amplitudes a_s and a_i. The remaining notation is standard. An actual light beam consists of a superposition of many radiation modes with close wavevectors. However, I will use a "two-modes approximation", which justifies using sharp wavevectors in Equation (26).

Within classical electrodynamics, the SPDC process may be described as follows [18]. The superposition of the incoming beam with frequency ω_s, that I will label "signal", and the laser with frequency ω_P, produces a resonant vibration of some elements (electrons) of the crystal, which radiate with frequency $\omega_P - \omega_s = \omega_i$ mainly in a particular direction, which should agree with the direction that we chose for the other incoming beam, labeled "idler". Similarly, the incoming idler beam with frequency ω_i together with the laser gives rise to radiation with frequency ω_s in the direction of the incoming signal beam. Thus, the equality $\omega_P = \omega_s + \omega_i$ is just a matching condition between the laser, the incoming fields and some vibration modes of the crystal. The relevant point is that the amplitude a_i^* that is superposed to the amplitude a_s of the "signal" is precisely the complex conjugate of the amplitude of the incoming "idler" beam, and similarly for the other superposition, see Equation (26). This is the reason for the strong correlation between the outgoing "signal" and "idler" beams.

Now, we may study within the WW the SPDC phenomenon involved in tests of Bell inequalities. It is similar to the macroscopic SPDC described above, but in this case, the incoming beams belong to ZPF radiation Equation (11). In practice, SPDC is used in the optical range, i.e., the EM fields involved belong to the visible or near-visible light. As stated above the strong correlation between the two outgoing beams is named "entanglement" in the standard HS quantum formalism. Indeed, in the HS formalism, the beams are viewed as two flows of pairs of entangled photons, as stated above. It is common to stress that entanglement is a typical quantum phenomenon, quite different from any classical correlation. In WW, we see that it is "quantum" because it involves the random radiation ZPF, Equation (11), which is indeed the characteristic trait of the EM quantized field, as shown in Section 3.

In practice, the effects of the spacetime dependence and the extra phases of Equation (26) play no role in most cases, and they can be ignored, writing simply

$$incoming : a_s, a_i, outgoing : (a_s + Da_i^*), (a_i + Da_s^*). \tag{27}$$

However, sometimes the phases are relevant and they shall be taken into account. With suitable devices, the signal and idler beams are combined, giving rise to two beams strongly correlated in polarization. These beams travel (maybe a long path) until Alice and Bob, respectively. Alice possesses a polarization analyzer and a detector in front of it, and similarly Bob. Thus, the beams arriving at these two detectors may be represented, in the two modes approximation, by the field amplitudes [11]

$$\begin{aligned} A &= a_s \cos\theta + ia_i \sin\theta + D[a_i^* \cos\theta + ia_s^* \sin\theta] \equiv A_0 + DA_1, \\ B &= a_i \cos\phi - ia_s \sin\phi + D[a_s^* \cos\phi - ia_i^* \sin\phi] \equiv B_0 + DB_1, \end{aligned} \tag{28}$$

where θ and ϕ are the polarizer's angles, A and B standing for Alice and Bob, respectively. The amplitudes complex conjugate to A and B will be labeled $A^*$and B^*, respectively. From Equation (28), it is straightforward to obtain the quantum predictions for the experiment within WW. We should take into account that

$$\langle |a_s|^2 \rangle = \langle |a_i|^2 \rangle = \langle |a_s|^4 \rangle = \langle |a_i|^4 \rangle = \frac{1}{2}, \langle a_s^m a_s^{*n} \rangle = 0 \text{ if } m \neq n, \tag{29}$$

where the averages, noted $\langle \rangle$, correspond to integrals involving W_0, Equation (11).

A rigorous theory of detection would be complex and it will not be attempted here. I shall just make an extremely simple proposal, at the same time plausible and fitting with the common approach in the standard HS treatments. That is, I will assume that both the Alice and Bob single detection rates are proportional to the radiation intensity arriving at their detectors, respectively, with the ZPF (vacuum) intensity subtracted. That intensity is just what would arrive at the detectors if the pumping laser was switched off, and the subtraction amounts to putting D = 0 in Equation (26). Modulo a proportionality constant (assumed equal for Alice and Bob), we obtain for the Alice detection rate

$$P_A = \langle | A |^2 \rangle - \langle | A_0 |^2 \rangle = D^2 \langle | A_1 |^2 \rangle + 2D \langle \mathrm{Re}(A_0 A_1^*) \rangle. \tag{30}$$

In the following, I will interpret P_A Equation (30) as the probability of a detection event within a given time window short enough so that more than one detection event is unlikely within the window. Thus, the detection rate will be the product of P_A times the rate of the time windows.

The second term of Equation (30) does not contribute because it is a linear combination of the averages $\langle a^2 \rangle = \langle a^{*2} \rangle = 0$ (see Equation (29)), whence we obtain

$$\begin{aligned} P_A &= D^2 \langle | A_1 |^2 \rangle = D^2 \langle |(a_i \cos\theta - i a_s \sin\theta)(a_i^* \cos\theta + i a_s^* \sin\theta)| \rangle \\ &= D^2 \Big(\langle |a_i| \rangle \cos^2\theta + \langle |a_s| \rangle \sin^2\theta \Big) = \frac{1}{2} |D|^2, \end{aligned} \tag{31}$$

A similar result may be obtained for the single detection rate of Bob, that is

$$P_A = P_B = \frac{1}{2} |D|^2. \tag{32}$$

The coincidence detection rate should obviously depend on the correlation between the signals arriving at Alice and Bob, respectively. However, it is not trivial to obtain a quantitative proposal from this qualitative idea, because some ambiguity arises. We might assume that the coincidence detection probability depends on the intensity correlation with the ZPF subtracted, but this would ignore the phase correlations, that in this case are relevant. Thus, I propose that the coincidence probability in a given detection window, P_{AB}, will depend on the correlation of the amplitudes A and B, Equation (28), arriving at Alice and Bob, respectively, as follows:

$$P_{AB} = |\langle AB \rangle|^2, \tag{33}$$

which is consistent with the single detection Equation (30). Here, the vacuum subtraction is not needed because the detection probability with the pumping laser switched off gives no contribution. In fact, from Equation (29), we obtain,

$$\langle A_0 B_0 \rangle = 0 \Rightarrow |\langle A_0 B_0 \rangle|^2 = 0.$$

Taking Equation (28) into account, we obtain

$$\begin{aligned} \langle AB \rangle &= \langle (A_0 + D A_1)(B_0 + D B_1) \rangle \\ &= \Big\langle A_0 B_0 \rangle + D \langle A_0 B_1 \rangle + D \langle A_1 B_0 \rangle + D^2 \langle A_1 B_1 \Big\rangle. \end{aligned} \tag{34}$$

The first and last terms are nil according to Equation (29). Thus, we tentatively obtain

$$P_{AB} = |\langle AB \rangle|^2 = D^2 |\langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle \exp(i\chi)|^2, \tag{35}$$

where in Equation (35), I have restored the phases which were ignored in Equation (34) (see comment below Equation (26)). Indeed, it is plausible that the phases of the two terms of Equation (35) are different and both may be assumed to be random variables with a homogeneous distribution in $[0, 2\pi]$ each. Thus, averaging over the relative phase χ in Equation (35), we obtain

$$P_{AB} = D^2 (|\langle A_0 B_1 \rangle|^2 + |\langle A_0 B_1 \rangle|^2).$$

These averages may be obtained taking Equation (28) into account and we obtain

$$\begin{aligned} \langle A_0 B_1 \rangle &= \Big\langle |a_s|^2 \cos\theta \cos\phi + |a_i|^2 \sin\theta \sin\phi \Big\rangle = \frac{1}{2} \cos(\theta - \phi), \\ \langle A_1 B_0 \rangle &= \Big\langle |a_i|^2 \cos\theta \cos\phi + |a_s|^2 \sin\theta \sin\phi \Big\rangle = \frac{1}{2} \cos(\theta - \phi). \end{aligned}$$

whence the coincidence probability will be

$$P_{AB} = \frac{1}{2} D^2 \cos^2(\theta - \phi). \tag{36}$$

The results Equations (32) and (36) violate a Bell inequality [19]. The result is arrived at via classical EM and ZPF, and then spooky action at a distance is not required to violate the Bell inequalities in this model.

5. Discussion

I have shown, in Section 3 of this article, that the classical (Maxwell) electromagnetic field may be quantized just assuming the existence of a random radiation filling space, the "zeropoint field" (ZPF). The ZPF is quantitatively characterized by the distribution Equation (13) of energies amongst the normal modes of the field. The formalism obtained by this method may be labeled the "Weyl–Wigner representation" (WW) of the quantum EM field. For a complete specification and understanding of WW, we must recall the following answers to three relevant questions.

The first question is whether WW is equivalent to the standard Hilbert space (HS) quantum theory of the EM field. The answer is affirmative provided that we assume that the ZPF contributes nil energy to the free field (i.e., the field without interactions). Thus, the ZPF represents the "vacuum state". The proof of the assertion is that there is a reversible Weyl transform that allows passing from HS to WW and from WW to HS, as studied in Section 2.

The second question is whether WW may be interpreted as local realistic, like classical theories. The answer might also be affirmative provided the following two points are clarified. Firstly, we must explain the paradoxical result that the strong ZPF possesses nil energy, that I have made in Section 3. Secondly, we must make a change in the formalism of Section 2, assuming now that the set of states in WW corresponds to *positive* functions of the amplitudes, so that these functions may be interpreted as probability distributions. In Section 3, I have argued that this change from the states in HS (mixtures of Fock states) to WW states, likely does not disturb the agreement of WW with experiments. Therefore, even if HS and WW are not fully equivalent, both allow a fair interpretation of the experiments, and WW has the advantage that it permits a realistic local understanding.

The third question is whether the WW quantization of the EM field may be extended to other fields, e.g., the Dirac electron-positron field. My answer is that I do not know, but I believe that the answer is affirmative for Bose fields. Certainly the absence of a generalization to other fields is a big difficulty that requires further research work. But at least the field in WW may be coupled to macroscopic bodies maintaining the possibility of a local realistic interpretation. This allows the interpretation of relevant experiments, as shown in Section 4. Namely, the crucial ("loophole free") tests of Bell inequalities [15–17], which proves that they may be interpreted maintaining locality. Those experiments had been taken as the death of local realism [20–22]. Further discussion about the results of Section 4 will be provided below.

The WW formalism for the EM field is a translation of the old Wigner representation for the quantum mechanics of particles (QM) in terms of functions in phase space. I believe that such translation has not been worked in detail for fields due to the big obstacle that the Wigner representation in QM does not admit a realistic interpretation because the Wigner functions are not positive in general. I argue that the case for the EM field is quite different. Indeed, as stated above, the formalism of Section 3 admits a local realistic interpretation in terms of fields without the need of particles (photons), which have a rather counterintuitive behavior, as commented in Section 2.

An interesting question is why the Wigner representation of QM (for systems of particles) does not admit a realistic interpretation. I believe that the reason is that quantum particles are dramatically different from classical particles. The latter may be treated as lacking structure, so that positions $\{\mathbf{r}_j\}$ and momenta $\{\mathbf{p}_j\}$ determine the state of a system of particles at a given time. They determine both the past and the future given the interactions, which exemplifies the celebrated Laplacian determinism of classical mechanics. Hence, the path of the particles is a line in phase space parametrized by time, which explains the relevance of phase space in classical mechanics. In contrast, quantum particles are complex structures dressed with many fields, in particular, the ZPF's. Hence, the particles' motion is governed by both the external forces and the dressing fields. Thus, an interesting research question is to attempt deriving the laws of motion of the particles from the properties of the fields plus the given forces. In the case of charged particles, it is plausible that the main contribution comes from the EM field, in particular, the ZPF. Indeed, it is the case that a line of research on the subject has been alive from long ago, with the name of *stochastic electrodynamics* [11,23–27]. This is a theory that attempts explaining phenomena taken as purely quantal, from the action on charged particles of an assumed EM random radiation field. Within this line, a recent article presents a derivation of the canonical commutation rules [28].

As a final part of this section, I shall discuss whether the analysis of Section 4 does provide a local realistic interpretation of the tests of Bell inequalities. Most of the development is a straightforward application of the WW formalism that fits in local realism as exposed in Section 3, but the photodetection involves extra assumptions that require discussion. Let us start with the single detection. The assumption that the probability of detection, in a one time window, is proportional to the arriving intensity is plausible and follows the usual practice. However, the subtraction of the ZPF suggests the following rhetorical question: How the Alice detector knows that it should act according to the field intensity $|A_1|^2$ when the actual arriving intensity is $|A|^2$? (see Equation (30) for the notation). Or alternatively, how the detector knows what intensity would arrive when the pumping laser was switched off in order to subtract it? My answer is as follows. We must assume that the intensity arriving at any point of space when there is no external action (i.e., for a free EM field) is always given by Equation (13), in particular, the radiation arriving at the said point is isotropic. Then, it is plausible to assume that only deviations from isotropy, due to signals, could excite a detector. It is similar to the effect of air on bodies that produces relevant consequences when the pressure of wind on the body is anisotropic. The intensity, $|A_0|^2$, that would arrive at Alice coming from the source (the nonlinear crystal) when the laser is switched off, is a part of the ZPF. Therefore, it would be balanced by a similar intensity coming from all other directions. In sharp contrast, $|A_1|^2$ is just the clean intensity arriving at the detector, which was originated by the action of the laser and superposed to the ZPF.

With respect to the coincidence detection, the choice Equation (33) is plausible because only the part of the radiation arriving at Alice which is correlated with the radiation arriving at Bob should contribute to the coincidence detection. And similarly for the radiation arriving to Bob.

In spite of these arguments, the possibility of a local realistic interpretation of the (claimed loophole-free) violations of Bell inequalities is not obvious. In fact, our study in Section 4 involves hypotheses that, although assumed plausible, might be flawed. There is also a contradiction which should be explained. The derivation of Bell inequalities with reference to the past light cone of the detection events [14] seems to be valid in the case analyzed in Section 4. Even if we assume that the ZPF is real, the field arriving at the detectors can be influenced by events in the past light cone and any possible action of the measurement of Alice (Bob) of the detection by Bob (Alice) is carefully excluded in the performed experiments. This contradiction between our results Equations (32) and (36) and the quite general derivation of the Bell inequalities [14] is worth study. I believe that a hint for the solution is the fact that detection of electromagnetic waves cannot be assumed as *an event*. It is a process with a duration much larger than the periods of the involved fields. This question will not be studied further in the present paper, but it certainly deserves future research.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Milonni, P.W. *The Quantum Vacuum. An Introduction to Quantum Electrodynamics*; Academic Press: San Diego, CA, USA, 1994.
2. Weyl, H. Quantenmechanik und gruppentheorie. *Z. Phys.* **1927**, *46*, 1.
3. Wigner, E.P. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.* **1932**, *40*, 749. [CrossRef]
4. Hillery, M.; O'Connell, R.F.; Scully, M.O.; Wigner, P.E. Distribution functions in physics. Fundamentals. *Phys. Rep.* **1984**, *106*, 121–168. [CrossRef]
5. Zachos, C.K.; Fairlie, D.B.; Curtright, T.L. *Quantum Mechanics in Phase Space*; World Scientic: Singapore, 2005.
6. Soto, F.; Claverie, P. Some properties of the smoothed Wigner function. *J. Math. Phys.* **1983**, *24*, 97. [CrossRef]
7. Einstein, A.; Podolski, B.; Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **1935**, *47*, 777–780. [CrossRef]
8. Santos, E. Local realistic interpretation of entangled photon pairs in the Weyl-Wigner formalism. *Front. Phys.* **2020**, *8*, 191. [CrossRef]
9. Santos, E. Local model of entangled photon experiments compatible with quantum predictions based on the reality of the vacuum fields. *Found. Phys.* **2020**, *50*, 1587–1607. [CrossRef]
10. Santos, E. On the analogy between stochastic electrodynamics and nonrelativistic quantum electrodynamics. *Eur. Phys. J. Plus* **2020**, *137*, 1302. [CrossRef]
11. Santos, E. *Realistic Interpretation of Quantum Mechanics*; Cambridge Scholars Publishing: Newcastle upon Tyne, UK, 2022.
12. Santos, E. Stochastic interpretation of quantum mechanics assuming that vacuum fields are real. *Foundations* **2022**, *2*, 409–442. [CrossRef]
13. Sakurai, J.J. *Advanced Quantum Mechanics*; Addison Wesley: Boston, MA, USA, 1968.
14. Bell, J.S. *Speakable and Unspeakable in Quantum Mechanics*; Cambridge University: Cambridge, UK, 1987; This Book Reproduces Most of Bell Articles on Foundations.
15. Shalm, L.K.; Meyer-Scott, E.; Christensen, B.G.; Bierhorst, P.; Wayne, M.A.; Stevens, M.J.; Gerrits, T.; Glancy, S.; Hamel, D.R.; Allman, M.S.; et al. Strong Loophole-Free Test of Local Realism. *Phys. Rev. Lett.* **2015**, *115*, 250402. [CrossRef] [PubMed]
16. Giustina, M.; Versteegh, M.A.; Wengerowsky, S.; Handsteiner, J.; Hochrainer, A.; Phelan, K.; Steinlechner, F.; Kofler, J.; Larsson, J.Å.; Abellán, C.; et al. Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons. *Phys. Rev. Lett.* **2015**, *115*, 250401. [CrossRef] [PubMed]
17. The BIG Bell Test Collaboration. Challenging local realism with human choices. *Nature* **2018**, *557*, 212216.
18. Dechoum, K.; Marshall, T.W.; Santos, E. Parametric up and dawn conversion in the Wigner representation of quantum optics. *J. Mod. Optics* **2000**, *47*, 1273–1287. [CrossRef]
19. Clauser, J.F.; Horne, M.A. Experimental consequences of objective local theories. *Phys. Rev. D* **1974**, *10*, 526. [CrossRef]
20. Wiseman, H. Death by experiment for local realism. *Nature* **2015**, *526*, 687. [CrossRef] [PubMed]
21. Aspect, A. Closing the door on Einstein and Bohr's quantum debate. *Physics* **2015**, *8*, 123. [CrossRef]
22. Freire, O. (Ed.) *The Oxford Handbook of the History of Quantum Interpretations*; Oxford U.P.: Oxford, UK, 2022.
23. Braffort, P.; Tzara, C. Quelques consequences de la theorie de laction a distance en electrodynamique classique. *Comptes Rendu Acad. Sci. Paris* **1954**, *239*, 1775.
24. Marshall, T.W. Random electrodynamics. *Proc. R. Soc. A* **1963**, *276*, 475.
25. de la Peña, L.; Cetto, A.M. *The Quantum Dice. An Introduction to Stochastic Electrodynmics*; Kluwer Academic Publishers: Dordrecht, The Netherland, 1996.
26. de la Peña, L.; Cetto, A.M.; Hernández, A.V. *The Emerging Quantum. The Physics Beyond Quantum Mechanics*; Springer: Heidelberg, Germany, 2015.
27. Boyer, T.H. Stochastic electrodynamics: The closest classical approximation to quantum theory. *Atoms* **2019**, *7*, 29. [CrossRef]
28. Cetto, A.M.; de la Peña, L. The radiation field, at the origin of the quantum canonical operators. *Found. Phys.* **2024**, *54*, 51. [CrossRef]