The multiassociahedron and p-adic symplectic geometry

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Tesis doctoral

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Contents

Resumen

La tesis consta de dos partes diferenciadas. La primera parte, formada por los capítulos 1, 2, 3 y 4, continúa la línea de investigación del TFM sobre realización del multiasociaedro.

Sean $k, n \in \mathbb{N}$ con $n > 2k$. Un *k-cruce* es un conjunto de k diagonales del n-gono convexo que se cruzan. Una *k-triangulación* es un subconjunto maximal de $\binom{[n]}{2}$ sin $(k+1)$-cruces. El *k-asociaedro* o *multiasociaedro* $\Delta_k(n)$ es el complejo simplicial de los subconjuntos de $\binom{[n]}{2}$ sin $(k+1)$-cruces, y cuyas facetas son las k-triangulaciones; ignorando las aristas de longitud $\leq k$, que no pueden formar parte de un $(k+1)$-cruce, se obtiene el complejo reducido $\overline{\Delta}_k(n)$.

Jonsson [70] demostró que $\overline{\Delta}_k(n)$ es una esfera shellable y conjeturó que era politopal. Esta conjetura se ha demostrado en los casos $n = 2k + 1$ (el politopo es un punto), $n = 2k + 2$ (es un símplice), $n = 2k + 3$ (es un politopo cíclico) y $(k, n) = (2, 8)$ [11]. En el capítulo 2 demostramos esta conjetura para $(k, n) \in \{(2, 9), (2, 10), (3, 10)\}$. La demostración usa técnicas de teoría de la rigidez. Concretamente, se demuestra que estas esferas son politopales tomando como posiciones de los puntos las filas de una matriz de rigidez de cofactores.

La matriz de rigidez de cofactores de n puntos en el plano, $\mathbf{q} = (q_1, \ldots, q_n) \subset \mathbb{R}^2$, es una matriz con una fila por cada par de puntos, y d columnas por cada punto, donde d es un parámetro. La forma exacta está en la ecuación (4). Para $d = 2k$, esta matriz tiene el mismo rango que la cardinalidad de una k-triangulación y, por lo tanto, sus filas pueden usarse como coordenadas de los vértices de $\Delta_k(n)$. Que esta matriz realice o no el complejo reducido $\overline{\Delta}_k(n)$ como un politopo depende de ciertas condiciones de signos y de la factibilidad de un programa lineal (véase sección 2.2.2).

Los resultados principales son los siguientes:

- Para $(k, n) \in \{(2, 9), (2, 10), (3, 10)\}$, $\overline{\Delta}_k(n)$ es una esfera politopal, y $\overline{\Delta}_4(13)$ se puede realizar como abanico simplicial completo (Teorema 2.1).
- Las 2-triangulaciones son isostáticas con $d = 4$ para posiciones genéricas en la cónica (Teorema 2.3).
- Si lo anterior se cumple para cualquier posición en la cónica, entonces todas esas posiciones realizan el multiasociaedro como un abanico (Teorema 2.5).
- Hay 3-triangulaciones en 9 puntos que no son isostáticas para $d = 6$ en posición arbitraria (Teorema 2.6).
- Ninguna posición convexa realiza el multiasociaedro como un abanico para $k \geq 3$ y $n \geq 2k + 6$ (Teorema 2.7).

El capítulo 3 explora relaciones entre el multiasociaedro y la variedad $\mathcal{P}f_k(n)$ formada por las matrices antisimétricas de rango menor o igual que $2k$. Esta variedad está generada por los polinomios llamados *pfaffianos*: hay uno por cada subconjunto de tamaño $2k+2$ de $[n]$ y tiene un término por cada emparejamiento perfecto de dicho subconjunto. Los vectores de pesos *fp-positivos*, es decir, aquellos que dan peso máximo al emparejamiento que forma un $(k+1)$-cruce en cada pfaffiano, hacen que estos polinomios sean base de Gröbner del ideal que generan, y el cono de Gröbner $\mathrm{Grob}_k(n)$ de los $(k+1)$ cruces contiene (estrictamente si $k \geq 2$) al cono de los vectores fp-positivos. Esto implica que las k-triangulaciones son bases de la matroide algebraica de $\mathcal{P}f_k(n)$ (Corolario 3.20), lo que a su vez tiene varias implicaciones:

- Por un lado, relaciona el hecho de no tener $(k+1)$-cruces, o de contener una k-triangulación, con el problema de completación de matrices de rango menor o igual que $2k$ [8, 77]: véase teorema 3.3.
- Por otro lado, esta matroide algebraica resulta ser la matroide de hiperconectividad genérica, de donde se deduce que las k-triangulaciones son isostáticas en dicha matroide de hiperconectividad en dimensión $2k$: véase corolario 3.22. La hiperconectividad es otra matroide de rigidez, definida por Kalai [74].

Después dirigimos la atención a la tropicalización de $\mathcal{P}f_k(n)$. La operación de tropicalización consiste en sustituir, en un polinomio, las sumas por máximos, los productos por sumas, y eliminar los coeficientes. Si se realiza esta operación con los pfaffianos, los puntos para los que el máximo se alcanza dos veces en cada pfaffiano forman la prevariedad tropical $\mathrm{Pf}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$. Este conjunto contiene la variedad tropical $\mathrm{trop}(\mathcal{P}f_k(n))$, que se obtiene tropicalizando todos los polinomios del ideal generado por los pfaffianos (en lugar de solo los generadores), pero en general no coincide (Teorema 3.29).

La parte de $\mathrm{Pf}_k(n)$ de mayor interés para nosotros es la contenida dentro del cono $\mathrm{Grob}_k(n)$, a la que llamamos $\mathrm{Pf}_k^+(n)$. Nuestros resultados al respecto son:

- $\mathrm{Pf}_k^+(n) = \mathrm{Grob}_k(n) \cap \mathrm{trop}(\mathcal{P}f_k(n)) \subset \mathrm{trop}^+(\mathcal{P}f_k(n))$ (Teorema 3.4).
- $\mathrm{Pf}_k^+(n)$ es isomorfo al multiasociaedro como abanico (Corolario 3.5).

El capítulo termina conectando el último resultado con la realización del asociaedro con **g**-vectores [66].

Finalmente, el capítulo 4 trata los mismos temas que los capítulos anteriores, pero desde una nueva perspectiva: se introduce una operación de *bipartización* que permite convertir las multitriangulaciones en grafos bipartitos. Estos grafos pueden tratarse con otra forma de rigidez, que vuelve a ser la hiperconectividad pero restringida a grafos bipartitos [75], y la dimensión es k en lugar de $2k$. Los resultados principales son:

- Las bipartizaciones de k-triangulaciones son bases de la matroide de hiperconectividad genérica en dimensión k (Teorema 4.2).
- Si la bipartización de un grafo E es independiente en la matroide de hiperconectividad genérica en dimensión k, entonces E es independiente en la matroide de hiperconectividad genérica en dimensión $2k$ (Teorema 4.3).

- Conocemos posiciones explícitas para $n = 2k + 2$ y $n = 2k + 3$ para las que las filas de la matriz de hiperconectividad bipartita realizan el multiasociaedro como un abanico completo (Teoremas 4.6 y 4.7).
- Hay 3-triangulaciones en 9 puntos cuyas bipartizaciones no son isostáticas para $d = 3$ en posición arbitraria (Teorema 4.4).
- Si $k = 3$ y $n \geq 12$, o si $k \geq 4$ y $n \geq 2k+4$, ninguna posición de puntos en la curva de momentos realiza el multiasociaedro como un abanico (Teorema 4.8).

También se aborda el problema desde un punto de vista algebraico: la matroide de hiperconectividad bipartita es también una matroide algebraica, cuyo ideal resulta ser un ideal inicial del ideal de los pfaffianos. Esto permite reutilizar la maquinaria introducida en el capítulo anterior.

La segunda parte de la tesis, también publicada como [27, versión 1] (salvo reordenamiento de algunos contenidos y cambios expositorios para hacer la tesis más accesible), consta de los capítulos 5, 6, 7, 8 y 9, y está dedicada a encontrar un equivalente p-ádico de la famosa clasificación de Weierstrass-Williamson de matrices y luego aplicar dicha clasificacion para determinar todos los modelos lineales posibles de las singularidades de un sistema integrable p-ádico en una variedad analítica simpléctica de dimensión 4.

En 1858 Weierstrass [136] demostró que toda matriz definida positiva puede ser diagonalizada por una matriz simpléctica, esto es, una matriz S tal que $S^T\Omega_0 S = \Omega_0$, donde Ω_0 es la matriz diagonal por bloques con todos los bloques iguales a

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Este resultado fue generalizado en 1936 por Williamson [142], quien dio un conjunto de formas normales de matrices tales que para cualquier matriz M existe una forma normal N y una matriz simpléctica S tales que $S^T M S = N$. Si bien no todas las formas normales de Williamson son explícitas, sí lo son en el caso particular en el que los valores propios de $\Omega_0^{-1}M$ son diferentes.

La clasificación de Weierstrass-Williamson puede usarse para clasificar puntos críticos de sistemas integrables. Concretamente, dada una variedad simpléctica (M, ω), un sistema integrable $F : (M, \omega) \to \mathbb{R}^n$ y un punto crítico m de F, siempre existen coordenadas simplécticas lineales $(x_1, \xi_1, \ldots, x_n, \xi_n)$ con origen en m tales que

$$F - F(m) = B \circ (g_1, \ldots, g_n) + \mathcal{O}(3),$$

donde las g_i pueden ser componentes *elípticas*, *hiperbólicas*, *foco-foco* o *regulares*.

Después de deducir algunos resultados auxiliares en el capítulo 6, en el capítulo 7 recuperamos la clasificación de Weierstrass-Williamson usando una nueva estrategia, consistente en resolver primero el problema en los complejos, donde es más fácil al ser algebraicamente cerrado, y luego “bajar” a los reales.

En el capítulo 8 se usa la misma estrategia para deducir el equivalente de esta clasificación en el caso donde el cuerpo $\mathbb{R}$ se sustituye por el cuerpo de los p-ádicos $\mathbb{Q}_p$, donde p es un número primo. El cuerpo $\mathbb{R}$ se puede construir como una completación de $\mathbb{Q}$ con el valor absoluto usual $|x| = \max\{x, -x\}$. El cuerpo $\mathbb{Q}_p$ es una completación de $\mathbb{Q}$ con un valor absoluto distinto, definido a partir del orden de p en x. En este capítulo se deduce la clasificación de Weierstrass-Williamson en el caso p-ádico para dimensiones 2 y 4, y en el capítulo 9 se aplica esta clasificación a sistemas integrables p-ádicos, con especial mención a la versión p-ádica del sistema de Jaynes-Cummings estudiada en [26].

Hay importantes diferencias entre la clasificación real de Weierstrass-Williamson y su equivalente p-ádico:

- En dimensión 2, en el caso real hay solo dos familias infinitas de formas normales con un grado de libertad (de la forma rM donde $r \in R$ y M es una matriz fija) y dos formas normales aisladas. En el caso p-ádico hay 5, 7 u 11 familias infinitas con un grado de libertad y 4 u 8 formas normales aisladas, dependiendo de p (Teorema 5.30).
- Como las familias infinitas corresponden a formas normales de puntos críticos no degenerados de sistemas integrables, el número de estas formas normales en el caso p-ádico también es de 5, 7 u 11 dependiendo de p (Corolario 9.6) frente a solo 2 en el caso real.
- En dimensión 4, en el caso real hay 4 familias infinitas de formas normales con dos grados de libertad (de la forma $r_1M_1 + r_2M_2$ donde $r_1, r_2 \in \mathbb{R}$ y M_1 y M_2 son matrices fijas), 7 familias con un grado de libertad y 5 formas normales aisladas. En el caso p-ádico hay 32, 49 o 211 familias con dos grados de libertad, 27, 35 o 103 con un grado de libertad, y 20 o 72 formas aisladas (Teorema 5.34).
- Una vez más, las familias infinitas con dos grados de libertad corresponden a formas normales de puntos críticos no degenerados de rango 0 de sistemas integrables, luego el número de estas formas normales en el caso p-ádico es de 32, 49 o 211 dependiendo de p (Teorema 5.22), frente a solo 4 en el caso real. El número de formas para puntos de rango 1 es de 5, 7 u 11, frente a 2 en el caso real.
- En dimensión arbitraria, en el caso real no degenerado, el número de familias de formas normales de matrices, y por lo tanto el de formas normales de sistemas integrables, es cuadrático en la dimensión. En el caso p-ádico, este número es casi exponencial en la dimensión (Teoremas 5.26 y 5.37).

Como se ve, la geometría simpléctica p-ádica resulta ser mucho más rica que la real: muchos de los fenómenos que aparecen en estos casos no tienen equivalente real.

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Part 1

Multitriangulations and the multiassociahedron

CHAPTER 1

Preliminaries

1.1. Multitriangulations

The starting point for this part is my master thesis [**24**], whose second part was about multitriangulations. The results in this part have been published in [**25, 29, 30**]. Let us recall the definition of a multitriangulation.

Triangulations of the convex n-gon P $(n > 2)$, regarded as sets of edges, are the facets of an abstract simplicial complex with vertex set

$$\binom{[n]}{2} := \{\{i,j\} : i,j \in [n], i < j\}$$

and defined by taking as simplices all the non-crossing sets of edges. This simplicial complex, ignoring the boundary edges $\{i, i+1\}$, is a polytopal sphere of dimension $n-4$ dual to the *associahedron*. (Here and all throughout the thesis, indices for vertices of the n-gon are regarded modulo n).

A similar complex can be defined if, instead of forbidding pairwise crossings, we forbid crossings of more than a certain number of edges. More precisely:

Definition 1.1 ([**24**, Definición 2.1.1]). Two disjoint elements $\{i,j\},\{k,l\} \in \binom{[n]}{2}$, with $i < j$ and $k < l$, of $\binom{[n]}{2}$ *cross* if $i < k < j < l$ or $k < i < l < j$. That is, if they cross when seen as diagonals of a cyclically labeled convex n-gon.

A *k-crossing* is a subset of k elements of $\binom{[n]}{2}$ such that every pair cross. A subset of $\binom{[n]}{2}$ is $(k+1)$*-crossing-free* if it doesn't contain any $(k+1)$-crossing. A *k-triangulation* is a maximal $(k+1)$-crossing-free set.

Observe that whether two pairs $\{i,j\},\{k,l\} \in \binom{[n]}{2}$ cross is a purely combinatorial concept, but it captures the idea that the corresponding diagonals of a convex n-gon geometrically cross.

The *length* of an edge $\{i,j\} \in \binom{[n]}{2}$, is $\min\{|j-i|, n-|j-i|\}$. That is, the distance from i to j measured cyclically in $[n]$. Edges of length at most k cannot participate in any $k+1$-crossing and, hence, all of them lie in every k-triangulation. We call edges of length at least $k+1$ *relevant* and those of length at most $k-1$ *irrelevant*. The "almost relevant" edges, those of length k, are called *boundary edges* and, although they lie in all k-triangulations, they still play an important role in the theory (see Proposition 2.13(3)).

By definition, $(k+1)$-crossing-free subsets form an abstract simplicial complex on the vertex set $\binom{[n]}{2}$, whose facets are the k-triangulations and whose minimal non-faces are the $(k+1)$-crossings. We denote this complex $\Delta_k(n)$. Since the kn irrelevant and boundary edges lie in every facet, it makes sense to consider also the *reduced complex* $\overline{\Delta}_k(n)$. Technically speaking, we have that $\Delta_k(n)$ is the join of $\overline{\Delta}_k(n)$ with the irrelevant face (the face consisting of irrelevant and boundary edges). We call $\overline{\Delta}_k(n)$ the *multiassociahedron* or k-associahedron. See Section 2.2.1 for more precise definitions, and [105, 106, 128] for additional information.

Multitriangulations were studied (under a different name) by Capoyleas and Pach [16], who showed that no $(k+1)$-crossing-free subset has more than $k(2n-2k-1)$ edges. Then it was proved in [88, 42] that every k-triangulation of the n-gon has exactly $k(2n-2k-1)$ edges. That is, $\Delta_k(n)$ is pure of dimension $k(2n-2k-1)-1$, hence $\overline{\Delta}_k(n)$ has dimension $k(n-2k-1)-1$. The main result about $\overline{\Delta}_k(n)$ for our purposes is the following theorem of Jonsson, also a particular case of a theorem of Knutson and Miller:

THEOREM 1.2 (Jonsson [70], Knutson-Miller [78]). *$\overline{\Delta}_k(n)$ is a vertex-decomposable (hence shellable) sphere of dimension $k(n-2k-1)-1$.*

Remember that all polytopal spheres are shellable, so shellability can be considered evidence in favor of polytopality. Vertex-decomposability is a stronger notion introduced by Provan and Billera [110] implying, for example, that the diameters of these spheres satisfy the Hirsch bound.

CONJECTURE 1.3 (Jonsson). *For every $n \geq 2k+1$ the complex $\overline{\Delta}_k(n)$ is a polytopal sphere. That is, there is a simplicial polytope of dimension $k(n-2k-1)-1$ and with $\binom{n}{2}-kn$ vertices whose lattice of proper faces is isomorphic to $\overline{\Delta}_k(n)$.*

The first appearance of this statement, as a question rather than a conjecture, is the 2003 preprint [70]. The conjecture then appeared explicitly in Jonsson's handwritten abstract after his talk in an Oberwolfach Workshop the same year [91, 71] (but it did not appear in the shorter abstract published in the Oberwolfach Reports). It was also included in the unpublished manuscript by Dress, Grünewald, Jonsson, and Moulton [41], before appearing in papers by other authors [106, 128].

REMARK 1.4. The question of polytopality of $\overline{\Delta}_k(n)$ is quite natural, since it generalizes the associahedron (the case $k=1$) which admits many different constructions as a polytope [20, 109]. One would expect that, as happens in the case of the associahedron, having explicit polytopal constructions of $\overline{\Delta}_k(n)$ would uncover interesting combinatorics. If, in the contrary, it turns out that $\overline{\Delta}_k(n)$ is not always polytopal, it would also be interesting to know it; it would probably be the first family of shellable spheres naturally arising from a combinatorial problem and that are proven not to be polytopal.

Interest in this question comes also from cluster algebras and Coxeter combinatorics. Let $w \in W$ be an element in a Coxeter group W and let Q be a word of a certain length N. Assume that Q contains as a subword a reduced expression for w. The *subword complex* of Q and w is the simplicial complex with vertex set $[N]$ and with faces the subsets of positions that can be deleted from Q and still contain a reduced expression for w. Knutson and Miller [78, Theorem 3.7 and Question 6.4] proved that every subword complex is either a vertex-decomposable (hence shellable) ball or sphere, and they asked whether all spherical subword complexes are polytopal. It was later proved by Stump [128, Theorem 2.1] that $\overline{\Delta}_k(n)$ is a spherical subword complex for the Coxeter system of type A_{n-2k-1} and, moreover, it is *universal*: every other spherical subword complex of type A appears as a link in some $\overline{\Delta}_k(n)$ [107, Proposition 5.6]. In particular, Conjecture 1.3 is equivalent to a positive answer (in type A) to the question of Knutson and Miller.

Versions of k-associahedra for the rest of finite Coxeter groups exist, with the same implications [19].

Conjecture 1.3 is easy to prove for $n \le 2k+3$. $\overline{\Delta}_k(2k+1)$ is indeed a -1-sphere (the complex whose only face is the empty set). $\overline{\Delta}_k(2k+2)$ is the face poset of a $(k-1)$-simplex, and $\overline{\Delta}_k(2k+3)$ is (the polar of) the cyclic polytope of dimension $2k-1$ with n vertices (Lemma 8.7 in [**106**]). The only additional case for which Jonsson's conjecture is known to hold is $k = 2$ and $n = 8$ [**11**]. In some additional cases $\overline{\Delta}_k(n)$ has been realized as a complete simplicial fan, but it is open whether this fan is polytopal. This includes the cases $n \le 2k+4$ [**7**], the cases $k = 2$ and $n \le 13$ [**84**] and the cases $k = 3$ and $n \le 11$ [**7**].

1.2. Rigidity

Several definitions and results here follow [**24**, section 1.2.1].

Let $\mathbf{p} = (p_1, \dots, p_n)$ be a configuration of n points in $\mathbb{R}^d$, labelled by $[n]$.[1] Their *bar-and-joint rigidity matrix* is the following $\binom{n}{2} \times nd$ matrix:

$$R(\mathbf{p}) := \begin{pmatrix} p_1 - p_2 & p_2 - p_1 & 0 & \dots & 0 & 0 \\ p_1 - p_3 & 0 & p_3 - p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_1 - p_n & 0 & 0 & \dots & 0 & p_n - p_1 \\ 0 & p_2 - p_3 & p_3 - p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_{n-1} - p_n & p_n - p_{n-1} \end{pmatrix}. \tag{1}$$

Since there is a row of the matrix for each pair $\{i, j\} \in \binom{[n]}{2}$, rows can be considered labeled by edges in the complete graph K_n. The matrix is a sort of "directed incidence matrix" of K_n, except instead of having one column for each vertex $i \in [n]$ we have a block of d columns, and instead of putting a single $+1$ and -1 in the row of edge $\{i, j\}$ we put the d-dimensional (row) vectors $p_i - p_j$ and $p_j - p_i$.

An important property of $R(\mathbf{p})$ (Lemma 11.1.3 in [**141**], see also [**24**, Teorema 1.2.5]) is that if the points $\mathbf{p}$ affinely span $\mathbb{R}^d$ then the rank of $R(\mathbf{p})$ equals

$$\begin{cases} \binom{n}{2} & \text{if } n \le d+1, \\ dn - \binom{d+1}{2} & \text{if } n \ge d. \end{cases} \tag{2}$$

(Observe that the two formulas give the same result for $n \in \{d, d+1\}$.) If the points span an r-dimensional affine subspace, the same formulas hold with r substituted for d.

A pair $(G, \mathbf{p})$ where G is a graph on n vertices and $\mathbf{p}$ is a set of n points in $\mathbb{R}^n$ (positions for the vertices) is usually called a framework. Seeing this framework as a bar-and-joint structure, the coefficients of any linear dependence among the rows of $R(\mathbf{p})$ can be interpreted as forces acting along the bars (edges) with the property that the resultant force on every vertex cancels out. These systems of forces are called *self-stresses* of *equilibrium stresses*. We will denote $Z(R(\mathbf{p}))$ the space of self-stresses of $\mathbf{p}$.

In the same manner, a linear dependence among the columns of $R(\mathbf{p})$ can be understood as an infinitesimal motion of the vertices (that is, an assignment of velocity vectors to the joints) that preserves the length of all bars. This is called an

[1] By a *configuration* we mean an ordered set of points or vectors, usually labelled by the first n positive integers. For this reason we write $\mathbf{p}$ as a vector rather than a set.

infinitesimal flex of the framework. We do not introduce a particular notation for flexes since our main interest is in the vector configuration, and matroid, of *rows* of $R(\mathbf{p})$. To this end, for any $E \subset \binom{[n]}{2}$ we denote by $R(\mathbf{p})|_E$ the restriction of $R(\mathbf{p})$ to the rows or elements indexed by E.

DEFINITION 1.5 (Rigidity). Let $E \subset \binom{[n]}{2}$ be a subset of edges of K_n (equivalently, of rows of $R(\mathbf{p})$). We say that E, or the corresponding subgraph of K_n, is *self-stress-free* or *independent* in the position $\mathbf{p}$ if the rows of $R(\mathbf{p})|_E$ are linearly independent, and *rigid* or *spanning* if they are linearly spanning (that is, if they have the same rank as the whole matrix $R(\mathbf{p})$).

Put differently, self-stress-free and rigid graphs are, respectively, the independent and spanning sets in the linear matroid of rows of $R(\mathbf{p})$. We call this matroid the *bar-and-joint rigidity matroid* of $\mathbf{p}$ and denote it $\mathcal{R}(\mathbf{p})$. It is a matroid with ground set $\binom{[n]}{2}$ and, for points affinely spanning $\mathbb{R}^d$, of rank given by Equation (2). See, for example, [**57, 141**] for more information on rigidity matrices and their matroids. Let us remark that, although rigidity theory usually deals only with $\mathcal{R}(\mathbf{p})$ as an (unoriented) matroid, its definition as the linear matroid of a configuration of real vectors produces in fact an *oriented matroid*. Orientations will be important for us in Section 2.4, in the light of the results of Section 2.2.2.

The following two matrices and matroids reminiscent of $R(\mathbf{p})$ are of interest:

- The *hyperconnectivity* matroid of the configuration $\mathbf{p} = (p_1, \dots, p_n)$ in $\mathbb{R}^d$, denoted $\mathcal{H}(\mathbf{p})$, is the matroid of rows of

$$H(\mathbf{p}) := \begin{pmatrix} p_2 & -p_1 & 0 & \dots & 0 & 0 \\ p_3 & 0 & -p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_n & 0 & 0 & \dots & 0 & -p_1 \\ 0 & p_3 & -p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_n & -p_{n-1} \end{pmatrix} \tag{3}$$

- For points $\mathbf{q} = (q_1, \dots, q_n)$ in $\mathbb{R}^2$ and a parameter $d \in \mathbb{N}$, the d-dimensional *cofactor rigidity*[2] matroid of the points $q_1, \dots, q_n$, which we denote $\mathcal{C}_d(\mathbf{q})$, is the matroid of rows of

$$C_d(\mathbf{q}) := \begin{pmatrix} \mathbf{c}_{12} & -\mathbf{c}_{12} & 0 & \dots & 0 & 0 \\ \mathbf{c}_{13} & 0 & -\mathbf{c}_{13} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{c}_{1n} & 0 & 0 & \dots & 0 & -\mathbf{c}_{1n} \\ 0 & \mathbf{c}_{23} & -\mathbf{c}_{23} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{c}_{n-1,n} & -\mathbf{c}_{n-1,n} \end{pmatrix}, \tag{4}$$

where the vector $\mathbf{c}_{ij} \in \mathbb{R}^d$ associated to $q_i = (x_i, y_i)$ and $q_j = (x_j, y_j)$ is

$$\mathbf{c}_{ij} := \left((x_i - x_j)^{d-1}, (y_i - y_j)(x_i - x_j)^{d-2}, \dots, (y_i - y_j)^{d-1}\right).$$

[2]This form of rigidity is usually called C^{d-2}_{d-1}-*cofactor rigidity*, since it is related to the existence of piecewise linear splines of degree $d-1$ and of type C^{d-2}. The definition we give here is not the same as in [**24**, section 1.3.1], but it is equivalent.

For $d = 1$ this is independent of the choice of $\mathbf{q}$ and equals the directed incidence matrix of K_n. For $d = 2$ we have $C_2(\mathbf{q}) = R(\mathbf{q})$.

The matroids $\mathcal{R}(\mathbf{p})$ and $\mathcal{C}_d(\mathbf{q})$ are invariant under affine transformation of the points, and $\mathcal{H}(\mathbf{p})$ under linear transformation. (More generally, although we do not need this, $\mathcal{R}(\mathbf{p})$ and $\mathcal{C}_d(\mathbf{q})$ are invariant under projective transformation in $\mathbb{RP}^d$ and $\mathbb{RP}^2$ as compactifications of $\mathbb{R}^d$ and $\mathbb{R}^2$, and $\mathcal{H}(\mathbf{p})$ under projective transformation in $\mathbb{RP}^{d-1}$ as a quotient of $\mathbb{R}^d \setminus \{0\}$). We say that the points chosen are in *general position* for $\mathcal{R}$ (respectively for $\mathcal{C}$ or for $\mathcal{H}$) if no $d+1$ of them lie in an affine hyperplane (respectively no three of them in an affine line or no d of them in a linear hyperplane). In the three cases, general position implies that the corresponding matroid has the rank stated in Equation (2) and that every copy of the graph K_{d+2} is a circuit. Nguyen [89] showed that the matroids on $\binom{[n]}{2}$ with these properties are exactly the *abstract rigidity matroids* introduced by Graver in [56].

Clearly, in the three cases and for each choice of the "dimension" d there is a unique most free matroid that can be obtained, in the sense that the independent sets in any other matroid will also be independent in this one, realized by sufficiently generic choices of the points. We call these the *generic bar-and-joint, hyperconnectivity, and cofactor matroids of dimension d on n points*, and denote them $\mathcal{R}_d(n), \mathcal{H}_d(n), \mathcal{C}_d(n)$. (Observe, however, that this generic matroid may stratify into several different generic oriented matroids; this is important for us since we will be concerned with the signs of circuits, by the results in Section 2.2.2).

In [28] we prove that these three rigidity theories coincide when the points $\mathbf{p}$ or $\mathbf{q}$ are chosen along the moment curve (for bar-and-joint and hyperconnectivity) and the parabola (for cofactor). More precisely:

THEOREM 1.6 ([28], [24, Teorema 1.3.11]). *Let $t_1 < \ldots < t_n \in \mathbb{R}$ be real parameters. Let*

$$p_i = (1, t_i, \ldots, t_i^{d-1}) \in \mathbb{R}^d, \quad p'_i = (t_i, t_i^2, \ldots, t_i^d) \in \mathbb{R}^d, \quad q_i = (t_i, t_i^2) \in \mathbb{R}^2.$$

Then, the three matrices $H(p_1, \ldots, p_n)$, $R(p'_1, \ldots, p'_n)$ and $C_d(q_1, \ldots, q_n)$ can be obtained from one another multiplying on the right by a regular matrix and then multiplying rows by some positive scalars.

In particular, the rows of the three matrices define the same oriented matroid.

PROOF. This follows from the proofs of Lemma 2.3 and Theorem 2.5 in [28]. Although the statements there speak only of the *matroids* of rows, the proofs show that dividing each row (i, j) of $R(p'_1, \ldots, p'_n)$ by $t_j - t_i$ and that of $C_d(q_1, \ldots, q_n)$ by $(t_j - t_i)^{d-1}$ one obtains matrices that are equivalent to $H(p_1, \ldots, p_n)$ under multiplication on the right by a regular matrix. □

DEFINITION 1.7 (Polynomial rigidity). We call the matrix $H(p_1, \ldots, p_n)$ in the statement of Theorem 1.6 the *polynomial d-rigidity matrix with parameters* $t_1, \ldots, t_n$. We denote it $P_d(t_1, \ldots, t_n)$, and denote $\mathcal{P}_d(t_1, \ldots, t_n)$ the corresponding matroid.

Among the polynomial rigidity matroids $\mathcal{P}_d(t_1, \ldots, t_n)$ there is again one that is the most free, obtained with a sufficiently generic choice of the t_i. We denote it $\mathcal{P}_d(n)$ and call it the *generic polynomial d-rigidity matroid on n points.* Theorem 1.6 implies that we can regard $\mathcal{P}_d(n)$ as capturing generic bar-and-joint rigidity along the moment curve, generic hyperconnectivity along the moment curve, or generic cofactor rigidity on a conic.

We do not know whether $\mathcal{P}_d(n)$ equals $\mathcal{H}_d(n)$, but we do know that $\mathcal{H}_d(n)$, $\mathcal{R}_d(n)$ and $\mathcal{C}_d(n)$ are different for $d \geq 4$ and n large enough. For example:

- $K_{d+1,d+1}$ is a circuit in $\mathcal{P}_d(n)$ and $\mathcal{H}_d(n)$, but independent in $\mathcal{R}_d(n)$ and $\mathcal{C}_d(n)$ for every $d \geq 2$ (More strongly, $K_{d+1,\binom{d+1}{2}}$ is a basis in both, [**141**, Theorem 9.3.6 and Example 11.3.12]).
- $K_{6,7}$ is a basis in $\mathcal{C}_4(n)$ and dependent in $\mathcal{R}_4(n)$ for $n \geq 13$ [**141**, Sections 11.4 and 11.5]. (The example generalizes to show that $\mathcal{C}_d(n) \neq \mathcal{R}_d(n)$ for $n - 9 \geq d \geq 4$).

See [**28**] for a recent account of these and other relations among these matroids, including some questions and conjectures. See [**90**] for a comprehensive study of bar-and-joint and cofactor rigidities.

1.3. Pfaffians and tropical varieties

In the case $k = 1$, one way of realizing the associahedron is as the positive part of the space of "tree metrics", which coincides with the tropicalization $\mathrm{trop}(\mathcal{G}r_2(n))$ of the Grassmannian $\mathcal{G}r_2(n)$ (see [**124, 125**], or Remark 3.34). More precisely:

Theorem 1.8 ([**125**, Section 5]). *The totally positive tropical Grassmannian* $\mathrm{trop}^+(\mathcal{G}r_2(n))$ *is a simplicial fan isomorphic to (a cone over) the extended associahedron* $\overline{\Delta}_1(n)$.

Let us briefly recall what the tropicalization of a variety, and its positive part, are. (See also [**13**]). Let $I \subset \mathbb{K}[x_1, \dots, x_N]$ be a polynomial ideal and let $V = V(I) \subset \mathbb{K}^N$ be its corresponding variety. Each vector $v \in \mathbb{R}^N$, considered as giving weights to the variables, defines an initial ideal $\mathrm{in}_v(I)$, consisting of the initial forms $\mathrm{in}_v(f)$ of the polynomials in f. For our purposes we take the following definitions. (These are not the standard definitions, but are equivalent to them as shown for example in [**125**, Propositions 2.1 and 2.2]):

Definition 1.9. The *tropical hypersurface* $\mathrm{trop}(f)$ of a polynomial $f \in \mathbb{K}[x_1, \dots, x_N]$ is the collection of weight vectors $v \in \mathbb{R}^N$ for which $\mathrm{in}_v(f)$ is not a monomial. Put differently, the weight vectors for which the maximum weight among monomials in f is attained at least twice. It is a polyhedral fan, namely the codimension one skeleton of the normal fan of the Newton polytope of f.

If V is the algebraic variety of an ideal I, the *tropicalization* of V equals

$$\mathrm{trop}(V) := \cap_{f \in I} \mathrm{trop}(f).$$

A finite subset $B \subset I$ such that $\mathrm{trop}(V) = \cap_{f \in B} \mathrm{trop}(f)$, which always exists, is called a *tropical basis of* I. Not every generating set of I (not even a universal Gröbner basis of I, see [**10**, Example 10] or [**83**, Example 2.6.1]) is a tropical basis. In general, a finite intersection of tropical hypersurfaces is called a *tropical prevariety*, while the tropicalization of a variety is a *tropical variety* [**83**, Definitions 3.1.1 and 3.2.1]. The tropical prevariety defined by a finite set of polynomials $\{f_1, \dots, f_n\}$ contains, but is sometimes not equal to, the tropical variety of the ideal $(f_1, \dots, f_n)$ generated by them.

Pachter and Sturmfels [**94**, p. 107] hint at the fact that the relation between the associahedron and $\mathcal{G}r_2(n)$ extends to a relation between the multiassociahedron $\Delta_k(n)$ and the tropical variety of Pfaffians of degree $k + 1$.

The complete graph on a set of vertices $U \subset [n]$ of size $2k$ has $(2k - 1)!!$ matchings (by which we always mean a *perfect* matching), one of which is the unique k-crossing with vertex set U. The *parity* of a matching E is the parity of the number of pairwise crossings among the edges in E. This parity coincides with the parity as a permutation, when the pairs of matched vertices are written one after another, in increasing order within each pair. By *swapping* two pairs $\{a,b\}$ and $\{c,d\}$ in a matching E we mean removing them and inserting one of the other two matchings of $\{a,b,c,d\}$ instead. Observe that one of the three matchings of $\{a,b,c,d\}$ has a crossing (that is, it is odd) and the other two are crossing-free (hence even).

LEMMA 1.10. *A swap changes parity if and only if one of the two pairs of edges in the swap (the pair removed or the pair inserted) is a crossing.*

PROOF. Let $\{a,b\}$ and $\{c,d\}$ be the initial pairs and $\{a,d\}$ and $\{b,c\}$ the new pairs. Any other edge from the matching crosses the cycle *abcda* an even number of times. Hence, the only change in the number of crossings comes from whether the edges in the swap cross. □

Recall that an antisymmetric matrix of odd size n has zero determinant because

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

For even size there is the following classical result:

THEOREM 1.11 (Cayley 1852 [18]). *Let M be a size $2k$ antisymmetric matrix. Then*

$$\det M = \left(\sum_{E \text{ matching}} s(E) \prod_{(i,j)\in E, i<j} m_{ij} \right)^2 \tag{5}$$

where the sum is taken over the matchings of $[2k]$ and $s(E) = \pm 1$ according to the parity of E.

The expression inside the parenthesis in this theorem, that is, the square root of the determinant of an antisymmetric matrix, is called the *Pfaffian* of M.

For each $n \geq 2k+2$, let $I_k(n)$ be the ideal in $\mathbb{K}[x_{i,j}, \{i,j\} \in \binom{[n]}{2}]$ generated by all the Pfaffians of degree $k+1$. Let $\mathcal{P}f_k(n) \subset \mathbb{K}^{\binom{n}{2}}$ be the corresponding algebraic variety. That is, points in $\mathcal{P}f_k(n)$ are antisymmetric $n \times n$ matrices with coefficients in $\mathbb{K}$ and of rank at most $2k$. It is well-known and easy to see that $\mathcal{P}f_1(n)$ equals the Grassmannian $\mathcal{G}r_2(n)$ in its Plücker embedding and, as pointed out in [94], $\mathcal{P}f_k(n)$ equals the k-th secant variety of it.

For $k = 1$, Pfaffians are a universal Gröbner basis of $I_k(n)$ [94, 124]. For $k > 1$ they are not (see Example 3.16), but it is known that they are a Gröbner basis for certain choices of monomial orders: in [61] it is proved that this happens for a v that selects as initial monomial in each Pfaffian the $(k+1)$-nesting and in [73] for one that selects the $(k+1)$-crossing.

For each subset U of $[n]$ of size $2k+2$, the corresponding Pfaffian has as tropical hypersurface the set of vectors $v \in \mathbb{R}^{\binom{[n]}{2}}$ for which the maximum

$$\Big\{ \sum_{\{i,j\}\in E} v_{ij} : E \text{ matching in } U \Big\},$$

is attained at least twice. We denote by $\mathrm{Pf}_k(n)$ the intersection of all these tropical hypersurfaces for the different $U \in \binom{[n]}{2k}$. We call it the *tropical Pfaffian prevariety.*

It contains the tropicalization $\mathrm{trop}(\mathcal{P}f_k(n))$ of $\mathcal{P}f_k(n)$ and it is known to coincide with it in the following cases:

- If $n = 2k + 2$, since then we have a single Pfaffian defining $\mathrm{trop}(\mathcal{P}f_k(n))$.
- If $k = 1$, by the results in [124] and the fact that $\mathcal{P}f_1(n)$ coincides with the Grassmannian $\mathcal{G}r_2(n)$ (see Remark 3.34 below).

CHAPTER 2

The multiassociahedron via rigidity

In this chapter we explore Conjecture 1.3 both in its polytopality version and in the weaker version where we want to realize $\overline{\Delta}_k(n)$ as a complete fan.

Our method is to use as rays for the fan the row vectors of a rigidity matrix of n points in dimension $2k$, which has exactly the required rank $k(2n-2k-1)$ for $\Delta_k(n)$. There are several versions of rigidity that can be used, most notably bar-and-joint, hyperconnectivity, and cofactor rigidity. Among these, cofactor rigidity seems the most natural one because it deals with points in the plane; the "dimension" $2k$ of this rigidity theory relates to the degree of the polynomials used.

Our results are of two types. On the one hand we show new cases of multiassociahedra $\overline{\Delta}_k(n)$ that can be realized, be it as fans or as polytopes, with cofactor rigidity taking points along the parabola (which is known to be equivalent to bar-and-joint rigidity with points along the moment curve). On the other hand we show that certain multiassociahedra, namely those with $k \geq 3$ and $n \geq 2k+6$ cannot be realized as fans with cofactor rigidity, no matter how we choose the points.

2.1. Statement of results

Using a (human guided) computer search, we find explicit embeddings of $\overline{\Delta}_k(n)$ for additional parameters, be it as a polytope or only as a complete fan. We list only the ones that were not previously known:

THEOREM 2.1. (1) *For $(k,n) \in \{(2,9),(2,10),(3,10)\}$, $\overline{\Delta}_k(n)$ is a polytopal sphere.*
(2) *$\overline{\Delta}_4(13)$ can be realized as a complete simplicial fan.*

Adding this to previous results, we have that $\overline{\Delta}_k(n)$ can be realized as a fan (which for us always means a complete fan) if $n \leq \max\{2k+4, 13\}$ except maybe for $(n,k) = (3,12)$ and $(3,13)$, and as a polytope if $n \leq \max\{2k+3, 10\}$.

Our method to realize $\overline{\Delta}_k(n)$ is via rigidity theory. We now explain the connection. The number $k(2n-2k-1) = 2kn - \binom{2k+1}{2}$ of edges in a k-triangulation of the n-gon happens to coincide with the rank of abstract rigidity matroids of dimension $2k$ on n elements. This numerical coincidence (plus some evidence) led [106] to conjecture that *all k-triangulations of the n-gon are bases in the generic bar-and-joint rigidity matroid of n points in dimension $2k$.*

Apart of its theoretical interest, knowing k-triangulations to be bases can be considered a step towards proving polytopality of $\overline{\Delta}_k(n)$, as follows. For any given choice of points $p_1, \dots, p_n \in \mathbb{R}^{2k}$ in general position, the rows of their rigidity matrix (see Section 1.2) give a real vector configuration $\mathcal{V} = \{p_{ij}\}_{i,j}$ of rank $k(2n-2k-1)$. The question then is whether using those vectors as generators makes $\overline{\Delta}_k(n)$ be a fan, and whether this fan is polytopal. Being bases is then a partial result: it says that at least the individual cones have the right dimension and are simplicial.

All the realizations of $\overline{\Delta}_k(n)$ that we construct use this strategy for positions of the points along the *moment curve* $\{(t, t^2, \ldots, t^{2k}) \in \mathbb{R}^{2k} : t \in \mathbb{R}\}$. The reason to restrict our search to the moment curve is that, as stated in the introduction, for points along the moment curve, the vector configuration obtained with bar-and-joint rigidity coincides (modulo linear isomorphism) with configurations coming from hyperconnectivity along the moment curve and cofactor rigidity along the parabola. This is useful in our proofs and it also makes our realizations more "natural", since they can be interpreted in the three versions of rigidity.

In fact, we pose the conjecture that positions along the moment curve realizing $\overline{\Delta}_k(n)$ as a basis collection exist for every k and n:

Conjecture 2.2. *k-triangulations of the n-gon are isostatic (that is, bases) in the bar-and-joint rigidity matroid of generic points along the moment curve in dimension $2k$.*

This conjecture implies the one from [106] mentioned above, but it would imply the same for the generic cofactor rigidity matroid and for the generic hyperconnectivity matroid. (The latter is known to hold by Corollary 3.22.) As evidence for the conjecture we prove the case $k = 2$, already in [24, Teorema 2.2.5]:

Theorem 2.3. *2-triangulations are isostatic in dimension 4 for generic positions along the moment curve.*

In fact, our experiments make us believe that in this statement the word "generic" can be changed to "arbitrary".

Conjecture 2.4. *2-triangulations of the n-gon are isostatic (that is, bases) in the bar-and-joint rigidity matroid of arbitrary (distinct) points along the moment curve in dimension 4.*

This conjecture has an apparently much stronger implication:

Theorem 2.5. *If Conjecture 2.4 is true, then all positions along the moment curve realize $\overline{\Delta}_2(n)$ as a fan (hence, Conjecture 1.3 would almost be true for $k = 2$).*

So far we have discussed whether k-triangulations are bases in the rigidity *matroid*, but for the polytopality question we are also interested in the *oriented matroid*, which tells us the orientation that each k-triangulation has as a basis of the vector configuration. The first thing to notice is that now there is a priori not a unique "generic" oriented matroid; different generic choices of points may lead to different orientations of the underlying generic matroid.

Since our points lie in the moment curve, we can refer to each point $(t, \ldots, t^{2k})$ via its parameter t. The parameters proving Theorem 2.1 are as follows:

- For $k = 2$, the standard positions ($t_i = i$ for each i) realize $\overline{\Delta}_2(n)$ as a polytope if and only if $n \leq 9$. For $k = 2$ and $n \in \{10, 11, 12, 13\}$ they still realize it as a fan, but not as a polytope. Modifying a bit the positions to $(-2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ we get a polytopal fan for $\overline{\Delta}_2(10)$ (Lemma 2.54).
- Equispaced positions along a circle, mapped to the moment curve via a birational map, realize $\overline{\Delta}_k(n)$ as a fan for every (k, n) with $2k + 2 \leq n \leq 13$ except $(3, 12)$ and $(3, 13)$, and they realize $\overline{\Delta}_3(10)$ as a polytope (Lemma 2.55).

Our experiments show a difference between the case $k = 2$, in which all the positions along the moment curve that we have tried realize $\overline{\Delta}_k(n)$ at least as a fan, and the case $k \geq 3$, in which we show that the standard positions do not realize $\overline{\Delta}_k(2k+3)$ as a fan (realizing $\overline{\Delta}_k(n)$ for $n < 2k+3$ is sort of trivial):

THEOREM 2.6. *The graph $K_9 - \{16, 37, 49\}$ is a 3-triangulation of the n-gon, but it is a circuit in the cofactor rigidity matroid $\mathcal{C}_6(\mathbf{q})$ if the position $\mathbf{q}$ makes the lines through 16, 37 and 49 concurrent. This occurs, for example, if we take points along the parabola with $t_i = i$.*

This shows that Conjecture 2.4 fails for $k \geq 3$, and we prove that it fails in the worst possible way. We consider this our second main result, after Theorem 2.1:

THEOREM 2.7. *If $k \geq 3$ and $n \geq 2k + 6$ then no choice of points $\mathbf{q} \in \mathbb{R}^2$ in convex position makes the cofactor rigidity $C_{2k}(\mathbf{q})$ realize the k-associahedron $\overline{\Delta}_k(n)$ as a fan. The same happens for bar-and-joint rigidity and for hyperconnectivity with any choice of points along the moment curve.*

Let us explain this statement. Cofactor rigidity, introduced by Whiteley following work of Billera on the combinatorics of splines, is related to the existence of $(d-2)$-continuous splines of degree $d-1$, for a certain parameter d. For this reason it is usually denoted C^{d-1}_{d-2}-rigidity, although we prefer to denote it C_d-rigidity since, as said above, it induces an example of abstract rigidity matroid of dimension d. Since this form of rigidity is based on choosing positions for n points in the plane, it is the most natural rigidity theory in the context of k-triangulations; for any choice $\mathbf{q}$ of n points in convex position in the plane, we have at the same time a convex n-gon on which we can model k-triangulations and a $2k$-dimensional rigidity matroid $C_{2k}(\mathbf{q})$ whose rows we can use as vectors to (try to) realize $\overline{\Delta}_k(n)$ as a fan. For $n = 2k + 3$ we show that this realization, taking as points the vertices of a regular n-gon, always realizes $\overline{\Delta}_k(n)$ as a fan (Corollary 2.40), but the above statement says that for $n \geq 2k + 6$ (and $k \geq 3$) no points in convex position do. As said above, $C_{2k}(\mathbf{q})$ with points along a parabola is equivalent to bar-and-joint rigidity and to hyperconnectivity with points along the moment curve in $\mathbb{R}^{2k}$.

REMARK 2.8. Theorem 2.7 still leaves open the possibility of realizing $\overline{\Delta}_k(n)$ for $n \geq 2k+6 \geq 12$ via bar-and-joint rigidity or via hyperconnectivity, but it would need to be with a choice of points not lying in the moment curve. We have not explored this possibility because we cannot think of a "natural" choice of n points in $\mathbb{R}^{2k}$.

Also, observe that for $k \in \{3, 4\}$ this theorem and Theorem 2.1 (or, rather, its more precise version Lemma 2.55) completely settle realizability of $\overline{\Delta}_k(n)$ as a fan via cofactor rigidity: it can be done for $n \leq 2k + 5$ and it cannot for $n \geq 2k + 6$.

From a computational viewpoint, our methods have three parts (see more details in Section 2.4.3):

(1) First, for given k, n, we enumerate all the k-triangulations of the n-gon. To do this we have adapted code by Vincent Pilaud which uses the relations between k-triangulations and sorting networks [105]. Although computationally easy, this is the bottleneck of the process because of the large number of k-triangulations. (Jonsson [70] proved that the number of k-triangulations of the n-gon is a Henkel determinant of Catalan numbers, hence growing as ${C_n}^k$ times a rational function of degree $2k$ in n, where

C_n denotes the n-th Catalan number). In all cases where we have been able to enumerate all k-triangulations, we have also been able to decide whether given positions realize the fan and/or the polytope.

(2) Then, our code tests whether, for given positions, the rigidity matrix realizes the fan or not. We have always used points along the parabola/moment curve (for which the three rigidity theories are equivalent), but the code would work for arbitrary positions and for the three theories. The running time is essentially linear in the number of k-triangulations, with a factor depending on k and n since we are doing linear algebra in $\mathbb{R}^{k(2n-2k-1)}$.

(3) Finally, when the second step works, deciding whether the fan (in the obtained realization) is polytopal is equivalent to feasibility of a linear program with one variable for each ray of the $\binom{n}{2}$ diagonals of the n-gon and $k(n-2k-1)$ constraints for each k-triangulation.

To choose the positions for the points we use a bit of trial and error. By default we start with equispaced points along the parabola and along the circle, and when both of them fail we modify the positions.

2.2. Auxiliary results

2.2.1. Multitriangulations. The following lemma shows that the realizability question we want to look at is *monotone*; if we have a realization of $\overline{\Delta}_k(n)$ then we also have it for all $\overline{\Delta}_{k'}(n')$ with $k' \le k$ and $n'-2k' \le n-2k$. Remember that the link of a face F in a simplicial complex Δ is

$$\mathrm{lk}_\Delta(F) := \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\} = \{\sigma \setminus F : \sigma \in \Delta, F \subset \sigma\}.$$

In a shellable d-sphere the link of any face of dimension d' is a shellable $d-d'-1$-sphere.

LEMMA 2.9 (Monotonicity). *Let $n \ge 2k+1$. Then, both $\overline{\Delta}_k(n)$ and $\overline{\Delta}_{k-1}(n-1)$ appear as links in $\overline{\Delta}_k(n+1)$. More precisely:*

(1) $\overline{\Delta}_k(n) = \mathrm{lk}_{\overline{\Delta}_k(n+1)}(B_{n+1})$*, where* $B_{n+1} := \{\{n-k+i, i\} : i = 1, \dots, k\} \in \binom{[n]}{2}$ *is the set of edges of length $k+1$ leaving $n+1$ in their short side.*

(2) $\overline{\Delta}_{k-1}(n-1) \cong \mathrm{lk}_{\overline{\Delta}_k(n+1)}(E_{n+1})$*, where* $E_{n+1} = \{\{i, n+1\} : i \in [k+1, n-k]\} \in \binom{[n+1]}{2}$ *is the set of relevant edges using $n+1$.*

PROOF. By Theorem 1.2, the three complexes $\overline{\Delta}_k(n)$, $\overline{\Delta}_{k-1}(n-1)$ and $\overline{\Delta}_k(n+1)$ are spheres, of the appropriate dimensions. For example,

$$\dim(\overline{\Delta}_k(n+1)) = k(n+1-2k-1)-1 = k(n-2k)-1.$$

Since the link of a face of size j in a shellable sphere is a sub-sphere of codimension j, the right-hand sides in both equalities are spheres of respective dimensions

$$\dim(\overline{\Delta}_k(n+1)) - k = k(n-2k)-1-k = k(n-2k-1)-1 = \dim(\overline{\Delta}_k(n))$$

in part (1) and

$$\begin{aligned}\dim(\overline{\Delta}_k(n+1)) - (n-2k) &= k(n-2k)-1-(n-2k)\\ &= (k-1)(n-2k)-1 \qquad = \dim(\overline{\Delta}_{k-1}(n-1))\end{aligned}$$

in part (2). Two simplicial spheres of the same dimension cannot be properly contained in one another, so in both equalities we only need to check one containment

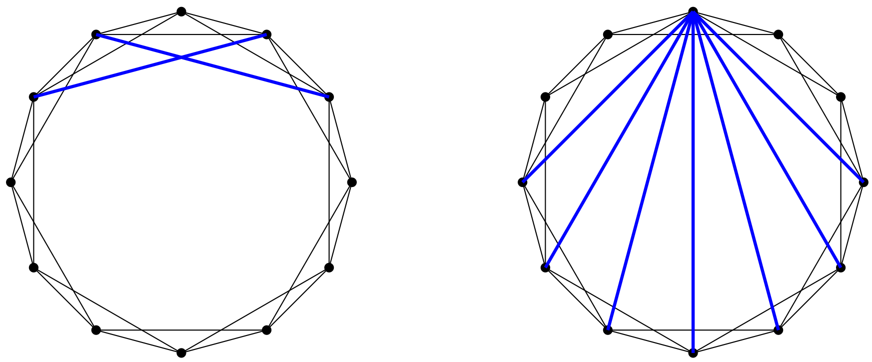


FIGURE 1. The black edges are the irrelevant edges in $\Delta_2(12)$. The link of the blue edges in the left is isomorphic to $\overline{\Delta}_2(11)$ and the link of those in the right is isomorphic to $\overline{\Delta}_1(10)$.

(with a relabelling of the complex allowed in part (2)) and the other containment then follows automatically.

In part (1) we show $\overline{\Delta}_k(n) \subset \mathrm{lk}_{\overline{\Delta}_k(n+1)}(B_{n+1})$. That is, for every k-triangulation T of the n-gon with vertices $[n]$ we need to check that $T \cup B_{n+1}$ is $(k+1)$-crossing-free. This is because all the edges in B_{n+1} have length $k+1$ and have $n+1$ in their short side, so any $(k+1)$-crossing involving one of them needs to use the vertex $n+1$. But $T \cup B_{n+1}$ has no (relevant) edge using $n+1$.

For part (2), we consider the following map

$$\begin{aligned} \phi : \binom{[n]}{2} &\to \binom{[n-1]}{2} \\ \{i,j\} &\mapsto \{i, j-1\}, \quad 1 \le i < j \le n. \end{aligned}$$

The map ϕ is a bijection between the k-relevant edges in $\binom{[n+1]}{2}$ not using $n+1$ and the $(k-1)$-relevant edges in $\binom{[n-1]}{2}$. Moreover, the map reduces crossing of pairs of edges. More precisely, $\phi(e)$ and $\phi(f)$ cross if and only if e and f crossed and were not of the form $\{i, j+1\}$, $\{j, \ell+1\}$ for some $1 \le i < j < \ell \le n$.

Hence, if T is a k-triangulation in $\overline{\Delta}_k(n+1)$ containing E_{n+1} then its image $\phi(T \setminus E_{n+1})$ is $(k+1)$-crossing-free. We need to check that it is also k-crossing-free. For this, consider a $(k+1)$-crossing C in $T \setminus E_{n+1}$. Two things can happen:

- C uses an edge of E_{n+1}, so it is no longer a $(k+1)$-crossing in $\phi(T \setminus E_{n+1})$.
- C does not use any edge of E_{n+1}. Then, C is of the form $\{\{a_i, b_i\} : i \in [k+1]\}$ with $1 \le a_1 < \ldots < a_{k+1} < b_1 < \ldots < b_{k+1} \le n$. But we need $b_1 = a_{k+1} + 1$, or otherwise $C \cup \{\{a_{k+1}+1, n+1\}\}$ is a $(k+2)$-crossing in $T \setminus E_{n+1}$. Hence, $\phi(C)$ is no longer a $(k+1)$-crossing because $\phi(\{a_1, b_1\})$ and $\phi(\{a_{k+1}, b_{k+1}\})$ do not cross.

□

This result, in particular part 2, is much easier to see on the subword complex interpretation of the multiassociahedron. See Figure *** for an illustration.

Being a sphere (more precisely, being a pseudo-manifold) has the following important consequence:

PROPOSITION 2.10 (Flips [**42, 88**], see also [**106**, Lemma 5.1], [**24**, Corolario 2.1.20]). *For every relevant edge f of a k-triangulation T there is a unique edge $e \in \binom{[n]}{2}$ such that*

$$T \triangle \{e, f\} := T \setminus \{f\} \cup \{e\}$$

is another k-triangulation.

We call the operation that goes from T to $T \triangle \{e, f\}$ a *flip*. The paper [**106**] gives a quite explicit description of flips using for this the so-called *stars*:

DEFINITION 2.11 (Stars). Let $s_0, s_1, \ldots, s_{2k} \in [n]$ be distinct vertices, ordered cyclically. The k-star S with this set of vertices is the cycle $\{\{s_i, s_{i+k}\} : 0 \le i \le 2k\}$, with indices taken modulo $2k + 1$.

In classical terms, a k-star is sometimes called a "star polygon of type $\{2k + 1/k\}$" [**31, 58**]. Observe that every k-star S is $(k+1)$-crossing-free but the set $S \cup \{t\}$ where t is a bisector of S is never $(k+1)$-crossing-free. Here, by *bisector* we mean the following:

DEFINITION 2.12 (Bisectors). An *angle* consists of two elements $\{a, b\}$ and $\{b, c\}$ in $\binom{[n]}{2}$ with a common end-point b. A *bisector* of the angle is any edge $\{b, d\}$ with d lying between a and c as seen cyclically from b. A *bisector of a star* is a bisector of any of its $2k + 1$ angles. That is, an edge of the form $\{s_i, t\}$ such that t lies between s_{i-k} and s_{i+k} for some s_i in the star (with the notation of Definition 2.11).

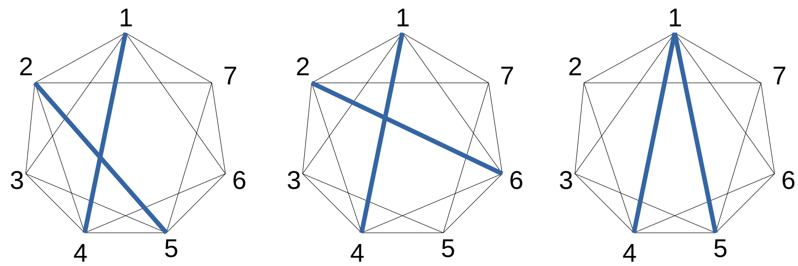
In terms of stars and their bisectors, flips can be described as follows:

PROPOSITION 2.13 (See also [**24**, section 2.1.2]). *Let T be a k-triangulation of the n-gon. Then:*

1. *T contains exactly $n - 2k$ k-stars* [**106**, Corollary 4.4 and Theorem 3.5].
2. *Each pair of k-stars in T have a unique common bisector* [**106**, Theorem 3.5].
3. *Every relevant edge e in T belongs to exactly two such k-stars, and every boundary edge belongs to exactly one* [**106**, Corollary 4.2].
4. *The k-triangulation obtained by flipping e in T is $T \triangle \{e, f\}$ where f is the common bisector of the two k-stars containing e* [**106**, Lemma 5.1].

In our next result we ask the following question. Suppose that F is a face in $\overline{\Delta}_k(n)$. That is, F is contained in some k-triangulation T. How big can the link of F be? By "how big" we here mean how many vertices (of $\overline{\Delta}_k(n)$, that is, diagonals of the n-gon) are used in the link.

EXAMPLE 2.14. The three graphs below represent codimension-two faces of $\overline{\Delta}_2(7)$. In each drawing, blue edges are relevant and two more relevant edges are needed to form a 2-triangulation. The link of the first face is a cycle of length 3, consisting of the edges $\{15, 26, 47\}$. In the second, the length is 4: $\{15, 25, 36, 47\}$. In the third the length is 5: $\{25, 26, 36, 37, 47\}$. (In each case, adding to the given graph any two consecutive edges from the list we get a 2-triangulation, and all 2-triangulations containing that graph have this form.)



Lemma 2.15. *Let $T \in \overline{\Delta}_k(n)$ be a k-triangulation and $F \subset T$ a subset of its edges. Let S be the set of k-stars of T containing an edge in $T \setminus F$. Then, all edges used in $\mathrm{lk}_{\overline{\Delta}_k(n)}(F)$ are either from $T \setminus F$ or common bisectors of two of the k-stars in S.*

Proof. Facets in the link correspond to k-triangulations containing F. As the graph of each link is connected (because the multiassociahedron is a sphere) they can all be obtained from T by iteratively flipping edges not belonging to F. The edges so obtained are bisectors perhaps not of the original stars in T but at least of new stars obtained along the way. However, we can give the following characterization of them: At each vertex i of the n-gon, consider (locally) the union of the angles of stars in S, seeing each angle as a sector of a small disk centered at i. The bisectors of such a union with an endpoint at i are either bisectors (at i) of one of the stars, or common edges of two stars. Now, this "union of stars" is unchanged by flipping, because each flip removes two stars and inserts another two but with the same union.

Thus, which possible edges are used can be prescribed by looking only at T. They either are bisectors of pairs of stars in T or common edges of pairs of stars in T. Among the latter we are only allowed to flip, or insert, those that are not in F. □

Corollary 2.16. *All links of dimension one in $\overline{\Delta}_k(n)$ are cycles of length at most five.*

Proof. Every such link is a sphere of dimension one, hence a cycle.

Let F be the face we are looking at, so that there are relevant edges $\{e, f\} \in \binom{[n]}{2} \setminus F$ and a k-triangulation T with $F = T \setminus \{e, f\}$. The set S of stars in the previous lemma has size at most four (two for e and two for f) but it may have size three or two if e and f belong to one or two common stars. By the lemma, if $|S|$ is two or three then the length of the cycle is (at most) three or five, respectively.

If $|S| = 4$, then each of the flips leaves the two stars corresponding to the other flip untouched. Hence, the two flips commute and the cycle is a quadrilateral, consisting of T, its two neighbors by the flip at e or f, and the k-triangulation obtained by performing both flips, in any order. □

2.2.2. Polytopality. Throughout this section Δ will denote a pure simplicial sphere of dimension $D - 1$ with vertex set V. We ask ourselves whether Δ can be realized as the normal fan of a polytope. That is, we ask whether there is a vector configuration $\mathcal{V} = \{v_i : i \in V\} \subset \mathbb{R}^D$ with the property that the family of cones

$$\{\mathrm{cone}(\mathcal{V}_F) : F \in \Delta\}$$

form a complete simplicial fan and whether this fan is the normal fan of a polytope. Here we denote

$$\mathcal{V}_X := \{v_i : i \in X\}$$

for each $X \subset V$ and

$$\text{cone}(X) := \{\lambda_1 x_1 + \cdots + \lambda_s x_s : s \in \mathbb{N}, \lambda_i \in [0, \infty), x_i \in \mathcal{V}_X\}$$

is the *cone* generated by X.

The first obvious necessary condition is that we need $\mathcal{V}_F$ to be linearly independent for each $F \in \Delta$. When this happens the cones $\text{cone}(\mathcal{V}_F)$ are *simplicial cones* and $\mathcal{V}$ naturally defines a continuous map $\phi_{\Delta,\mathcal{V}} : |\Delta| \to S^{D-1}$ where $|\Delta|$ denotes the topological realization of Δ and $S^{D-1} \subset \mathbb{R}^D$ is the unit sphere. The precise definition of $\phi_{\Delta,\mathcal{V}}$ is as follows: First map $|\Delta|$ to $\mathbb{R}^D$ by sending each vertex i to its corresponding vector v_i, and extend this map to $|\Delta|$ linearly within each $F \in \Delta$. The fact that each F is linearly independent ensures that the image of this map does not contain the origin, so we can compose the map with the normalization map $\mathbb{R}^D \to S^{D-1}$ which divides each vector by its L_2-norm. We call this map $\phi_{\Delta,\mathcal{V}}$ a *pre-embedding* of Δ. Slightly abusing notation we will also say that $\mathcal{V}$ *is a pre-embedding of* Δ.

If the pre-embedding happens to be injective, then it is a continuous injective map between two spheres of the same dimension, hence bijective, hence a homeomorphism. This implies that Δ is a *triangulation of* $\mathcal{V}$ in the sense of [34] (see, e.g., Theorem 4.5.20 in that book). We can also say that, in this case, $\mathcal{V}$ is an *embedding* of Δ or that it *realizes* Δ *as a complete fan*.

2.2.2.1. *Conditions for a complete fan.* Remember that the *contraction* of a vector configuration $\mathcal{V} \subset \mathbb{R}^D$ at an independent subset I is the image of $\mathcal{V} \setminus I$ under the quotient linear map $\mathbb{R}^D \to \mathbb{R}^D/\text{lin}(I) \cong \mathbb{R}^{D-|I|}$; it is denoted $\mathcal{V}/I$. If $\mathcal{V}$ is a pre-embedding (resp. an embedding) of Δ then $\mathcal{V}/F$ is also a pre-embedding (resp. an embedding) of $\text{lk}_\Delta(F)$ for every $F \in \Delta$. We can then consider a hierarchy of embedding properties by asking $\mathcal{V}/F$ to be an embedding only for faces of at least a certain dimension. The case where F is a facet is trivial. The next level in the hierarchy, when F is a ridge, has received some attention in [34]:

Definition 2.17 (ICoP property). Let $\mathcal{V} \subset \mathbb{R}^D$ be a pre-embedding of a pure $(D-1)$-complex Δ. We say that the pre-embedding has the the *intersection cocircuit property (ICoP)* if the pre-embedding $\mathcal{V}_\tau$ of $\text{lk}_\Delta(\tau)$ is an embedding for every ridge τ. That is to say, if the following two properties hold:

- τ is contained in exactly two facets σ_1 and σ_2.
- The cones $\mathcal{V}_{\sigma_1}$ and $\mathcal{V}_{\sigma_2}$ lie in opposite sides of the hyperplane spanned by $\mathcal{V}_\tau$ (which exists and is unique since τ is independent of size $D-1$). This is equivalent to saying that the unique (modulo a scalar multiple) linear dependence in $\mathcal{V}_{\sigma_1 \cup \sigma_2}$ has coefficients of the same sign in the two vectors indexed by $\sigma_1 \setminus \tau$ and $\sigma_2 \setminus \tau$.

Observe that the first condition is independent of $\mathcal{V}$. When it holds, Δ is said to be a *pseudo-manifold*. The pseudo-manifold is *strongly connected* if its dual graph is connected.[1]

[1]Sometimes strong-connectedness is considered part of the definition of pseudo-manifold, but we do not take this approach.

Every link in a pseudo-manifold is itself a pseudo-manifold. For example, the link of a codimension-two face ρ is a disjoint union of cycles. We say that ρ is *non-singular* if $\mathrm{lk}_\Delta(\rho)$ is a single cycle and in this case we call this cycle the *elementary cycle with center at* ρ.

When this happens for every ρ we say that the pseudo-manifold Δ *has no singularities of codimension two*. Being a pseudo-manifold with no singularities of codimension two is computationally easy to check: In a pure complex the link of every codimension-two face is a graph, and we only need to check that each of these graphs is a cycle. All manifolds, hence all spheres, hence $\overline{\Delta}_k(n)$ have these properties.

As seen in [34, Theorem 4.5.20], the (ICoP) property is almost sufficient for Δ to be a triangulation of $\mathcal{V}$, but something else is needed. We here express this "something else" in topological terms, in two ways.

Our first characterization is in terms of links of codimension-two faces. Suppose that Δ has no singularities of codimension two, so that every face ρ of codimension two defines an elementary cycle. Then, we have that $\mathcal{V}/\rho$ embeds $\mathrm{lk}_\Delta(\rho)$ as a cyclic collection of cones in $\mathbb{R}^2$, for which we can define its *winding number*: the number of times the cycle wraps around $\mathbb{R}^2 \setminus \{0\}$. Homologically, this number is the image in $H_1(\mathbb{R}^2 \setminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$ of the elementary cycle as a generator of its homology group $H_1(\mathrm{lk}_\Delta(\rho), \mathbb{Z}) \cong \mathbb{Z}$. We say that an elementary cycle is *simple* in $\mathcal{V}$ if its winding number is ± 1.

The second characterization is in terms of the degree of the pre-embedding, which is a generalization of winding number to higher dimensions. The *degree* of a continuous map $\phi : |\Delta| \to S^{D-1}$ from an orientable $(D-1)$-dimensional pseudo-manifold Δ to the sphere S^{D-1} can be defined as the image of the fundamental cycle of $H_{D-1}(M, \mathbb{Z}) \cong \mathbb{Z}$ in $H_{D-1}(S^{D-1}, \mathbb{Z}) \cong \mathbb{Z}$. If ϕ is injective in each facet (for example, if it is a pre-embedding as defined above), the degree of ϕ can be computed as the number (with sign) of preimages in $\phi^{-1}(y)$ for a sufficiently generic point $y \in S^{D-1}$; "with sign" means that each preimage $x \in \phi^{-1}(y)$ counts as $+1$ or -1 depending on whether ϕ preserves or reverses orientation in the facet containing x.

Observe that being a pre-embedding with the (ICoP) property implies Δ to be an orientable pseudo-manifold.

THEOREM 2.18. *Let $\mathcal{V} \subset \mathbb{R}^D$ be a pre-embedding of Δ with the (ICoP) property. Let $\phi_{\Delta,\mathcal{V}} : |\Delta| \to S^{D-1}$ be the associated map. Then, the following conditions are equivalent:*

(1) *$\phi_{\Delta,\mathcal{V}}$ is a homeomorphism; that is, $\mathcal{V}$ is a triangulation of Δ or, equivalently, $\mathcal{V}$ embeds Δ as a complete simplicial fan in $\mathcal{R}^D$.*
(2) *Every sufficiently generic vector $v \in \mathbb{R}^D$ is contained in only one of the facet cones $\{\mathrm{cone}(\mathcal{V}_\sigma) : \sigma \in \mathrm{facets}(\Delta)\}$.*
(3) *There is some vector $v \in \mathbb{R}^D$ that is contained in only one of the facet cones $\{\mathrm{cone}(\mathcal{V}_\sigma) : \sigma \in \mathrm{facets}(\Delta)\}$.*
(4) *Δ has no singularities of codimension two and all its elementary cycles are simple in $\mathcal{V}$.*
(5) *$\phi_{\Delta,\mathcal{V}}$ has degree ± 1.*

PROOF. We only need to show that any of (4) and (5) implies one of (1), (2) or (3), since the implications (1)⇒(2)⇒(3), (1)⇒(4) and (1)⇒(5) are obvious and (3)⇒(1) is part of [34, Corollary 4.5.20].

Let us see the implication (5)⇒(3). The property (ICoP) implies that the map $\phi_{\Delta,\mathcal{V}}$ is consistent with orientations: either it preserves orientations of all facets or reverses orientation of all facets. This implies that when we compute the degree via a generic fiber there are no cancellations and, since the map has degree one, every generic fiber has a single point. That is, $\phi_{\Delta,\mathcal{V}}$ is injective except perhaps in a subset of measure zero (the $(D-2)$-skeleton of $|\Delta|$), so (3) holds.

For the implication (4)⇒(1) we use induction on D. If $D \leq 2$ there is nothing to prove, so we assume $D \geq 3$. Since elementary cycles are preserved by taking links/contractions, the inductive hypothesis implies that $\mathrm{lk}_\Delta(i)$ is a triangulation of $\mathcal{V}/v_i$ for every $i \in V$. In particular, $\mathrm{lk}_\Delta(i)$ is topologically a sphere and, hence, Δ is a manifold. Moreover, again by the inductive hypothesis, the map $\phi_{\Delta,\mathcal{V}} : |\Delta| \to S^{D-1}$ is a local homeomorphism. Every local homeomorphism between two manifolds is a covering map. Now, $D-1 \geq 2$ implies that S^{D-1} is simply connected and, since $|\Delta|$ is connected, the covering map $\phi_{\Delta,\mathcal{V}}$ is a global homeomorphism. □

Now, by Corollary 2.16, elementary cycles in $\overline{\Delta}_k(n)$ have length bounded by five. This suggests we study Theorem 2.18 in more detail for such cycles:

PROPOSITION 2.19. *Let $\mathcal{V}$ be a pre-embedding of Δ with the (ICoP) property. Then:*

(1) *All cycles of length ≤ 4 are automatically simple.*
(2) *Let ρ be a codimension-two face whose elementary cycle Z has length five. Let $i_1, \dots, i_5 \in V$ be the vertices of Z, in their cyclic order. Then, Z is simple if and only if there are three consecutive elements $i_1, i_2, i_3 \in Z$ such that the unique linear dependence among the vectors $\{v_i : i \in \rho \cup \{i_1, i_2, i_3\}\}$ has opposite sign in i_2 than the sign it takes in i_1 and i_3.*

PROOF. Let us first explain the condition in part two. The (ICoP) property implies that for every three consecutive vertices i_1, i_2, i_3 in the elementary cycle (of arbitrary length) of a codimension-two face ρ we have that i_1 and i_3 lie in opposite sides of the hyperplane spanned by $\rho \cup \{i_2\}$. By elementary linear algebra (or oriented matroid theory), this translates to the fact that the unique dependence contained in $\rho \cup \{i_1, i_2, i_3\}$ has the same sign in i_1 and i_3. Similarly, whether this sign equals the one at i_2 or not expresses whether i_3 lies on the same or different side of $\rho \cup \{i_1\}$ as i_2. Put differently, it tells us whether the dihedral angles of $i_1 i_2$ and $i_2 i_3$, as seen from ρ, add up to more or less than π. (If the dependence vanishes at i_2 then the angle is exactly π).

In general, if $Z = i_1 i_2 \dots i_n i_1$ is a cycle with center ρ and (after contraction of the vector configuration at ρ) it is embedded in $\mathbb{R}^2$ with vectors $w_1, \dots, w_n \in \mathbb{R}^2 \setminus \{0\}$ for its generators, we can compute the winding number of Z by adding the dihedral angles $w_i w_{i+1}$, taken with sign. This sum of angles is necessarily going to be a multiple $2\pi\alpha$ of 2π, and the the winding number equals the integer α.

Since each individual angle is, in absolute value, smaller than π, it is impossible to get a sum of at least 4π with four angles or less. With five angles it is possible, but not if two of them add up to less than π, as expressed by the condition in part (2). Conversely, if no three consecutive elements in Z satisfy this condition, then the sum of any two consecutive angles in the cycle is at least π, the sum of four of them is at least 2π, and the sum of the five of them is more than 2π. □

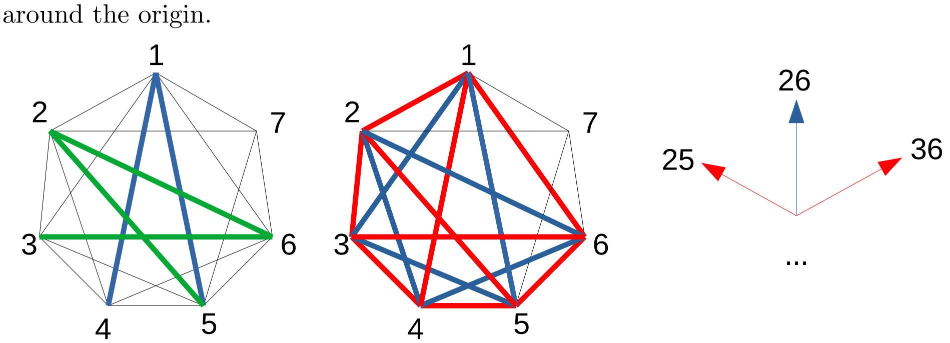
Summing up, we have an easy way of checking whether a vector configuration embeds $\overline{\Delta}_k(n)$ as a fan:

COROLLARY 2.20. *Let $\mathcal{V} = \{v_{ij}\}_{\{i,j\}\in\binom{[n]}{2}} \subset \mathbb{R}^{k(2n-2k-1)}$ be a vector configuration. $\mathcal{V}$ embeds $\overline{\Delta}_k(n)$ as a complete fan in $\mathbb{R}^{k(n-2k-1)}$ if and only if it satisfies the following properties:*

(1) *(Basis collection) For every facet (k-triangulation) T, the vectors $\{v_{ij} : \{i,j\} \in T\}$ are a linear basis.*
(2) *(ICoP) For every flip between two k-triangulations T_1 and T_2, the unique linear dependence among the vectors $\{v_{ij} : \{i,j\} \in T_1 \cup T_2\}$ has the same sign in the two elements involved in the flip (the unique elements in $T_1 \setminus T_2$ and $T_2 \setminus T_1$).*
(3) *(Elementary cycles of length 5) Every elementary cycle of length five has three consecutive elements satisfying the sign condition in part (2) of Proposition 2.19.*

□

EXAMPLE 2.21. The pictures below illustrate part (3) of Corollary 2.20. The left picture shows a flip (the union of two triangulations) belonging to the elementary cycle of the codimension-two face ρ from the third picture of Example 2.14. In the centre picture, blue and red represent the signs of the coefficients in the circuit, that in this case is a K_6, for a generic vector configuration $\mathcal{V}$ (see Section 2.3.1 to understand why the signs have to be like this). The sign of 26 is opposite to 25 and 36, and this implies that, as two-dimensional vectors in $\mathcal{V}/\rho \subset \mathbb{R}^2$, the vector 26 is a positive combination of 25 and 36, as in the right part of the figure. Thus, the angles in $(25, 26)$ and $(26, 36)$ add to less than π and the cycle cannot wind twice around the origin.



Observe that, computationally, what we need to do to apply the corollary is to check that the determinant corresponding to any k-triangulation is nonzero and to compute (the signs of) the linear dependence corresponding to each flip (plus some book-keeping to identify which flips form an elementary cycle). We emphasize that only the signs are needed because computing signs may sometimes be easier than computing actual values.

2.2.2.2. *Conditions for polytopality.* Once we have a collection of vectors $\mathcal{V} = \{v_i : i \in v\} \subset \mathcal{R}^D$ that embeds a simplicial complex Δ as a complete simplicial fan in $\mathbb{R}^D$, saying that the fan is the normal fan of a polytope is equivalent to saying the Δ is a *regular triangulation* of $\mathcal{V}$. (This is Theorem 9.5.6 in [34]).

Regular here means that there is a choice of lifting heights $f_i \in (0, \infty)^V$ for the vertices $i \in V$ of Δ such that Δ is the boundary complex of the cone in $\mathbb{R}^{D+1}$ generated by lifting the vectors $v_i \in \mathbb{R}^D$ to vectors $(v_i, f_i) \in \mathbb{R}^{D+1}$. That is to say, we need that for every facet $\sigma \in \Delta$ the linear hyperplane containing the lift of σ lies strictly below all the other lifted vectors.

We call such lifting vectors $(f_i)_{i\in V}$ *valid*. The following lemma is a version of [115, Theorem 3.7], which in turn is closely related to [34, Proposition 5.2.6(i)].

LEMMA 2.22. *Let Δ be a simplicial complex with vertex set V and dimension $D-1$, and assume it is a triangulation of a vector configuration $\mathcal{V} \subset \mathbb{R}^D$ positively spanning $\mathbb{R}^D$. Then, a lifting vector $(f_i)_{i\in V}$ is valid if and only if for every facet $\sigma \in \Delta$ and element $a \in V \setminus \sigma$ the inequality*

$$\sum_{i\in C} \omega_i(C) f_i > 0 \tag{6}$$

holds, where $C = \sigma \cup \{a\}$ and $\omega(C)$ is the vector of coefficients in the unique (modulo multiplication by a positive scalar) linear dependence in $\mathcal{V}$ with support in C, with signs chosen so that $\omega_a(C) > 0$ for the extra element.

PROOF. This is similar to Proposition 5.2.6(i) in [34]. For a facet σ and an extra element that forms a circuit C, we need to prove that the extra element is in the correct side via the lifting vector f. Let i be the extra element and f_i' the last coordinate of the intersection point of $v_i \times \mathbb{R}$ with the hyperplane spanned by $(v_j, f_j)_{j\in C}$. We want that $f_i > f_i'$. But obviously

$$\sum_{j\in\sigma} \omega_j(C) f_j + \omega_i(C) f_i' = 0$$

so the condition is equivalent to $f_i > f_i'$. □

REMARK 2.23. Two remarks are in order:

(1) If we already know Δ to be a triangulation of $\mathcal{V}$, it is enough to check the inequalities for the case when i is a neighbor of σ, because a locally convex cone is globally convex. That is, checking validity amounts to checking one linear inequality for each ridge in Δ: if τ is a ridge and $\tau \cup \{i\}$ and $\tau \cup \{j\}$ are the two facets containing it, we need to check inequality (6) for the circuit $C = \tau \cup \{i, j\}$ contained in $\tau \cup \{i, j\}$. (See, e.g., [34, Theorem 2.3.20 and Lemma 8.2.3]).

(2) If $(f_i)_{i\in V}$ is a valid lifting vector and $(w_i)_{i\in V}$ is the vector of values that a certain linear functional takes in $\mathcal{V}$ then $(f_i + w_i)_{i\in V}$ is also valid. (See, e.g., [34, Proposition 5.4.1]). In particular, when looking for valid vectors we can assume, without loss of generality, that $f_i = 0$ for all i in a certain independent set S (we here say that S is independent if the vectors $\{v_i\}_{i\in S}$ are linearly independent).

2.3. Obstructions to realizability with cofactor rigidity

Our main goal in this chapter is to study whether one of the three forms of rigidity from Section 1.2 provides, by choosing the configurations $\mathbf{p}$ in $\mathbb{R}^{2k}$ or $\mathbf{q}$ in $\mathbb{R}^2$ adequately, realizations of the k-associahedron $\overline{\Delta}_k(n)$. For positive results (realizations) the strongest possible setting goes via the polynomial rigidity of Definition 1.7, since that is a special case of the other three. For negative results (obstructions to realization) we are going to use cofactor rigidity. This is stronger than using polynomial rigidity, and is also the most natural setting for studying k-associahedra since, after all, the combinatorics of a k-associahedron comes from thinking about crossings in the complete graph embedded with vertices in convex position in the plane.

2.3.1. Some results on cofactor rigidity. In this section we present some results about cofactor rigidity that we need later.

We first show that cofactor rigidity is invariant under projective transformation. This, as some other results from this section, was already noticed by Whiteley [**141**], but we develop things from scratch since we will not only be interested in the cofactor rigidity *matroid* but also in the *oriented matroid*. Notice also that the same projective invariance of the matroid is true and well-known for bar-and-joint rigidity (see again [**141**]).

Throughout this section we work primarily with a vector configuration $\mathbf{Q} = (Q_1, \dots, Q_n)$ in dimension three, that is, with $Q_i = (X_i, Y_i, Z_i) \in \mathbb{R}^3 \setminus \{0\}$. We normally assume that $\mathbf{Q}$ is in general position (every three of its vectors form a linear basis) and sometimes that it is also in *convex position*: (a) each of the vectors Q_i generates a ray of

$$\operatorname{cone}(\mathbf{Q}) = \{\sum_i \lambda_i Q_i : \lambda_i \geq 0\},$$

and all these rays are different, and (b) the cyclic order of $Q_1, \dots, Q_n$ equals their order as rays of $\operatorname{cone}(\mathbf{Q})$.

In this setting, let us redefine the vectors $\mathbf{c}_{ij}$ that appear in the matrix $C_d(\mathbf{q})$ in terms of the vectors Q_i as follows. We let

$$\mathbf{c}_{ij} = \left(x_{ij}^{d-1}, y_{ij}x_{ij}^{d-2}, \dots, y_{ij}^{d-1}\right),$$

where

$$x_{ij} = X_iZ_j - Z_iX_j, \quad y_{ij} = Y_iZ_j - Z_iY_j.$$

We define the matrix $C_d(\mathbf{Q})$ exactly as in Equation (4), but with these new vectors $\mathbf{c}_{ij}$:

$$C_d(\mathbf{q}) := \begin{pmatrix} \mathbf{c}_{12} & -\mathbf{c}_{12} & 0 & \dots & 0 & 0 \\ \mathbf{c}_{13} & 0 & -\mathbf{c}_{13} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{c}_{1n} & 0 & 0 & \dots & 0 & -\mathbf{c}_{1n} \\ 0 & \mathbf{c}_{23} & -\mathbf{c}_{23} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{c}_{n-1,n} & -\mathbf{c}_{n-1,n} \end{pmatrix}. \tag{7}$$

Observe that the original definition of $C_d(\mathbf{q})$ is a special case of this one, obtained when we take all the Z_i's equal to 1 and we let $q_i = (X_i, Y_i)$. Observe also that, if d is odd, $\mathbf{c}_{ij} = \mathbf{c}_{ji}$, and if d is even, $\mathbf{c}_{ij} = -\mathbf{c}_{ji}$.

With the new definition, we have the following invariance:

PROPOSITION 2.24. *Let $\mathbf{Q} = (Q_1, \dots, Q_n)$ be a vector configuration in $\mathbb{R}^3 \setminus \{0\}$. Then,*

1. *The column-space of $C_d(\mathbf{Q})$, hence the oriented matroid $\mathcal{C}_d(\mathbf{Q})$ of its rows, is invariant under a nonsingular linear transformation of $\mathbf{Q}$.*
2. *The matroid $\mathcal{C}_d(\mathbf{Q})$ is also invariant under rescaling (that is, multiplication by non-zero scalars) of the vectors Q_i. If the scalars are all positive or d is odd then the same holds for the oriented matroid.*

PROOF. For each vector $Q \in \mathbb{R}^3 \setminus \{0\}$ let $C^{d-2}_{d-1}(Q)$ be the set of all three-variate polynomials in $\mathbb{R}[X,Y,Z]$ that are homogeneous of degree $d-1$ and such that all their partial derivatives up to order $d-2$ vanish at Q. This is a vector space of dimension d. In fact, if we fix a $Q_i = (X_i, Y_i, Z_i)$ and consider $Q_j = (X, Y, Z)$ as a vector of variables, then the d entries in the vector $\mathbf{c}_{ij}$ are a basis for the space $C^{d-2}_{d-1}(Q_i)$. In particular, the i-th block in the matrix $C_d(\mathbf{Q})$ has as rows the vectors obtained by evaluating that basis of $C^{d-2}_{d-1}(Q_i)$ either at 0 (if the row does not use the point i) or at one of the Q_j's (if the row corresponds to the edge $\{i,j\}$).

Now, let $\mathbf{Q} = (Q_1, \ldots, Q_n)$. A nonsingular linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ induces, for each vector Q_i, a linear map $\tilde{L}_i$ from the space $C^{d-2}_{d-1}(L(Q_i))$ to the space $C^{d-2}_{d-1}(Q_i)$, defined by $\tilde{L}_i(f) = f \circ L$.

Let $M_i \in \mathbb{R}^{d\times d}$ be the matrix of $\tilde{L}_i$ in the bases of $C^{d-2}_{d-1}(L(Q_i))$ and $C^{d-2}_{d-1}(Q_i)$ described above. Let $M \in \mathbb{R}^{dn\times dn}$ be the block-diagonal matrix with blocks of size $d \times d$ and having in the i-th diagonal block the matrix M_i. Then we have that

$$C_d(L(\mathbf{Q})) = C_d(\mathbf{Q})M^{-1}.$$

As L is nonsingular, this proves part (1).

For part (2): the effect of multiplying a Q_i by a scalar λ_i is to multiply all the rows of edges using i by the scalar λ_i^{d-1}. Hence, although the column space $C_d(\mathbf{Q})$ changes by rescaling, the matroid $\mathcal{C}_d(\mathbf{Q})$ does not, and the oriented matroid does not either as long as the rescaling factors are all positive or d is odd. $\square$

We now translate the above result to the original setting of a point configuration $\mathbf{q} = (q_1, \ldots, q_n)$ in $\mathbb{R}^2$:

COROLLARY 2.25. *The matroid $\mathcal{C}_d(\mathbf{q})$ is invariant under projective transformation of $\mathbf{q}$. If d is odd or the projective transformation sends* $\operatorname{conv}(\mathbf{q})$ *to lie in the affine chart of $\mathbb{RP}^2$ (the subset of the projective points $[X : Y : Z]$ with $Z \neq 0$), the same is true for the oriented matroid.*

PROOF. Starting with a point configuration $\mathbf{q} = (q_1, \ldots, q_n)$ in the (affine) plane we can consider the vector configuration $\mathbf{Q} = (Q_1, \ldots, Q_n)$ with $Q_i = (q_i, 1) \in \mathbb{R}^3 \setminus \{0\}$. A projective transformation in $\mathbf{q}$ amounts to a linear transformation in $\mathbf{Q}$. Moreover, if the projective transformation sends $\operatorname{conv}(\mathbf{q})$ to lie in the affine chart of $\mathbb{RP}^2$ then all the Z'_i in the transformed vector configuration are positive, so they can be brought back to the form $(x, y, 1)$ by a positive rescaling. $\square$

Our next result is essentially [141, Theorem 11.3.3] and shows how examples and properties of cofactor rigidity in dimension d can be lifted to dimension $d+1$ via coning. Recall that the contraction $\mathbf{p}/p_i$ of a vector configuration at an element p_i was defined in Section 2.2.2.

PROPOSITION 2.26 (Coning Theorem, [141, Theorem 11.3.3]). *Let $\mathbf{Q} = (Q_1, \ldots, Q_{n+1})$ be a vector configuration in general position in $\mathbb{R}^3$. Then, $\mathcal{C}_d(Q_1, \ldots, Q_n)$ is the contraction to $\binom{[n]}{2}$ of the matroid $\mathcal{C}_{d+1}(\mathbf{Q})$. If the vectors are in convex position, the same is true for the oriented matroids.*

Let us mention that the same result holds for the other two forms of rigidity, $\mathcal{R}$ and $\mathcal{H}$ [141, Theorem 9.3.11], and [74, Theorem 5.1]. We call this statement "coning theorem" because it implies that a graph G with vertex set $[n]$ is d-independent or d-rigid when realized on $(Q_1, \dots, Q_n)$ if and only if its cone $G * \{n+1\}$ is $(d+1)$-independent or $(d+1)$-rigid on $(Q_1, \dots, Q_n, Q_{n+1})$. Here, the *cone over a graph* $G = ([n], E)$ is defined as the graph with vertex set $[n+1]$ and with edges

$$E * \{n+1\} := E \cup \{\{i, n+1\} : i \in [n]\}.$$

Proof. By a linear transformation we can assume without loss of generality that $Q_{n+1} = (0,1,0)$ and that no other Q_i lies in the "hyperplane at infinity" $\{Z = 0\}$; hence, we can rescale them to have $Z_i = 1$ for $i = 1, \dots, n$. This linear transformation and rescaling do not affect the matroids. Moreover, if the original vectors are in convex position, all of them are in a half-space whose delimiting plane contains Q_{n+1}. This implies that we can further assume that the linear transformation sends this plane to $Z_i = 0$ and after this step $Z_i > 0$ for every i, so that the rescaling is positive and does not affect the oriented matroids either.

Under these assumptions we have that

$$\mathbf{c}_{i,n+1} = (0, 0, \dots, (-1)^{d-1}).$$

In particular, the contraction of the elements $\{i, n+1\}$ in the matroid $\mathcal{C}_{d+1}(\mathbf{Q})$ can be performed in the matrix $C_{d+1}(\mathbf{Q})$ as follows: first, forget the last block of columns (the one corresponding to Q_{n+1}). This does not affect the oriented matroid since the sum of the n blocks of $\mathcal{C}_{d+1}(\mathbf{Q})$ equals zero (that is, the columns in each one block are linear combinations of the other blocks). After the block of Q_{n+1} is deleted, the rows $\{i, n+1\}$ that we want to contract have a single non-zero entry, so the contraction is equivalent to deleting those rows and their corresponding columns, namely the last column in the block of each Q_i, $i = 1, \dots, n$.

The resulting matrix coincides with $C_d(Q_1, \dots, Q_n)$ except that the row corresponding to each edge $\{i, j\}$ has been multiplied by the factor $x_{ij} := X_i Z_j - Z_i X_j = X_i - X_j$. General position (under the assumption $Q_{n+1} = (0,1,0)$ and $Z_i = 1$ for every other i) implies $X_i \neq X_j$ for $i \neq j$, so this factor x_{ij} does not affect the matroid.

The factor could a priori affect the oriented matroid, but our assumption that the vectors are in convex position with $Q_{n+1} = (0,1,0)$ and with $Z_1 = \dots = Z_n = 1$ implies that $X_1 < \ldots < X_n$. Hence, the spurious factors x_{ij} are all of the same sign (all negative) and do not change the oriented matroid. □

We now look at what happens if the point we add/delete is not the last one Q_{n+1} but an intermediate one Q_i. This is a mere cyclic reordering of the points with respect to the previous result, but reordering has a non-trivial effect in the cofactor matrix, because of a lack of symmetry in its definition. Indeed, the row of an edge $\{i, j\}$ with $i < j$ has the shape

$$(\dots, \mathbf{c}_{ij}, \dots, -\mathbf{c}_{ij}, \dots).$$

If the reordering keeps i before j the row does not change; its entries simply get moved around as indicated by the reordering. In contrast, if after reordering we end up having j before i then the new row equals

$$(\dots, \mathbf{c}_{ji}, \dots, -\mathbf{c}_{ji}, \dots).$$

That is, we get $\mathbf{c}_{ji}$ where the "moving around" should give $-\mathbf{c}_{ij}$ and $-\mathbf{c}_{ji}$ where we should get $\mathbf{c}_{ij}$. The effect of this depends on the parity of d. If d is even, then $\mathbf{c}_{ji} = -\mathbf{c}_{ij}$ and the relabelling does not affect the oriented matroid. If d is odd, however, then $\mathbf{c}_{ji} = \mathbf{c}_{ij}$, so the relabelling globally changes the sign of that row of the matrix. This implies:

Proposition 2.27. *Let $\mathbf{Q} = (Q_1, \ldots, Q_{n+1})$ be vectors in $\mathbb{R}^3$ in general position. Then the oriented matroid $\mathcal{C}_d(Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_{n+1})$ is obtained from $\mathcal{C}_{d+1}(\mathbf{Q})$ by contracting at the elements $\{i, j\}$ with $j \in [n+1] \setminus \{i\}$, and reorienting the elements $\{j, k\}$ with $1 \leq j < i < k \leq n+1$.*

Proof. Let us first relabel points cyclically so that the point i becomes $n+1$, then apply Proposition 2.26, and finally relabel the points back to their original labels. As noted above, relabelling does nothing if the dimension is even or to the edges that keep their direction, but it reorients the edges that change their direction (that is, the edges $\{j, k\}$ with $j < i < k$) if the dimension is odd. Since we are relabelling first in dimension $d+1$ and then in dimension d, exactly one of them is odd. □

Now we prove a result about the number of sign changes in any dependence in $\mathcal{C}_d(\mathbf{Q})$, with elements in convex position:

Lemma 2.28. *Let $\mathbf{Q} = (Q_1, \ldots, Q_n)$ be vectors in convex position. Let $\lambda \in \mathbb{R}^{\binom{n}{2}}$ be a linear dependence among the rows of $C_d(\mathbf{Q})$. For each i, consider the sequence formed by $\{\lambda_{ij}\}_{j \neq i}$, with values of j ordered cyclically from i. That is, with the order $(i+1, i+2, \ldots, n, 1, \ldots, i-1)$. Then:*

1. *If d is even, the sequence changes sign at least d times.*
2. *If d is odd, the same happens for the sequence $\{-\lambda_{ij}\}_{j>i} \cup \{\lambda_{ij}\}_{j<i}$.*

Proof. Let us first assume that d is even. In this case, as mentioned above, a cyclic relabelling does not change the oriented matroid, so we can assume without loss of generality that $i = n$. Also, by linear transformation and positive rescaling we can assume that $Q_n = (0, 0, 1)$ and that $Z_j = 1$ and $X_j > 0$ for $j \neq i$. Observe that under these assumptions we have

$$\mathbf{c}_{jn} = (X_j^{d-1}, X_j^{d-2} Y_j, \ldots, Y_j^{d-1}) = X_j^{d-1}(1, m_j, \ldots, m_j^{d-1}),$$

where $m_j := Y_j / X_j$ is the slope of the segment from $q_n = (0, 0)$ to $q_j = (X_j, Y_j)$. Since the X_j are positive we can neglect them without changing the oriented matroid, and convex position implies that the m_j are increasing: $m_1 < \ldots < m_{n-1}$.

Hence, the sequence $(\lambda_{jn})_{j \in [n-1]}$ that we want to study is (at least regarding its signs) a linear dependence among the vectors $(1, m_j, \ldots, m_j^{d-1})$ for an increasing sequence of m_j's. Put differently, it is an affine dependence among the vertices of a cyclic $(d-1)$-polytope. It is well-known that the circuits in the cyclic polytope are alternating sequences with $d+1$ points [**34**, Theorem 6.1.11], hence they have d sign changes. Since every dependence is a composition of circuits [**34**, Lemma 4.1.12, Corollary 4.1.13], it has at least the same number of changes.

For the case where d is odd all of the above remains true except the initial cyclic relabelling reverses the sign of all the λ_{ij} with $j > i$. □

2.3.2. The Morgan-Scott obstruction in cofactor rigidity. In this section we show that the graph obtained from K_6 by removing a perfect matching (that is, the graph of an octahedron) is a circuit or a basis in the three-dimensional cofactor rigidity $\mathcal{C}_3$, depending on whether the points are in "Desargues position" or not. This is well-known in the theory of splines, and usually called the *Morgan-Scott split* or Morgan-Scott configuration [87]. We here rework it, following [139, Example 4], since we need an oriented version of it. See also [90, Example 41, p. 90].

DEFINITION 2.29 (Desargues position). Let $\mathbf{q} = (q_1, \dots, q_6)$ be a configuration of six points in convex position in the plane. Let us call upper side of the line 25 the side containing the points 1 and 6, and lower side the other one. We say that $\mathbf{q}$ is *positively (resp. negatively) oriented* if the intersection of the lines 14 and 36 lies in the lower (resp. upper) side of 25. We say that $\mathbf{q}$ is in *Desargues position* if none of the two happens, that is, if the lines 14, 25 and 36 are concurrent.

See Figure 2 for an illustration of this concept, with points chosen along the standard parabola. We call the concurrent case *Desargues position* since Desargues theorem says that this concurrency is equivalent to the triangles $q_1q_3q_5$ and $q_2q_4q_6$ being axially perspective.

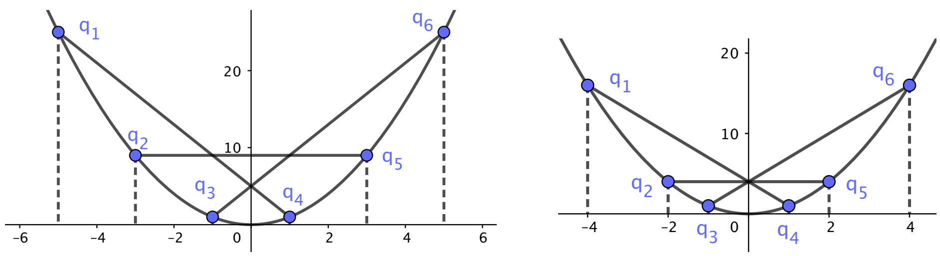


FIGURE 2. Two configurations of six points in convex position, chosen along the parabola. The configuration in the left is positively oriented, the one on the right is in Desargues position

THEOREM 2.30. *Consider the graph $G = K_6 \setminus \{25, 36\}$ embedded with six points $\mathbf{q}$ in general position. Then,*

(1) *G is spanning in $\mathcal{C}_3(q)$, hence it contains a unique dependence. This dependence may not vanish at any edge other than* 14.
(2) *Assume $\mathbf{q}$ in convex position, and let $(\lambda_{ij})_{ij} \in \mathbb{R}^{\binom{6}{2}}$ be this unique dependence. Then $\lambda_{15} \neq 0$ and the sign of $\lambda_{14}/\lambda_{15}$ is positive, negative, or zero, if $\mathbf{q}$ is positively oriented, negatively oriented, or in Desargues position, respectively.*

This statement immediately implies:

COROLLARY 2.31. *Consider the graph $G = K_6 \setminus \{14, 25, 36\}$ embedded with six points $\mathbf{q}$ in convex position. Then, G is a circuit in $\mathcal{C}_3(q)$ if the points are in Desargues position, and a basis otherwise.*

The proof of the first part of Theorem 2.30 is easy. Let $G' = G \setminus \{ij\}$ for an edge ij different from 14. Without loss of generality assume $i \notin \{1, 4\}$. Then G' has degree three at vertex i and $G' \setminus i$ equals K_5 minus one edge. Since K_5 is a circuit in $\mathcal{C}_3$ (for any choice of points in general position), $G' \setminus i$ is a basis, and hence G' is a basis too. In particular, G is spanning and contains a unique circuit, and this circuit does not vanish at the edge ij.

To prove part two, we follow the derivation in [139, Example 4]. There, the following concept is introduced as a way to express the determinant of a submatrix in the cofactor matrix C_3 of a triangulated sphere.

DEFINITION 2.32 (3-fan). Given a graph $G = ([n], E_0)$ and a bipartition of the vertices $[n] = V_0 \cup V_1$, a *3-fan* in (V_0, V_1, E_0) is an orientation of G such that the vertices in V_0 have out-degree 3 and those in V_1 have out-degree 0.

For a 3-fan π and a vertex $i \in V_0$, let π^i be the set of three edges that start at the vertex i. The *sign of* π, denoted $\sigma(\pi)$, is the sign of $(\pi^1, \pi^2, \dots)$ as a permutation of E_0, with the three elements of each π^i in increasing order, multiplied by $(-1)^r$ where r is the number of edges oriented from a vertex to another with lower index.

In what follows, we will denote by $C_d(\mathbf{q})|_{(E,V)}$ the restriction of the cofactor matrix of $\mathbf{q}$ to the rows indexed by E and the column blocks indexed by V.

DEFINITION 2.33 (Notation $[q_i; q_j q_k q_l]$). For $\mathbf{q} : V \to \mathbb{R}^2$, we define $[q_i; q_j q_k q_l] = \det C_3(\mathbf{q})|_{(\{(i,j),(i,k),(i,l)\},i)}$. This determinant can be shown to be equal to $|q_i q_j q_k| \cdot |q_i q_j q_l| \cdot |q_i q_k q_l|$, where $|abc|$ denotes the determinant of the three points (written as $(x, y, 1)$).

The following statement summarizes the derivations in [139, pp.15–17]:

LEMMA 2.34. *Let (V, E) be the graph of a triangulated sphere and $\mathbf{q} : V \to \mathbb{R}^2$ a position for the vertices that realizes the graph as planar. Let V_0 the set of internal vertices, V_1 the three external vertices and E_0 the internal edges of the graph. Then*

$$\det C_3(\mathbf{q})|_{(E_0,V_0)} = \sum_{\pi \text{ 3-fan in } \{V_0,V_1,E_0\}} \sigma(\pi)[\pi^1][\pi^2]\dots[\pi^{n-3}]$$

where $[\pi^i]$ stands for $[q_i; q_j q_k q_l]$ with $\pi^i = \{(i, j), (i, k), (i, l)\}$.

With this we can finish the proof of Theorem 2.30:

PROOF OF THEOREM 2.30. The coefficients of a row dependence in an $(N + 1) \times N$ matrix are the complementary minors of each row with alternating signs. In our case, our initial matrix $C_3(\mathbf{q})$ has size 13×18 and rank 12, but we can by an affine transformation fix the positions of vertices 1, 2 and 3 (which implies no loss of generality) and then delete the nine columns of their three blocks, plus the three rows of the triangle they form. This leaves us with a 10×9 matrix whose unique row-dependence we want to study. The coefficients λ_{14} and λ_{15} have the same sign in the dependence if and only if their complementary minors have opposite sign.

To compute these two signs we use Lemma 2.34 with $V_0 = \{4, 5, 6\}, V_1 = \{1, 2, 3\}$.

For the edge 15 this is easy because the remaining edges form $K_6 \setminus \{12, 13, 23, 15, 25, 36\}$, in which the only possible 3-fan is $\{41, 42, 43, 53, 54, 56, 61, 62, 64\}$. This is an even permutation in which there are 8 "reversed" edges, hence the sign of the 3-fan is positive.

By Lemma 2.34, the determinant is

$$[4; 123][5; 346][6; 124] = |412|\,|413|\,|423|\,|534|\,|536|\,|546|\,|612|\,|614|\,|624|$$

where i stands for q_i. A determinant of three points is positive if they are ordered counter-clockwise and negative otherwise. In this case the result is positive because there are two negative determinants, 536 and 546. (Here and in the rest of the proof we assume without loss of generality that our points are not only in convex position but also placed in counter-clockwise order along their convex hull.)

Now we compute the determinant for $K_6 \setminus \{12, 13, 23, 14, 25, 36\}$. There are two 3-fans here:

$$\{42, 43, 46, 51, 53, 54, 61, 62, 65\} \text{ and } \{42, 43, 45, 51, 53, 56, 61, 62, 64\}$$

Both are even permutations, the first one with 8 reversed edges and the second with 7. By Lemma 2.34 the determinant is

$$\begin{aligned}
&[4; 236][5; 134][6; 125] - [4; 235][5; 136][6; 124] = \\
=&|423|\,|426|\,|436|\,|513|\,|514|\,|534|\,|612|\,|615|\,|625| \\
&\qquad - |423|\,|425|\,|435|\,|513|\,|516|\,|536|\,|612|\,|614|\,|624| \\
=&|126|\,|135|\,|145|\,|156|\,|234|\,|246|\,|256|\,|345|\,|346| \\
&\qquad - |126|\,|135|\,|146|\,|156|\,|234|\,|245|\,|246|\,|345|\,|356| \\
=&|126|\,|135|\,|156|\,|234|\,|246|\,|345|\,(|145|\,|256|\,|346| - |146|\,|245|\,|356|)
\end{aligned}$$

The factor $|126|\,|135|\,|156|\,|234|\,|246|\,|345|$ is positive: since the points are in convex counter-clockwise position every determinant $|abc|$ with $a < b < c$ is positive. Hence, we ignore it. To further simplify the rest we use the Plücker relations

$$|145|\,|256| = |125|\,|456| + |245|\,|156|, \qquad |146|\,|356| = |346|\,|156| - |546|\,|136|.$$

Hence, the last factor becomes:

$$\begin{aligned}
&|145|\,|256|\,|346| - |146|\,|245|\,|356| \\
=&|125|\,|456|\,|346| + |245|\,|156|\,|346| - |346|\,|245|\,|156| + |546|\,|245|\,|136| \\
=&|456|\,(|125|\,|346| - |136|\,|245|)
\end{aligned}$$

Dividing again by the positive factor $|456|$ and by $|245|\,|346|$ we get that the sign of the determinant equals

$$\frac{|125|}{|245|} - \frac{|136|}{|346|} = \frac{|120|}{|240|} - \frac{|130|}{|340|},$$

where we call q_0 (and abbreviate as 0) the intersection point of 25 and 36. This last expression can be rewritten in term of the angles around q_0, as follows:

$$\frac{|120|}{|240|} - \frac{|130|}{|340|} = \frac{\sin\angle 201}{\sin\angle 402} - \frac{\sin\angle 301}{\sin\angle 403} = \frac{\sin\alpha}{\sin\beta} - \frac{\sin\alpha'}{\sin\beta'}, \tag{8}$$

where α, α', β and β' are displayed in Figure 3.

Looking at the figure we see that the configuration is positively oriented if, and only if, $\alpha < \beta$ and $\alpha' > \beta'$, and it is negatively oriented if the opposite inequalities hold. Hence:

$$\begin{aligned}
\lambda_{14}\lambda_{15} > 0 &\Leftrightarrow \text{the complementary determinants have opposite sign} \\
&\Leftrightarrow \text{the complementary determinant to 14 is negative} \\
&\Leftrightarrow \text{the value of (8) is negative} \\
&\Leftrightarrow \alpha < \beta \text{ and } \alpha' > \beta' \\
&\Leftrightarrow \text{the configuration is positively oriented.}
\end{aligned}$$

□

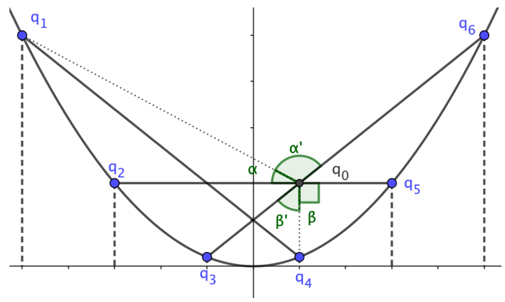


FIGURE 3. Explanation of the last equivalence in the proof of Theorem 2.30. Each of the inequalities $\alpha < \beta$ and $\alpha' > \beta'$ is equivalent to the configuration being positively oriented.

Consider now the graph $K_6 \setminus \{14, 25, 36\}$, with vertices in convex position. That is, K_6 minus a perfect matching, which is the graph of an octahedron. Since this graph equals $G \setminus \{14\}$, Corollary 2.31 implies that $K_6 \setminus \{14, 25, 36\}$ is a circuit in $\mathcal{C}_3$ in Desargues position, and a basis in non-Desargues position. In particular:

COROLLARY 2.35. *The graph K_6 without the matching $\{14, 25, 36\}$ is a circuit in the polynomial rigidity matroid $\mathcal{P}_3(1, 3, 4, 6, 7, 9)$, and a basis in the generic matroid $\mathcal{P}_3$.*

PROOF. Applying a translation to the parameters t_i along the parabola produces an affine transformation of the point configuration, hence it does not change the oriented matroid $\mathcal{P}$. So, the statement for $\mathcal{P}_3(1, 3, 4, 6, 7, 9)$ is equivalent to the same statement for the more symmetric $\mathcal{P}_3(-4, -2, -1, 1, 2, 4)$. The latter is in Desargues position, as seen in Figure 2. □

We can now prove that, for $k = 3$ and $n = 9$, there are positions where the rows of the cofactor matrix do not realize the multiassociahedron as a basis collection:

PROOF OF THEOREM 2.6. We start with the graph $K_6 - \{16, 37, 49\}$, with vertices labelled $\{1, 3, 4, 6, 7, 9\}$. Its coning at three vertices labelled 2, 5 and 8 is the graph $K_9 - \{16, 37, 49\}$. The statement in the first sentence then follows from Proposition 2.27 and Corollary 2.31. The second sentence follows from Corollary 2.35. □

Even more strongly, we can show that in this "realization" not even the map $\phi_{\overline{\Delta}_3(9),\mathcal{V}} : |\overline{\Delta}_3(9)| \to S^5$ of Theorem 2.18 is well-defined:

EXAMPLE 2.36. Let T be the 3-triangulation $K_9 \setminus \{16, 37, 49\}$, and locate it with a $\mathbf{p}$ in Desargues position, so that it contains a circuit. For example, embedding it via $P(1, 2, 3, 4, 5, 6, 7, 8, 9)$ in the parabola. By Theorem 2.6 T is a circuit in this embedding. The vertices 1, 3, 4, 6, 7 and 9 have degree 7, so the edges incident to each of them must have alternating signs in the circuit by Lemma 2.28. As a result, the six relevant edges in T, which are $\{1, 5\}$, $\{5, 9\}$, $\{3, 8\}$, $\{4, 8\}$, $\{2, 7\}$ and $\{2, 9\}$, all have the same sign.

The fact that the circuit is positive in all relevant edges implies that the map sending $\overline{\Delta}_3(9)$ to $\mathbb{R}^{k(n-2k)} = \mathbb{R}^6$ (mapping each vertex to the corresponding row vector of $\mathcal{P}(t_1, \ldots, t_9)$ and extending linearly in each face) contains the origin in its image. Hence, it does not produce a well-defined map $\phi_{\overline{\Delta}_3(9),\mathcal{V}} : |\overline{\Delta}_3(9)| \to S^5$. Moreover, if we choose positions $\mathbf{p}'$ and $\mathbf{p}''$ close to $\mathbf{p}$ that are sufficiently generic to make T a basis but with opposite orientations, then the degrees of the maps $|\overline{\Delta}_3(9)| \to S^5$ obtained will differ by one unit.

2.3.3. Cofactor rigidity does not realize $\overline{\Delta}_k(n)$, for $n \geq 2k+6$, $k \geq 3$. Although our main result in this section deals with the case $n \geq 2k+6$, for most of the section we assume $n = 2k+3$ and characterize exacty when does cofactor rigidity C_{2k} realize $\overline{\Delta}_k(2k+3)$ as a complete fan. (We already saw in Theorem 2.6 that it not always does).

With $n = 2k+3$ there are exactly $2k+3$ relevant edges, namely those of the form $\{a, a+k+1\}$, for $a \in [n]$. These edges form a $(k+1)$-star S that we call the *relevant star*. We will normally consider the relevant edges in their "star order": the cyclic order in which $\{a, a+k+1\}$ is placed right after $\{a-k-1, a\}$. Put differently, the edges of the relevant star, in their star order, are

$$S = \{\{1, k+2\}, \{1, k+3\}, \{2, k+3\}, \{2, k+4\}, \dots, \{k+1, 2k+3\}, \{k+2, 2k+3\}\}$$

Removing a number ℓ of edges of the relevant star results in ℓ paths counting as a "path of length zero" the empty path between two consecutive edges removed.

A simple counting shows that k-triangulations with $2k+3$ vertices are of the form $K_{2k+3} \setminus \{3 \text{ edges}\}$. However, K_{2k+3} minus three relevant edges is not always a k-triangulation. The necessary and sufficient condition is that the three paths obtained removing these edges are even. This "evenness criterion" is the reason why $\overline{\Delta}_k(2k+3)$ is combinatorially a cyclic polytope of dimension $2k$ in $2k+3$ vertices.

In a similar way, the union of two adjacent k-triangulations is obtained removing two relevant edges from K_{2k+3}. We call such unions *circuits* since we want to realize them as circuits in the vector configuration. The two relevant paths in a circuit C necessarily have different parity, and the k-triangulations contained in C are obtained removing an edge that splits the odd path into two even paths. (For this to be doable in more than one way the odd path in C must be of length at least three. But if C equals K_{2k+3} minus two edges and the odd relevant path in C has length one then C contains a $(k+1)$-crossing, so we are not interested in it).

Any codimension-two face F of $\overline{\Delta}_k(2k+3)$, in turn, is of the form "K_{2k+3} minus five relevant edges". Let $\{a, b, c, d, e\}$ be the edges in star order, so that the relevant star minus $\{a, b, c, d, e\}$ consists of five paths (some of which may have length zero, as remarked above). The length of the elementary cycle of F depends on the parity of the five paths, as follows (see Example 2.14 for the first three cases with $k = 2$):

- If the five paths are even, then the cycle has length five and the vertices of the cycle (that is, the k-triangulations containing F) are obtained adding $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, e\}$, and $\{e, a\}$.
- If two consecutive paths, say (a, b) and (b, c), are odd, then the cycle has length three and its vertices are formed with $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$.
- If two non-consecutive paths, say (a, b) and (c, d), are odd, then the cycle has length four and the vertices are formed with $\{a, c\}$, $\{c, b\}$, $\{b, d\}$ and $\{d, a\}$.
- If only one path is even then no multitriangulation contains F. F is not really a face.

Hence, the only case with length 5 is the one with all intervals even.

We call a k triangulation of the $(2k+3)$-gon *octahedral* if its three missing edges have six distinct endpoints; equivalently, if the three relevant paths in it have positive length. (Observe that this needs $2k+3 \geq 9$, hence $k \geq 3$). The reason for this name is that any such k-triangulation is, as a graph, an iterated cone (an odd number of times, greater than 1) over the graph of an octahedron.

LEMMA 2.37. *Consider a configuration* $\mathbf{q}$ *in convex position. Let* $1 \leq i_1 < i_2 < i_3 < i_4 \leq 2k+3$ *be such that* $C := K_{2k+3} \setminus \{\{i_1, i_3\}, \{i_2, i_4\}\}$ *is a circuit and let* λ *be its unique (modulo a scalar factor) dependence in the cofactor matric* $C_{2k}(\mathbf{q})$*; in particular, we assume that* $i_3 - i_1, i_4 - i_2 \in \{k+1, k+2\}$*. Let* $\{i'_1, i'_3\}$ *be the first edge in the odd path of* $S \setminus \{\{i_1, i_3\}, \{i_2, i_4\}\}$ *(which can be* $\{i_1 \pm 1, i_3\}$ *or* $\{i_1, i_3 \pm 1\}$*) and let* $\{j_1, j_2\}$ *be another edge in the same path at a distance* d *from the first. Then, we have that*

$$\operatorname{sign}(\lambda_{j_1 j_2}) = (-1)^d \operatorname{sign}(\lambda_{i'_1 i'_3})$$

if and only if the triangle formed by the three edges $\{i_1, i_3\}$*,* $\{i_2, i_4\}$ *and* $\{j_1, j_2\}$ *is in the inner side of* $\{i_1, i_3\}$ *and* $\{i_2, i_4\}$ *(the side of length* $k+2$*).*

In particular, if all edges with d *even satisfy the condition, then the condition ICoP is satisfied by all flips contained in* C*.*

PROOF. Without loss of generality we can suppose that $j_1 < i_1 < i_2 < j_2 < i_3 = i_1 + k + 1 < i_4 = i_2 + k + 2$. Then $i'_1 = i_1 - 1$, $i'_3 = i_3$, and by the definition of star order, $j_1 + j_2 = i'_1 + i'_3 - d = i_1 + i_3 - d - 1$.

In the circuit, the degree of i_3 is $2k+1$ and by Lemma 2.28,

$$\operatorname{sign}(\lambda_{i'_1 i'_3}) = \operatorname{sign}(\lambda_{i_1 - 1, i_3}) = (-1)^{i_1 - j_1 - 1} \operatorname{sign}(\lambda_{j_1 i_3})$$

so the condition to be checked reduces to

$$\operatorname{sign}(\lambda_{j_1 j_2} \lambda_{j_1 i_3}) = (-1)^{d+1+i_1-j_1} = (-1)^{i_3 - j_2}$$

The circuit is obtained by a repeated coning from the K_6 without two edges, so that the original six vertices become $\{j_1, i_1, i_2, j_2, i_3, i_4\}$. The sign of an edge is inverted whenever we make a cone with the new vertex between the endpoints of that edge. As a result, the sign of $\lambda_{j_1 j_2} \lambda_{j_1 i_3}$ is inverted $i_3 - j_2 - 1$ times exactly, so in the graph at the beginning, we should have $\lambda_{14} \lambda_{15} < 0$.

By Theorem 2.30, this happens when the triangle formed by 14, 25 and 36 is negatively oriented. After making the cones, the configuration $\{j_1, i_1, i_2, j_2, i_3, i_4\}$ is negatively oriented and the triangle is in the side between i_3 and i_4, which is the inner side of the two edges. □

Observe that a relevant edge with $n = 2k+3$ leaves k points of the configuration on one side and $k+1$ on the other side. We call the one with $k+1$ points *the big half-plane* defined by the relevant edge.

THEOREM 2.38. *Let* $\mathbf{q} = (q_1, q_2, \dots, q_{2k+3})$ *be a configuration in convex position in* $\mathbb{R}^2$*. The following are equivalent:*

(1) $C_{2k}(\mathbf{q})$ *realizes* $\overline{\Delta}_k(2k+3)$ *as a complete fan.*
(2) *For every octahedral* k*-triangulation* T *the big half-planes defined by the three edges not in* T *have non-empty intersection.*
(3) *The relevant star has "non-empty interior" (that is, the big half-planes of all relevant edges have non-empty intersection).*

REMARK 2.39. It is interesting to note that, when condition (3) holds, any point o taken in the "interior" of the relevant star makes the vector configuration $\{q_1 - o, \dots q_{2k+3} - o\}$ be a Gale transform of the cyclic $2k$-polytope with $2k+3$ vertices. That is to say, the theorem says that $\mathbf{q}$ realizes $\overline{\Delta}_k(2k+3)$ as a fan if and only if there is a point $o \in \mathbb{R}^2$ such that $\mathbf{q} - o$ is the Gale transform of a cyclic polytope. It seems to be a coincidence that the cyclic polytope in question is in fact isomorphic to $\overline{\Delta}_k(2k+3)$.

PROOF. The implication (3)⇒(2) is trivial. Let us see the converse. First observe that, by Helly's Theorem, the intersection of all half-planes is non-empty if, and only if, the intersection of every three of them is non-empty. So, we only need to show that, when condition (3) is restricted to three half-planes, only the case where the half-planes come from the missing edges in an octahedral triangulation matters.

So, consider three relevant edges $\{\{i_1, i_4\}, \{i_2, i_5\}, \{i_3, i_6\}\}$ and their corresponding big half-planes. We look at the three paths obtained in the relevant star when removing these three edges. If at least one path (and hence exactly two) has odd length, then the intersection of the three big half-planes is automatically non-empty: let the even interval be (i_6, i_1). Then the edge $\{i_2, i_5\}$ crosses the other two and leaves both i_1 and i_6 in its big half-plane, so we can always find a point in the intersection of the three half-planes in a neighborhood of the intersection of the lines containing $\{i_1, i_4\}$ and $\{i_3, i_6\}$.

Similarly, if two of the three edges are consecutive (say $i_6 = i_1$), then the intersection of their two half-planes is an angle of the relevant star. This angle necessarily meets both of the half-planes defined by the third edge $\{i_2, i_5\}$, so the intersection is again non-empty.

That is, the only case of three edges whose big half-planes might perhaps produce an empty intersection is when the three paths they produce are even and non-empty. This is exactly the same as saying that they are the three missing edges of an octahedral k-triangulation, which proves (2)⇒(3).

Now, the implication (1)⇒(2) follows from the previous lemma: if the complete fan is realized, the condition ICoP is satisfied in the flips from an octahedral triangulation, in particular, the flip from $K_{2k+3} \setminus \{\{i_1, i_3\}, \{i_2, i_4\}, \{j_1, j_2\}\}$ that removes $\{i'_1, i'_3\}$ and inserts $\{j_1, j_2\}$, with d even. By Lemma 2.37, this is equivalent to saying that the big half-planes of $\{i_1, i_3\}$, $\{i_2, i_4\}$ and $\{j_1, j_2\}$ intersect, which covers all the cases in (2). So it only remains to show that (2)⇒(1).

If (2), or equivalently (3), holds then we know, by the previous argument, that flips from an octahedral triangulation satisfy ICoP. These are exactly the flips whose two missing edges do not share a vertex. The other flips must be of the form K_{2k+3} minus two edges with a common end-point. Hence, they contain a K_{2k+2}, in which the signs are as predicted by Lemma 2.28 and the ICoP property also holds in them.

To finish the proof, we just need to check the condition about elementary cycles of length 5. Given one of these cycles, adding three consecutive edges of the five in the cycle gives the graph of a flip. If two edges in the cycle share a vertex, we can add the other three edges to get the graph of a flip that contains a K_{2k+2}, which has the two flipping edges as diameters and the other edge with length k, so it has opposite sign. Otherwise, the five edges are disjoint.

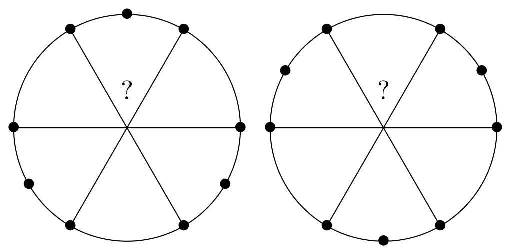


FIGURE 4. Two subsets of nine vertices of the convex 12-gon. By Theorem 2.38, in order to realize the face of the 3-associahedron corresponding to each subset of nine points, the three lines in the left should intersect in the opposite way to the three lines in the right, which means it is impossible to realize both at the same time.

In this case, let $\{a, b, c, d, e\}$ be the edges. By the condition (3), their five big half-planes have non-empty intersection. Without loss of generality, suppose that b is a side of that intersection. Adding a, b and c, we get the graph of a flip. As b is a side of the intersection, the triangle formed by b, d and e is inside the big half-planes of d and e, so we can again apply Lemma 2.37 to get that b has opposite sign to a and c, as we wanted to prove. □

COROLLARY 2.40. *For every k there are point configurations $\mathbf{q}$ such that $C_{2k}(\mathbf{q})$ realizes $\overline{\Delta}_k(2k+3)$ as a fan. For example, the vertices of a regular $(2k+3)$-gon.*

PROOF. The barycenter of the $(2k+3)$-gon lies in the interior of the relevant star. □

This theorem also implies Theorem 2.7. Cofactor rigidity with points in convex position cannot realize $\overline{\Delta}_k(n)$ as a fan for $n \geq 2k+6$ and $k \geq 3$:

PROOF OF THEOREM 2.7. Let $\mathbf{q} = (q_1, \ldots, q_n)$ be a configuration in convex position. By Lemma 2.9 we only need to show the case $n = 2k+6$.

Let $I_1 = [n] \setminus \{4, k+5, k+9\}$ and $I_2 = [n] \setminus \{2, 6, k+7\}$. Then $\mathbf{q}|_{I_1}$ and $\mathbf{q}|_{I_2}$ are configurations with $2k+3$ points, to which we can apply Theorem 2.38. We consider their respective k-triangulations $T_1 = K_{I_1} \setminus \{\{1, k+4\}, \{3, k+6\}, \{5, k+8\}\}$ and $T_2 = K_{I_2} \setminus \{\{1, k+4\}, \{3, k+6\}, \{5, k+8\}\}$. This theorem tells us that in order for $\mathbf{q}_{I_1}$ to realize $\overline{\Delta}_k(2k+3)$ we need $(q_1, q_3, q_5, q_{k+4}, q_{k+6}, q_{k+8})$ to be negatively oriented, and in order for $\mathbf{q}_{I_2}$ to realize it we need the same configuration to be positively oriented. This is a contradiction, so one of the two does not realize $\overline{\Delta}_k(2k+3)$. Lemma 2.9 implies that $\mathbf{q}$ does not realize $\overline{\Delta}_k(2k+6)$. □

See Figure 4 for an illustration of the proof.

2.4. Positive results on realizability, for $k = 2$

In this section, we prove Conjecture 2.2 for $k = 2$ and Theorem 2.5.

2.4.1. 2-triangulations are bases in $\mathcal{P}_2(n)$. To prove Theorem 2.3 (that is, Conjecture 2.2 for $k = 2$) we need operations that send a 2-triangulation in n vertices to one in $n + 1$ vertices, and viceversa. These operations are called the *inflation of a* 2*-crossing*, and the *flattening of a star*. They are defined for arbitrary k in [**106**, Section 7], and also studied in [**24**, section 2.1.4], but we use only the case $k = 2$.

An *external* 2*-crossing* is a 2-crossing with two of the end-points consecutive. An *external* 2*-star* is one with three consecutive points. Equivalently, one using a boundary edge (an edge of length two). Let u, v, w be three consecutive vertices of the $(n + 1)$-gon, and consider the n-gon obtained by removing the middle vertex v. The identity map on vertices then induces a bijection between external 2-stars in the $(n + 1)$-gon using the boundary edge $\{u, w\}$ and external 2-crossings in the n-gon using the vertices u, w. If, to simplify notation, we let $[n + 1]$ and $[n]$ be our vertex sets, with $u = n$, $v = n + 1$ and $w = 1$, the bijection is

$$S = \{\{n, 1\}, \{1, c\}, \{c, n + 1\}, \{n + 1, b\}, \{b, n\}\} \quad \leftrightarrow \quad C = \{\{n, b\}, \{1, c\}\}.$$

From now on let us fix an external star $S \subset \binom{[n+1]}{2}$ and its corresponding external crossing $C \in \binom{[n]}{2}$, of the above form. Consider the set $\mathrm{lk}_2(S)^0$ (respectively $\mathrm{lk}_2(C)^0$) of relevant edges that do not form a 3-crossing with S, (respectively, with C); they are, respectively, the sets of vertices in $\mathrm{lk}_{\overline{\Delta}_2(n+1)}(S)$ and in $\mathrm{lk}_{\overline{\Delta}_2(n)}(C)$.

THEOREM 2.41 ([**106**, Section 7]). *The following map is a bijection*

$$\begin{array}{rrcl} \phi: & \mathrm{lk}_2(S)^0 & \to & \mathrm{lk}_2(C)^0 \\ & \{i,j\} & \mapsto & \{i,j\} \quad \text{if } n+1 \notin \{i,j\} \\ & \{i,n+1\} & \mapsto & \begin{cases} \{i,n\} & \text{if } 1 \le i < b \\ \{1,i\} & \text{if } c < i \le n \end{cases} \end{array}$$

and it induces an isomorphism of simplicial complexes

$$\hat{\phi} : \mathrm{lk}_{\overline{\Delta}_2(n+1)}(S) \xrightarrow{\cong} \mathrm{lk}_{\overline{\Delta}_2(n)}(C).$$

PROOF. The map is well defined because $\{i, n + 1\}$ is in $\mathrm{lk}_2(S)^0$ if and only if $i \notin [b, c]$.

It is injective because the only edges that could have the same image are $\{i, n\}$ and $\{i, n + 1\}$ if $1 \le i < b$ or $\{1, j\}$ and $\{j, n + 1\}$ if $c < j \le n$. But in the first case $\{i, n\}$ would form a 3-crossing with $\{b, n + 1\}$ and $\{c, 1\}$, and in the second case $\{1, j\}$ would form a 3-crossing with $\{b, n\}$ and $\{c, n + 1\}$.

It is surjective because if $\{i, j\} \in \binom{[n]}{2}$ is not in the image of ϕ then the first case in the definition implies $\{i, j\} \notin \mathrm{lk}_2(S)^0$, but the only edges in $\binom{[n]}{2} \setminus \mathrm{lk}_2(S)^0$ are those with $1 \le i < b$ and $c \le j < n$. Among these, the only ones in $\mathrm{lk}_2(C)^0$ are those with $i = b$ or $j = c$, which are in the image of ϕ.

To show that it induces an isomorphism of the complexes we need to check that if $T \subset \binom{[n+1]}{2}$ is 3-crossing-free and contains S then $\phi(T \setminus S) \cup C$ is also 3-crossing-free, and vice versa. These are essentially Lemmas 7.3 and 7.6 in [**106**]. □

DEFINITION 2.42 (Flattening of a star, inflation of a crossing). Let $e \in \binom{[n+1]}{2}$ be a boundary edge, let $S \subset \binom{[n+1]}{2}$ be an external 2-star using e, and let T be a 2-triangulation containing S.

Let $\hat{\phi}$ be the isomorphism of Theorem 2.41 (after a cyclic relabelling sending e to $\{n,1\}$). We say that the 2-triangulation $\hat{\phi}(T)$ is the *flattening of e in T*, and denote it $\underline{T}_e$. We also say that T is the *inflation* of C in $\hat{\phi}(T)$.

The crucial fact that we need is, under certain conditions, *inflation of a 2-crossing* is a particular case of a *vertex split*.

DEFINITION 2.43 (Vertex d-split). A *vertex d-split* in a graph $G = (V, E)$ consists in changing a vertex $u \in V$, with degree at least $d-1$, into two vertices u_1 and u_2 joined by an edge and joining all neighbours of u to at least one of u_1 or u_2, and exactly $d-1$ of the neighbors to both.

Put differently, the graph G' with vertex set $V \setminus \{u\} \cup \{u_1, u_2\}$ is a vertex d-split of a graph G on V if, and only if: G' contains the edge u_1u_2, the contraction of that edge produces G, and u_1 and u_2 have exactly $d-1$ common neighbors in G'.

LEMMA 2.44 ([106, proof of Theorem 8.7]). *Inflation of a "doubly external" 2-crossing of the form $C = \{\{n,b\},\{1,n-1\}\}$ in a 2-triangulation T is an example of vertex 4-split at n, with new vertices n and $n+1$. The inflated star has four consecutive vertices $n-1, n, n+1$ and 1.*

PROOF. Let T' be the inflated 2-triangulation. Plugging $c = n-1$ in Theorem 2.41 gives

$$\begin{array}{rrcll} \phi: & \mathrm{lk}_2(S)^0 & \to & \mathrm{lk}_2(C)^0 & \\ & \{i,j\} & \mapsto & \{i,j\} & \text{if } n+1 \notin \{i,j\} \\ & \{i,n+1\} & \mapsto & \{i,n\} & \text{if } 1 \le i < b \\ & \{n,n+1\} & \mapsto & \{1,n\}, & \end{array}$$

which implies that the relevant edges of T are indeed obtained from those of T' by identifying the vertices n and $n+1$. The same happens for the irrelevant and boundary edges (which are independent of T and T'). It also implies that all the neighbors of n in T are neighbors of at least one of n and $n+1$ in T'.

Hence, we only need to check that n and $n+1$ have exactly three common neighbors in T'. This holds since $n-1$, 1 and b are common neighbors of n and $n+1$ in T', and any additional common neighbour b' would create a 3-crossing with $\{n-1,1\}$ and either $\{n,b\}$ if $b' > b$ or $\{n+1,b\}$ if $b' < b$. ☐

That vertex d-splits preserve independence in both $\mathcal{R}_d(n)$ and $\mathcal{C}_d(n)$ is a classical result [141, pp. 68 and Remark 11.3.16]. Preserving independence holds also in $\mathcal{H}_d(n)$ and $\mathcal{P}_d(n)$:

PROPOSITION 2.45 ([28, Proposition 4.10], [24, Teorema 1.2.13]). *Corank does not increase under vertex d-split neither in $\mathcal{H}_d$ nor in $\mathcal{P}_d$. Hence, vertex d-splits of independent graphs are independent, both in $\mathcal{H}_d$ and $\mathcal{P}_d$.*

This has the following consequence. We only state and prove it for $\mathcal{H}_d(\mathbf{q})$ (which includes the case of $\mathcal{P}_d(\mathbf{t})$ when points are chosen along the moment curve) but the same statement, with the same proof, holds also for $\mathcal{R}_d(\mathbf{q})$ and $\mathcal{C}_d(\mathbf{q})$.

COROLLARY 2.46. *Let $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be a configuration in $\mathbb{R}^d$ (resp. in the moment curve) and assume that a certain graph G is a circuit in $\mathcal{H}_d(\mathbf{q})$. Let G' be a vertex d-split of G and consider it embedded in positions $\mathbf{q}'$ that are generic (resp. generic along the moment curve) and sufficiently close to $\mathbf{q}$.*

Then, G' is either independent in $\mathcal{H}_d(\mathbf{q}')$ or it contains a unique circuit. If the latter happens, then the signs of the non-splitting edges are preserved.

PROOF. First perturb $\mathbf{q}$ to be generic, which either makes it independent or maintains it being a circuit. Proposition 2.45 implies that G' is independent in the first case and that it is either independent or it contains a unique circuit in the second case.

So, we only need to show that if the latter happens then all the non-spliting edges are part of the circuit and that they preserve their signs. That they are part of the circuit follows again from the proposition: if $e \in G'$ is not a spliting edge then it comes from an edge of G. Now, since G was a circuit, $G \setminus e$ was independent; hence $G' \setminus e$ is also independent, so e belongs to the circuit.

To see that the signs are preserved, consider moving the points continuously from $\mathbf{q}'$ back to $\mathbf{q}$. (At the end, the two vertices created in the split collide into the same position but we can still consider them two different vertices of the graph G', with a degenerate embedding). Since the positions of $\mathbf{q}'$ are taken sufficiently close to those of $\mathbf{q}$, this continuous motion can be made through positions at which G' always has a unique dependence, and such that the signs of the dependence do not change except perhaps at the end of the path, when we get to $\mathbf{q}$. At the end of the path the dependence must degenerate to the original (unique) dependence of G, in the sense that the coefficients of non-splitting edges are the same in G and G', and the coefficients of the splitting edges in G' add up to those in G. Now, since the signs of the non-splitting edges are never zero along the path and still non-zero at the end, by continuity they must be preserved. □

Dependence, however, is not preserved. See example after [28, Proposition 4.10].

DEFINITION 2.47 (Ears). A star in a 2-triangulation is *doubly external* if it has four consecutive vertices, like the ones that can be obtained in Lemma 2.44.

An *ear* of the 2-triangulation is an edge of length 3.

For every ear $\{a, a+3\}$ in a 2-triangulation T there is a unique star in T using the vertices a, $a+1$, $a+2$ and $a+3$ (hence, a doubly external star), and it has $\{a, a+2\}$, $\{a, a+3\}$ and $\{a+1, a+3\}$ as three consecutive sides. We call this star the star *bounded by the edge* $\{a, a+3\}$.

THEOREM 2.48 ([106, Corollary 6.2], [24, Corolario 2.1.15]). *The number of ears in a 2-triangulation is exactly 4 more than the number of internal stars.*

PROOF OF THEOREM 2.3. This is proved by induction in n. For $n = 5$, the only 2-triangulation is K_5, that is a basis in 4 dimensions.

Suppose the statement is true for n and take a 2-triangulation T' on $n+1$ vertices. By Theorem 2.48, T' has at least four ears. Without loss of generality, suppose one of them is $\{n-1, 1\}$. Then it bounds a doubly external star with the vertices $n-1, n, n+1$ and 1. Let b be the remaining vertex in the star. Then, T' can be obtained from a 2-triangulation T in n vertices inflating the 2-crossing $\{\{n, b\}, \{1, n-1\}\}$. By Lemma 2.44, this operation is a vertex 4-split, that preserves independence by Proposition 2.45. □

2.4.2. Realizing individual elementary cycles. We now prove some results for realizability as a fan in the case $k = 2$. Among other things, we show that for $n \leq 7$ any position of the points in the plane will realize the multiassociahedron $\overline{\Delta}_2(n)$ as a fan via the cofactor rigidity matrix.

COROLLARY 2.49. *For $n = 2k+2$, any choice of $q_1, \dots, q_{2k+2} \in \mathbb{R}^2$ in convex position makes the rows of the cofactor matrix realize $\overline{\Delta}_k(n)$ as a polytopal fan.*

PROOF. In this case, all the k-triangulations are K_{2k+2} minus a diameter. The simplicial complex is in this case the boundary of a $(k-1)$-simplex. We also know that K_{2k+2} is a circuit, therefore all the k-triangulations are bases.

In this circuit, Lemma 2.28 implies that the sign of each edge coincides with the parity of its length. The flipped edges in this case are two of the diameters, that have all the same sign. Hence, the condition ICoP in Corollary 2.20 is true. The condition on the elementary cycles is trivial because all of them have length three.

This implies that every position of the points realizes the boundary of the simplex as a complete fan. But realizing a simplex as a complete fan is equivalent to realizing it as a polytope, so the corollary is proved. □

COROLLARY 2.50 ([24, Proposición 2.2.20]). *For $k = 2$ and $n = 7$, any choice of $q_1, \dots, q_7 \in \mathbb{R}^2$ in convex position realizes $\overline{\Delta}_2(7)$ as a fan.*

PROOF. By Theorem 2.38, the fact that a position realizes the fan is equivalent to the interior of the 3-star formed by the seven points being non-empty. This is equivalent to saying that the big half-planes of every three edges without common vertices have non-empty intersection, which is trivial, because any three such edges are consecutive in the circle, and the seventh vertex will always be in the intersection. □

We now look at the case $k = 2$ and $n \geq 8$. We are going to show that for each elementary cycle there are positions that make that cycle simple (and, in particular, for every flip there is an embedding that makes ICoP hold for that flip). Of course, this does not imply that $\overline{\Delta}_2(n)$ can be realized as a fan; for that we would need fixed positions that work for all cycles, not one position for each cycle. But this implication would hold if 2-triangulations were basis at arbitrary positions (Theorem 2.5).

We need the fact that, in this case, neither a flip nor an elementary cycle of length 5 can use all the doubly external stars in a 2-triangulation.

LEMMA 2.51. *Let T be a 2-triangulation on at least 8 vertices.*

1. *For any relevant edge e in T there is a doubly external star in $T \setminus \{e\}$.*
2. *If e and f are two relevant edges in T and the elementary cycle $\mathrm{lk}_{\overline{\Delta}_2(n)}(T \setminus \{e, f\})$ has length five then there is a doubly external star in $T \setminus \{e, f\}$.*

PROOF. A star cannot be bounded by more than two ears, and if a star S is bounded by two ears then its five vertices are consecutive. That is, the edges of S are the two ears plus three boundary edges. We call such stars *triply external*.

Two triply external stars cannot have a common edge (except for $n = 6$, but we are assuming $n \geq 8$). This, together with the fact that T has at least four ears (Theorem 2.48) implies part (1): if T has two triply external stars then (at least) one of them does not use e, and if T has one triply external star, or none, then the existence of four ears implies that there are at least three doubly external stars in T, and only two of them can use e.

We now look at part (2) of the statement. By the proof of Corollary 2.16, for the link of $T \setminus \{e, f\}$ to have length five we need that there is a star S_0 in T using both e and f, plus another two stars S_e and S_f using each one of e and f. We want to show that there is a doubly external star that is not any of these three. (These three may or may not be doubly external, or external).

Suppose, to seek a contradiction, that no star other than these three is doubly external. Then every ear bounds one of these three stars. Since only triply external stars are bounded by two (and then only two) ears, the total number of ears is at most three plus the number of triply external stars among S_0, S_e and S_f. The three cannot be triply external (because S_0 shares edges with both S_e and S_f), so the number of ears is at most five. But five ears would imply S_e and S_f to be triply external, in particular e and f to be ears bounding them, and S_0 to be doubly external, bounded by the fifth ear, different form e and f. In particular, the five edges of S_0 would have to be two boundary edges and three ears, which can only happen with $n = 7$ (because two of the ears would need to share a vertex and have their other end-points consecutive).

So, there are at most four ears in T and, by Theorem 2.48, exactly four. Moreover, they all bound S_0, S_e or S_f. By Theorem 2.48 again, this implies that each of the other $n-7$ stars in T is an external star, but not a doubly external one. That is, each of these $n-7$ stars contains one and only one of the $n-7$ boundary edges, and the other seven boundary edges are distributed among S_0, S_e and S_f.

Let T_0 be the 2-triangulation of the 7-gon obtained by flattening one by one the $n-7$ boundary edges not in S_0, S_e or S_f. Observe that, when we flatten a singly external star, all other stars have the same number of boundary edges before and after the flattening. In particular, the last star that was flattened was still singly external before the flattening, so it becomes a singly external 2-crossing (that is, a crossing of two relevant edges) in T_0. At the end, in T_0 only S_0, S_e and S_f survive, and the edges e and f are such that their link is a cycle of length five (because all throughout the process the link of $T \setminus \{e, f\}$ preserves its length, by Theorem 2.41).

The contradiction is that for the cycle to be of length five we need the two relevant edges in $T \setminus \{e, f\}$ to be non-crossing, as in the third picture of Example 2.14, but those two edges must cross because they are the 2-crossing obtained form the last star that was flattened. □

THEOREM 2.52. (1) *For each pair of adjacent facets in* $\overline{\Delta}_2(n)$ *there is a choice of parameters for* $P_4(t_1, \ldots, t_n)$ *that makes the corresponding circuit of the polynomial rigidity matrix satisfy ICoP.*
(2) *For each elementary cycle with length 5 in* $\overline{\Delta}_2(n)$ *there is a choice of parameters for* $P_4(t_1, \ldots, t_n)$ *that makes the cycle simple.*

This is stated in [24] as Teorema 2.2.11 and Teorema 2.2.17, but the proof contains errors.

PROOF. The proof goes by induction in n. For $n \leq 7$ it is already proved in corollaries 2.49 and 2.50, so suppose it is true for n and prove it for $n+1 \geq 8$.

For the first part, let T_1 and T_2 be two 2-triangulations we are looking at, and let e, f be the edges in $T_1 \setminus T_2$ and $T_2 \setminus T_1$, respectively. Part (1) of the previous lemma implies that there is a doubly external star $S \subset T_1 \setminus \{e\} = T_1 \cap T_2$. By Theorem 2.41, flattening S in T_1 and T_2 we get 2-triangulations T_1' and T_2' that still differ by a flip, and by the inductive hypothesis the sign condition ICoP will hold in the circuit $T_1' \cup T_2'$ for certain choice of the parameters t_i. Now we return to the flip graph in $n+1$ vertices by the reverse operation of flattening a doubly external star, which is a vertex split by Lemma 2.44. If we keep the two split vertices close enough, the signs of the non-split edges (which include the flip edges e and f) will not be altered (Corollary 2.46), and the ICoP condition still holds.

For the second part, let our elementary cycle be (the link of) $T \setminus \{e, f\}$, for a 2-triangulation T and relevant edges $e, f \in T$. By the previous lemma, there is a doubly external star $S \subset T \setminus \{e, f\}$.

Again, we can flatten S, assume by the inductive hypothesis that the sign condition in part (3) of Corollary 2.20 holds in the flattened 5-cycle, and return to $n+1$ vertices by a vertex split that will preserve the sign condition if the split vertices are kept close enough. □

Proof of Theorem 2.5. Suppose, for a contradiction, that all positions realize $\overline{\Delta}_2(n)$ as a basis collection, but there is a position $\mathbf{t}$ that does not realize it as a fan. This implies that, at $\mathbf{t}$, there is an elementary cycle with wrong signs. But, by Theorem 2.52, there is another position $\mathbf{t}'$ giving the right signs in that cycle.

Consider now a continuous transition between $\mathbf{t}$ and $\mathbf{t}'$. At some point, the signs need to change, either by attaining condition ICoP at a flip, or by making the cycle simple. But any of the two ways would involve collapsing some cone to lower dimension at that point, which does not happen by hypothesis. □

2.4.3. Experimental results. In this section we report on some experimental results. In all of them we choose real parameters $\mathbf{t} = \{t_1 < t_2 < \ldots < t_n\}$ (actually we choose them integer, so that they are exact) and computationally check whether the configuration of rows of $P_{2k}(t_1, \ldots, t_n)$ realizes $\overline{\Delta}_k(n)$ first as a collection of bases, then as a complete fan, and finally as the normal fan of a polytope.

For the experiments we have written Python code which, with input k, n and the parameters $\mathbf{t}$, first constructs the set of all k-triangulations and then checks the three levels of realizability as follows:

(1) Realizability as a collection of bases amounts to computing the rank of the submatrix $P_{2k}(\mathbf{t})|_T$ corresponding to each k-triangulation T.

(2) For realizability as a fan we first check the ICoP property, which amounts to computing the signs of certain dependences among rows of $P_{2k}(\mathbf{t})$. There is one such dependence for each ridge in the complex, so the total number of them is $ND/2$ where N is the number of k-triangulations on n points and $D = k(n - 2k - 1)$ is the dimension of $\overline{\Delta}_k(n)$.

    If ICoP holds then we check that a certain vector lies in the positive span of a unique facet of the complex. We do this for the sum of rays corresponding of a particular k-triangulation, the so-called *greedy* one. This property, once we have ICoP, is equivalent to being realized as a fan by parts (2) and (3) of Theorem 2.18.

    The *greedy* k-triangulation is the (unique) one containing all the irrelevant edges and the edges in the complete bipartite graph $[1, k] \times [k+1, n]$, and only those. It is obvious that these edges do not contain any $(k+1)$-crossing and we leave it to the reader to verify that the number of relevant ones is indeed $k(n - 2k - 1)$.

(3) For realizability as a polytope we then need to check feasibility of the linear system of inequalities (6) from Lemma 2.22.

Here, without loss of generality we can assume that the lifting vector f_{ij} is zero in all edges of a particular k-triangulation, and we again use the greedy one. This reduces the number of variables in the feasibility problem from $n(n-2k-1)/2$ to $(n-2k)(n-2k-1)/2$, a very significant reduction for the values of (n, k) where we can computationally construct $\overline{\Delta}_k(n)$. Apart of the computational advantage, it saves space when displaying a feasible solution; in all the tables in this section we show only the non-zero values of f_{ij}, which are those of relevant edges contained in $[k+1, n]$. Note that taking all the f_{ij}'s of a particular k-triangulation equal to zero makes the rest strictly positive.

Remark 2.53. If a choice of parameters realizes $\overline{\Delta}_k(n)$ (at any of the three levels) for a certain pair (k, n) then deleting any j of the parameters the same choice realizes $\overline{\Delta}_{k'}(n-j)$ for any k' with $k - j/2 \leq k' \leq k$. This follows from Lemma 2.9 plus the fact that each of the three levels of realization is preserved by taking links.

Our first experiment is taking equispaced parameters. Since an affine transformation in the space of parameters produces a linear transformation in the rows of $P_{2k}(\mathbf{t})$, we take without loss of generality $\mathbf{t} = (1, 2, 3, \ldots, n)$. We call these the *standard positions* along the parabola.

For $k \geq 3$ and $n \geq 2k+3$ we show in Theorem 2.6 that standard positions do not even realize $\overline{\Delta}_k(n)$ as a collection of bases. Hence, we only look at $k = 2$.

Lemma 2.54. *Let $\mathbf{t} = \{1, 2, \ldots, n\}$ be standard positions for the parameters. Then:*

(1) *Standard positions for $P_4(\mathbf{t})$ realize $\overline{\Delta}_2(n)$ as the normal fan of a polytope if and only if $n \leq 9$.*
(2) *The non-standard positions $\mathbf{t} = (-2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ for $P_4(\mathbf{t})$ realize $\overline{\Delta}_2(10)$ as the normal fan of a polytope.*
(3) *Standard positions for $P_4(\mathbf{t})$ realize $\overline{\Delta}_2(n)$ as a complete fan for all $n \leq 13$.*

Proof. For part (1), by Lemma 2.54 we only need to check that $n = 9$ works and $n = 10$ does not. For $n = 8, 9$ Table 1 shows values of $(f_{ij})_{i,j}$ that prove the fan polytopal. For $n = 10$ the computer said that the system is not feasible (which finishes the proof of part (1)), but modifying the standard positions to the ones in part (2) it gave the feasible solution displayed in Table 2.

For part (3), the computer checked the conditions for a complete fan for $n = 8, 9, 10, 11, 12, 13$. Only the last one would really be needed; this last one took about 7 days of computing in a standard laptop. □

Let us mention that for $n = 11$ we have tried several positions besides the standard ones. All of them realize the complete fan but none realizes it as polytopal. Among the positions we tried are the "equispaced positions along a circle" that we now explain.

The standard parabola is projectively equivalent to the unit circle. Since $P_{2k}(\mathbf{t})$ is linearly equivalent to the cofactor rigidity matrix $C_{2k}(\mathbf{q})$ for the points $\mathbf{q}$ of the parabola corresponding to the parameters $\mathbf{t}$, it makes sense to look at the values of $\mathbf{t}$ that produce equispaced points (that is, a regular n-gon) when the parabola is mapped to the circle. We call those values of $\mathbf{t}$, "equispaced along the circle".

i,j	f_{ij}
3,6	3
3,7	14
3,8	36
4,7	3
4,8	16
5,8	6

i,j	f_{ij}	i,j	f_{ij}
3,6	7	4,8	33
3,7	29	4,9	95
3,8	76	5,8	6
3,9	165	5,9	42
4,7	9	6,9	16

TABLE 1. Height vectors $(f_{ij})_{i,j}$ realizing $\overline{\Delta}_2(8)$ (left) and $\overline{\Delta}_2(9)$ (right) as a polytopal fan with rays in $P_4(1,2,\ldots,n)$ (standard positions).

i,j	f_{ij}	i,j	f_{ij}	i,j	f_{ij}
3,6	44	4,7	45	5,9	1062
3,7	161	4,8	260	5,10	42019
3,8	424	4,9	1722	6,9	196
3,9	1733	4,10	60296	6,10	13048
3,10	46398	5,8	106	7,10	6146

TABLE 2. A lifting vector that leads to a polytopal realization of the multiassociahedron for $k=2$ and $n=10$, with $(t_i)_{i=1,\ldots,10} = (-2,1,2,3,4,\,5,6,7,9,20)$.

They are

$$t_i = \tan\left(\alpha_0 + \frac{i}{n}\pi\right), \quad i = 1,\ldots,n \tag{9}$$

for any choice of α_0, with a symmetric choice being $\alpha_0 = -(n+1)\pi/2n$.

For $k=3$, $n=10$, we already know that standard positions do not even give a basis collection. We have tried two strategies to realize $\overline{\Delta}_3(10)$: perturbing the standard positions slightly we have been able to recover independence (that is, a basis collection), but not the fan (the ICoP condition was not satisfied). Using equispaced positions along the circle via Equation (9) the positions realize the polytope.

For $k=3$, $n=11$ and for $k=4$ and $n=12,13$ equispaced positions realize the fan but not the polytope, even after trying several perturbations.

LEMMA 2.55. (1) *For the same positions* $\mathbf{t}$ *of Table 2,* $P_6(\mathbf{t})$ *realizes* $\overline{\Delta}_3(10)$ *as the normal fan of a polytope. The following are valid values of* f*:*

i,j	$f_{i,j}$
4,8	*4*
4,9	*69*
4,10	*16074*
5,9	*14*
5,10	*10281*
6,10	*3948*

(2) *Equispaced positions along the circle realize* $\overline{\Delta}_k(n)$ *as a fan for* $(n,k) \in \{(3,11),(4,12),(4,13)\}$. ☐

CHAPTER 3

The multiassociahedron and tropical Pfaffians

In this chapter we explore relations between k-triangulations and the variety $\mathcal{P}f_k(n)$. Our starting point is restricting Gröbner bases and tropicalization to weight vectors satisfying the following "four-point positivity" conditions.

Definition 3.1. We say that a weight vector $v \in \mathbb{R}^{\binom{[n]}{2}}$ is *four-point positive* (abbreviated *fp-positive*) if for all $1 \le a < a' < b < b' \le n$ we have that

$$v_{a,b} + v_{a',b'} \ge \max\{v_{a,a'} + v_{b,b'}, v_{a,b'} + v_{a',b}\}. \tag{10}$$

We denote by FP_n the subset of $\mathbb{R}^{\binom{[n]}{2}}$ consisting of fp-positive vectors. That is to say, $v \in \mathrm{FP}_n$ if the maximum weight given by v to the three matchings among four points is attained always for the matching that forms a 2-crossing.

Although the polyhedron $\mathrm{FP}_n \subset \mathbb{R}^{\binom{[n]}{2}}$ of fp-positive vectors (the solution set of equations (10)) is defined by $2\binom{n}{4}$ inequalities, the following $\binom{n}{2} - n$ alone are an irredundant description of it, with indices considered cyclically:

$$v_{a,b} + v_{a+1,b+1} - v_{a,b+1} - v_{a+1,b} \ge 0, \quad \forall \{a,b\} \in \binom{[n]}{2} \text{ with } |a-b| > 1, \tag{11}$$

The left-hand side coefficient vectors (that is, the facet normals of FP_n) are linearly independent, so that FP_n is linearly isomorphic to an orthant plus a lineality space of dimension n. We like to think of FP_n as the "positive orthant" of $\mathbb{R}^{\binom{[n]}{2}}$ regarding Pfaffians. It can be interpreted as the space of weights that represent *separation vectors* among sides of the n-gon, or as the weights that are monotone with respect to crossings among perfect matchings of each fixed even set $U \subset [n]$. See Proposition 3.10 and Corollary 3.11 for details.

Algebraically, fp-positive vectors are the monomial weight vectors for which the leading form of every 3-term Plücker relation

$$x_{a,b}x_{a',b'} - x_{a,a'}x_{b,b'} - x_{a,b'}x_{a',b}, \qquad 1 \le a < a' < b < b' \le n,$$

contains the crossing monomial. These relations generate the ideal of the Grassmannian $\mathcal{G}r_2(n)$. In particular, fp-positive vectors are the (closed) Gröbner cone of $\mathcal{G}r_2(n)$ producing as initial ideal the one generated by 2-crossings $x_{a,b}x_{a',b'}$.

Extending this, we denote by $\mathrm{Grob}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$ the Gröbner cone consisting of weights that select the $(k+1)$-crossing as the leading monomial (or as one of them) in every Pfaffian of degree $k+1$. What we say above can then be stated as $\mathrm{Grob}_1(n) = \mathrm{FP}_n$, and the result of [73] says that $\mathrm{Grob}_k(n)$ has non-empty interior. In Section 3.1 we show that $\mathrm{FP}_n \subset \mathrm{Grob}_k(n)$ (Theorem 3.13) and give an explicit description of $\mathrm{Grob}_k(n)$, both by inequalities and by generators (Theorem 3.14).

Theorem 3.2 (Theorem 3.14). *For any $k > 2n + 2$, $\mathrm{Grob}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$ is a simplicial cone with a lineality space of dimension n.*

(1) *It is generated by:*
   - *(lineality space) For each $i \in [n]$, the line generated by the indicator vector of the set $\{\{i,j\} : j \in [n] \setminus i\}$.*
   - *("short" generators) For each $\{i,j\} \in [n]$ with $1 \le |i-j| \le k$, the negative basis vector corresponding to $\{i,j\}$.*
   - *("long" generators) For each $\{i,j\} \in [n]$ with $|i-j| \ge k+2$, the ray of FP_n corresponding to $\{i,j\}$.*

(2) *An irredundant facet description of it is given by the following $\binom{[n]}{2} - n$ inequalities:*
   - *("long" inequalities) For each $\{i,j\} \in [n]$ with $|i-j| \ge k+1$, the inequality (11) corresponding to $\{i,j\}$.*
   - *("short" inequalities) For each $\{i,j\} \in [n]$ with $2 \le |i-j| \le k$, the sum of the inequalities (11) corresponding to all the $\{i',j'\}$ with $|i'-j'| \le k+1$ and with $\{i,j\}$ contained in the short side of $\{i',j'\}$.*

In particular, $\mathrm{Grob}_k(n)$ *contains* FP_n *for every k and n.*

This description has the following combinatorial interpretation: modulo its lineality space (of dimension n, equal to that of FP_n), $\mathrm{Grob}_k(n)$ is a simplicial cone with one facet and generator corresponding to each of the $\binom{[n]}{2} - n$ edges of length at least two. The "long" facet-inequalities (those corresponding to relevant edges) are the same as the corresponding ones in FP_n, and the "short" ones are looser in $\mathrm{Grob}_k(n)$ than in FP_n.

Moreover, we show that the monomial initial ideal of $I_k(n)$ produced by any generic weight vector $v \in \mathrm{Grob}_k(n)$ equals the Stanley-Reisner ideal of $\Delta_k(n)$. That is, to say, the ideal in $\mathbb{K}[x_{i,j}, \{i,j\} \in \binom{[2k]}{2}]$ generated by $(k+1)$-crossings. This, in turn, implies that k-triangulations are bases of the algebraic matroid of $\mathcal{P}f_k(n)$ (Corollary 3.20). We find this of interest for two reasons (see Section 3.1.3 for details):

On the one hand, the algebraic matroid $\mathcal{M}(\mathcal{P}f_k(n))$ of $\mathcal{P}f_k(n)$ is closely related to *low-rank completion* of antisymmetric matrices [8, 77]: given a subset $T \subset \binom{[n]}{2}$ of positions for entries in an antisymmetric matrix M of size $n \times n$, a generic choice of values for those entries can be extended to an antisymmetric matrix of rank $\le 2k$ if and only if T is independent in $\mathcal{M}(\mathcal{P}f_k(n))$. Thus:

THEOREM 3.3. *Let $T \subset \binom{[n]}{2}$.*

(1) *If T is $(k+1)$-free and $\mathbb{K}$ is algebraically closed, then for any generic choice of values $v \in \mathbb{K}^T$ there is at least one skew-symmetric matrix of rank $\le 2k$ with the entries prescribed by v.*

(2) *If T contains a k-triangulation then for any choice of values $v \in \mathbb{K}^T$ there is only a finite number (maybe zero) of skew-symmetric matrices of rank $\le 2k$ with those prescribed entries.*

On the other hand, the algebraic matroid of $\mathcal{P}f_k(n)$ coincides with the generic *hyperconnectivity matroid* in dimension $2k$ defined by Kalai [74]. The fact that k-triangulations are bases in it is closely related to the conjecture by Pilaud and Santos [106] that they are bases in the generic bar-and-joint rigidity matroid in dimension $2k$.

In Section 3.2 we turn our attention to the tropicalization of $\mathcal{P}f_k(n)$. More precisely, we denote $\mathrm{Pf}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$ the intersection of the tropical hypersurfaces corresponding to Pfaffians of degree k. This is by definition a tropical *prevariety*. It contains the tropical variety $\mathrm{trop}(\mathcal{P}f_k(n))$ but it does not, in general, coincide with it, as we show in Theorem 3.29.

In the light of Theorem 3.2, it makes sense to look at the part of $\mathrm{Pf}_k(n)$ contained in the Gröbner cone $\mathrm{Grob}_k(n)$. That is, we define

$$\mathrm{Pf}_k^+(n) := \mathrm{Pf}_k(n) \cap \mathrm{Grob}_k(n).$$

Since the crossing monomial is the only positive monomial in each 3-term Plücker relation, for $k = 1$ we have

$$\mathrm{trop}^+(\mathcal{P}f_1(n)) = \mathrm{trop}(\mathcal{P}f_1(n)) \cap \mathrm{FP}_n = \mathrm{Pf}_1^+(n).$$

One of our main results partially generalizes this to higher k:

THEOREM 3.4 (See Theorem 3.31 and Corollary 3.33).

(1) $\mathrm{Pf}_k^+(n) = \mathrm{Grob}_k(n) \cap \mathrm{trop}(\mathcal{P}f_k(n)) \subset \mathrm{trop}^+(\mathcal{P}f_k(n))$.
(2) $\mathrm{Pf}_k^+(n)$ *is the union of the faces of* $\mathrm{Grob}_k(n)$ *corresponding to* $(k+1)$*-free graphs.*

This theorem says that for a $v \in \mathrm{Grob}_k(n)$, being in $\mathrm{Pf}_k(n)$ is equivalent to the fact that the "long inequalities" of Theorem 3.2 (that is, the inequalities (11) for $|a-b| \geq k+1$) are satisfied with equality except in a $(k+1)$-free set. Moreover, when this happens v can be proved to be in $\mathrm{trop}(\mathcal{P}f_k(n))$, and in fact in $\mathrm{trop}^+(\mathcal{P}f_k(n))$.

In part (2), by *the face corresponding to* a certain graph $G \subset \binom{[n]}{2}$ we mean the intersection of the facets of $\mathrm{Grob}_k(n)$ corresponding to $\binom{[n]}{2} \setminus G$ in the description of Theorem 3.2. That is, we consider $\mathrm{Grob}_k(n)$ as (a cone over) the simplex with vertex set $\binom{[n]}{2}$, so that every simplicial complex on $\binom{[n]}{2}$ is a subcomplex of its face complex. Hence, Theorem 3.4 has the following interpretation:

COROLLARY 3.5. *As a simplicial fan and modulo its lineality space,* $\mathrm{Pf}_k^+(n) = \mathrm{Grob}_k(n) \cap \mathrm{trop}(\mathcal{P}f_k(n))$ *is isomorphic to (the cone over) the extended multiassociahedron* $\Delta_k(n)$.

REMARK 3.6. $\mathrm{Pf}_k^+(n)$ is *not* equal to $\mathrm{trop}^+(\mathcal{P}f_k(n))$. Put differently, "four point positivity" implies but is *not the same* as positivity in the sense of Definition 1.9. See Example 3.35.

Theorem 3.4 suggests that one way to realize the multiassociahedron as a polytope would be to find a projection $\mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{k(2n-2k-1)}$ that is injective on $\mathrm{Pf}_k^+(n)$. This would embed $\Delta_k(n)$ as a full-dimensional simplicial fan in $\mathbb{R}^{k(2n-2k-1)}$ whose link at the irrelevant face would necessarily realize the multiassociahedron $\overline{\Delta}_k(n)$ as a complete fan in $\mathbb{R}^{k(n-2k-1)}$. A second step is needed in order to show polytopality: to find appropriate right-hand sides showing that the complete fan is polytopal.

We have achieved both steps for $k = 1$. We show that, for any seed triangulation T, the projection $\mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{2n-3}$ that keeps only the coordinates corresponding to edges in T is injective on $\mathrm{Pf}_1^+(n)$ (Corollary 3.42). That is, we have an explicit projection sending $\mathrm{Pf}_1^+(n)$ to (the normal fan of) the associahedron. It was pointed out to us by Vincent Pilaud that the embedding that we obtain is exactly the so-called **g**-*vector fan* associated to the seed triangulation. **g**-vector fans can be defined in an arbitrary cluster algebra of finite type and starting with any seed cluster, and they were shown to be polytopal by Hohlweg, Pilaud and Stella [66]. See Section 3.3 for details.

THEOREM 3.7 (Corollary 3.47). *For each seed triangulation T of the n-gon, projection of $\mathrm{Pf}_1^+(n)$ to the $n-3$ coordinates of the edges in T gives a realization of the $(n-3)$-associahedron in $\mathbb{R}^{n-3}$ isomorphic to the $\mathbf{g}$-vector fan of T.*

This would seem to open up the possibility of using these same ideas to find polytopal realizations of $\overline{\Delta}_k(n)$ for higher k, by adapting to k-triangulations the (quite simple) procedure used to define the $\mathbf{g}$-vectors from a seed triangulation. Unfortunately, our final result Corollary 3.50 says that this approach is doomed to fail, under certain natural assumptions.

3.1. The variety of antisymmetric matrices of bounded rank

3.1.1. Four-point positive weight vectors. We now introduce certain term orders for the variables that produce as initial ideal of $I_k(n)$ the monomial ideal generated by $(k+1)$-crossings. For this, we need to introduce a change of basis in $\mathbb{R}^{\binom{[n]}{2}}$, and a change of point of view on the n-gon.

Let us call a-th side of the n-gon the edge $\{a-1, a\}$ (with indices taken modulo n). Then, any choice of real numbers $w_{i,j}$ (with $\{i,j\} \in \binom{[n]}{2}$) for the edges connecting *vertices* of the n-gon induces a "separation" distance between each pair of *sides*, as the sum of w's of the edges separating those sides. That is:

DEFINITION 3.8. Given a vector $w \in \mathbb{R}^{\binom{[n]}{2}}$, the *separation vector* $d(w) \in \mathbb{R}^{\binom{[n]}{2}}$ induced by w is defined as

$$d_{a,b}(w) = \sum_{\substack{(i,j)\in\binom{[n]}{2} \\ a\le i<b\le j<a}} w_{ij}, \qquad \forall \{a,b\} \in \binom{[n]}{2}, \tag{12}$$

Here the order symbols "$<$" and "$\le$" for indices are considered cyclically. E.g., $a<b<c<a$ means that a,b,c are different and they appear in that cyclic order along the n-gon.

Figure 1 shows an example of this transformation. To compute $d_{26}(w)$, where $a=2$ and $b=6$ denote two sides of the octagon, we have to sum the w_{ij}s in the complete bipartite graph on the two subsets of vertices separated by a and b.

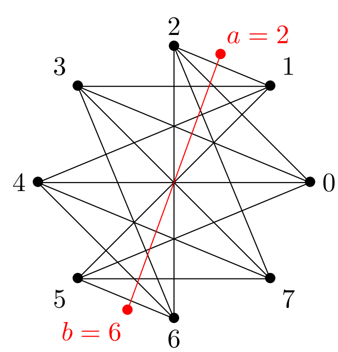


FIGURE 1

The entries of $d(w)$ are going to be used as weights for variables in our monomial orders, but we want to have in mind the weight vector w from which they come. This is well-defined thanks to the following result, which implies that the transformation from w to $d(w)$ is a linear isomorphism in $\mathbb{R}^{\binom{[n]}{2}}$:

Proposition 3.9. *For any* $w \in \mathbb{R}^{\binom{[n]}{2}}$, *and every* $\{a,b\} \in \binom{[n]}{2}$, *we have*

$$2w_{a,b} = d_{a,b}(w) + d_{a+1,b+1}(w) - d_{a,b+1}(w) - d_{a+1,b}(w), \tag{13}$$

where $d_{a,a}(w) = 0$ *by convention.*

Hence, each $v \in \mathbb{R}^{\binom{[n]}{2}}$ *can be expressed uniquely as* $d(w)$ *for a certain* $w \in \mathbb{R}^{\binom{[n]}{2}}$.

Proof. It is enough to check that the rest of w_{ij} cancel out when $d_{a,b}(w) + d_{a+1,b+1}(w) - d_{a,b+1}(w) - d_{a+1,b}(w)$ is computed via Eq.(12). □

That is, we can think of $d(w)$ and w as different choices of linear coordinates for $\mathbb{R}^{\binom{[n]}{2}}$.

Proposition 3.10. *Let* $v \in \mathbb{R}^{\binom{[n]}{2}}$ *be a weight vector. The following conditions are equivalent:*

(1) $v \in \mathrm{FP}_n$. *That is, it satisfies the positive four-point conditions* (10) *in Definition 3.1.*
(2) v *satisfies the* $\binom{n}{2} - n$ *inequalities* (11).
(3) $v = d(w)$ *in the sense of Definition 3.8 for a* w *with* $w_{a,b} \geq 0$ *for all* $\{a,b\} \in \binom{[n]}{2}$ *with* $|a-b| > 1$.
(4) *For every* $k \geq 1$ *and every* $U \in \binom{[n]}{2k}$ *the weights given by* v *to matchings in* U *are monotone with respect to swaps that create crossings.*
(5) *For every* $k \geq 1$ *and every* $U \in \binom{[n]}{2k}$ *the maximum weight given by* v *to matchings in* U *is attained by the* k*-crossing.*

Proof. The equivalence of parts 1 and 4 is obvious and the equivalence of 2 and 3 follows from Proposition 3.9. The implications $5 \Rightarrow 1 \Rightarrow 2$ are also trivial because the inequalities in condition 1 are nothing but the case $k = 2$ of condition 5, and they contain the inequalities in condition 2 as a subset.

The implication $4 \Rightarrow 5$ follows from the fact that every matching can monotonically be converted into a full crossing by swaps that create crossings.

Finally, the implication $3 \Rightarrow 1$ follows from the fact that if $1 \leq a < a' < b < b' \leq n$ then Equations (12) give

$$\begin{aligned} v_{a,b} + v_{a',b'} &= W_1 + W_2 + W_3, \\ v_{a,a'} + v_{b,b'} &= W_1 + W_2, \\ v_{a,b'} + v_{a',b} &= W_1 + W_3, \end{aligned}$$

where

$$W_1 = \sum_{a \leq i < a' \leq j < b} w_{ij} + \sum_{a' \leq i < b \leq j < b'} w_{ij} + \sum_{b \leq i < b' \leq j < a} w_{ij} + \sum_{b' \leq i < a \leq j < a'} w_{ij},$$

$$W_2 = \sum_{\substack{a \leq i < a', \\ b \leq j < b'}} w_{ij},$$

$$W_3 = \sum_{\substack{a' \leq i < b, \\ b' \leq j < a}} w_{ij}.$$

Since w is nonnegative (except perhaps for consecutive indices) we have that $W_2, W_3 \geq 0$ and hence $v_{a,b} + v_{a',b'}$ is greater or equal than both of $v_{a,a'} + v_{b,b'}$ and $v_{a,b'} + v_{a',b}$. $\square$

That is to say, FP_n is essentially the positive orthant in the w coordinates, except for one detail. Proposition 3.9 implies that the inequalities (11) from the introduction are equivalent to

$$w_{a,b} \geq 0 \qquad \forall \{a,b\} \in \binom{[n]}{2} \text{ with } |a-b|>1;$$

but the inequalities $w_{a,a+1} \geq 0$ are not valid in FP_n. The n-dimensional subspace generated by the vectors with $w_{a,b} = 0$ if $|a-b|>1$ and $w_{a,a+1}$ arbitrary can be thought of as the "irrelevant" part of the w coordinates; in fact, is the lineality space of FP_n. This suggests we give it a name. We denote:

$$L_n := \left\{ d(w) : w \in \mathbb{R}^{\binom{[n]}{2}} \text{ and } w_{i,j} = 0 \text{ if } |i-j|>1 \right\} \cong \mathbb{R}^n,$$
$$\mathrm{FP}_n^+ := \left\{ d(w) : w \in \mathbb{R}_{\geq 0}^{\binom{[n]}{2}} \right\} \cong \mathbb{R}_{\geq 0}^{\binom{[n]}{2}}.$$

Corollary 3.11. $\mathrm{FP}_n = L_n + \mathrm{FP}_n^+$, *and it is linearly isomorphic to* $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^{\binom{[n]}{2}-n}$.

Proof. By Proposition 3.9 the map $w \to d(w)$ is a linear automorphism in $\mathbb{R}^{\binom{[n]}{2}}$; by Proposition 3.10, FP_n is the image FP_n^+ of the positive orthant plus the linear subspace L_n. $\square$

3.1.2. Pfaffians as a Gröbner basis for four-point positive weights. The following is the main result of J. Jonsson and V. Welker [73], although it is also stated without proof in [94, p. 107].

Theorem 3.12 ([73]). *There is a (lexicographical) term order for which* $\mathrm{in}_v(I_k(n))$ *is the monomial ideal generated by all* $(k+1)$*-crossings.*

The term order of Jonsson and Welker necessarily selects in each Pfaffian the monomial corresponding to the $(k+1)$-crossing (in fact, it is designed to have that property), and Pfaffians are a Gröbner basis for it since each Pfaffian contains one and only one of the generators in the initial ideal. Once we know this, any term order that selects this same monomial in each Pfaffian will produce the same initial ideal by, for example, Exercise 8.4 in [23, p. 435]. Proposition 3.10(5) says that this includes the order induced by any (generic) fp-positive vector $v \in \mathrm{FP}_n$. Hence, we have the following statement, a bit more general than Theorem 3.12:

Theorem 3.13. *Pfaffians of degree* $2k+2$ *are a Gröbner basis for* $I_k(n)$ *with respect to any monomial order that selects the* k*-crossing in each Pfaffian. In particular, for the ordering of any fp-positive vector* $v \in \mathrm{FP}_n$.

If the fp-positive vector is sufficiently generic then $\mathrm{in}_v(I_k(n))$ *is the monomial ideal generated by all* $(k+1)$*-crossings.*

The case $k=1$ of this theorem is classical, via the equality $\mathcal{P}f_1(n) = \mathcal{G}r_2(n)$, see [94, Theorem 3.20] and Remark 3.34 below). In fact, in this case the last sentence in the theorem is an "if and only if". Indeed, FP_n is, by definition, the closed Gröbner cone of $I_1(n)$ corresponding to the initial ideal generated by 2-crossings.

In general, let $\mathrm{Grob}_k(n)$ be the Gröbner cone of $I_k(n)$ corresponding to the ideal of $(k+1)$-crossings. For higher k it is no longer true that $\mathrm{FP}_n = \mathrm{Grob}_k(n)$, we only have the containement $\mathrm{FP}_n \subset \mathrm{Grob}_k(n)$ which follows from the previous theorem. Our next result explicitly describes $\mathrm{Grob}_k(n)$.

For arbitrary k, the Gröbner cone is the intersection of the normal cones of each $(k+1)$-crossing in the Newton polytope of the corresponding Pfaffian. A priori, this intersection is described by the following family of linear inequalities, running over all the even cycles $(i_0, i_1 \dots, i_{2l-1}, i_0)$ of length $2l$ that contain an l-crossing contained in a $(k+1)$-crossing, for $l \leq k+1$:

$$v_{i_0 i_1} - v_{i_1 i_2} + \ldots - v_{i_{2l-1} i_0} \geq 0 \tag{14}$$

But most of these inequalities are redundant. For example, for $k = 1$, $\mathrm{Grob}_1(n) = \mathrm{FP}_n$ which is defined by $2\binom{n}{4}$ four-point conditions, but only the $\binom{n}{2} - n$ in Eqs.(11) are irredundant. In fact, it turns out that for every k and every $n \geq 2k+3$, the Gröbner cone is simplicial:

Theorem 3.14. *For $n \geq 2k+3$, $\mathrm{Grob}_k(n)$ is, modulo the lineality space L_n, a simplicial cone given by the following inequalities, one for each $\{i,j\}$ with $|j-i| \geq 2$:*

$$w_{ij} \geq 0 \quad \text{if } |j-i| > k \qquad \text{(long inequalities)} \tag{15}$$

$$\sum_{i' \leq i < j \leq j' \leq i'+k+1} w_{i'j'} \geq 0 \quad \text{if } 2 \leq |j-i| \leq k \qquad \text{(short inequalities)} \tag{16}$$

The ray opposite to the facet indexed by $\{i,j\}$ is generated by:

- *The basis vector indexed by $\{i,j\}$ in the w coordinates if $|j-i| \geq k+2$, and*
- *The negative basis vector indexed by $\{i+1, j\}$ in the v coordinates if $|j-i| < k+2$.*

Observe that the "long" inequalities are also facet-defining for FP_n and the "short" ones are sums of facet-defining "short" inequalities in FP_n.

Proof. First let us see that the inequalities are valid in the cone. The first group (15) is obvious, because the $(k+1)$-crossing has higher weight than any swap. For the second group, let $\{i, i+\ell\}$ be an edge with $\ell \leq k$. For each set U of $2k+2$ *sides* of the n-gon and each edge $e \in T$ we call *length of e with respect to U* and denote it $\ell_U(e)$ the smallest size of the two parts of U separated by e. (Equivalently, it is the usual length of the edge as a diagonal of the n-gon when all the sides not in U are contacted). For a matching M of U and an edge e we denote by $c_M(e)$ the number of edges of M that cross e.

Consider the matching

$$M = \{\{i+1, i+\ell\}, \{i+2, i+k+3\}, \{i+3, i+k+4\}, \ldots, \{i+\ell-1, i+k+\ell\},$$
$$\{i-k+\ell-1, i+\ell+1\}, \{i-k+\ell, i+\ell+2\}, \ldots, \{i, i+k+2\}\}$$

This is a k-crossing plus the edge $\{i+1, i+\ell\}$. The coefficient of a w will be the same in this matching than in the $(k+1)$-crossing, that is, $\ell_U(e) = c_M(e)$, except for the edges $\{i', j'\}$ with $i' \leq i < i+l \leq j' \leq i'+k+1$, for which $c_M(e) = \ell_U(e) - 2$. Hence the left hand side of (16) is half the difference between the weights, which proves the inequalities.

Once we know that the inequalities are valid, let G_{ij} be the ray defined in the statement. We only need to show that at each G_{ij} all inequalities are equalities, except for the one of index ij, and that the G_{ij} indeed lie in $\mathrm{Grob}_k(n)$.

Indeed, if $|i-j| > k+1$ then G_{ij} has all w coordinates equal zero except $w_{ij} > 0$. it is clear that all inequalities of the form (16) are equalities (since they only involve w's of length $\leq k+1$) and all of type (15) except the one for ij are equalities (by construction). If $|i-j| \leq k+1$, in G_{ij} we have that the the only nonzero v coordinate is $v_{i+1,j}$, which is negative. We take it equal to -1. Proposition 3.9 implies that in the w coordinates the only non-zero ones are

$$w_{i+1,j} = w_{i,j-1} = -\frac{1}{2}, \qquad w_{i+1,j-1} = w_{i,j} = \frac{1}{2}.$$

Now, if $j-i \leq k$, (15) gives always 0 and (16) gives 1/2 exactly for one sum, the corresponding to $\{i,j\}$, and 0 for the rest. If $j-i = k+1$, (16) gives always 0 and (15) gives 1 only for $w_{i,j}$.

It remains to see that these rays are in $\mathrm{Grob}_k(n)$:

- For the w basis vectors this follows from the fact that they are in FP_n.
- For the negative v basis vectors, we are giving weight -1 to an irrelevant edge and 0 to all the other edges; it is clear that every $(k+1)$-crossing gets weight zero, and every other matching gets nonpositive weight.

□

REMARK 3.15. Theorem 3.14 fails for $n = 2k+2$, but in this case it is easy to describe $\mathrm{Grob}_k(n)$. Since we have a single Pfaffian, the Gröbner fan is simply the normal fan of its Newton polytope. In particular, none of the equalities (14) is redundant and $\mathrm{Grob}_k(n)$ has as many facets as there are matchings of $[2k+2]$ whose symmetric difference with the $k+1$-crossing is a single cycle. For example:

- For $k=2$, $n=6$, all matchings differ from the 3-crossing in a single cycle. Thus, the $\mathrm{Grob}_2(6)$ has (modulo its lineality space) dimension $\binom{6}{2} - 6 = 9$ and 14 facets.
- For $k=3$, $n=8$, there are matchings differing from the 4-crossing in two cycles of length four. There are exactly 12 of them, coming from the three ways of partitioning the 4-crossing into two pairs of edges and the two ways of completing each pair of edges into a four-cycle. Hence, $\mathrm{Grob}_3(8)$ has dimension $\binom{8}{2} - 8 = 20$ and it has $105 - 1 - 12 = 92$ facets.

One difference between $k=1$ and $k>1$ is that for $k=1$ Pfaffians are a universal Gröbner basis for the ideal $I_1(n)$ (one proof is that every other Gröbner cone of $I_1(n)$ can be sent to FP_n by a permutation of $[n]$, see [124, Theorem 4.3]). The same is known to fail for higher Grassmannians (see e.g., [124, Section 7] or [83, Example 4.3.10]) and it also fails for higher Pfaffians:

EXAMPLE 3.16 (Pfaffians are not a universal Gröbner basis). Let $n=9$ and $k=2$. Consider the vector with

$$\begin{aligned} &v_{12} = v_{34} = v_{56} = v_{47} = v_{89} = 2,\\ &v_{58} = v_{69} = 1,\\ &v_{17} = v_{28} = v_{39} = 10, \end{aligned}$$

and the rest of entries equal to zero. We are going to show that, regardless of the field $\mathbb{K}$, Pfaffians are not a Gröbner basis for this choice of v (or any small perturbation of it).

Call f and g the Pfaffians on the sets $U = \{1,2,3,4,5,6\}$ and $V = \{4,5,6,7,8,9\}$, which have as matchings of highest weight $\{12,34,56\}$ and $\{56,47,89\}$, both of weight 6. That is,

$$in(f) = x_{12}x_{34}x_{56}, \qquad in(g) = x_{56}x_{47}x_{89}.$$

The following polynomial, which is nothing but the S-polynomial of f and g that arises in Buchberger's algorithm, lies in $I_3(9)$

$$h := x_{12}x_{34}\, g - x_{47}x_{89}\, f.$$

The only monomials of weight > 6 in h are the initial terms of the two parts $x_{12}x_{34}\, g$ and $x_{47}x_{89}\, f$, which cancel out, and $x_{12}x_{34}\, x_{47}x_{58}x_{69}$, of weight 8. Hence, we have that $in(h) = x_{12}x_{34}\, x_{47}x_{58}x_{69}$.

In particular, if Pfaffians were a universal Gröbner basis, there should be a Pfaffian whose leading monomial divides $in(h)$. That is, there should be a set $W \subset [9]$ of six elements whose matching M of maximum weight is contained in $\{12,34,47,58,69\}$. This W does not exist. Indeed, W cannot contain any of the pairs $\{1,7\}$, $\{2,8\}$ or $\{3,9\}$, because then its highest matching would have weight ≥ 10. And every set of three edges among $\{12,34,47,58,69\}$ not containing any of those pairs of vertices contains the edges $\{58,69\}$, which cannot be in the leading term of any Pfaffian since they produce smaller weight than their swap $\{56,89\}$.

REMARK 3.17. That Pfaffians are a Gröbner basis for the ideal $I_k(n)$ they generate was known before [**73**]. The earliest proof we are aware of is by Herzog and Trung [**61**], who construct a lexicographic order for which the initial ideal $in_<(I_k(n))$ is generated by the $(k+1)$-nestings. Here $\{a,d\}$ and $\{b,c\}$ are nested if $1 \leq a < b < c < d \leq n$.

This result was recovered by Sturmfels and Sullivant [**129**] as a special case of a more general behaviour; Sturmfels and Sullivant study the relation between the Gröbner bases of an ideal I and those of its secant ideals $I^{\{k\}}$, and call a monomial order "delightful" if the initial ideal of $I^{\{k\}}$ can be obtained from that of I by the following simple combinatorial rule: the standard monomials in $in_<(I^{\{k\}})$ are the products of k standard monomials of $in_<(I)$. They then consider $I_k(n) = I_1(n)^{\{k\}}$ as an example [**129**, Example 4.13], and show that the lexicographic order of Herzog and Trung [**61**] is delightful.

It is worth noticing that fp-positive orders are not "delightful" in the sense of [**129**]. Indeed, the maximal square-free standard monomials in our initial ideal are the k-triangulations of the n-gon, and not every k-triangulation is the union of k triangulations. For a trivial example observe that the complete graph on 5 vertices is a 2-triangulation but it is not the union of two triangulations of the pentagon. Related to this, see [**106**, Section 6].

Theorem 3.13 has a natural interpretation via $(k+1)$-free sets and multitriangulations. Observe that $k(2n-2k-1)$, the dimension of $\Delta_k(n)$, coincides with that of $\mathcal{P}f_k(n)$.

COROLLARY 3.18. *If the weight vector* v *for the variables in* $\mathbb{K}[x_{i,j}, \{i,j\} \in \binom{[n]}{2}]$ *lies in* $\mathrm{Grob}_k(n)$ *(for example, if it is fp-positive) and generic then the initial ideal of* $I_k(n)$ *equals the Stanley-Reisner ideal of the extended* k*-associahedron* $\Delta_k(n)$*.*

That is: it is the radical monomial ideal whose square-free standard monomials form, as a simplicial complex, $\Delta_k(n)$.

3.1.3. The algebraic matroid of $\mathcal{P}f_k(n)$ and low-rank matrix completion. Let $I \subset \mathbb{K}[x_1, \dots, x_N]$ be a prime ideal. The *algebraic matroid* of I, which we denote $\mathcal{M}(I)$, has the variables $E := \{x_1, \dots, x_N\}$ as elements and a subset $S \subset E$ is independent if I does not contain any non-zero polynomial in $\mathbb{K}[S]$. If $\mathbb{K}$ is algebraically closed and $V = V(I)$ is the irreducible variety of V, then dependence and independence of a subset S of variables can be told via the natural projection map $\pi_S : V \subset \mathbb{K}^N \to \mathbb{K}^S$, as follows. A set is independent in $\mathcal{M}(I)$ if, and only if, the corresponding projection map $\pi_S : V \to K^S$ is dominant; that is, its image is (Zariski) dense. We use [**77, 113, 114**] as our main sources for algebraic matroids.

THEOREM 3.19. *Let $\mathbb{K}$ be an algebraically closed field, $I \subset \mathbb{K}[x_1, \dots, x_N]$ a prime ideal and V its algebraic variety. For each $S \subset [N]$ denote by $\pi_S : \mathbb{K}^{[N]} \to \mathbb{K}^S$ the coordinate projection to S. Then:*

(1) *S is independent in $\mathcal{M}(I)$ if and only if $\pi_S(V)$ is Zariski dense in $\mathbb{K}^S$.*
(2) *The rank of S is equal to the dimension of $\pi_S(V)$.*
(3) *S is spanning if and only if π_S is finite-to-one: for every $x \in \mathbb{K}^S$ the fiber $\pi_S^{-1}(x)$ is finite (perhaps empty).*

PROOF. The first part is Theorem 15 in [**114**]. For the second, the rank of S is the maximum size among independent subsets of S, which are the subsets T for which $\pi_T(V) = \pi_T(\pi_S(V))$ has dimension $|T|$. The maximal ones are those which have the same size as the dimension of $\pi_S(V)$, so this is the rank.

The third part is a consequence of the second, because a projection has the same dimension than the variety if and only if the fiber has dimension zero, and a fiber has dimension zero if and only if it is finite. □

This statement has as a consequence that, over an algebraically closed field, we can speak of the algebraic matroid of the irreducible variety V, and denote it $\mathcal{M}(V)$, instead of looking at the ideal.

We now turn our attention to the case of $\mathcal{P}f_k(n)$.

COROLLARY 3.20. *$(k+1)$-free subsets of edges are independent in the algebraic matroid of $\mathcal{P}f_k(n)$ and k-triangulations are bases.*

After Proposition 3.21 we show examples of non-planar graphs that are independent in $\mathcal{P}f_1(n)$. This implies that the converse of Corollary 3.20 is false; not every basis of $\mathcal{P}f_1(n)$ is, after relabelling vertices, a triangulation.

PROOF. Let S be a dependent set in the matroid. Then there is a polynomial f in $I_k(n)$ using only the variables in S and the initial monomial of f according to any fp-positive weight also uses only variables in S. By Corollary 3.18 $I_k(n)$ has an initial ideal consisting only of $(k+1)$-crossing monomials, hence f has a monomial with a $(k+1)$-crossing, and S is not $(k+1)$-free.

For the second part, it is enough to see that the rank of the matroid equals $2nk - \binom{2k+1}{2}$. This is because points in $\mathcal{P}f_k(n)$ are antisymmetric matrices of rank $\leq 2k$. In order to construct one such matrix M we can choose generic elements in the first $2k$ rows above the diagonal and every other element $M_{i,j}$ $(i, j > 2k)$ is uniquely determined by them. Indeed, the Pfaffian of the rows and columns indexed by $[2k] \cup \{i,j\}$ has the form $AM_{i,j} + B$ where A is the Pfaffian of $[2k]$. Since our choice was generic, $A \neq 0$. □

This proof already shows the relation between independence in the algebraic matroid of $\mathcal{P}f_k(n)$ and low-rank completion of partially known antisymmetric matrices. Suppose that we are given a matrix $M \in \mathbb{K}^{n\times n}$, of which we only know a subset T of entries, we want to deduce the rest of entries with the restriction that M needs to be antisymmetric and have at most range $2k$. Corollary 3.20 then immediately allows us to prove Theorem 3.3:

Proof of Theorem 3.3. Consider the projection $\pi_T : \mathbb{K}^{\binom{[n]}{2}} \to \mathbb{K}^T$ that keeps only the coordinates of T. In part (1) we are saying that π_T is almost surjective (any element has a preimage except for a zero measure set) and in part (2) that it is finite-to-one (every point $x \in \mathbb{K}^T$ has a finite fiber $\pi^-1(x)$). Both parts follow from Corollary 3.20, via the characterization of algebraic matroids in Theorem 3.19. □

It is worth mentioning that the algebraic matroid of $\mathcal{P}f_k(n)$ coincides with the *generic hyperconnectivity matroid in dimension* $2k$ introduced by Kalai [74]. Let us review this relation.

If an algebraic variety V is parametrized by a polynomial map $T : \mathbb{R}^M \to V \subset \mathbb{R}^N$, then the algebraic matroid of V equals the linear matroid of rows of the Jacobian of T at a sufficiently generic point of $\mathbb{R}^M$ [113, Proposition 2.5]. In our case, $\mathcal{P}f_k(n)$ is parametrized by the following linear map:

$$T : (\mathbb{R}^n)^{2k} \to \mathcal{P}f_k(n) \subset \mathbb{R}^{\binom{n}{2}}$$

$$(\mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{a}_k, \mathbf{b}_k) \mapsto \sum_{l=1}^{k} (a_{l,i}b_{l,j} - a_{l,j}b_{l,i})_{1\le i<j\le n}\,, \tag{17}$$

where $\mathbf{a}_l = (a_{l,1}, \dots, a_{l,n})$ and $\mathbf{b}_l = (b_{l,1}, \dots, b_{l,n})$. The Jacobian of T at a point $(\mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{a}_k, \mathbf{b}_k)$ then coincides with the hyperconnectivity matrix of the configuration $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ where

$$\mathbf{p}_i = (b_{1,i}, -a_{1,i}, \dots, b_{k,i}, -a_{k,i}).$$

As a consequence we get the following (known) result, which is implicit for example in [94, Theorem 3.23]:

Proposition 3.21. *The algebraic matroid of $\mathcal{P}f_k(n)$ coincides with the generic hyperconnectivity matroid in dimension $2k$.*

With this result it is easy to construct non-planar graphs that are bases in $\mathcal{P}f_1(n) = \mathcal{H}_2(n)$. Start with any non-planar graph and subdivide every edge into two parts. The graph G obtained is independent in every two-dimensional rigidity matroid, in particular in $\mathcal{H}_2(n)$, because iteratively removing the new vertices, which all have degree two, we get the empty graph. Hence, G can be extended to a non-planar basis of $\mathcal{H}_2(n)$. As a consequence, not every basis of $\mathcal{P}f_1(n)$ is a triangulation.

Corollary 3.22. *k-triangulations are bases in the generic hyperconnectivity matroid of dimension $2k$.*

3.2. The tropicalization of $\mathcal{P}f_k(n)$

3.2.1. The tropical Pfaffian variety and prevariety. As already said, for $k = 1$ and for $n = 2k + 2$ Pfaffians are a tropical basis of the ideal they generate. The following example looks at the first open case:

EXAMPLE 3.23. For $k = 2$ and $n = 7$, using `Gfan` [69] we have computed $\mathrm{Pf}_2(7)$ as the intersection of the seven hypersurfaces corresponding to Pfaffians. The result is a non-simplicial fan of pure dimension 18 with 77 rays and a lineality space of dimension 7 (as expected). It has 73395 maximal cones, all of them with multiplicity 1. These cones correspond to 33 classes of symmetry via permutations of variables. The 77 rays are:

- The 21 vectors in the standard basis of the coordinates v, and their 21 opposites. That is, for each $\{i, j\} \in \binom{7}{2}$, the two vectors with $v_{ij} = \pm 1$ and $v_{i'j'} = 0$ otherwise.
- The 35 vectors obtained as follows: for each $\{i, j, k\} \in \binom{7}{3}$, the vector with $v_{ij} = v_{ik} = v_{jk} = 1$ and $v_{i'j'} = 0$ otherwise.

7 of the 14 extremal rays of FP_7 are among these vectors. In the w coordinates these are the vectors with $w_{ij} = 1$ and all other entries equal to zero, for the fourteen choices of non-consecutive i and j. The seven with $i = j - 2$ coincide (modulo the lineality space) with the v-basis vectors with $v_{j-1,j} = -1$, which are rays, and the seven with $i = j - 3$ are the vectors with $v_{j-2,j} = v_{j-1,j} = v_{j-2,j-1} = -1$, that is, the opposites to some rays, but they are not rays themselves. None of the other 77 rays computed by `Gfan` lie in FP_7.

The cone corresponding to a given 2-triangulation cannot be in this prevariety, because its rays are not among those rays. But it can be the result of intersecting a cone from the prevariety with FP_7, because, by Remark 3.15, the Gröbner cone in which it is contained is a bit greater than FP_7. In fact, a v coming from a 2-triangulation is in the cone spanned by the rays $v_{j-1,j} = -1$ for all j and $v_{j-2,j} = -1$ for $\{j - 3, j\}$ in the 2-triangulation. The intersection of this cone with FP_7 is the cone in the 2-associahedron.

In this case, we want to check whether the tropical prevariety $\mathrm{Pf}_2(7)$ coincides with the variety $\mathrm{trop}(\mathcal{P}f_2(7))$. To do that, we need to compute the tropical variety as a subfan of the Gröbner fan. However, it is not enough to check that the cones in both fans are the same, because the tropical prevariety may not be a subfan of the Gröbner fan.

$\mathrm{trop}(\mathcal{P}f_2(7))$ is a simplicial fan with 84420 cones, that belong to 35 equivalence classes. The equality as sets for the two fans can now be checked by showing that all the simplicial cones in $\mathrm{trop}(\mathcal{P}f_2(7))$ are contained in a cone of $\mathrm{Pf}_2(7)$ and the union of the cones contained in the same cone gives the whole cone.

The prevariety contains 71820 simplicial and 1575 non-simplicial cones. The simplicial ones are also cones of the variety, so that part is correct. Now there are 12600 remaining cones in the variety, that correspond to the non-simplicial part. The non-simplicial cones can be triangulated in two ways: in 8 cones and in 3 cones. The triangulation in 8 cones of all them can be shown to match exactly the cones of the variety, and we are done.

To better understand the difference between $\mathrm{Pf}_k(n)$ and $\mathrm{trop}(\mathcal{P}f_k(n))$ we are now going to relate them to two different notions of rank for a tropical matrix. For this, it is convenient to extend $\mathbb{R}$ to the *tropical semiring* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$, with the operations max as "addition" and + as "multiplication". By a tropical $n_1 \times n_2$-matrix we mean an $n_1 \times n_2$-matrix with entries in $\overline{\mathbb{R}}$. To distinguish between tropical (pre)-varieties in $\mathbb{R}^n$ and $\overline{\mathbb{R}}^n$ we denote $\overline{V}$ the extension to $\overline{\mathbb{R}}^n$ of a tropical variety or prevariety $V \in \mathbb{R}^n$.

Clearly, for every family F of polynomials, the prevariety of F in $\overline{\mathbb{R}}^n$ is topologically closed, so it contains the closure of the prevariety in $\mathbb{R}^n$, and the same holds for varieties. The converse is not always true, as the following example shows:

EXAMPLE 3.24. Let $I = (x_1x_3 + x_2, x_2x_3 + x_1)$. The tropical variety it defines in $\mathbb{R}^3$ equals $\{(a, a, 0) : a \in \mathbb{R}\}$, while the variety it defines in $\overline{\mathbb{R}}^3$ contains that plus the points $\{(-\infty, -\infty, b) : b \in \overline{\mathbb{R}}\}$.

Observe that this ideal is not prime, since it contains $x_1(x_3^2 - 1)$ but it does not contain any of its factors x_1, $x_3 + 1$ or $x_3 - 1$. We do not know whether for prime ideals it is always true that the closure of V equals $\overline{V}$.

The following two notions of rank were introduced in [36].

DEFINITION 3.25 (Tropical rank, [83, Def. 5.3.3]). A square matrix $M \in \overline{\mathbb{R}}^{r\times r}$ is *tropically singular* if the maximum in the tropical determinant

$$\operatorname{trop}\det(M) := \max_{\sigma \in S_r} \sum_{i=1}^{r} m_{i\sigma(i)}$$

is attained at least twice, and *tropically regular* otherwise.

The *tropical rank* of a tropical matrix is the largest size of a tropically regular minor in it.

Stated differently, the tropical rank of M is the largest r such that M is not in the tropical prevariety of the $r \times r$ minors or, equivalently, the smallest r such that M is in the tropical prevariety of the $(r + 1) \times (r + 1)$ minors.

DEFINITION 3.26 (Kapranov rank, [83, Def. 5.3.2]). Let $M \in \overline{\mathbb{R}}^{n_1\times n_2}$ be a tropical matrix. The *Kapranov rank* of M over a valuated field $\mathbb{K}$ is the smallest rank of a lift of the matrix, that is, a matrix $\widetilde{M} \in \mathbb{K}$ such that the degree of $\widetilde{M}_{ij}$ is m_{ij}.

The tropical variety of the $(r+1) \times (r+1)$ minors is the tropicalization of the (classical) variety of the matrices with rank at most r. Hence, the Kapranov rank is the smallest r such that M is in the tropical variety of the $(r+1)\times(r+1)$ minors, or the largest r such that M is not in the tropical variety of the $r \times r$ minors.

Observe that the Kapranov rank of M depends on the field $\mathbb{K}$ under consideration, while the tropical rank does not. The relation of the two notions of rank to the tropical variety and prevariety of minors readily shows that the Kapranov rank is greater or equal than the tropical rank [36, Theorem 1.4]. Two small examples where the two notions do not coincide appear in [36, Section 7] (a 7×7 matrix of tropical rank three and Kapranov rank four) and [120] (a 6×6 matrix of tropical rank four and Kapranov rank five). The two examples are reproduced in [121, Section 4] where Shitov, completing work of Develin-Santos-Sturmfels [36], Chan-Jensen-Rubei [21], and himself [122] shows that these two examples are the smallest possible:

LEMMA 3.27 ([121]). *For given positive integers r, n_1, n_2 the following are equivalent:*

(1) *The* $(r+1)\times(r+1)$ *minors are a tropical basis for the variety of* $n_1 \times n_2$ *matrices of rank* r *(over any of the complex, real, or rational fields).*
(2) $r \le 2$, *or* $r = \min\{n_1, n_2\}$, *or* $r = 3$ *and* $\min\{n_1, n_2\} \le 6$.

Since these notions of rank distinguish between the variety and prevariety of minors, antisymmetric versions of them will distinguish between the variety and prevariety of Pfaffians. (The same idea for the *symmetric* case is explored in [**145**]).

Let $M \in \mathbb{R}^{n_1\times n_2}$ be a tropical matrix and let $n = n_1 + n_2$. Let $K \in \mathbb{R}$ be a sufficiently big constant. From M and K we construct the following $n \times n$ matrix:

$$\mathrm{Sym}(M,K) := \begin{pmatrix} N_1 & M \\ M^t & N_2 \end{pmatrix} \in \overline{\mathbb{R}}^{n\times n},$$

where $(N_1)_{ij} = m_{i1} + m_{j1} - K$ and $(N_2)_{ij} = m_{1i} + m_{1j} - K$ for $i \ne j$, and $(N_1)_{ii} = (N_2)_{ii} = -\infty$. We have a corresponding vector $v(M,K) \in \mathbb{R}^{\binom{[n]}{2}}$ of entries of $\mathrm{Sym}(M,K)$:

$$v_{ij} := \begin{cases} m_{i,j-n_1} & \text{if } 1 \le i \le n_1 < j. \\ m_{i1} + m_{j1} - K & \text{if } 1 \le i, j \le n_1. \\ m_{1,i-n_1} + m_{1,j-n_1} - K & \text{if } i, j > n_1. \end{cases}$$

For example, for the 2×3 matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

we have

$$\mathrm{Sym}(M,10) = \begin{pmatrix} -\infty & -5 & 1 & 2 & 3 \\ -5 & -\infty & 4 & 5 & 6 \\ 1 & 4 & -\infty & -7 & -6 \\ 2 & 5 & -7 & -\infty & -5 \\ 3 & 6 & -6 & -5 & -\infty \end{pmatrix}$$

and

$$v(M,10) = (-5, 1, 2, 3, 4, 5, 6, -7, -6, -5),$$

where the negative entries are obtained subtracting 10 from the sum of the two corresponding elements from the first row or from the first column of M.

We also consider the matrix and vector $\mathrm{Sym}(M,\infty)$ and $v(M,\infty) \in \overline{\mathbb{R}}^{\binom{[n]}{2}}$ obtained using ∞ instead of K. That is:

$$\mathrm{Sym}(M,\infty) := \begin{pmatrix} -\infty & M \\ M^t & -\infty \end{pmatrix} \in \overline{\mathbb{R}}^{n\times n},$$

Lemma 3.28. *Let* $M \in \overline{\mathbb{R}}^{n_1\times n_2}$ *be a tropical matrix and* $K \in \overline{\mathbb{R}}$. *For the vector* $v(M,K) \in \overline{\mathbb{R}}^{\binom{[n]}{2}}$ *defined above we have:*
(1) *For* K *sufficiently large,* $v(M,K) \in \mathrm{Pf}_k(n)$ *if and only if the tropical rank of* M *is at most* k.
(2) $v(M,\infty) \in \mathrm{trop}(\mathcal{P}f_k(n))$ *if and only if the Kapranov rank of* M *is at most* k.

Proof. For part (1), assume first that $v(M,K) \in \mathrm{Pf}_k(n)$, and consider a $(k+1)\times(k+1)$ minor of M. This corresponds to a set $U \in \binom{[n]}{2k+2}$ with half of the elements in $[1,\dots,n_1]$ and the other half in $[n_1+1,\dots,n]$. Since $v(M,K) \in \mathrm{Pf}_k(n)$, there are at least two perfect matchings in U of maximum weight. Since we chose K very big, none of these matchings come from the N_1 or N_2 parts of $\mathrm{Sym}(M,K)$. This implies that the minor of M that we started with is tropically singular.

Conversely, assume that $\operatorname{trop\,rank} M \leq k$. Let $U \in \binom{[n]}{2k+2}$ and consider a perfect matching E in U with maximal weight, which is a term in the Pfaffian of U. We have three cases:

- If all the edges in E are between $[n_1]$ and $[n_1+1,n]$, E corresponds to a permutation in M attaining the tropical determinant. As $\operatorname{trop\,rank} M \leq k$, there must be another permutation with the same weight.
- If all the edges in E except one are between $[n_1]$ and $[n_1+1,n]$, suppose $E = \{\{i_1,j_1\},\ldots,\{i_{k+1},j_{k+1}\}\}$, and $i_1,\ldots,i_{k+1},j_1 \leq n_1 < j_2,\ldots,j_{k+1}$ (the other case is symmetric). Then

$$w(E) = v_{i_1j_1}+\cdots+v_{i_{k+1}j_{k+1}} = m_{i_11}+m_{j_11}+m_{i_2,j_2-n_1}+\ldots+m_{i_{k+1},j_{k+1}-n_1}-K.$$

  We have now two cases:
  - If $j_l = n_1+1$ for some l, for example $j_2 = n_1+1$, then
  $$w(E) = v_{i_1i_2}+v_{j_1,n_1+1}+v_{i_3j_3}+\ldots+v_{i_{k+1}j_{k+1}}.$$
  - If $j_l > n_1+1$ for all l, $w(E)-m_{j_11}+K$ is the weight of the permutation $\{i_1,1\},\{i_2,j_2-n_1\},\ldots,\{i_{k+1},j_{k+1}-n_1\}$ in M. Since the tropical rank of M is smaller than $k+1$, there is another permutation with weight greater or equal than $w(E)-m_{j_11}+K$. That is,
  $$w(E)-m_{j_11}+K \leq m_{i'_11}+m_{i'_2,j_2-n_1}+\ldots+m_{i'_{k+1},j_{k+1}-n_1}$$
  where $(i'_1,\ldots,i'_{k+1})$ is a permutation of $(i_1,\ldots,i_{k+1})$. Equivalently
  $$\begin{aligned} w(E) \leq & (m_{i'_11}+m_{j_11}-K)+m_{i'_2,j_2-n_1}+\ldots+m_{i'_{k+1},j_{k+1}-n_1} = \\ = & v_{i'_1j_1}+v_{i'_2j_2}+\ldots+v_{i'_{k+1}j_{k+1}}. \end{aligned}$$
  As E is maximal, this is an equality, and we have another matching in U with the same weight.
- If there is more than one edge inside $[n_1]$ or inside $[n_1+1,n]$, suppose for example we have the edges $\{a,b\}$ and $\{c,d\}$ with $a,b,c,d \leq n_1$. Then any of the two swaps among these four elements preserves weight, indeed:

$$v_{a,b}+v_{c,d} = m_{a1}+m_{b1}+m_{c1}+m_{d1}-2K = v_{a,c}+v_{b,d} = v_{a,d}+v_{b,c}$$

In any case, there is another matching with the same weight as E, and this finishes part (1).

For part (2), if M has Kapranov rank at most k then there is a lift $\widetilde{M}$ of M of rank k. Thus,

$$\begin{pmatrix} 0 & \widetilde{M} \\ \widetilde{M}^t & 0 \end{pmatrix}$$

is an antisymmetric lift of $\mathrm{Sym}(M,\infty)$ of rank $2k$.

Conversely, if $v(M,\infty) \in \mathrm{trop}(\mathcal{P}f_k(n))$, consider an antisymmetric matrix in $\mathcal{P}f_k(n)$ projecting to it, hence of rank $2k$. This matrix necessarily has zero entries in the places where $v(M,\infty)$ has $-\infty$, so it is of the form

$$\begin{pmatrix} 0 & \widetilde{M} \\ \widetilde{M}^t & 0 \end{pmatrix},$$

where $\widetilde{M}$ is a matrix of rank at most k and projecting to M. □

THEOREM 3.29. *If there is a matrix $M \in \mathbb{R}^{n_1 \times n_2}$ of tropical rank $\leq k$ and Kapranov rank $> k$ then $\mathrm{Pf}_k(n) \neq \mathrm{trop}(\mathcal{P}f_k(n))$, where $n = n_1 + n_2$.*

This happens, for example, for $k = 3$ and any $n \geq 14$ and for any $k \geq 4$ and $n \geq 2k + 4$.

PROOF. Let $M \in \mathbb{R}^{n_1 \times n_2}$ be a matrix of tropical rank $\leq k$ and Kapranov rank $> k$. By Part (1) of Lemma 3.28 we have that $v(M, K) \in \mathrm{Pf}_k(n)$ for every sufficiently big K.

Also, by Part (2) of the Lemma, $v(M, \infty) \notin \overline{\mathrm{trop}(\mathcal{P}f_k(n))}$. In particular, $v(M, \infty)$ is not in the closure of $\mathrm{trop}(\mathcal{P}f_k(n))$, which implies it is not true that $v(M, K) \in \mathrm{trop}(\mathcal{P}f_k(n))$ for all sufficiently big K.

Thus, $\mathrm{Pf}_k(n) \neq \mathrm{trop}(\mathcal{P}f_k(n))$. □

Summing up, the cases where we do not know whether $\mathrm{Pf}_k(n) = \mathrm{trop}(\mathcal{P}f_k(n))$ are:

- $k = 2$ and $n \geq 8$,
- $k = 3$ and $n \in \{9, 10, 11, 12, 13\}$,
- $k \geq 4$ and $n = 2k + 3$.

3.2.2. The k-associahedron as the fp-positive part of the tropical Pfaffian variety. We are interested in the part of $\mathrm{Pf}_k(n)$ contained in $\mathrm{Grob}_k(n)$:

DEFINITION 3.30. We define

$$\mathrm{Pf}_k^+(n) := \mathrm{Pf}_k(n) \cap \mathrm{Grob}_k(n).$$

We call it the $(k+1)$-*free part* of the tropical Pfaffian variety of parameters n and k for two reasons. On the one hand, the initial ideal corresponding to $\mathrm{Grob}_k(n)$ is the Stanley-Reisner ideal of the complex of $(k + 1)$-free sets. But, more significantly, our results in this section say that $\mathrm{Pf}_k^+(n)$ coincides with the points of $\mathrm{Grob}_k(n)$ which, expressed in the w-coordinates, have $(k + 1)$-free support.

THEOREM 3.31. *Let $v = d(w) \in \mathrm{Grob}_k(n)$ be a vector in the Gröbner cone. This includes the case where w is non-negative (or, equivalently, $v \in \mathrm{FP}_n$). Then,*

1. *$v \in \mathrm{Pf}_k^+(n)$ if and only if the support of w is $(k + 1)$-free.*
2. *If the above holds, then for every subset $U \subset \binom{[n]}{2}$ of size $2k + 2$ one of the maximal matchings of U for v is the one producing a $(k+1)$-crossing, and a second one is obtained from it by a swap of two consecutive edges in the $(k + 1)$-crossing.*

PROOF. Let $U = \{a_0, a_1, \ldots, a_{2k+1}\}$ written in cyclic order, and let E_0 be the $(k+1)$-crossing in it, that is, the matching that pairs a_i with a_{k+1+i}. As we already know, the maximum weight given by v to matchings of U is attained at E_0.

If the support of w is $(k + 1)$-free, there must be an l such that no edge in the support of w has an end between sides a_l and a_{l+1} and the other between a_{l+k+1} and a_{l+k+2}. Then, let $E_1 = E_0 \setminus \{\{a_l, a_{l+k+1}\}, \{a_{l+1}, a_{l+k+2}\}\} \cup \{\{a_l, a_{l+k+2}\}, \{a_{l+1}, a_{l+k+1}\}\}$ has the same weight as E_0, so that $v \in \mathrm{Pf}_k^+(n)$ and part (2) holds.

Conversely, if the support of w contains a $(k + 1)$-crossing then there is a $U = \{a_0, a_1, \ldots, a_{2k+1}\}$ such that each a_i lies in one of the $2k + 2$ regions defined by that crossing, and then the matching E_0 of U has weight strictly larger than any other matching. In particular, $v \notin \mathrm{Pf}_k(n)$. □

We now want to show that $\mathrm{Pf}_k^+(n)$ is contained in $\mathrm{trop}(\mathcal{P}f_k(n))$. That is to say, even if the tropical Pfaffian variety and prevariety may not coincide, their "$(k+1)$-free parts" coincide. We need the following Lemma, the proof of which we postpone to Section 3.2.3:

LEMMA 3.32. *Let $v = d(w) \in \mathrm{Grob}_k(n)$ be sufficiently generic. Then, for every subset $U \in \binom{[n]}{2k+2}$ we have that U has the same number of positive and negative matchings of maximum weight with respect to v.*

COROLLARY 3.33. $\mathrm{Pf}_k^+(n) \subset \mathrm{trop}(\mathcal{P}f_k(n))$. *Moreover,* $\mathrm{Pf}_k^+(n) \subset \mathrm{trop}^+(\mathcal{P}f_k(n))$.

Let us point out that $\mathrm{Pf}_k(n)$ and $\mathrm{Pf}_k^+(n)$ are independent of the field $\mathbb{K}$, while $\mathrm{trop}(\mathcal{P}f_k(n))$ and $\mathrm{trop}^+(\mathcal{P}f_k(n))$ are (probably) not. The first statement is over an arbitrary field. The second statement is stronger, but it makes sense only over fields of characteristic zero.

PROOF. Let $v \in \mathrm{Pf}_k^+(n)$. We want to show that $v \in \mathrm{trop}(\mathcal{P}f_k(n))$. In fact, it is enough to show this under the assumption that v is sufficiently generic (within $\mathrm{Pf}_k^+(n)$), since $\mathrm{trop}(\mathcal{P}f_k(n))$ is closed. By Theorem 3.31, being generic in $\mathrm{Pf}_k^+(n)$ implies that $v = d(w)$ for a w with support equal to a k-triangulation. By Lemma 3.32 the latter implies that the initial form of every Pfaffian for the weight vector v vanishes at the point $(1, \dots, 1)$. Since Pfaffians are a Gröbner basis for v by Theorem 3.13, we have that

$$(1, \dots, 1) \in V(\mathrm{in}_v(I_k(n))).$$

This clearly implies that $\mathrm{in}_v(I_k(n))$ contains no monomials (over an arbitrary field) and that it does not contain polynomials with all coefficients real and of the same sign (over fields of characteristic zero). □

Putting together Theorem 3.31 and Corollary 3.33 we conclude Theorem 3.4.

REMARK 3.34. Since Pfaffians of degree two coincide with the 3-term Plücker relations that generate the Grassmannian $\mathcal{G}r_2(n)$, we have that $\mathcal{P}f_1(n) = \mathcal{G}r_2(n)$ and that $\mathrm{Pf}_1(n)$ equals the *Dressian* $\mathcal{D}r_2(n)$ (the tropical prevariety defined by quadratic Plücker relations [83, Section 4.4]).

It was proven in [124] that $\mathcal{D}r_2(n) = \mathrm{trop}(\mathcal{G}r_2(n))$ (equivalently, $\mathrm{Pf}_1(n) = \mathrm{trop}(\mathcal{P}f_1(n))$, by showing that $\mathrm{trop}(\mathcal{G}r_2(n))$ also coincides with the space $\mathcal{T}ree_n$ of *tree metrics* for trees with n leaves. The proof is reproduced in [83, Theorem 4.3.3] and the idea of it is the following: The tropical hypersurface corresponding to the Pfaffian of degree two (or the 3-term Plücker relation) of a certain $U \subset \binom{[n]}{4}$ equals the solution set of:

$$v_{i,j} + v_{k,l} \leq \max\{v_{i,k} + v_{j,l},\ v_{i,l} + v_{j,k}\}, \quad \forall \{i,j\} \in \binom{U}{2}.$$

These relations (taken for all U) are exactly the *four-point conditions* that characterize tree metrics [14]. Hence, $\mathrm{trop}(\mathcal{P}f_1(n)) \subset \mathrm{Pf}_1(n) = \mathcal{T}ree_n$. For the converse, for any given (generic) $v \in \mathcal{T}ree_n = \mathrm{Pf}_1(n)$ there is a ternary tree T with nonnegative weights w on its edges and realizing v as a tree metric. By relabelling its leaves, we can assume that T is the dual tree of a certain triangulation of the n-gon. Hence, v coincides (after relabelling, but this does not change $\mathrm{trop}(\mathcal{P}f_1(n))$) with the $d(w)$

of Definition 3.8 for this choice of weights. Theorem 3.31 and Corollary 3.33 then imply that $v \in \mathrm{Pf}_1^+(n) \subset \mathrm{trop}(\mathcal{P}f_1(n))$.

We do not have a concrete example showing that $\mathrm{Pf}_2(n) \neq \mathrm{trop}(\mathcal{P}f_2(n))$ for any n, nor $\mathrm{Pf}_k(2k+3) \neq \mathrm{trop}(\mathcal{P}f_k(2k+3))$ for any k, but the above proof cannot work for $k \geq 2$ since not every cone in $\mathrm{Pf}_k(n)$ can be sent to $\mathrm{Pf}_k^+(n)$ by a relabelling of the vertices. This is illustrated in the following example.

Example 3.35. Let $n = 6$ and $k = 2$. Observe that $\mathrm{Pf}_2(6) = \mathrm{trop}(\mathcal{P}f_2(6))$ since it is a hypersurface.

Consider the $v \in \mathbb{R}^{\binom{[6]}{2}}$ defined by

$$v_{1,3} = v_{2,3} = v_{2,4} = v_{4,5} = v_{5,6} = v_{1,6} = 1,$$

and $v_{i,j} = 0$ for every other i, j. This v lies in $\mathrm{Pf}_2(6)$ since it gives maximum weight to (exactly) two matchings, namely $\{13, 24, 56\}$ and $\{23, 45, 16\}$.

Since the first matching is negative and the second one is positive, we have that $v \in \mathrm{trop}^+(\mathcal{P}f_2(6))$. Since the two matchings do not differ by a single swap, part (2) of Theorem 3.31 implies that no relabelling sends v to $\mathrm{Pf}_2^+(6)$.

The example also shows that $\mathrm{trop}^+(\mathcal{P}f_k(n))$ is not contained in the Gröbner cone of $k+1$-crossings, but that is also easy to achieve with the following simpler example: let $v_{13} = 1$ and every other $v_{ij} = 0$. For any $k \geq 2$ and every $n \geq 6$ this gives a point in $\mathrm{trop}^+(\mathcal{P}f_k(n))$ (in every maximum matching of size 3 we can swap the two edges of weight zero to get a maximum matching of the opposite sign) that is not in the Gröbner cone (in any U containing $\{1, 3\}$ the matching using $\{1, 3\}$ has weight larger than the 3-crossing).

3.2.3. Proof of Lemma 3.32. In the following result we call an *accordion* any sequence $e_1, \dots, e_m$ of edges from $\binom{[n]}{2}$ such that: (a) For every $i = 1, \dots, n-1$, e_i and e_{i+1} share a vertex; (b) For every $i = 2, \dots, n-1$, the endpoints of e_{i-1} and e_{i+1} that are not in e_i lie on opposite sides of the line containing e_i.

The only property of k-triangulations that we need in what follows (apart from the fact that they are $(k+1)$-free) is:

Lemma 3.36. *Let T be a k-triangulation of the n-gon, for some k. Then, every two edges of T that do not cross are part of an accordion contained in T.*

Proof. Let $e = \{a, b\}$ and $e' = \{a', b'\}$ be the two edges of T; we assume without loss of generality that $1 \leq a \leq a' < b' \leq b \leq n$. We will use induction on $\min\{a' - a, b - b'\}$, taking as base cases those with $a = a'$ or $b = b'$, which are trivial. Hence, for the inductive step we suppose that e and e' have no endpoints in common.

If $\{a, b'\} \in T$, we are done, so we assume that $\{a, b'\} \notin T$. Then there is a k-crossing K in T that crosses that edge. That is, $K \cup \{a, b'\}$ is a $(k+1)$-crossing contained in $T \cup \{a, b'\}$. Let e'' be the edge next to $\{a, b'\}$ in the positive direction in this $(k+1)$-crossing. If e'' crossed e (resp. e'), then every edge in K would cross e (resp. e'), which would imply that T contains the $(k+1)$-crossing $K \cup \{e\}$ (resp. $K \cup \{e'\}$). Thus, e'' does not cross any of e or e'. Inductive hypothesis implies that T contains an accordion from e to e'' and an accordion from e'' to e', and the union of these two accordions is an accordion from e to e'. $\square$

We now consider a subset $U \in \binom{[n]}{2k+2}$ (as a set of *sides*, not *vertices*, of the n-gon) and $v = d(w) \in \mathrm{Grob}_k(n)$ sufficiently generic. Genericity implies, by Theorem 3.31, that the support of w is a certain k-triangulation T. For each edge $e \in T$ we call *length of* e *with respect to* U and denote it $\ell_U(e)$ the smallest size of the two parts of U separated by e. If both parts are equal (that is, if $\ell_U(e) = k+1$) we say that e is a diameter of U.

For a matching M of U and an edge e of T we denote by $c_M(e)$ the number of edges of M that cross e. Remember that, v being in the Gröbner cone, the maximum weight among matchings of U is the weight of the $(k+1)$-crossing.

LEMMA 3.37. *Let M be a matching of U. Then, M is of maximum weight with respect to v if, and only if, for every $e \in T$ we have that $\ell_U(e) = c_M(e)$.*

PROOF. Observe that the equality $\ell_U(e) = c_M(e)$ holds for the case when M is the $(k+1)$-crossing, and that, for arbitrary M, knowing which edges of T cross each edge of M is enough to compute the weight of M. This shows the sufficiency of $\ell_U(e) = c_M(e)$.

Now suppose that $\ell_U(e) > c_M(e)$ for some edge $e \in T$. Take a vector w' obtained setting w_e to its minimum possible value while staying in $\mathrm{Grob}_k(n)$. For $v' = d(w')$, the $(k+1)$-crossing is still the maximum weight matching, so

$$\sum_{e\in T} w'_e c_M(e) \le \sum_{e\in T} w'_e l_U(e) \Rightarrow \sum_{e\in T} w'_e (l_U(e) - c_M(e)) \ge 0$$

Our condition in w implies that $w_e > w'_e$, so

$$\sum_{e\in T} w_e (l_U(e) - c_M(e)) > 0 \Rightarrow \sum_{e\in T} w_e c_M(e) < \sum_{e\in T} w_e l_U(e)$$

Hence, M is not of maximum weight. ☐

For the rest of this section, we collapse the n-gon to a $(2k+2)$-gon by leaving only the sides labelled by U; that is, by contracting all edges e with $\ell_U(e) = 0$. We denote T_U the subgraph of K_{2k+2} obtained from T after this collapse. We introduce the following partial order among edges of T_U (or, in fact, among edges of K_{2k+2}): e and f are incomparable if they either cross or are separated by a diameter of U, and if they are comparable then they are ordered according to their ℓ_U.

Observe that both $\ell_U(e)$ and $c_M(e)$ depend only on the class of e in T_U. Thus, Lemma 3.37 needs only to be checked in T_U and not in T. (That is, only one representative edge of T for each class in T_U needs to be checked). But even more is true. Let $T_U^{\max}$ be the set of edges of T_U that are maximal (within T_U) for this order.

LEMMA 3.38. *Let M be a matching of U. If $\ell_U(e) = c_M(e)$ holds for the edges in $T_U^{\max}$ then it holds for all edges in T_U, hence in T.*

PROOF. Let $e < e'$ be two edges of T_U and suppose that $\ell_U(e') = c_M(e')$. Then, the edges of M that cross e' match the $\ell_U(e')$ edges of the $(2k+2)$-gon on the shorter side of e' to the same number of edges on the longer side (if e' is a diameter it does not matter which side we call "short"). By definition of $e < e'$, the smaller side of e is contained in the smaller side of e', so the same holds for e and $\ell_U(e) = c_M(e)$. ☐

This last lemma suggests we should look at properties of $T_U^{\max}$:

Lemma 3.39. (1) *Every two edges in $T_U^{\max}$ either cross each other or share a vertex.*
(2) *There is a vertex of the $(2k+2)$-gon not used in $T_U^{\max}$.*

Proof. For part (1) we use Lemma 3.36 and the observation that the passage from T to T_U preserves accordions. In particular, every two edges of T_U that do not cross are part of an accordion in T_U. Only two of the edges of an accordion contained in T_U can be in $T_U^{\max}$, and they share a vertex; hence, every two edges in $T_U^{\max}$ that do not cross share a vertex.

This finishes the proof of part (1) and gives us two possibilities:

- If all the edges in $T_U^{\max}$ mutually cross, then $T_U^{\max}$ is a j-crossing for some $j < k+1$. Hence, at least one (in fact at least two) of the $2k+2$ vertices of the $(2k+2)$-gon are not used.
- If two edges e and e' of $T_U^{\max}$ share a vertex p, then none of them is a diameter and, in fact, they are on opposite sides of the diameter using p. Then the opposite vertex q of that diameter is not used in $T_U^{\max}$ because it is impossible for an edge with an end-point in q other than the diameter itself to cross or share a vertex with both of e and e'.

In both cases we have a proof of part (2). □

Lemma 3.40. *Let p be a vertex of the $(2k+2)$-gon not used in $T_U^{\max}$. Let a and b be the elements of U next to p. Then, no maximal matching of U matches a to b.*

Proof. To seek a contradiction, suppose that M is a maximal matching and that $\{a, b\} \in M$. We claim that, for any other edge $\{c, d\} \in M$, no edge of $T_U^{\max}$ has a and b on one side and c and d on the other side. Suppose that there is such an edge e. Then, by Lemmas 3.37 and 3.38, we have $c_M(e) = l_U(e)$, and the swaps $\{a, c\}, \{b, d\}$ and $\{a, d\}, \{b, c\}$ cross $T_U^{\max}$ more often than the original pair of edges $\{a, b\}, \{c, d\}$; that is, more often than the single edge $\{c, d\}$ (since $\{a, b\}$ does not cross $T_U^{\max}$). This implies that, after swapping, we have $c_M(e) > l_U(e)$, which is not possible. This proves the claim.

Now, since all edges of $T_U^{\max}$ have a and b on the same side, we conclude that this side must contain one of c or d for every $\{c, d\} \in M$ other than $\{a, b\}$. In particular, for every $e \in T_U^{\max}$ the side of e containing a and b has length at least $k+2$ (it contains a, b and one vertex of each of the other k edges in M). This gives the following contradiction: Let p' be one of the vertices of the $(2k+2)$-gon next to p. The edge $\{p, p'\}$ is in T_U, since every boundary edge of the $2k+2$-gon is. Hence, there must be an edge in T_U that is greater than $\{p, p'\}$ in the partial order, and that edge can have length at most $k+1$ on the side containing a and b. □

We are now ready to prove Lemma 3.32:

Proof of Lemma 3.32. Let p be a vertex of the $(2k+2)$-gon not used in $T_U^{\max}$, which exists by Lemma 3.39. Let a and b be the first elements of U on both directions starting at p.

Let us denote by $\mathcal{M}$ the set of matchings of U not using the edge $\{a, b\}$. This contains all matchings of maximum weight by Lemma 3.40. Consider the map $\phi : \mathcal{M} \to \mathcal{M}$ that takes each matching $M \in \mathcal{M}$ and swaps in it the edges that contain a and b in the way that does not produce the pair $\{a, b\}$. This map is well-defined since there are three possible matchings among four vertices and we are excluding one of them. We have that:

- The map ϕ is obviously an involution.
- The map ϕ sends matchings of maximum weight to matchings of maximum weight by Lemmas 3.37 and 3.38, since every edge of $T_U^{\max}$ leaves a and b on the same side.
- If a' and b' are the elements of U matched to a and b in a certain matching M then the matching of of a, b, a', b' that has a crossing is involved in the swap from M to $\phi(M)$ (because the matching that is *not* involved in the swap is $\{a, b\}, \{a', b'\}$, which does *not* have a crossing). Hence, M and $\phi(M)$ have opposite parity, by Lemma 1.10.

Putting these facts together we conclude that ϕ restricts to a bijection between the odd and the even matchings of U of maximum weight. $\square$

3.3. Recovering the g-vector fan for $k = 1$.

In this section we look at the case $k = 1$ and show how to project $\mathrm{Pf}_1^+(n)$ isomorphically to the associahedron $\overline{\Delta}_1(n)$. In doing so we recover the so-called **g**-vector fan of the associahedron defined in the context of cluster algebras. Throughout the section let $T \subset \binom{[n]}{2}$ be an arbitrary triangulation of the n-gon, that we call the *seed triangulation*. Then:

Lemma 3.41. *For every $(v_{i,j})_{i,j} \in \mathrm{Pf}_1^+(n)$, knowing the entries of v corresponding to T we can recover all other entries. That is, the projection $\pi : \mathrm{Pf}_1^+(n) \to \mathbb{R}^T \cong \mathbb{R}^{2n-3}$ that restricts each vector $(v_{i,j})_{i,j}$ to the entries with $\{i, j\} \in T$ is injective.*

Proof. Let $v \in \mathrm{Pf}_1^+(n)$ and let us see that we can recover the entry $v_{i,j}$ for any $\{i, j\} \in \binom{[n]}{2}$, knowing the entries of v corresponding to edges of T.

The proof is by induction on the number of triangles of T crossed by $\{i, j\}$. If only two triangles are crossed, then $\{i, j\}$ is the only unknown entry from the quadruple $U = \{i, j, k, l\}$ consisting of those two triangles, and the edges $\{i, j\}$ and $\{k, l\}$ cross. Since $d \in \mathrm{Pf}_1^+(n)$, we have that the maximum weight among the three matchings in U is attained by $\{ij, kl\}$ and at least one of the other two matchings, so we can write:

$$v_{i,j} = \max\{v_{i,k} + v_{j,l}, v_{i,l} + v_{j,k}\} - v_{k,l}.$$

If $\{i, j\}$ crosses more than two triangles, let $\{k, i, l\}$ be the triangle incident to i and crossed by $\{i, j\}$. By inductive hypothesis, all the entries among the 4-tuple $\{i, j, k, l\}$ are known except for the entry $\{i, j\}$, so we can recover $v_{i,j}$ with the same formula as above. $\square$

That is, π embeds $\mathrm{Pf}_1^+(n)$ as a full-dimensional fan $\pi(\mathrm{Pf}_1^+(n)) \subset \mathbb{R}^T \cong \mathbb{R}^{2n-3}$. If we now compose it with a second projection

$$\phi : \mathbb{R}^T \to \mathbb{R}^{\overline{T}} \cong \mathbb{R}^{n-3}$$

that sends the irrelevant face of $\pi(\mathrm{Pf}_1^+(n))$ to zero we will automatically have that $\phi(\pi(\mathrm{Pf}_1^+(n)))$ is a fan isomorphic to the link of the irrelevant face in $\pi(\mathrm{Pf}_1^+(n))$, that is, isomorphic to $\overline{\Delta}_1(n)$, the normal fan of the associahedron. Here, $\overline{T}$ denotes the relevant part (the $n-3$ diagonals) of T.

Corollary 3.42. *The projection*

$$\phi \circ \pi : \mathrm{Pf}_1^+(n) \to \mathbb{R}^{\overline{T}} \cong \mathbb{R}^{n-3}$$

gives a realization of the associahedron $\overline{\Delta}_1(n)$ as a complete fan. $\square$

PROOF. This projection is conewise linear (linear in each cone). After normalizing, it becomes a continuous map from the $(n-4)$-dimensional sphere $\overline{\Delta}_1(n)$ to the unit sphere in $\mathbb{R}^{n-3}$ and, by Lemma 3.41, it is injective. Since every injective continuous map from a sphere to itself is a homeomorphism, $\phi(\pi(\mathrm{Pf}_1^+(n)))$ is complete. □

REMARK 3.43. Lemma 3.41 and its Corollary 3.42 do not hold for $k \geq 2$. In fact, suppose we take T to be any k-triangulation containing all the edges of the form $(1, i)$ and $(2, i)$, which exists since $k \geq 2$. Consider now the cone corresponding to a k-triangulation T' that does not use a certain edge $(1, i)$. In this cone we have $w_{1,i} = 0$ and hence

$$v_{1,i} + v_{2,i+1} = v_{1,i+1} + v_{2,i}.$$

Thus, the projection π is not injective; it collapses the cone of T' to lower dimension.

We now want to give a more explicit description of the fan in 3.42; that is, explicit coordinates in $\mathbb{R}^{n-3}$ for the ray corresponding to each edge $\{i, j\} \in \binom{[n]}{2}$. For this, remember that T is embedded as a true triangulation using the vertices of our n-gon, while the edges $\{a, b\} \in \binom{[n]}{2}$ corresponding to coordinates in our ambient space correspond to pairs of sides. For any given edge δ we define the following *crossing sign* of $\{a, b\}$ with respect to δ and the **g**-vector of $\{a, b\}$ with respect to T as follows:

DEFINITION 3.44 (See [66, Proposition 33] or [67, Definition 1.1]). Let δ be an edge in $\overline{T}$ and $\{a, b\} \in \binom{[n]}{2}$. Let $q(\delta)$ be the quadrilateral in T consisting of δ and its two adjacent triangles. We define the *crossing sign of* $\{i, j\}$ *with respect to* δ *in* T

$$\varepsilon(\delta \in T, \{a, b\}) := \begin{cases} +1 & \text{if } \{a, b\} \text{ crosses } q(\delta) \text{ as a } \mathsf{Z} \text{ (``zig'')} \\ -1 & \text{if } \{a, b\} \text{ crosses } q(\delta) \text{ as a } \Sigma \text{ (``zag'')} \\ 0 & \text{otherwise} \end{cases}$$

We define the **g**-*vector of* $\{a, b\}$ *with respect to* T as

$$\mathbf{g}(T, \{a, b\}) := (\varepsilon(\delta \in T, \{a, b\}))_{\delta \in \overline{T}}$$

REMARK 3.45. $\mathbf{g}(T, \{a, b\})$ has the following interpretation: The edges of T crossed by $\{i, j\}$ form an *accordion* in the sense of Section 3.2.3. The signs in the vector $\mathbf{g}(T, \{a, b\})$ record at which edges the accordion turns left or right. In particular, the **g**-vector is zero for edges of T that are not in the accordion, but also for those in which the accordion 'does not turn'.

This definition of **g**-vectors, which we have taken from Hohlweg, Pilaud and Stella [66], is a specialization to the disc of the *shear coordinates* described for arbitrary surfaces by Fomin and Thurston in [52]. They consider the **g**-*vector fan* obtained considering as cones all the possible clusters (which, in type A are the triangulations) and taking as generators the **g**-vectors for a fixed but arbitrary seed triangulation T. The main result of [66] is that these fans are polytopal. It turns out that these fans are linearly isomorphic to the ones of Corollary 3.42:

THEOREM 3.46. *In the basis of* $\mathbb{R}^{n-3}$ *consisting of the rays corresponding to the edges of* $\overline{T}$ *we have that for every* $\{a, b\} \in \binom{[n]}{2}$*, the vector* $\mathbf{g}(T, \{a, b\})$ *spans the ray of* $im(\phi \circ \pi)$ *corresponding to* $\{a, b\}$.

PROOF. For each $(i,j) \in \binom{[n]}{2}$ let $W_{i,j}$ be the generator of FP_n corrresponding to a certain $\{i,j\}$. That is, $W_{i,j} = d(w)$ for the vector w with $w_{i,j} = 1$ and $w_{i',j'} = 0$ if $\{i',j'\} \neq \{i,j\}$. We think of $W_{i,j}$ as the standard basis vector in the coordinates $w_{i,j}$, and let $V_{i,j}$ be the standard basis vector in the coordinates $v_{i,j}$ that we have been using so far. The $W_{i,j}$ are also the generators for the fan structure in $\mathrm{Pf}^+_k(n)$, so that $\phi \circ \pi(W_{i,j})$ is the corresponding generator of $\phi \circ \pi(\mathrm{Pf}^+_1(n))$.

The relations in Definition 3.8, which express the coordinates v in terms of the coordinates w, get transposed to the following relations among the vectors $W_{i,j}$ and $V_{a,b}$:

$$W_{i,j} = \sum_{\substack{\{a,b\} \in \binom{[n]}{2} \\ i<b\leq j<a\leq i}} V_{a,b}. \tag{18}$$

Observe that the projections π and ϕ are defined by their images at the vectors V and W, respectively. π sends $V_{i,j}$ to zero if $\{i,j\} \notin T$, and ϕ sends $\pi(W_{i,i+1})$ to zero for every i. For simplicity, for each vector $V \in \mathbb{R}^{\binom{[n]}{2}}$ we will denote by $\overline{V} := \phi(\pi(V)) \in \mathbb{R}^{n-3}$, and the same for $\overline{W}$.

Let $\{i,j\}$ be an edge of T. We then have

$$\overline{W}_{i,j} + \overline{W}_{i+1,j+1} = \overline{W}_{i,i+1} + \overline{W}_{j,j+1} = 0,$$

where the first equality comes from Equations (18) taking into account that the only edges of T crossing $\{i,j\}$ or $\{i+1,j+1\}$ are those with an end-point in i or j, and each of them crosses $\{i,j\}$ and $\{i+1,j+1\}$ the same number of times as it crosses $\{i,i+1\}$ or $\{j,j+1\}$. (Namely, they all cross once except for the edge $\{i,j\}$ which crosses twice). The second equality comes from the fact that $\phi(\pi(W_{i,i+1}) = 0$ for every i. Thus we have

$$\overline{W}_{i,j} = -\overline{W}_{i+1,j+1}$$

for each edge $\{i,j\}$ of T.

Now, let a and b be two sides of the n-gon and consider the accordion in T between a and b. Let $\{i_1,j_1\},\dots,\{i_\ell,j_\ell\}$ be the edges of T at which the accordion has an "inflection point" (it changes from turning left to turning right, or viceversa, that is, $\{a,b\}$ crosses $\{i_m,j_m\}$ as a Z or a Σ alternatively). The statement we want to prove is that

$$\overline{W}_{a,b} = \sum_{\delta \in \overline{T}} \varepsilon(\delta \in T, \{a,b\})\overline{W}_\delta = \sum_m \varepsilon(\{i_m,j_m\} \in T, \{a,b\})\overline{W}_{i_m,j_m} \tag{19}$$

Note that $-\overline{W}_{i_m,j_m}$ equals $\overline{W}_{i_m+1,j_m+1}$, so we are taking the sum of the edges in the zigzag turned in the direction of the path. Indeed, the sum in the right-hand side includes three times the edges $\{i_m,j_m\}$, twice the rest of edges in the accordion and once the rest of edges with an end-point in vertices where an $\{i_m,j_m\}$ meets the next one. Subtracting the irrelevant $\overline{W}$'s for these vertices, we get exactly once the edges separating a and b, and only them. □

COROLLARY 3.47. *Let T be any triangulation of the n-gon. The associahedral fan $im(\phi \circ \pi)$ in $\mathbb{R}^{n-3}$ of Corollary 3.42 equals the* **g***-vector fan of T. Hence, it is polytopal.*

REMARK 3.48. From the perspective of cluster algebras, associahedra are the type A case of the *generalized associahedra* that Fomin and Zelevinsky defined as simplicial spheres and F. Chapoton, S. Fomin and A. Zelevinsky [22] constructed as polytopes, using the so-called $\mathbf{d}$-vector fans for certain seed clusters. In type A, this construction was generalized by Santos [20, Section 5] to obtain Catalan-many associahedra by showing that any triangulation works as seed triangulation in the $\mathbf{d}$-vector construction.

The construction of generalized associahedra via $\mathbf{g}$-vectors instead of $\mathbf{d}$-vectors was first achieved in various special cases by, among others, Hohlweg-Lange-Thomas [65], Pilaud-Stump [108] and Stella [127], before the general case was settled by Hohlweg, Pilaud and Stella in [66].

The associahedral fans obtained by Santos via $\mathbf{d}$-vector fans and by Hohlweg-Pilaud-Stella via $\mathbf{g}$-vector fans from a seed triangulation T have certain similarities:

(1) For each of the $n-3$ edges $\{i,j\} \in \overline{T}$, the ray corresponding to $\{i,j\}$ is opposite to another ray. That is, the corresponding facets in the associahedron are parallel.
(2) Every other ray can be expressed as a $\{+1,0,-1\}$ combination in the basis given by those $n-3$ rays.

However, they are not the same. In the $\mathbf{g}$-vector fan the ray opposite to an edge $\{i,j\}$ of T is $\{i+1,j+1\}$ while in the $\mathbf{d}$ construction it is the edge inserted in T by the flip of $\{i,j\}$.

One could think that there is a variant of $\mathbf{g}$-vectors for $k>1$. For example, for $k=2$ it is known that multitriangulations are complexes of 5-sided stars [106], and a $\mathbf{g}$-vector can be defined assigning different values for $\varepsilon(\{i,j\} \in T, \{a,b\})$ depending on the position of $\{a,b\}$ with respect to the two stars incident to $\{i,j\}$. A priori, the problem would be how to define these $\varepsilon(\{i,j\} \in T, \{a,b\})$ so that they work. If the two edges cross, there are 4 possible positions for a and the same number for b, giving 16 different positions, and the idea would be to use different coefficients as ε depending on which of the 16 possibilities (or 10, if we mod out symmetry) we are in.

However, this idea can not work for n big enough.

THEOREM 3.49. *For a k-triangulation T, with $k>1$, if there is an edge whose two endpoints are not vertices in the same pair of adjacent stars of T, it is impossible to realize the k-associahedron as a $\mathbf{g}$-vector fan with seed T, independently of the values chosen for ε.*

PROOF. Let $\{a,b\}$ be the edge. We will show that $\mathbf{g}(a,b)+\mathbf{g}(a+1,b+1) = \mathbf{g}(a,b+1)+\mathbf{g}(a+1,b)$. Then, we can choose a k-triangulation that contains these edges (for $k>1$ it will exist), and its cone will not have full dimension.

This equality can be checked one coordinate at a time. For an edge $\{i,j\} \in T$, either a is not in the two stars delimited by $\{i,j\}$ or b is not. In the first case, $\varepsilon(\{i,j\} \in T, \{a,c\}) = \varepsilon(\{i,j\} \in T, \{a+1,c\})$ for any c, concretely for $c=b$ and $c=b+1$, and the equality holds for this component. The same happens if b is not in the two stars. □

COROLLARY 3.50. *It is impossible to realize the k-associahedron as a $\mathbf{g}$-vector fan, independently of the values chosen for ε, for $k>1$ and n big enough.*

PROOF. Suppose it is possible. Then all edges must be contained in a pair of adjacent stars. There are as many pairs of adjacent stars as relevant edges in T, that is, $k(n-2k-1)$. Each pair contains at most $4k$ vertices that form $\binom{4k}{2}$ edges, so we get

$$\binom{n}{2} \leq k(n-2k-1)\binom{4k}{2}$$

which is false for n big enough. □

CHAPTER 4

A bipartite formalism

The topic of this chapter is to study a bipartized version of multitriangulations which can be studied using these same tools as the previous chapters, and reproduce for them most of those results. The main advantage is that the "dimension" of the involved rigidity is halved from the previous study.

For bipartite graphs, the vertex set will be denoted as $[n_1] \cup [n_2']$, where $[n_1] = \{1, 2, \ldots, n_1\}$ are the vertices at one side of the graph and $[n_2'] = \{1', 2', \ldots, n_2'\}$ the vertices at the other side. The edge set can be identified now with a subset of $[n_1] \times [n_2]$, so the pair (i, j) corresponds to the edge between i and j'.

Definition 4.1. Let $G = ([n], E)$ be a simple graph on the vertex set $[n]$. The *bipartization of* G is the bipartite graph with vertex set $[n] \cup [n']$ and edge set

$$\{(i, n + 1 - j) : \{i, j\} \in E, i < j\}.$$

That is to say: each vertex i of G is split into two vertices i and $(n + 1 - i)'$, the first incident to the edges that (in G) go from i to higher labels and the second to the lower labels. As an example, in Figure 1 we show a 2-triangulation of the 7-gon and its bipartization.

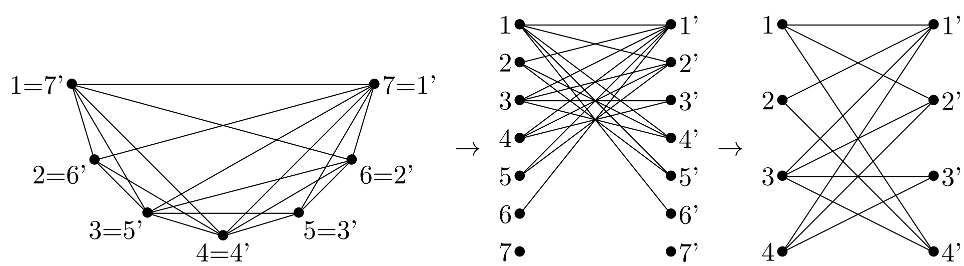


Figure 1. A 2-triangulation (left), its bipartization (center) and the reduced version of the latter (right).

The readers are encouraged to convince themselves that the bipartization of a triangulation of the n-gon is a tree on the vertices $[n-1] \cup [(n-1)']$, together with the two isolated vertices n and n'. The trees obtained can easily be related to the *non-crossing alternating trees* on $n - 2$ vertices, known to be in bijection with triangulations (see, e.g., [115]). Observe that trees are the bases of the 1-dimensional rigidity matroid, which hints at the main topic of this chapter: relating bipartized k-triangulations with rigidity in dimension k, as opposed to the dimension $2k$ from the original setting.

The bipartization of a k-triangulation is a bipartite graph with $2n$ vertices in which the vertices $n - i$ and $(n - i)'$ have degree i, for $i \leq k$. Since vertices of degree $\leq k$ are irrelevant in k-dimensional rigidity, we can delete them and obtain the *reduced bipartization* of a k-triangulation. As an example, Figure 1 (right) shows the reduced bipartization of the 2-triangulation in the left. In general, the reduced bipartization of a k-triangulation is a graph on the vertex set $[n-k] \cup [(n-k)']$ and with

$$2kn - \binom{2k}{2} - k(k+1) = 2kn - 3k^2 - 2k = k(2n-2k-2) - k^2$$

edges. This number coincides with the rank of the *bipartite (k,k)-rigidity* introduced in [75], which is nothing but k-hyperconnectivity restricted to bipartite graphs. We call it *bipartite k-hyperconnectivity*. Further exploiting this connection, in this chapter we prove that:

THEOREM 4.2. *Reduced bipartizations of k-triangulations are bases in the generic bipartite k-hyperconnectivity matroid, hence independent in the generic k-rigidity matroid.*

We also prove the following result relating $2k$-dimensional rigidity of a graph and k-dimensional rigidity of its bipartization. Observe that this result makes Corollary 3.22 a corollary of Theorem 4.2.

THEOREM 4.3. *Let $E \subset \binom{[n]}{2}$. If the bipartization of E is independent in the generic hyperconnectivity matroid in dimension k, then E is free in the generic hyperconnectivity matroid in dimension $2k$.*

However, the converse is not true: $K_{2k+2} \backslash \{1,2\}$ is a basis in $2k$-hyperconnectivity, while its reduced bipartization is $K_{k+1,k+1}$, which is a circuit in bipartite k-hyperconnectivity.

We then undertake the study of $\overline{\Delta}_k(2k+3)$ in the context of bipartite rigidity. For the usual rigidity the key to understanding which positions make all k-triangulations bases was the so-called Morgan-Scott obstruction, related to 6-tuples of points being in Desargues position or not. Here we have a similar obstruction. The base case is $k = 3$ and $n = 9$. Observe that hyperconnectivity is projectively invariant, so it makes sense to speak of the cross-ratio of four linear hyperplanes with a common codimension-two intersection.

THEOREM 4.4. *The graph $K_9 - \{16, 37, 49\}$ is a 3-triangulation of the 9-gon, but its reduced bipartization is dependent in the bipartite rigidity matroid if and only if the cross-ratio between the hyperplanes $(12, 23; 24, 25)$ equals $(2'4', 2'3'; 1'2', 2'5')$. This occurs, for example, if we take points along the moment curve with $\mathbf{t} = (1, 3, 4, 5, 7; 1, 3, 4, 5, 7)$.*

In order to have some control over the signs of the dependence, we will use *cyclic positions* for the vectors, which include the positions in the moment curve as a particular case. The reason for this choice is that every cyclic position realizes the case $2k+2$.

DEFINITION 4.5. A vector configuration $\mathbf{p} \subset \mathbb{R}^k$ is in cyclic position if its oriented matroid is the same than that of the cyclic polytope, that is, if a linear dependence between any $k+1$ of them has alternating signs.

THEOREM 4.6. *For $n = 2k+2$, every configuration in cyclic position realizes $\overline{\Delta}_k(2k+2)$ as a polytopal fan.*

Now, we can state exactly when $\overline{\Delta}_k(2k+3)$ is realized with this kind of positions:

THEOREM 4.7. *Let* $\mathbf{p}$ *be a configuration in cyclic position in* $\mathbb{R}^k$*. The following are equivalent:*

(1) $H_k(\mathbf{p})$ *realizes* $\overline{\Delta}_k(2k+3)$ *as a complete fan.*
(2) *For every* k*-triangulation* T *such that the three edges not in* T *do not share a vertex, those three edges are correctly located (see Definition 4.48).*

This leads to the following bipartite analog of Theorem 2.7. Observe that it is a bit stronger, since for $k \geq 4$ it applies to every $n \geq 2k+4$ instead of $n \geq 2k+6$.

THEOREM 4.8. *If* $k = 3$ *and* $n \geq 12$*, or* $k \geq 4$ *and* $n \geq 2k+4$*, then no choice of points* $\mathbf{t} \in \mathbb{R}^{2(n-k-1)}$ *in the moment curve makes the bar-and-joint rigidity matrix* $P_k(\mathbf{t})$ *realize the* k*-associahedron* $\overline{\Delta}_k(n)$ *as a fan.*

We think that, in cyclic positions outside the moment curve, this obstruction can be overcome.

4.1. Bipartite hyperconnectivity

4.1.1. Definition. For $n_1, n_2, d \in \mathbb{N}$, let $\mathbf{p} = (p_1, \dots, p_{n_1}; p'_1, \dots, p'_{n_2})$ be a configuration[1] of $n_1 + n_2$ points in $\mathbb{R}^d$. Their *bipartite hyperconnectivity matrix* (or, less precisely, *bipartite rigidity matrix*) is the following $n_1 n_2 \times (n_1+n_2)d$ matrix:

$$H(\mathbf{p}) := \begin{pmatrix} p'_1 & 0 & \dots & 0 & p_1 & 0 & \dots & 0 \\ p'_2 & 0 & \dots & 0 & 0 & p_1 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ p'_{n_2} & 0 & \dots & 0 & 0 & 0 & \dots & p_1 \\ 0 & p'_1 & \dots & 0 & p_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & p'_{n_2} & 0 & 0 & \dots & p_{n_1} \end{pmatrix} \tag{20}$$

In this matrix, there is a row for each pair (i, j), for $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$. This means that we can identify the rows with edges of the complete bipartite graph K_{n_1,n_2}. Given a realization of a bipartite graph $([n_1] \cup [n'_2], E, \mathbf{p})$, we call $H_E(\mathbf{p})$ the submatrix of $H(\mathbf{p})$ with the rows corresponding to E.

DEFINITION 4.9. We call *bipartite hyperconnectivity matroid* (or *bipartite rigidity matroid*), and denote by $\mathcal{H}(\mathbf{p})$, the linear matroid of the rows of this matrix. In particular, we say that a subset $E \subset [n_1] \times [n_2]$ of edges of K_{n_1,n_2} is *independent* or that it is *spanning* if it is so in this matroid. That is, E is independent if $H_E(\mathbf{p})$ has rank $|E|$, and spanning if it has rank equal to the rank of $H(\mathbf{p})$.

Independent and spanning sets of edges are also called *self-stress-free* and *rigid*, respectively.

The terminology used above coincides with the usual of rigidity theory. In fact, our setting is coincides with Kalai's d-hyperconnectivity [74, Sect. 6] restricted to bipartite graphs, and with Kalai et al's *bipartite rigidity* [75, Definition 3.2], restricted to the case where both parts are embedded in the same dimension ($k = l$, in the notation of [75]).

Note also that, for a fixed E, the matrix $H_E(\mathbf{p})$ will attain its maximum possible rank for generic $\mathbf{p}$. So we can define the *generic bipartite hyperconnectivity matroid*

[1] By a *configuration* we mean an ordered and labeled set of points or vectors. For this reason we use vector notation for $\mathbf{p}$ rather than set notation.

as the matroid obtained taking as $\mathbf{p}$ any generic matrix in $\mathbb{R}^{(n_1+n_2)\times d}$, and call it $\mathcal{H}_d(n_1,n_2)$. (However, the oriented matroid will depend on the concrete choice of $\mathbf{p}$, even in the generic case.)

4.1.2. Properties. In [28] we prove that, for bipartite graphs, hyperconnectivity is a special case of bar-and-joint rigidity, more concretely:

THEOREM 4.10 ([28, Theorem 4.4]). *Let $G = ([n_1] \cup [n_2'], E, \mathbf{p})$ be a realization of a bipartite graph, with $\mathbf{p} \subset \mathbb{R}^{d-1}$. Then, the following two d-dimensional rigidity matroids coincide:*

1. *the bar-and-joint matroid of the points $\{(p_i, 0) : i \in [n_1]\} \cup \{(p_j', 1) : j \in [n_2]\}$.*
2. *the hyperconnectivity matroid of the points $\{(p_i, 1) : i \in [n_1] \cup [n_2']\}$.*

Note that, for a generic $\mathbf{p}$, the first matroid mentioned is less free than generic bar-and-joint rigidity in dimension d but more free than the same matroid for dimension $d-1$ (because it is a projection to the first $d-1$ coordinates). On the other hand, due to the scaling invariance of hyperconnectivity (Proposition 4.17), the second matroid coincides with bipartite d-rigidity. This implies

COROLLARY 4.11. *Generic bipartite d-rigidity is more free than bar-and-joint rigidity in dimension $d-1$ and less free than the same matroid in dimension d.*

Our next property relates linear dependences in the bipartite rigidity matrix to linear dependences between the vectors.

LEMMA 4.12. *Given a framework $([n_1] \cup [n_2], E, \mathbf{p})$, the tensor product of two linear dependences between the points in $\mathbf{p}$ gives a dependence in $H_d(\mathbf{p})$.*

PROOF. Let $(c_i)_{i\in[n_1]} \cup (c_j')_{j\in[n_2]}$ be the dependence. Then, for $i \in [n_1]$,

$$\sum_{j\in[n_2]} c_i c_j' p_j = c_i \sum_{j\neq i} c_j' p_j = 0$$

and the same holds for $j \in [n_2]$. □

COROLLARY 4.13. *The $K_{d+1,d+1}$ graph on the nodes $p_1, p_2, \ldots, p_{d+1}$ and $p_1', p_2', \ldots, p_{d+1}'$ is a circuit in the hyperconnectivity matrix, with the coefficients given as the tensor product of the two linear dependences in the p_i and the p_i'.*

With this, we can compute the rank of the bipartite rigidity matroid:

THEOREM 4.14 ([74, Theorem 6.1], [28, Lemma 4.7]). *Let $G = K_{n_1,n_2}$ be a complete bipartite graph. Assuming that the positions chosen for each part of vertices are in general position, the rank of G in the d-hyperconnectivity matroid equals:*

1. *$n_1 n_2$ (that is, G is independent) if $\min\{n_1, n_2\} \le d$.*
2. *$dn - d^2$ if $\min\{n_1, n_2\} \ge d$, where $n = n_1 + n_2$ is the number of vertices.*

PROOF. For part (1), the edges of a vertex of degree d or less are independent of the rest, in an arbitrary graph. This is a common property of all abstract rigidity matroids.

For part (2), as the matrix has $n_1 n_2$ rows, we just have to find $(n_1-d)(n_2-d)$ dependences among them; that is, vectors orthogonal to every column. We consider columns as living in $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$ and observe that every tensor product $l \otimes m$ of a linear dependence l among the points on one side and a linear dependence m among the points on the other side is such an orthogonal vector. That is, we have the tensor product of two linear spaces of dimensions $n_1 - d$ and $n_2 - d$, which indeed has dimension $(n_1 - d)(n_2 - d)$. $\square$

REMARK 4.15. It is also easy to find d^2 linear dependences among the nd columns. Let $A = \{x_1, x_2, \ldots, x_{n_1}\} \subset \mathbb{R}^d, B = \{y_1, y_2, \ldots, y_{n_2}\} \subset \mathbb{R}^d$ be the positions for our vertices. For each pair $(l, l') \in [d]^2$, let $U \in \mathbb{R}^{dn}$ be the vector with coordinates

$$u_{il} = x_{il'}, i \in [n_1]$$

$$u'_{jl'} = -y_{jl}, j \in [n_2],$$

where the u_{il} are the coefficients for the first half of the columns and u'_{jl} for the second half. The coefficients u_{ih} or $u'_{jh'}$ with $h \neq l$ or $h' \neq l'$ are zero. Then, U is the vector of coefficients in the sought dependence.

REMARK 4.16. The rank computed in Theorem 4.14 is strictly less than the rank of an abstract rigidity matroid. This means that there are no bipartite bases of the hyperconnectivity matroid, unlike both bar-and-joint and cofactor matroids, in which $K_{d+1,\binom{d+1}{2}}$ is a basis.

The next result is a bipartite version of the linear invariance of hyperconnectivity ([28, Lemma 2.1]):

PROPOSITION 4.17. *Let $n_1, n_2 \in \mathbb{N}$ and $\mathbf{p} = (p_1, \ldots, p_{n_1}, p'_1, \ldots, p'_{n_2})$ be a vector configuration in $\mathbb{R}^d$. Then,*

(1) *The column-space of $H(\mathbf{p})$, hence the oriented matroid of its rows, that is, the bipartite hyperconnectivity matroid, is invariant under a linear transformation of the points $(p_1, \ldots, p_{n_1})$ in one side of the graph.*
(2) *The matroid $\mathcal{H}_d(\mathbf{p})$ is also invariant under rescaling (that is, multiplication by non-zero scalars) of the vectors p_i. If the scalars are all positive then the same holds for the oriented matroid.*

PROOF. Both transformations consist in making linear operations in the matrix: for the first, multiplying at the right by

$$\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}$$

where I is the identity matrix with size $n_1 d$ and M is a block-diagonal matrix with n_2 blocks, all equal to the matrix of the linear transformation. As we are multiplying at the right, this does not change the column-space.

For the second, as already noted in [28, Lemma 2.1], if p_i is rescaled to $\alpha_i p_i$, the effect in the matrix is multiplying the rows for the edges in the vertex i by α_i and then dividing the columns for the vertex i by the same scalar. If the scalars are positive, it does not affect the oriented matroid. $\square$

We also need a bipartite version of the operation called *coning*. In this operation, a new vertex is added to the graph and connected to all the rest, but obviously we cannot do this with a bipartite graph. However, there is an equivalent operation with similar properties.

Definition 4.18. Given a graph $G = ([n_1] \cup [n_2]', E)$, the *bipartite coning* of G is the graph with vertex set $[n_1+1] \cup [n_2+1]'$ where the two new vertices are joined to all the previous ones in the opposite side.

It is proved in [75, Lemma 3.12] that the bipartite coning of G is independent in the generic hyperconnectivity matroid in dimension $d+1$ if and only if G is independent in dimension d. Our next result is more precise in two aspects: we deal with arbitrary, not necessarily generic, positions, and we include information about the *oriented* matroid, not just the matroid. In fact, it is a bipartite version of Proposition 2.26.

Theorem 4.19 (Coning). *For any vector configuration V, the contraction of the oriented matroid for K_{n_1+1,n_2+1} in dimension $d+1$ by the edges in vertices $n_1+1, (n_2+1)'$ gives the oriented matroid for K_{n_1,n_2} in dimension d, where the vector configuration is contracted by the vectors for a and b.*

Proof. Let G' be the bipartite coning of G. Let us represent the two vertex additions in the rigidity matrix as follows: Start with the $(d+1, d+1)$-rigidity matrix and make a linear transformation to send the new vertex b to $(1, 0, \dots, 0)$, that we will suppose to be in the right side. Now delete a block of columns in that side, which is redundant for the matroid because of the linear dependences in the columns, so that the rows for the edges in b have only a nonzero entry. Thus, we can delete those rows and the associated columns, which are the first columns in all the blocks in the left side. The resulting matrix is the $(d+1, d)$-rigidity matrix for $G' \setminus b$ and the configuration where the right vectors have lost the first coordinate, that is, they have been contracted by b. Repeating this transformation with the other side, we obtain the result. □

4.1.3. Points along the moment curve and other cyclic positions. We are particularly interested in the bipartite rigidity with points in cyclic position (see Definition 4.5). A particular case of it is the *bipartite polynomial rigidity*, as in [28]: using (20) as a matrix but taking points along the moment curve. That is, $p_i = (1, t_i, \dots, t_i^{d-1})$ and $p'_j = (1, t'_j, \dots, (t'_j)^{d-1})$, where t_i, for $i \in [n_1]$, and t'_j, for $j \in [n_2]$, are real parameters. It is easy to see that changing the order of the points in $\mathbf{p}$ does not alter the matroid, not even the oriented one (it just permutes the elements in the ground set), so we can suppose without loss of generality that $t_1 < \ldots < t_{n_1}$ and $t'_1 < \ldots < t'_{n_2}$.

By the main result in [28], when points are chosen along the moment curve the bar-and-joint and hyperconnectivity rigidities coincide. In particular, we have that the rank stated in Theorem 4.14 is the rank of bar-and-joint rigidity of a complete graph with generic points along the moment curve. In fact, it is asked in [28, Question 4.6] whether $\mathcal{P}_d$ and $\mathcal{H}_d$ coincide; that is, whether for hyperconnectivity the moment curve is generic enough.

We want to see what happens with the oriented matroid of bipartite rigidity, in cyclic position, in the coning procedure depending on the position where the new point is inserted (not necessarily the last).

Corollary 4.20. *Let C be a bipartite circuit in $\mathcal{H}_d(n_1, n_2)$. The bipartite coning of C where the vertices i_0 and j'_0 are added is a circuit in $\mathcal{H}_{d+1}(n_1+1, n_2+1)$. If the initial graph is embedded in cyclic position, the final circuit has the same signs as if being in cyclic position, but multiplying the sign of an edge (i, j) by that of $(i - i_0)(j - j_0)$.*

PROOF. This is a particular case of Theorem 4.19. Starting with a cyclic embedding of K_{n_1+1,n_2+1}, we contract the left vertices by p_{i_0} and the right vertices by p'_{j_0}. The result at each side is not a cyclic embedding, but it becomes cyclic when we change sign the vectors with $i < i_0$ and $j < j_0$: the alternating sequences of signs are again alternating after losing one element and reversing the sign of the elements before it. This sign change multiplies the sign of an edge in the circuit by that of $(i - i_0)(j - j_0)$. □

The following result is about the number of sign changes in the sequence of coefficients for the edges in a given vertex.

LEMMA 4.21. *Let $\lambda \in \mathbb{R}^{n_1 n_2}$ be a linear dependence in the bipartite rigidity matroid in cyclic position and i a left vertex (resp. j' a right vertex). The sequence $\{\lambda_{ij}\}_{1\le j\le n_2}$ (resp. $\{\lambda_{ij}\}_{1\le i\le n_1}$) of signs of the edges in i (resp. j') change at least d times.*

PROOF. As the points are in cyclic position, the linear circuits between the points are alternating sequences with $d + 1$ points. The circuits in the matroid are a product of linear circuits between the points, hence the signs of the edges in a vertex have d changes. Finally, every dependence is a composition of circuits, hence it has at least the same number of changes. □

COROLLARY 4.22. *In that setting, the signs of the edges in a vertex of degree $d + 1$ alternate. Moreover, for the circuit $K_{d+1,d+1}$, we have $\operatorname{sign}(\lambda_{ij}) = (-1)^{i+j}$.*

4.1.4. The graph of the 3-cube. Our next result is a bipartite analogue of the Morgan-Scott obstruction [87] and its signed version from Section 2.3.2, but using hyperconnectivity instead of cofactor rigidity and the cube instead of the octahedron. Consider the bipartite graph obtained removing the perfect matching $11', 22', 33', 44'$ from $K_{4,4}$ in the vertices $\{1, 2, 3, 4\}$ and $\{1', 2', 3', 4'\}$, that we suppose embedded in that order along an affine line in $\mathbb{R}^2$. This is the graph of the cube and it is a basis in the bipartite hyperconnectivity matroid $\mathcal{H}_2(4, 4)$: it has one edge less than needed to be a basis in the hyperconnectivity matroid, but none of the four diagonals of the cube can be added keeping independence, only the diagonals of the faces, which would make the graph not bipartite.[2]

If we add the diagonal $11'$ to the cube, we get a circuit in $\mathcal{H}_2$. In this circuit, all the signs are fixed except for that of $11'$.

THEOREM 4.23. *Let $G = K_{4,4} \setminus \{22', 33', 44'\}$. In the hyperconnectivity matroid $\mathcal{H}_2$ (for positions in linear general position), the coefficient of the edge $11'$ in the circuit has the same sign (respectively, opposite sign) as that of $12'$ if and only if the cross-ratio $(1, 2; 3, 4)$ is less than (respectively greater than) the cross-ratio of $(1', 2'; 3', 4')$. In particular, the coefficient of $11'$ vanishes if and only if the two cross-ratios coincide.*

PROOF. Let the position of the vertex i be x_i and that of i' be y_i.

To compute the circuit, we will cancel the edge $22'$ between the circuits in the graphs $\{1, 2, 3\} \times \{1', 2', 4'\}$ and $\{1, 2, 4\} \times \{1', 2', 3'\}$. Each one is a $K_{3,3}$ in which we know how to compute the signs: they are the tensor product of the two

[2] This is a difference with bar-and-joint rigidity, in which a cube plus any edge is a basis in the generic 2-dimensional rigidity matroid.

dependences. In this case, the dependences themselves have an easy expression: in $\{1,2,3\}$, for example, the coefficients are the distances 23, 31 and 12, respectively.

Applying this to the first graph, the coefficient of $11'$ is $(x_3-x_2)(y_4-y_2)$, and that of $22'$ is $(x_1-x_3)(y_1-y_4)$. Normalizing the coefficient of $22'$ to 1, that of $11'$ becomes

$$\frac{(x_3-x_2)(y_4-y_2)}{(x_1-x_3)(y_1-y_4)} = \frac{(x_3-x_2)(y_4-y_2)}{(x_3-x_1)(y_4-y_1)}$$

Doing the same in the second graph, we get analogously

$$\frac{(x_4-x_2)(y_3-y_2)}{(x_4-x_1)(y_3-y_1)}$$

The difference between the two dependences gives the circuit we want, because the coefficients of $22'$ are now the same. The sign of $31'$ is the same as in the first dependence (because this edge is not in the second one), and if we normalized $22'$ to 1, $31'$ will also be positive. So here we must have $21'$ negative and $12'$ positive.

Respect to $11'$, we get

$$\begin{aligned} &\frac{(x_3-x_2)(y_4-y_2)}{(x_3-x_1)(y_4-y_1)} - \frac{(x_4-x_2)(y_3-y_2)}{(x_4-x_1)(y_3-y_1)} \\ = &\frac{(x_3-x_2)(y_3-y_2)}{(x_3-x_1)(y_3-y_1)}\left(\frac{(y_3-y_1)(y_4-y_2)}{(y_4-y_1)(y_3-y_2)} - \frac{(x_3-x_1)(x_4-x_2)}{(x_4-x_1)(x_3-x_2)}\right) \\ &= \frac{(x_3-x_2)(y_3-y_2)}{(x_3-x_1)(y_3-y_1)}((1',2';3',4')-(1,2;3,4)) \end{aligned}$$

as we wanted. □

Note that the cross-ratio is the only projective invariant between four points in a projective line. Hence:

COROLLARY 4.24. *$G = K_{4,4} \setminus \{11', 22', 33', 44'\}$ (that is, the graph of a 3-cube) is a circuit in $\mathcal{H}_2$ if, and only if, the two sets of four points are projectively equivalent.*

This, in turn, has Theorem 4.4 as a consequence:

PROOF OF THEOREM 4.4. Let G' be the bipartization of $K_9 - \{16, 37, 49\}$. G' coincides with the bipartite coning of G in the vertices 2 and $2'$. Hence, G' will be a circuit if and only if G is a circuit when embedded as the contraction of G' by the points 2 and $2'$. This happens exactly when the two sets of four points in G are projectively equivalent, which, in terms of G', means that the cross-ratios $(12, 23; 24, 25)$ and $(2'4', 2'3'; 1'2', 2'5')$ coincide. □

4.2. Hyperconnectivity of bipartized multitriangulations

4.2.1. Bipartized multitriangulations and stack polyominoes. As stated in the introduction, the reduced bipartization of a triangulation is a bipartite graph with $n-k-1$ vertices in each side and with $2kn - k(2k+1) - k(k+1) = 2kn - 3k^2 - 2k = k(2n-2k-2) - k^2$ edges. This is the rank of the bipartite hyperconnectivity matroid, so there are chances that they are basis. In this section we show that they indeed are. We will work in a more general framework that will allow us to prove the independence of a larger class of graphs.

For $n_1, n_2 \in \mathbb{N}$, consider the Cartesian product $[n_1]\times[n_2]$ as a poset, taking the increasing order in $[n_1]$ and $[n_2]$. Recall that an order ideal is a subset $S \subset [n_1]\times[n_2]$

so that for all $(a, b) \in S$ we have that $(a', b') \in S$ if $(a', b') \leq (a, b)$, that is, $a' \leq a$ and $b' \leq b$.

DEFINITION 4.25. A *Ferrers diagram* is an order ideal in the poset $[n_1] \times [n_2]$.

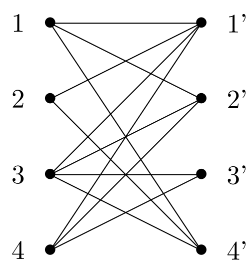
Ferrers diagrams are a particular case of *stack polyominoes* [72].

In what follows, let S be a Ferrers diagram. Observe that this is equivalent to the existence of positions $x_1 < \ldots < x_{n_1}$ and $y_1 > \ldots > y_{n_2}$ of the vertices $a \in [n_1], b \in [n_2]$ of K_{n_1,n_2} along a line such that $x_a < y_b$ if and only if $(x_a, y_b) \in S$. If we consider the elements of $[n_1] \times [n_2]$ as the edges in the complete bipartite graph, and locate the vertex a in the position x_a and b' in the position y_b, the edges in S are exactly those going right from a to b'.

For the reduced bipartization of a k-triangulation, $n_1 = n_2 = n - k - 1$, an edge (a, b) can appear in the bipartized graph if and only if $\{a, n+1-b\}$ is an edge in the original graph, that is, $a < n + 1 - b$. So the set of valid edges is

$$S_k(n) := \{(a, b) : a, b \in [n - k - 1], a + b \leq n\}$$

For the reduced bipartized 2-triangulation in Figure 1, we can represent $S_2(7)$ as a Ferrers diagram, formed by the "possible" edges, which will contain the edges in the bipartized 2-triangulation itself. Here the edges in the graph are marked with dots and the edges in $S_2(7)$ but not in the graph are empty squares.



$4'$	•	•	•	
$3'$			•	•
$2'$	•		•	•
$1'$	•	•	•	•
	1	2	3	4

Basically, the reason why the graph contains no 3-crossings, despite its appearance, is that the vertex 4 should be at the right of $4'$, or, said differently, $(4, 4) \notin S_2(7)$. This leads us to the following definition.

DEFINITION 4.26. A $(k+1)$-*crossing* in S is a set of $k+1$ incomparable elements whose supremum (in $[n_1] \times [n_2]$) belongs to S, that is, a set $\{(a_i, b_i)\}_{i=1,\ldots,k+1} \subset S$ such that $a_i < a_{i+1}$ and $b_i > b_{i+1}$ for all i, and $(a_{k+1}, b_1) \in S$. A set, or bipartite graph, is $(k+1)$-*free* if it has no $(k+1)$-crossing.

If we consider the elements of S as edges in a bipartite graph with the vertices in a parabola and all edges going to the right, this definition of crossing is equivalent to saying that the edges in question cross. In this way, $(k+1)$-crossings and $(k+1)$-free sets generalize our previous definitions, that are obtained when $S = S_k(n)$, and a (bipartized) k-triangulation is a maximal $(k+1)$-free graph in $S_k(n)$.

Every Ferrers diagram S can be written as

$$S = \{(a, b) : 1 \leq a \leq n_1, 1 \leq b \leq s_a\}$$

for adequate integers $n_2 \geq s_1 \geq s_2 \geq \ldots \geq s_{n_1} \geq 1$. With this notation, S has a "greedy" subset T_0 that is a maximal subset without $(k+1)$-crossings, namely

$$T_0 := \{(a, b) \in S : a \leq k \text{ or } b \geq s_a - k + 1\}.$$

Jonsson [72, Section 3] showed the following:

THEOREM 4.27. *For every Ferrers diagram S, all maximal $(k+1)$-free sets have exactly the same size as T_0.*

Observe that the biggest possible size for T_0 is $k(n_1+n_2)-k^2$, attained when $(k,n_2),(n_1,k) \in S$; that is, $s_k = n_2$ and $s_{n_1} \geq k$. When this happens we will say that S is *k-full*. In what follows, we suppose that S is k-full, because the elements in T contained in any row or column of S with less than k squares will not affect $(k+1)$-freeness. (Essentially, that is what we are doing when we reduce a bipartized graph.)

This result of Jonsson can be interpreted as saying that any induced subgraph of a $(k+1)$-free bipartite graph has the adequate number of edges to be itself independent. In fact, taking a subset of the vertices is equivalent to delete rows or columns of S, which will leave us with a Ferrers diagram of $[n_3]\times[n_4]$, for $n_3 \leq n_1$ and $n_4 \leq n_2$, to which we can apply Theorem 4.27.

COROLLARY 4.28. *A maximal 3-free bipartite graph plus any edge is a basis in the generic 2-rigidity matroid.*

PROOF. Any maximal 3-free set has $2(n_1+n_2)-4$ edges, and any subgraph induced by n_3+n_4 vertices has at most $2(n_3+n_4)-4$ edges. Adding any edge to the graph, the resulting graph satisfies the so-called Laman condition, which is equivalent to generic rigidity in dimension 2. □

This implies, of course, that 3-free graphs are independent in generic rigidity in dimension 2. It does not imply that they are independent in generic hyperconnectivity.

However, precisely that will be the main result of this section:

THEOREM 4.29. *$(k+1)$-free bipartite graphs defined in an arbitrary Ferrers diagram S are independent in the bipartite hyperconnectivity matroid in dimension k. Hence, $(k+1)$-free bipartite graphs with the maximum number of edges ($k(n_1+n_2)-k^2$) are bases.*

Applying Theorem 4.10, we get that these graphs are independent in the rigidity matroid with the positions of the vertices in two hyperplanes. This has as a consequence the generic independence of these graphs.

COROLLARY 4.30. *$(k+1)$-free bipartite graphs defined in an arbitrary Ferrers diagram S are independent in the bar-and-joint rigidity matroid in dimension k.*

Theorem 4.29 and Corollary 4.30 immediately imply Theorem 4.2. It also implies, using Theorem 4.3, that the original multitriangulations are bases in the $2k$-dimensional hyperconnectivity matroid (Theorem 3.22). It would be interesting to prove the same for bar-and-joint rigidity, but we know no analogue to Corollary 4.3 that relates the bar-and-joint rigidity of a graph in dimension $2k$ and that of its bipartization in dimension k.

4.2.2. Bipartite hyperconnectivity as an algebraic matroid. To take up the proof of Theorem 4.29, we need some algebraic background. What we do is similar to Section 3.1, but as we are now dealing with bipartite graphs, our matrices have no symmetry, and we have something more similar to the determinantal variety than to the Pfaffian variety. Also, we are only interested in a part of the matrix, that corresponds to the Ferrers diagram S.

Let $\mathcal{M}_k(n_1, n_2) \subset \mathbb{R}^{n_1 \times n_2}$ be the k-th *determinantal variety*, consisting of the $n_1 \times n_2$ matrices with rank at most k. Let $I_k^B(n_1, n_2) \subset \mathbb{R}[x_{ij} : i \in [n_1], j \in [n_2]]$ the corresponding ideal, which is generated by the $k+1$-size minors.

THEOREM 4.31. *The bipartite generic hyperconnectivity matroid for $n_1 + n_2$ vertices in dimension k coincides with the algebraic matroid of $\mathcal{M}_k(n_1, n_2)$.*

PROOF. Recall that if a variety $V \subset \mathbb{R}^N$ is parametrized as the image of a polynomial map $f : \mathbb{R}^M \to \mathbb{R}^N$ then the algebraic matroid of V coincides with the linear matroid of the Jacobian of f at a generic point [113, Proposition 2.5].

In our case, $\mathcal{M}_k(n_1, n_2)$ is the image of

$$T : \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{k \times n_2} \to \mathbb{R}^{n_1 \times n_2}$$

where T is the matrix product. The entries in the Jacobian of T are as follows. Let $(A, B) \in \mathbb{R}^{n_1 \times k} \times \mathbb{R}^{k \times n_2}$ and denote as $a_1, \dots, a_{n_1}$ the rows of A and $b_1, \dots, b_{n_2}$ the columns of B. All of them are vectors in $\mathbb{R}^k$. Then:

$$\frac{\partial T(A,B)_{ij}}{\partial (a_r)_s} = \frac{\partial (a_i \cdot b_j)}{\partial (a_r)_s} = \delta_{ir}(b_j)_s, \qquad \frac{\partial T(A,B)_{ij}}{\partial (b_t)_s} = \frac{\partial (a_i \cdot b_j)}{\partial (b_t)_s} = \delta_{jt}(a_i)_s,$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. That is, the row for the element ij is exactly the corresponding row of the hyperconnectivity matroid, with the coordinates of A sorted by rows and the coordinates of B by columns. □

Now let $\mathcal{M}_k(S)$ be the restriction of $\mathcal{M}_k(n_1, n_2)$ to a subset S of coordinates that is a Ferrers diagram, and $I_k^B(S)$ the ideal of this variety (which consists in the $(k+1)$-size minors contained in S). Then, the algebraic matroid of $\mathcal{M}_k(S)$, for any S, is precisely our bipartite hyperconnectivity matroid, and what we want to prove is that the graphs lacking certain sets of edges (in this case, $(k+1)$-crossings), are independent in the matroid.

We also define $\mathcal{M}_k(n) := \mathcal{M}_k(S_k(n))$ and $I_k^B(n) := I_k^B(S_k(n))$. This result is the core of the proofs in Chapter 3 and will be useful again in this setting.

PROPOSITION 4.32. *Let $I \subset \mathbb{K}[x_1, \dots, x_N]$ be a prime ideal, v a weight vector, $\mathcal{B}$ a Gröbner basis for I with respect to v and $\mathcal{F} \subset 2^{[N]}$ the set of the supports of the leading terms (with respect to v) of the polynomials in $\mathcal{B}$. Then, all subsets of $[N]$ that do not contain any element of $\mathcal{F}$ are independent in the algebraic matroid of I.*

PROOF. Suppose that T is dependent in the algebraic matroid. Then there is a polynomial $f \in I$ using only variables in T. In particular, T contains the support of the leading term of f. As $\mathcal{B}$ is a Gröbner basis, this leading term is multiple of a leading term of a polynomial in $\mathcal{B}$, so T contains an element of $\mathcal{F}$, as we wanted. □

That is, to prove independence of the $(k+1)$-free sets, we need weight vectors for which the minors are a Gröbner basis of the ideal they generate and that select in each minor the term with the $(k+1)$-crossing. For the standard set $S_k(n)$, it turns out that our ideal is a relabelling of an initial ideal of the Pfaffian ideal.

Let ϕ be the morphism $\mathbb{K}[x_{ij}]_{(i,j) \in S_k(n)} \to \mathbb{K}[x_{ij}]_{1 \le i < j \le n}$ given by $\phi(x_{ij}) = x_{i,n+1-j}$. That is, a relabelling that reverses the order of the second indices.

LEMMA 4.33. *Let v_{Pf} be the weight vector that assigns to each edge (a, b) weight equal to $b - a$. Then, $\phi(I_k^B(n)) = \mathrm{in}_{v_{Pf}}(I_k(n))$.*

PROOF. It is enough to see that the initial form of a Pfaffian of degree $k+1$, respect to v_{Pf}, coincides with the image by ϕ of a size $k+1$ minor contained in $S_k(n)$, and all the possible minors are obtained this way. Let f be the Pfaffian of $\{i_1, \ldots, i_{2k+2}\}$, with $i_1 < \ldots < i_{2k+2}$. Then, every term f_1 in f corresponds to a perfect matching in this set: $\{\{a_1, b_1\}, \ldots, \{a_{k+1}, b_{k+1}\}\}$.

$$v_{Pf}(f_1) = \sum_{j=1}^{k+1}(b_j - a_j) \leq \sum_{j=k+2}^{2k+2} i_j - \sum_{j=1}^{k+1} i_j$$

with equality if and only if $\{a_1, \ldots, a_{k+1}\} = \{i_1, \ldots, i_{k+1}\}$ and $\{b_1, \ldots, b_{k+1}\} = \{i_{k+2}, \ldots, i_{2k+2}\}$. The terms that attain this maximal weight are exactly the permutations in the minor for the rows $\{i_1, \ldots, i_{k+1}\}$ and columns $\{i_{k+2}, \ldots, i_{2k+2}\}$, so that minor is the initial form of f. As all the column indices are greater than $k+1$ and $i_{k+2} > i_{k+1}$, this minor is the image by ϕ of a minor contained in $S_k(n)$.

On the other hand, if we start with a minor contained in $S_k(n)$, after applying ϕ, the first column is greater than the last row, and as already shown, it is the initial form of a Pfaffian. $\square$

Recall that the algebraic matroid of the Pfaffian ideal $I_k(n)$ coincides with the generic hyperconnectivity matroid of n vectors in dimension $2k$. With this in mind, Proposition 4.32 and Lemma 4.33 have as a consequence Theorem 4.3.

PROOF OF THEOREM 4.3. Let E be dependent in dimension $2k$ and E_1 the bipartization of E. As we know, E is algebraically dependent in $I_k(n)$, so there is a polynomial f in $I_k(n)$ using only variables in E. Then $\text{in}_{v_{Pf}}(f) = \phi(g)$ where g is a polynomial in $I_k^B(n)$. If a variable x_{ij} is used in g, $\phi(g)$ uses $x_{i,n+1-j}$, so f also uses it and $\{i, n+1-j\} \in E$. By the definition of bipartization, $(i,j) \in E_1$. Hence, all the variables in g are in E_1, and E_1 is algebraically dependent in $I_k^B(n)$, that is, it is dependent in dimension k. $\square$

In what follows, we will call $\text{Grob}_k^B(S)$ the Gröbner cone of the $(k+1)$-crossing terms of $I_k^B(S)$ (that is, the cone of the vectors for which the $(k+1)$-crossing attains the maximum weight in each A and B with size $k+1$), and $\text{Grob}_k^B(n)$ analogously.

Also, let ϕ^* be the linear projection

$$\begin{array}{rcl} \phi^* : \mathbb{R}^{\binom{[n]}{2}} & \to & \mathbb{R}^{S_k(n)} \\ e_{ij} & \mapsto & \begin{cases} e_{i,n+1-j} & \text{if } (i, n+1-j) \in S_k(n) \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Note that, for $v \in \mathbb{R}^{\binom{[n]}{2}}$, an ideal $I \in K[x_{ij}]_{(i,j)\in S_k(n)}$, and a polynomial $f \in I$, $\text{in}_v(\phi(f))$ is formed by the terms in $\phi(f)$ that have greater weight respect to v. These terms come from corresponding terms in f, and the weight of the new terms respect to v is the weight of the original terms respect to $\phi^*(v)$. This implies that

$$\text{in}_v(\phi(f)) = \phi(\text{in}_{\phi^*(v)}(f))$$

and

$$\text{in}_v(\phi(I)) = \phi(\text{in}_{\phi^*(v)}(I))$$

THEOREM 4.34. *For $v \in \mathbb{R}^{\binom{[n]}{2}}$, $\phi^*(v) \in \text{Grob}_k^B(n)$ if and only if $v_{Pf} + \epsilon v \in \text{Grob}_k(n)$ for $\epsilon > 0$ small enough. Also, for weights in the interior of the cone, the $(k+1)$-size minors are a Gröbner basis.*

PROOF. For the first part,

$$\begin{aligned}
\phi^*(v) \in \mathrm{Grob}_k^B(n) &\Leftrightarrow \mathrm{in}_{\phi^*(v)}(I_k^B(n)) \text{ is the ideal of (bipartite) } (k+1)\text{-crossings} \\
&\Leftrightarrow \phi(\mathrm{in}_{\phi^*(v)}(I_k^B(n))) \text{ is the ideal of (circular) } (k+1)\text{-crossings} \\
&\Leftrightarrow \mathrm{in}_v(\phi(I_k^B(n))) \text{ is the ideal of } (k+1)\text{-crossings} \\
&\Leftrightarrow \mathrm{in}_v(\mathrm{in}_{v_{Pf}}(I_k(n))) \text{ is the ideal of } (k+1)\text{-crossings} \\
&\Leftrightarrow \mathrm{in}_{v_{Pf}+\epsilon v}(I_k(n)) \text{ is the ideal of } (k+1)\text{-crossings} \\
&\Leftrightarrow v_{Pf} + \epsilon v \in \mathrm{Grob}_k(n)
\end{aligned}$$

For the second, let $f \in I_k^B(n)$ and v an interior vector of the cone. We need to see that the leading term f_0 of f (that is a single monomial because v is interior) is multiple of a $(k+1)$-crossing monomial. As $\phi(I_k^B(n))$ is the initial ideal of $I_k(n)$ with weight vector v_{Pf}, $\phi(f) = \mathrm{in}_{v_{Pf}}(g)$ for a $g \in I_k(n)$. Also, $v = \phi^*(w)$, for some $w \in \mathbb{R}^{\binom{[n]}{2}}$. That is,

$$\phi(f_0) = \phi(\mathrm{in}_v(f)) = \phi(\mathrm{in}_{\phi^*(w)}(f)) = \mathrm{in}_w(\phi(f)) = \mathrm{in}_w(\mathrm{in}_{v_{Pf}}(g)) = \mathrm{in}_{v_{Pf}+\epsilon w}(g)$$

for ϵ small enough. Now we know that $v_{Pf} + \epsilon w \in \mathrm{Grob}_k(n)$ and by Theorem 3.13, Pfaffians are a Gröbner basis with respect to $v_{Pf} + \epsilon w$, so $\phi(f_0)$ is multiple of a circular $(k+1)$-crossing monomial, and f_0 is multiple of a bipartite one. □

Now the arguments will follow a similar path than in the case of Pfaffians: fp-positive weight vectors are in the Gröbner cone.

DEFINITION 4.35. For a Ferrers diagram S in $[n_1] \times [n_2]$, let $\overline{S}$ be the Ferrers diagram in $[n_1 - 1] \times [n_2 - 1]$ obtained by deleting the elements of the form $(1, j)$ or $(i, 1)$ and decrementing the rest of indices.

Let $\delta : \mathbb{R}^S \to \mathbb{R}^{\overline{S}}$ given by

$$\delta(v)_{ij} = v_{i,j+1} + v_{i+1,j} - v_{ij} - v_{i+1,j+1}$$

PROPOSITION 4.36. *$\delta(v)$ determines v except for a constant term added to each row or column:*

$$v_{ij} = v_{i1} + v_{1j} - v_{11} - \sum_{r=1}^{i-1}\sum_{s=1}^{j-1} \delta(v)_{rs}$$

THEOREM 4.37. *Let $v \in \mathbb{R}^S$ be a weight vector for the variables of $I_k^B(S)$. The following are equivalent:*

1. *All coordinates of $\delta(v)$ are nonnegative.*
2. *For $(i, j), (i', j') \in S$, $i < i'$ and $j < j'$, $v_{ij'} + v_{i'j} \geq v_{ij} + v_{i'j'}$.*
3. *For any subsets $A \subset [n_1]$ and $B \subset [n_2]$ with the same size such that $A \times B \subset S$, the weights given by v to perfect matchings between them are monotone with respect to swaps that increase crossings.*
4. *For any subsets $A \subset [n_1]$ and $B \subset [n_2]$ with the same size such that $A \times B \subset S$, the maximum weight given by v to perfect matchings between them is attained at the complete crossing.*

A vector v satisfying any of these conditions will be called four-point positive or fp-positive.

PROOF. Obviously we have $4 \Rightarrow 3 \Leftrightarrow 2 \Rightarrow 1$: the part 2 is a particular case of 4 when $|A| = |B| = 2$, and it is equivalent to 3 by the definitions of swap and crossing, and in turn, part 1 is a particular case of 2 when $i' = i+1$ and $j' = j+1$.

Suppose that part 1 holds. Then, by Proposition 4.36 or directly from the definition,

$$v_{ij'} + v_{i'j} - v_{ij} - v_{i'j'} = \sum_{r=i}^{i'-1} \sum_{s=j}^{j'-1} \delta(v)_{rs} \geq 0$$

and we have part 2.

Suppose now that part 3 holds. Then we can go from any matching between A and B to the complete crossing by swaps that create crossings, and by part 3 this process increases the weight in each step, so the complete crossing will have higher weight and part 4 holds. □

REMARK 4.38. This definition of fp-positive vectors is similar, but not exactly equal, to the one found in Chapter 2. There, we define a coordinate w for each edge $\{i, j\}$, and then show that the ones corresponding to sides of the polygon are in the lineality space of the four-point positive cone, that is, there are $\binom{n}{2} - n$ inequalities defining four-point positive vectors, that have dimension $\binom{n}{2}$.

For $\delta(v)$ as defined here, it has no coordinates in the lineality space by Theorem 4.37, and its dimension is $n_1 + n_2 - 1$ less than that of v, that is exactly the dimension of the lineality space (given by constants added to each row or column). For the standard case, the fp-positive vectors have dimension $\binom{n}{2} - k(k+1)$ and are defined by $\binom{n-2}{2} - k(k-1)$ inequalities. The lineality space is larger: it has dimension $2(n-k) - 3$. That is, the projection of that cone is strictly contained in this cone.

Let $\Delta : \mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{\binom{[n]}{2}}$ be the morphism that sends the v coordinates to the w coordinates as defined for the circular crossings (except by a factor 2 that appears there):

$$\Delta(v)_{ij} = v_{ij} + v_{i+1,j+1} - v_{i,j+1} - v_{i+1,j}$$

where i and j are here taken modulo n, and $v_{ii} = 0$.

LEMMA 4.39. *For $v \in \mathbb{R}^{\binom{[n]}{2}}$, $\delta(\phi^*(v)) = \phi^*(\pi(\Delta(v)))$, where π is the projection $\mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{\binom{[n-2]}{2}}$ that deletes the first and last elements, $\{i, i+1\}$ and $\{i, n\}$, in row i, and decrements the second index for the rest. That is, the operation δ is the bipartite analog of the circular operation Δ.*

PROOF. First note that the left-hand side is in $\mathbb{R}^{\overline{S_k(n)}}$ and the right-hand side is in $\mathbb{R}^{S_{k-1}(n-2)}$, which coincide because $\overline{S_k(n)} = S_{k-1}(n-2)$.

Now take $(i, j) \in S_{k-1}(n-2)$. Then

$$\begin{aligned}
\delta(\phi^*(v))_{ij} &= \phi^*(v)_{i,j+1} + \phi^*(v)_{i+1,j} - \phi^*(v)_{ij} - \phi^*(v)_{i+1,j+1} \\
&= v_{i,n-j} + v_{i+1,n-j+1} - v_{i,n-j+1} - v_{i+1,n-j} \\
&= \Delta(v)_{i,n-j} = \pi(\Delta(v))_{i,n-j-1} = \phi^*(\pi(\Delta(v)))_{ij}
\end{aligned}$$

where, in the last step, $n - j - 1$ gets changed to j because the size of $\pi(\Delta(v))$, seen as a matrix, is $n - 2$, not n. □

In the standard case where $S = S_k(n)$, we can characterize exactly which vectors are in the Gröbner cone. Let e_{ij}, for $(i, j) \in S$, be a vector in the standard basis, and f_{ij}, for $(i, j) \in \overline{S}$, the vector whose image by δ is e_{ij}.

THEOREM 4.40. $\mathrm{Grob}_k^B(n)$ *is given by the following inequalities:*

$$\sum_{r\geq i, s\geq j, s-r\leq k+1} \delta(v)_{rs} \geq 0 \quad \text{if } 2 \leq i+j \leq k \tag{21}$$

$$\delta(v)_{ij} \geq 0 \quad \text{if } k+1 \leq i+j \leq n-k-1 \tag{22}$$

$$\sum_{r\leq i, s\leq j, s+r\geq n} \delta(v)_{rs} \geq 0 \quad \text{if } n-k \leq i+j \leq n-2 \tag{23}$$

Its rays are:

- $-e_{ij}$ *with* $2 \leq i+j \leq k+1$
- f_{ij} *with* $k+2 \leq i+j \leq n-k-2$
- $-e_{i+1,j+1}$ *with* $n-k-1 \leq i+j \leq n-2$

PROOF. The cone we are looking for is the "link" of the smallest face containing v_{Pf} in $\mathrm{Grob}_k(n)$. The inequalities in the expression for $\mathrm{Grob}_k(n)$ in Theorem 3.14 are

$$\Delta(w)_{ij} \geq 0 \quad \text{if } |j-i| > k$$
$$\sum_{i'\leq i<j\leq j'\leq i'+k+1} \Delta(w)_{i'j'} \geq 0 \quad \text{if } 2 \leq |j-i| \leq k$$

Now we must take $w := v_{Pf} + \epsilon v$. Using the formula to compute $\Delta(v_{Pf})$, we get that $\Delta(v_{Pf})_{ij} = 0$ except if $j = n$, in which case

$$\Delta(v_{Pf})_{in} = n - i + (i+1) - 1 - n + (i+1) - i + 1 = 2.$$

That is, taking a point $v_{Pf} + \epsilon v$ close to v_{Pf} is equivalent to setting the coordinates $\{i, n\}$ of $\Delta(w)$ to a value much bigger than the rest. By doing this, all inequalities including a $\Delta(w)_{in}$ become irrelevant, and for the rest we have $\Delta(w) = \epsilon\Delta(v)$. Hence the inequalities reduce to

$$\Delta(v)_{ij} \geq 0 \quad \text{if } |j-i| > k, \quad i, j \leq n-1$$
$$\sum_{i'\leq i<j\leq j'\leq i'+k+1} \Delta(v)_{i'j'} \geq 0 \quad \text{if } 2 \leq |j-i| \leq k, \quad i, j \leq n-1$$

The $\Delta(v)$ appearing here can be replaced by $\pi(\Delta(v))$, because the remaining elements are not used. Using that $\delta(\phi^*(v)) = \phi^*(\pi(\Delta(v)))$, the first line gives (22), and the second one gives (21) and (23).

The rays are deduced by setting all the inequalities to hold with equality except one. If we do this with a coordinate in the range $k+2 \leq i+j \leq n-k-2$, we get just $\delta(v)_{ij} > 0$ and the rest equal to 0, which is f_{ij}. If the coordinate is outside this range, the result is

$$\delta(v)_{ij} = -\delta(v)_{i,j-1} = -\delta(v)_{i-1,j} = \delta(v)_{i-1,j-1} > 0$$

and the rest 0, which leads exactly to $-e_{ij}$, or

$$\delta(v)_{ij} = -\delta(v)_{i,j+1} = -\delta(v)_{i+1,j} = \delta(v)_{i+1,j+1} > 0$$

which leads to $-e_{i+1,j+1}$. □

THEOREM 4.41. *For any k-full set S, $\mathrm{Grob}_k^B(S)$ contains the four-point positive cone. For weights in the interior of the cone, the $(k+1)$-size minors are a Gröbner basis.*

Proof. The first sentence is a consequence of the point 4 of Theorem 4.37.

For the second one, Theorem 4.34 already proves it for $S = S_k(n)$. Now we will prove it for any other k-full S. In first place, it is easy to see that S can be obtained from $S_k(n)$ for some n by deleting rows and columns. In terms of $I_k^B(n)$, this is equivalent to restrict the ideal to a subset of variables.

Now, we can extend the weight vector v from S to $S_k(n)$ interpolating linearly the values in the new rows and columns. The effect of this in $\delta(v)$ is to divide the value in a coordinate between several ones. This preserves positivity of the coordinates, so we are still in the Gröbner cone. Applying Theorem 4.34 to this case, we have that the minors are a Gröbner basis, so the restriction of the ideal also is. □

Putting together Lemma 4.32 and Theorem 4.41, we get:

Corollary 4.42. *$(k+1)$-free bipartite graphs defined in a k-full set S are independent in the algebraic matroid of $\mathcal{M}_k(S)$. Hence, the maximal of these graphs (that have $k(n_1+n_2)-k^2$ edges) are bases.*

This has two consequences. On one hand, together with Theorem 4.31, it implies Theorem 4.29. On the other hand, we can apply to this case Theorem 3.19 to conclude

Corollary 4.43. *Let $T \subset [n_1] \times [n_2]$.*

1. *If T is $(k+1)$-free and $\mathbb{K}$ is algebraically closed, then for any generic choice of values $v \in \mathbb{K}^T$ there is at least one matrix in $\mathbb{K}^{n_1 \times n_2}$ of rank $\leq k$ with the entries prescribed by v.*
2. *If T contains a maximal $(k+1)$-free graph then for any choice of values $v \in \mathbb{K}^T$ there is only a finite number (maybe zero) of matrices in $\mathbb{K}^{n_1 \times n_2}$ of rank $\leq k$ with those prescribed entries.*

4.2.3. Tropical geometry. In Chapter 3, we prove that the tropical prevariety of antisymmetric matrices with size n and rank at most $2k$ contains a fan isomorphic to the k-associahedron in n vertices. It turns out that the tropical prevariety of matrices with size $n-k-1$ and rank at most k (projected to a subset of the coordinates) also contains a fan isomorphic, in this case, to the $(k-1)$-associahedron in $n-2$ vertices.

For the determinantal prevariety, we have defined $\mathcal{M}_k(S)$ as the projection of the determinantal variety into a subset S of coordinates, and its particular case $\mathcal{M}_k(n)$. So we can define $M_k(n)$ as the tropical prevariety of $I_k^B(n)$, $\mathrm{Grob}_k^B(n)$ the Gröbner cone for the $(k+1)$-crossings, and $M_k^+(n) = M_k(n) \cap \mathrm{Grob}_k^B(n)$. Then, the following is true:

Theorem 4.44. *Let v be a vector in $\mathrm{Grob}_k^B(n)$. Then, $v \in M_k^+(n)$ if and only if the support of $\delta(v)$ is k-free.*

To prove this, we need the following lemma:

Lemma 4.45. *For $v \in \mathbb{R}^{\binom{[n]}{2}}$, $\Delta(v_{Pf}) + \epsilon v$ has $(k+1)$-free support if and only if $\phi^*(\pi(v))$ has k-free support.*

Proof. We already know that $\Delta(v_{Pf})_{ij} = 0$ except if $j = n$, in which case $\Delta(v_{Pf})_{in} = 2$ (see proof of Theorem 4.40).

Suppose that there is a (circular) $(k+1)$-crossing in $\Delta(v_{Pf})+\epsilon v$, $\{a_1,b_1\},\{a_2,b_2\},\dots,\{a_{k+1},b_{k+1}\}$ with $a_1<\dots<a_{k+1}<b_1<\dots<b_{k+1}\le n$. Then $b_k<n$, and the first k elements $\{a_1,b_1\},\{a_2,b_2\},\dots,\{a_k,b_k\}$ are zero in $\Delta(v_{Pf})$ and consequently nonzero in v. This is a k-crossing in v that will subsist when applying π, and will be converted by ϕ^* to a bipartite k-crossing.

Conversely, if there is a k-crossing in $\phi^*(\pi(v))$, it translates to a (circular) k-crossing in $\pi(v)$. Let this be $\{a_1,b_1\},\{a_2,b_2\},\dots,\{a_k,b_k\}$. This makes $\{a_1,b_1+1\},\{a_2,b_2+1\},\dots,\{a_k,b_k+1\}$ a k-crossing in v that does not include n. This in turn implies that $\{a_1,b_1+1\},\{a_2,b_2+1\},\dots,\{a_k,b_k+1\},\{b_1,n\}$ is a $(k+1)$-crossing in $\Delta(v_{Pf})+\epsilon v$. □

Proof of Theorem 4.44. Let $v\in \mathrm{Grob}_k^B(n)$ and take w such that $\phi^*(w)=v$. As we already know, $v=\phi^*(w)\in \mathrm{Grob}_k^B(n)$ is equivalent to $v_{Pf}+\epsilon w\in \mathrm{Grob}_k(n)$ for small ϵ.

$$
\begin{aligned}
v\in M_k^+(n) &\Leftrightarrow \mathrm{in}_v(f) \text{ is not a monomial for any } (k+1)\text{-size minor } f\\
&\Leftrightarrow \mathrm{in}_{v_{Pf}+\epsilon w}(g) \text{ is not a monomial for any Pfaffian } g\\
&\Leftrightarrow v_{Pf}+\epsilon w\in \mathrm{Pf}_k^+(n)\\
&\Leftrightarrow \mathrm{Supp}\,\Delta(v_{Pf}+\epsilon w) \text{ is } (k+1)\text{-free (as a subset of } \binom{[n]}{2})\\
&\Leftrightarrow \mathrm{Supp}(\Delta(v_{Pf})+\epsilon\Delta(w)) \text{ is } (k+1)\text{-free}\\
&\Leftrightarrow \mathrm{Supp}\,\phi^*(\pi(\Delta(w))) \text{ is } k\text{-free (by the previous lemma)}\\
&\Leftrightarrow \mathrm{Supp}\,\delta(\phi^*(w)) \text{ is } k\text{-free}\\
&\Leftrightarrow \mathrm{Supp}\,\delta(v) \text{ is } k\text{-free.}
\end{aligned}
$$
□

Theorem 4.46. *$M_k^+(n)\subset \mathrm{trop}(\mathcal{M}_k(n))$. Moreover, $M_k^+(n)\subset \mathrm{trop}^+(\mathcal{M}_k(n))$.*

Proof. If $v\in M_k^+(n)$, $v_{Pf}+\epsilon w\in \mathrm{Pf}_k^+(n)$ for all w with $\phi^*(w)=v$ and some $\epsilon>0$. By Corollary 3.33, $v_{Pf}+\epsilon w\in \mathrm{trop}(\mathcal{P}f_k(n))$, so $\mathrm{in}_{v_{Pf}+\epsilon w}(I_k(n))$ contains no monomials. But $\mathrm{in}_{v_{Pf}+\epsilon w}(I_k(n))=\mathrm{in}_w(\mathrm{in}_{v_{Pf}}(I_k(n)))=\phi(\mathrm{in}_{\phi^*(w)}(I_k^B(n)))=\phi(\mathrm{in}_v(I_k^B(n)))$, so $v\in \mathrm{trop}(\mathcal{M}_k(n))$. The proof of the second part is analogous. □

Theorems 4.44 and 4.46 are also true if we replace $S_k(n)$ by any k-full S, because all them can be obtained from $S_k(n)$ for some n.

To conclude this section, we see that projecting the $M_k^+(n)$ to a complete fan, in order to realize the multiassociahedron, is equivalent to projecting a link of $\mathrm{Pf}_k^+(n)$ into a complete fan. This confirms Lemma 2.9, saying that the $(k-1)$-associahedron in $n-2$ vertices is a link of the k-associahedron in n vertices.

4.3. Realizability of the multiassociahedron with bipartite rigidity in cyclic position

In this section we turn our attention to realizability of $\overline{\Delta}_k(n)$ as a complete fan with the rows of the bipartite rigidity matrix in cyclic position (as a particular case, in the moment curve) as vectors. As proved in Theorem 2.20, realizing a simplicial complex as a fan with a given configuration of vectors is equivalent to checking some sign conditions.

4.3.1. The case $n \leq 2k+3$. The case $n = 2k+2$ is easy with what we know:

Proof of Theorem 4.6. In this case all the triangulations are K_{2k+2} minus a diameter. Bipartizing the complete graph, we get the complete bipartite graph $K_{k+1,k+1}$. The diameters of the original graph become $(1, k+1), (2, k), \ldots, (k+1, 1)$ after bipartizing. The condition ICoP means here that, in the linear dependence given by the complete graph, those edges must have the same sign.

For a configuration in the moment curve, Corollary 4.22 predicts all the signs of the circuit $K_{k+1,k+1}$: the sign of (i, j) is $(-1)^{j-1}$ times the sign of $(i, 1)$, which is $(-1)^{i-1}$ times the sign of $(1, 1)$, giving a final sign of $(-1)^{i+j}$. This implies that the flipping edges have the same sign, so the condition ICoP is satisfied.

Respect to the condition about elementary cycles, there is nothing to prove here because all them have length 3: removing two edges from a triangulation, the resulting graph can be completed only by adding two of the three edges. So the fan is realized in any position, and it is automatically polytopal because it is the fan of a simplex. □

Now we will find necessary and sufficient conditions so that the complete fan is realized for $n = 2k+3$. The bipartized graph for $n = 2k+3$ has $k+2$ vertices in each side: only the central vertex $k+2$ is common to both sides (as the last in each side), each vertex $i < k+2$ becomes i in the left side, and each $j > k+2$ becomes $2k+4-j$ in the right side. The bipartization of K_{2k+3} gives $K_{k+2,k+2}$ minus the edge $(k+2, k+2)$. The k-triangulations are formed by removing three more edges from this graph. These edges of course need to be relevant, that is, $\{i, i+k+1\}$ or $\{i, i+k+2\}$, which become after bipartizing $(i, k+3-i)$ and $(i, k+2-i)$ respectively. The $2k+3$ relevant edges form initially a cycle

$$k+2, 1, k+3, 2, \ldots, 2k+2, k+1, 2k+3, k+2,$$

which becomes a path

$$(k+2)', 1, (k+1)', 2, \ldots, 2', k+1, 1', k+2.$$

However, not every three relevant edges can be removed to obtain a triangulation. As stated in Section 2.3.3, the necessary and sufficient condition is that the three paths resulting from removing them (in the graph before bipartizing, where the relevant edges form a cycle) have even length (in what follows, "even length" includes zero). After bipartizing, this condition still holds, as long as we identify vertices $k+2$ and $(k+2)'$.

The union of two bipartized multitriangulations differing in one edge is $K_{k+2,k+2}$ minus $(k+2, k+2)$ and other two edges. Any two relevant edges can be removed, and between the two paths of relevant edges (identifying $k+2$ with $(k+2)'$) that remain one will have odd length. The multitriangulations contained in this graph are obtained by removing one edge from that path so that it becomes two even paths.

Respecting to the elementary cycles, they are obtained by removing two edges in a triangulation, so that the result is $K_{k+2,k+2}$ minus six edges. In Section 2.3.3, it is discussed how the length of this cycle depends on the different positions of the edges. The conclusion is that the length is five if and only if the lengths of the five paths that remain are all even.

Returning to the cyclic positions, we need now some definitions.

DEFINITION 4.47. Let S be a set of $k+2$ points in the projective space $\mathbb{RP}^{k-1}$ in general position and let a, b, c and d be four of them. Let $T = S \setminus \{a, b, c, d\}$. The *relative cross-ratio* of $a, b, c, d \in S$, denoted as $(a, b; c, d)_S$, is the cross-ratio of the four points obtained by conical projection of a, b, c, d with center at T onto a projective line. Equivalently, it equals the cross-ratio of the four planes spanned by Ta, Tb, Tc, Td. That is,

$$(a, b; c, d)_S = \frac{|acT| \cdot |bdT|}{|adT| \cdot |bcT|}$$

where $|xyT|$ denotes the $k \times k$ determinant whose rows are x, y and the points in T.

In the case of points $\{t_i : i \in [n]\}$ in the moment curve, this formula simplifies to

$$(a, b; c, d)_{\mathbf{t}} = \frac{(t_c - t_a)(t_d - t_b)}{(t_d - t_a)(t_c - t_b)}$$

DEFINITION 4.48. Given three relevant edges $\{i_l, j_l\}, l = 1, 2, 3$ of the $(2k+3)$-gon, where $i_1 < i_2 < i_3 < k+2 < j_1 < j_2 < j_3$, and a position $\mathbf{p}$, we will say that the three edges are *correctly located* if $(i_1, i_2; i_3, k+2)_{\mathbf{p}} > (j'_1, j'_2; j'_3, (k+2)')_{\mathbf{p}}$, where $j'_l = 2k + 4 - j_l$ is the new index of vertex j_l.

Note that, despite its asymmetric appearance, being correctly located is a symmetric relation: if we reverse the order of the vertices, the relation becomes $(j'_3, j'_2; j'_1, (k+2)')_{\mathbf{p}} > (i_3, i_2, i_1, k+2)_{\mathbf{p}}$, which is equivalent to the previous one.

LEMMA 4.49. *Given four relevant edges $e_l = \{i_l, j_l\}, l = 1, 2, 3, 4$ of the $(2k+3)$-gon, where $i_1 < i_2 < i_3 < i_4 < k+2 < j_1 < j_2 < j_3 < j_4$, if $\{e_1, e_2, e_3\}$ and $\{e_2, e_3, e_4\}$ are correctly located, $\{e_1, e_2, e_4\}$ are also correctly located (and, by symmetry, $\{e_1, e_3, e_4\}$, but we do not need that).*

PROOF. We can make a conical projection by the points not in $\{i_1, i_2, i_3, i_4, k+2\}$ to the projective plane, followed by a projective transformation so that the positions p_{i_j}, for $j = 1, 2, 3, 4$, become $[0 : 0 : 1], [1 : 1 : 1], [1 : 0 : 0], [0 : 1 : 0]$, and the same for the other side with $\{j'_1, j'_2, j'_3, j'_4, (k+2)'\}$. After that, the only point that is in different position is $k+2$.

Now let $p_{k+2} = [x : y : 1], p'_{k+2} = [x' : y' : 1]$. By convexity we have $y < x < 0$ and $y' < x' < 0$.

As $\{e_1, e_2, e_3\}$ are correctly located,

$$([0 : 1], [1 : 1]; [1 : 0], [x : 1]) > ([0 : 1], [1 : 1]; [1 : 0], [x' : 1])$$

which gives $x > x'$, and as $\{e_2, e_3, e_4\}$ are correctly located,

$$([1 : 1], [1 : 0]; [0 : 1], [x : y]) > ([1 : 1], [1 : 0]; [0 : 1], [x' : y'])$$

which gives $y/x > y'/x'$. These two imply $y > y'$, that is

$$([0 : 1], [1 : 1]; [1 : 0], [y : 1]) > ([0 : 1], [1 : 1]; [1 : 0], [y : 1])$$

and $\{e_1, e_2, e_4\}$ are correctly located. □

LEMMA 4.50. *Let $1 \leq i_1 < i_2 < k+2 < j_1 < j_2 \leq 2k+3$ be such that the bipartization of $C := K_{2k+3} \setminus \{\{i_1, j_1\}, \{i_2, j_2\}\}$ is a circuit. Consider a configuration $\mathbf{p}$ in cyclic position for the vertices of this bipartization and let λ be the unique (modulo a scalar factor) dependence for C in the bipartite rigidity matrix $H_k(\mathbf{p})$.*

Let $\{i_3, j_3\}$ *be the first edge in the odd path delimited by* $\{i_1, j_1\}$ *and* $\{i_2, j_2\}$ *(that is,* $\{i_1 \pm 1, j_1\}$ *or* $\{i_1, j_1 \pm 1\}$*), and let* $\{i_0, j_0\}$ *be another edge in the same path at a distance* d *from the first. Then, we have that*

$$\operatorname{sign}(\lambda_{i_0 j_0}) = (-1)^d \operatorname{sign}(\lambda_{i_3 j_3})$$

if and only if $\{\{i_0, j_0\}, \{i_1, j_1\}, \{i_2, j_2\}\}$ *are correctly located.*

Proof. Suppose first that $i_0 < i_1 < i_2 < j_0 < j_1 = i_1 + k + 1 < j_2 = i_2 + k + 2$. Then, by the structure of the path, $i_3 = i_1 - 1$, $j_3 = j_1$, $i_0 + j_0 = i_1 + j_1 - d - 1$.

Bipartizing the circuit, we get $K_{k+2,k+2}$ minus three edges. In the resulting graph, the degree of j_1 is $k + 1$ and by Lemma 4.21,

$$\operatorname{sign}(\lambda_{i_1 - 1, j_1}) = (-1)^{i_1 - i_0 - 1} \operatorname{sign}(\lambda_{i_0 j_1})$$

so the condition to be checked reduces to

$$\operatorname{sign}(\lambda_{i_0 j_0} \lambda_{i_0 j_1}) = (-1)^{d+1+i_1-i_0} = (-1)^{j_1 - j_0}$$

The circuit can be obtained by a repeated bipartite coning from the graph of $K_{4,4}$ minus three edges, so that the original eight vertices become $\{i_0, i_1, i_2, k+2\}$ in the left side and $\{j_2', j_1', j_0', (k+2)'\}$ in the right side. That is, the three missing edges are $22'$, $31'$ and $44'$ before the coning. By Corollary 4.20, the condition reduces to $\operatorname{sign}(\lambda_{12}\lambda_{13}) = -1$, because this sign is inverted $i_3 - j_2 - 1$ times (once for each vertex added between j_2 and i_3).

Rearranging (via a projective transformation of the line) the vertices $1'$ and $3'$ will not alter the signs in the first three vertices, because this is a linear transformation in dimension 2 followed by a rescaling that has the same sign in these three vertices (Proposition 4.17). After the change, we are in the situation of Theorem 4.23, and the condition is now $\operatorname{sign}(\lambda_{11}\lambda_{12}) = -1$. This happens if and only if $(1, 2; 3, 4) > (1', 2'; 3', 4')$. As contracting does not change the relative cross-ratios, this means $(i_1, i_2; i_3, k+2)_{\mathbf{p}} > (j_1', j_2'; j_3', (k+2)')_{\mathbf{p}}$, as we want.

The remaining cases lead, analogously, to $(i_2, i_3; k{+}2, i_1)_{\mathbf{p}} > (j_2', j_3'; (k{+}2)', j_1')_{\mathbf{p}}$ and $(i_3, k+2; i_1, i_2)_{\mathbf{p}} > (j_3', (k+2)'; j_1', j_2')_{\mathbf{p}}$. The third condition is the same as the first one, and the second one is also equivalent, because of the order of the vertices. □

We are now ready to prove the main result for $n = 2k + 3$. We call *octahedral* a triangulation whose three missing edges are disjoint.

Proof of Theorem 4.7. If (1) holds, the condition ICoP is satisfied in all the flips. In particular, with the notations of the previous Lemma, it is satisfied in the flip from $K_{2k+3} \setminus \{\{i_0, j_0\}, \{i_1, j_1\}, \{i_2, j_2\}\}$ that removes $\{i_3, j_3\}$ and inserts $\{i_0, j_0\}$, with d even, that is, the edges $\{i_3, j_3\}$ and $\{i_0, j_0\}$ have the same sign in the dependence contained in the union of the two triangulations. By Lemma 4.50, this implies that the three edges $\{i_0, j_0\}, \{i_1, j_1\}, \{i_2, j_2\}$ are correctly located and (2) holds.

Now suppose that (2) holds. Then, there are two types of flips: those whose two missing edges share a vertex and the rest. For the first case, the bipartization contains a $K_{k+1,k+1}$ and we already know that this graph is a circuit where the signs of the edges in the only $(k{+}1)$-crossing coincide, so this case is solved. For the second, the two missing edges are disjoint. If one of them contains the vertex $k+2$, the bipartization contains again a $K_{k+1,k+1}$. Otherwise, we are in the conditions of Lemma 4.50. Applying it to all edges with d even, that are exactly the edges that can be removed from the graph to get a triangulation, we obtain that all of them have the same sign, hence the condition ICoP also holds here.

It only remains to see the condition about elementary cycles. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be an elementary cycle of length 5, which means that all the paths left between the edges are even. If two edges share a vertex, for example e_1 and e_2, the flip $K_{2k+3} \setminus \{e_1, e_2\}$ contains a $K_{k+1,k+1}$, and the signs of the other three edges are automatically correct. The same happens if $k+2$ is a vertex of any edge.

In the remaining cases, $\{e_1, e_2, e_3\}$ and $\{e_2, e_3, e_4\}$ are correctly located by hypothesis. By Lemma 4.49, $\{e_1, e_2, e_4\}$ is also correctly located. By Lemma 4.50, in the flip $K_{2k+3} \setminus \{e_1, e_2\}$, e_4 has opposite sign to e_3 and e_5, and we are done. □

Corollary 4.51.
- *For $k = 2$ and $n = 7$, any choice of points $(p_1, p_2, p_3, p_4; p'_1, p'_2, p'_3, p'_4)$ in cyclic position realizes $\overline{\Delta}_2(7)$ as a fan.*
- *For any $k > 2$, there is a choice of points in cyclic position that realizes $\overline{\Delta}_k(2k+3)$ as a fan, and a choice that does not.*

Proof. For $k = 2$, the condition of the previous theorem trivially holds in any position because there are no octahedral triangulations.

For the second part, it is easy to find a position that does not realize the fan: just choose an octahedral triangulation, and locate the points in such a way that they violate the condition in Theorem 4.7. To find a position that realizes the fan, take the positions in the moment curve with t_{k+2} and t'_{k+2} very big, $t_{k+1} = t'_{k+1} = 1$, $t_k = t'_k = 0$ and t_i lexicographically smaller than t_{i+1}. Then, for any octahedral triangulation, in the corresponding inequality the left hand side $(i_1, i_2; i_3, k+2)_{\mathbf{t}}$ is large and the right hand side $(j'_1, j'_2; j'_3, (k+2)')_{\mathbf{t}}$ is close to 1 (because j'_1 and j'_2 are relatively close). □

For $k \geq 3$, we can now prove Theorem 4.8, because, if $2k+4$ vertices realize the fan, any subset with $2k+3$ vertices should also realize it.

Proof of Theorem 4.8. First suppose that $k = 3$ and $n = 12$. This will account for all cases with $n \geq 12$, because all of them contain $\overline{\Delta}_3(12)$.

Consider the subconfiguration formed by the 9 vertices $1, 2, 3, 5, 6, 7, 8, 9, 10$, where 6 is the central vertex and $\{1, 7\}$, $\{3, 8\}$ and $\{5, 10\}$ are three missing edges in an octahedral 3-triangulation. If there is a position which realizes the fan, by Theorem 4.7, the three edges are correctly located, that is:

$$(1, 3; 5, 6) > (6', 5'; 3', 7')$$

(the numbers refer to positions of vertices of $K_{8,8}$, so $6', 5', 3', 7'$ are respectively $7, 8, 10, 6$ before bipartizing). In terms of the parameters,

$$\frac{(t_5 - t_1)(t_6 - t_3)}{(t_6 - t_1)(t_5 - t_3)} > \frac{(t'_7 - t'_5)(t'_6 - t'_3)}{(t'_7 - t'_6)(t'_5 - t'_3)}$$

Using that the parameters are in increasing order, this implies

$$\frac{t_6 - t_3}{t_5 - t_3} > \frac{t'_6 - t'_3}{t'_5 - t'_3}$$

Now we repeat the argument with the subconfiguration symmetric to the previous one, so the t's get swapped with the t''s, and we get

$$\frac{t'_6 - t'_3}{t'_5 - t'_3} > \frac{t_6 - t_3}{t_5 - t_3}$$

This is a contradiction.

Now we will get a similar contradiction for $k \geq 4$ and $n \geq 2k+4$. As before, we can take $n = 2k+4$.

Consider the subconfiguration formed by the first $2k+3$ vertices. The complete graph in those vertices minus the three edges $\{2, k+3\}$, $\{3, k+5\}$ and $\{k+1, 2k+2\}$ is an octahedral k-triangulation (note that, as $k \geq 4$, $2k+2 > k+5$).

Same as before, the three edges are correctly located and

$$(2,3; k+1, k+2) > ((k+2)', k'; 3', (k+3)')$$
$$\frac{(t_{k+1} - t_2)(t_{k+2} - t_3)}{(t_{k+2} - t_2)(t_{k+1} - t_3)} > \frac{(t'_{k+3} - t'_k)(t'_{k+2} - t'_3)}{(t'_{k+3} - t'_{k+2})(t'_k - t'_3)}$$

This implies

$$\frac{t_{k+2} - t_3}{t_{k+1} - t_3} > \frac{t'_{k+2} - t'_3}{t'_k - t'_3} > \frac{t'_{k+2} - t'_3}{t'_{k+1} - t'_3}$$

and we are done by applying symmetry. □

This proof does not apply in general to cyclic positions outside the moment curve. The reason for this is that the cross-ratios $(a, b; c, d)_{\mathbf{p}}$ involved depend in all the parameters in $\mathbf{p}$. In the moment curve, most of them cancel out, and only t_a, t_b, t_c, t_d remain, but in general the points in $\mathbf{p}$ different from a, b, c, d may affect the value of $(a, b; c, d)_{\mathbf{p}}$.

4.3.2. Experimental results. In this section we report on some experimental results. In all of them we choose real parameters $\mathbf{t} = \{t_1 < t_2 < \ldots < t_{n-k-1}, t'_1 < t'_2 < \ldots < t'_{n-k-1}\}$ and computationally check whether the configuration of rows of $P_k(\mathbf{t})$ realizes $\overline{\Delta}_k(n)$ first as a collection of bases, then as a complete fan, and finally as the normal fan of a polytope, as in Section 2.4.3.

We first look at $k = 2$. Our first experiment is taking equispaced parameters. Since an affine transformation in the space of parameters produces a linear transformation in the rows of $P_k(\mathbf{t})$, we take without loss of generality $\mathbf{t} = (1, 2, 3, \ldots, n)$. We call these the *standard positions* along the parabola.

LEMMA 4.52. *Let* $\mathbf{t} = \{1, 2, \ldots, n\}$ *be standard positions for the parameters. Then:*

(1) *Standard positions for* $P_2(\mathbf{t})$ *realize* $\overline{\Delta}_2(n)$ *as the normal fan of a polytope if and only if* $n \leq 8$.
(2) *The near-lexicographic positions* $t_i = 2^{(i-1)^2}$ *for* $P_2(\mathbf{t})$ *realize* $\overline{\Delta}_2(10)$ *as the normal fan of a polytope.*
(3) *Standard positions for* $P_2(\mathbf{t})$ *realize* $\overline{\Delta}_2(n)$ *as a complete fan for all* $n \leq 13$.

PROOF. (1) For $n = 8$, Table 1 shows values of $(f_{ij})_{i,j}$ that prove the fan polytopal. This implies the same for $n < 8$.[3] For $n = 9$ the computer said that the system is not feasible, and that implies the same for $n > 9$.

(2) Near-lexicographic positions for $n = 10$ gave the feasible solution displayed in Table 1.

(3) The computer checked the conditions for a complete fan for $n = 13$. This took about 2 days of computing in a standard laptop. □

[3]In order to check polytopality we can arbitrarily fix the right-hand side variables $f_{i,j}$ corresponding to one particular k-triangualtion. In both cases we have chosen $f_{1,j} = f_{2,j} = 0$ for all j, corresponding to the greedy 2-triangulation

i,j	f_{ij}
3,6	16
3,7	35
3,8	59
4,7	11
4,8	36
5,8	37

i,j	f_{ij}	i,j	f_{ij}	i,j	f_{ij}
3,6	2097152	4,7	1094032	5,9	6427739
3,7	2116197	4,8	1133231	5,10	6575138
3,8	2116816	4,9	1136343	6,9	166833470
3,9	2116875	4,10	1137410	6,10	237316440
3,10	2116899	5,8	5949943	7,10	120253293274

TABLE 1. Height vectors $(f_{ij})_{i,j}$ realizing $\overline{\Delta}_2(8)$ (left) as a polytopal fan with rays in standard positions ($P_2(1,2,3,4,5,1,2,3,4,5)$), and $\overline{\Delta}_2(10)$ (right) with rays in $P_2(\mathbf{t})$, with $t_i = 2^{(i-1)^2}$.

For $k \geq 4$ our results 4.6, 4.7 and 4.8 completely describe what choices of parameters realize the associahedron as a complete fan: any choice if $n \leq 2k+2$, no choice if $n \geq 2k+4$, and the choices specified in Theorem 4.7 if $n = 2k+3$.

For $k = 3$ our results are a bit less complete; for $n \in \{10, 11\}$ we cannot characterize what particular choices realize the fan. But we do know by Theorem 4.7 that there are choices that do not realize it, and we have verified that the choice $(0, 1, 31, 32, 42, 67, 100)$ at both sides realizes the fan for $n = 11$ and, hence, also for $n = 10$ (deleting any of the points).

Part 2

p-adic symplectic geometry of integrable systems and Weierstrass-Williamson theory

CHAPTER 5

Preliminaries

In a recent lecture [82] Lurie commented that *"roughly speaking p-adic geometry, or rigid analytic geometry, is a version of the theory of complex manifolds where instead of using complex numbers you use something like p-adic numbers"*. In this part of my thesis, also published as a paper in [27], we attempt to start studying this direction for the closely related class of symplectic manifolds: we will concentrate on the *local linear theory* of integrable systems with p-adic coefficients, on p-adic analytic symplectic manifolds.[1]

More concretely, for any prime number p, the p-adic numbers $\mathbb{Q}_p$ form an extension field of the rational numbers $\mathbb{Q}$ which plays a prominent role in various parts of geometry, as seen for instance in the aforementioned recent lecture by Lurie and Scholze-Weinstein's lectures [118].

Here we take a first step in introducing p-adic methods in symplectic geometry of integrable systems. We focus on p-adic matrix theory and p-adic linear symplectic geometry and applying them to fully describe the local linear theory of p-adic integrable systems $F = (f_1, f_2) : (M, \omega) \to (\mathbb{Q}_p)^2$ on p-adic analytic symplectic 4-manifolds (M, ω), as part of a general approach to this new field proposed ten years ago by Pelayo, Voevodsky and Warren [98, Section 7]. Our techniques are primarily algebraic and rely on the theory of p-adic extension fields.

We have two closely related goals. One of the goals is to fully describe the local symplectic geometry of p-adic integrable systems on symplectic 4-manifolds, that is, we explicitly classify their local models and give a concrete list of their formulas. In order to this, we first need to extend the seminal theory of Weierstrass [136] and Williamson [142] concerning the diagonalization of real symmetric matrices by means of symplectic matrices, to p-adic 4-by-4 matrices, which is our second goal.

More concretely, by the work of Weierstrass [136] any real symmetric positive definite matrix is diagonalizable by a symplectic matrix. This was generalized in an influential paper [142] by Williamson from 1936, where he shows that any symmetric matrix is reducible to a normal form by a symplectic matrix, and gives a classification of all matrix normal forms. In Williamson's approach, matrix reductions take place in the base field. In this paper we recover the real Wierstrass-Williamson classification with a genuinely different strategy: *we lift the problem to suitable extension fields* where the solution is simpler.

It is well known [119] that all Galois extensions over $\mathbb{Q}_p$ have a solvable Galois group, that is, the roots of all polynomials with p-adic coefficients can be expressed algebraically by extraction of successive radicals. However, since the general equation of degree 5 is not solvable by radicals, there are no formulas, from degree 5

[1]The relation between complex and symplectic structures on manifolds already appears implicitly in the pioneering work of Kodaira [79] and in a well known paper by Thurston [130], as well as in many other contributions including for instance [6, 35, 44, 45, 51].

onwards, to express the roots in terms of successive radicals. In other words, all our results are generalizable to any dimension with the same method we use here, but without explicit formulas for the local normal forms on symplectic manifolds of dimension 10 or higher. These forms will be explicit for dimensions 6 and 8, though we do not carry this out because it includes hundreds and even thousands of possibilities for the local models even in dimension 6.

All of the above results concerning matrices are stated in Section 5.3.2 ("Main results concerning matrices: Theorems 5.28–5.37") of the paper. These results can be used as stepping stones to derive a complete classification of the local linear models of p-adic analytic integrable systems in dimension 4, this being the second main goal of this part of the thesis.

As an application of our results and the Hardy-Ramanujan formula [**60**] (obtained also by Uspensky [**131**]) in number theory, we confirm that the number of p-adic $(2n)$-by-$(2n)$ matrix normal forms grows at least with $\mathrm{e}^{\pi\sqrt{2n/3}}/4\sqrt{3}n$, which in particular implies that the number of local linear normal forms of p-adic integrable systems on $2n$-dimensional symplectic manifolds at a rank 0 critical point grows in the same way. This is in strong contrast with the real case, where the number of normal forms of integrable systems at a rank 0 critical point is quadratic in the dimension.

5.1. The p-adic numbers

5.1.1. Definition of $\mathbb{Q}_p$. The field of real numbers $\mathbb{R}$ is defined as a completion of $\mathbb{Q}$ with respect to the normal absolute value on $\mathbb{Q}$. Analogously, the field of p-adic numbers $\mathbb{Q}_p$ can be defined as a completion of $\mathbb{Q}$ with respect to a non-archimedean absolute value. *Throughout this section we fix a prime number $p \in \mathbb{Z}$.*

Following [**55**, Definitions 2.1.2 and 2.1.4], the *p-adic valuation* on $\mathbb{Z}$ is the function

$$\mathrm{ord}_p : \mathbb{Z} \setminus \{0\} \to \mathbb{Z}$$

defined as follows: for each integer $n \in \mathbb{Z}$, $n \neq 0$, let $\mathrm{ord}_p(n)$ be the unique positive integer satisfying

$$n = p^{\mathrm{ord}_p(n)} n', \quad \text{with } p \nmid n'.$$

We extend ord_p to the field of rational numbers as follows: if $x = a/b \in \mathbb{Q} \setminus \{0\}$, then

$$\mathrm{ord}_p(x) = \mathrm{ord}_p(a) - \mathrm{ord}_p(b).$$

Also, for any $x \in \mathbb{Q}$, we define the *p-adic absolute value* of x by

$$|x|_p = p^{-\mathrm{ord}_p(x)}$$

if $x \neq 0$, and we set $|0|_p = 0$.

One can check that $|\cdot|_p$ is a non-archimedean absolute value:

- $|x|_p > 0$ for all $x \neq 0$,
- $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$,
- $|xy|_p = |x|_p\, |y|_p$ for all $x, y \in \mathbb{Q}$.

THEOREM 5.1 ([**55**, Theorem 3.2.13]). *There exists a field $\mathbb{Q}_p$ with a non-archimedean absolute value $|\cdot|_p$, such that the following statements hold.*

(1) *There exists an inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, and the absolute value induced by $|\cdot|_p$ on $\mathbb{Q}$ via this inclusion is the p-adic absolute value.*

(2) *The image of* $\mathbb{Q}$ *under this inclusion is dense in* $\mathbb{Q}_p$ *with respect to the absolute value* $|\cdot|_p$.
(3) $\mathbb{Q}_p$ *is complete with respect to the absolute value* $|\cdot|_p$.

The field $\mathbb{Q}_p$ *satisfying (1), (2) and (3) is unique up to isomorphism of fields preserving the absolute values.*

Following [55, Definition 3.3.3], the ring of p-adic integers $\mathbb{Z}_p$ is defined by:

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

PROPOSITION 5.2 ([55, Proposition 3.3.4]). *For any* $x \in \mathbb{Z}_p$, *there exists a Cauchy sequence* α_n *converging to* x, *of the following type:*

- $\alpha_n \in \mathbb{Z}$ *satisfies* $0 \leq \alpha_n \leq p^n - 1$;
- *for every* n *we have* $\alpha_n \equiv \alpha_{n-1} \mod p^{n-1}$.

The sequence (α_n) *with these properties is unique.*

Proposition 5.2 implies that any p-adic number a can be written uniquely as $a = \sum_{n=n_0}^{\infty} a_n p^n$ where $0 \leq a_n \leq p-1$ and $a_{n_0} > 0$, which is called *p-adic expansion of a*. We have that the absolute value defined in Theorem 5.1, $|a|_p$, coincides with p^{-n_0}. This motivates to define $\operatorname{ord}_p(a) := n_0$ and call it *order* of a.

We also need some properties about squares in $\mathbb{Q}_p$. We start with a result known as Hensel's lifting:

THEOREM 5.3 (Hensel's lifting, [55, Theorem 3.4.1 and Problem 112]). *Let* f *be a polynomial in* $\mathbb{Z}_p[x]$. *Let* α_1 *be a p-adic integer,* $r = \operatorname{ord}(f(\alpha_1))$ *and* $s = \operatorname{ord}(f'(\alpha_1))$. *If* $r > 2s$, *there exists* $\alpha \in \mathbb{Z}_p$ *such that* $\operatorname{ord}(\alpha - \alpha_1) \geq r - s$ *and* $f(\alpha) = 0$.

A consequence which is useful for us is the following:

COROLLARY 5.4. *For* $a, b \in \mathbb{Z}_2$, *such that* $2 \nmid a, b$, $a \equiv b \mod 2^n$ *or* $a \equiv -b \mod 2^n$ *if and only if* $a^2 \equiv b^2 \mod 2^{n+1}$.

PROOF. If $a \equiv \pm b \mod 2^n$, $a = \pm b + 2^n t$ for some $t \in \mathbb{Z}_2$, and

$$a^2 = b^2 \pm 2^{n+1} bt + 2^{2n} t^2 \equiv b^2 \mod 2^{n+1}.$$

Suppose now that $a^2 \equiv b^2 \mod 2^{n+1}$. We apply Hensel's lifting to $f(x) = x^2 - a^2$ and $\alpha_1 = b$. We have

$$r = \operatorname{ord}(b^2 - a^2) \geq n + 1$$

and

$$s = \operatorname{ord}(2b) = 1,$$

so there is α with $\operatorname{ord}(\alpha - b) \geq n$ and $\alpha^2 - a^2 = 0$. This implies $\alpha = \pm a$, so $\operatorname{ord}(\pm a - b) \geq n$, as we wanted. $\square$

Another consequence is the characterization of squares in $\mathbb{Q}_p$:

COROLLARY 5.5. (1) *If* $p \neq 2$, $a \in \mathbb{Q}_p$ *is a square if and only if* $\operatorname{ord}(a)$ *is even and the digit of* x *at the position* $\operatorname{ord}(a)$ *is a square modulo* p.
(2) *If* $p = 2$, $a \in \mathbb{Q}_p$ *is a square if and only if* $\operatorname{ord}(a)$ *is even and* a *ends in* 001 *(that is,* $a/2^{\operatorname{ord}(a)} \equiv 1 \mod 8$*).*

PROOF. (1) For the implication to the right: if $a = b^2$, $\operatorname{ord}(a) = 2\operatorname{ord}(b)$, and the leading digit of a is the square of that of b modulo p.

For the implication to the left: If $\operatorname{ord}(a)$ is even and the leading digit is square, let $a_0 = a/p^{\operatorname{ord}(a)}$. Let $c \in \mathbb{Z}$ such that $c^2 \equiv a_0 \mod p$. We apply Hensel's lifting to $x^2 - a_0$ with $\alpha = c$. We have $r = \operatorname{ord}(c^2 - a_0) > 0$, because the leading digits of c^2 and a_0 coincide, and $s = \operatorname{ord}(2c) = 0$. Then, there is $b_0 \in \mathbb{Z}_p$ such that $b_0^2 = a_0$. Taking $b = b_0 p^{\operatorname{ord}(a)/2}$, we have $b^2 = a_0 p^{\operatorname{ord}(a)} = a$.

(2) If $a = b^2$, we have again $\operatorname{ord}(a) = 2\operatorname{ord}(b)$. Let $t = \operatorname{ord}(b)$. Then $b = 2^t(1 + 2c)$ for some $c \in \mathbb{Z}_2$ and

$$a = b^2 = 2^{2t}(1 + 4c + 4c^2) = 2^{2t} + 2^{2t+2}c(c + 1)$$

As c is integer, $c(c + 1)$ is even, and this ends in 001.

If $\operatorname{ord}(a)$ is even and a ends in 001, let $a_0 = a/2^{\operatorname{ord}(a)}$. We apply Hensel's lifting to $x^2 - a_0$ with $\alpha = 1$. We have $r = \operatorname{ord}(1 - a_0) \geq 3$, $s = \operatorname{ord}(2) = 1$, and there is $b_0 \in \mathbb{Z}_p$ such that $b_0^2 = a_0$. Taking $b = b_0 2^{\operatorname{ord}(a)/2}$, we have $b^2 = a_0 2^{\operatorname{ord}(a)} = a$. □

The topology of the p-adic field is very different from the reals, despite both being completions of the rationals with different metrics.

THEOREM 5.6 ([55, Corollaries 3.3.6 and 3.3.7]). *The following statements hold.*

- *The p-adic metric on $\mathbb{Q}_p$ given by $d_p(a, b) = |a - b|$ satisfies the inequality $d_p(a, c) \leq \max\{d_p(a, b), d_p(b, c)\}$. This makes $\mathbb{Q}_p$ an* ultrametric space.
- *$\mathbb{Q}_p$ is totally disconnected, that is, all sets with more than one element are disconnected.*
- *The balls in the ultrametric space $\mathbb{Q}_p$ are given by*

$$\mathrm{B}_\epsilon(x_0) = \{x \in \mathbb{Q}_p \mid |x - x_0|_p \leq \epsilon\}.$$

  Replacing x_0 by any other point in the ball does not change the ball.
- *All balls are compact and open (in particular, $\mathbb{Z}_p$ is compact and open). $\mathbb{Q}_p$ is locally compact.*

COROLLARY 5.7. *An open subset of $\mathbb{Q}_p$ is a disjoint union of balls.*

PROOF. This is a consequence of the previous theorem and [117, Lemma 1.4]. □

The following result is a consequence of the absolute value being non-archimedean.

PROPOSITION 5.8 ([55, Corollary 4.1.2]). *A series in $\mathbb{Q}_p$ converges if and only if the sequence of its terms converges to zero.*

Now we define some concepts we need concerning the topology of $(\mathbb{Q}_p)^n$.

- For any $n \in \mathbb{N}$, we define the *p-adic norm* on $(\mathbb{Q}_p)^n$ by

$$\|v\| = \max_{1 \leq i \leq n} |v_i|.$$

- The *balls* in $(\mathbb{Q}_p)^n$ are defined with this norm:

$$\mathrm{B}_\epsilon(x_0) = \{x \in (\mathbb{Q}_p)^n \mid \|x - x_0\| \leq \epsilon\}.$$

  The resulting topology in $(\mathbb{Q}_p)^n$ is the n-th product of the topology in $\mathbb{Q}_p$.

- For any $n, m \in \mathbb{N}$, the *limit* of a function $f : U \to (\mathbb{Q}_p)^m$, where U is an open set in $(\mathbb{Q}_p)^n$, at a point $x_0 \in \overline{U}$, is equal to y_0 if and only if, for any $\epsilon > 0$, there is $\delta > 0$ such that $f(\mathrm{B}_\delta(x_0) \cap U) \subset \mathrm{B}_\epsilon(y_0)$; we denote this by $\lim_{x \to x_0} f(x) = y_0$.
- f is *continuous* at $x_0 \in U$ if $\lim_{x \to x_0} f(x) = f(x_0)$.
- f is *continuous* at U if it is continuous in each $x_0 \in U$ (this is equivalent to the standard definition of continuous function between two topological spaces).

Because of Theorem 5.6, continuous functions look very different from their real counterparts. For example, the functions $x \mapsto \operatorname{ord}(x)$ and $x \mapsto |x|$ are both continuous in $\mathbb{Q}_p \setminus \{0\}$, despite having discrete images.

p-adic differentiation is defined in analogy to the real case. Let $U \subset (\mathbb{Q}_p)^n$ be an open set. (Actually, by Corollary 5.7, we can take U to be a ball.) A function $f : U \to (\mathbb{Q}_p)^m$ is *differentiable at* $x \in U$ if there is a linear map $\mathrm{d}f(x) : (\mathbb{Q}_p)^n \to (\mathbb{Q}_p)^m$ such that

$$\lim_{v \to 0} \frac{\|f(x+v) - f(x) - \mathrm{d}f(x)(v)\|}{\|v\|} = 0.$$

It is easy to check that if $f : U \to (\mathbb{Q}_p)^m$ is differentiable at x, then the limit

$$\frac{\partial f}{\partial x_i}(x) := \lim_{t \to 0} \frac{f(x + t\mathrm{e}_i) - f(x)}{t}$$

exists and $\mathrm{d}f(x)(v) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) v_i$. The derivatives of elementary functions give the same result in the real and p-adic cases. For example, $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ and $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{x} = \frac{1}{2\sqrt{x}}$. The easiest way of seeing this is just taking the limits:

$$\lim_{t \to 0} \frac{(x+t)^n - x^n}{t} = \lim_{t \to 0} (n x^{n-1} + \binom{n}{2} x^{n-2} t + \ldots) = n x^{n-1};$$

$$\lim_{t \to 0} \frac{\sqrt{x+t} - \sqrt{x}}{t} = \lim_{t \to 0} \frac{x + t - x}{t(\sqrt{x+t} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

The previous results convey, at a formal level, that there is a high degree of similarity between the real and p-adic cases. However, upon closer analysis, one realizes that this is not necessarily the case. Indeed, consider the function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ given by $f(x) = \sum_{n=\operatorname{ord}(x)}^{\infty} a_n p^{2n}$ where $x = \sum_{n=\operatorname{ord}(x)}^{\infty} a_n p^n$ is the p-adic expansion of x. We can check that $|f(x+t) - f(x)| = p^{-\operatorname{ord}(f(x+t) - f(x))} = p^{-2\operatorname{ord}(t)} = |t|^2$ which implies that the function is continuous, and also that the function has zero derivative everywhere. In the real case, such a function would necessarily be constant. However, f is not only non-constant, but it is actually injective.

5.1.2. p-adic initial value problems. It is not a good idea, at least in principle, to use differentiable functions in general in the context of p-adic symplectic geometry: for any differential equation, the solution will not be unique, not even locally, because we could add an injective function with zero derivative to the solution and we will have another solution. The workaround is to restrict to analytic functions.

A *power series* in $(\mathbb{Q}_p)^n$ is given by $f(x) = \sum_{I \in \mathbb{N}^n} a_I (x - x_0)^I$ where x^I means $x_1^{i_1} \ldots x_n^{i_n}$ and a_I are coefficients in $\mathbb{Q}_p$. The following result is well-known and will be useful to us later.

PROPOSITION 5.9 ([55, Proposition 4.2.1]). *Consider a power series f in one variable in $\mathbb{Q}_p$. The convergence radius of the series is given by*

$$\rho = \frac{1}{\limsup \sqrt[i]{|a_i|}} = p^{-r} \quad \text{where the convergence order is} \quad r = -\liminf \frac{\operatorname{ord}(a_i)}{i}$$

Then:

- *If $\rho = 0$ (that is, $r = \infty$), then $f(x)$ converges only when $x = x_0$.*
- *If $\rho = \infty$ (that is, $r = -\infty$), then $f(x)$ converges for every $x \in \mathbb{Q}_p$.*
- *If $0 < \rho < \infty$ and $\lim_{i\to\infty} |a_i|\rho^i = 0$ (that is, $\lim_{i\to\infty} \operatorname{ord}(a_i) + ir = \infty$), then $f(x)$ converges if and only if $|x| \le \rho$ (that is, $\operatorname{ord}(x) \ge r$).*
- *If $0 < \rho < \infty$ and $|a_i|\rho^i$ does not tend to zero, then $f(x)$ converges if and only if $|x| < \rho$ (that is, $\operatorname{ord}(x) > r$).*

Let $U \subset (\mathbb{Q}_p)^n$ be an open set. A function $f : U \to \mathbb{Q}_p$ is *analytic* [117, page 38] if U can be expressed as $U = \bigcup_{i\in I} U_i$ where $U_i = x_i + p^{r_i}(\mathbb{Z}_p)^n$, for some $x_i \in (\mathbb{Q}_p)^n$ and $r_i \in \mathbb{Z}$, and there is a power series f_i converging in U_i such that $f(x) = f_i(x)$ for every $x \in U_i$.

PROPOSITION 5.10 (p-adic analytic initial value problem). *Let U, V be open subsets of $\mathbb{Q}_p$. An initial value problem, of the form*

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

where $f : U \times V \to \mathbb{Q}_p$ is analytic, $x_0 \in U$ and $y_0 \in V$, has an analytic solution in a neighborhood of x_0. The solution is locally unique among analytic functions, that is, any other solution coincides with it near x_0.

PROOF. We may assume without loss of generality (shrinking U if necessary) that f is given by a power series in U, centered at x_0. Take $y(x) = \sum_{i=0}^{\infty} a_i(x-x_0)^i$. The initial value implies that $a_0 = y_0$. The differential equations give

$$\sum_{i=0}^{\infty} a_i i(x - x_0)^{i-1} = f(x, \sum_{i=0}^{\infty} a_i(x - x_0)^i).$$

The degree k part at the left-hand side gives $(k + 1)a_{k+1}$ and the right-hand side gives a polynomial in $a_0, \ldots, a_k$. Hence a_{k+1} is uniquely determined from the previous ones. The resulting $y(x)$ is a solution in a neighborhood of the origin (the intersection of U with the convergence domain of the series), and it is locally unique because any other analytic solution would have the same power series around x_0 and so it coincides with y near this point. □

REMARK 5.11. Proposition 5.10 implies that we can now speak of "the solution" of an analytic differential equation, maybe not in the sense of "the unique solution", but in the sense of "the germ of every solution".

5.1.3. p-adic analytic manifolds. Now we review some concepts for p-adic differential geometry, which can be found in the literature (see for example [117]), starting with the concept of a p-adic manifold. The following definitions are straightforward extensions of the real case. Following [117, Sections 7-8], given a Hausdorff topological space M and an integer n, an *n-dimensional p-adic analytic atlas* is a set of functions $A = \{\phi : U_\phi \to V_\phi\}$, where $U_\phi \subset M$ and $V_\phi \subset (\mathbb{Q}_p)^n$ are open subsets, such that

- ϕ is a homeomorphism between U_ϕ and V_ϕ;
- for any $\phi, \psi \in A$, the change of charts $\psi \circ \phi^{-1} : \phi(U_\phi \cap U_\psi) \to \psi(U_\phi \cap U_\psi)$ is bi-analytic, i.e. it is analytic with analytic inverse.

Such an M together with such an atlas is called an *n-dimensional p-adic analytic manifold*. A *maximal atlas* for M has a chart for each open set. The integer n is called the *dimension* of M.

Now let M and N be p-adic analytic manifolds of dimensions m and n respectively, a map $F : M \to N$ is *analytic* if, for any $u \in M$, there are neighborhoods U_ϕ of u and U_ψ of $F(u)$ such that $\psi \circ F \circ \phi^{-1}$ is analytic (as a function from a subset of $(\mathbb{Q}_p)^m$ to a subset of $(\mathbb{Q}_p)^n$). F is *bi-analytic*, or an *isomorphism of p-adic analytic manifolds*, if it is bijective and F and F^{-1} are analytic.

THEOREM 5.12 ([117, Proposition 8.6]). *Let p be a prime number. For a p-adic analytic manifold M the following conditions are equivalent.*

(1) *M is paracompact (any open covering can be refined to a locally finite one).*
(2) *M is strictly paracompact (any open covering can be refined to one consisting in pairwise disjoint sets).*
(3) *M is an ultrametric space (its topology can be defined by a metric that satisfies the strict triangle inequality).*

COROLLARY 5.13. *Let p be a prime number. Any paracompact p-adic analytic manifold is isomorphic to a disjoint union of p-adic analytic balls. Hence, a compact p-adic analytic manifold is isomorphic to a finite disjoint union of p-adic analytic balls.*

PROOF. This is a consequence of Theorem 5.12 and Corollary 5.7. □

Corollary 5.13 implies that, when defining an atlas for a manifold, we can take the open sets in the atlas as disjoint, and the charts sending them to balls in $(\mathbb{Q}_p)^n$.

The last part of Corollary 5.13 was strengthened by Serre [119]: two finite disjoint unions of balls are isomorphic if and only if the corresponding numbers of balls differ by a multiple of $p-1$. That is, there are exactly $p-1$ compact p-adic manifolds, modulo isomorphism.

5.1.4. p-adic analytic functions, vector fields and forms. Throughout this section p is a fixed prime number. The content of this section is directly analogous to the real case, we include it here for completeness and also because it gives us the chance to discuss some peculiarities of the p-adic case which do not appear in the real case.

Given a p-adic analytic manifold M as defined in Section 5.1.3, a function $f : M \to \mathbb{Q}_p$ is (p-adic) *analytic* [117, page 49] if it is analytic as a map between manifolds, that is, for the charts ϕ of M, $f|_{U_\phi} \circ \phi^{-1}$ is analytic on $\phi(U_\phi)$. Let $\Omega^0(M)$ be the space of analytic maps $M \to \mathbb{Q}_p$.

A *tangent vector* to $q \in M$ is a linear map $v : \Omega^0(M) \to \mathbb{Q}_p$ such that $v(fg) = v(f)g(q)+f(q)v(g)$ for all $f, g \in \Omega^0(M)$. Let T_qM be the space of tangent vectors to q. If M is a p-adic analytic manifold, then $\mathrm{T}M$ has also the structure of an analytic manifold. An *analytic vector field* on M is an analytic map $M \to \mathrm{T}M$ that assigns a tangent vector to each point, or equivalently, a linear map $X : \Omega^0(M) \to \Omega^0(M)$ such that $X(fg) = X(f)g + fX(g)$ for all $f, g \in \Omega^0(M)$. Let $\mathfrak{X}(M)$ be the space of vector fields in M. An *analytic k-form* in M is a linear antisymmetric map $\alpha : \mathfrak{X}(M)^k \to \Omega^0(M)$. Let $\Omega^k(M)$ be the space of k-forms in M.

The *pullback* $F^*(f)$ of $f \in \Omega^0(N)$ by F and the *push-forward* $F_*(v)$ of a vector $v \in \mathrm{T}_q M$ are defined exactly as in the real case. Similarly, if F is bi-analytic, the *push-forward* of a vector field $X \in \mathfrak{X}(M)$ is defined as in the real case and denoted by $F_*(X) \in \mathfrak{X}(N)$. Similarly for the *pullback* of a form $\alpha \in \Omega^k(N)$, denoted as usual by $F^*(\alpha) \in \Omega^k(M)$.

In the linear case where M is an open subset of $(\mathbb{Q}_p)^n$, as $(\mathbb{Q}_p)^n$ is paracompact, analytic functions can be given by a family of power series, each one converging in an element U of a partition of M in open sets (actually, balls):

$$f(x_1, \ldots, x_n) = \sum_{I \in \mathbb{N}^n} a_I (x - x_0)^I$$

for any $x_0 \in U$ and some coefficients a_I. This allows us to define the vector field $\partial/\partial x_i \in \mathfrak{X}(M)$, for M open in $(\mathbb{Q}_p)^n$ is given by

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial f}{\partial x_i} = \sum_{I \in \mathbb{N}^n} a_I i_j (x - x_0)^{I_j}$$

for $f \in \Omega^0(M)$, where $I = (i_1, \ldots, i_n)$ and I_j is defined as $(i_1, \ldots, i_j - 1, \ldots, i_n)$. It follows that for any function $f \in \Omega^0(M)$ and $x_0 \in M$, there are functions $g_i \in \Omega^0(M)$ such that

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x_i - x_{0i}) g(x) \tag{24}$$

and $g(x_0) = \frac{\partial f}{\partial x}(x_0)$. Also, for any $X \in \mathfrak{X}(M)$ and $f \in \Omega^0(M)$, we have that $X(f) = \sum_{i=1}^{n} X(x_i) \frac{\partial f}{\partial x_i}$. This says that the vector fields $\partial/\partial x_i$ form a basis of the space $\mathfrak{X}(M)$, or locally, that the vectors $(\partial/\partial x_i)_q$ form a basis of the vector space $\mathrm{T}_q(M)$.

In the real case, the proof of the formula (24) is usually done by a method involving integrals (as for example in [54, Theorem 3.4]). One has to be careful if one wants to derive it for p-adic smooth functions (which we have not defined, since we are restricting to the analytic case, but the definition is the natural one) because it is more delicate to work with integrals. However, the formula can still be derived without integrals, by using induction on n. Supposing it is true for n, to prove it for $n+1$ we only need to find g_{n+1} such that

$$f(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n, x_{0,n+1}) + (x_{n+1} - x_{0,n+1}) g_{n+1}(x_1, \ldots, x_n, x_{n+1}).$$

The formula already gives the value of g_{n+1} for $x_{n+1} \neq x_{0,n+1}$. For $x_{n+1} = x_{0,n+1}$ we take g_{n+1} to be the partial derivative of f with respect to x_{n+1} in those points. By smoothness of f, this function is also continuous on those points, and in fact smooth.

Finally, if $M \subset (\mathbb{Q}_p)^n$ and $f \in \Omega^0(M)$, the differential form $\mathrm{d}f$ is defined as the map sending a vector field X to $X(f)$. In particular, if

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad f_i : M \to \mathbb{Q}_p,$$

we have $\mathrm{d}x_i(X) = X(x_i) = f_i$. Hence, the 1-forms $\mathrm{d}x_i$ are a basis of $\Omega^1(M)$, dual to the basis of $\mathfrak{X}(M)$.

All of the previous definitions generalize to the context of p-adic analytic manifolds. Indeed, let M be a p-adic analytic manifold and $U = U_\phi$ an open set of M. The vector field $\partial/\partial x_i \in \mathfrak{X}(U)$ is defined as the push-forward by ϕ of the corresponding vector field in $(\mathbb{Q}_p)^n$, and the 1-forms $\mathrm{d}x_i \in \Omega^1(U)$ are defined as the pullback of the corresponding forms in $(\mathbb{Q}_p)^n$. (With this definition, the vector fields $\partial/\partial x_i$ and the 1-forms $\mathrm{d}x_i$ are again bases of their respective spaces.)

The wedge operation $\wedge$ is defined as usual, and similarly for the differential operator

$$\mathrm{d}(f \cdot \mathrm{d}x_I) := \mathrm{d}f \wedge \mathrm{d}x_I,$$

extending linearly to all forms, where $\mathrm{d}x_I$ is shorthand for $\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$. A form is *closed* if its differential is 0. Also, given a k-form ω and a vector field X, $\imath(X)\omega$ is defined as usual.

We can see $\mathrm{d}f(m)$ as the vector whose coordinates are the partial derivatives of f:

$$\mathrm{d}f(m)_i = \frac{\partial f(m)}{\partial x_i}$$

In the same way, we define the Hessian of f, which we denote as $\mathrm{d}^2 f$, the matrix with the second derivatives of f as entries:

$$\mathrm{d}^2 f(m)_{ij} = \frac{\partial^2 f(m)}{\partial x_i \partial x_j}.$$

A *critical point* of $f : M \to \mathbb{Q}_p$ is a point $m \in M$ such that $\mathrm{d}f(m) = 0$.

5.2. Symplectic geometry and the Weierstrass-Williamson classification

5.2.1. Real and p-adic symplectic geometry.

DEFINITION 5.14. Let n be a positive integer.

- Given a field F, a *symplectic vector space* over F is a pair (V, ω) where V is a $2n$-dimensional vector space over F and $\omega : V \times V \to F$ is a non-degenerate bilinear map such that $\omega(v, v) = 0$ for all $v \in V$. We say that ω is a *linear symplectic form.* If the characteristic of F is not 2, the last condition is equivalent to ω being antisymmetric: $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$.
- A *real symplectic manifold* is a pair (M, ω) where M is a real manifold and ω is a closed non-degenerate 2-form on M. We say that ω is a *symplectic form.* At each point, ω gives a linear symplectic form.
- Let p be a prime number. A *p-adic analytic symplectic manifold* is a pair (M, ω) where M is a p-adic analytic manifold and ω is a closed non-degenerate 2-form on M. We say that ω is a *symplectic form.* At each point, ω gives a linear symplectic form. For example, if S is a p-adic analytic manifold, then the canonical symplectic form on $M = \mathrm{T}^*S$ is also analytic by construction.

Let us call Ω_0 the matrix of the standard symplectic form ω_0 on $\mathbb{R}^{2n}$, that is, a block-diagonal matrix of size $2n$ with all blocks equal to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

DEFINITION 5.15. Let n be a positive integer.

- Given a field F, a *linear symplectomorphism* between two symplectic vector spaces (V_1, ω_1) and (V_2, ω_2) over F is a linear isomorphism $\phi : V_1 \to V_2$ that preserves the symplectic form, that is, $\phi^*\omega_2 = \omega_1$. In this case we say that ω_1 and ω_2 are *linearly symplectomorphic*.
- A matrix $S \in \mathcal{M}_{2n}(F)$ is *symplectic* if $S^T\Omega_0 S = \Omega_0$, that is, it leaves invariant the standard symplectic form.
- Let (M_1, ω_1) and (M_2, ω_2) be real (resp. p-adic analytic) symplectic manifolds. Let $m \in M$. A *local linear symplectomorphism* $\phi : U_1 \to U_2$ *centered at* m is a diffeomorphism (resp. p-adic analytic diffeomorphism) between some open sets $U_1 \subset M_1$ and $U_2 \subset M_2$, such that $m \in U_1$, and which yields a linear symplectomorphism $\mathrm{T}_m\phi : \mathrm{T}_m M_1 \to \mathrm{T}_{\phi(m)} M_2$.
- Let (M, ω) be a real or p-adic symplectic manifold. By *linear symplectic coordinates* $(x_1, \xi_1, ..., x_n, \xi_n)$ *with the origin a point* $m \in M$ we mean coordinates given by a local linear symplectomorphism centered at m, that is, $\phi^*\omega_m = \omega_0$. In terms of matrices this last condition can be formulated as $S^T\Omega S = \Omega_0$, where S, Ω and Ω_0 are the matrices of ϕ, ω_m and ω_0.

REMARK 5.16. By definition, an automorphism of a symplectic space is a linear symplectomorphism if and only if its matrix is a symplectic matrix.

Given a symplectic manifold (M, ω) (real or p-adic) and a function $H : M \to \mathbb{Q}_p$, there is a unique vector field that satisfies

$$\imath(X_H)\omega = \mathrm{d}H. \tag{25}$$

As in the real case, X_H is called the *Hamiltonian vector field* associated to H. We recall the proof of this fact, which is the same in the real and the p-adic case. Let $q \in M$. We may assume that ω_q has the form

$$\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$$

in coordinates $(x_1, y_1, \ldots, x_n, y_n)$ near q, and $\mathrm{d}H(q) = \sum_{i=1}^n \left(\frac{\partial H}{\partial x_i}(q)\mathrm{d}x_i + \frac{\partial H}{\partial y_i}(q)\mathrm{d}y_i\right)$. Hence $X_H(q) = \sum_{i=1}^n \left(\frac{\partial H}{\partial y_i}(q)\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}(q)\frac{\partial}{\partial y_i}\right)$.

The *Poisson bracket* $\{\cdot, \cdot\}$ of two p-adic analytic functions $f, g : M \to \mathbb{Q}_p$ is defined by

$$\{f, g\} = \omega(X_f, X_g).$$

DEFINITION 5.17 (Pelayo-Voevodsky-Warren [98, Definition 7.1], with a slight change in item (2), for the p-adic case). Let p be a prime number and let (M, ω) be a symplectic manifold (real or p-adic). We say that a map

$$F := (f_1, \ldots, f_n) : (M, \omega) \to (\mathbb{Q}_p)^n$$

is an *integrable system* if two conditions hold:

(1) The functions $f_1, \ldots, f_n$ satisfy $\{f_i, f_j\} = 0$ for all $1 \le i \le j \le n$;
(2) The set where the n differential 1-forms $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are linearly independent is dense in M.

In [98, Definition 7.1], the definition used was slightly different because the second condition was that the set where the n differential 1-forms are linearly dependent has p-adic measure zero instead, as it is usually assumed in the real case. We now think that this condition is too restrictive, because sets with measure zero are less frequent in the p-adic case than in the real case, which explains the change in the definition.

5.2.2. The real Weierstrass-Williamson classification. For a symmetric matrix $M \in \mathcal{M}_{2n}(\mathbb{R})$ such that the eigenvalues of $\Omega_0^{-1}M$ are pairwise distinct, the Weierstrass-Williamson classification says that there is a symplectic matrix S such that $S^T MS = N$, where N is a block-diagonal matrix with each block equal to

$$\begin{pmatrix} r_i & 0 \\ 0 & r_i \end{pmatrix}, \begin{pmatrix} 0 & r_i \\ r_i & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & r_{i+1} & 0 & r_i \\ r_{i+1} & 0 & -r_i & 0 \\ 0 & -r_i & 0 & r_{i+1} \\ r_i & 0 & r_{i+1} & 0 \end{pmatrix},$$

for some $r_i \in \mathbb{R}, 1 \leq i \leq n$, which are called *elliptic block*, *hyperbolic block* and *focus-focus block*.

Quite often the Weierstrass-Williamson classification is stated only for positive-definite matrices: in fact the condition that all eigenvalues of $\Omega_0^{-1}M$ are pairwise distinct is implied by M being positive definite, and in this case only the elliptic block appears. (For applications to integrable systems the condition on $\Omega_0^{-1}M$ "translates" to the notion of a critical point being *non-degenerate*; we'll see this later.) In fact, this is the particular case of what is often called Williamson's theorem (that is, what we call the Weierstrass-Williamson classification) which is due to Weierstrass [136] (we learned this fact from the book by Hofer and Zehnder [62, Theorem 8]).

Of course, the case which Williamson treats is much more complicated and interesting: he is able to deal with the completely general situation in which the eigenvalues are not necessarily pairwise distinct. The problem with this case is that it is not feasible to write all the possibilities, for arbitrary dimension, in a compact form because the size of the blocks which are needed can increase without bound. However Williamson does provide a complete list of 4-by-4 matrix normal forms at the end of his paper [142]. These are the possibilities, expressed in a different way to align with the conventions of our paper:

$$\begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & s & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & s & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

$$\begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & r & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & r \\ 0 & 0 & -r & 0 \\ 0 & -r & a & 0 \\ r & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & s & 0 & r \\ s & 0 & -r & 0 \\ 0 & -r & 0 & s \\ r & 0 & s & 0 \end{pmatrix},$$

where $r, s \in \mathbb{R}$ and $a, b \in \{1, -1\}$. Since in the present paper we only provide a classification of p-adic 4-by-4 matrices (also of p-adic 2-by-2 matrices, but this case is simpler), it is precisely the list above which is most relevant to us.

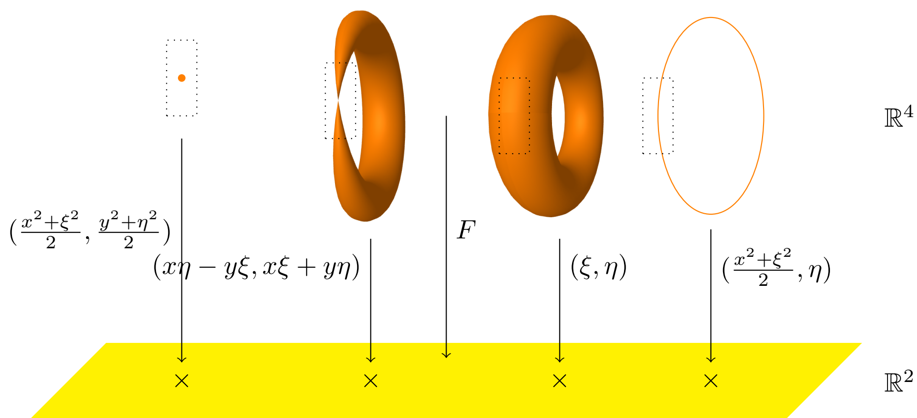


FIGURE 1. Normal forms of regular and critical points of elliptic-elliptic, focus-focus and elliptic-regular type of an integrable system $F : \mathbb{R}^4 \to \mathbb{R}^2$. Some of these can be normal forms of Theorem 5.19 (see Remark 5.24).

5.3. Our results

5.3.1. Main results about integrable systems: Theorems 5.19, 5.22 and 5.26. In order to state our classifications we need to define the following special sets of numbers. Recall that a *quadratic residue modulo* p is an integer which is congruent to a perfect square modulo p; if this does not hold, then the integer is called a *quadratic non-residue*[2] *modulo* p. Quadratic non-residues play a crucial role in our main theorems, for which we will need the following definition.

DEFINITION 5.18 (Non-residue sets and coefficient functions). Let p be a prime number. If $p \equiv 1 \mod 4$, let c_0 be the smallest quadratic non-residue modulo p. We define the *non-residue sets*

$$X_p = \begin{cases} \{1, c_0, p, c_0p, c_0^2p, c_0^3p, c_0p^2\} & \text{if } p \equiv 1 \mod 4; \\ \{1, -1, p, -p, p^2\} & \text{if } p \equiv 3 \mod 4; \\ \{1, -1, 2, -2, 3, -3, 6, -6, 12, -18, 24\} & \text{if } p = 2. \end{cases}$$

$$Y_p = \begin{cases} \{c_0, p, c_0p\} & \text{if } p \equiv 1 \mod 4; \\ \{-1, p, -p\} & \text{if } p \equiv 3 \mod 4; \\ \{-1, 2, -2, 3, -3, 6, -6\} & \text{if } p = 2. \end{cases}$$

We also define the *coefficient functions* $\mathcal{C}_i^k : Y_p \times (\mathbb{Q}_p)^4 \to \mathbb{Q}_p$ and $\mathcal{D}_i^k : Y_p \times (\mathbb{Q}_p)^4 \to \mathbb{Q}_p$, for $k \in \{1, 2\}$, $i \in \{0, 1, 2\}$, by

$$\mathcal{C}_0^1(c, t_1, t_2, a, b) = \frac{ac}{2(c - b^2)}, \; \mathcal{C}_1^1(c, t_1, t_2, a, b) = \frac{b}{b^2 - c}, \; \mathcal{C}_2^1(c, t_1, t_2, a, b) = \frac{1}{2a(c - b^2)},$$

$$\mathcal{C}_0^2(c, t_1, t_2, a, b) = \frac{abc}{2(b^2 - c)}, \; \mathcal{C}_1^2(c, t_1, t_2, a, b) = \frac{c}{c - b^2}, \; \mathcal{C}_2^2(c, t_1, t_2, a, b) = \frac{b}{2a(b^2 - c)},$$

[2]For any prime $p > 2$, the number of quadratic non-residues modulo p is $(p-1)/2$. For example, if $p = 17$, the quadratic non-residues modulo 17 are $3, 5, 6, 7, 10, 11, 12$ and 14.

$$\mathcal{D}_0^1(c,t_1,t_2,a,b) = -\frac{t_1+bt_2}{2a},\ \mathcal{D}_1^1(c,t_1,t_2,a,b) = -bt_1-ct_2,\ \mathcal{D}_2^1(c,t_1,t_2,a,b) = -\frac{ac(t_1+bt_2)}{2},$$

$$\mathcal{D}_0^2(c,t_1,t_2,a,b) = -\frac{bt_1+ct_2}{2a},\ \mathcal{D}_1^2(c,t_1,t_2,a,b) = -c(t_1+bt_2),\ \mathcal{D}_2^2(c,t_1,t_2,a,b) = -\frac{ac(bt_1+ct_2)}{2}.$$

The choice of c_0 as the least quadratic residue is not essential, in the sense that the normal forms that appear in the upcoming forms are "equivalent" (in the precise sense of Propositions 8.3 and 8.14).

Now we state our main results about integrable systems, which concern non-degenerate systems (for results about degenerate systems, which are more technical, see Sections 9.2.2 and 9.2.4). We refer to Figures 2 and 3 for a depiction of some of the cases covered by Theorem 5.19 and to Figure 1 for an illustration of the real case.

Let (M,ω) be a p-adic analytic symplectic 4-dimensional manifold. In the results below we use the following terminology. By *linear symplectic coordinates* (x,ξ,y,η) *with the origin at a point* $m\in M$ we mean coordinates given by a local linear symplectomorphism $\phi:((\mathbb{Q}_p)^4,\omega_0)\to(M,\omega)$, centered at $(0,0,0,0)$ (that is, such that $\phi(0,0,0,0)=m$ and $\phi^*\omega_m=\omega_0$), where ω_0 is the standard symplectic form on $(\mathbb{Q}_p)^4$. In terms of matrices this means that $S^T\Omega S=\Omega_0$, where S, Ω and Ω_0 are the matrices of ϕ, ω_m and the standard symplectic form ω_0 (see Definition 5.14 for details). For the notion of critical point of p-adic integrable system and its rank see Definition 9.10.

THEOREM 5.19 (p-adic integrable local linear models in dimension 4). *Let p be a prime number. Let $X_p, Y_p, \mathcal{C}_i^k, \mathcal{D}_i^k$ be the non-residue sets and coefficient functions in Definition 5.18. Let (M,ω) be a p-adic analytic symplectic manifold of dimension 4 and let $F:(M,\omega)\to(\mathbb{Q}_p)^2$ be a p-adic analytic integrable system. Let m be a non-degenerate critical point of F. Then there exist linear symplectic coordinates (x,ξ,y,η) with the origin at m and an invertible matrix $B\in\mathcal{M}_2(\mathbb{Q}_p)$ such that in these coordinates we have:*

$$B\circ(F-F(m)) = (g_1,g_2)+\mathcal{O}(3), \tag{26}$$

where the expression of (g_1,g_2) depends on the rank of $m\in\{0,1\}$. If m is a rank 0 critical point then one of the following situations occurs:

1. *There exist $c_1,c_2\in X_p$ such that $g_1(x,\xi,y,\eta)=x^2+c_1\xi^2, g_2(x,\xi,y,\eta)=y^2+c_2\eta^2$;*
2. *There exists $c\in Y_p$ such that $g_1(x,\xi,y,\eta)=x\eta+cy\xi, g_2(x,\xi,y,\eta)=x\xi+y\eta$;*
3. *There exist c,t_1 and t_2 corresponding to one row of Table 1 and $(a,b)\in\{(1,0),(a_1,b_1)\}$, where (a_1,b_1) is given in the row in question, such that*

$$g_k(x,\xi,y,\eta)=\sum_{i=0}^{2}\mathcal{C}_i^k(c,t_1,t_2,a,b)x^iy^{2-i}+\sum_{i=0}^{2}\mathcal{D}_i^k(c,t_1,t_2,a,b)\xi^i\eta^{2-i},$$

   for $k\in\{1,2\}$.

Otherwise, if m is a rank 1 point, then there exists $c\in X_p$ such that $g_1(x,\xi,y,\eta)=x^2+c\xi^2$ and $g_2(x,\xi,y,\eta)=\eta$.

Furthermore, if there are two sets of linear symplectic coordinates in which F has one of these forms, then the pair (g_1,g_2) corresponding to the first set of linear symplectic coordinates and the pair (g_1',g_2') corresponding to the second set of linear *symplectic coordinates are in the same case; if it is case (2) or (3), or a rank* 1 *point, they coincide, and in case (1) they coincide up to ordering.*

Remark 5.20. There are many more local linear models of p-adic integrable systems than real ones, so from a physical viewpoint they should be able to model physical or other phenomena beyond the applications with real coefficients, see for example [**59, 68, 76**] and the references therein. For applications in biology see [**3, 40**].

Remark 5.21. Identifying a Hessian with its quadratic form, the formula (26) would be written $\mathrm{d}^2F = (g_1, g_2)$. In the expression for g_1 and g_2 in any of the cases above, if we change the values of the parameters, the resulting functions still form an integrable system, but they are not a normal form. We can apply the theorem to these new functions, resulting in a new normal form (g_1', g_2') linearly symplectomorphic to (g_1, g_2), in the same or a different case.

Theorem 5.22 (Number of p-adic integrable local linear models, in dimension 4). *Let p be a prime number. Let $X_p, Y_p, \mathcal{C}_i^k, \mathcal{D}_i^k$ be the non-residue sets and coefficient functions in Definition 5.18. Let (M,ω) be a p-adic analytic symplectic 4-manifold. Then the following statements hold:*

(1) *If $p \equiv 1 \mod 4$, there are exactly 49 normal forms for a rank 0 non-degenerate critical point, and exactly 7 normal forms of a rank 1 non-degenerate critical point, of a p-adic analytic integrable system $F : (M,\omega) \to (\mathbb{Q}_p)^2$ up to local linear symplectomorphisms centered at the critical point;*
(2) *If $p \equiv 3 \mod 4$, there are exactly 32 normal forms for a rank 0 non-degenerate critical point, and exactly 5 normal forms of a rank 1 non-degenerate critical point, of a p-adic analytic integrable system $F : (M,\omega) \to (\mathbb{Q}_p)^2$ up to local linear symplectomorphisms centered at the critical point;*
(3) *If $p = 2$, there are exactly 211 normal forms for a rank 0 non-degenerate critical point, and exactly 11 normal forms of a rank 1 non-degenerate critical point, of a p-adic analytic integrable system $F : (M,\omega) \to (\mathbb{Q}_p)^2$ up to local linear symplectomorphisms centered at the critical point.*

In the three cases above, the normal forms for a rank 0 point are given by

$$\Big\{(x^2 + c_1\xi^2, y^2 + c_2\eta^2) : c_1, c_2 \in X_p\Big\} \cup \Big\{(x\eta + cy\xi, x\xi + y\eta) : c \in Y_p\Big\}$$

$$\cup\Big\{\Big(\sum_{i=0}^{2}\mathcal{C}_i^1(c,t_1,t_2,a,b)x^iy^{2-i} + \sum_{i=0}^{2}\mathcal{D}_i^1(c,t_1,t_2,a,b)\xi^i\eta^{2-i},$$

$$\sum_{i=0}^{2}\mathcal{C}_i^2(c,t_1,t_2,a,b)x^iy^{2-i} + \sum_{i=0}^{2}\mathcal{D}_i^2(c,t_1,t_2,a,b)\xi^i\eta^{2-i}\Big) :$$

$$(a,b) \in \Big\{(1,0),(a_1,b_1)\Big\}, c,t_1,t_2,a_1,b_1 \textit{ in one row of Table 1}\Big\}$$

and those for a rank 1 point are given by

$$\Big\{(x^2 + c\xi^2, \eta) : c \in X_p\Big\}.$$

$p \equiv 1 \mod 4$				
c	t_1	t_2	a_1	b_1
c_0	p	0	p	$1/p$
	0	1	p	0
	0	p	p	0
p	c_0	0	1	1
	0	1	c_0	0
	0	c_0	c_0	0
c_0p	c_0	0	1	1
	0	1	c_0	0
	0	c_0	c_0	0
$p \equiv 3 \mod 4$				
c	t_1	t_2	a_1	b_1
-1	p	0	b_0	a_0/b_0
	a_0	b_0	p	0
	pa_0	pb_0	p	0
p	-1	0	1	1
	0	1	-1	0
$-p$	-1	0	1	1
	0	1	-1	0

$p = 2 \wedge c = -1$			
t_1	t_2	a_1	b_1
2	0	1	2
3	0	1	1
6	0	1	1
1	1	3	0
3	3	3	0
1	2	2	0
2	4	2	0
-1	3	2	0
-2	6	2	0

$p = 2 \wedge c = 2$			
t_1	t_2	a_1	b_1
-1	0	1	2
3	0	1	1
-3	0	1	1
0	1	-1	0
0	3	-1	0
1	1	-1	0
3	3	-1	0
2	1	3	0
-2	-1	3	0
6	3	3	0
-6	-3	3	0

$p = 2 \wedge c = -2$			
t_1	t_2	a_1	b_1
-1	0	1	1
3	0	1	-2
-3	0	1	1
0	1	3	0
0	3	3	0
1	1	-1	0
-1	-1	-1	0
-2	1	-1	0
2	-1	-1	0

$p = 2 \wedge c = 3$			
t_1	t_2	a_1	b_1
-1	0	1	1
2	0	1	1
-2	0	1	3
0	1	2	0
0	2	2	0
1	1	-1	0
-1	-1	-1	0
3	1	-1	0
-3	-1	-1	0

$p = 2 \wedge c = -3$			
t_1	t_2	a_1	b_1
-1	0	2	$1/2$
2	0	2	$1/2$
-2	0	1	-6
0	1	-1	0
0	2	-1	0
1	2	2	0
-1	-2	2	0
2	4	2	0
-2	-4	2	0
-6	1	-1	0
-12	2	-1	0

$p = 2 \wedge c = 6$			
t_1	t_2	a_1	b_1
-1	0	1	1
3	0	1	6
-3	0	1	1
0	1	3	0
0	3	3	0
1	1	-1	0
-1	-1	-1	0
6	1	-1	0
-6	-1	-1	0

$p = 2 \wedge c = -6$			
t_1	t_2	a_1	b_1
-1	0	1	-6
3	0	1	1
-3	0	1	1
0	1	-1	0
0	3	-1	0
1	1	-1	0
3	3	-1	0
-6	1	3	0
6	-1	3	0
-18	3	3	0
18	-3	3	0

TABLE 1. Parameters for the normal form (3) of Theorem 5.19. In the table, for $p \equiv 1 \mod 4$, c_0 is the smallest quadratic non-residue modulo p. For $p \equiv 3 \mod 4$, a_0 and b_0 are such that $a_0^2 + b_0^2 \equiv -1 \mod p$. For $p = 2$ there are many more possible parameters, and they are separated by the value of c.

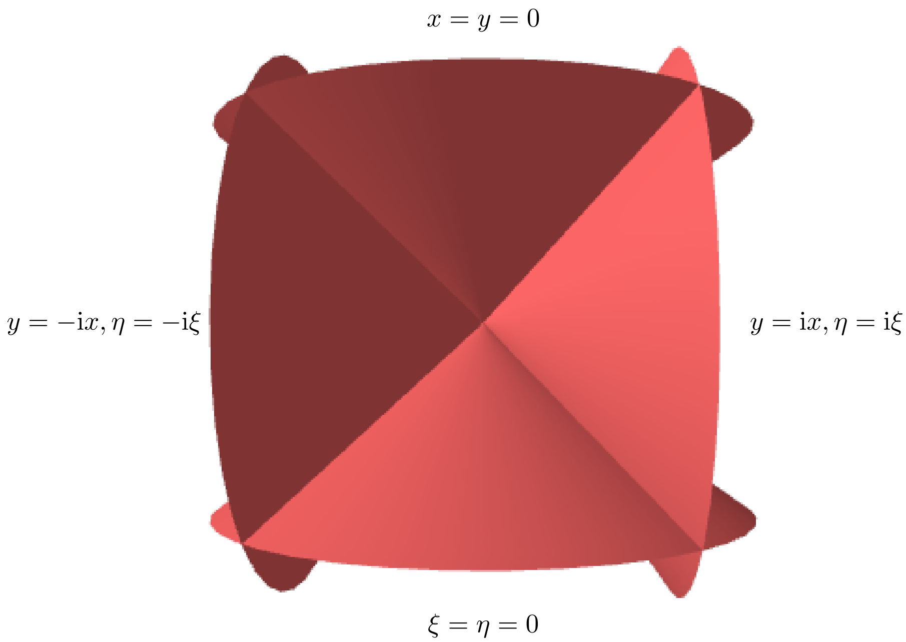


FIGURE 2. Symbolic representation of 2-dimensional fiber of focus-focus model if $p \equiv 1 \mod 4$, as a case of point (1) of Theorem 5.19, which coincides with the elliptic-elliptic model. The four "cones" are 2-dimensional planes in 4-dimensional space.

REMARK 5.23. In the real case, there are exactly 4 normal forms for a rank 0 non-degenerate critical point:

$$\Big\{(x^2+\xi^2, y^2+\eta^2), (x^2+\xi^2, y\eta), (x\xi, y\eta), (x\eta - y\xi, x\xi + y\eta)\Big\}$$

and exactly 2 normal forms for a rank 1 non-degenerate critical point:

$$\Big\{(x^2+\xi^2, \eta), (x\xi, \eta)\Big\}.$$

REMARK 5.24. In the p-adic category, the elliptic-elliptic and elliptic-regular points are normal forms of Theorem 5.19: the former is a rank 0 point in case (1) with $c_1 = c_2 = 1$, and the latter is a rank 1 point with $c = 1$. The focus-focus point may also appear as a normal form, in case (2) with $c = -1$, but only if $-1 \in Y_p$, which happens if $p \not\equiv 1 \mod 4$. Actually, if $p \equiv 1 \mod 4$, the normal form of a focus-focus point is elliptic-elliptic; hence, the focus-focus and elliptic-elliptic points are linearly symplectomorphic if and only if $p \equiv 1 \mod 4$.

DEFINITION 5.25. Let p be a prime number and let n be a positive integer. To each partition $P = (a_1, \ldots, a_k)$ of n we associate the function

$$f_{P,p}(x_1, \xi_1, \ldots, x_n, \xi_n) = \sum_{i=1}^{k} \left(\frac{x_{b_{i-1}+1}^2}{2} + \sum_{j=b_{i-1}+1}^{b_i - 1} \xi_j x_{j+1} + \frac{p\xi_{b_i}^2}{2} \right),$$

where $b_0 = 0$ and $b_i = \sum_{j=0}^{i} a_j$, for $i \in \{1, \ldots, k\}$.

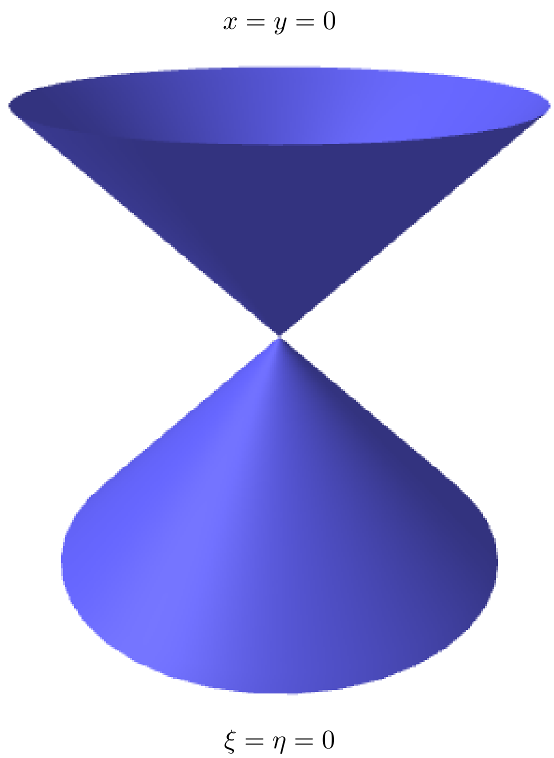


FIGURE 3. Symbolic representation of 2-dimensional fiber of focus-focus model if $p \not\equiv 1 \mod 4$, as a case of point (2) of Theorem 5.19, which coincides with the fiber in the real case. The two "cones" are actually 2-dimensional planes in 4-dimensional space that meet at a point.

THEOREM 5.26 (Number of p-adic integrable local linear models, arbitrary dimension). *Let n be a positive integer. Let p be a prime number. The number of local linear normal forms of p-adic analytic integrable systems on $2n$-dimensional p-adic analytic symplectic manifolds at a rank 0 non-degenerate critical point, up to local linear symplectomorphisms centered at the critical point, grows at least with*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

Explicitly, for any two partitions P and Q of n, any two p-adic analytic integrable systems on a $2n$-dimensional p-adic analytic manifold containing $f_{P,p}$ and $f_{Q,p}$, respectively, as a component of their corresponding local linear normal forms, are not equivalent by local linear symplectomorphisms centered at the origin, where $f_{P,p}$ is as given in Definition 5.25.

REMARK 5.27. These results indicate that a global theory of p-adic integrable systems, which will probably be based on gluing local models, will include a large number of phenomena which do not occur in the real case. We have computed explicit lower bounds of the number of normal forms in Table 2.

$2n$	$\mathbb{R}$	$\mathbb{Q}_2$	$\mathbb{Q}_3$	$\mathbb{Q}_5$	$\mathbb{Q}_7$
2	2	11	5	7	5
4	4	211	32	49	32
6	6	1883	123	234	129
8	9	21179	495	1054	525
10	12	161343	1595	4021	1787

TABLE 2. The number of families of normal forms of integrable systems on $\mathbb{R}^{2n}$ and $(\mathbb{Q}_p)^{2n}$ at a rank 0 critical point. Data is extracted from Table 3. For the p-adic case in dimension greater than 4, the numbers are only lower bounds. The actual number of forms might be even larger.

5.3.2. Main results concerning matrices: Theorems 5.28–5.37. In this section we state our main classification results concerning normal forms of p-adic matrices in dimensions 2 and 4: Theorems 5.28–5.37. We will use these results as stepping stones to prove the results concerning integrable systems (stated in Section 5.3.1), but they are also of independent interest and they can be read independently of all the material concerning p-adic analytic integrable systems and p-adic analytic functions.

The classification is completely different for $p = 2$ than $p \neq 2$; for the latter case, in turn, it depends on the class of p modulo 4.

THEOREM 5.28 (p-adic classification, 2-by-2 case). *Let p be a prime number. Let $M \in \mathcal{M}_2(\mathbb{Q}_p)$ be a symmetric matrix. Let X_p, Y_p be the non-residue sets in Definition 5.18. Then, there exists a symplectic matrix $S \in \mathcal{M}_2(\mathbb{Q}_p)$ and either $c \in X_p$ and $r \in \mathbb{Q}_p$, or $c = 0$ and $r \in Y_p \cup \{1\}$, such that*

$$S^T M S = \begin{pmatrix} r & 0 \\ 0 & cr \end{pmatrix}.$$

Furthermore, if two symplectic matrices S and S' reduce M to the normal form of the right hand side of the equality above, then the two normal forms are equal.

In Theorem 5.28 we note that we are not saying that the value of (c, r) is unique (which essentially is, but not quite), but that the canonical matrix obtained at the right-hand side is unique.

In the statement below, and also Theorem 5.34, we use the following terminology.

DEFINITION 5.29. Let n be a positive integer. Let p be a prime number. We say that two $(2n)$-by-$(2n)$ matrices M and M' with coefficients in $\mathbb{Q}_p$ are *equal up to multiplication by a symplectic matrix* if there exists a $(2n)$-by-$(2n)$ symplectic matrix S with coefficients in $\mathbb{Q}_p$ such that $S^T M S = M'$ (same definition works for arbitrary fields).

THEOREM 5.30 (Number of inequivalent p-adic 2-by-2 matrix normal forms). *Let p be a prime number. Let X_p, Y_p be the non-residue sets in definition 5.18. Then the following statements hold.*

(1) *If $p \equiv 1 \mod 4$, there are exactly 7 infinite families of normal forms of 2-by-2 p-adic matrices with one degree of freedom up to multiplication by*

a symplectic matrix:

$$\left\{\left\{r\begin{pmatrix}1 & 0\\ 0 & c\end{pmatrix} : r \in \mathbb{Q}_p\right\} : c \in X_p\right\},$$

and exactly 4 *isolated normal forms, which correspond to* $c = 0$*:*

$$\left\{\begin{pmatrix}r & 0\\ 0 & 0\end{pmatrix} : r \in Y_p \cup \{1\}\right\}.$$

(2) *If* $p \equiv 3 \mod 4$*, there are exactly* 5 *infinite families of normal forms of* 2*-by-*2 p*-adic matrices with one degree of freedom up to multiplication by a symplectic matrix, with the same formula as above, and exactly* 4 *isolated normal forms.*

(3) *If* $p = 2$*, there are exactly* 11 *infinite families of normal forms of* 2*-by-*2 p*-adic matrices with one degree of freedom up to multiplication by a symplectic matrix, also with the same formula, and exactly* 8 *isolated normal forms.*

This is in contrast with the real case, where there are exactly 2 *families, elliptic and hyperbolic, and* 2 *isolated normal forms. Here by "infinite family" we mean a family of normal forms of the form* $r_1M_1 + r_2M_2 + \ldots + r_kM_k$*, where* $r_i \in \mathbb{Q}_p$ *are parameters and* k *is the number of degrees of freedom.*

Hence, already in dimension 2, the p-adic situation is much richer than its real counterpart. The situation is even more surprising in dimension 4. This is the classification in the case where $\Omega_0^{-1}M$ has all eigenvalues distinct, where Ω_0 is the same matrix as before for dimension 4:

$$\Omega_0 = \begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}$$

THEOREM 5.31 (p-adic classification, 4-by-4, non-degenerate case). *Let* p *be a prime number. Let* Ω_0 *be the matrix of the standard symplectic form on* $(\mathbb{Q}_p)^4$*. Let* X_p, Y_p *be the non-residue sets in Definition 5.18. Let* $M \in \mathcal{M}_4(\mathbb{Q}_p)$ *be a symmetric matrix such that all the eigenvalues of* $\Omega_0^{-1}M$ *are distinct. Then there exists a symplectic matrix* $S \in \mathcal{M}_4(\mathbb{Q}_p)$ *and* $r, s \in \mathbb{Q}_p$ *such that one of the following three possibilities holds:*

(1) *There exist* $c_1, c_2 \in X_p$ *such that*

$$S^TMS = \begin{pmatrix}r & 0 & 0 & 0\\ 0 & c_1r & 0 & 0\\ 0 & 0 & s & 0\\ 0 & 0 & 0 & c_2s\end{pmatrix}.$$

(2) *There exists* $c \in Y_p$ *such that*

$$S^TMS = \begin{pmatrix}0 & s & 0 & r\\ s & 0 & cr & 0\\ 0 & cr & 0 & s\\ r & 0 & s & 0\end{pmatrix}.$$

(3) *There exist c, t_1 and t_2 corresponding to one row of the Table 1 such that $S^T MS$ is equal to the matrix*

$$\begin{pmatrix} \frac{ac(r-bs)}{c-b^2} & 0 & \frac{sc-rb}{c-b^2} & 0 \\ 0 & \frac{-r(t_1+bt_2)-s(bt_1+ct_2)}{a} & 0 & -r(bt_1+ct_2)-sc(t_1+bt_2) \\ \frac{sc-rb}{c-b^2} & 0 & \frac{r-bs}{a(c-b^2)} & 0 \\ 0 & -r(bt_1+ct_2)-sc(t_1+bt_2) & 0 & ac(-r(t_1+bt_2)-s(bt_1+ct_2)) \end{pmatrix}$$

where (a, b) are either $(1, 0)$ or (a_1, b_1) of the corresponding row.

Furthermore, if two matrices S and S' reduce M to one of the normal forms in the right-hand side of the three equalities above, then the two normal forms are in the same case; if it is case (2) or (3), they coincide, and in case (1) they coincide up to exchanging the 2 by 2 diagonal blocks.

See Figure 4 for a diagram of the classes in the statement of Theorem 5.31.

Definition 5.32 (Non-residue function). Let p be a prime number. If $p \equiv 1 \mod 4$, let c_0 be the smallest quadratic non-residue modulo p. We define the *non-residue function*:

$$h_p : Y_p \to \mathbb{Q}_p \text{ given by } \begin{cases} h_p(c_0) = p, h_p(p) = h_p(c_0 p) = c_0 & \text{if } p \equiv 1 \mod 4; \\ h_p(-1) = p, h_p(p) = h_p(-p) = -1 & \text{if } p \equiv 3 \mod 4; \\ h_p(-1) = h_p(-2) = h_p(3) = h_p(6) = -1, & \\ h_p(-3) = h_p(-6) = 2, h_p(2) = 3 & \text{if } p = 2. \end{cases}$$

For the cases where the eigenvalues of $\Omega_0^{-1} M$ are not pairwise distinct, we have:

Theorem 5.33 (p-adic classification, 4-by-4, degenerate case). *Let p be a prime number. Let Ω_0 be the matrix of the standard symplectic form on $(\mathbb{Q}_p)^4$. Let X_p, Y_p be the non-residue sets in Definition 5.18. Let $h_p : Y_p \to \mathbb{Q}_p$ be the non-residue function in Definition 5.32. Let $M \in \mathcal{M}_4(\mathbb{Q}_p)$ a symmetric matrix such that $\Omega_0^{-1} M$ has at least one multiple eigenvalue. Then there exists a symplectic matrix $S \in \mathcal{M}_4(\mathbb{Q}_p)$ such that one of the following three possibilities holds:*

(1) *There exist $r, s \in \mathbb{Q}_p$ and $c_1, c_2 \in X_p \cup \{0\}$ such that $S^T MS$ has the form in the case (1) of Theorem 5.31. Moreover, if $c_1 = 0$ then $r \in Y_p \cup \{1\}$, and if $c_2 = 0$ then $s \in Y_p \cup \{1\}$.*

(2) *There exists $r \in \mathbb{Q}_p$ such that*

$$S^T MS = \begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & r & 0 \end{pmatrix}$$

(3) *There exist $r \in \mathbb{Q}_p$, $c \in Y_p$ and $a \in \{1, h_p(c)\}$ such that*

$$S^T MS = \begin{pmatrix} a & 0 & 0 & r \\ 0 & 0 & cr & 0 \\ 0 & cr & a & 0 \\ r & 0 & 0 & 0 \end{pmatrix}.$$

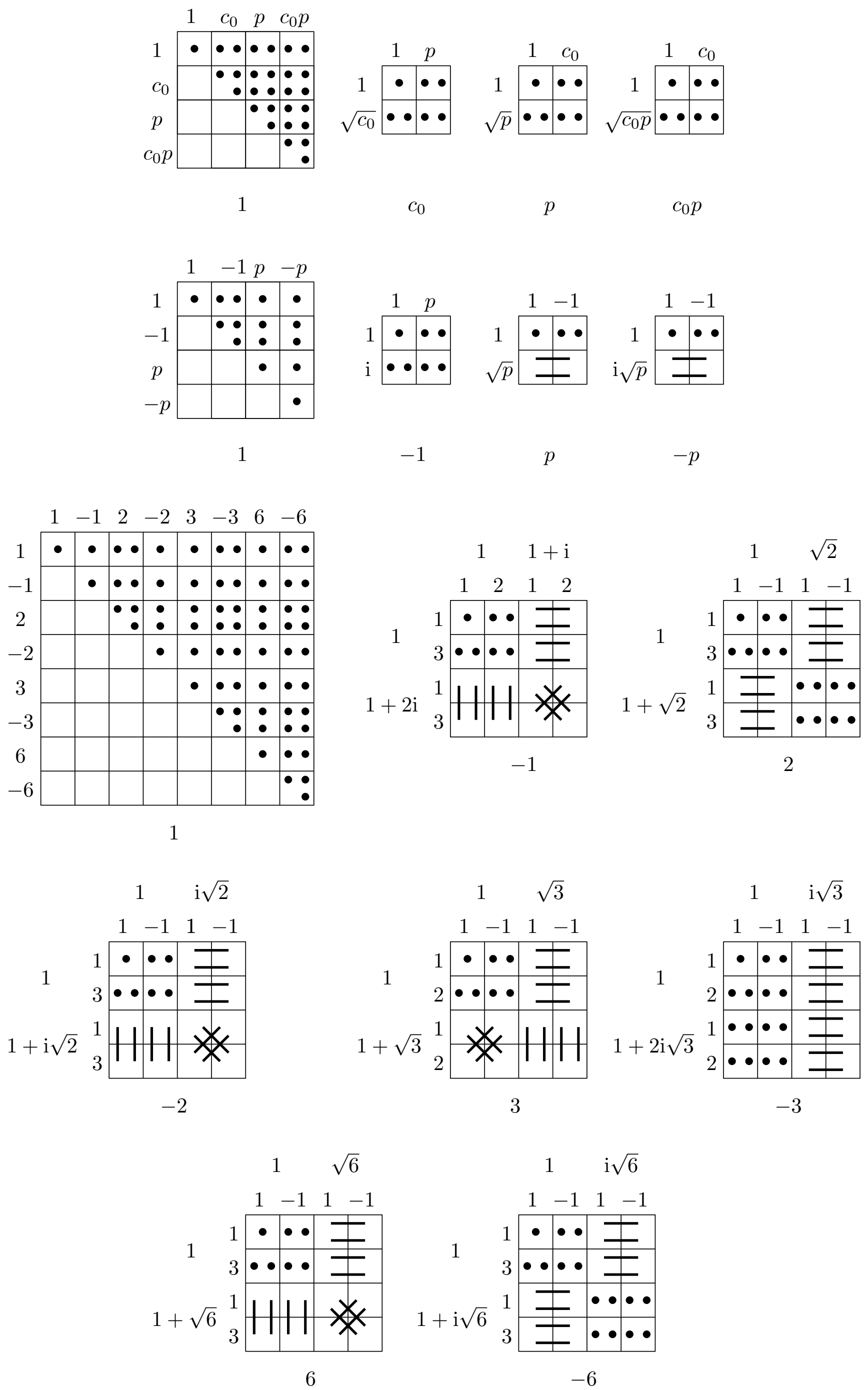


FIGURE 4. A diagram of the normal forms of Theorem 5.31 for $p \equiv 1 \mod 4$ (first row), $p \equiv 3 \mod 4$ (second row) and $p = 2$ (third to fifth row). Each point (if it is in a single cell) or line (if it is in two cells) represents a normal form. The numbers below the tables represent the first extension of $\mathbb{Q}_p$ (the one containing the squares of the eigenvalues of $\Omega_0^{-1}M$) and those in the rows and columns represent the second extension (for the eigenvalues themselves). In the first table in each block, there are two such extensions corresponding to the row and the column; in the other ones, there is one extension obtained multiplying the numbers in the row and column.

(4) *There exists* $c \in Y_p \cup \{1\}$ *such that*

$$S^T M S = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{pmatrix}.$$

Furthermore, if two matrices S *and* S' *reduce* M *to one of these normal forms on the right-hand side of the three equalities above, then the two normal forms are in the same case; if it is case (2), (3) or (4), they coincide completely, and in case (1) they coincide up to exchanging the* 2 *by* 2 *diagonal blocks.*

Our proof method is different from Williamson's method and in particular gives another proof (self-contained, while Williamson's is not, as his proof relies on some applications of other substantial works which he cites in his paper [142]) of the classical Weierstrass-Williamson classification in any dimension. We carry this out in Chapter 7.

Theorem 5.34 (Number of 4-by-4 p-adic matrix normal forms). *Let* p *be a prime number. Let* X_p, Y_p *be the non-residue sets in Definition 5.18. Let* $h_p : Y_p \to \mathbb{Q}_p$ *be the non-residue function in Definition 5.32.*

(1) *If* $p \equiv 1 \mod 4$*, there are exactly* 49 *infinite families of normal forms of p-adic* 4*-by-*4 *matrices with two degrees of freedom, exactly* 35 *infinite families with one degree of freedom, and exactly* 20 *isolated normal forms, up to multiplication by a symplectic matrix.*
(2) *If* $p \equiv 3 \mod 4$*, there are exactly* 32 *infinite families of normal forms of p-adic* 4*-by-*4 *matrices with two degrees of freedom, exactly* 27 *infinite families with one degree of freedom, and exactly* 20 *isolated normal forms, up to multiplication by a symplectic matrix.*
(3) *If* $p = 2$*, there are exactly* 211 *infinite families of normal forms of p-adic* 4*-by-*4 *matrices with two degrees of freedom, exactly* 103 *infinite families with one degree of freedom, and exactly* 72 *isolated normal forms, up to multiplication by a symplectic matrix.*

In the three cases above, the infinite families with two degrees of freedom are given by

$$\left\{\left\{\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & c_1 r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & c_2 s \end{pmatrix} : r, s \in \mathbb{Q}_p\right\} : c_1, c_2 \in X_p\right\} \cup \left\{\left\{\begin{pmatrix} 0 & s & 0 & r \\ s & 0 & cr & 0 \\ 0 & cr & 0 & s \\ r & 0 & s & 0 \end{pmatrix} : r, s \in \mathbb{Q}_p\right\} : c \in Y_p\right\}$$

$$\cup \left\{\left\{\begin{pmatrix} \frac{ac(r-bs)}{c-b^2} & 0 & \frac{sc-rb}{c-b^2} & 0 \\ 0 & \frac{-r(t_1+bt_2)-s(bt_1+ct_2)}{a} & 0 & -r(bt_1+ct_2)-sc(t_1+bt_2) \\ \frac{sc-rb}{c-b^2} & 0 & \frac{r-bs}{a(c-b^2)} & 0 \\ 0 & -r(bt_1+ct_2)-sc(t_1+bt_2) & 0 & ac(-r(t_1+bt_2)-s(bt_1+ct_2)) \end{pmatrix} :\right.\right.$$

$$\left.\left. r, s \in \mathbb{Q}_p\right\} : (a,b) \in \Big\{(1,0),(a_1,b_1)\Big\}, c, t_1, t_2, a_1, b_1 \text{ in one row of Table 1}\right\},$$

those with one degree of freedom are

$$\left\{\left\{\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & c_1 r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : r \in \mathbb{Q}_p\right\} : c_1 \in X_p, s \in Y_p \cup \{1\}\right\} \cup \left\{\left\{\begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & r & 0 \end{pmatrix} : r \in \mathbb{Q}_p\right\}\right\}$$

$$\cup \left\{\left\{\begin{pmatrix} a & 0 & 0 & r \\ 0 & 0 & cr & 0 \\ 0 & cr & a & 0 \\ r & 0 & 0 & 0 \end{pmatrix} : r \in \mathbb{Q}_p\right\} : c \in Y_p, a \in \{1, h_p(c)\}\right\},$$

and the isolated forms are

$$\left\{\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : r, s \in Y_p \cup \{1\}\right\} \cup \left\{\begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{pmatrix} : c \in Y_p \cup \{1\}\right\}.$$

This is in contrast with the real case, where there are exactly 4 *infinite families with two degrees of freedom, exactly* 7 *infinite families with one degree of freedom and exactly* 5 *isolated normal forms. Here by "infinite family" we mean a family of normal forms of the form* $r_1 M_1 + r_2 M_2 + \ldots + r_k M_k$*, where* r_i *are parameters in* $\mathbb{Q}_p$ *and* k *is the number of degrees of freedom, and by "isolated" we mean a form that is not part of any family.*

Remark 5.35. Note that Theorems 5.30 and 5.34 refer to infinite families of matrices, but Theorem 5.22 does not mention families. This is because the equivalence between integrable systems we consider allows for a matrix B to appear, as in equation (26).

We can give a lower bound for the number of blocks of size $2n$ that can appear in the normal forms of the matrices. In the real case, taking into account only the "non-degenerate case" where the eigenvalues of $\Omega_0^{-1} M$ are pairwise distinct, there are two infinite families of blocks with size two (each one with one degree of freedom), one family with size four (with two degrees of freedom) and no blocks with size greater than four: this is due to the fact that all irreducible polynomials in $\mathbb{R}$ have degree at most two, and has as a consequence that the number of families of normal forms of size $2n$ is quadratic in n. For the p-adic case, however, one has:

Definition 5.36. Let p be a prime number and let n be a positive integer. For each partition $P = (a_1, \ldots, a_k)$ of n, we define $M(P, p) \in \mathcal{M}_{2n}(\mathbb{Q}_p)$ as the block diagonal matrix whose blocks have sizes $(2a_1, \ldots, 2a_k)$ and each block has the form

$$\begin{pmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ & & & & 1 & & & \\ & & & 1 & & \ddots & & \\ & & & & \ddots & & & 1 \\ & & & & & & 1 & \\ & & & & & & & p \end{pmatrix}.$$

$2n$	$\mathbb{R}$		$\mathbb{Q}_2$		$\mathbb{Q}_3$		$\mathbb{Q}_5$		$\mathbb{Q}_7$	
	blocks	forms	blocks	forms	blocks	forms	blocks	forms	blocks	forms
2	2	2	11	11	5	5	7	7	5	5
4	1	4	145	211	17	32	21	49	17	32
6	0	6	2	1883	3	123	3	234	9	129
8	0	9	1	21179	2	495	4	1054	2	525
10	0	12	2	161343	3	1595	3	4021	3	1787
12	0	16	1	1374427	2	5111	4	14493	6	5874
14	0	20	2	9232171	3	14491	3	47462	3	17586
16	0	25	1	65570626	2	40244	4	148087	2	50614
18	0	30	2	397086458	3	103484	3	433330	9	137311
20	0	36	1	2469098766	2	259712	4	1217761	2	359463

TABLE 3. The number of families of real and p-adic normal forms of matrices of order $2n$. The numbers in the real case are exact, and come from Remark 8.18. The numbers for the p-adic case, for dimensions 2 and 4, come from Theorems 5.30 and 5.34. For the rest of dimensions, the numbers of blocks are only lower bounds coming from Lemma 8.17, and the numbers of forms are obtained summing over the partitions of n. The actual number of forms might be even larger.

THEOREM 5.37 (Number of $(2n)$-by-$(2n)$ p-adic matrix normal forms). *Let p be a prime number. Let n be a positive integer. The number of p-adic families of non-degenerate normal forms of $(2n)$-by-$(2n)$ matrices up to multiplication by a symplectic matrix, each family being of the form $r_1M_1 + \ldots + r_nM_n$, where r_i are parameters in $\mathbb{Q}_p$, grows at least with*

$$\frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

Explicitly, if P and P' are distinct partitions of n then the matrices $M(P,p)$ and $M(P',p)$ in Definition 5.36 are not equivalent by multiplication by a symplectic matrix.

We will later prove this result using the Hardy-Ramanujan formula [60]. We have computed explicit lower bounds for the number of families in Table 3.

The field of p-adic geometry is extensive, see [117, 118] and the references therein. p-adic geometry is also fundamental in mathematical physics and the theory of integrable systems, see for example [26, 37, 38, 39, 59, 98, 133]. For an introduction to different aspects of symplectic geometry, including its relations to mechanics and Poisson geometry, we refer to the survey articles [48, 97, 137, 138] and the books [5, 15, 62, 85, 86, 93]. The Weierstrass-Williamson theory of matrices has crucial applications in many areas including the theory of quantum states in quantum physics [32, 123, 135], hence this paper provides a new tool to further explore p-adic analogues of these applications in symplectic geometry and beyond.

CHAPTER 6

Results for general fields

We start by studying the classification problem of normal forms of matrices in an algebraically closed field. This case is an essential ingredient of our strategy to obtain general classifications in the real and p-adic cases later on. As we will see, the problem in this case reduces to an equality of eigenvalues, and to an equality of normal forms if there are multiple eigenvalues.

DEFINITION 6.1. Let n be a positive integer. Let F be a field with multiplicative identity element 1. We define Ω_0 as the $(2n)$-by-$(2n)$ matrix whose blocks are all

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that is,

$$\Omega_0 = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}.$$

(It is also common to take Ω_0 to have the blocks in the "other" diagonal.) Ω_0 is called the *standard symplectic form on* F^{2n}.

PROPOSITION 6.2. *Let n be a positive integer. Let F be an algebraically closed field. Let Ω_0 be the matrix of the standard symplectic form on F^{2n}. Let $M_1, M_2 \in \mathcal{M}_{2n}(F)$ be symmetric matrices,*

$$A_i = \Omega_0^{-1} M_i,$$

J_i the Jordan form of A_i, and let Ψ_i be such that $\Psi_i^{-1} A_i \Psi_i = J_i$, for $i \in \{1, 2\}$. Then there exists a symplectic matrix S such that $S^T M_1 S = M_2$ if and only if $J_1 = J_2$. Moreover, in that case S must have the form $\Psi_1 D \Psi_2^{-1}$, where D is a matrix that commutes with $J_1 = J_2$ and satisfies $D^T \Psi_1^T \Omega_0 \Psi_1 D = \Psi_2^T \Omega_0 \Psi_2$.

PROOF. Let $A_i = \Omega_0^{-1} M_i$. Suppose first that such a S exists. Then,

$$S^{-1} A_1 S = S^{-1} \Omega_0^{-1} M_1 S = S^{-1} \Omega_0^{-1} (S^T)^{-1} S^T M_1 S = \Omega_0^{-1} M_2 = A_2$$

hence A_1 and A_2 are equivalent, and $J_1 = J_2$.

Let $D = \Psi_1^{-1} S \Psi_2$. We have that $S^T \Omega_0 S = \Omega_0$, which implies

$$D^T \Psi_1^T \Omega_0 \Psi_1 D = \Psi_2^T \Omega_0 \Psi_2.$$

Also,

$$J_1 D = \Psi_1^{-1} A_1 \Psi_1 D = \Psi_1^{-1} A_1 S \Psi_2 = \Psi^{-1} S A_2 \Psi_2 = D \Psi_2^{-1} A_2 \Psi_2 = D J_2 = D J_1.$$

Now suppose that $J_1 = J_2$, let D be a matrix which satisfies the conditions and let $S = \Psi_1 D \Psi_2^{-1}$. The condition $D^T \Psi_1^T \Omega_0 \Psi_1 D = \Psi_2^T \Omega_0 \Psi_2$ implies that $S^T \Omega_0 S = \Omega_0$, that is, S is symplectic. Moreover,

$$S^{-1} A_1 S = \Psi_2 D^{-1} \Psi_1^{-1} A_1 \Psi_1 D \Psi_2^{-1} = \Psi_2 D^{-1} J_1 D \Psi_2^{-1} = \Psi_2 J_1 \Psi_2^{-1} = A_2$$

which implies

$$S^T M_1 S = S^T \Omega_0 A_1 S = S^T \Omega_0 S S^{-1} A_1 S = \Omega_0 A_2 = M_2$$

as we wanted. □

In the case where the eigenvalues are pairwise distinct, the proposition above can be simplified:

LEMMA 6.3. *Let n be a positive integer. Let F be an algebraically closed field with characteristic different from 2. Let Ω_0 be the matrix of the standard symplectic form on F^{2n} and let $M \in \mathcal{M}_{2n}(F)$ be a symmetric matrix such that the eigenvalues of $\Omega_0^{-1} M$ are pairwise distinct. Then there exists a basis $\{u_1, v_1, \ldots, u_n, v_n\}$ of F^{2n} such that u_i and v_i are eigenvectors of $\Omega_0^{-1} M$ with opposite eigenvalues and in which Ω_0 is block-diagonal with blocks of size two.*

PROOF. For ease of notation, we write the proof for $n = 2$, but the proof is the same for any n.

Let $A = \Omega_0^{-1} M$. We have that

$$\begin{aligned}
\det(\lambda I - A) &= \det(\lambda I - \Omega_0^{-1} M) \\
&= \det(\lambda I - (\Omega_0^{-1} M)^T) \\
&= \det(\lambda I + M \Omega_0^{-1}) \\
&= \det(\lambda \Omega_0 + M) \det(\Omega_0^{-1}) \\
&= \det(\lambda I + \Omega_0^{-1} M) \\
&= \det(\lambda I + A).
\end{aligned}$$

This implies that the eigenvalues of A must come in pairs, that is, if λ is an eigenvalue, $-\lambda$ also is. In particular, 0 is not an eigenvalue, because it would be at least double, contradicting the hypothesis. So the eigenvalues are $\lambda, -\lambda, \mu, -\mu$, for $\lambda, \mu \in F^*$.

Now let w_1 and w_2 be two eigenvectors of A with eigenvalues α_1 and α_2, such that $\alpha_1 \neq -\alpha_2$. Then

$$\begin{aligned}
\alpha_1 w_1^T \Omega_0 w_2 &= (\Omega_0^{-1} M w_1)^T \Omega_0 w_2 \\
&= w_1^T M (-\Omega_0^{-1}) \Omega_0 w_2 \\
&= -w_1^T M w_2 \\
&= -w_1^T \Omega_0 \Omega_0^{-1} M w_2 \\
&= -\alpha_2 w_1^T \Omega_0 w_2.
\end{aligned}$$

As $\alpha_1 \neq -\alpha_2$, this implies $w_1^T \Omega_0 w_2 = 0$.

Let u_1, v_1, u_2, v_2 be the eigenvectors of $\lambda, -\lambda, \mu, -\mu$, respectively. By the previous result, $w_1^T \Omega_0 w_2 = 0$ for any two vectors $w_1, w_2 \in \{u_1, v_1, u_2, v_2\}$ which are not a pair u_i, v_i. We call Ψ the matrix with (u_1, v_1, u_2, v_2) as columns. Then,

$$\Psi^T \Omega_0 \Psi = \begin{pmatrix} 0 & u_1^T \Omega_0 v_1 & 0 & 0 \\ v_1^T \Omega_0 u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2^T \Omega_0 v_2 \\ 0 & 0 & v_2^T \Omega_0 u_2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & u_1^T \Omega_0 v_1 & 0 & 0 \\ -u_1^T \Omega_0 v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2^T \Omega_0 v_2 \\ 0 & 0 & -u_2^T \Omega_0 v_2 & 0 \end{pmatrix},$$

as we wanted. □

The basis in Lemma 6.3 is almost but not quite a symplectic basis:

Definition 6.4. Let n be a positive integer. Let F be a field. We say that a basis $\{u_1, v_1, \ldots, u_n, v_n\}$ of F^{2n} is *symplectic* if, for any two vectors w_1, w_2 in the basis, $w_1^T \Omega_0 w_2 = 1$ if $w_1 = u_i$ and $w_2 = v_i$ for some i with $1 \leq i \leq n$, and otherwise $w_1^T \Omega_0 w_2 = 0$. This condition is equivalent to saying that the matrix in $\mathcal{M}_{2n}(F)$ with $u_1, v_1, \ldots, u_n, v_n$ as columns is symplectic.

Actually, we can rescale the vectors v_i such that the basis becomes symplectic. But this may break the structure of the eigenvectors: for example, if $-\lambda = \bar{\lambda}$ for some definition of the conjugate, we can take $v_1 = \bar{u}_1$, which will no more hold after rescaling v_1. We leave the lemma as such because we do not need that rescaling.

Corollary 6.5. *Let n be a positive integer. Let F be an algebraically closed field with characteristic different from 2. Let Ω_0 be the matrix of the standard symplectic form on F^{2n}. Given symmetric matrices $M_1, M_2 \in \mathcal{M}_{2n}(F)$ such that $\Omega_0^{-1} M_i$ has pairwise distinct eigenvalues for $i \in \{1, 2\}$, there is a symplectic matrix S such that $S^T M_1 S = M_2$ if and only if $\Omega_0^{-1} M_1$ and $\Omega_0^{-1} M_2$ have the same eigenvalues. Moreover, in this case S must have the form $\Psi_1 D \Psi_2^{-1}$, where D is a diagonal matrix such that*

$$d_{2i-1,2i-1} d_{2i,2i} = \frac{(u_i^2)^T \Omega_0 v_i^2}{(u_i^1)^T \Omega_0 v_i^1}, \tag{27}$$

u_i^1 and v_i^1 are those of Lemma 6.3 for the first form, u_i^2 and v_i^2 for the second form, and for $j \in \{1, 2\}$,

$$\Psi_j = \begin{pmatrix} u_1^j & v_1^j & \ldots & u_n^j & v_n^j \end{pmatrix}.$$

Proof. By Lemma 6.3, there are Ψ_1 and Ψ_2 such that $\Psi_i^{-1} A_i \Psi_i = J_i$, where J_i is the matrix in Proposition 6.2, and $\Psi_i^T \Omega_0 \Psi_i$ has all elements zero except those of the form $(2i-1, 2i)$ and $(2i, 2i-1)$. Moreover, in this case matrices J_1 and J_2 are diagonal and with all elements in the diagonal different. They are equal if and only if A_1 and A_2 have the same eigenvalues.

A matrix D that commutes with $J_1 = J_2$ is necessarily diagonal, and the relation

$$D^T \Psi_1^T \Omega_0 \Psi_1 D = \Psi_2^T \Omega_0 \Psi_2$$

in this case reduces to (27). □

In the case where the eigenvalues are not all different, the situation is not so simple, but we can do something similar to Lemma 6.3.

LEMMA 6.6. *Let n be a positive integer. Let F be an algebraically closed field with characteristic different from 2 and let $M \in \mathcal{M}_{2n}(F)$ be a symmetric matrix. Let Ω_0 be the matrix of the standard symplectic form on F^{2n}. Let $A = \Omega_0^{-1}M$. Then, the number of nonzero eigenvalues of A is even, that is, $2m$ for some integer m with $0 \le m \le n$, and there exists a set*

$$\Big\{u_1, v_1, \ldots, u_m, v_m\Big\} \subset F^{2n}$$

which satisfies the following properties:

- *$Au_i = \lambda_i u_i + \mu_i u_{i-1}$ and $Av_i = -\lambda_i v_i + \mu_{i+1} v_{i+1}$ for $1 \le i \le m$, where $\lambda_i \in F$, $\mu_i = 0$ or 1, $\mu_1 = \mu_{m+1} = 0$, and $\mu_i = 1$ only if $\lambda_i = \lambda_{i-1}$. (That is to say, the vectors are a "Jordan basis".)*
- *The vectors can be completed to a symplectic basis: given w_1, w_2 in the set, $w_1^T \Omega_0 w_2 = 1$ if $w_1 = u_i, w_2 = v_i$ for some i with $1 \le i \le m$, and otherwise $w_1^T \Omega_0 w_2 = 0$.*

PROOF. Let J be the Jordan form of A and Ψ such that $\Psi^{-1}A\Psi = J$. We have that

$$\begin{aligned} J &= \Psi^{-1}A\Psi \\ &= \Psi^{-1}\Omega_0^{-1}M\Psi \\ &= \Psi^{-1}\Omega_0^{-1}M\Omega_0^{-1}\Omega_0\Psi \\ &= \Psi^{-1}\Omega_0^{-1}(-\Omega_0^{-1}M)^T\Omega_0\Psi \\ &= \Psi^{-1}\Omega_0^{-1}(-\Psi J\Psi^{-1})^T\Omega_0\Psi \\ &= (\Psi^T\Omega_0\Psi)^{-1}(-J^T)\Psi^T\Omega_0\Psi. \end{aligned}$$

But J can only be similar to $-J^T$ if for each block having λ in the diagonal there is another having $-\lambda$ in the diagonal, and with the same size. We can split the blocks in three parts, that is, there is a Φ such that

$$\Phi^{-1}A\Phi = \begin{pmatrix} J_+ & 0 & 0 \\ 0 & J_- & 0 \\ 0 & 0 & J_0 \end{pmatrix}, \tag{28}$$

for some matrices J_+, J_- and J_0 in Jordan form, such that J_+ and J_- have opposite eigenvalues and J_0 has only 0 as eigenvalue. Let m be the size of J_+ and J_-. Now $2m$ is the number of nonzero eigenvalues.

Let the first m columns of Φ be $u_1, \ldots, u_m$. Because of (28), we have

$$Au_i = \lambda_i u_i + \mu_i u_{i-1},$$

for adequate λ_i and μ_i, with $\mu_i = 0$ or 1.

Now we change sign and transpose, getting

$$-(\Phi^{-1}A\Phi)^T = \begin{pmatrix} -J_+^T & 0 & 0 \\ 0 & -J_-^T & 0 \\ 0 & 0 & -J_0^T \end{pmatrix} \tag{29}$$

The left-hand side equals

$$\begin{aligned}-\Phi^T A^T (\Phi^{-1})^T &= \Phi^T M \Omega_0^{-1} (\Phi^{-1})^T \\ &= \Phi^T \Omega_0 A \Omega_0^{-1} (\Phi^{-1})^T \\ &= \Phi^T \Omega_0 A (\Phi^T \Omega_0)^{-1}.\end{aligned}$$

Again, let the first m columns of $(\Phi^T \Omega_0)^{-1}$ be $v_1, \dots, v_m$. By (29), we have

$$Av_i = -\lambda_i v_i - \mu_{i+1} v_{i+1}.$$

It is left to prove that the $2m$ vectors form a partial symplectic basis. To do this, first we see that $u_i^T \Omega_0 u_j = 0$ for any i, j, by induction on $i + j$. The base case is when $i = j = 1$, which is trivial. Supposing it true for $(i-1, j)$ and $(i, j-1)$, we prove it for (i, j):

$$\begin{aligned}\lambda_i u_i^T \Omega_0 u_j &= (\Omega_0^{-1} M u_i - \mu_i u_{i-1})^T \Omega_0 u_j \\ &= u_i^T M (-\Omega_0^{-1}) \Omega_0 u_j - \mu_i \cdot 0 \\ &= -u_i^T M u_j \\ &= -u_i^T \Omega_0 \Omega_0^{-1} M u_j \\ &= -u_i^T \Omega_0 (\lambda_j u_j + \mu_j u_{j-1}) \\ &= -\lambda_j u_i^T \Omega_0 u_j\end{aligned}$$

which implies that $u_i^T \Omega_0 u_j = 0$ because $\lambda_i \neq -\lambda_j$ (the opposites of the eigenvalues in the part J_+ are all in J_-). Analogously we prove that $v_i^T \Omega_0 v_j = 0$, making the induction backwards.

Finally, $u_i^T \Omega_0 v_j$ is the element in the position (i, j) of $\Phi^T \Omega_0 (\Phi^T \Omega_0)^{-1}$, which is 1 if $i = j$ and 0 otherwise, so the vectors are a partial symplectic basis. □

Lemma 6.6 allows us to separate the part of a matrix with nonzero eigenvalues from the part with zero eigenvalues. Choosing a symplectic basis for the latter to complete the partial basis in a way that we obtain a complete "Jordan basis" is more complicated, but it can also be done, as the following theorem shows.

Definition 6.7. A tuple $K = (k_1, k_2, \dots, k_t)$ of natural numbers is called *good* if the numbers are in non-increasing order, that is, $k_i \geq k_{i+1}$ for every $1 \leq i \leq t-1$, and each odd number appears an even number of times in K. We define $f_K : \{1, \dots, t\} \to \{1, \dots, t\}$ as follows:

- If k_i is even, $f_K(i) = i$.
- If k_i is odd, let i_0 be the first index such that $k_{i_0} = k_i$. Then, $f_K(i) = i + (-1)^{i-i_0}$.

If K is good, f_K is an involution that fixes the indices of even elements and pairs those of odd elements.

Theorem 6.8. *Let n be a positive integer. Let F be a field with characteristic different from 2 and let $M \in \mathcal{M}_{2n}(F)$ be a symmetric matrix. Let Ω_0 be the matrix of the standard symplectic form on F^{2n}. Let $A = \Omega_0^{-1} M$. Suppose that the eigenvalues of A are all zero. Then, there is an integer $t \geq 1$, a good tuple $K = (k_1, \dots, k_t)$ and a basis of F^{2n} of the form*

$$\left\{u_{11}, u_{12}, \dots, u_{1k_1}, \dots, u_{t1}, u_{t2}, \dots, u_{tk_t}\right\} \tag{30}$$

such that the following two conditions hold:

- *For every* $1 \le i \le t, 1 \le j \le k_t$, $Au_{ij} = u_{i,j-1}$, *where* $u_{i0} = 0$.
- *Given* $u_{ij}, u_{i'j'}$ *in this basis,* $u_{ij}^T \Omega_0 u_{i'j'} \neq 0$ *if and only if* $i' = f_K(i)$ *and* $j + j' = k_i + 1$.

See Table 1, left, for an example of such a basis.

To prove this theorem, we need several intermediate results.

Definition 6.9. Given a tuple $K = (k_1, \dots, k_t)$ such that $k_i \ge k_{i+1}$ for $1 \le i \le t-1$, we call R_K the set of tuples of the form (ℓ, i, j) where $1 \le \ell \le t, \ell \le i \le t, 1 \le j \le k_i$.

Given $R \subset R_K$, we say that a basis is R-*acceptable* if, for each $(\ell, i, j) \in R$, one of these alternatives holds:

- $j = 1$, $i = f_K(\ell)$, $\ell = f_K(i)$, and $u_{\ell k_\ell}^T \Omega_0 u_{ij} \neq 0$;
- $j \neq 1$ or $i \neq f_K(\ell)$, and $u_{\ell k_\ell}^T \Omega_0 u_{ij} = 0$.

(Note that, if there exists $(\ell, i, j) \in R$ with $j = 1$, $i = f_K(\ell)$ but $\ell \neq f_K(i)$, no basis is R-acceptable.)

Given $(\ell, i, j), (\ell', i', j') \in R_K$, we say that $(\ell, i, j) < (\ell', i', j')$ if

- $\ell < \ell'$,
- $\ell = \ell'$ and $j < j'$, or
- $\ell = \ell'$, $j = j'$ and $i < i'$.

Note that $<$ is a total order in R_K.

For each s integer with $0 \le s \le |R_K|$, we call $R_{K,s}$ the set of the first s elements of R_K according to the order $<$.

Let J be the Jordan form of A. Since all eigenvalues of A are zero, J has all entries equal to zero except some in the first diagonal above the main diagonal, which are 1. That is, J is a block-diagonal matrix. Let $k_1, \dots, k_t$ be the sizes of the blocks, with $k_i \ge k_{i+1}$, and $K = (k_1, \dots, k_t)$. For this K, any Jordan basis satisfies the first condition.

Lemma 6.10. *If* $\{u_{ij}\}$ *is a Jordan basis for* A *in the form* (30), $u_{ij}^T \Omega_0 u_{i'j'} = -u_{i,j-1}^T \Omega_0 u_{i',j'+1}$. *That is, the product* $u_{ij}^T \Omega_0 u_{i'j'}$ *only depends on* $j + j'$, *except for the sign.*

Proof. Using that M is symmetric and Ω_0 is antisymmetric,

$$\begin{aligned} u_{ij}^T \Omega_0 u_{i'j'} &= u_{ij}^T \Omega_0 A u_{i',j'+1} \\ &= u_{ij}^T M u_{i',j'+1} \\ &= -u_{ij}^T A^T \Omega_0 u_{i',j'+1} \\ &= -u_{i,j-1}^T \Omega_0 u_{i',j'+1}. \end{aligned}$$ □

Corollary 6.11. *Let us assume the conditions of Lemma 6.10. If* $j + j' \le \max\{k_i, k_{i'}\}$, $u_{ij}^T \Omega_0 u_{i'j'} = 0$.

Proof. Without loss of generality suppose that $k_i \le k_{i'}$. Applying the previous lemma j times, we have

$$u_{ij}^T \Omega_0 u_{i'j'} = u_{i0}^T \Omega_0 u_{i',j+j'} = 0.$$ □

Corollary 6.12. *If a basis* $\{u_{ij}\}$ *of the form* (30) *is* R*-acceptable for some* R *which contains* $(\ell, f_K(\ell), 1)$ *or* $(f_K(\ell), \ell, 1)$, *then* $u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1} \neq 0$.

u_{16}					
u_{15}	u_{25}	u_{35}			
u_{14}	u_{24}	u_{34}	u_{44}		
u_{13}	u_{23}	u_{33}	u_{43}	u_{53}	u_{63}
u_{12}	u_{22}	u_{32}	u_{42}	u_{52}	u_{62}
u_{11}	u_{21}	u_{31}	u_{41}	u_{51}	u_{61}

$\rightarrow$

u_{16}					
u_{15}	u_{25}	u_{35}			
u_{14}	u_{24}	u_{34}	$u_{44}+cu_{23}$		
u_{13}	u_{23}	u_{33}	$u_{43}+cu_{22}$	u_{53}	u_{63}
u_{12}	u_{22}	u_{32}	$u_{42}+cu_{21}$	u_{52}	u_{62}
u_{11}	u_{21}	u_{31}	u_{41}	u_{51}	u_{61}

TABLE 1. Above: a basis B of F^{26} in the form (30) for $K = (6,5,5,4,3,3)$. Below: the basis $B[2,4,2,c]$, for $c \in F$.

PROOF. If $(\ell, f_K(\ell), 1) \in R$, then the conclusion holds by definition.
If $(f_K(\ell), \ell, 1) \in R$, then $f_K(\ell) \leq \ell$, which implies $k_\ell = f_K(\ell)$, and $u^T_{\ell k_\ell} \Omega_0 u_{f_K(\ell),1} = (-1)^{k_\ell - 1} u^T_{\ell 1} \Omega_0 u_{f_K(\ell),k_\ell} \neq 0$. □

We will need this operation to change a basis:

DEFINITION 6.13. Let B be a basis of F^{2n} of the form (30), $1 \leq \ell \leq t, 1 \leq i \leq t, 1 \leq j \leq k_i$ and $c \in F$. We call $B[\ell, i, j, c]$ the basis

$$\left\{u'_{11}, u'_{12}, \ldots, u'_{1k_1}, \ldots, u'_{t1}, u'_{t2}, \ldots, u'_{tk_t}\right\}$$

where

$$u'_{i'j'} = \begin{cases} u_{i'j'} & \text{if } i' \neq i, \\ u_{ij'} & \text{if } i' = i, j' < j, \\ u_{ij'} + cu_{\ell, j'-j+1} & \text{if } i' = i, j' \geq j. \end{cases}$$

See Table 1 for an example.

LEMMA 6.14. (1) *If B is a Jordan basis for A, $B[\ell, i, j, c]$ is also a Jordan basis.*

(2) *The only products of the form $u^T_{\ell' k_{\ell'}} \Omega_0 u_{i'j'}$ which may be different in B and $B[\ell, i, j, c]$ are those with $i' = i \neq \ell'$ and $j' \geq j$, which vary in $cu^T_{\ell' k_{\ell'}} \Omega_0 u_{\ell, j'-j+1}$, those with $\ell' = i \neq i'$, which vary in $cu^T_{\ell, k_i - j+1} \Omega_0 u_{i'j'}$, and those with $\ell' = i' = i$, which vary in $cu^T_{ik_i} \Omega_0 u_{\ell, j'-j+1} + cu^T_{\ell, k_i-j+1} \Omega_0 u_{ij'} + c^2 u^T_{\ell, k_i - j+1} \Omega_0 u_{\ell, j'-j+1}$ (understanding $u_{ij} = 0$ if $j \leq 0$ in the last equality).*

PROOF. (1) We have that $Au_{i'j'} = u_{i', j'-1}$ and we need to see that $Au'_{i'j'} = u'_{i',j'-1}$ for all indices $1 \leq i' \leq t, 1 \leq j' \leq k_{i'}$. If $i' \neq i$, the conclusion follows because the vectors do not change. If $i' = i$ and $j' < j$, the same happens. Otherwise, $i' = i$ and $j' \geq j$, and

$$Au'_{ij'} = A(u_{ij'} + cu_{\ell, j'-j+1}) = u_{i,j'-1} + cu_{\ell, j'-j} = u'_{i,j'-1}$$

where the last equality also holds if $j' = j$ and $u_{\ell, j'-j} = 0$.

(2) This follows from the definition of the new basis: to change the product we need to change any of the two vectors. □

Lemma 6.15. *For every s with $0 \le s \le |R_K|$, there is a Jordan basis for A which is $R_{K,s}$-acceptable.*

Proof. We prove this by induction in s. For $s = 0$, $R_{K,0}$ is empty and the problem reduces to the existence of a Jordan basis.

Supposing it is true for s, we prove it for $s+1$. Let $\{(\ell, i, j)\} = R_{K,s+1} \setminus R_{K,s}$, that is, (ℓ, i, j) is the $(s+1)$-th element of R_K according to the order $<$. Let $B = \{u_{ij}\}$ be the $R_{K,s}$-acceptable basis given by the inductive hypothesis.

There are several cases to consider.

(1) $i = \ell$ and $k_\ell - j$ is even. By Lemma 6.10, we have that

$$u^T_{\ell k_\ell}\Omega_0 u_{\ell j} = u^T_{\ell,(k_\ell+j)/2}\Omega_0 u_{\ell,(k_\ell+j)/2} = 0.$$

Either $j > 1$ and we want a zero, or $j = 1$ with k_ℓ odd and we also want a zero; in any case B itself is $R_{K,s+1}$-acceptable.

(2) $i = f_K(\ell), j = 1$. This means that i is ℓ if k_ℓ is even and $\ell + 1$ if k_ℓ is odd (it cannot be $\ell - 1$ because $(\ell, i, j) \in R_K$). If $u^T_{\ell k_\ell}\Omega_0 u_{i1} \ne 0$, B is $R_{K,s+1}$-acceptable and we are done. Otherwise, since B is a basis, there are $1 \le i_1 \le t, 1 \le j_1 \le k_{i_1}$ such that $u^T_{\ell 1}\Omega_0 u_{i_1 j_1} \ne 0$. By Corollary 6.11, $j_1 + 1 > \max\{k_\ell, k_{i_1}\}$ or equivalently $j_1 \ge \max\{k_\ell, k_{i_1}\}$. But this implies $k_{i_1} \ge j_1 \ge \max\{k_\ell, k_{i_1}\} \ge k_\ell$, so $j_1 \ge k_{i_1}$, which implies $j_1 = k_{i_1}$ and $u^T_{\ell 1}\Omega_0 u_{i_1 k_{i_1}} \ne 0$.

By Lemma 6.10, $u^T_{\ell 1}\Omega_0 u_{ik_i} = (-1)^{k_i - 1}u^T_{\ell k_\ell}\Omega_0 u_{i1} = 0$, so $i \ne i_1$. If $i_1 < i$, either $i_1 < \ell$ or $i_1 = \ell$ with $i = \ell+1$. In any case $(i_1, \ell, 1) < (\ell, i, 1)$, and since B is $R_{K,s}$-acceptable, $u^T_{\ell 1}\Omega_0 u_{i_1 k_{i_1}} = 0$, a contradiction. Hence, $i_1 > i \ge \ell$. As the sequence K is non-increasing and $k_{i_1} \ge k_\ell$, we must have $k_{i_1} = k_i = k_\ell$, and again by Lemma 6.10, $u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} \ne 0$. If $i = \ell+1$, $k_i = k_\ell$ together with $f_K(\ell) = i$ implies that $f_K(i) = \ell$.

Let $B' = \{u'_{ij}\} = B[i_1, i, 1, 1]$. This basis is $R_{K,s}$-acceptable: by Lemma 6.14, the conditions in $R_{K,s}$ that may break with this change are (ℓ', i, j') with $\ell' < \ell$ (the only condition which may be in $R_{K,s}$ with $\ell' = \ell$ is $(\ell, \ell, 1)$). For these tuples,

$$(u'_{\ell' k_{\ell'}})^T\Omega_0 u'_{ij'} = u^T_{\ell' k_{\ell'}}\Omega_0 u_{ij'} + u^T_{\ell' k_{\ell'}}\Omega_0 u_{i_1 j'} = 0$$

because (ℓ', i, j') and (ℓ', i_1, j') are in $R_{K,s}$. Moreover, if $i = \ell + 1$,

$$(u'_{\ell k_\ell})^T\Omega_0 u'_{i1} = u^T_{\ell k_\ell}\Omega_0 u_{i1} + u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} = u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} \ne 0,$$

so B' is $R_{K,s+1}$-acceptable, as we wanted.

If $i = \ell$, we also define $B'' = \{u''_{ij}\} = B[i_1, i, 1, -1]$. Analogously to what we said for B', B'' is $R_{K,s}$-acceptable. Now we have

$$(u'_{\ell k_\ell})^T\Omega_0 u'_{i1} = u^T_{\ell k_\ell}\Omega_0 u_{i1} + u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} + u^T_{i_1 k_\ell}\Omega_0 u_{i1} + u^T_{i_1 k_\ell}\Omega_0 u_{i_1 1} = 2u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} + u^T_{i_1 k_\ell}\Omega_0 u_{i_1 1}$$

and

$$(u''_{\ell k_\ell})^T\Omega_0 u''_{i1} = u^T_{\ell k_\ell}\Omega_0 u_{i1} - u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} - u^T_{i_1 k_\ell}\Omega_0 u_{i1} + u^T_{i_1 k_\ell}\Omega_0 u_{i_1 1} = -2u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} + u^T_{i_1 k_\ell}\Omega_0 u_{i_1 1}$$

where the second equality in each line is due to Lemma 6.10. If both results were zero at the same time, that would imply $u^T_{\ell k_\ell}\Omega_0 u_{i_1 1} = 0$, so one of them must be nonzero, and one of B' and B'' must be $R_{K,s+1}$-acceptable.

(3) $i = \ell, j > 1$ and $k_\ell - j$ is odd. We set

$$c = -\frac{u_{\ell k_\ell}^T \Omega_0 u_{\ell j}}{2u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1}}$$

and $B' = B[f_K(\ell), \ell, j, c]$ (the denominator is not zero by Corollary 6.12, because $(\ell, f_K(\ell), 1) \in R_{K,s}$ or $(f_K(\ell), \ell, 1) \in R_{K,s}$). By Lemma 6.14, the conditions that can break are (ℓ, i', j') with $\ell \le i' \le t$ and $1 \le j' \le j-1$, and (ℓ', ℓ, j') with $1 \le \ell' \le \ell - 1$ and $j \le j' \le k_i$. For the first type, we see that

$$(u'_{\ell k_\ell})^T \Omega_0 u'_{i'j'} = u_{\ell k_\ell}^T \Omega_0 u_{i'j'} + c u_{f_K(\ell), k_\ell - j + 1}^T \Omega_0 u_{i'j'}$$

The second term is 0 because $k_\ell - j + 1 + j' \le k_\ell = k_{f_K(\ell)}$ (we are using here that $f_K(f_K(\ell)) = \ell$ because the basis is acceptable). For the second type,

$$(u'_{\ell' k_{\ell'}})^T \Omega_0 u'_{\ell j'} = u_{\ell' k_{\ell'}}^T \Omega_0 u_{\ell j'} + c u_{\ell' k_{\ell'}}^T \Omega_0 u_{f_K(\ell), j'-j+1}$$

Both terms are zero because (ℓ', ℓ, j') and $(\ell', f_K(\ell), j'-j+1)$ are in $R_{K,s}$, $f_K(\ell') \neq f_K(\ell)$ and $j' > 1$. It is only left to show that the new condition is satisfied:

$$(u'_{\ell k_\ell})^T \Omega_0 u'_{\ell j} = u_{\ell k_\ell}^T \Omega_0 u_{\ell j} + c u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1} + c u_{f_K(\ell), k_\ell - j+1}^T \Omega_0 u_{\ell j} + c^2 u_{f_K(\ell), k_\ell - j + 1}^T \Omega_0 u_{f_K(\ell),1}$$

The last term is zero because $k_\ell - j + 1 + 1 \le k_\ell$. The second and the third are equal because, by Lemma 6.10,

$$u_{f_K(\ell), k_\ell - j+1}^T \Omega_0 u_{\ell j} = -u_{\ell j}^T \Omega_0 u_{f_K(\ell), k_\ell - j + 1} = (-1)^{k_\ell - j + 1} u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1} = u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1}.$$

Hence

$$(u'_{\ell k_\ell})^T \Omega_0 u'_{\ell j} = u_{\ell k_\ell}^T \Omega_0 u_{\ell j} + 2c u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1} = 0$$

as we wanted.

(4) $i > \ell$ and either $j > 1$ or $i \neq f_K(\ell)$. We set

$$c = -\frac{u_{\ell k_\ell}^T \Omega_0 u_{ij}}{u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1}}$$

and $B' = B[f_k(\ell), i, j, c]$ (the denominator is not zero by Corollary 6.12, because $(\ell, f_K(\ell), 1) \in R_{K,s}$ or $(f_K(\ell), \ell, 1) \in R_{K,s}$). By Lemma 6.14, the conditions that can break are (ℓ', i, j') with $1 \le \ell' \le \ell - 1$ and $j \le j' \le k_i$. In these cases we have

$$(u'_{\ell' k_{\ell'}})^T \Omega_0 u'_{ij'} = u_{\ell' k_{\ell'}}^T \Omega_0 u_{ij'} + c u_{\ell' k_{\ell'}}^T \Omega_0 u_{f_K(\ell), j'-j+1}$$

Both terms are zero because (ℓ', i, j') and $(\ell', f_K(\ell), j'-j+1)$ are in $R_{K,s}$, $f_K(\ell') \neq f_K(\ell)$, and either $j' > 1$ or $i \neq f_K(\ell)$. It is left to show the new condition:

$$(u'_{\ell k_\ell})^T \Omega_0 u'_{ij} = u_{\ell k_\ell}^T \Omega_0 u_{ij} + c u_{\ell k_\ell}^T \Omega_0 u_{f_K(\ell),1} = 0. \qquad \square$$

With these results, we are ready to prove Theorem 6.8.

Proof of Theorem 6.8. We apply Lemma 6.15 with $s = |R_K|$ to obtain a basis B which is R_K-acceptable. Now we show that this B is the basis we want. The first condition holds because B is a Jordan basis. For the second, if

$j+j' \leq \max\{k_i, k_{i'}\}$, the product is zero by Corollary 6.11, as we want. Otherwise, suppose $i \leq i'$ and $k_i \geq k_{i'}$. By Lemma 6.10,

$$u_{ij}^T \Omega_0 u_{i'j'} = u_{ik_i}^T \Omega_0 u_{i',j+j'-k_i} = 0,$$

and the second condition also holds.

It is left to show that K is good, but this is an immediate consequence of B being R_K-acceptable: by definition, if $f_K(l) = i$ but $f_K(i) \neq l$ for some $(l,i,j) \in R_K$, there would not be any R_K-acceptable basis. □

We can use Lemma 6.6 and Theorem 6.8 to give a classification in the case where the field is algebraically closed. In particular, we can apply this to $\mathbb{C}$.

THEOREM 6.16. *Let n be a positive integer. Let F be an algebraically closed field with characteristic different from 2 and let $M \in \mathcal{M}_{2n}(F)$ be a symmetric matrix. There exists a positive integer k, $r \in F$, $a \in \{0,1\}$, with $a = 1$ only if $r = 0$, and a symplectic matrix $S \in \mathcal{M}_{2n}(F)$ such that $S^T M S$ is a block-diagonal matrix whose blocks are of the form*

$$M_{\mathrm{h}}(k,r,a) = \begin{pmatrix} & r & & & & & \\ r & & 1 & & & & \\ & 1 & & r & & & \\ & & r & & \ddots & & \\ & & & \ddots & & 1 & \\ & & & & 1 & & r \\ & & & & & r & a \end{pmatrix}$$

with $2k$ rows. Furthermore, if there are two matrices S and S' which reduce M to this form, then the two forms only differ in the order in which the blocks are arranged.

PROOF. (a) First we prove existence. We start applying Lemma 6.6. This gives us a partial symplectic basis

$$\Big\{u_1, v_1, \ldots, u_m, v_m\Big\},$$

which is also a partial Jordan basis of A corresponding to the nonzero eigenvalues, and u_i and v_i correspond to opposite eigenvalues λ_i and $-\lambda_i$. If $\{u_1, v_1, \ldots, u_k, v_k\}$ are the vectors of a block with eigenvalues r and $-r$, for $r \in F$, these same vectors taken as columns of S give the matrix $M_{\mathrm{h}}(k,r,0)$.

For the other part of the Jordan form, we apply Theorem 6.8 to the eigenspace of 0, resulting in a good multiset $K = \{k_1, \ldots, k_t\}$ and a basis

$$\Big\{u_{11}, u_{12}, \ldots, u_{1k_1}, \ldots, u_{t1}, u_{t2}, \ldots, u_{tk_t}\Big\}.$$

This is not necessarily a symplectic basis as such, but it allows us to construct one with the required properties:

- If $k_i = 2\ell_i$ is even, we have that

$$u_{ij}^T \Omega_0 u_{i,k_i+1-j} = (-1)^{j+1} c_i$$

for some $c_i \in F$. Let

$$c_i = b_i^2$$

for $b_i \in F$. After dividing all the chain by b_i, we can assume that $c_i = (-1)^{\ell_i}$. Now

$$\left\{u_{i1}, (-1)^{\ell_i} u_{i,2\ell_i}, -u_{i2}, (-1)^{\ell_i} u_{i,2\ell_i-1}, \dots, (-1)^{\ell_i-1} u_{i\ell_i}, (-1)^{\ell_i} u_{i,\ell_i+1}\right\}$$

is a partial symplectic basis which gives the form $M_{\mathrm{h}}(\ell_i, 0, 1)$.

- If k_i is odd and $f_K(i) = i+1$, we have instead

$$u_{ij}^T \Omega_0 u_{i+1,k_i+1-j} = (-1)^{j+1} c_i$$

for some $c_i \in F$. After dividing the elements of the second chain by c_i, we can assume that $c_i = 1$. Now

$$\left\{u_{i1}, u_{i+1,k_i}, -u_{i2}, u_{i+1,k_i-1}, \dots, (-1)^{k_i-1} u_{ik_i}, u_{i+1,1}\right\}$$

is a partial symplectic basis which gives the form $M_{\mathrm{h}}(k_i, 0, 0)$.

(b) Uniqueness follows from Proposition 6.2, because two matrices in normal form have the same Jordan form if and only if they differ in the order of the blocks. □

In the cases of greatest interest to us the matrices in the statement of Theorem 6.16 have coefficients in $\mathbb{Q}_p$. We can take $F = \mathbb{C}_p$, but the resulting matrix S will have the entries in $\mathbb{C}_p$ and not necessarily in $\mathbb{Q}_p$, and we want a symplectomorphism of $(\mathbb{Q}_p)^n$, which is given by a symplectic matrix with entries in $\mathbb{Q}_p$. To avoid this, we need to manipulate adequately the symplectic basis, which translates to adjusting the matrix D in Proposition 6.2. This problem also happens in the real case, but, as we will see, the matrix S can always be adapted to have the entries in $\mathbb{R}$ instead of $\mathbb{C}$.

In dimension 2, the eigenvalues of $\Omega_0^{-1} M$ are the roots of a degree 2 polynomial with coefficients in F, so they have the form λ and $-\lambda$. The case where the eigenvalues are in the base field is the easiest one.

Proposition 6.17. *Let F be a field of characteristic different from 2. Let Ω_0 be the matrix of the standard symplectic form on F^2. Let $M \in \mathcal{M}_2(F)$ symmetric and invertible such that the eigenvalues of $\Omega_0^{-1} M$ are in F. Then there is a symplectic matrix $S \in \mathcal{M}_2(F)$ and $a \in F$ such that*

$$S^T M S = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Proof. If the eigenvalues $\pm\lambda$ of $\Omega_0^{-1} M$ are in F, its eigenvectors are also in F, and the matrix Ψ_1 of Corollary 6.5 is in F. In order to have the same eigenvalues for

$$\Omega_0^{-1} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$

we need $a = \lambda$. The matrix S that we need is precisely Ψ_1. □

If the matrix is not invertible, the case of the null matrix is already covered by the previous result, with $a = 0$. The other case is solved in the next proposition.

Proposition 6.18. *Let F be a field of characteristic different from 2. Let*

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathcal{M}_2(F)$$

where $b^2 = ac$. *Let* $d \in F$. *There exists a symplectic matrix* S *such that*

$$S^T M S = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$$

if and only if ad *is a square.*

PROOF. The second column of S must be $(kb, -ka)$ for some $k \in F$. Let the first column be (x, y), for $x, y \in F$. Then

$$d = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + cy^2 + 2bxy$$

and

$$ad = a^2x^2 + acy^2 + 2abxy = a^2x^2 + b^2y^2 + 2abxy$$

is a square. Conversely, if ad is a square, we can take

$$S = \begin{pmatrix} -kd & kb \\ 0 & -ka \end{pmatrix}$$

where k is chosen so that $k^2ad = 1$. □

For the cases where $\lambda \notin F$, we need some definitions.

DEFINITION 6.19. Given an Abelian group G, we call $\mathrm{Sq}(G)$ the subgroup formed by the squares in G.

DEFINITION 6.20. Given a field F with additive identity 0 and $c \in F$, we call

$$\mathrm{DSq}(F, c) = \Big\{x^2 + cy^2 : x, y \in F\Big\} \setminus \{0\}$$

and

$$\overline{\mathrm{DSq}}(F, c) = \mathrm{DSq}(F, c) / \mathrm{Sq}(F^*).$$

LEMMA 6.21. *Let* F *be a field and* $c \in F$. *Then* $\mathrm{DSq}(F, c)$ *is a group with respect to multiplication in* F.

PROOF. We just need to see that the product of two elements of $\mathrm{DSq}(F, c)$ is in $\mathrm{DSq}(F, c)$ and the inverse of an element of $\mathrm{DSq}(F, c)$ is in $\mathrm{DSq}(F, c)$:

$$(x_1^2 + cy_1^2)(x_2^2 + cy_2^2) = (x_1x_2 + c^2y_1y_2)^2 + c(x_1y_2 - x_2y_1)^2$$

and

$$\frac{1}{x^2 + cy^2} = \left(\frac{x}{x^2 + cy^2}\right)^2 + c\left(\frac{y}{x^2 + cy^2}\right)^2.$$ □

The group $\mathrm{DSq}(F, c)$ can also be defined in terms of the Hilbert symbol:

$$(a, b)_p = \begin{cases} 1 & \text{if } ax^2 + by^2 = 1 \text{ for some } x, y \in \mathbb{Q}_p; \\ -1 & \text{otherwise.} \end{cases}$$

With this definition, we have that

$$\mathrm{DSq}(F, c) = \Big\{a \in \mathbb{Q}_p : (a, -c)_p = 1\Big\}.$$

Lemma 6.21 is a consequence of the multiplicativity of the Hilbert symbol.

We have that

$$\begin{aligned} F^* / \mathrm{DSq}(F, c) &\cong (F^* / \mathrm{Sq}(F^*))/(\mathrm{DSq}(F, c) / \mathrm{Sq}(F^*)) \\ &= (F^* / \mathrm{Sq}(F^*))/\overline{\mathrm{DSq}}(F, c), \end{aligned}$$

that is, we have a group isomorphism in the rightmost part of the commutative diagram

$$\begin{array}{ccccc} \mathrm{DSq}(F,c) & \hookrightarrow & F^* & \twoheadrightarrow & F^*/\,\mathrm{Sq}(F,c) \\ \downarrow & & \downarrow & & \downarrow\cong \\ \overline{\mathrm{DSq}}(F,c) & \hookrightarrow & F^*/\,\mathrm{Sq}(F^*) & \twoheadrightarrow & (F^*/\,\mathrm{Sq}(F^*))/\overline{\mathrm{DSq}}(F,c). \end{array}$$

By definition, $\overline{\mathrm{DSq}}(F,c)$ is the subset of the classes in $F^*/\,\mathrm{Sq}(F^*)$ which contain elements of the form x^2+c, for $x \in F$. Note also that $\mathrm{DSq}(F,c)$, and hence $\overline{\mathrm{DSq}}(F,c)$, only depend on the class of c modulo $\mathrm{Sq}(F^*)$.

Now we can give a necessary and sufficient condition for a matrix to be symplectomorphic to a normal form.

PROPOSITION 6.22. *Let F be a field of characteristic different from 2. Let Ω_0 be the matrix of the standard symplectic form on F^2. Let $M \in \mathcal{M}_2(F)$ be a symmetric matrix such that the eigenvalues of $\Omega_0^{-1}M$ are of the form $\pm\lambda$ with $\lambda \notin F$ but $\lambda^2 \in F$. Let u be the eigenvector of value λ in $\Omega_0^{-1}M$. Then for any $a, b \in F$ there is a symplectic matrix S such that*

$$S^T M S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

if and only if $\lambda^2 = -ab$ and

$$\frac{2\lambda a}{u^T \Omega_0 \bar{u}} \in \mathrm{DSq}(F, -\lambda^2).$$

PROOF. The eigenvalues of

$$\Omega_0^{-1}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}$$

are $\pm\sqrt{-ab}$, so we must have $\lambda^2 = -ab$.

The matrix Ψ_2 of Corollary 6.5 has the form

$$\begin{pmatrix} \lambda & -\lambda \\ a & a \end{pmatrix}.$$

The formula for S gives that

$$S = \Psi_1 \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2\lambda} & \frac{1}{2a} \\ -\frac{1}{2\lambda} & \frac{1}{2a} \end{pmatrix}.$$

The two columns of Ψ_1 are the eigenvectors of λ and $-\lambda$. The first is u, and the second is the conjugate $\bar{u}$ (or, more precisely, can be taken as the conjugate).

If S has entries in F, let c_1 and c_2 be its columns. We get

$$c_1 = \frac{d_1 u - d_2 \bar{u}}{2\lambda} \in F^2;$$

$$c_2 = \frac{d_1 u + d_2 \bar{u}}{2a} \in F^2;$$

$$d_1 u = a c_2 + \lambda c_1, d_2 \bar{u} = a c_2 - \lambda c_1 = \overline{d_1 u} \Rightarrow d_2 = \bar{d}_1.$$

The numbers d_1 and d_2 must also satisfy (27), that is

$$d_1 d_2 = \frac{(\lambda, a)\Omega_0(-\lambda, a)^T}{u^T \Omega_0 \bar{u}} = \frac{2\lambda a}{u^T \Omega_0 \bar{u}}. \tag{31}$$

Taking into account that $d_2 = \bar{d}_1$, we can substitute $d_1 = r + s\lambda$ and $d_2 = r - s\lambda$ in 31, giving

$$r^2 - s^2\lambda^2 = \frac{2\lambda a}{u^T \Omega_0 \bar{u}}$$

so this is in $\mathrm{DSq}(F, -\lambda^2)$. □

We can apply Proposition 6.22 to the real elliptic case, that is, the case of a matrix in $\mathcal{M}_2(\mathbb{R})$ whose eigenvalues are $\pm\lambda = \pm r\mathrm{i}$. There, we want to achieve $a = b$. λ is purely imaginary, so $\lambda^2 = -ab = -a^2$ determines exactly $|a| = r$. $\mathrm{DSq}(\mathbb{R}, -\lambda^2)$ consists of all positive reals, so there is always a solution for a (we know $|a|$ and can take the adequate sign to make $2\lambda a / u^T \Omega_0 \bar{u}$ positive). This is why the real Weierstrass-Williamson classification in dimension 2 has just two cases.

For dimension 4 we have the analogous to the focus-focus normal form:

PROPOSITION 6.23. *Let F be a field of characteristic different from 2. Let Ω_0 be the matrix of the standard symplectic form on F^4. Let $M \in \mathcal{M}_4(F)$ be a symmetric matrix such that the eigenvalues of $\Omega_0^{-1}M$ are of the form $\pm\lambda, \pm\mu$ where $\lambda, \mu \notin F$ but all of them are in a degree 2 extension $F[\alpha]$ for some α. Then there are $r, s \in F$ and a symplectic matrix $S \in \mathcal{M}_4(F)$ such that $S^T M S$ has the form of Theorem 5.31(2) for $c = \alpha^2$.*

PROOF. The condition in the statement is equivalent to say that $\mu = \bar{\lambda}$ and both are in $F[\alpha]$. Let $\lambda = s + r\alpha$ and $\mu = s - r\alpha$, for $r, s \in F$.

Let

$$N = \begin{pmatrix} 0 & s & 0 & r \\ s & 0 & r\alpha^2 & 0 \\ 0 & r\alpha^2 & 0 & s \\ r & 0 & s & 0 \end{pmatrix},$$

which is the matrix in Theorem 5.31(2) for $c = \alpha^2$. The eigenvalues of

$$\Omega_0^{-1} N = \begin{pmatrix} -s & 0 & -r\alpha^2 & 0 \\ 0 & s & 0 & r \\ -r & 0 & -s & 0 \\ 0 & r\alpha^2 & 0 & s \end{pmatrix}$$

are precisely $\pm s \pm r\alpha$, that is, $\pm\lambda$ and $\pm\mu$, so we can apply Corollary 6.5. The matrix Ψ_2 has the form

$$\begin{pmatrix} 0 & \alpha & 0 & -\alpha \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \alpha & 0 & -\alpha & 0 \end{pmatrix}$$

with the values in the order $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ as needed by Corollary 6.5, and the resulting matrix S is

$$S = \Psi_1 \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{\alpha} \\ -\frac{1}{\alpha} & 0 & 1 & 0 \end{pmatrix}.$$

The condition (27) of Corollary 6.5 implies that

$$d_1 d_2 = \frac{(0, 1, 0, \alpha)\Omega_0(\alpha, 0, 1, 0)^T}{u^T \Omega_0 v} = \frac{-2\alpha}{u^T \Omega_0 v} \tag{32}$$

and

$$d_3 d_4 = \frac{(0,1,0,-\alpha)\Omega_0(-\alpha,0,1,0)^T}{\bar{u}^T\Omega_0\bar{v}} = \frac{2\alpha}{\bar{u}^T\Omega_0\bar{v}}, \tag{33}$$

where u and v are the eigenvectors for λ and $-\lambda$ respectively.

We also want S to have entries in F. Let c_1, c_2, c_3, c_4 be its columns, which should be vectors in F^4:

$$\begin{cases} c_1 = \dfrac{d_2 v - d_4\bar{v}}{\alpha}; \\ c_2 = d_1 u + d_3\bar{u}; \\ c_3 = d_2 v + d_4\bar{v}; \\ c_4 = \dfrac{d_1 u - d_3\bar{u}}{\alpha}. \end{cases}$$

These expressions imply that

$$2d_1 u = c_2 + \alpha c_4, 2d_3\bar{u} = c_2 - \alpha c_4 = \overline{2d_1 u} \Rightarrow d_3 = \bar{d}_1$$

and

$$2d_2 v = c_3 + \alpha c_1, 2d_4\bar{v} = c_3 - \alpha c_1 = \overline{2d_2 v} \Rightarrow d_4 = \bar{d}_2.$$

Now we can take d_1 and d_2 arbitrary such that (32) holds, and (33) will hold automatically because $d_3 d_4 = \overline{d_1 d_2}$. □

In the real case, this is enough to complete the Weierstrass-Williamson classification in all dimensions if the eigenvalues of $\Omega_0^{-1}M$ are pairwise distinct. Indeed, these eigenvalues can be associated in pairs of the form $\{a, -a\}$ or $\{\mathrm{i}a, -\mathrm{i}a\}$ and quadruples of the form

$$\Big\{a + \mathrm{i}b, a - \mathrm{i}b, -a + \mathrm{i}b, -a - \mathrm{i}b\Big\}.$$

We can apply respectively Propositions 6.17, 6.22 (we already explained why this is always possible) and 6.23, giving the hyperbolic, elliptic and focus-focus normal forms in Section 5.3.2.

In the p-adic case, such a classification is still not complete, even for 4-by-4 matrices. The reason for this difference is that, if $\alpha \notin \mathbb{R}$ with $\alpha^2 \in \mathbb{R}$, this means that α is an imaginary number and $\mathbb{R}[\alpha] = \mathbb{C}$, which is algebraically closed. But if $\alpha \notin \mathbb{Q}_p$ with $\alpha^2 \in \mathbb{Q}_p$, $\mathbb{Q}_p[\alpha]$ is not algebraically closed. So it is possible that $\lambda^2 \notin \mathbb{Q}_p$ and simultaneously $\lambda \notin \mathbb{Q}_p[\lambda^2]$.

To fix this issue, consider a degree four polynomial of the form $t^4 + At^2 + B$ (at the moment in an arbitrary field F). Its roots are of the form $\lambda, -\lambda, \mu, -\mu$. If λ^2 and μ^2 are not in F, they are conjugate in some degree 2 extension, that is, $\lambda^2 = a + b\alpha$ and $\mu^2 = a - b\alpha$ for some $\alpha \in F[\lambda^2]$. In turn, if λ and μ are not in $F[\lambda^2]$, we have a hierarchy of fields:

$$\begin{array}{c} F[\lambda,\mu] \\ \Big| \, 2 \\ F[\alpha] = F[\lambda^2] = F[\mu^2] \\ \Big| \, 2 \\ F \end{array}$$

There is an automorphism of $F[\lambda,\mu]$ that fixes F and moves α to $-\alpha$ (an extension of the conjugation in $F[\alpha]$). We will denote this as $x \mapsto \bar{x}$. $\bar{\lambda}$ must be μ or $-\mu$, without loss of generality we suppose that it is μ. There is another automorphism of $F[\lambda,\mu]$ that fixes $F[\alpha]$ and changes λ to $-\lambda$ and μ to $-\mu$, which we will denote as $x \mapsto \hat{x}$.

PROPOSITION 6.24. *Let F be a field with characteristic different from 2. Let $F[\alpha]$ be a degree two extension of F and let $F[\gamma,\bar{\gamma}]$ be an extension of $F[\alpha]$ such that $\gamma^2 \in F[\alpha]$. Let Ω_0 be the matrix of the standard symplectic form on F^4. Let $t_1, t_2 \in F$ such that*

$$\begin{cases} \gamma^2 = t_1 + t_2\alpha; \\ \bar{\gamma}^2 = t_1 - t_2\alpha. \end{cases}$$

Let $M \in \mathcal{M}_4(F)$ be a symmetric matrix such that the eigenvalues of $\Omega_0^{-1}M$ are of the form $\pm\lambda, \pm\mu$ with

$$\lambda = (r + s\alpha)\gamma$$

and

$$\mu = (r - s\alpha)\bar{\gamma},$$

for $r, s \in F$. Let $a, b \in F$. Let u be the eigenvector of λ. Then, there is a symplectic matrix $S \in \mathcal{M}_4(F)$ such that $S^T M S$ has the form of Theorem 5.31(3) with $c = \alpha^2$ if and only if

$$\frac{a\alpha\gamma(b+\alpha)}{u^T\Omega_0\hat{u}} \in \mathrm{DSq}(F[\alpha], -\gamma^2).$$

PROOF. Let N be the matrix

$$\begin{pmatrix} \frac{a\alpha^2(r-bs)}{\alpha^2-b^2} & 0 & \frac{s\alpha^2-rb}{\alpha^2-b^2} & 0 \\ 0 & \frac{-r(t_1+bt_2)-s(bt_1+\alpha^2t_2)}{a} & 0 & -r(bt_1+\alpha^2t_2)-s\alpha^2(t_1+bt_2) \\ \frac{s\alpha^2-rb}{\alpha^2-b^2} & 0 & \frac{r-bs}{a(\alpha^2-b^2)} & 0 \\ 0 & -r(bt_1+\alpha^2t_2)-s\alpha^2(t_1+bt_2) & 0 & a\alpha^2(-r(t_1+bt_2)-s(bt_1+\alpha^2t_2)) \end{pmatrix}.$$

We have that $\Omega_0^{-1}N$ is equal to

$$\begin{pmatrix} 0 & \frac{r(t_1+bt_2)+s(bt_1+\alpha^2t_2)}{a} & 0 & r(bt_1+\alpha^2t_2)+s\alpha^2(t_1+bt_2) \\ \frac{a\alpha^2(r-bs)}{\alpha^2-b^2} & 0 & \frac{s\alpha^2-br}{\alpha^2-b^2} & 0 \\ 0 & r(bt_1+\alpha^2t_2)+s\alpha^2(t_1+bt_2) & 0 & a\alpha^2(r(t_1+bt_2)+s(bt_1+\alpha^2t_2)) \\ \frac{s\alpha^2-br}{\alpha^2-b^2} & 0 & \frac{r-bs}{a(\alpha^2-b^2)} & 0 \end{pmatrix}$$

which has as set of eigenvalues

$$\Big\{\pm(r+s\alpha)\gamma, \pm(r-s\alpha)\bar{\gamma}\Big\} = \Big\{\lambda, -\lambda, \mu, -\mu\Big\},$$

and the condition of Corollary 6.5 is satisfied. The matrix

$$\Psi_2 = \begin{pmatrix} (b+\alpha)\gamma & -(b+\alpha)\gamma & (b-\alpha)\bar{\gamma} & -(b-\alpha)\bar{\gamma} \\ a\alpha & a\alpha & -a\alpha & -a\alpha \\ a\alpha\gamma(b+\alpha) & -a\alpha\gamma(b+\alpha) & -a\alpha\bar{\gamma}(b-\alpha) & a\alpha\bar{\gamma}(b-\alpha) \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and the columns of Ψ_1 are the eigenvectors of $\lambda, -\lambda, \mu$ and $-\mu$ in that order, which means that they are of the form $u, \hat{u}, \bar{u}$ and $\hat{\bar{u}}$. Let c_1, c_2, c_3 and c_4 be the columns of S, which we want to be in F. We have that $\Psi_2 S = \Psi_1 D$, that is,

$$\Psi_2 \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix} = \begin{pmatrix} u & \hat{u} & \bar{u} & \hat{\bar{u}} \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix} = \begin{pmatrix} d_1 u & d_2\hat{u} & d_3\bar{u} & d_4\hat{\bar{u}} \end{pmatrix}$$

which expands to

$$d_1 u = (b+\alpha)\gamma c_1 + a\alpha c_2 + a\alpha\gamma(b+\alpha)c_3 + c_4;$$
$$d_2 \hat{u} = -(b+\alpha)\gamma c_1 + a\alpha c_2 - a\alpha\gamma(b+\alpha)c_3 + c_4;$$
$$d_3 \bar{u} = (b-\alpha)\bar{\gamma} c_1 - a\alpha c_2 - a\alpha\bar{\gamma}(b-\alpha)c_3 + c_4;$$
$$d_4 \hat{\bar{u}} = -(b-\alpha)\bar{\gamma} c_1 - a\alpha c_2 + a\alpha\bar{\gamma}(b-\alpha)c_3 + c_4;$$

that is

$$d_2\hat{u} = \widehat{d_1 u} \Rightarrow d_2 = \hat{d}_1;$$
$$d_3\bar{u} = \overline{d_1 u} \Rightarrow d_3 = \bar{d}_1;$$
$$d_4\hat{\bar{u}} = \widehat{d_3\bar{u}} \Rightarrow d_4 = \hat{d}_3 = \hat{\bar{d}}_1.$$

It only remains to apply the condition about D in Corollary 6.5. The result is

$$d_1 d_2 = d_1\hat{d}_1 = \frac{4a\alpha\gamma(b+\alpha)}{u^T\Omega_0\hat{u}};$$

$$d_3 d_4 = \bar{d}_1\hat{\bar{d}}_1 = \frac{-4a\alpha\bar{\gamma}(b-\alpha)}{\bar{u}^T\Omega_0\hat{\bar{u}}}.$$

Obviously, both conditions are equivalent. Putting $d_1 = 2(z_1 + z_2\gamma)$ with $z_1, z_2 \in F[\alpha]$, the first condition becomes

$$z_1^2 - z_2^2\gamma^2 = \frac{a\alpha\gamma(b+\alpha)}{u^T\Omega_0\hat{u}}$$

so this must be in $\mathrm{DSq}(F[\alpha], -\gamma^2)$. □

Proposition 6.24 reduces the problem of classifying this type of matrices to classifying (adequate) degree 4 extensions of F. In our case $F = \mathbb{Q}_p$.

The values of α^2 are given by

$$\mathbb{Q}_p^* / \operatorname{Sq}(\mathbb{Q}_p^*),$$

because two values whose quotient is a square are equivalent. The squares in $\mathbb{Q}_p$ are the numbers with even order and with a leading digit in $\operatorname{Sq}(\mathbb{F}_p)$, so the quotient is $\{1, c_0, p, c_0 p\}$, except if $p = 2$, where squares have even order and end in 001, and the quotient is $\{1, -1, 2, -2, 3, -3, 6, -6\}$. The possible values of α^2 are all except 1 (because $\alpha \notin \mathbb{Q}_p$). In each case, to find the normal form of M we still need to know γ, a and b.

In turn, the values of γ are given by the quotient

$$\mathbb{Q}_p[\alpha]^* / \operatorname{Sq}(\mathbb{Q}_p[\alpha]^*).$$

So we need to determine which numbers are squares in $\mathbb{Q}_p[\alpha]$.

LEMMA 6.25. *Let F be a field with characteristic different from 2. Let $F[\alpha]$ be a degree 2 extension of F. Let $a, b \in F$. Then $a + b\alpha$ is a square in $F[\alpha]$ if and only if $a^2 - b^2\alpha^2$ is a square in F and one of the numbers*

$$\frac{a \pm \sqrt{a^2 - b^2\alpha^2}}{2}$$

is also a square in F.

PROOF. Suppose that $a + b\alpha = (r + s\alpha)^2$. Then also $a - b\alpha = (r - s\alpha)^2$, multiplying $a^2 - b^2\alpha^2 = (r^2 - s^2\alpha^2)^2$, and finally

$$\frac{a \pm \sqrt{a^2 - b^2\alpha^2}}{2} = \frac{r^2 + s^2\alpha^2 + r^2 - s^2\alpha^2}{2} = r^2.$$

Reciprocally, if both numbers are squares, they give r and s such that $a + b\alpha = (r + s\alpha)^2$: we get r from the previous formula, and then s from $b = 2rs$. □

By Proposition 6.24, for fixed values of α and γ, two normal forms for (a, b) and (a', b') are equivalent if and only if the quotient between $a(b+\alpha)$ and $a'(b'+\alpha)$ is an element of $\operatorname{DSq}(\mathbb{Q}_p[\alpha], -\gamma^2)$. Hence, the normal forms for fixed α and γ are given by the classes in

$$\mathbb{Q}_p[\alpha]^* / \operatorname{DSq}(\mathbb{Q}_p[\alpha], -\gamma^2),$$

or equivalently in

$$(\mathbb{Q}_p[\alpha]^* / \operatorname{Sq}(\mathbb{Q}_p[\alpha]^*)) / \overline{\operatorname{DSq}}(\mathbb{Q}_p[\alpha], -\gamma^2).$$

That is, we have reduced the problem of determining subgroups of $\mathbb{Q}_p[\alpha]$ to determining the subgroup

$$\overline{\operatorname{DSq}}(\mathbb{Q}_p[\alpha], -\gamma^2),$$

which is easier because this is a subgroup of a finite group (and in all cases of interest, isomorphic to $\mathbb{F}_2^n$ for some n).

CHAPTER 7

The real Weierstrass-Williamson classification

In this chapter we give a new proof of the most general case of the Weierstrass-Williamson classification theorem using the new strategy introduced in the previous sections of this paper. In the simplest case, that is, for positive definite symmetric matrices, the proof reduces only to a few lines.

Invertible and without multiple eigenvalues
Possible blocks: $M_{\mathrm{h}}(1, r, 0), M_{\mathrm{e}}(1, r, 1)$ and $M_{\mathrm{ff}}(1, r, s)$, with $r, s \in \mathbb{R}, r, s \neq 0$
All blocks are different

Invertible and diagonalizable
Possible blocks: $M_{\mathrm{h}}(1, r, 0), M_{\mathrm{e}}(1, r, 1)$ and $M_{\mathrm{ff}}(1, r, s)$, with $r, s \in \mathbb{R}, r, s \neq 0$
Blocks may be repeated

Invertible
Possible blocks: $M_{\mathrm{h}}(k, r, 0), M_{\mathrm{e}}(k, r, a)$ and $M_{\mathrm{ff}}(k, r, s)$, with $k \in \mathbb{N}, a \in \{-1, 1\}, r, s \in \mathbb{R}, r, s \neq 0$
Blocks may be repeated

General
Possible blocks: $M_{\mathrm{h}}(k, r, 0), M_{\mathrm{h}}(k, 0, a), M_{\mathrm{e}}(k, r, a)$ and $M_{\mathrm{ff}}(k, r, s)$, with $k \in \mathbb{N}, a \in \{-1, 1\}, r, s \in \mathbb{R}$
Blocks may be repeated

FIGURE 1. Hierarchy of degeneracy levels of real matrices, according to the properties of the block decomposition of their normal forms (Theorems 7.2 and 7.3).

7.1. The general case

In the real case, as already explained after Proposition 6.23, in the case where the eigenvalues of $\Omega_0^{-1}M$ are different, the blocks up to dimension 4 are enough to classify the matrix. Actually, a weaker condition is sufficient: see Theorem 7.2. If the matrix is not diagonalizable or not invertible, the size of the blocks is not limited to 2 or 4, but instead can grow indefinitely: see Theorem 7.3. See Figure 1 for a hierarchy of properties of the decomposition.

The explicit form of the blocks is as follows:

DEFINITION 7.1. A *diagonal block of hyperbolic type* is any matrix of the form

$$M_{\mathrm{h}}(k,r,a) = \begin{pmatrix} & r & & & & & \\ r & & 1 & & & & \\ & 1 & & r & & & \\ & & r & & \ddots & & \\ & & & \ddots & & 1 & \\ & & & & 1 & & r \\ & & & & & r & a \end{pmatrix},$$

for some positive integer k, $r \in \mathbb{R}$ and $a \in \{-1, 0, 1\}$ with $a = 0$ if $r \neq 0$, and which has a total of $2k$ rows. A *diagonal block of elliptic type* is any matrix of the form

$$M_{\mathrm{e}}(k,r,a) = \begin{pmatrix} M_{\mathrm{e}1}(r) & M_{\mathrm{e}2}(1,a) & & & \\ M_{\mathrm{e}2}(1,a) & M_{\mathrm{e}1}(r) & M_{\mathrm{e}2}(2,a) & & \\ & M_{\mathrm{e}2}(2,a) & \ddots & & \\ & & & M_{\mathrm{e}1}(r) & M'_{\mathrm{e}2}(\ell,a) \\ & & & M'_{\mathrm{e}2}(\ell,a)^T & M'_{\mathrm{e}1}(r) \end{pmatrix}$$

if $k = 2\ell + 1$ is odd, and

$$M_{\mathrm{e}}(k,r,a) = \begin{pmatrix} M_{\mathrm{e}1}(r) & M_{\mathrm{e}2}(1,a) & & & \\ M_{\mathrm{e}2}(1,a) & M_{\mathrm{e}1}(r) & M_{\mathrm{e}2}(2,a) & & \\ & M_{\mathrm{e}2}(2,a) & \ddots & & \\ & & & M_{\mathrm{e}1}(r) & M_{\mathrm{e}2}(\ell-1,a) \\ & & & M_{\mathrm{e}2}(\ell-1,a) & M_{\mathrm{e}1}(r) + M_{\mathrm{e}2}(\ell,a) \end{pmatrix}$$

if $k = 2\ell$ is even, for some positive integer k, $r \in \mathbb{R}$ and $a \in \{-1, 1\}$, and which has a total of $2k$ rows. A *diagonal block of focus-focus type* is any matrix of the form

$$M_{\mathrm{ff}}(k,r,s) = \begin{pmatrix} M_{\mathrm{ff}1}(r,s) & M_{\mathrm{e}2}(1,1) & & & \\ M_{\mathrm{e}2}(1,1) & M_{\mathrm{ff}1}(r,s) & M_{\mathrm{e}2}(1,1) & & \\ & M_{\mathrm{e}2}(1,1) & \ddots & & \\ & & & M_{\mathrm{ff}1}(r,s) & M_{\mathrm{e}2}(1,1) \\ & & & M_{\mathrm{e}2}(1,1) & M_{\mathrm{ff}1}(r,s) \end{pmatrix},$$

for some positive integer k and $r, s \in \mathbb{R}$, and which has a total of $4k$ rows. In the previous blocks the following sub-blocks are used:

$$M_{\mathrm{e}1}(r) = \begin{pmatrix} 0 & 0 & 0 & r \\ 0 & 0 & -r & 0 \\ 0 & -r & 0 & 0 \\ r & 0 & 0 & 0 \end{pmatrix}, M'_{\mathrm{e}1}(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, M_{\mathrm{ff}1}(r,s) = \begin{pmatrix} 0 & s & 0 & r \\ s & 0 & -r & 0 \\ 0 & -r & 0 & s \\ r & 0 & s & 0 \end{pmatrix},$$

$$M_{\mathrm{e}2}(j,a)=\begin{pmatrix}a&0&0&0\\0&0&0&0\\0&0&a&0\\0&0&0&0\end{pmatrix}\text{ if } j \text{ is odd, }\begin{pmatrix}0&0&0&0\\0&a&0&0\\0&0&0&0\\0&0&0&a\end{pmatrix}\text{ if } j \text{ is even,}$$

$$M'_{\mathrm{e}2}(j,a)=\begin{pmatrix}a&0\\0&0\\0&a\\0&0\end{pmatrix}\text{ if } j \text{ is odd, }\begin{pmatrix}0&0\\a&0\\0&0\\0&a\end{pmatrix}\text{ if } j \text{ is even.}$$

THEOREM 7.2. *Let n be a positive integer. Let Ω_0 be the matrix of the standard symplectic form in $\mathbb{R}^{2n}$. Let $M \in \mathcal{M}_{2n}(\mathbb{R})$ be a symmetric and invertible matrix such that $\Omega_0^{-1}M$ is diagonalizable. Then, there exists a symplectic matrix $S \in \mathcal{M}_{2n}(\mathbb{R})$ such that S^TMS is a block-diagonal matrix with blocks of hyperbolic, elliptic type or focus-focus type with $k = 1$.*

PROOF. Applying Lemma 6.6, we can take a symplectic basis $\{u_1, v_1, \dots, u_n, v_n\}$ of $\mathbb{R}^{2n}$ such that all the vectors in the basis are eigenvectors of A, and u_i and v_i have opposite eigenvalues λ_i and $-\lambda_i$. We can sort these vectors in such a way that two λ_i's which are conjugate appear with consecutive indices.

Taking as Ψ_1 the matrix with these vectors as columns, the problem decomposes into finding normal forms for each block of columns associated to eigenvalues of the form $\{r, -r\}$, $\{\mathrm{i}r, -\mathrm{i}r\}$ or

$$\{r + \mathrm{i}s, -r - \mathrm{i}s, r - \mathrm{i}s, -r + \mathrm{i}s\},$$

for $r, s \in \mathbb{R}^*$. The first block, by Proposition 6.17, gives the hyperbolic block. The second block, by Proposition 6.22 with $a = b = r$ or $a = b = -r$ (one of them will always work), gives the elliptic block. The third block, by Proposition 6.23, gives the focus-focus block. □

THEOREM 7.3. *Let n be a positive integer and let $M \in \mathcal{M}_{2n}(\mathbb{R})$ be a symmetric matrix. Then, there exists a symplectic matrix $S \in \mathcal{M}_{2n}(\mathbb{R})$ such that S^TMS is a block diagonal matrix with each of the diagonal blocks being of hyperbolic, elliptic or focus-focus type, as in Definition 7.1.*

Furthermore, if there are two matrices S and S' such that $N = S^TMS$ and $N' = S'^TMS'$ are normal forms, then $N = N'$ except by the order of the blocks.

PROOF. (a) First we prove existence. The proof starts as in Theorems 6.16 and 7.2, applying Lemma 6.6. This gives us a partial symplectic basis $\{u_1, v_1, \dots, u_m, v_m\}$. However, unlike Theorem 6.16, these vectors only work for the hyperbolic blocks, giving $M_{\mathrm{h}}(k, r, 0)$; for the rest, the blocks would not be real, so we need to recombine the vectors. We can sort the blocks of the Jordan form in such a way that the blocks of two λ_i's which are conjugate appear with consecutive indices.

For an elliptic block, $\{u_1, v_1, \dots, u_k, v_k\}$ are the vectors corresponding to the values $\mathrm{i}r$ and $-\mathrm{i}r$, for $r \in \mathbb{R}$. We see that $M_{\mathrm{e}}(k, r, 1)$ and $M_{\mathrm{e}}(k, r, -1)$ have this block as Jordan form, so we can apply Proposition 6.2. The columns of Ψ_1 are $\{u_1, v_1, \dots, u_k, v_k\}$, which are part of a symplectic basis, and u_1 and v_k are eigenvectors with values $\mathrm{i}r$ and $-\mathrm{i}r$, so $v_k = c\bar{u}_1$ for some $c \in \mathbb{C}$. Using that $Au_j = \mathrm{i}ru_j + u_{j-1}$ and

$Av_j = -\mathrm{i}rv_j - v_{j+1}$, we deduce that $v_{k+1-j} = (-1)^{j-1}c\bar{u}_j$. Concretely,

$$\begin{aligned} cu_1^T\Omega_0\bar{u}_k &= u_1^T\Omega_0 v_1 \\ &= 1 \\ &= \overline{u_k^T\Omega_0 v_k} \\ &= \overline{(-1)^{k-1}cu_k^T\Omega_0\bar{u}_1} \\ &= (-1)^{k-1}\bar{c}\bar{u}_k^T\Omega_0 u_1 \end{aligned}$$

which implies $c = (-1)^k\bar{c}$, that is, c is real if k is even and imaginary if k is odd.

The columns of Ψ_2 have two nonzero entries, of the form

$$(\ldots, \pm 1, 0, \pm\mathrm{i}, \ldots)$$

or

$$(\ldots, \pm a, 0, \pm\mathrm{i}a, \ldots),$$

except the two central ones if k is odd, which are $(\ldots, 1, \mathrm{i})$ or a similar form. In any case, we have $u_j'^T\Omega_0 v_j' = 2a$ if k is even, and $2\mathrm{i}a$ if k is odd.

We can take as D a diagonal matrix with the entries alternating between d_1 and d_2, that commutes with J. The condition

$$D^T\Omega_0 D = D^T\Psi_1^T\Omega_0\Psi_1 D = \Psi_2\Omega_0\Psi_2$$

implies that $d_1d_2 = 2a$ for k even, and $2\mathrm{i}a$ for k odd. As S must be a real matrix, in $S\Psi_2 = \Psi_1 D$ the first and last columns are conjugate and we get $\bar{d}_1 = cd_2$, that is, $d_1\bar{d}_1 = 2ac$ for k even and $2\mathrm{i}ac$ for k odd. We just need to take $a \in \{1, -1\}$ so that this is positive: note that a is unique.

For the focus-focus case, we have in the symplectic basis a block of vectors

$$\Big\{u_1, v_1, \ldots, u_k, v_k, u_1', v_1', \ldots, u_k', v_k'\Big\},$$

where u_1, v_k, u_1' and v_k' are eigenvectors for $\lambda, -\lambda, \bar{\lambda}$ and $-\bar{\lambda}$ respectively. If $\lambda = s + \mathrm{i}r$, the matrix $M_{\mathrm{f}}(k, r, s)$ has the same Jordan form.

The columns of Ψ_1 are the vectors u_i, v_i, u_i' and v_i'; we have that $u_1' = c\bar{u}_1$ for some $c \in \mathbb{C}$, which implies $u_i' = c\bar{u}_i$ for all i and, from $u_i^T\Omega_0 v_i = u_i'^T\Omega_0 v_i' = 1$, we deduce $v_i' = \bar{v}_i/c$ for all i. The columns of Ψ_2 have now the form $(0, 1, 0, \mathrm{i}, \ldots)$, $(\mathrm{i}, 0, 1, 0, \ldots)$, $(\ldots, 0, 1, 0, \mathrm{i}, \ldots)$, and so on, for a total of $2k$, followed by their conjugates.

We can take as D a diagonal matrix with the values

$$d_1, d_2, \ldots, d_1, d_2, d_3, d_4, \ldots, d_3, d_4,$$

which commutes with J. The condition

$$D^T\Omega_0 D = D^T\Psi_1^T\Omega_0\Psi_1 D = \Psi_2\Omega_0\Psi_2$$

implies that $d_1d_2 = -2\mathrm{i}$ and $d_3d_4 = 2\mathrm{i}$. Using that S is a real matrix, the left and right halves of $S\Psi_2 = \Psi_1 D$ are conjugate, which implies $\bar{d}_1 = cd_3$ and $c\bar{d}_2 = d_4$, so the condition $d_3d_4 = 2\mathrm{i}$ reduces to a consequence of $d_1d_2 = -2\mathrm{i}$, and we can take for example $d_1 = 1$ and $d_2 = -2\mathrm{i}$.

This finishes the treatment of the nonzero eigenspaces. For the other part, we can use the same treatment as in Theorem 6.16, but in the even case we cannot always make $c_i = 1$; instead we make $c_i = a_i$. Now

$$\left\{u_{i1}, a_i u_{i,2\ell_i}, -u_{i2}, a_i u_{i,2\ell_i-1}, \dots, (-1)^{\ell_i-1} u_{i\ell_i}, a_i u_{i,\ell_i+1}\right\}$$

is a partial symplectic basis which gives the form $M_{\mathrm{h}}(\ell_i, 0, (-1)^{\ell_i} a_i)$.

(b) Finally we prove uniqueness. If N and N' are two normal forms which are equivalent, by Proposition 6.2, they have the same Jordan form. This means that the set of blocks is the same except for the values of a_i. But in the elliptic case we already saw that a is unique. This only leaves the hyperbolic case with $r = 0$.

In this case, if a can be 1 and -1 at the same time, there is a chain

$$\left\{u_1, u_2, \dots, u_{2k}\right\}$$

such that $Au_i = u_{i-1}$, $Au_1 = 0$, and $u_i \Omega_0 u_{2k+1-i} = (-1)^i$, and another one

$$\left\{u'_1, u'_2, \dots, u'_{2k}\right\}$$

with the same properties except that

$$u'_i \Omega_0 u'_{2k+1-i} = (-1)^{i+1}.$$

In the space generated by these vectors there is only one vector in the kernel, so $u'_1 = ku_1$ for some $k \in \mathbb{R}$. As $Au_i = u_{i-1}$ and $Au'_i = u'_{i-1}$, we have that $u'_i = ku_i$ for all i. This together implies that

$$(-1)^{i+1} = u'_i \Omega_0 u'_{2k+1-i} = k^2 u_i \Omega_0 u_{2k+1-i} = k^2 (-1)^i$$

and $k^2 = -1$, a contradiction. □

The matrix Ψ_1 in the previous proof gives a complex symplectic basis in which M has the block diagonal form of Theorem 6.16. The relation between this and the final matrix S can be written in terms of vectors. In the hyperbolic case, they are the same matrix. In the elliptic case, we first multiply the vectors by the corresponding d_j: $u_j := \sqrt{|2c|} u_j$, $v_j := \sqrt{2/|c|} v_j$. Then, the matrix Ψ_2 indicates how the vectors in the final basis relate to these u_j and v_j: each column has a ± 1 entry, a $\pm \mathrm{i}$ entry and the rest are 0, so we have $\pm u'_h + \pm \mathrm{i} u'_\ell = u_j$, for some indices h and ℓ. As the new vectors u'_h and u'_ℓ must be real, this has a unique solution, and the new vectors are the real and imaginary parts of the old ones (maybe with the sign changed).

In the focus-focus case, we also start multiplying the vectors by the d_j:

$$v_j := -2\mathrm{i} v_j, u'_j := u'_j / c = \bar{u}_j, v'_j := 2\mathrm{i} c v'_j = \bar{v}_j.$$

Then we apply Ψ_2^{-1}: now each column of the left half of Ψ_2 has a 1 and an i, and the right half has a 1 and a $-\mathrm{i}$ in the same positions. For every j with $1 \le j \le k$, the equations are $v''_{2j-1} + \mathrm{i} v''_{2j} = u_j$ and $\mathrm{i} u''_{2j-1} + u''_{2j} = v_j$ for the first half and their conjugates for the second half, where u''_j and v''_j are the new vectors. The solution consists of taking the real and imaginary parts of the new vectors.

COROLLARY 7.4. *Let $M \in \mathcal{M}_4(\mathbb{R})$ be a symmetric matrix. Then there exists $r, s \in \mathbb{R}$, $a, b \in \{-1, 1\}$, and a symplectic matrix $S \in \mathcal{M}_4(\mathbb{R})$ such that $S^T M S$ is one of the following ten matrices:*

$$\begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & s & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & s & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

$$\begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 1 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & r & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & r & 0 & 0 \\ r & 0 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & r \\ 0 & 0 & -r & 0 \\ 0 & -r & a & 0 \\ r & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & s & 0 & r \\ s & 0 & -r & 0 \\ 0 & -r & 0 & s \\ r & 0 & s & 0 \end{pmatrix},$$

Furthermore, if there are two matrices of this form equivalent to M, they are both in the first or in the eighth case swapping r and s, or in the third case swapping a and b.

PROOF. They are in this order: two hyperbolic blocks with $k = 1$ and $a = 0$, one with $a = 0$ and one with $a \neq 0$, two with $a \neq 0$, one hyperbolic block with $k = 2$ and $a = 0$, the same with $a \neq 0$, one hyperbolic with $a = 0$ and one elliptic, the same with $a \neq 0$, two elliptic blocks with $k = 1$, one elliptic block with $k = 2$, and one focus-focus block. □

These are the same (up to symplectic transformations) that Williamson gives in his paper [**142**, page 24] and which we gave in Section 5.3.2.

7.2. Example of application of our method for matrices of arbitrary order

Consider the matrix

$$M = \begin{pmatrix} 1 & & & & & & & \\ & & 1 & & & & & \\ & 1 & & & & & & \\ & & & & 1 & & & \\ & & & 1 & & \ddots & & \\ & & & & \ddots & & 1 & \\ & & & & & 1 & & \\ & & & & & & & 1 \end{pmatrix}$$

We have that

$$
\Omega_0^{-1}M = \begin{pmatrix}
 & & -1 & & & & & \\
1 & & & & & & & \\
 & & & & -1 & & & \\
 & 1 & & & & \ddots & & \\
 & & & & & & -1 & \\
 & & & 1 & & & & \\
 & & & & \ddots & & & -1 \\
 & & & & & 1 & &
\end{pmatrix}
$$

whose characteristic polynomial is $\lambda^{2n} + (-1)^n$, and the eigenvalues are $\lambda = \mathrm{e}^{\pi \mathrm{i} k/n}$ for $0 \le k \le 2n-1$, if n is odd, and $\lambda = \mathrm{e}^{\pi \mathrm{i}(2k+1)/2n}$ for $0 \le k \le 2n-1$, if n is even.

If $n = 2m+1$ is odd, the normal form of this matrix contains a hyperbolic block, which corresponds to the eigenvalues $\{1, -1\}$, and m focus-focus blocks, for the eigenvalues

$$
\left\{ \mathrm{e}^{\pi \mathrm{i} k/n}, \mathrm{e}^{\pi \mathrm{i}(n-k)/n}, \mathrm{e}^{\pi \mathrm{i}(n+k)/n}, \mathrm{e}^{\pi \mathrm{i}(2n-k)/n} \right\},
$$

for $1 \le k \le m$:

$$
S^T M S = \begin{pmatrix}
0 & 1 & & & \\
1 & 0 & & & \\
 & & N(1) & & \\
 & & & \ddots & \\
 & & & & N(m)
\end{pmatrix}, N(k) = \begin{pmatrix}
0 & \cos\frac{\pi k}{n} & 0 & \sin\frac{\pi k}{n} \\
\cos\frac{\pi k}{n} & 0 & -\sin\frac{\pi k}{n} & 0 \\
0 & -\sin\frac{\pi k}{n} & 0 & \cos\frac{\pi k}{n} \\
\sin\frac{\pi k}{n} & 0 & \cos\frac{\pi k}{n} & 0
\end{pmatrix}.
$$

On the other hand, if $n = 2m$ is even, the normal form contains only m focus-focus blocks $N(k+1/2)$, for $0 \le k \le m-1$. In this case, if we change any 1 entry in M to -1, the normal form contains again the hyperbolic block, the focus-focus blocks $N(k)$ for $1 \le k \le m-1$ and also an elliptic block, corresponding to the eigenvalues i and $-$i:

$$
S^T M S = \begin{pmatrix}
0 & 1 & & & & & \\
1 & 0 & & & & & \\
 & & 1 & 0 & & & \\
 & & 0 & 1 & & & \\
 & & & & N(1) & & \\
 & & & & & \ddots & \\
 & & & & & & N(m-1)
\end{pmatrix}.
$$

7.3. Positive-definite case

If the matrix is positive-definite, only the elliptic case may appear: this happens because, if u is an eigenvector of A with value λ,

$$
\bar{u}^T M u = \bar{u}^T \Omega_0 A u = \lambda \bar{u}^T \Omega_0 u
$$

The left-hand side is a positive real, and for $\bar{u}^T \Omega_0 u$ we have

$$
\overline{\bar{u}^T \Omega_0 u} = u^T \Omega_0 \bar{u} = -\bar{u}^T \Omega_0 u,
$$

so it is imaginary, and λ must be imaginary.

Also, it is impossible to obtain for these matrices the elliptic blocks with size greater than 2: if such a block appeared, there would be an element in the diagonal of $S^T M S$ equal to 0. Let k be its index and u the k-th column of S. Then $u^T M u = 0$, which contradicts M being positive definite. This means that any positive definite symmetric matrix can be diagonalized by a symplectic matrix, which is the result most often referred to as “Williamson theorem”.

CHAPTER 8

The p-adic Weierstrass-Williamson classification

8.1. Dimension 2

In this section we prove Theorems 5.28 and 5.30, that is, the p-adic version of the Weierstrass-Williamson matrix classification in dimension 2. The strategy consists of lifting the problem, when needed, to an extension field of order 2, and then using the theorems in Chapter 6 to go back to the p-adic field.

8.1.1. Preparatory results. In the p-adic case, Propositions 6.17, 6.18 and 6.22 are still enough to achieve a complete Weierstrass-Williamson classification in dimension 2, but it is more complicated than the real case. First we need to compute $\mathrm{DSq}(\mathbb{Q}_p, c)$ for all values of c. Of course, "all values" means "all classes modulo $\mathrm{Sq}(\mathbb{Q}_p^*)$": by Corollary 5.5, this quotient is

$$\Big\{1, c_0, p, c_0 p\Big\},$$

where c_0 is a quadratic non-residue modulo p, except if $p = 2$, in which case the quotient is

$$\Big\{1, -1, 2, -2, 3, -3, 6, -6\Big\}.$$

We use the notation $\mathrm{digit}_i(x)$ for the digit in the p-adic expansion of x which is i positions to the left of the leading digit, that is, the digit of order $\mathrm{ord}(x) + i$. The value of $\mathrm{DSq}(\mathbb{Q}_p, c)$ can also be deduced from known facts about the Hilbert symbol; however, this does not seem simpler than a direct proof.

Proposition 8.1. *Let p be a prime number such that $p \neq 2$ and $c \in \mathbb{Q}_p^*$. Then $\mathrm{DSq}(\mathbb{Q}_p, c)$ is given as follows (see Figure 1):*

1. *If $p \equiv 3 \mod 4$, then $\mathrm{DSq}(\mathbb{Q}_p, 1) = \{u \in \mathbb{Q}_p : \mathrm{ord}_p(u) \equiv 0 \mod 2\}$ and $\mathrm{DSq}(\mathbb{Q}_p, 1) = \mathbb{Q}_p$ otherwise;*
2. *if $p \equiv 1 \mod 4$, then $\mathrm{DSq}(\mathbb{Q}_p, c_0) = \{u \in \mathbb{Q}_p : \mathrm{ord}_p(u) \equiv 0 \mod 2\}$ and $\mathrm{DSq}(\mathbb{Q}_p, c_0) = \mathbb{Q}_p$ otherwise;*
3. *for any value of the prime p, $\mathrm{DSq}(\mathbb{Q}_p, p) = \{u \in \mathbb{Q}_p : \mathrm{digit}_0(u) \in \mathrm{Sq}(\mathbb{F}_p^*)\}$;*
4. *for any value of the prime p, $\mathrm{DSq}(\mathbb{Q}_p, c_0 p) = \{u \in \mathbb{Q}_p : \mathrm{ord}_p(u) \equiv 0 \mod 2, \mathrm{digit}_0(u) \in \mathrm{Sq}(\mathbb{F}_p^*)\} \cup \{u \in \mathbb{Q}_p : \mathrm{ord}_p(u) \not\equiv 0 \mod 2, \mathrm{digit}_0(u) \notin \mathrm{Sq}(\mathbb{F}_p^*)\}$.*

Proof. For the first point, we need to look at the possibilities modulo $\mathrm{Sq}(\mathbb{Q}_p^*)$ of numbers of the form x^2+1. We can immediately get 1 and c_0, the first by taking x with high order and the second by taking it with order 0 and an adequate leading digit. We can only get p and $c_0 p$ if x^2 can have -1 as a leading digit, which only happens if $p \equiv 1 \mod 4$.

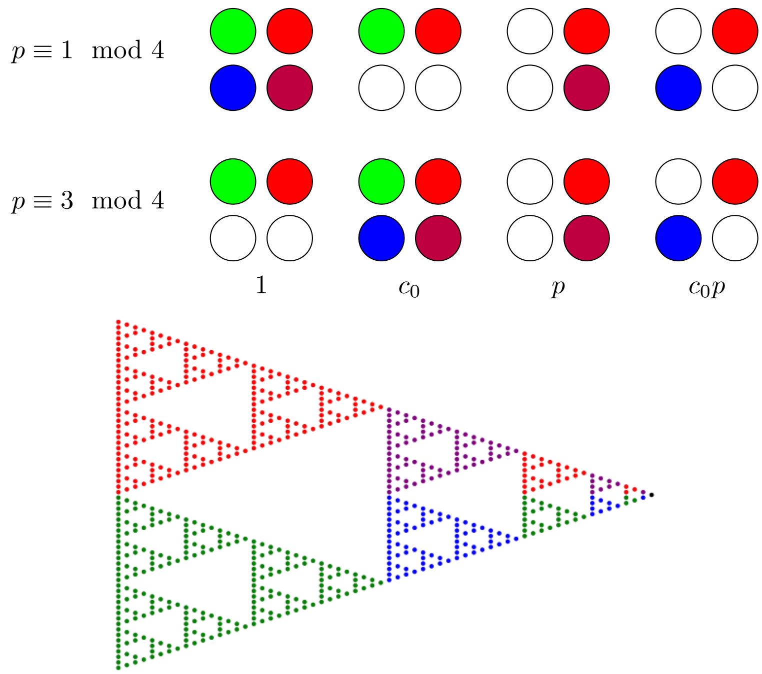


FIGURE 1. Top: $\mathrm{DSq}(\mathbb{Q}_p, c)$ for $c \in \mathbb{Q}_p$ and $p \neq 2$. In each group of four circles, the upper circles represent even order numbers and the lower circles odd order, and the right circles represent square leading digits and the left circles non-square digits. Bottom: these four classes depicted for $p = 3$. Each circle "contains" the points with the same color and the black point at the right is 0.

For the second point, we need, analogously, to look at numbers of the form $x^2 + c_0$. We immediately get the classes 1 and c_0. We can only get p and $c_0 p$ if $-c_0$ is a square modulo p, that is, if $p \equiv 3 \mod 4$.

For the last two points, numbers of the form $x^2 + p$ are in the classes 1 and p, depending only in the order of x, and those of the form $x^2 + c_0 p$ are in 1 and $c_0 p$. $\square$

PROPOSITION 8.2. $\mathrm{DSq}(\mathbb{Q}_2, c)$ *is given as follows (see Figure 2):*

(1) $\mathrm{DSq}(\mathbb{Q}_2, 1) = \{u \in \mathbb{Q}_2 : \mathrm{digit}_1(u) = 0\}$;
(2) $\mathrm{DSq}(\mathbb{Q}_2, -1) = \mathbb{Q}_2$;
(3) $\mathrm{DSq}(\mathbb{Q}_2, 2) = \{u \in \mathbb{Q}_2 : \mathrm{digit}_2(u) = 0\}$;
(4) $\mathrm{DSq}(\mathbb{Q}_2, -2) = \{u \in \mathbb{Q}_2 : \mathrm{digit}_1(u) = \mathrm{digit}_2(u)\}$;
(5) $\mathrm{DSq}(\mathbb{Q}_2, 3) = \{u \in \mathbb{Q}_2 : \mathrm{ord}_2(u) \equiv 0 \mod 2\}$;
(6) $\mathrm{DSq}(\mathbb{Q}_2, -3) = \{u \in \mathbb{Q}_2 : \mathrm{ord}_2(u) + \mathrm{digit}_1(u) \equiv 0 \mod 2\}$;
(7) $\mathrm{DSq}(\mathbb{Q}_2, 6) = \{u \in \mathbb{Q}_2 : \mathrm{ord}_2(u) + \mathrm{digit}_1(u) + \mathrm{digit}_2(u) \equiv 0 \mod 2\}$;
(8) $\mathrm{DSq}(\mathbb{Q}_2, -6) = \{u \in \mathbb{Q}_2 : \mathrm{ord}_2(u) + \mathrm{digit}_2(u) \equiv 0 \mod 2\}$.

PROOF. Table 1 indicates the leading digits and the parity of the order of $a^2 + cb^2$ depending on c and the difference $\mathrm{ord}_2(b) - \mathrm{ord}_2(a)$. The result follows by

c	cb^2	$\text{ord}_2(b) - \text{ord}_2(a)$				
		≤ -2	-1	0	1	≥ 2
1	001 even	001 even	101 even	01 odd	101 even	001 even
-1	111 even	111 even	011 even	anything	101 even	001 even
2	001 odd	001 odd	011 odd	011 even	001 even	001 even
-2	111 odd	111 odd	001 odd	111 even	001 even	001 even
3	011 even	011 even	111 even	1 even	101 even	001 even
-3	101 even	101 even	001 even	11 odd	101 even	001 even
6	011 odd	011 odd	101 odd	111 even	001 even	001 even
-6	101 odd	101 odd	111 odd	011 even	001 even	001 even

TABLE 1. Leading digits and parity of the order of $a^2 + cb^2$ depending on c and the difference $\text{ord}_2(b) - \text{ord}_2(a)$. The number a^2 is always described as 001 even, cb^2 depends exclusively on c, and the result of the addition of both terms depends in the offset between these digits. Note that the leading 1's will add up to 0 if the offset is 0, hence making the second digit the leading one in the cases "01 odd" and "11 odd" (in these cases the order increases in 1), the third in the case "1 even" (the order increases in 2), and giving any possible result when adding 001 and 111 at the same position.

putting together the cases in the same row of the table. Note that a case such as "011 even" covers *all* 2-adic numbers with even order and ending in 011. □

8.1.2. Proof of Theorems 5.28 and 5.30. First, note that Theorem 5.30 follows directly from Theorem 5.28: the isolated normal forms correspond to the different values of r for $c = 0$, and each family of normal forms corresponds to a value of $c \in X_p$ with $c \neq 0$. Now we prove Theorem 5.28.

(a) First we prove existence. The characteristic polynomial of $\Omega_0^{-1}M$ has two opposite roots, which may or may not be in $\mathbb{Q}_p$. If they are in $\mathbb{Q}_p$ and are not 0, Proposition 6.17 implies that M can be converted to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

except by a constant factor. But the matrix

$$N = \begin{pmatrix} r & 0 \\ 0 & cr \end{pmatrix}$$

in Theorem 5.28, where $c = 1$ if $p \equiv 1 \mod 4$ and $c = -1$ otherwise, has also two eigenvalues in $\mathbb{Q}_p$, so it can be converted to the same matrix.

If the eigenvalues are 0, either the matrix is zero, and we are in the same situation but with $r = 0$, or they are not zero, and Proposition 6.18 gives the same matrix but with $c = 0$. In this case r must be such that ar is a square, where a is one of the coefficients. There is one and only one $r \in Y_p \cup \{1\}$ such that this happens.

Now suppose that the roots of the characteristic polynomial of $\Omega_0^{-1}M$ are $\pm\lambda$ for $\lambda \notin \mathbb{Q}_p$. In this case, we must have $\lambda^2 \in \mathbb{Q}_p$. We have $N = S^T MS$ for a symplectic matrix S and some r and c if the two

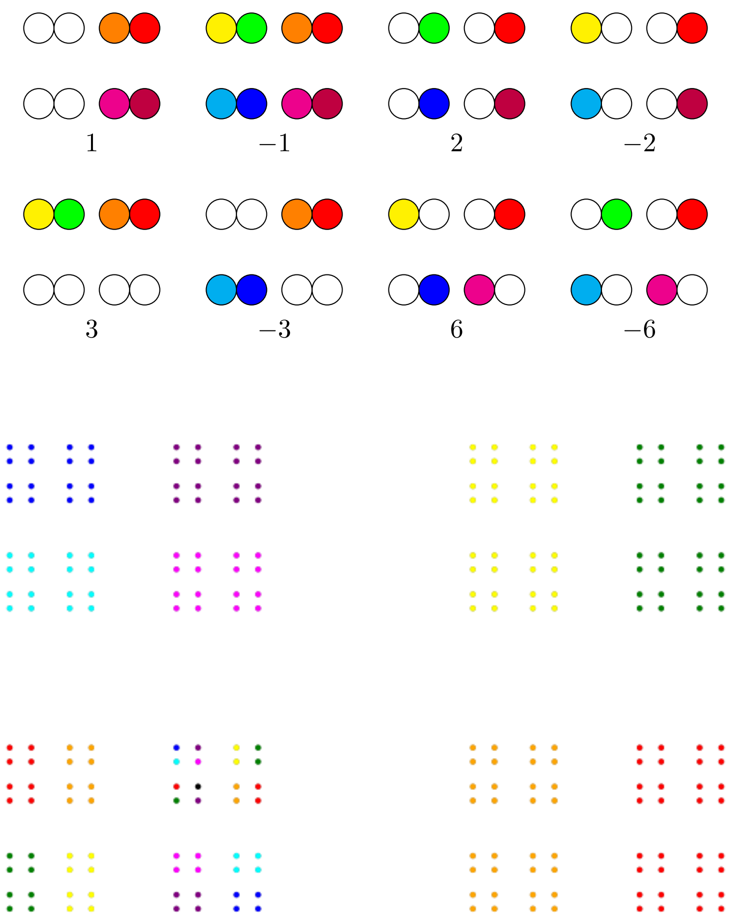


FIGURE 2. Top: $\mathrm{DSq}(\mathbb{Q}_2, c)$ for $c \in \mathbb{Q}_2$. In each group of eight circles, the upper circles represent even order numbers and the lower circles odd order, the two rightmost circles in the row represent a 0 as second digit and the two leftmost circles a 1, and in each pair of circles, the rightmost one has 0 as third digit and the leftmost one has 1. Bottom: a depiction of the eight classes. Each circle "contains" the points with the same color and the black point in the lower left is 0.

conditions of Proposition 6.22 hold for some a and $b = ac$. The first condition reads

$$\lambda^2 = -a^2 c \Rightarrow a^2 = -\frac{\lambda^2}{c}.$$

We must now split the proof into several cases.

- λ^2 has even order. In order for $-\lambda^2/c$ to be a square, we need c of even order. We also know that λ^2 is not a square, so $-c$ must not be a square. The elements of X_p which satisfy these conditions are

$\{c_0, c_0p^2\}$ if $p \equiv 1 \mod 4$, $\{1, p^2\}$ if $p \equiv 3 \mod 4$, and $\{1, 3, -3, 12\}$ if $p = 2$.

For $p = 2$, we also need that $-\lambda^2/c$ ends in 001, so the three last digits in c must agree with those in $-\lambda^2$, which will be 001, 101 or 011 (not 111, which would make λ^2 a square). This narrows further down the options to $\{1\}$, $\{-3\}$ or $\{3, 12\}$, respectively.

Let C be the current set of options for c, which contains only the singleton $\{c_1\}$ or the set of two values $\{c_1, c_1p^2\}$. All of them satisfy that $-\lambda^2/c$ is a square. We still need to apply the second condition, that is, we need to choose a such that

$$\frac{2\lambda a}{u^T\Omega_0\bar{u}} \in \mathrm{DSq}(\mathbb{Q}_p, -\lambda^2) = \mathrm{DSq}(\mathbb{Q}_p, c_1).$$

Let

$$a_1 = \sqrt{\frac{-\lambda^2}{c_1}}.$$

In the two cases where $C = \{c_1\}$, by Proposition 8.2(1) and (6), for any $x \in \mathbb{Q}_p$, either x or $-x$ is in $\mathrm{DSq}(\mathbb{Q}_p, c_1)$. So, either $a = a_1$ or $a = -a_1$ satisfies the condition, and $c = c_1$ is valid.

If

$$C = \{c_1, c_1p^2\},$$

by Proposition 8.1(1) and (2) and Proposition 8.2(5),

$$\mathrm{DSq}(\mathbb{Q}_p, c_1) = \Big\{u \in \mathbb{Q}_p : \mathrm{ord}_p(u) \equiv 0 \mod 2\Big\}.$$

This implies that either $a = a_1$ or $a = a_1/p$ satisfies the condition. Hence, only one between c_1 and c_1p^2 is a valid value of c.

- λ^2 has odd order. Now we need c to have odd order instead of even. What happens next depends on p.
  - If $p \equiv 1 \mod 4$, the values of c with odd order are $c_0^k p$ for $k = 0, 1, 2, 3$. The first condition implies that $-\lambda^2/c_0^k p$ is a square, which is true for $k = 0$ and 2 or for $k = 1$ or 3, depending on the leading digit of $-\lambda^2/p$. Let c_1 be the value which satisfies this between p and c_0p. Now the two candidates for c are c_1 and $c_1c_0^2$.
    We define again $a_1 = \sqrt{-\lambda^2/c_1}$. In this case, by Proposition 8.1(3) and (4), for any $x \in \mathbb{Q}_p$, either x or c_0x is in $\mathrm{DSq}(\mathbb{Q}_p, c_1)$. Hence, either $a = a_1$ or $a = a_1/c_0$ satisfies the condition, and either $c = c_1$ or $c = c_1c_0^2$ is valid.
  - If $p \equiv 3 \mod 4$, the values of c with odd order are p and $-p$. As -1 is not a square, only one will make $-\lambda^2/c$ a square. For this value of c, we set $a_1 = \sqrt{-\lambda^2/c}$. By Proposition 8.1(3) and (4), for any $x \in \mathbb{Q}_p$, either x or $-x$ is in $\mathrm{DSq}(\mathbb{Q}_p, c_1)$. Hence, either $a = a_1$ or $a = -a_1$ is valid, and c is valid in any case.
  - If $p = 2$, the values of c with odd order are 2, -2, 6, -6, -18 and 24. $-\lambda^2/c$ must end in 001, so c must agree with $-\lambda^2$ in the last three digits, that in this case can have all possible

values: 001, 101, 011 and 111. The valid c's in each case are, respectively,

$$\{2\}, \{-6\}, \{6, 24\}, \{-2, -18\}.$$

Let C be this set, c_1 the element of less absolute value (in the real sense) in C, and $a_1 = \sqrt{-\lambda^2/c_1}$.
If C has only one element, by Proposition 8.2(3) and (8), for any $x \in \mathbb{Q}_p$, either x or $-x$ is in $\mathrm{DSq}(\mathbb{Q}_p, c_1)$, and again c_1 is valid in any case.
If $C = \{-2, -18\}$, by Proposition 8.2(4), for any $x \in \mathbb{Q}_p$, either x or $3x$ is in $\mathrm{DSq}(\mathbb{Q}_p, -2)$ (because $\mathrm{digit}_1(x) = 1 - \mathrm{digit}_1(3x)$ and $\mathrm{digit}_2(x) = \mathrm{digit}_2(3x)$). Hence, either $a = a_1$ or $a = a_1/3$ works and $c = -2$ or $c = -18$, respectively, works.
Finally, if $C = \{6, 24\}$, by Proposition 8.2(7), for any $x \in \mathbb{Q}_p$, either x or $2x$ is in $\mathrm{DSq}(\mathbb{Q}_p, 6)$. So either $a = a_1$ or $a = a_1/2$ works and $c = 6$ or $c = 24$, respectively, works.

(b) Now we prove uniqueness. In the case where the roots of the characteristic polynomial are in $\mathbb{Q}_p$, the rest of c's in the lists do not lead to a matrix with the eigenvalues in $\mathbb{Q}_p$, because their opposites are not squares. In all the other cases, we have seen that there is one and only one valid value of c. If two matrices S and S' bring M to normal form, c must coincide, hence r also coincides because the eigenvalues of the normal forms must be the same.

Recall that, in the real case, there are two normal forms: elliptic and hyperbolic. To give them the form

$$\begin{pmatrix} r & 0 \\ 0 & cr \end{pmatrix},$$

we need $c = 1$ and $c = -1$, respectively. In the p-adic case, these two matrices are equivalent by multiplication by a symplectic matrix if and only if $p \equiv 1 \mod 4$: the list for this case is the only one that does not contain -1.

PROPOSITION 8.3. *All choices of quadratic residue in Definition 5.18 (including the least of all, which is used in the definition) lead to the same normal form in Theorem 5.28, up to multiplication by a symplectic matrix. That is, if c_0, c_0' are two quadratic residues modulo p then the normal forms corresponding to c_0 and the normal forms corresponding to c_0' are equivalent by multiplication by a symplectic matrix. (The order of the forms, however, may vary: for example, taking $c_0' \equiv c_0^3 \mod p$ results in the new form with $c = c_0'p$ being taken as the one which had previously $c = c_0^3 p$.)*

PROOF. By Theorem 5.28, the normal forms of matrices in the first set are equivalent to one and only one normal form in the second set, so the two sets are different representatives of the same classes. □

8.2. Dimension 4

We will now provide the p-adic classification of 4-by-4 matrices. Some cases are a consequence of the previous results: the characteristic polynomial of $\Omega_0^{-1}M$ has the form

$$At^4 + Bt^2 + C$$

and the roots are λ, $-\lambda$, μ and $-\mu$. If λ^2 is in $\mathbb{Q}_p$, then μ^2 is also in $\mathbb{Q}_p$. If $\lambda \neq \mu$, we can multiply M by a symplectic matrix to separate it into two components, one with the eigenvalues λ and $-\lambda$, and the other with μ and $-\mu$, and apply to each component Theorem 5.28. In the real case this results in three rank 0 normal forms: elliptic-elliptic, elliptic-hyperbolic and hyperbolic-hyperbolic. In the p-adic case, we have to combine analogously the normal forms for dimension 2, getting a total of $\binom{8}{2} = 28$ forms for $p \equiv 1 \mod 4$, $\binom{6}{2} = 15$ for $p \equiv 3 \mod 4$, and $\binom{12}{2} = 66$ for $p = 2$.

The other case is when $\lambda^2 \notin \mathbb{Q}_p$. In this case, λ^2 and μ^2 are conjugate roots in a degree two extension L of $\mathbb{Q}_p$. If λ^2 is a square in L, that is, $\lambda \in L$, we have $\mu^2 = \overline{\lambda^2}$ and $\mu = \bar{\lambda}$ is also in L. We will now see that, in this case, the necessary condition of having the same eigenvalues is also sufficient to be linearly symplectomorphic.

8.2.1. Case $p \neq 2$. We start with the case $p \equiv 1 \mod 4$. This subdivides in three cases, depending on whether α^2 is c_0, p or $c_0 p$. The following result gives a characterization of the squares in $\mathbb{Q}_p[\alpha]$ in each case.

PROPOSITION 8.4. *Let p be a prime number such that $p \equiv 1 \mod 4$. Let c_0 be a quadratic non-residue modulo p. Then the following statements hold.*

(1) $\mathrm{Sq}(\mathbb{Q}_p[\sqrt{c_0}]^*) = \{a + b\sqrt{c_0} : a, b \in \mathbb{Q}_p, \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{ord}_p(a) \equiv 0 \mod 2, a^2 - b^2 c_0 \in \mathrm{Sq}(\mathbb{Q}_p^*)\}$.
(2) $\mathrm{Sq}(\mathbb{Q}_p[\sqrt{p}]^*) = \{a + b\sqrt{p} : a, b \in \mathbb{Q}_p, \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{digit}_0(a) \in \mathrm{Sq}(\mathbb{F}_p^*)\}$.
(3) $\mathrm{Sq}(\mathbb{Q}_p[\sqrt{c_0 p}]^*) = \{a + b\sqrt{c_0 p} : a, b \in \mathbb{Q}_p, \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{ord}_p(a) \equiv 0 \mod 2, \mathrm{digit}_0(a) \in \mathrm{Sq}(\mathbb{F}_p^*)\} \cup \{a + b\sqrt{c_0 p} : \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{ord}_p(a) \not\equiv 0 \mod 2, \mathrm{digit}_0(a) \notin \mathrm{Sq}(\mathbb{F}_p^*)\}$.

PROOF. (1) Suppose that $a + b\sqrt{c_0}$ is a square in $\mathbb{Q}_p$. By Lemma 6.25,

$$a^2 - b^2 c_0 = (r^2 - s^2 c_0)^2$$

for some $r, s \in \mathbb{Q}_p$. In particular, $a^2 - b^2 c_0$ is a square in $\mathbb{Q}_p$. If $\mathrm{ord}(a)$ was higher than $\mathrm{ord}(b)$, that would make

$$\mathrm{digit}_0(a^2 - b^2 c_0) = \mathrm{digit}_0(-b^2 c_0) \notin \mathrm{Sq}(\mathbb{F}_p^*).$$

So $\mathrm{ord}(a) \leq \mathrm{ord}(b)$. We also have that $a = r^2 + s^2 c_0$, and by Proposition 8.1(2), $\mathrm{ord}(a)$ is even.

Suppose now that a and b satisfy the three conditions. Then the first condition in Lemma 6.25 is satisfied. Let t_1 and t_2 be the two candidates for r^2. Note that $t_1 t_2 = b^2 c_0 / 4$.

The leading terms cannot cancel simultaneously in $a + \sqrt{a^2 - b^2 c_0}$ and $a - \sqrt{a^2 - b^2 c_0}$. Without loss of generality, suppose that t_1 has no cancellation. Then $\mathrm{ord}(t_1) = \mathrm{ord}(a)$, which is even, and $\mathrm{ord}(t_2) = \mathrm{ord}(b^2 c_0 / 4t_1)$ is also even. But their product $t_1 t_2 = b^2 c_0 / 4$ is a non-square, hence one of t_1 and t_2 is a square (because the product of two even-order non-squares is a square).

(2) Suppose that $a + b\sqrt{p}$ is a square in $\mathbb{Q}_p$. By Lemma 6.25,

$$a^2 - b^2 p = (r^2 - s^2 p)^2$$

for some $r, s \in \mathbb{Q}_p$. This implies that $\mathrm{ord}(a^2 - b^2 p)$ is even, so $\mathrm{ord}(a) \leq \mathrm{ord}(b)$. Here $a = r^2 + s^2 p$, so by Proposition 8.1(3), $\mathrm{digit}_0(a) \in \mathrm{Sq}(\mathbb{F}_p^*)$.

Reciprocally, if a and b satisfy the conditions, $a^2 - b^2 p$ is a square because a^2 is. $t_1 = (a + \sqrt{a^2 - b^2 p})/2$ has the same order and leading digit than a, so if a is a square, t_1 is also a square. Otherwise, a is p times a square and the same applies to t_1, and $t_2 = b^2 p / 4t_1$ is a square.

(3) Suppose that $a + b\sqrt{c_0 p}$ is a square in $\mathbb{Q}_p$. By Lemma 6.25,

$$a^2 - b^2 c_0 p = (r^2 - s^2 c_0 p)^2$$

for some $r, s \in \mathbb{Q}_p$. This implies that $\operatorname{ord}(a^2 - b^2 c_0 p)$ is even, so $\operatorname{ord}(a) \le \operatorname{ord}(b)$. Now we have $a = r^2 + s^2 c_0 p$, which by Proposition 8.1(4) implies that either the order of a is even and its leading digit is square, or the order is odd and the leading digit is non-square.

Reciprocally, if a and b satisfy the conditions, $a^2 - b^2 c_0 p$ is a square because a^2 is. In the first case

$$t_1 = \frac{a + \sqrt{a^2 - b^2 c_0 p}}{2}$$

is a square. In the second case, a is p times an even order non-square, t_1 is the same, and

$$t_2 = \frac{b^2 c_0 p}{4t_1}$$

is a square. □

Corollary 8.5. *Let p be a prime number such that $p \equiv 1 \mod 4$. Let c_0 be a quadratic non-residue modulo p. Then the following statements hold.*

(1) $\mathbb{Q}_p[\sqrt{c_0}]^* / \operatorname{Sq}(\mathbb{Q}_p[\sqrt{c_0}]^*) = \{1, p, \sqrt{c_0}, p\sqrt{c_0}\}$.
(2) $\mathbb{Q}_p[\sqrt{p}]^* / \operatorname{Sq}(\mathbb{Q}_p[\sqrt{p}]^*) = \{1, c_0, \sqrt{p}, c_0\sqrt{p}\}$.
(3) $\mathbb{Q}_p[\sqrt{c_0 p}]^* / \operatorname{Sq}(\mathbb{Q}_p[\sqrt{c_0 p}]^*) = \{1, c_0, \sqrt{c_0 p}, c_0\sqrt{c_0 p}\}$.

Proof. (1) Given an element $a + b\sqrt{c_0}$ in $\mathbb{Q}_p[\sqrt{c_0}]^*$, if $\operatorname{ord}(a) > \operatorname{ord}(b)$ or they are equal and $a^2 - b^2 c_0$ is not a square, we multiply it by $\sqrt{c_0}$. This guarantees that $\operatorname{ord}(a) \le \operatorname{ord}(b)$ and $a^2 - b^2 c_0$ is a square, because

$$\sqrt{c_0}(a + b\sqrt{c_0}) = a\sqrt{c_0} + bc_0$$

and

$$b^2 c_0^2 - a^2 c_0 = c_0(a^2 - b^2 c_0)$$

so if $a^2 - b^2 c_0$ was non-square, it is now square. Hence, if the order of a is odd, we multiply the element by p, and we obtain a square.

(2) Given an element $a + b\sqrt{p}$ in $\mathbb{Q}_p[\sqrt{p}]^*$, if $\operatorname{ord}(a) > \operatorname{ord}(b)$, we multiply it by $\sqrt{p}$, so that it has $\operatorname{ord}(a) \le \operatorname{ord}(b)$. Now, multiplying it by c_0 if needed, we ensure that $\operatorname{digit}_0(a)$ is a square.

(3) Given an element $a + b\sqrt{c_0 p}$ in $\mathbb{Q}_p[\sqrt{c_0 p}]^*$, if $\operatorname{ord}(a) > \operatorname{ord}(b)$, we multiply it by $\sqrt{c_0 p}$, so that it has $\operatorname{ord}(a) \le \operatorname{ord}(b)$. Now, multiplying it by c_0 if needed, we ensure that $\operatorname{digit}_0(a)$ is a square or a non-square, depending on the order. □

The element γ is the square root of an element in this set, but different from 1, which would lead to the case of Proposition 6.23. So there are three possible γ's for each α. Also, note that γ^2 always is in $\mathbb{Q}_p$ or α times an element of $\mathbb{Q}_p$: this means that $\bar{\gamma}^2$ is γ^2 or $-\gamma^2$, that is, $\bar{\gamma}$ is γ or $\mathrm{i}\gamma$. In any case, $\bar{\gamma} \in \mathbb{Q}_p[\gamma]$ (here it is important that $p \equiv 1 \mod 4$ so that $\mathrm{i} \in \mathbb{Q}_p$), or in other words, the extension $\mathbb{Q}_p[\gamma, \bar{\gamma}]$ is the same as $\mathbb{Q}_p[\gamma]$, which is different for each γ.

α^2	all classes	γ^2	attainable classes	a	b	$a(b+\alpha)$	$[a(b+\alpha)]$
c_0	$1, p, \sqrt{c_0}, p\sqrt{c_0}$	p	$1, p$	1	0	$\sqrt{c_0}$	$\sqrt{c_0}$
				p	$1/p$	$1+p\sqrt{c_0}$	1
		$\sqrt{c_0}$	$1, \sqrt{c_0}$	1	0	$\sqrt{c_0}$	$\sqrt{c_0}$
				p	0	$p\sqrt{c_0}$	$p\sqrt{c_0}$
		$p\sqrt{c_0}$	$1, p\sqrt{c_0}$	1	0	$\sqrt{c_0}$	$\sqrt{c_0}$
				p	0	$p\sqrt{c_0}$	$p\sqrt{c_0}$
p	$1, c_0, \sqrt{p}, c_0\sqrt{p}$	c_0	$1, c_0$	1	0	$\sqrt{p}$	$\sqrt{p}$
				1	1	$1+\sqrt{p}$	1
		$\sqrt{p}$	$1, \sqrt{p}$	1	0	$\sqrt{p}$	$\sqrt{p}$
				c_0	0	$c_0\sqrt{p}$	$c_0\sqrt{p}$
		$c_0\sqrt{p}$	$1, c_0\sqrt{p}$	1	0	$\sqrt{p}$	$\sqrt{p}$
				c_0	0	$c_0\sqrt{p}$	$c_0\sqrt{p}$
c_0p	$1, c_0, \sqrt{c_0p}, c_0\sqrt{c_0p}$	c_0	$1, c_0$	1	0	$\sqrt{c_0p}$	$\sqrt{c_0p}$
				1	1	$1+\sqrt{c_0p}$	1
		$\sqrt{c_0p}$	$1, \sqrt{c_0p}$	1	0	$\sqrt{c_0p}$	$\sqrt{c_0p}$
				c_0	0	$c_0\sqrt{c_0p}$	$c_0\sqrt{c_0p}$
		$c_0\sqrt{c_0p}$	$1, c_0\sqrt{c_0p}$	1	0	$\sqrt{c_0p}$	$\sqrt{c_0p}$
				c_0	0	$c_0\sqrt{c_0p}$	$c_0\sqrt{c_0p}$

TABLE 2. Values of a and b for Proposition 6.24 with $F = \mathbb{Q}_p, p \equiv 1 \mod 4$. The second column shows the classes of $\mathbb{Q}_p[\alpha]$ modulo a square, the fourth shows the classes attainable as $x^2 + \gamma^2$, the fifth and sixth show values of a and b, the seventh shows the resulting $a(b+\alpha)$, and the eighth shows its class. These classes, multiplied by the "attainable classes", should cover the set of "all classes". (It is interesting that the attainable classes are always 1 and γ^2. This may have to do with the field being non-archimedean.)

The next step is to determine

$$\overline{\mathrm{DSq}}(\mathbb{Q}_p[\alpha], -\gamma^2),$$

or equivalently

$$\overline{\mathrm{DSq}}(\mathbb{Q}_p[\alpha], \gamma^2),$$

because -1 is a square. This consists of seeing which classes of $\mathbb{Q}_p[\alpha]^*$ modulo a square are attainable by elements of the form $x^2 + \gamma^2$ for different x. Once this is done, the quotient of $\mathbb{Q}_p[\alpha]^* / \mathrm{Sq}(\mathbb{Q}_p[\alpha]^*)$ by this subgroup will give us the necessary a and b. The computations are shown in Table 2.

Now we make the analogous treatment with $p \equiv 3 \mod 4$. The values of α are the same as in the previous case, but now we can take $c_0 = -1$ to simplify the formulas, so we have $\alpha = \mathrm{i}, \sqrt{p}$ or $\mathrm{i}\sqrt{p}$.

PROPOSITION 8.6. *Let p be a prime number such that $p \equiv 3 \mod 4$. Then the following statements hold.*

(1) $\mathrm{Sq}(\mathbb{Q}_p[\mathrm{i}]^*) = \{a + \mathrm{i}b : a, b \in \mathbb{Q}_p, \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\} \equiv 0 \mod 2, a^2 + b^2 \in \mathrm{Sq}(\mathbb{Q}_p^*)\}$.

(2) $\mathrm{Sq}(\mathbb{Q}_p[\sqrt{p}]^*) = \{a + b\sqrt{p} : a, b \in \mathbb{Q}_p, \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{digit}_0(a) \in \mathrm{Sq}(\mathbb{F}_p^*)\}$.

(3) $\mathrm{Sq}(\mathbb{Q}_p[\mathrm{i}\sqrt{p}]^*) = \{a + \mathrm{i}b\sqrt{p} : a, b \in \mathbb{Q}_p, \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{ord}_p(a) \equiv 0 \mod 2, \mathrm{digit}_0(a) \in \mathrm{Sq}(\mathbb{F}_p^*)\} \cup \{a + \mathrm{i}b\sqrt{p} : \mathrm{ord}_p(a) \leq \mathrm{ord}_p(b), \mathrm{ord}_p(a) \not\equiv 0 \mod 2, \mathrm{digit}_0(a) \notin \mathrm{Sq}(\mathbb{F}_p^*)\}$.

PROOF. Parts (2) and (3) have the same proof as the corresponding parts of Proposition 8.4, so we focus on part (1).

Suppose that $a + \mathrm{i}b$ is a square. By Lemma 6.25 $a^2 + b^2 = (r^2 + s^2)^2$ for $r, s \in \mathbb{Q}_p$. In particular, $a^2 + b^2$ is a square. By Proposition 8.1(1), $r^2 + s^2$ has even order, so $4 \mid \mathrm{ord}(a^2 + b^2)$. As $p \equiv 3 \mod 4$, we cannot have a cancellation in $a^2 + b^2$, so

$$\mathrm{ord}(a^2 + b^2) = \min\Big\{\mathrm{ord}(a^2), \mathrm{ord}(b^2)\Big\} = 2\min\Big\{\mathrm{ord}(a), \mathrm{ord}(b)\Big\}.$$

As this is a multiple of 4, $\min\{\mathrm{ord}(a), \mathrm{ord}(b)\}$ is even.

Now suppose that a and b satisfy the conditions. We have the first condition in Lemma 6.25. To check the second, let t_1 and t_2 be the two candidates for r^2 and suppose, without loss of generality, that $t_1 = (a + \sqrt{a^2 + b^2})/2$ does not cancel the leading terms. Then

$$\mathrm{ord}(\sqrt{a^2 + b^2}) = \min\Big\{\mathrm{ord}(a), \mathrm{ord}(b)\Big\}$$

is even. The order of a is either higher than this (if b has lower order) or the same, and in any case $\mathrm{ord}(t_1)$ is even. $t_2 = -b^2/4t_1$ has also even order, and their product $-b^2/4$ is not a square (because $p \equiv 3 \mod 4$). This implies that either t_1 or t_2 is a square. ☐

COROLLARY 8.7. *Let p be a prime number such that $p \equiv 3 \mod 4$. Let $a_0, b_0 \in \mathbb{Z}_p$ such that $a_0^2 + b_0^2 \equiv -1 \mod p$. (This pair exists by Proposition 8.1(1).) Then the following statements hold.*

(1) $\mathbb{Q}_p[\mathrm{i}]^* / \mathrm{Sq}(\mathbb{Q}_p[\mathrm{i}]^*) = \{1, p, a_0 + \mathrm{i}b_0, p(a_0 + \mathrm{i}b_0)\}$.
(2) $\mathbb{Q}_p[\sqrt{p}]^* / \mathrm{Sq}(\mathbb{Q}_p[\sqrt{p}]^*) = \{1, -1, \sqrt{p}, -\sqrt{p}\}$.
(3) $\mathbb{Q}_p[\mathrm{i}\sqrt{p}]^* / \mathrm{Sq}(\mathbb{Q}_p[\mathrm{i}\sqrt{p}]^*) = \{1, -1, \mathrm{i}\sqrt{p}, -\mathrm{i}\sqrt{p}\}$.

PROOF. Again, parts (2) and (3) are similar to the corresponding ones in Corollary 8.5, so we focus on part (1).

Given a number in the form $a + \mathrm{i}b$, we first ensure that $a^2 + b^2$ is a square multiplying by $a_0 + \mathrm{i}b_0$ if needed (this changes $a^2 + b^2 \mod p$ to the opposite). Then we have to ensure that $\min\{\mathrm{ord}(a), \mathrm{ord}(b)\}$ is even, multiplying by p if needed, and we have a square because multiplying by p multiplies $a^2 + b^2$ by p^2 and it will still be a square. ☐

Now we have determined the possible γ's. In this case, it is not always true that $\bar\gamma \in \mathbb{Q}_p[\gamma]$:

- If $\gamma^2 = -1$ or p, then $\bar\gamma = \gamma$.
- If $\gamma^2 = a_0 + \mathrm{i}b_0$ or $p(a_0 + \mathrm{i}b_0)$, the product $\gamma^2\bar\gamma^2$ is $a_0^2 + b_0^2$ or $p^2(a_0^2 + b_0^2)$ respectively. But $a_0^2 + b_0^2 \equiv -1 \mod p$ implies that $-\gamma^2\bar\gamma^2$ is a square in $\mathbb{Q}_p$, so $\gamma\bar\gamma$ is i times an element of $\mathbb{Q}_p$, and in this case we also have $\bar\gamma \in \mathbb{Q}_p[\gamma]$.
- Otherwise, $\bar\gamma = -\gamma$, which is a different class.

Hence, the five cases p, -1, $a_0 + \mathrm{i}b_0$ and $p(a_0 + \mathrm{i}b_0)$ give different extensions $\mathbb{Q}_p[\gamma, \bar\gamma]$ and the other four cases give only two extensions, one for $\pm\sqrt{p}$ and the other for $\pm\mathrm{i}\sqrt{p}$.

The next step is to determine $\overline{\mathrm{DSq}}(\mathbb{Q}_p[\alpha], -\gamma^2)$ for each possible α and γ, and make the quotients. The computation is in Table 3.

α^2	all classes	γ^2	attainable classes	a	b	$a(b+\alpha)$	$[a(b+\alpha)]$
-1	$1, p, a_0 + \mathrm{i}b_0, p(a_0 + \mathrm{i}b_0)$	p	$1, p$	1	0	i	1
				b_0	a_0/b_0	$a_0 + \mathrm{i}b_0$	$a_0 + \mathrm{i}b_0$
		$a_0 + \mathrm{i}b_0$	$1, a_0 + \mathrm{i}b_0$	1	0	i	1
				p	0	$\mathrm{i}p$	p
		$p(a_0 + \mathrm{i}b_0)$	$1, p(a_0 + \mathrm{i}b_0)$	1	0	i	1
				p	0	$\mathrm{i}p$	p
p	$1, -1, \sqrt{p}, -\sqrt{p}$	-1	$1, -1$	1	0	$\sqrt{p}$	$\sqrt{p}$
				1	1	$1 + \sqrt{p}$	1
		$\sqrt{p}$	$1, -\sqrt{p}$	1	0	$\sqrt{p}$	$\sqrt{p}$
				-1	0	$-\sqrt{p}$	$-\sqrt{p}$
$-p$	$1, -1, \mathrm{i}\sqrt{p}, -\mathrm{i}\sqrt{p}$	-1	$1, -1$	1	0	$\mathrm{i}\sqrt{p}$	$\mathrm{i}\sqrt{p}$
				1	1	$1 + \mathrm{i}\sqrt{p}$	1
		$\mathrm{i}\sqrt{p}$	$1, -\mathrm{i}\sqrt{p}$	1	0	$\mathrm{i}\sqrt{p}$	$\mathrm{i}\sqrt{p}$
				-1	0	$-\mathrm{i}\sqrt{p}$	$-\mathrm{i}\sqrt{p}$

TABLE 3. Values of a and b for Proposition 6.24 with $F = \mathbb{Q}_p, p \equiv 3 \mod 4$. The second column shows the classes of $\mathbb{Q}_p[\alpha]$ modulo a square, the fourth shows the classes attainable as $x^2 - \gamma^2$, the fifth and sixth show values of a and b, the seventh shows the resulting $a(b + \alpha)$, and the eighth shows its class.

8.2.2. Case $p = 2$. It only remains to make the analysis for $p = 2$. This case is different from the rest because we now have seven values of α^2 instead of three: $-1, 2, -2, 3, -3, 6$ and -6.

LEMMA 8.8. *Let $k, \ell \in \mathbb{N}$ with $k \geq \ell$. Let $a, b, r \in \mathbb{Z}_2$ such that $\operatorname{ord}_2(2r - a) = \ell$. Then we have that*

$$\frac{a \pm \sqrt{a^2 - b^2\alpha^2}}{2} \equiv r \mod 2^k$$

if and only if

$$\left(\frac{b}{2}\right)^2 \alpha^2 \equiv r(a - r) \mod 2^{k+\ell}.$$

PROOF. The first equation is equivalent to $a \pm \sqrt{a^2 - b^2\alpha^2} \equiv 2r \mod 2^{k+1}$, which itself is equivalent to

$$\frac{\pm\sqrt{a^2 - b^2\alpha^2}}{2^\ell} \equiv \frac{2r - a}{2^\ell} \mod 2^{k+1-\ell}.$$

Since the right-hand side is odd, this is equivalent to all the following identities:

$$\frac{a^2 - b^2\alpha^2}{2^{2\ell}} \equiv \frac{(2r - a)^2}{2^{2\ell}} \mod 2^{k+2-\ell} \Leftrightarrow$$

$$a^2 - b^2\alpha^2 \equiv (2r - a)^2 \mod 2^{k+\ell+2} \Leftrightarrow$$

$$-b^2\alpha^2 \equiv 4r^2 - 4ra \mod 2^{k+\ell+2} \Leftrightarrow$$

$$\left(\frac{b}{2}\right)^2 \alpha^2 \equiv r(a - r) \mod 2^{k+\ell},$$

as we wanted. ☐

PROPOSITION 8.9. *The following statements hold.*

(1) $\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}]^*) = \{a + \mathrm{i}b : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 2, \mathrm{ord}_2(a) \equiv 0 \mod 2, b/4a + \mathrm{digit}_1(a) + \mathrm{digit}_2(a) \equiv 0 \mod 2\} \cup \{a + \mathrm{i}b : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(a) - \mathrm{ord}_2(b) \geq 2, \mathrm{ord}_2(b) \equiv 1 \mod 2, a/4b + \mathrm{digit}_1(b) + \mathrm{digit}_2(b) \equiv 0 \mod 2\}$.
(2) $\mathrm{Sq}(\mathbb{Q}_2[\sqrt{2}]^*) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 1, \mathrm{digit}_2(a) = 0, b/2a + \mathrm{digit}_1(a) \equiv 0 \mod 2\}$.
(3) $\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{2}]^*) = \{a + \mathrm{i}b\sqrt{2} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 1, \mathrm{digit}_1(a) = \mathrm{digit}_2(a), b/2a + \mathrm{ord}_2(a) + \mathrm{digit}_1(a) \equiv 0 \mod 2\}$.
(4) $\mathrm{Sq}(\mathbb{Q}_2[\sqrt{3}]^*) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 2, \mathrm{ord}_2(a) \equiv 0 \mod 2, b/4a + \mathrm{digit}_2(a) \equiv 0 \mod 2\} \cup \{a + b\sqrt{3} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(a) - \mathrm{ord}_2(b) = 1, \mathrm{ord}_2(a) \equiv 0 \mod 2, \mathrm{digit}_1(a) + \mathrm{digit}_1(b) + \mathrm{digit}_2(b) \equiv 0 \mod 2\}$.
(5) $\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{3}]^*) = \{a + \mathrm{i}b\sqrt{3} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 2, \mathrm{ord}_2(a) \equiv 0 \mod 2, \mathrm{digit}_1(a) = 0\} \cup \{a + \mathrm{i}b\sqrt{3} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(a) = \mathrm{ord}_2(b), \mathrm{ord}_2(a) \equiv 1 \mod 2, \mathrm{digit}_1(a) = 1, a^2 + 3b^2 \in \mathrm{Sq}(\mathbb{Q}_2^*)\}$.
(6) $\mathrm{Sq}(\mathbb{Q}_2[\sqrt{6}]^*) = \{a + b\sqrt{6} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 1, \mathrm{ord}_2(a) + \mathrm{digit}_1(a) + \mathrm{digit}_2(a) \equiv 0 \mod 2, b/2a + \mathrm{digit}_2(a) \equiv 0 \mod 2\}$.
(7) $\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{6}]^*) = \{a + \mathrm{i}b\sqrt{6} : a, b \in \mathbb{Q}_p, \mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 1, \mathrm{ord}_2(a) + \mathrm{digit}_2(a) \equiv 0 \mod 2, b/2a + \mathrm{digit}_1(a) \equiv 0 \mod 2\}$.

PROOF. The first condition in Lemma 6.25 implies that $a^2 - \alpha^2 b^2$ is a square, so it has even order and ends in 001. This depends on $\alpha^2 \in \{-1, 2, -2, 3, -3, 6, -6\}$ as well as in the difference $\mathrm{ord}_2(b) - \mathrm{ord}_2(a)$, in the way described in Table 1 (where $c = -\alpha^2$). The valid values are as follows:

(i) α^2 is odd and $\mathrm{ord}_2(b) - \mathrm{ord}_2(a) \geq 2$. Then $\sqrt{a^2 - b^2\alpha^2}$ has the same order than a, and without loss of generality we suppose that $\mathrm{digit}_1(\sqrt{a^2 - b^2\alpha^2}) = \mathrm{digit}_1(a)$ (otherwise choose the other square root).

We have that t_1 or t_2 is a square, so it has even order, and $t_1 t_2 = b^2\alpha^2/4$, which has even order, so both t_1 and t_2 have even order. But, as $\mathrm{ord}_2(\sqrt{a^2 - b^2\alpha^2}) = \mathrm{ord}_2(a)$ and $\mathrm{digit}_1(\sqrt{a^2 - b^2\alpha^2}) = \mathrm{digit}_1(a)$,

$$\mathrm{ord}_2(2t_1) = \mathrm{ord}_2(a + \sqrt{a^2 - b^2\alpha^2}) = \mathrm{ord}_2(a) + 1$$

which implies $\mathrm{ord}_2(t_1) = \mathrm{ord}_2(a)$. As this is even, a has even order.

To simplify the computation, we divide a and b by some power of 4 so that $\mathrm{ord}_2(a) = 0$. (Obviously, dividing by 4 does not affect being a square.) Now $\mathrm{ord}_2(t_1) = 0$.

If t_1 is a square, it must be 1 modulo 8. By Lemma 8.8, using that in this case $\mathrm{ord}_2(2 - a) = 0$, this is equivalent to

$$\left(\frac{b}{2}\right)^2 \alpha^2 \equiv a - 1 \mod 8.$$

If $\mathrm{ord}_2(b) = 2$, this implies $4\alpha^2 \equiv a - 1 \mod 8$, and, using that α is odd, $4 \equiv a - 1 \mod 8$, so $a \equiv 5 \mod 8$. Otherwise, $\mathrm{ord}_2(b) \geq 3$ and $a \equiv 1 \mod 8$. In any case, t_1 is square if and only if $\mathrm{digit}_1(a) = 0$ and $b/4a + \mathrm{digit}_2(a)$ is even.

If t_2 is a square, as $t_2 = b^2\alpha^2/4t_1$, α^2/t_1 is also a square and has order 0, so it must be 1 modulo 8 and $t_1 \equiv \alpha^2 \mod 8$. Now $\mathrm{ord}_2(2\alpha^2 - a) = 0$

again, so this is equivalent to

$$\left(\frac{b}{2}\right)^2 \alpha^2 \equiv \alpha^2(a-\alpha^2) \mod 8,$$

that is

$$\left(\frac{b}{2}\right)^2 \equiv a-\alpha^2 \mod 8.$$

The left-hand side is equivalent to 4 modulo 8 if $\mathrm{ord}_2(b)=2$ and 0 otherwise. If $\alpha^2=-1$, a must be 3 or 7 respectively, so $\mathrm{digit}_1(a)=1$ and $b/4a+\mathrm{digit}_2(a)$ is odd. If $\alpha^2=3$, a is 7 or 3 respectively, so $\mathrm{digit}_1(a)=1$ and $b/4a+\mathrm{digit}_2(a)$ is even. Finally, if $\alpha^2=-3$, a is 5 or 1 respectively, $\mathrm{digit}_1(a)=0$ and $b/4a+\mathrm{digit}_2(a)$ is odd. Putting this together with the results for t_1, we obtain the first set in the cases (1), (4) and (5).

(ii) $\alpha^2=-1$ and $\mathrm{ord}_2(b)-\mathrm{ord}_2(a)\leq -2$. Now a^2+b^2 has the order of b^2, and $a\pm\sqrt{a^2+b^2}$ has the same order as b, so $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(t_2)=\mathrm{ord}_2(b)-1$ and $\mathrm{ord}_2(b)$ is odd. By dividing a and b by a power of 4, we assume that $\mathrm{ord}_2(b)=1$ and $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(t_2)=0$.

If t_i is square, $t_i\equiv 1 \mod 8$. Now $\mathrm{ord}_2(2-a)=1$ (because a is multiple of 4) and Lemma 8.8 implies that this is equivalent to

$$\left(\frac{b}{2}\right)^2 \equiv 1-a \mod 16.$$

The left-hand side is 1 if $\mathrm{digit}_3(b^2)$ is 0, that is, if $\mathrm{digit}_1(b)+\mathrm{digit}_2(b)$ is even, and 9 otherwise, so a is 0 or 8 modulo 16, respectively. Putting this together, we obtain the second set in case (1).

(iii) $\alpha^2=3$ and $\mathrm{ord}_2(b)-\mathrm{ord}_2(a)=-1$. a^2-3b^2 has the order of b^2, and $a\pm\sqrt{a^2-3b^2}$ has the order of b, so again $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(t_2)=\mathrm{ord}_2(b)-1$ and $\mathrm{ord}_2(b)$ is odd. By dividing a and b by a power of 4, we assume that $\mathrm{ord}_2(b)=1$, $\mathrm{ord}_2(a)=2$ and $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(t_2)=0$.

If t_i is square, $t_i\equiv 1 \mod 8$. Now $\mathrm{ord}_2(2-a)=1$ again, and this is equivalent to

$$3\left(\frac{b}{2}\right)^2 \equiv a-1 \mod 16.$$

The left-hand side is 3 if $\mathrm{digit}_3(b^2)$ is 0, that is, if $\mathrm{digit}_1(b)+\mathrm{digit}_2(b)$ is even, and 11 otherwise, so a is 4 or 12 modulo 16, respectively. Putting this together, we obtain the second set in case (4).

(iv) $\alpha^2=-3$ and $\mathrm{ord}_2(b)=\mathrm{ord}_2(a)$. In this case $\mathrm{ord}_2(\sqrt{a^2+3b^2})=\mathrm{ord}_2(a)+1$ and $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(t_2)=\mathrm{ord}_2(a)-1$, so $\mathrm{ord}_2(a)$ is odd. By dividing a and b by a power of 4, we assume that $\mathrm{ord}_2(a)=\mathrm{ord}_2(b)=1$ and $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(t_2)=0$.

If t_i is square, $t_i\equiv 1 \mod 8$. In this case, as $\mathrm{ord}_2(a)=1$, $\mathrm{ord}_2(2-a)$ is at least 2, and

$$3\left(\frac{b}{2}\right)^2 \equiv 1-a \mod 32.$$

The left-hand side is $3, 27, 11$ or 19 if b is ± 2, ± 6, ± 10 or ± 14 modulo 32 respectively, so a is $30, 6, 22$ or 14 modulo 32. These are exactly the cases where a^2+3b^2 is a square and $\mathrm{digit}_1(a)=1$ (the opposite remainder modulo 32 for a also makes a^2+3b^2 a square, but has $\mathrm{digit}_1(a)=0$). So we obtain the second set in case (5).

(v) α^2 is even and $\mathrm{ord}_2(b)-\mathrm{ord}_2(a)\geq 1$. $\sqrt{a^2-b^2\alpha^2}$ has the same order than a, and without loss of generality we suppose that $\mathrm{digit}_1(\sqrt{a^2-b^2\alpha^2})=\mathrm{digit}_1(a)$ (otherwise choose the other square root).

We have that $t_1t_2=b^2\alpha^2/4$, which has odd order, so one of t_1 and t_2 has even order and the other has odd order. But, as $\mathrm{ord}_2(\sqrt{a^2-b^2\alpha^2})=\mathrm{ord}_2(a)$ and $\mathrm{digit}_1(\sqrt{a^2-b^2\alpha^2})=\mathrm{digit}_1(a)$,

$$\mathrm{ord}_2(2t_1)=\mathrm{ord}_2(a+\sqrt{a^2-b^2\alpha^2})=\mathrm{ord}_2(a)+1$$

which implies $\mathrm{ord}_2(t_1)=\mathrm{ord}_2(a)$. That is, if a has even order t_1 is a square and otherwise t_2 is a square.

To simplify the computation, we divide a and b by some power of 4 so that $\mathrm{ord}_2(a)$ is 0 or 1.

If t_1 is a square, then $\mathrm{ord}_2(a)=\mathrm{ord}_2(t_1)=0$, so $t_1\equiv 1 \mod 8$, and $\mathrm{ord}_2(2-a)=0$, so

$$\left(\frac{b}{2}\right)^2\alpha^2\equiv a-1 \mod 8.$$

If $\mathrm{ord}_2(b)=1$, this implies $\alpha^2\equiv a-1 \mod 8$: for $\alpha^2=2$ or -6, $a\equiv 3 \mod 8$, and otherwise $a\equiv 7 \mod 8$. If $\mathrm{ord}_2(b)\geq 2$, as α is even, we get $a\equiv 1 \mod 8$. In any case, t_1 is square if and only if $\mathrm{digit}_2(a)=0$ and $b/2a+\mathrm{digit}_1(a)$ is even, if $\alpha^2=2$ or -6, or $\mathrm{digit}_1(a)=\mathrm{digit}_2(a)$ and $b/2a+\mathrm{digit}_1(a)$ is even, otherwise.

If t_2 is a square, then $\mathrm{ord}_2(a)=\mathrm{ord}_2(t_1)=1$, $t_2=b^2\alpha^2/4t_1$, α^2/t_1 is also a square and has order 0, so it must be 1 modulo 8 and $t_1\equiv\alpha^2 \mod 16$. Now $\mathrm{ord}_2(2\alpha^2-a)=1$, so

$$\left(\frac{b}{2}\right)^2\alpha^2\equiv\alpha^2(a-\alpha^2) \mod 32,$$

that is,

$$\left(\frac{b}{2}\right)^2\equiv a-\alpha^2 \mod 16.$$

The left-hand side is equivalent to 4 modulo 16 if $\mathrm{ord}_2(b)=2$ and 0 otherwise. If $\alpha^2=2$, a must be 6 or 2 respectively, so $\mathrm{digit}_2(a)=0$ and $b/2a+\mathrm{digit}_1(a)$ is even. If $\alpha^2=-2$, a must be 2 or 14 respectively, so $\mathrm{digit}_1(a)=\mathrm{digit}_2(a)$ and $b/2a+\mathrm{digit}_1(a)$ is odd. If $\alpha^2=6$, a is 10 or 6 respectively, so $\mathrm{digit}_1(a)\neq\mathrm{digit}_2(a)$ and $b/2a+\mathrm{digit}_2(a)$ is even. Finally, if $\alpha^2=-6$, a is 14 or 10 respectively, $\mathrm{digit}_2(a)=1$ and $b/2a+\mathrm{digit}_1(a)$ is even. Putting this together with the results for t_1, we obtain the result for the cases (2), (3), (6) and (7). □

Now we need the analogous result to Corollaries 8.5 and 8.7. As it turns out, the quotient group has now 16 elements, instead of 4 like for the other primes, so it will be given as the list of generators. The notation $G=\langle g_1,\ldots,g_n\rangle$ means that G is generated by the elements $g_1,\ldots,g_n\in G$. For example, the group $\{1,c_0,p,c_0p\}$ can be described as $\langle c_0,p\rangle$.

Corollary 8.10. *The following statements hold.*

(1) $\mathbb{Q}_2[\mathrm{i}]^*/\operatorname{Sq}(\mathbb{Q}_2[\mathrm{i}]^*) = \langle 2, 3, 1+\mathrm{i}, 1+2\mathrm{i}\rangle$.
(2) $\mathbb{Q}_2[\sqrt{3}]^*/\operatorname{Sq}(\mathbb{Q}_2[\sqrt{3}]^*) = \langle -1, 2, \sqrt{3}, 1+\sqrt{3}\rangle$.
(3) $\mathbb{Q}_2[\mathrm{i}\sqrt{3}]^*/\operatorname{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{3}]^*) = \langle -1, 2, \mathrm{i}\sqrt{3}, 1+2\mathrm{i}\sqrt{3}\rangle$.
(4) $\mathbb{Q}_2[\alpha]^*/\operatorname{Sq}(\mathbb{Q}_2[\alpha]^*) = \langle -1, 3, \alpha, 1+\alpha\rangle$, *for* $\alpha^2 \in \{2, -2, 6, -6\}$.

Proof. In all cases, the quotient can be computed, as with other primes, by starting with an arbitrary number in $\mathbb{Q}_p[\alpha]$ and proving that it can be multiplied by some generators to make it a square.

(1) First we ensure that $\operatorname{ord}_2(a) \neq \operatorname{ord}_2(b)$ multiplying by $1+\mathrm{i}$ if needed. Then we ensure that the orders are not consecutive, multiplying by $1+2\mathrm{i}$ if they are (such operation will increment the highest order). Then we ensure that the order of a is even, if $\operatorname{ord}_2(a) < \operatorname{ord}_2(b)$, and that the order of b is odd, otherwise, multiplying by 2 if needed. Finally, we ensure the condition on the digits of a or b, multiplying by 3 if needed (in general, multiplying a 2-adic number x by 3 preserves $\operatorname{digit}_2(x)$ and inverts $\operatorname{digit}_1(x)$).
(2) First we ensure that $\operatorname{ord}_2(a) \neq \operatorname{ord}_2(b)$ multiplying by $1+\sqrt{3}$ if needed. Then we ensure that the difference $\operatorname{ord}_2(b) - \operatorname{ord}_2(a)$ is correct (-1 or at least 2), multiplying by $\sqrt{3}$ if needed (this inverts the difference). Then we multiply by 2 if needed so that the order of a is even. Finally, we ensure the condition on digits by multiplying by -1: both conditions involve an odd number of digits, so they will invert on multiplication by -1.
(3) This will be split in two cases.
   (a) If $\operatorname{ord}_2(a) = \operatorname{ord}_2(b)$, we first make $\operatorname{digit}_2(a^2+3b^2) = 0$ multiplying by $1+2\mathrm{i}\sqrt{3}$:
   $$(1+2\mathrm{i}\sqrt{3})(a+b\mathrm{i}\sqrt{3}) = a - 6b + (b+2a)\sqrt{3}$$
   and
   $$(a-6b)^2 + 3(b+2a)^2 = 13(a^2+3b^2).$$
   As $13 \equiv 5 \mod 8$, this inverts $\operatorname{digit}_2(a^2+3b^2)$. Now we make $\operatorname{digit}_1(a^2+3b^2) = 0$ multiplying by $\mathrm{i}\sqrt{3}$, which will analogously multiply a^2+3b^2 by 3. So now a^2+3b^2 is a square. Next we multiply by 2 to make the order odd, and by -1 to make $\operatorname{digit}_1(a) = 1$, all without affecting a^2+3b^2.
   (b) If $\operatorname{ord}_2(a) \neq \operatorname{ord}_2(b)$, we first make $\operatorname{ord}_2(a) < \operatorname{ord}_2(b)$ multiplying by $\mathrm{i}\sqrt{3}$ if needed, then $\operatorname{ord}_2(b) - \operatorname{ord}_2(a) \geq 2$ multiplying by $1+2\mathrm{i}\sqrt{3}$, then by 2 to make $\operatorname{ord}_2(a)$ even, and finally by -1 to make $\operatorname{digit}_1(a) = 0$.
(4) First we ensure $\operatorname{ord}_2(b) \geq \operatorname{ord}_2(a)$ multiplying by α and then $\operatorname{ord}_2(b) > \operatorname{ord}_2(a)$ multiplying by $1+\alpha$. In each of the four cases, there are two conditions left, both related to $\operatorname{digit}_1(a)$ and $\operatorname{digit}_2(a)$. We set $\operatorname{digit}_2(a)$ to the required value, multiplying by -1 if needed, and finally $\operatorname{digit}_1(a)$, multiplying by 3. □

A depiction of the 16 classes can be found at Figure 3 for $\alpha^2 = -1$, at Figure 4 for $\alpha^2 = 2$, at Figure 5 for $\alpha^2 = -2$, at Figure 6 for $\alpha^2 = 3$, at Figure 7 for $\alpha^2 = -3$, at Figure 8 for $\alpha^2 = 6$, and at Figure 9 for $\alpha^2 = -6$.

Now we have to compute which of the classes are "paired" in the sense of being the classes of γ^2 and $\bar{\gamma}^2$, so that they give the same extension $\mathbb{Q}_p[\gamma, \bar{\gamma}]$. In general, if $\gamma = t_1 + t_2\alpha$,

$$\gamma^2\bar{\gamma}^2 = (t_1 + t_2\alpha)(t_1 - t_2\alpha) = t_1^2 - t_2^2\alpha^2$$

which is always in $\mathbb{Q}_2$, so two paired classes differ in a factor in $\mathbb{Q}_2$. In the last column of Tables 4 to 10 we give the pair of each class. After identifying the paired classes, if $\alpha^2 \in \{-1, -2, 3, 6\}$, 9 classes remain (not counting 1), and if $\alpha^2 \in \{2, -3, -6\}$, 11 classes remain.

The next step in order to achieve the classification is to compute the classes of

$$\overline{\mathrm{DSq}}(\mathbb{Q}_2[\alpha], -\gamma^2)$$

for each possible α and γ. For the other primes this meant three different α's and two or three γ's for each one, but here we need seven α's and nine or eleven γ's for each one. To simplify what would otherwise be a very long and error-prone computation, we will now use the Hilbert symbol for $\mathbb{Q}_2[\alpha]$.

LEMMA 8.11. *The Hilbert symbol $(a, b)_F$ in any field F (concretely $F = \mathbb{Q}_2[\alpha]$) has the following properties:*

(1) $(1, u)_F = (u, -u)_F = 1$ *for any* u.
(2) $(u, v)_F = (v, u)_F$.
(3) $(u, v)_F = 1$ *if and only if* $v \in \overline{\mathrm{DSq}}(F, -u)$.
(4) $(u, v_1v_2)_F = (u, v_1)_F(u, v_2)_F$.

We define a subset

$$S_\alpha \subset (\mathbb{Q}_2[\alpha]^* / \mathrm{Sq}(\mathbb{Q}_2[\alpha]^*))^2$$

for different values of α: S_{i} is defined in Table 4, $S_{\sqrt{2}}$ in Table 5, $S_{\mathrm{i}\sqrt{2}}$ in Table 6, $S_{\sqrt{3}}$ in Table 7, $S_{\mathrm{i}\sqrt{3}}$ in Table 8, $S_{\sqrt{6}}$ in Table 9, and $S_{\sqrt{-6}}$ in Table 10.

LEMMA 8.12. *Let $F = \mathbb{Q}_2[\alpha]$ be a degree 2 extension of $\mathbb{Q}_2$. If $(u, v)_F = 1$ for all $(u, v) \in S_\alpha$ and there exists $(u, v) \in (\mathbb{Q}_2[\alpha]^*)^2$ such that $(u, v)_F = -1$, then $(u, v)_F = -1$ for all $(u, v) \notin S_\alpha$.*

PROOF. In all cases, the set $\{v : (u, v) \in S_\alpha\}$ for a fixed $u \neq 1$ is a multiplicative subgroup of the quotient $\mathbb{Q}_2[\alpha]^* / \mathrm{Sq}(\mathbb{Q}_2[\alpha]^*)$ with eight elements. If all them have $(u, v)_F = 1$ and other v has $(u, v)_F = -1$, then by multiplicativity all the other v have $(u, v)_F = -1$. □

PROPOSITION 8.13. *For all degree 2 extensions $F = \mathbb{Q}_2[\alpha]$, $(u, v)_F = 1$ if and only if $(u, v) \in S_\alpha$.*

PROOF. We use Lemma 8.12 for each possible α. For some values $(u, v) \in S_\alpha$, it can be easily computed that they have $(u, v)_F = 1$, and this can be deduced for the rest of S_α by Lemma 8.11. Then we just need to prove that there is (u, v) such that $(u, v)_F = -1$, and we are done.

- Case $\alpha = \mathrm{i}$: since we have that $3 - 2 = 1$, $2(1 + 2\mathrm{i}) - 2(1 + \mathrm{i}) = 2\mathrm{i}$, $6(-1 + 3\mathrm{i}) + 6(1 + 2\mathrm{i}) = 30\mathrm{i}$, $(2 + 2\mathrm{i}) - 2 = 2\mathrm{i}$ and $4(1 + 2\mathrm{i}) - 3 = 1 + 8\mathrm{i}$, all of which are squares, all elements of S_{i} have $(u, v)_F = 1$. Now we prove that $(2, 1 + 2\mathrm{i})_F = -1$, for which we have to see that $x^2 - 2$ will never be $1 + 2\mathrm{i}$ times a square, for any x. Suppose this happens. Then, if $x^2 = a + b\mathrm{i}$, the orders of a and b must differ in at least 2 and the orders of $a - 2$ and b differ in 1. This implies that $a - 2$ and a have different order. There are two possibilities:

- $\text{ord}_2(a-2) = 1$ and $\text{ord}_2(a) > 1$. Then, b has order 0 or 2. Since $a + b\text{i}$ is a square, $\text{ord}_2(a)$ is at most 0, a contradiction.
- $\text{ord}_2(a-2) > 1$ and $\text{ord}_2(a) = 1$. Since $a + b\text{i}$ is a square, b has odd order at most -1, but it should differ in 1 with $\text{ord}_2(a-2)$, also a contradiction.

- Case $\alpha = \sqrt{2}$: now the pairs that add up to a square are $3-2 = 1$, $5-4 = 1$, $(1+\sqrt{2}) - \sqrt{2} = 1$, $(2+\sqrt{2}) - (1+\sqrt{2}) = 1$, $4(1+\sqrt{2}) - 3 = 1+4\sqrt{2}$ and $3+2\sqrt{2} = 3+2\sqrt{2}$. We prove that $(-1, \sqrt{2})_F = -1$, for which we have to see that x^2+1 will never be $\sqrt{2}$ times a square. Suppose it is. Let $x^2 = a + b\sqrt{2}$. We have $\text{ord}_2(a) < \text{ord}_2(b) < \text{ord}_2(a+1)$, which is possible only if a has order 0 and ends in 11. But then b must have order 1 and a ends in 011, so $a+1$ has order 2, and $a+1+b\sqrt{2}$ cannot be $\sqrt{2}$ times a square (the difference in order between $a+1$ and b should be at least 2).
- Case $\alpha = \text{i}\sqrt{2}$: now the pairs that add up to a square are $2-1 = 1$, $5-4 = 1$, $(1+\text{i}\sqrt{2}) - \text{i}\sqrt{2} = 1$, $3(1+\text{i}\sqrt{2}) - 3(-2+\text{i}\sqrt{2}) = 9$, $-1+2\text{i}\sqrt{2} = -1+2\text{i}\sqrt{2}$ and $2(1+\text{i}\sqrt{2}) - 3 = -1+2\text{i}\sqrt{2}$. We prove that $(-1, 1+\text{i}\sqrt{2})_F = -1$, for which we have to see that x^2+1 will never be $1+\text{i}\sqrt{2}$ times a square. Suppose it is. Let $x^2 = a + b\text{i}\sqrt{2}$. We have $\text{ord}_2(a) < \text{ord}_2(b) = \text{ord}_2(a+1)$, which is possible only if $\text{ord}_2(a) = 0$. Also, $b/2a + \text{digit}_1(a)$ is odd, so that $\text{ord}_2(a+1) = \text{ord}_2(b)$. This makes $a + b\text{i}\sqrt{2}$ not a square.
- Case $\alpha = \sqrt{3}$: now the pairs that add up to a square are $2-1 = 1$, $5-4 = 1$, $(1+\sqrt{3}) - \sqrt{3} = 1$, $2(3+\sqrt{3}) - 2(1+\sqrt{3}) = 4$, $2(3+\sqrt{3}) - 2 = 4+2\sqrt{3}$ and $-1+\sqrt{3}(4+2\sqrt{3}) = 5+4\sqrt{3}$. We prove that $(-1, 1+\sqrt{3})_F = -1$, for which we have to see that x^2+1 will never be $1+\sqrt{3}$ times a square. Suppose it is. Let $x^2 = a + b\sqrt{3}$.
  - If $\text{ord}_2(b) - \text{ord}_2(a) \geq 2$, $\text{ord}_2(a) = 0$ and a must end in 11 so that $\text{ord}_2(a+1) = \text{ord}_2(b)$. Also, $b/4a + \text{digit}_2(a)$ is odd. This makes $a + b\sqrt{3}$ not a square.
  - If $\text{ord}_2(a) - \text{ord}_2(b) = 1$, $\text{ord}_2(a)$ must be even, so it is impossible that $a+1$ and b have the same order.
- Case $\alpha = \text{i}\sqrt{3}$: now the pairs that add up to a square are $2-1 = 1$, $3-2 = 1$, $(1+2\text{i}\sqrt{3}) - 2\text{i}\sqrt{3} = 1$, $(1+2\text{i}\sqrt{3}) - 2(-6+\text{i}\sqrt{3}) = 13$, $2+2\text{i}\sqrt{3} = 2+2\text{i}\sqrt{3}$ and $2(-6+\text{i}\sqrt{3}) + 14 = 2+2\text{i}\sqrt{3}$. We prove that $(-1, \text{i}\sqrt{3})_F = -1$, for which we have to see that x^2+1 will never be $\text{i}\sqrt{3}$ times a square. Suppose it is. Let $x^2 = a + b\text{i}\sqrt{3}$.
  - If $\text{ord}_2(b) - \text{ord}_2(a) \geq 2$, $\text{ord}_2(a) = 0$ and a must end in 11 so that $\text{ord}_2(a+1) \geq \text{ord}_2(b)$, but then $a + b\text{i}\sqrt{3}$ is not a square.
  - If $\text{ord}_2(a) = \text{ord}_2(b)$, they must be odd, and $\text{ord}_2(a+1)$ cannot be greater than $\text{ord}_2(b)$, so they must be equal, and $\text{ord}_2(a) = \text{ord}_2(a+1) = \text{ord}_2(b) < 0$. If the order is -3 or less,

    $$(a+1)^2 + 3b^2 = a^2 + 3b^2 + 2a + 1$$

    must be three times a square in $\mathbb{Q}_2$, but $\text{ord}_2(a^2+3b^2) \leq -4$ and $\text{digit}_1((a+1)^2+3b^2) = \text{digit}_1(a^2+3b^2) = 0$, so it is impossible.
    If the order is -1, let $a_1 = 4a$, $b_1 = 4b$. We want $a_1 + 4 + b_1\text{i}\sqrt{3}$ to be $\text{i}\sqrt{3}$ times a square. Let $a_2 + b_2\text{i}\sqrt{3}$ be this square. All of a_1, b_1, a_2 and b_2 have order 1, and

    $$a_1 + 4 + b_1\text{i}\sqrt{3} = \text{i}\sqrt{3}(a_2 + b_2\text{i}\sqrt{3})$$

		1				$1+\mathrm{i}$				$1+2\mathrm{i}$				$-1+3\mathrm{i}$						
		1	2	3	6	1	2	3	6	1	2	3	6	1	2	3	6	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	2	•	•	•	•	•	•	•	•									1	2	2
	3	•	•	•	•					•	•	•	•					1	1	3
	6	•	•	•	•									•	•	•	•	1	1	6
$1+\mathrm{i}$	1	•	•			•	•					•	•			•	•	3	0	2
	2	•	•			•	•			•	•			•	•			3	0	1
	3	•	•					•	•			•	•	•	•			3	0	6
	6	•	•					•	•	•	•					•	•	3	0	3
$1+2\mathrm{i}$	1	•		•			•		•	•		•			•		•	2	0	3
	2	•		•			•		•		•		•	•		•		2	0	6
	3	•		•		•		•		•		•		•		•		2	0	1
	6	•		•		•		•			•		•		•		•	2	0	2
$-1+3\mathrm{i}$	1	•			•		•	•			•	•		•			•	2	0	6
	2	•			•		•	•		•			•		•	•		2	0	3
	3	•			•	•			•		•	•			•	•		2	0	2
	6	•			•	•			•	•			•	•			•	2	0	1

TABLE 4. S_i as a subset of $(\mathbb{Q}_2[\mathrm{i}]^*/\,\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}]^*))^2$. For each row except the first, we need two normal forms to cover all the classes: $a=1, b=0$ and $a=a_1, b=b_1$. The last column indicates the second index of the class that pairs with each class; the first index is always the same.

implies that $a_1+4=-3b_2$ and $b_1=a_2$. Hence, $\mathrm{digit}_1(b_1)=\mathrm{digit}_1(a_2)=1$. a_1 is $30, 6, 22$ or 14 modulo 32 exactly when b_1 can take those remainders, so $a_1 \equiv b_1 \mod 32$, and

$$-3b_2 = a_1+4 \equiv b_1+4 = a_2+4 \mod 32$$

But now a_2 is $30, 6, 22$ and 14 when b_2 is $\pm 2, \pm 6, \pm 10$ and ± 14 respectively, and the equation does not hold in any case.

- Case $\alpha=\sqrt{6}$: now the pairs that add up to a square are $2-1=1$, $5-4=1$, $(1+\sqrt{6})-\sqrt{6}=1$, $3(1+\sqrt{6})-3(6+\sqrt{6})=-15$, $-1+2\sqrt{6}=-1+2\sqrt{6}$ and $2(1+\sqrt{6})-3=-1+2\sqrt{6}$. We prove that $(-1,1+\sqrt{6})_F=-1$, for which we have to see that x^2+1 will never be $1+\sqrt{6}$ times a square. Suppose it is. Let $x^2=a+b\sqrt{6}$. We have $\mathrm{ord}_2(a)<\mathrm{ord}_2(b)=\mathrm{ord}_2(a+1)$, which is possible only if $\mathrm{ord}_2(a)=0$. Also, $b/2a+\mathrm{digit}_1(a)$ is odd, so that $\mathrm{ord}_2(a+1)=\mathrm{ord}_2(b)$. This makes $a+b\sqrt{6}$ not a square.
- Case $\alpha=\mathrm{i}\sqrt{6}$: now the pairs that add up to a square are $2-1=1$, $3-2=1$, $(1+\mathrm{i}\sqrt{6})-\mathrm{i}\sqrt{6}=1$, $(-6+\mathrm{i}\sqrt{6})-(1+\mathrm{i}\sqrt{6})=-7$, $3+2\mathrm{i}\sqrt{6}=3+2\mathrm{i}\sqrt{6}$ and $6(1+\mathrm{i}\sqrt{6})-3=3+6\mathrm{i}\sqrt{6}$. We prove that $(-1,\mathrm{i}\sqrt{6})_F=-1$, for which we have to see that x^2+1 will never be $\mathrm{i}\sqrt{6}$ times a square. Suppose it is. Let $x^2=a+b\mathrm{i}\sqrt{6}$. We have $\mathrm{ord}_2(a)<\mathrm{ord}_2(b)<\mathrm{ord}_2(a+1)$, which is possible only if a has order 0 and ends in 11. But then b must have order 1 and a ends in 011, so $a+1$ has order 2, and $a+1+b\mathrm{i}\sqrt{6}$ cannot be $\mathrm{i}\sqrt{6}$ times a square (the difference in order between $a+1$ and b should be at least 2). □

After finding the values of the Hilbert symbol for each α, we have $\overline{\mathrm{DSq}}(\mathbb{Q}_2[\alpha],-\gamma^2)$: it is the row indexed by γ^2 of the corresponding table. It always contains eight classes, so we need two pairs (a,b) to cover all classes: we can always take one of

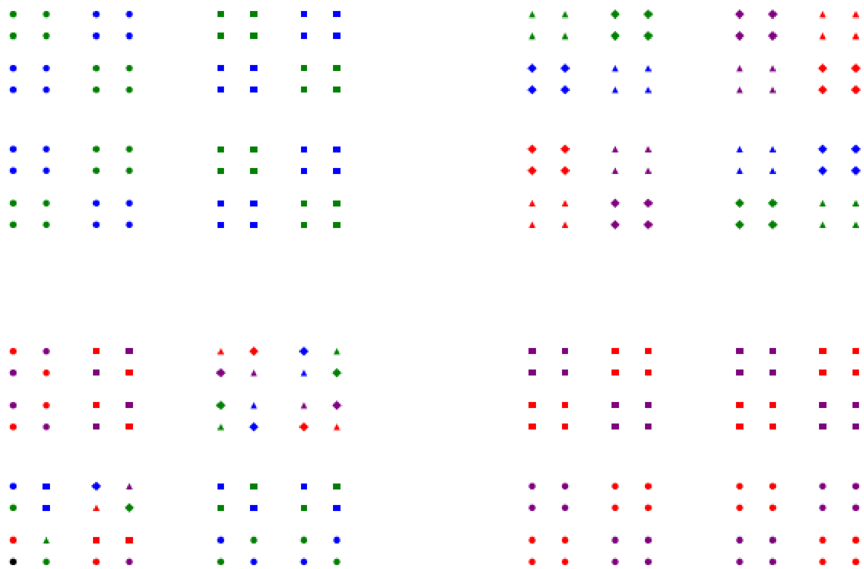


FIGURE 3. The 16 classes of Table 4. Each class contains the points $x + y\mathrm{i}$ with a given symbol, where x and y are the horizontal and vertical coordinates. The circles, triangles, squares and diamonds correspond to the four values of the first index (here 1, $1+\mathrm{i}$, $1+2\mathrm{i}$ and $-1+3\mathrm{i}$), and the colors red, green, purple and blue to the four values of the second index (here 1, 2, 3 and 6).

		1				$\sqrt{2}$				$1+\sqrt{2}$				$2+\sqrt{2}$						
		1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	-1	•	•	•	•									•	•	•	•	1	2	-1
	3	•	•	•	•	•	•	•	•									1	1	3
	-3	•	•	•	•					•	•	•	•					1	1	-3
$\sqrt{2}$	1	•		•			•		•		•		•	•		•		-1	0	-1
	-1	•		•		•		•		•		•		•		•		-1	0	1
	3	•		•			•		•	•		•			•		•	-1	0	-3
	-3	•		•		•		•			•		•		•		•	-1	0	3
$1+\sqrt{2}$	1	•			•		•	•			•	•		•			•	-1	0	-1
	-1	•			•	•			•	•			•	•			•	-1	0	1
	3	•			•		•	•		•			•		•	•		-1	0	-3
	-3	•			•	•			•		•	•			•	•		-1	0	3
$2+\sqrt{2}$	1	•	•			•	•			•	•			•	•			3	0	1
	-1	•	•					•	•			•	•	•	•			3	0	-1
	3	•	•			•	•					•	•			•	•	3	0	3
	-3	•	•					•	•	•	•					•	•	3	0	-3

TABLE 5. $S_{\sqrt{2}}$ as a subset of $(\mathbb{Q}_2[\sqrt{2}]^* / \operatorname{Sq}(\mathbb{Q}_2[\sqrt{2}]^*))^2$.

them $(1,0)$, and the other is (a_1, b_1) such that the class of $a_1(b_1+\alpha)$ is marked (in the row of γ^2) if and only if that of α is unmarked. A possibility is included at the right of the corresponding row.

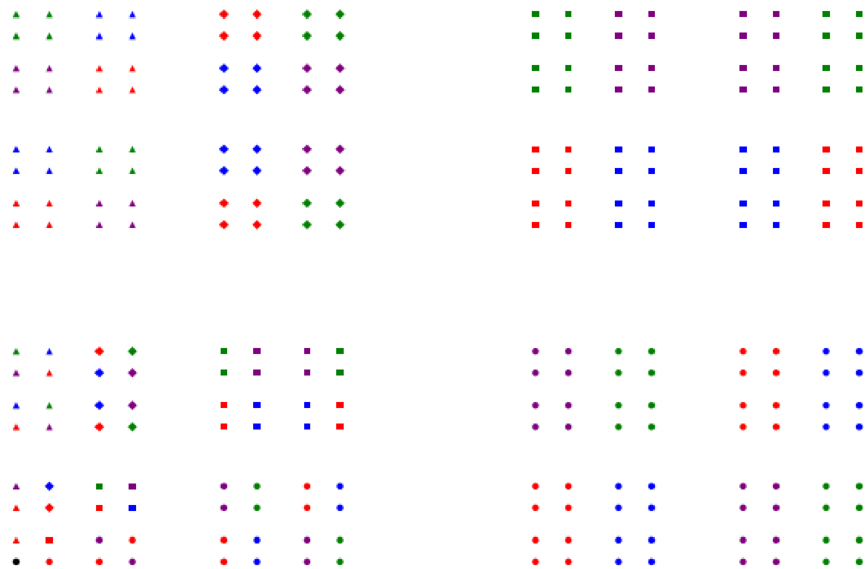


FIGURE 4. The 16 classes of Table 5.

		1				$\mathrm{i}\sqrt{2}$				$1+\mathrm{i}\sqrt{2}$				$-2+\mathrm{i}\sqrt{2}$						
		1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	-1	•	•	•	•	•	•	•	•									1	1	-1
	3	•	•	•	•									•	•	•	•	1	-2	3
	-3	•	•	•	•					•	•	•	•					1	1	-3
$\mathrm{i}\sqrt{2}$	1	•	•			•	•					•	•			•	•	3	0	-1
	-1	•	•			•	•			•	•			•	•			3	0	1
	3	•	•					•	•	•	•					•	•	3	0	-3
	-3	•	•					•	•			•	•	•	•			3	0	3
$1+\mathrm{i}\sqrt{2}$	1	•			•		•	•			•	•		•			•	-1	0	3
	-1	•			•		•	•		•			•		•	•		-1	0	-3
	3	•			•	•			•	•			•	•			•	-1	0	1
	-3	•			•	•			•		•	•			•	•		-1	0	-1
$-2+\mathrm{i}\sqrt{2}$	1	•		•			•		•	•		•			•		•	-1	0	-3
	-1	•		•			•		•		•		•	•		•		-1	0	3
	3	•		•		•		•			•		•		•		•	-1	0	-1
	-3	•		•		•		•		•		•		•		•		-1	0	1

TABLE 6. $S_{\mathrm{i}\sqrt{2}}$ as a subset of $(\mathbb{Q}_2[\mathrm{i}\sqrt{2}]^*/\,\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{2}]^*))^2$.

8.2.3. Proof of Theorem 5.31.

(a) First we prove existence. Let $\lambda, -\lambda, \mu, -\mu$ be the eigenvalues of $\Omega_0^{-1}M$. If λ^2 is in $\mathbb{Q}_p$, μ^2 is also in $\mathbb{Q}_p$. Let $\{u_1, v_1, u_2, v_2\}$ be the associated basis. By Corollary 6.5, there is a matrix S with entries in $\mathbb{Q}_p[\lambda, \mu]$ such that

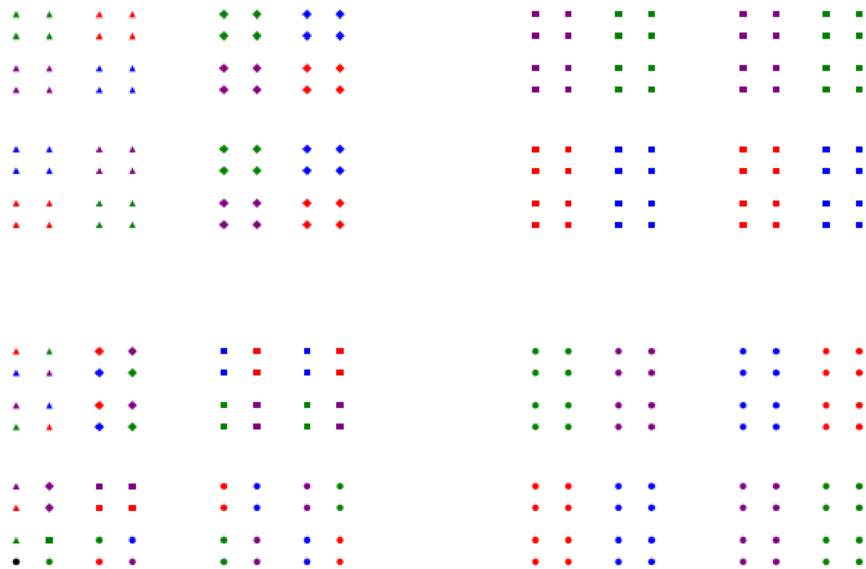


FIGURE 5. The 16 classes of Table 6.

		1				$\sqrt{3}$				$1+\sqrt{3}$				$3+\sqrt{3}$						
		1	−1	2	−2	1	−1	2	−2	1	−1	2	−2	1	−1	2	−2	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	−1	•	•	•	•	•	•	•	•									1	1	−1
	2	•	•	•	•					•	•	•	•					1	1	2
	−2	•	•	•	•									•	•	•	•	1	3	−2
$\sqrt{3}$	1	•	•			•	•					•	•			•	•	2	0	−1
	−1	•	•			•	•			•	•			•	•			2	0	1
	2	•	•					•	•			•	•	•	•			2	0	−2
	−2	•	•					•	•	•	•					•	•	2	0	2
$1+\sqrt{3}$	1	•		•			•		•		•		•	•		•		−1	0	−2
	−1	•		•			•		•	•		•			•		•	−1	0	2
	2	•		•		•		•			•		•		•		•	−1	0	−1
	−2	•		•		•		•		•		•		•		•		−1	0	1
$3+\sqrt{3}$	1	•			•		•	•		•			•		•	•		−1	0	2
	−1	•			•		•	•			•	•		•			•	−1	0	−2
	2	•			•	•			•	•			•	•			•	−1	0	1
	−2	•			•	•			•		•	•			•	•		−1	0	−1

TABLE 7. $S_{\sqrt{3}}$ as a subset of $(\mathbb{Q}_2[\sqrt{3}]^*/\,\mathrm{Sq}(\mathbb{Q}_2[\sqrt{3}]^*))^2$.

$S^T\Omega_0 S = \Omega_0$ and

$$S^T M S = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & c_1 r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & c_2 s \end{pmatrix}$$

if and only if $\lambda = r\sqrt{-c_1}$ and $\mu = s\sqrt{-c_2}$. There are always $r, s \in \mathbb{Q}_p$ and $c_1, c_2 \in X_p$ such that this is possible; moreover, there may be two valid values of c_1 or c_2.

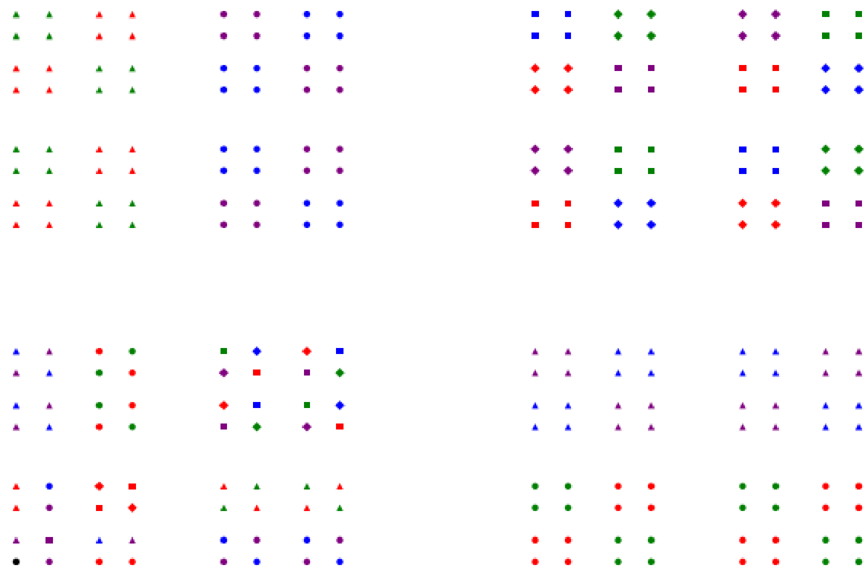


FIGURE 6. The 16 classes of Table 7.

		1				$\mathrm{i}\sqrt{3}$				$1+2\mathrm{i}\sqrt{3}$				$-6+\mathrm{i}\sqrt{3}$						
		1	-1	2	-2	1	-1	2	-2	1	-1	2	-2	1	-1	2	-2	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	-1	•	•	•	•					•	•	•	•					2	$1/2$	-1
	2	•	•	•	•	•	•	•	•									2	$1/2$	2
	-2	•	•	•	•									•	•	•	•	1	-6	-2
$\mathrm{i}\sqrt{3}$	1	•		•			•		•		•		•	•		•		-1	0	-1
	-1	•		•		•		•			•		•		•		•	-1	0	1
	2	•		•			•		•	•		•			•		•	-1	0	-2
	-2	•		•		•		•		•		•		•		•		-1	0	2
$1+2\mathrm{i}\sqrt{3}$	1	•	•					•	•	•	•					•	•	2	0	1
	-1	•	•			•	•			•	•			•	•			2	0	-1
	2	•	•					•	•			•	•	•	•			2	0	2
	-2	•	•			•	•					•	•			•	•	2	0	-2
$-6+\mathrm{i}\sqrt{3}$	1	•			•	•			•		•	•			•	•		-1	0	-1
	-1	•			•		•	•			•	•		•			•	-1	0	1
	2	•			•	•			•	•			•	•			•	-1	0	-2
	-2	•			•		•	•		•			•		•	•		-1	0	2

TABLE 8. $S_{\mathrm{i}\sqrt{3}}$ as a subset of $(\mathbb{Q}_2[\mathrm{i}\sqrt{3}]^*/\,\mathrm{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{3}]^*))^2$.

A matrix S with this property must have the form $\Psi_1 D \Psi_2^{-1}$, where

$$\Psi_1 = \begin{pmatrix} u_1 & v_1 & u_2 & v_2 \end{pmatrix}, \Psi_2 = \begin{pmatrix} \lambda & -\lambda & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \mu & -\mu \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and D is a diagonal matrix. As Ψ_2^{-1} is "box-diagonal", the first two columns of S come from the first two of Ψ_1 and the last two of S come

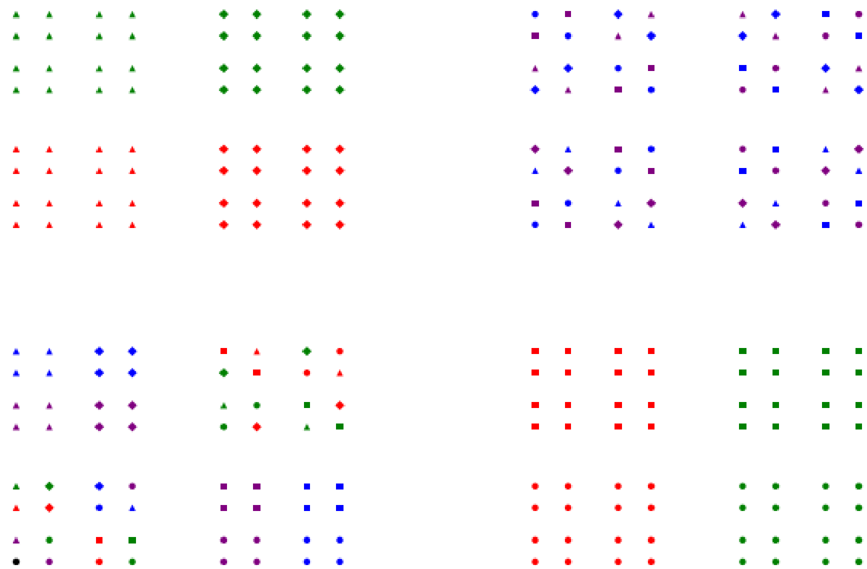


FIGURE 7. The 16 classes of Table 8.

		1				$\sqrt{6}$				$1+\sqrt{6}$				$6+\sqrt{6}$						
		1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	-1	•	•	•	•	•	•	•	•									1	1	-1
	3	•	•	•	•									•	•	•	•	1	6	3
	-3	•	•	•	•					•	•	•	•					1	1	-3
$\sqrt{6}$	1	•	•			•	•					•	•			•	•	3	0	-1
	-1	•	•			•	•			•	•			•	•			3	0	1
	3	•	•					•	•	•	•					•	•	3	0	-3
	-3	•	•					•	•			•	•	•	•			3	0	3
$1+\sqrt{6}$	1	•			•		•	•			•	•		•			•	-1	0	3
	-1	•			•		•	•		•			•		•	•		-1	0	-3
	3	•			•	•			•	•			•	•			•	-1	0	1
	-3	•			•	•			•		•	•			•	•		-1	0	-1
$6+\sqrt{6}$	1	•		•			•		•	•		•			•		•	-1	0	-3
	-1	•		•			•		•		•		•	•		•		-1	0	3
	3	•		•		•		•			•		•		•		•	-1	0	-1
	-3	•		•		•		•		•		•		•		•		-1	0	1

TABLE 9. $S_{\sqrt{6}}$ as a subset of $(\mathbb{Q}_2[\sqrt{6}]^*/\operatorname{Sq}(\mathbb{Q}_2[\sqrt{6}]^*))^2$.

from the last two of Ψ_1. That is to say, the existence of a D for which $\Psi_1 D \Psi_2^{-1}$ has entries in $\mathbb{Q}_p$ is equivalent to the existence of D_1 and D_2 such that

$$\begin{pmatrix} u_1 & v_1 \end{pmatrix} D_1 \begin{pmatrix} \lambda & -\lambda \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u_2 & v_2 \end{pmatrix} D_2 \begin{pmatrix} \mu & -\mu \\ 1 & 1 \end{pmatrix}$$

have entries in $\mathbb{Q}_p$.

By Theorem 5.28, there are always c_1 and c_2 in X_p for which D_1 and D_2 exist, so this leads to case (1).

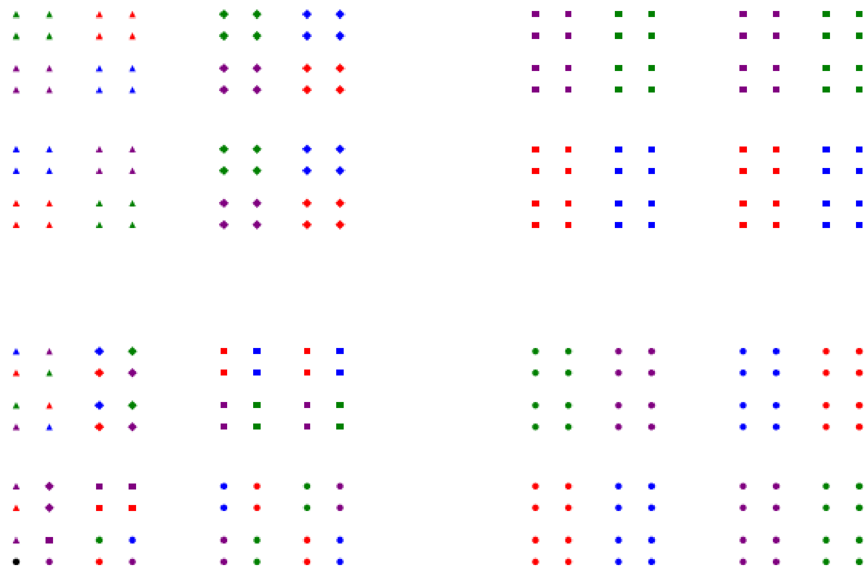


FIGURE 8. The 16 classes of Table 9.

		1				$\mathrm{i}\sqrt{6}$				$1+\mathrm{i}\sqrt{6}$				$-6+\mathrm{i}\sqrt{6}$						
		1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	1	-1	3	-3	a_1	b_1	pair
1	1	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•			1
	-1	•	•	•	•									•	•	•	•	1	-6	-1
	3	•	•	•	•	•	•	•	•									1	1	3
	-3	•	•	•	•					•	•	•	•					1	1	-3
$\mathrm{i}\sqrt{6}$	1	•		•			•		•		•		•	•		•		-1	0	-1
	-1	•		•		•		•		•		•		•		•		-1	0	1
	3	•		•			•		•	•		•			•		•	-1	0	-3
	-3	•		•		•		•			•		•		•		•	-1	0	3
$1+\mathrm{i}\sqrt{6}$	1	•			•		•	•			•	•		•			•	-1	0	-1
	-1	•			•	•			•	•			•	•			•	-1	0	1
	3	•			•		•	•		•			•		•	•		-1	0	-3
	-3	•			•	•			•		•	•			•	•		-1	0	3
$-6+\mathrm{i}\sqrt{6}$	1	•	•			•	•			•	•			•	•			3	0	1
	-1	•	•					•	•			•	•	•	•			3	0	-1
	3	•	•			•	•					•	•			•	•	3	0	3
	-3	•	•					•	•	•	•					•	•	3	0	-3

TABLE 10. $S_{\mathrm{i}\sqrt{6}}$ as a subset of $(\mathbb{Q}_2[\mathrm{i}\sqrt{6}]^*/\operatorname{Sq}(\mathbb{Q}_2[\mathrm{i}\sqrt{6}]^*))^2$.

Now suppose that $\lambda^2 \notin \mathbb{Q}_p$. If $\lambda \in \mathbb{Q}_p[\lambda^2]$, we are in the situation of Proposition 6.23. λ and μ are in a degree 2 extension $\mathbb{Q}_p[\alpha]$, and M is equivalent by multiplication by a symplectic matrix to the matrix of case (2) for some $r, s \in \mathbb{Q}_p$. The possible values of α^2 are the classes of $\mathbb{Q}_p^*$ modulo squares, that is, precisely the elements of Y_p, and we have case (2).

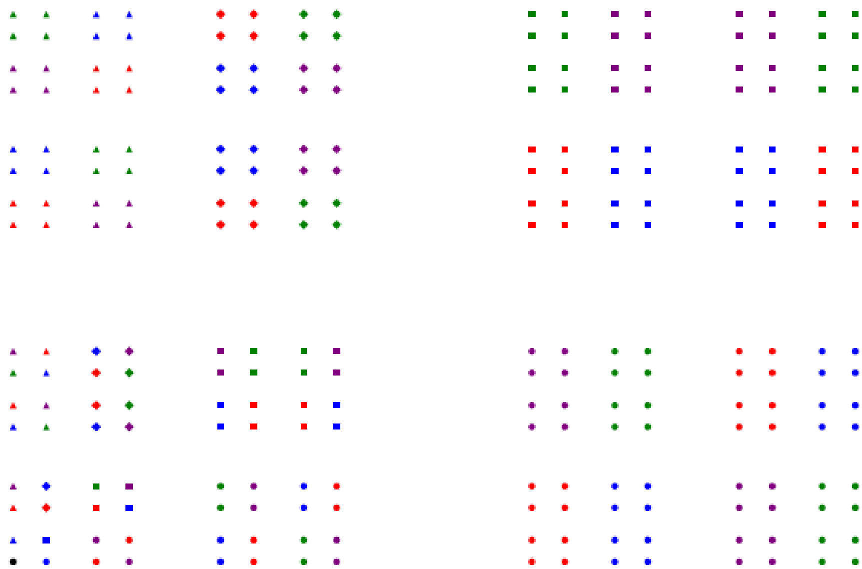


FIGURE 9. The 16 classes of Table 10.

Finally, if $\lambda \notin \mathbb{Q}_p[\lambda^2]$, we are in the situation of Proposition 6.24: we have a hierarchy of extensions

$$\mathbb{Q}_p \subsetneq \mathbb{Q}_p[\alpha] \subsetneq \mathbb{Q}_p[\gamma, \bar{\gamma}],$$

and M is equivalent by multiplication by a symplectic matrix to the matrix in case (3), for some $r, s \in \mathbb{Q}_p$, which depends on the parameters α^2, t_1, t_2, a and b. The only ones that are not fixed by the extension are a and b: a choice of them is valid if and only if

$$\frac{a\alpha\gamma(b+\alpha)}{u^T \Omega_0 \hat{u}} \in \mathrm{DSq}(F[\alpha], -\gamma^2).$$

The denominator is a constant, which implies that the valid values form a class modulo $\mathrm{DSq}(F[\alpha], -\gamma^2)$ in $F[\alpha]$. These classes are as described in Tables 2 to 10, depending on p and α^2. After substituting $c = \alpha^2$ and extracting t_1, t_2, a_1 and b_1 from these tables, we obtain case (3) of the theorem.

(b) Finally we prove uniqueness. Let N and N' be the two normal forms. The case (1), (2) or (3) of the normal form is determined uniquely by the eigenvalues of $A = \Omega_0^{-1} M$, so both N and N' are in the same case. Now we split between the three cases.
   - In case (1), there are two eigenvalues of A composing the first block of the normal form and two eigenvalues composing the second one. Hence, by Theorem 5.28, the normal form is unique up to changing the order of the blocks.
   - In case (2), the extension which contains the eigenvalues is different for each c, hence $N = N'$.

- In case (3), analogously, the extension is different for each c, t_1 and t_2, so these parameters must coincide. If a and b do not coincide, we have a number in $\mathbb{Q}_p[\alpha]$ which is in $\mathrm{DSq}(\mathbb{Q}_p[\alpha], -\gamma^2)$ and which cannot be there by previous results (Tables 2 to 10), so a and b must also coincide and $N = N'$.

8.2.4. Proof of Theorem 5.33.

(a) First we prove existence. Suppose first that the eigenvalues of A are λ, λ, $-\lambda$ and $-\lambda$, with $\lambda \neq 0$. If A is diagonalizable, Lemma 6.6 implies that there is a symplectic basis $\{u_1, v_1, u_2, v_2\}$ such that $Au_i = \lambda u_i$ and $Av_i = -\lambda v_i$. This means we are in the first case of Theorem 5.31, that is, case (1) of this theorem, and we can proceed from there.

If A is not diagonalizable, we can also apply Lemma 6.6, getting a symplectic basis $\{u_1, v_1, u_2, v_2\}$, or equivalently a symplectic matrix Ψ_1, such that

$$\Psi_1^{-1} A \Psi_1 = J = \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & -1 & 0 & -\lambda \end{pmatrix}.$$

If $\lambda \in \mathbb{Q}_p$, we can rearrange the coordinates to make J equal to $\Omega_0^{-1} M_2$, where M_2 is the matrix in case (2) with $r = \lambda$. As Ψ_1 is symplectic, rearranging its columns in the same way gives the S we need.

Otherwise, we can write $\lambda = r\alpha$, with $\alpha = \sqrt{c}$ for some $c \in Y_p$. Let M_2 be the matrix in case (3) and $A_2 = \Omega_0^{-1} M_2$. We have $\Psi_2^{-1} A_2 \Psi_2 = J$, where

$$\Psi_2 = \begin{pmatrix} 0 & \alpha z_1 & -\alpha z_1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & z_1 & z_1 & 0 \\ \alpha & -t_1 & t_1 & -\alpha \end{pmatrix},$$

where

$$z_1 = \frac{2\alpha}{a(1-\alpha^2)}, t_1 = \frac{1+\alpha^2}{r(1-\alpha^2)}.$$

A matrix that commutes with J has the form

$$D = \begin{pmatrix} d_1 & 0 & d_2 & 0 \\ 0 & d_3 & 0 & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & d_4 & 0 & d_3 \end{pmatrix}.$$

We apply the condition $D^T \Psi_1^T \Omega_0 \Psi_1 D = \Psi_2^T \Omega_0 \Psi_2$ of Proposition 6.2. As Ψ_1 is symplectic, $\Psi_1^T \Omega_0 \Psi_1 = \Omega_0$ and the condition becomes $d_1 d_3 = -2\alpha z_1$ and $d_1 d_4 + d_2 d_3 = 0$.

We also want that S has the entries in $\mathbb{Q}_p$. The first and fourth columns of Ψ_1 are the eigenvectors of A with value λ and $-\lambda$, which are conjugate up to a multiplicative constant: we call them u and $k\bar{u}$. The second and third columns correspond to v and v' such that $Av = \lambda v + u$

and $Av' = -\lambda v' - k\bar{u}$. This implies $A\bar{v} = -\lambda\bar{v} + \bar{u}$, that is, $v' = -k\bar{v}$, and

$$\begin{aligned} S\Psi_2 &= \Psi_1 D \\ &= \begin{pmatrix} u & -k\bar{v} & v & k\bar{u} \end{pmatrix} D \\ &= \begin{pmatrix} d_1 u & -d_3 k\bar{v} + d_4 k\bar{u} & d_2 u + d_1 v & d_3 k\bar{u} \end{pmatrix}. \end{aligned}$$

If we call c_i the i-th column of S, we have

$$c_2 + \alpha c_4 = d_1 u, c_2 - \alpha c_4 = d_3 k\bar{u} \Rightarrow d_3 k = \bar{d}_1;$$
$$\alpha z_1 c_1 + z_1 c_3 - t_1 c_4 = -d_3 k\bar{v} + d_4 k\bar{u} = \bar{d}_1\bar{v} + d_4 k\bar{u};$$
$$-\alpha z_1 c_1 + z_1 c_3 + t_1 c_4 = d_2 u + d_1 v.$$

Changing sign and conjugating

$$\alpha z_1 c_1 + z_1 c_3 - t_1 c_4 = \bar{d}_1\bar{v} - \bar{d}_2\bar{u},$$

so we have $d_4 k = -\bar{d}_2$. We can take $d_2 = d_4 = 0$, and the condition reduces to find d_1 such that

$$d_1\bar{d}_1 = \frac{-4\alpha^2 k}{a(1-\alpha^2)}. \tag{34}$$

So we need (34) to be in

$$\mathrm{DSq}(\mathbb{Q}_p, -\alpha^2) = \mathrm{DSq}(\mathbb{Q}_p, -c),$$

which is possible for a value of a in $\mathbb{Q}_p^*/\,\mathrm{DSq}(\mathbb{Q}_p, -c)$. This quotient is exactly the set called $\{1, h_p(c)\}$ in the statement.

Now suppose that the eigenvalues are $\lambda, -\lambda, 0, 0$ for $\lambda \neq 0$. By Lemma 6.6, we can choose u_1 and v_1 as eigenvectors with values λ and $-\lambda$ such that $u_1^T\Omega_0 v_1 = 1$ and they are Ω_0-complementary to the kernel of A. We then complete to a symplectic basis $\{u_1, v_1, u_2, v_2\}$, with $Au_2 = Av_2 = 0$. At this point we are again in case (1) of Theorem 5.31, with $c_2 = 0$.

The only case left is that all the eigenvalues of A are 0. Then Theorem 6.8 gives a good tuple K with sum 4 and a basis. The possible cases for K are $(4), (2,2), (2,1,1)$ or $(1,1,1,1)$.

- If $K = (1,1,1,1)$, $M = 0$ and the result follows trivially.
- If $K = (2,1,1)$, the basis is $\{u_{11}, u_{12}, u_{21}, u_{31}\}$. We can multiply u_{11} and u_{12} by a constant so that $u_{11}^T\Omega_0 u_{12} = 1/r$ for $r \in Y_p \cup \{1\}$, and u_{31} so that $u_{21}^T\Omega_0 u_{31} = 1$. Taking as S the matrix with the columns $\{ru_{12}, u_{11}, u_{21}, u_{31}\}$, we are in case (1) of Theorem 5.31, with $c_1 = c_2 = 0$.
- If $K = (2,2)$, the basis is $\{u_{11}, u_{12}, u_{21}, u_{22}\}$. We multiply u_{11} and u_{12} by a constant and u_{21} and u_{22} by another constant so that $u_{i1}^T\Omega_0 u_{i2} = 1/r_i$ for $r_i \in Y_p \cup \{1\}, i = 1,2$. Taking as S the matrix with columns $\{r_1 u_{12}, u_{11}, r_2 u_{22}, u_{21}\}$, we are in the same case as before.
- If $K = (4)$, the basis is $\{u_1, u_2, u_3, u_4\}$. Let $k = u_1^T\Omega_0 u_4 = -u_2^T\Omega_0 u_3$. We can multiply the four vectors by a constant so that $1/k \in Y_p \cup \{1\}$. Taking S with the columns $\{u_3/k, u_2, ku_1, u_4/k^2\}$, we are in case (4) of this theorem, with $c = 1/k$.

(b) Finally we prove uniqueness. If there are two normal forms N and N', by Proposition 6.2, they must have the same Jordan form. The matrices in each case have different Jordan forms, so N and N' are in the same case.

- If it is case (1), by Theorem 5.28 we have $N = N'$, except perhaps for the order of the blocks.
- If it is case (2), the equality of eigenvalues implies $N = N'$.
- If it is case (3), each c corresponds to a different extension, so the equality of eigenvalues implies $c = c'$ and $r = r'$. It is left to prove $a = a'$. Suppose on the contrary that $a = 1$ and $a' = h_p(c)$. Applying the proof of existence, we have that
$$\frac{-4\alpha^2 k}{1-\alpha^2} \in \mathrm{DSq}(\mathbb{Q}_p, -c) \text{ and } \frac{-4\alpha^2 k}{h_p(c)(1-\alpha^2)} \in \mathrm{DSq}(\mathbb{Q}_p, -c).$$
As $\mathrm{DSq}(\mathbb{Q}_p, -c)$ is a group, we also have $h_p(c) \in \mathrm{DSq}(\mathbb{Q}_p, -c)$, which is a contradiction.
- If it is case (4), again by equality of eigenvalues we have $c = c'$.

PROPOSITION 8.14. *Proposition 8.3 also holds for dimension 4, that is, the choice of c_0 only affects the choice of representatives of each class of matrices up to multiplication by a symplectic matrix. The same happens for a_0 and b_0.*

PROOF. Applying Theorems 5.31 and 5.33 to the normal forms of one set gives for each one and only one form of the other set which is equivalent. □

8.2.5. Proof of Theorem 5.34. From Theorem 5.31, if $p \equiv 1 \mod 4$, case (1) leads to $\binom{8}{2} = 28$ normal forms (there are seven possible values for c_1 and c_2), case (2) to 3 normal forms, and case (3) has 9 possibilities for c, t_1 and t_2, each one with two possible a and b. Hence, there is a total of 49 normal forms if $p \equiv 1 \mod 4$. Analogously, there is a total of $\binom{6}{2} + 3 + 7 \cdot 2 = 32$ normal forms if $p \equiv 3 \mod 4$, and a total of $\binom{12}{2} + 7 + (9 + 11 + 9 + 9 + 11 + 9 + 11) \cdot 2 = 211$ normal forms if $p = 2$ (7 possibilities for c, some of them with 9 options for t_1 and t_2 and others with 11, and 2 for a and b).

From Theorem 5.33, case (1) produces $7 \cdot 4 = 28$ families of normal forms with one degree of freedom if $p \equiv 1 \mod 4$, $5 \cdot 4 = 20$ if $p \equiv 3 \mod 4$ and $11 \cdot 8 = 88$ if $p = 2$, case (2) produces one such family and case (3) produces 6, 6 and 14 families, respectively. Case (1) produces 16, 16 and 64 isolated forms, and case (4) produces 4, 4 and 8 such forms.

8.3. Comments on the p-adic classification in higher dimensions and proof of Theorem 5.37

Our strategy for the 4-dimensional case extends to any dimension, using the fact that all algebraic extensions of $\mathbb{Q}_p$ are extensions by radicals (though it would be needed to take higher order radicals) but for brevity we do not deal with those cases (we expect hundreds or even thousands of possibilities for the model matrices already in dimension 6, see Table 3). In dimension 10 or higher, however, the class of a given matrix cannot be determined by a formula involving radicals, as the general equation of degree five or greater is not solvable by radicals.

8.3.1. Preparatory lemmas. In order to prove Theorem 5.37, we need some lemmas.

LEMMA 8.15. *Let p be a prime number and let n be a positive integer. The polynomial*
$$P(x) = x^n - ap,$$

where $\mathrm{ord}_p(a) = 0$*, is irreducible in* $\mathbb{Q}_p$*.*

PROOF. The roots of P have order $1/n$. If P was reducible, a factor should have a subset of the roots whose product has integer order, but this would need all the n roots. □

LEMMA 8.16. *Let* p *be a prime number and let* n *be a positive integer. Let* $a, b \in \mathbb{Q}_p$ *such that* $\mathrm{ord}_p(a) = 0$ *and* $\mathrm{ord}_p(b) = 1$*. If* a *is an* n*-th power in* $\mathbb{Q}_p[b^{1/n}]$*, then it is an* n*-th power in* $\mathbb{F}_p$*.*

PROOF. Suppose that $a = c^n$ for $c \in \mathbb{Q}_p[b^{1/n}]$. We can write

$$c = c_0 + c_1 b^{\frac{1}{n}} + \ldots + c_{n-1} b^{\frac{n-1}{n}}$$

where $c_i \in \mathbb{Q}_p$ for all i. Raising this to the n-th power, we have $c^n = a$ at the left, and c_0^n plus terms of positive order at the right. Then, $a - c_0^n$ has positive order, and as it is in $\mathbb{Q}_p$ the order must be at least 1, and $a \equiv c_0^n \mod p$, as we wanted. □

LEMMA 8.17. *Let* p *be a prime number and let* n *be a positive integer. There are at least* $\gcd(2n, p-1) + \gcd(n, p-1)$*, if* n *is odd, and* $\gcd(2n, p-1)$*, if* n *is even, infinite families of blocks of size* $2n$ *in the normal form of a matrix up to multiplication by a symplectic matrix, where each family is of the form* $r_1 M_1 + \ldots + r_n M_n$*.*

PROOF. Consider the polynomial $P(x) = x^{2n} - ap$ where $\mathrm{ord}_p(a) = 0$. This is irreducible by Lemma 8.15, so it will give a block of size $2n$ in the normal form. This block may not be unique up to multiplication by a symplectic matrix (as happens in Propositions 6.22 and 6.24), but, in analogy with the proofs of those results, two blocks corresponding to different a will be in the same family only if the roots of the polynomials are in the same extension of $\mathbb{Q}_p$. Suppose that this happens for a_1 and a_2. In particular, $(a_1 p)^{1/2n}$ and $(a_2 p)^{1/2n}$ are in the same extension, that is,

$$\left(\frac{a_2}{a_1}\right)^{\frac{1}{2n}} \in \mathbb{Q}_p[(a_1 p)^{\frac{1}{2n}}]$$

By Lemma 8.16, a_2/a_1 must be a $2n$-th power in $\mathbb{F}_p$. This implies that the number of families of blocks is at least the cardinality of $\mathbb{F}_p^*$ modulo $2n$-th powers, which is $\gcd(2n, p-1)$, because that group is cyclic of order $p-1$.

If n is odd, we also consider

$$Q(x) = x^{2n} - a^2 p^2 = (x^n + ap)(x^n - ap).$$

The two factors are again irreducible and it also gives a block of size $2n$ (one factor comes from changing the sign of x in the other). Two blocks for a_1 and a_2 are in the same family only if $(a_1 p)^{1/n}$ and $(a_2 p)^{1/n}$ are in the same extension, that is,

$$\left(\frac{a_2}{a_1}\right)^{\frac{1}{n}} \in \mathbb{Q}_p[(a_1 p)^{\frac{1}{n}}]$$

(note that choosing $-ap$ instead of ap gives the same extension because $(-1)^{1/n} = -1$). Again by Lemma 8.16, a_2/a_1 is an n-th power in $\mathbb{F}_p$. So the number of families is now the cardinality of $\mathbb{F}_p^*$ modulo n-th powers, which is $\gcd(n, p-1)$. □

REMARK 8.18. Concerning the number of families of normal forms (instead of just blocks), in the real case, supposing that there are k focus-focus blocks, there are $2n-4k$ variables left, which can be distributed between hyperbolic and elliptic blocks in $n-2k+1$ ways. The total number of forms is

$$\sum_{k=0}^{m} 2m-2k+1 = 2m^2 - m(m+1) + m + 1 = m^2+1$$

if $n = 2m$ and

$$\sum_{k=0}^{m} 2m+1-2k+1 = m(2m+1) - m(m+1) + m + 1 = m^2+m+1$$

if $n = 2m+1$.

8.3.2. Proof of Theorem 5.37. Lemma 8.17 tells us that there is at least one block with each even size. Hence, the number of normal forms is at least the number of partitions of n in positive integers, that grows with $\mathrm{e}^{\pi\sqrt{2n/3}}/4n\sqrt{3}$ by the Hardy-Ramanujan formula [60]. In order to find the exact formulas of the matrices, we need to devise, for each partition, a matrix in $\mathcal{M}_{2n}(\mathbb{Q}_p)$ with the product of the corresponding factors as characteristic polynomial. This can be done with the same strategy as in Section 7.2, and gives the matrix $M(P,p)$ for each partition P.

8.3.3. Remarks and applications. Theorem 5.37 could be strengthened by using that there is not only *one* block of each size, but this would imply making a sum over the partitions. We do not know how to make that for general n, but we can do it for small n, obtaining the results in Table 3.

From the point of view of symplectic geometry and topology of integrable systems, which is the main motivation of the authors to write [27] and this part of the thesis, currently the only known global *symplectic* classifications of integrable systems which include physically intriguing local models (that is, essentially non-elliptic models) concern dimension 4 [95, 99, 100] in the real case. These real classifications include for example the coupled angular momentum [81] and the Jaynes-Cummings model [102]. Hence, in the p-adic case, with hundreds of local models (Theorem 5.22), we expect that the 4-dimensional case is already extremely complicated and that the $2n$-dimensional case, $n \geq 3$, is out of reach (since it is out of reach in the real case with only a very small proportion of local models in comparison, see Theorem 5.26).

In dimension 4 the authors analyzed one of these systems, the p-adic Jaynes-Cummings model [26], whose treatment is very extensive compared to its real counterpart, as expected. Although as we said, a classification of p-adic integrable systems in dimension 4, extending [95, 99, 100], seems out of reach, the present paper settles completely the first step: understanding explicitly p-adic local models. The proofs of [95, 99, 100] are based on gluing local models.

CHAPTER 9

Application to p-adic singularities and integrable systems

9.1. Application to normal forms of p-adic singularities

The Weierstrass-Williamson's classification of matrices can be used to classify critical points of p-adic analytic functions.

THEOREM 9.1. *Let F be a field and n be a positive integer. Every symplectic form on F^{2n} is linearly symplectomorphic to the form ω_0 which has as matrix*

$$\Omega_0 = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Hence, every two linear symplectic forms on F^{2n} are actually symplectomorphic.

PROOF. What we want to prove is that there is a basis with respect to which ω has the matrix Ω_0. Then, the symplectomorphism that sends this basis to the canonical one will send ω to ω_0.

We go by induction on n. As ω is non-degenerate, there are u_1 and v_1 with $\omega(u_1, v_1) = 1$. Of course, $\langle u_1, v_1\rangle$ is symplectic, so its complement $\langle u_1, v_1\rangle^\omega$ is also symplectic. Applying induction to this complement, we get a basis $\{u_2, v_2, \ldots, u_n, v_n\}$. Now, $\{u_1, v_1, \ldots, u_n, v_n\}$ is the basis we are looking for. □

We refer to section 5.1.4 for the definition of analytic function and critical point of a function on a p-adic manifold. It does not make sense to talk about the rank of such a critical point, because there is only one function and consequently only one differential form.

DEFINITION 9.2 (Non-degenerate critical point of p-adic analytic function, symplectic sense). Let n be a positive integer. Let p be a prime number. Let (M, ω) be a $2n$-dimensional p-adic analytic symplectic manifold. Let $f : M \to \mathbb{Q}_p$ be a p-adic analytic function and let $m \in M$ be a critical point of f (i.e. $\mathrm{d}f(m) = 0$). Let Ω be the matrix of ω. We say that m is *non-degenerate* if the eigenvalues of $\Omega^{-1}\mathrm{d}^2 f(m)$ are all distinct.

This is not the usual notion of non-degenerate critical point (which states that the Hessian of f is invertible), but the two are related as the following proposition shows.

Proposition 9.3. *Let n be a positive integer. Let p be a prime number. Let M be a p-adic analytic $2n$-dimensional manifold, $f : M \to \mathbb{Q}_p$ a p-adic analytic function, and m a critical point of f. Then the following are equivalent:*

1. *m is a non-degenerate critical point of f in the usual sense;*
2. *There exists a linear symplectic form ω such that m is a non-degenerate critical point of $f : (M, \omega) \to \mathbb{Q}_p$ in the symplectic sense (Definition 9.2).*
3. *There exist infinitely many linear symplectic forms ω such that m is a non-degenerate critical point of $f : (M, \omega) \to \mathbb{Q}_p$ in the symplectic sense.*

Proof. Suppose (2) holds. Let Ω be the matrix of ω. Then $\Omega^{-1}\mathrm{d}^2 f(m)$ has all eigenvalues distinct. Applying Lemma 6.3 to the Hessian, if zero was an eigenvalue, it would be at least double, contradicting (2). So $\Omega^{-1}\mathrm{d}^2 f(m)$ is invertible, which implies $\mathrm{d}^2 f(m)$ is invertible and (1) holds.

Now suppose (1) holds. Let $H = \mathrm{d}^2 f(m)$. We first solve the problem for H diagonal and then the general case.

If H is diagonal, (1) means that all diagonal elements are nonzero. Let h_i be the i-th diagonal element. We will take Ω^{-1} with a_i in the row $2i-1$ and column $2i$, $-a_i$ in the row $2i$ and column $2i-1$, and 0 the rest. After multiplying Ω^{-1} by H, a_i becomes $a_i h_{2i-1}$ and $-a_i$ becomes $-a_i h_{2i}$. The eigenvalues are

$$\pm a_i\sqrt{-h_{2i-1}h_{2i}} \text{ for } 1 \le i \le n.$$

Since $\mathbb{Q}_p$ is infinite, it is always possible to choose a_i so that these values are all different, independently of whether they are in $\mathbb{Q}_p$ or not (just choose each a_i in turn, and there will always be a possible value).

In the general case, we can apply Gram-Schmidt orthogonalization to the canonical basis to obtain a basis $\{v_1, \ldots, v_n\}$ such that $v_i^T H v_j \neq 0$ if and only if $i = j$. Taking these vectors as columns, we have a matrix M such that $M^T H M$ is diagonal. Applying the diagonal case to this matrix, we obtain an antisymmetric Ω such that $\Omega^{-1} M^T H M$ has all eigenvalues different. This matrix is similar to $M\Omega^{-1}M^T H$, so this has also all eigenvalues different, and $M\Omega^{-1}M^T$ is the matrix we want.

That (3) implies (2) is trivial. If (2) holds, $\Omega^{-1}M$ has all eigenvalues different and the same happens for any perturbation of Ω, hence (3) holds. □

In the following when we speak of non-degenerate critical points of a function we always de it in the symplectic sense of Definition 9.2.

In the real case, as a consequence of the Weierstrass-Williamson classification, it is always possible to choose linear symplectic coordinates $(x_1, \xi_1, \ldots, x_n, \xi_n)$ with the origin at any $m \in M$ such that

$$f = \sum_{i=1}^{n} r_i g_i + \mathcal{O}(3)$$

for some $r_i \in \mathbb{R}$, where $g_i : V \to \mathbb{R}$ has one of the following forms: $(x_i^2 + \xi_i^2)/2$ (*elliptic component*), $x_i\xi_i$ (*hyperbolic component*), or $x_i\xi_{i+1} - x_{i+1}\xi_i$ with the next function equal to $x_i\xi_i + x_{i+1}\xi_{i+1}$ (*focus-focus component*). In the p-adic case we are also able to make this conclusion.

Lemma 9.4. *Let n be a positive integer. Let p be a prime number. Let M be a p-adic analytic manifold of dimension $2n$ and let m be a non-degenerate critical point of a p-adic analytic function $f : M \to \mathbb{Q}_p$. Then there exists an open neighborhood*

U of m and coordinates $(x_1, \ldots, x_n)$ with the origin at m such that the restriction of f to U, that is, $f|_U : U \to \mathbb{Q}_p$ is given by a power series

$$\sum_{I \in \mathbb{N}^n, i_1+\ldots+i_n \geq 2} a_I x_1^{i_1} \ldots x_n^{i_n}.$$

Moreover, the matrix in $\mathcal{M}_{2n}(\mathbb{Q}_p)$ with the coefficient of $x_i x_j$ in the row i and column j, for $i \neq j$, and the coefficient of x_i^2 multiplied by 2 in the row and column i, is exactly the Hessian of f at m in the coordinates $x_1, \ldots, x_n$.

Proof. By definition, f is given by a power series converging in some open set U which contains m. The center of the power series can be arbitrarily chosen in U, which means that we can choose m as center. The degree 1 terms are 0 because m is a critical point, and the degree 2 terms are of the form $x^T Hx/2$ for some matrix $H \in \mathcal{M}_{2n}(\mathbb{Q}_p)$. Differentiating twice, we get that the Hessian of f is precisely H. □

Corollary 9.5 (Normal form of critical points in dimension 2). *Let p be a prime number. Let (M, ω) be a p-adic symplectic manifold of dimension 2 and let $f : M \to \mathbb{Q}_p$ be a p-adic analytic function. Let $m \in M$ be a critical point of f. Then there are linear symplectic coordinates (x, ξ) with the origin at m such that $f - f(m)$ coincides with $r(x^2 + c\xi^2)$ up to order 2, for some $r \in \mathbb{Q}_p$ and $c \in X_p$, or $r \in Y_p \cup \{1\}$ and $c = 0$. Furthermore, if $f - f(m)$ has this form for two different linear symplectic coordinates with the origin at m, then the two forms coincide.*

Proof. By Theorem 9.1, we can assume without loss of generality that $\omega_m = \omega_0$.

Applying Theorem 5.28 to $\mathrm{d}^2 f$, we get a symplectic matrix S, which is the matrix of a linear symplectomorphism ϕ, such that

$$\mathrm{d}^2(\phi^* f) = S^T \mathrm{d}^2 f S = \begin{pmatrix} r & 0 \\ 0 & cr \end{pmatrix} = \mathrm{d}^2(rx^2 + cr\xi^2).$$

By Lemma 9.4, f has the desired form. Uniqueness follows from the same theorem. □

Corollary 9.6. *Let p be a prime number. Let (M, ω) be a p-adic analytic symplectic 2-manifold. Let X_p, Y_p be the non-residue sets in Definition 5.18. Then the following statements hold.*

(1) *If $p \equiv 1 \mod 4$, there are exactly 7 families of local linear normal forms for a non-degenerate critical point of a p-adic analytic analytic function $f : (M, \omega) \to \mathbb{Q}_p$ up to local linear symplectomorphisms centered at the critical point, where the normal forms in each family differ by multiplication by a constant r, and exactly 4 normal forms for a degenerate critical point which only differ in the constant r.*
(2) *If $p \equiv 3 \mod 4$, there are exactly 5 families of local linear normal forms for a non-degenerate critical point of a p-adic analytic analytic function $f : (M, \omega) \to \mathbb{Q}_p$ up to local linear symplectomorphisms centered at the critical point, where the normal forms in each family differ by multiplication by a constant r, and exactly 4 normal forms for a degenerate critical point which only differ in the constant r.*

(3) *If $p = 2$, there are exactly 11 families of local linear normal forms for a non-degenerate critical point of a p-adic analytic analytic function $f : (M, \omega) \to \mathbb{Q}_p$ up to local linear symplectomorphisms centered at the critical point, where the normal forms in each family differ by multiplication by a constant r, and exactly 8 normal forms for a degenerate critical point which only differ in the constant r.*

In the three cases the above normal forms for a non-degenerate point are given by

$$\left\{\left\{r(x^2 + c\xi^2) : r \in \mathbb{Q}_p\right\} : c \in X_p\right\}$$

and those for a degenerate point are given by

$$\left\{rx^2 : r \in Y_p \cup \{1\}\right\}.$$

Proof. This follows from Theorem 5.30 and Corollary 9.5. □

Corollary 9.7 (Normal form of non-degenerate critical points in dimension 4). *Let p be a prime number. Let $X_p, Y_p, \mathcal{C}_i^k, \mathcal{D}_i^k$ be the non-residue sets and coefficient functions in Definition 5.18. Let (M, ω) be a p-adic symplectic manifold of dimension 4 and let $f : M \to \mathbb{Q}_p$ be a p-adic analytic function. Let $m \in M$ be a non-degenerate critical point of f. Then there are linear symplectic coordinates (x, ξ, y, η) with the origin at m such that in these cordinates we have:*

$$f - f(m) = rg_1 + sg_2 + \mathcal{O}(3),$$

where g_1 and g_2 have one of the following forms:

(1) $g_1(x, \xi, y, \eta) = x^2 + c_1\xi^2$ *and* $g_2(x, \xi, y, \eta) = y^2 + c_2\eta^2$*, for* $c_1, c_2 \in X_p$.
(2) $g_1(x, \xi, y, \eta) = x\eta + cy\xi$ *and* $g_2(x, \xi, y, \eta) = x\xi + y\eta$*, for* $c \in Y_p$.
(3)

$$g_k(x, \xi, y, \eta) = \sum_{i=0}^{2} \mathcal{C}_i^k(c, t_1, t_2, a, b)x^i y^{2-i} + \sum_{i=0}^{2} \mathcal{D}_i^k(c, t_1, t_2, a, b)\xi^i \eta^{2-i},$$

for $k \in \{1, 2\}$, where c, t_1 and t_2 correspond to one row of Table 1 and (a, b) are either $(1, 0)$ or (a_1, b_1) of the corresponding row.

If $f - f(m)$ has this form for two different linear symplectic coordinates, then the two forms coincide, except perhaps for swapping g_1 and g_2 at point (1).

Proof. Analogously to proof of Corollary 9.5, we apply Theorem 5.31 to the Hessian of f, and by Lemma 9.4, f has the desired form. The three cases (1), (2) and (3) for the resulting matrix correspond to the three cases of this corollary, because

$$\mathrm{d}^2(\phi^* f) = S^T \mathrm{d}^2 f S = r\mathrm{d}^2 g_1 + s\mathrm{d}^2 g_2 = \mathrm{d}^2(rg_1 + sg_2). \qquad \square$$

Corollary 9.8 (Normal form of degenerate critical points in dimension 4). *Let p be a prime number. let X_p, Y_p be the non-residue sets in Definition 5.18. Let $h_p : Y_p \to \mathbb{Q}_p$ be the non-residue function in Definition 5.32. Let (M, ω) be a p-adic symplectic manifold of dimension 4 and let $f : M \to \mathbb{Q}_p$ be a p-adic analytic function. Let $m \in M$ be a degenerate critical point of f. Then there are linear symplectic coordinates (x, ξ, y, η) with the origin at m such that $f - f(m)$ coincides with one of the following forms up to order 2:*

(1) $r(x^2+c_1\xi^2)/2+s(y^2+c_2\eta^2)/2$, *for some* $c_1,c_2\in X_p\cup\{0\}$ *and* $r,s\in\mathbb{Q}_p$. *If* $c_1=0$, r *can be taken in* $Y_p\cup\{1\}$, *and if* $c_2=0$, s *can be taken in* $Y_p\cup\{1\}$.
(2) $r(x\xi+y\eta)+y\xi$, *for some* $r\in\mathbb{Q}_p$.
(3) $r(x\eta+cy\xi)+a(x^2+y^2)/2$, *for some* $r\in\mathbb{Q}_p, c\in Y_p, a\in\{1,h_p(c)\}$.
(4) $c(x^2/2+\xi\eta)$, *for some* $c\in Y_p\cup\{1\}$.

Furthermore, if $f-f(m)$ *has this form for two different linear symplectic coordinates, then the two forms coincide, except perhaps for swapping* (r,c_1) *and* (s,c_2) *at point (1).*

Proof. It is analogous the previous two proofs, but with Theorem 5.33. ☐

Corollary 9.9. *Let* p *be a prime number. Let* (M,ω) *be a* p*-adic analytic symplectic* 4*-manifold. Let* $X_p, Y_p, \mathcal{C}_i^k, \mathcal{D}_i^k$ *be the non-residue sets and coefficient functions in Definition 5.18. Let* $h_p : Y_p\to\mathbb{Q}_p$ *be the non-residue function in Definition 5.32. Then the following statements hold.*

(1) *If* $p\equiv 1 \mod 4$*, there are exactly* 49 *infinite families of local linear normal forms with two degrees of freedom for a critical point of a* p*-adic analytic function on a* 4*-dimensional* p*-adic symplectic manifold* $f:(M,\omega)\to\mathbb{Q}_p$ *up to local linear symplectomorphisms centered at the critical point, exactly* 35 *infinite families with one degree of freedom, and exactly* 20 *isolated normal forms.*
(2) *If* $p\equiv 3 \mod 4$*, there are exactly* 32 *infinite families of local linear normal forms with two degrees of freedom for a critical point of a* p*-adic analytic function on a* 4*-dimensional* p*-adic symplectic manifold* $f:(M,\omega)\to\mathbb{Q}_p$ *up to local linear symplectomorphisms centered at the critical point, exactly* 27 *infinite families with one degree of freedom, and exactly* 20 *isolated normal forms.*
(3) *If* $p=2$*, there are exactly* 211 *infinite families of local linear normal forms with two degrees of freedom for a critical point of a* p*-adic analytic function on a* 4*-dimensional* p*-adic symplectic manifold* $f:(M,\omega)\to\mathbb{Q}_p$ *up to local linear symplectomorphisms centered at the critical point, exactly* 103 *infinite families with one degree of freedom, and exactly* 72 *isolated normal forms.*

In the three cases above, the infinite families with two degrees of freedom are given as

$$\Big\{\Big\{r(x^2+c_1\xi^2)+s(y^2+c_2\eta^2):r,s\in\mathbb{Q}_p\Big\}:c_1,c_2\in X_p\Big\}$$
$$\cup\Big\{\Big\{r(x\eta+cy\xi)+s(x\xi+y\eta):r,s\in\mathbb{Q}_p\Big\}:c\in Y_p\Big\}$$
$$\cup\Big\{\Big\{r\left(\sum_{i=0}^{2}\mathcal{C}_i^1(c,t_1,t_2,a,b)x^iy^{2-i}+\sum_{i=0}^{2}\mathcal{D}_i^1(c,t_1,t_2,a,b)\xi^i\eta^{2-i}\right)$$
$$+s\left(\sum_{i=0}^{2}\mathcal{C}_i^2(c,t_1,t_2,a,b)x^iy^{2-i}+\sum_{i=0}^{2}\mathcal{D}_i^2(c,t_1,t_2,a,b)\xi^i\eta^{2-i}\right):$$
$$r,s\in\mathbb{Q}_p\Big\}:(a,b)\in\Big\{(1,0),(a_1,b_1)\Big\},c,t_1,t_2,a_1,b_1 \text{ in one row of Table 1}\Big\},$$

those with one degree of freedom are

$$\Big\{\Big\{r(x^2+c_1\xi^2)+sy^2/2 : r\in\mathbb{Q}_p\Big\} : c_1\in X_p, s\in Y_p\cup\{1\}\Big\}\cup\{\{r(x\xi+y\eta)+y\xi : r\in\mathbb{Q}_p\}\}$$
$$\cup\Big\{\Big\{r(x\eta+cy\xi)+a(x^2+y^2)/2 : r\in\mathbb{Q}_p\Big\} : c\in Y_p, a\in\{1,h_p(c)\}\Big\},$$

and the isolated forms are

$$\Big\{(rx^2+sy^2)/2 : r,s\in Y_p\cup\{1\}\Big\}\cup\Big\{c(x^2/2+\xi\eta) : c\in Y_p\cup\{1\}\Big\}.$$

Here by "infinite family" we mean a family of normal forms of the form $r_1f_1+r_2f_2+\ldots+r_kf_k$*, where* r_i *are parameters and* k *is the number of degrees of freedom, and by "isolated" we mean a form that is not part of any family.*

9.2. Application to normal forms of singularities of integrable systems

The Weierstrass-Williamson classification is one of the foundational results used in the symplectic theory of integrable systems (in particular in Eliasson's linearization theorems [**49, 50**]). A consequence of the Weierstrass-Williamson classification states that, given an integrable system $F=(f_1,\ldots,f_n):(M,\omega)\to\mathbb{R}^n$ and a *non-degenerate* critical point m of F (in a precise sense which we will define shortly), it is always possible to choose linear symplectic coordinates $(x_1,\xi_1,\ldots,x_n,\xi_n)$ with the origin at m such that in these coordinates

$$B\circ(F-F(m))=(g_1,\ldots,g_n)+\mathcal{O}(3),$$

where B is a n-by-n matrix of reals and each $g_i, i\in\{1,\ldots,n\}$ has one of the following forms: ξ_i (*regular component*), $(x_i^2+\xi_i^2)/2$ (*elliptic component*), $x_i\xi_i$ (*hyperbolic component*), or $x_i\xi_{i+1}-x_{i+1}\xi_i$ with the next function equal to $x_i\xi_i+x_{i+1}\xi_{i+1}$ (*focus-focus component*). See Figure 1 for a representation.

9.2.1. Non-degenerate critical points of integrable systems.

As we see next, the classification theorems for critical points of functions on symplectic manifolds can be applied to classify critical points of integrable systems. In order to do this, first we recall the notion of non-degeneracy for a critical point of an integrable system $F:(M,\omega)\to(\mathbb{Q}_p)^n$ on a p-adic analytic symplectic manifold which we use in the paper, and which in the real case is equivalent to the usual definition in Vey's paper [**132**], see for example [**101**, Section 4.2.1] and [**46**, Lemma 2.5].

Definition 9.10. Let n be a positive integer. Let p be a prime number. Let (M,ω) be a p-adic analytic symplectic manifold of dimension $2n$. Let $F=(f_1,\ldots,f_n):(M,\omega)\to(\mathbb{Q}_p)^n$ be a p-adic analytic integrable system. A point $m\in M$ is a *critical point* of m if the 1-forms $\mathrm{d}f_1(m),\ldots,\mathrm{d}f_n(m)$ are linearly dependent. The number of linearly independent forms among $\mathrm{d}f_1(m),\ldots,\mathrm{d}f_n(m)$ is called the *rank* of the critical point.

In the following definition, a subspace U of a symplectic vector space (V,ω) is said to be *isotropic* if $\omega(u,v)=0$ for any $u,v\in U$.

Definition 9.11 (Non-degenerate critical point of p-adic analytic integrable system). Let n be a positive integer. Let p be a prime number. Let (M,ω) be a p-adic analytic symplectic manifold. Let Ω be the matrix of the linear symplectic form ω_m on the vector space T_mM. A rank 0 critical point m of a p-adic analytic integrable system $F = (f_1, \ldots, f_n) : (M, \omega) \to (\mathbb{Q}_p)^n$ is *non-degenerate* if the Hessians evaluated at m:

$$\mathrm{d}^2 f_1(m), \ldots, \mathrm{d}^2 f_n(m)$$

are linearly independent and if there exist $a_1, \ldots, a_n \in \mathbb{Q}_p$ such that the matrix

$$\Omega^{-1} \sum_{i=1}^{n} a_i \mathrm{d}^2 f_i(m)$$

has n different eigenvalues.

If m has rank r, then the vectors $X_{f_1}(m), \ldots, X_{f_n}(m)$ obtained by evaluating the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ of $f_1, \ldots, f_n$ at m, form an isotropic linear subspace L of $\mathrm{T}_m M$, whose dimension is r; suppose that $X_{f_1}(m), \ldots, X_{f_r}(m)$ are linearly independent. Then $\mathrm{d}f_{r+1}, \ldots, \mathrm{d}f_n$ descend to L^ω / L in such a way that the origin is a rank 0 critical point of the integrable system induced by F on L^ω/L. We say that the point is *non-degenerate* if the origin is a non-degenerate critical point of this induced integrable system.

Remark 9.12. Definition 9.11 in the p-adic case is motivated by the fact that in the real case the notion of being non-degenerate for a critical point on an integrable system can also be defined in this way. Indeed, the usual definition is given in terms of Cartan subalgebras as follows: if (M, ω) is a real symplectic manifold of dimension $2n$ and $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ is an integrable system, a critical point m of F of rank 0 is *non-degenerate* if the Hessians $\mathrm{d}^2 f_1(m), \ldots, \mathrm{d}^2 f_n(m)$ span a Cartan subalgebra of the symplectic Lie algebra of quadratic forms on the tangent space $(\mathrm{T}_m M, \omega_m)$.

A p-adic analytic integrable system $(f_1, \ldots, f_n) : (M, \omega) \to (\mathbb{Q}_p)^n$ is *non-degenerate* if all of its critical points are non-degenerate.

9.2.2. Degenerate critical points in the real case. In the real case, very little is known about *degenerate* singularities. In the literature only some results are available for some special kinds of degenerate points, see for instance [**143, 144**]. We give here a partial classification for real integrable systems (which must be smooth but need not be analytic).

Theorem 9.13. *Let (M, ω) be a real symplectic 4-manifold and let $F = (f_1, f_2) : (M, \omega) \to \mathbb{R}^2$ be a real integrable system. Let $m \in M$ be a degenerate critical point of F. Then one of the following two statements holds:*

(1) *There exist $a_1, a_2 \in \mathbb{R}$ and a linear combination $g = a_1 f_1 + a_2 f_2$, such that the four eigenvalues of $\Omega^{-1}\mathrm{d}^2 g(m)$ are zero.*
(2) *For all linear combinations $g = a_1 f_1 + a_2 f_2$, where $a_1, a_2 \in \mathbb{R}$, two of the eigenvalues of $\Omega^{-1}\mathrm{d}^2 g(m)$ are zero.*

Proof. As m is a degenerate critical point, $\Omega^{-1}\mathrm{d}^2 f_1(m)$ has a multiple eigenvalue. There are three cases to consider:

- All the eigenvalues of $\Omega^{-1}\mathrm{d}^2 f_1(m)$ are zero. We are immediately in case (1).
- Two eigenvalues of $\Omega^{-1}\mathrm{d}^2 f_1(m)$ are zero. Let $g_k = f_1 + k f_2$. The eigenvalues of $\Omega^{-1}\mathrm{d}^2 g_k(m)$ vary continuously with k, and they must include a multiple eigenvalue for any k. For $k = 0$, two eigenvalues are zero and the other two are opposite and nonzero. This implies that, for $|k|$ sufficiently small, $\Omega^{-1}\mathrm{d}^2 g_k(m)$ has also two zero eigenvalues. The values of k

for which this happens are either finitely many or all $\mathbb{R}$, hence, it must happen for all $k \in \mathbb{R}$.

This means that the condition of case (2) holds when $a_1 = 1$, which trivially implies the same for all $a_1 \neq 0$. For $a_1 = 0$, the result follows by continuity.

- $\Omega^{-1}\mathrm{d}^2 f_1(m)$ is invertible. By Lemma 6.6, the eigenvalues of $\Omega^{-1}\mathrm{d}^2 f_1(m)$ are either of the form
$$\{r, r, -r, -r\}$$
or
$$\{r\mathrm{i}, r\mathrm{i}, -r\mathrm{i}, -r\mathrm{i}\},$$
for some $r \in \mathbb{R}$ with $r \neq 0$, and the matrix can be brought by a transformation Ψ to Jordan form, which is one of the forms
$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix},$$
where λ is r or $r\mathrm{i}$. We call this matrix D_1. The same Ψ must bring $\Omega^{-1}\mathrm{d}^2 f_1(m)$ to a form that commutes with D_1: we call it D_2. We have that
$$D_2 = \begin{pmatrix} \mu_1 & 0 & \mu_2 & 0 \\ 0 & \mu_5 & 0 & \mu_6 \\ \mu_3 & 0 & \mu_4 & 0 \\ 0 & \mu_7 & 0 & \mu_8 \end{pmatrix}$$
and
$$\Psi^T \mathrm{d}^2 f_2 \Psi = \Psi^T \Omega_0 \Psi \Psi^{-1} \Omega_0^{-1} \mathrm{d}^2 f_2 \Psi = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix} D_2$$
for some $a, b \in \mathbb{C}$, must be a symmetric matrix, which implies $\mu_1 = -\mu_5$, $\mu_4 = -\mu_8$, $a\mu_2 = -b\mu_7$ and $b\mu_3 = -a\mu_6$. The last two equalities together imply $\mu_2\mu_3 = \mu_6\mu_7$.

If D_1 has the non-diagonal form, $D_1 D_2 = D_2 D_1$ implies that $\mu_3 = \mu_6 = 0$, $\mu_1 = \mu_4$ and $\mu_5 = \mu_8$. We can make a linear combination of the two matrices where the diagonal is 0, and whose eigenvalues will all be zero, and we are done. Now we suppose that D_1 is diagonal.

If D_1 is real, the matrix Ψ is and consequently D_2 are real. Otherwise, we have $\Omega_0^{-1}\mathrm{d}^2 f_2 \Psi = \Psi D_2$ and the columns of Ψ have the form $(u, \bar{u}, v, \bar{v})$, this gives
$$\Omega_0^{-1}\mathrm{d}^2 f_2 \begin{pmatrix} u & \bar{u} & v & \bar{v} \end{pmatrix} = \begin{pmatrix} \mu_1 u + \mu_3 v & \mu_5 \bar{u} + \mu_7 \bar{v} & \mu_2 u + \mu_4 v & \mu_6 \bar{u} + \mu_8 \bar{v} \end{pmatrix}$$
which implies that
$$\overline{\mu_1 u + \mu_3 v} = \mu_5 \bar{u} + \mu_7 \bar{v}$$
and
$$\overline{\mu_2 u + \mu_4 v} = \mu_6 \bar{u} + \mu_8 \bar{v},$$

and as u and v are linearly independent, we must have $\bar{\mu}_1 = \mu_5 = -\mu_1$ and $\bar{\mu}_4 = \mu_8 = -\mu_4$. That is, the diagonal of D_2 is real if D_1 is real, and imaginary if D_1 is imaginary.

Let $\mu_1 = r_1\alpha$ and $\mu_4 = r_2\alpha$, for r_1, r_2 in $\mathbb{R}$ and $\alpha \in \{1, \mathrm{i}\}$. By adding a multiple of D_1, we can make $r_2 = -r_1$, so that D_2 has the form

$$\begin{pmatrix} r_1\alpha & 0 & \mu_2 & 0 \\ 0 & -r_1\alpha & 0 & \mu_6 \\ \mu_3 & 0 & -r_1\alpha & 0 \\ 0 & \mu_7 & 0 & r_1\alpha \end{pmatrix},$$

and its characteristic polynomial is

$$(t^2 - r_1^2\alpha^2 - \mu_2\mu_3)(t^2 - r_1^2\alpha^2 - \mu_6\mu_7) = (t^2 - r_1^2\alpha^2 - \mu_2\mu_3)^2.$$

We now prove that $r_1^2\alpha^2 + \mu_2\mu_3$ must be 0, that is, the characteristic polynomial is t^4 and the conclusion follows. Suppose, for a contradiction, that it is not 0. Then D_2 has double eigenvalues different from zero, in the form

$$\{s, s, -s, -s\}$$

or

$$\{s\mathrm{i}, s\mathrm{i}, -s\mathrm{i}, -s\mathrm{i}\},$$

and each pair of double eigenvalues has an element coming from each factor.

Consider the matrix $D_3 = D_2 + kD_1$, where k is a real number close to zero. The characteristic polynomial of D_3 is still a product of two factors, but now they are different. For small enough k, the only way to have multiple eigenvalues is to have the same form as in D_2, that is, two real or imaginary pairs with an element from each factor. But this is not possible if the two factors are different. □

9.2.3. Proof of Theorems 5.19, 5.22 and 5.26. By Theorem 9.1, we may assume without loss of generality that $M = (\mathbb{Q}_p)^4$ and $\omega_m = \omega_0$.

For a rank 1 critical point, we can take $\mathrm{d}\eta$ in the direction of the nonzero differential, and the problem reduces to classify the critical point in L^{ω_0}/L. This is a system with one function (the linear combination of f_1 and f_2 with differential 0) in dimension 2, so we apply Corollary 9.5 and get the result.

For a rank 0 critical point, first we prove existence. The fact that f_1 and f_2 form an integrable system implies that $\{f_1, f_2\} = 0$, that is,

$$(\mathrm{d}f_1)^T \Omega_0^{-1} \mathrm{d}f_2 = 0.$$

Differentiating this twice, evaluating at m and using that $\mathrm{d}f_1(m) = \mathrm{d}f_2(m) = 0$,

$$\mathrm{d}^2 f_1(m) \Omega_0^{-1} \mathrm{d}^2 f_2(m) = \mathrm{d}^2 f_2(m) \Omega_0^{-1} \mathrm{d}^2 f_1(m)$$

We define $A_i = \Omega_0^{-1}\mathrm{d}^2 f_i(m)$. The previous expression implies that A_1 and A_2 commute.

Let $u \in (\mathbb{C}_p)^4$ be an eigenvector of A_1, where $\mathbb{C}_p$ is the field of complex p-adic numbers. Then

$$A_1 A_2 u = A_2 A_1 u = \lambda A_2 u$$

for $\lambda \in \mathbb{C}_p$. This implies that $A_2 u$ is also an eigenvector of A_1 with value λ. But the critical point is non-degenerate, which means that the only eigenvector with value λ is u. Hence, $A_2 u = \mu u$ for some $\mu \in \mathbb{C}_p$, and u is also eigenvector of A_2. So A_1 and A_2 have the same eigenvectors.

In the proof of Theorem 5.31 (Section 8.2.3), the case in which the Hessian of f falls and the values of the parameters (c_1, c_2, c, a, etc.) in the normal form are determined by the eigenvectors. This means that S is the same matrix for f_1 than for f_2, and the resulting normal forms are $r_1 g_1 + s_1 g_2$ and $r_2 g_1 + s_2 g_2$, for some $r_1, r_2, s_1, s_2 \in \mathbb{Q}_p$ and g_1 and g_2 are among the possibilities of Corollary 9.7. As the Hessians are linearly independent, the matrix

$$\begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}$$

changing (g_1, g_2) to (f_1, f_2) can be inverted, giving the matrix B that we need, and the proof of existence of Theorem 5.19 is complete. Uniqueness follows directly from Corollary 9.7.

Theorems 5.22 and 5.26 follow from applying Theorems 5.34 and 5.37, respectively, to the Hessians of the components of the system: each normal form of the matrix gives a normal form of the integrable system, and the Hessian of $f_{P,p}$ is exactly the matrix $M(P, p)$.

9.2.4. Degenerate critical points in the p-adic case. Based on Theorem 5.33, a version of Theorem 5.19 can also be deduced for degenerate singularities (a topic of growing interest in real symplectic geometry; see for instance [**46, 47, 63**] and the references therein) but the statement will be more complicated than that of Theorem 5.19. The reason is that, while for non-degenerate singularities the types (in the sense of Corollary 9.7) of f_1 and f_2 must coincide if these functions Poisson-commute, this does not happen for degenerate singularities. For example, the function of type (1) in Corollary 9.8 with $r = s = 1$ and $c_1 = c_2 = -1$ Poisson-commutes with the one of type (2) in the same list with $r = 1$. Hence, a full classification will need many more cases. We can give a partial result analogous to Theorem 9.13:

THEOREM 9.14. *Let p be a prime number. Let (M, ω) be a 4-dimensional p-adic analytic symplectic manifold. Let $F = (f_1, f_2) : (M, \omega) \to (\mathbb{Q}_p)^2$ be a p-adic analytic integrable system and let $m \in M$ be a degenerate rank 0 critical point of F. Then one of the following two statements holds:*

(1) *There are $a_1, a_2 \in \mathbb{Q}_p$ and a linear combination $g = a_1 f_1 + a_2 f_2$, such that all four eigenvalues of $\Omega^{-1}\mathrm{d}^2 g$ are zero;*
(2) *For all linear combinations $g = a_1 f_1 + a_2 f_2$, where $a_1, a_2 \in \mathbb{Q}_p$, two eigenvalues of $\Omega^{-1}\mathrm{d}^2 g$ are zero.*

PROOF. This proof follows the same strategy as in the real case (Theorem 9.13), we include it here for completeness.

As m is degenerate, for every linear combination $g = a_1 f_1 + a_2 f_2$, m is a critical point of g with repeated eigenvalues. Concretely this happens for f_1 and f_2 themselves. There are three cases to consider:

- $\Omega^{-1}\mathrm{d}^2 f_1$ has all eigenvalues zero, and we are in case (1).
- $\Omega^{-1}\mathrm{d}^2 f_1$ has exactly two eigenvalues zero. Since the other two are nonzero, they are opposites.

  Consider $g_k = f_1 + k f_2$, for $k \in \mathbb{Q}_p$. Since m is degenerate, $\Omega^{-1}\mathrm{d}^2 g_k$ has a multiple eigenvalue for any k. If k is small enough, the only way to have a multiple eigenvalue is that the same two that coincide for $k = 0$ keep coinciding. But as they must be opposites, they must still coincide at zero. This means that $\Omega^{-1}\mathrm{d}^2 g_k$ has two eigenvalues equal to zero for all k small enough. But the values of k for which this happens are either finitely many or all $\mathbb{Q}_p$, so it must happen for all k.

It follows that case (2) holds when $a_1 = 1$, and can be extended trivially to all $a_1 \neq 0$. For $a_1 = 0$, it must also hold by continuity.

- $\Omega^{-1}\mathrm{d}^2 f_1$ is invertible. By Corollary 9.8, we can bring f_1 to a normal form where $\omega_m = \omega_0$. The ones with an invertible Hessian are in case (1) with $c_1 \neq 0$ and $c_2 \neq 0$, case (2) with $r \neq 0$, and case (3) with $r \neq 0$. In the three cases, the eigenvalues are of the form $\{\lambda, \lambda, -\lambda, -\lambda\}$, where $\lambda^2 \in \mathbb{Q}_p$ (in some of them $\lambda \in \mathbb{Q}_p$ and in others it is in a degree two extension).

By Lemma 6.6, we can bring the matrix $\Omega_0^{-1}\mathrm{d}^2 f_1$ to Jordan form by a transformation Ψ, which is one of the two forms

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{pmatrix}.$$

We call this matrix D_1. The matrix $\Omega_0^{-1}\mathrm{d}^2 f_2$ will become, by the same transformation Ψ, a matrix D_2 that commutes with D_1, so it has the form

$$\begin{pmatrix} \mu_1 & 0 & \mu_2 & 0 \\ 0 & \mu_5 & 0 & \mu_6 \\ \mu_3 & 0 & \mu_4 & 0 \\ 0 & \mu_7 & 0 & \mu_8 \end{pmatrix}.$$

We have that $\Psi^T \mathrm{d}^2 f_2 \Psi$ is symmetric, which leads to $\mu_1 = -\mu_5$, $\mu_4 = -\mu_8$, and $\mu_2\mu_3 = \mu_6\mu_7$.

If D_1 has the second form above, $D_1 D_2 = D_2 D_1$ also implies $\mu_3 = \mu_6 = 0$, $\mu_1 = \mu_4$ and $\mu_5 = \mu_8$. Adding an adequate multiple of D_1 to D_2, we can make $\mu_1 = \mu_4 = \mu_5 = \mu_8 = 0$, so that this new matrix has all eigenvalues zero, and we are done. Now we suppose that D_1 has the first form.

If $\lambda \in \mathbb{Q}_p$, the matrix Ψ is in $\mathbb{Q}_p$, so D_2 is also in $\mathbb{Q}_p$. If $\lambda \notin \mathbb{Q}_p$, we have $\Omega_0^{-1}\mathrm{d}^2 f_2 \Psi = \Psi D_2$. Taking the columns of Ψ in the form $(u, \bar{u}, v, \bar{v})$, as happens in the real case but now $\bar{u}$ means the conjugate in $\mathbb{Q}_p[\lambda]$, we arrive at $\bar{\mu}_1 = \mu_5 = -\mu_1$ and $\bar{\mu}_4 = \mu_8 = -\mu_4$. This means that the elements in the diagonal of D_2 are multiples of λ with coefficients in $\mathbb{Q}_p$, independently of whether λ is in $\mathbb{Q}_p$ or not.

Let $\mu_1 = r_1\lambda$ and $\mu_4 = r_2\lambda$, for r_1, r_2 in $\mathbb{Q}_p$. By adding a multiple of D_1, we can make $r_2 = -r_1$, and D_2 has now the form

$$\begin{pmatrix} r_1\lambda & 0 & \mu_2 & 0 \\ 0 & -r_1\lambda & 0 & \mu_6 \\ \mu_3 & 0 & -r_1\lambda & 0 \\ 0 & \mu_7 & 0 & r_1\lambda \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$(t^2 - r_1^2\lambda^2 - \mu_2\mu_3)(t^2 - r_1^2\lambda^2 - \mu_6\mu_7) = (t^2 - r_1^2\lambda^2 - \mu_2\mu_3)^2.$$

We now prove that $r_1^2\lambda^2+\mu_2\mu_3$ must be 0, from which the conclusion will follow. Suppose, for a contradiction, that it is not 0. Then D_2 has double eigenvalues different from zero, and each pair of double eigenvalues has an element coming from each factor.

Consider the matrix $D_3=D_2+kD_1$, where k is a small p-adic number. The characteristic polynomial of D_3 is still a product of two factors. Since the point m is degenerate, for small enough k we must have the same behavior of the eigenvalues as in D_2, that is, two pairs with an element from each factor. But this cannot hold because the two factors are not equal. □

COROLLARY 9.15. *Let p be a prime number. Let X_p, Y_p be the non-residue sets in Definition 5.18. If the p-adic analytic integrable system $F:(M,\omega)\to(\mathbb{Q}_p)^2$ falls in the first case of Theorem 9.14, then the linear combination $g=a_1f_1+a_2f_2$ therein can be brought to one of the following forms by a linear symplectomorphism:*

(1) $(rx^2+sy^2)/2+\mathcal{O}(3)$, *for* $r,s\in Y_p\cup\{0,1\}$.
(2) $y\xi+\mathcal{O}(3)$.
(3) $c(x^2/2+\xi\eta)+\mathcal{O}(3)$, *for* $c\in Y_p\cup\{1\}$.

PROOF. It follows directly from Theorem 9.14 and Corollary 9.8. The selected forms are precisely those having all eigenvalues zero. □

In principle, Corollary 9.15 can be a first step towards making a classification analogous to Theorem 5.19 for degenerate critical points, but such a classification will be more complicated than the non-degenerate one, so we will not follow this direction in the paper.

REMARK 9.16. There is an extensive theory of quadratic forms over different types of fields, we refer to the classical treatment [92] and the more recent works by Alsina-Bayer [2], Bhargava [9], Casselman [17] and Lam [80] and the references therein. In Theorem 5.19 and Theorem 9.14 we have presented a list of local normal forms of integrable systems up to linear symplectic transformations, given by sums of binary quadratic forms, but we have not carried out a further analysis of the structure/properties of these forms since this does not appear to us as applicable in our context of symplectic geometry of integrable systems.

9.2.5. Symplectic dynamics of integrable systems and their level sets. We now calculate the vector fields generated by the integrable systems of Theorem 5.19.

PROPOSITION 9.17. *The vector fields generated by the integrable systems of Theorem 5.19 are as follows:*

(1) $X_{g_1}=(2c_1\xi,-2x,0,0), X_{g_2}=(0,0,2c_2\eta,-2y)$.
(2) $X_{g_1}=(cy,-\eta,x,-c\xi), X_{g_2}=(x,-\xi,y,-\eta)$.
(3)

$$X_{g_1}=\left(-\frac{t_1+bt_2}{a}\xi-(bt_1+ct_2)\eta,\frac{acx-by}{b^2-c},-ac(t_1+bt_2)\eta-(bt_1+ct_2)\xi,\frac{y-abx}{a(b^2-c)}\right),$$

$$X_{g_2}=\left(-\frac{bt_1+ct_2}{a}\xi-c(t_1+bt_2)\eta,\frac{cy-abcx}{b^2-c},-ac(bt_1+ct_2)\eta-c(t_1+bt_2)\xi,\frac{acx-by}{a(b^2-c)}\right).$$

PROOF. All fields are calculated applying directly the equation $\imath_{X_f}\omega_0 = \mathrm{d}f$. □

We can check that, in each system, g_1 and g_2 Poisson commute, which is equivalent to checking that, if $A_i = \Omega_0^{-1}\mathrm{d}^2 g_i$, we have $A_1A_2 = A_2A_1$. This matrix is zero in case (1),

$$\begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}$$

in case (2), and

$$\begin{pmatrix} ct_2 & 0 & \frac{t_1}{a} & 0 \\ 0 & ct_2 & 0 & act_1 \\ act_1 & 0 & ct_2 & 0 \\ 0 & \frac{t_1}{a} & 0 & ct_2 \end{pmatrix}$$

in case (3).

We can also calculate the fibers of the systems.

PROPOSITION 9.18. *The fibers $F^{-1}(0,0)$ of the integrable systems of Theorem 5.19 with a critical point of rank 0 are as follows:*

1. $\{(\pm d_1\xi, \xi, \pm d_2\eta, \eta) : \xi, \eta \in \mathbb{Q}_p\}$ *if* $-c_1 = d_1^2$ *and* $-c_2 = d_2^2$, $\{(\pm d_1\xi, \xi, 0, 0) : \xi \in \mathbb{Q}_p\}$ *if* $-c_1 = d_1^2$ *and* $-c_2$ *is not a square, and* $\{(0,0,0,0)\}$ *if* $-c_1$ *and* $-c_2$ *are not squares.*
2. $\{(x, 0, y, 0) : x, y \in \mathbb{Q}_p\} \cup \{(0, \xi, 0, \eta) : \xi, \eta \in \mathbb{Q}_p\}$.
3. $\{(0,0,0,0)\}$.

For those in which the origin has rank 1, the fibers are $\{(\pm d\xi, \xi, y, 0) : \xi, y \in \mathbb{Q}_p\}$ *if* $-c = d^2$ *and* $\{(0,0,y,0) : y \in \mathbb{Q}_p\}$ *otherwise.*

PROOF. The part about rank 1 is immediate from the formula $(x^2 + c\xi^2, \eta)$. For the rank 0 part, we have:

1. This follows from making $x^2 + c_1\xi^2 = y^2 + c_2\eta^2 = 0$.
2. We make $x\eta + cy\xi = x\xi + y\eta = 0$. Considering this a system in (ξ, η), we have two possible cases:
   - The determinant of the coefficient matrix is 0. Then $cy^2 - x^2 = 0$. Since c is not a square, $x = y = 0$.
   - The determinant of the coefficient matrix is not 0. Then $\xi = \eta = 0$.
3. We consider the coordinate change given by Ψ_2 in Proposition 6.24:

$$\begin{pmatrix} x \\ \xi \\ y \\ \eta \end{pmatrix} = \Psi_2 \begin{pmatrix} x' \\ \xi' \\ y' \\ \eta' \end{pmatrix}.$$

   We have that the first column of Ψ_2 is the hat-conjugate of the second, the same happens for the third and fourth, and the original coordinates are all in $\mathbb{Q}_p$ and they are their own conjugates, hence

$$\begin{pmatrix} x \\ \xi \\ y \\ \eta \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{\xi} \\ \hat{y} \\ \hat{\eta} \end{pmatrix} = \hat{\Psi}_2 \begin{pmatrix} \hat{x}' \\ \hat{\xi}' \\ \hat{y}' \\ \hat{\eta}' \end{pmatrix} = \Psi_2 \begin{pmatrix} \hat{\xi}' \\ \hat{x}' \\ \hat{\eta}' \\ \hat{y}' \end{pmatrix}$$

which implies $\xi' = \hat{x}'$ and $\eta' = \hat{y}'$. Now, if (x, ξ, y, η) is a point in the fiber,

$$0 = \begin{pmatrix} x & \xi & y & \eta \end{pmatrix} M \begin{pmatrix} x \\ \xi \\ y \\ \eta \end{pmatrix} = \begin{pmatrix} x' & \xi' & y' & \eta' \end{pmatrix} \Psi_2^T M \Psi_2 \begin{pmatrix} x' \\ \xi' \\ y' \\ \eta' \end{pmatrix}$$

where M is the matrix of the normal form. We know that Ψ_2 diagonalizes $\Omega_0^{-1} M$, so

$$\Psi_2^T M \Psi_2 = \Psi_2^T \Omega_0 \Psi_2 \Psi_2^{-1} \Omega_0^{-1} M \Psi_2$$

$$= \begin{pmatrix} 0 & 4a\alpha\gamma(b+\alpha) & 0 & 0 \\ -4a\alpha\gamma(b+\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & -4a\alpha\bar{\gamma}(b-\alpha) \\ 0 & 0 & 4a\alpha\bar{\gamma}(b-\alpha) & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}$$

$$= -4a\alpha \begin{pmatrix} 0 & \gamma(b+\alpha)\lambda & 0 & 0 \\ \gamma(b+\alpha)\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\gamma}(b-\alpha)\mu \\ 0 & 0 & \bar{\gamma}(b-\alpha)\mu & \end{pmatrix}.$$

Putting this together, we get

$$0 = \gamma(b+\alpha)\lambda x'\xi' + \bar{\gamma}(b-\alpha)\mu y'\eta' = \gamma^2(b+\alpha)(r+s\alpha)x'\hat{x}' + \bar{\gamma}^2(b-\alpha)(r-s\alpha)y'\hat{y}'$$

This must hold for all $r, s \in \mathbb{Q}_p$. Putting $(r, s) = (1, 0)$ and $(0, 1)$,

$$\gamma^2(b+\alpha)x'\hat{x}' + \bar{\gamma}^2(b-\alpha)y'\hat{y}' = \alpha\gamma^2(b+\alpha)x'\hat{x}' - \alpha\bar{\gamma}^2(b-\alpha)y'\hat{y}' = 0$$

These two equations imply $x'\hat{x}' = 0$ and $y'\hat{y}' = 0$, that is, $x' = y' = 0$, which in turn implies $\xi' = \eta' = 0$ and the vector is zero. □

In analogy with the real case, a submanifold N of a $2n$-dimensional p-adic analytic symplectic manifold (M, ω) is said to be *isotropic* if the tangent space at each point of N is an isotropic subspace of the tangent space of M, that is, if $\omega(u, v) = 0$ for any two vectors $u, v \in \mathrm{T}_m N$. It is called *Lagrangian* if it is isotropic and with dimension n.

The fibers of regular points of real integrable systems are Lagrangian, and homeomorphic to tori (for this reason, it is called a singular Lagrangian torus fibration). For p-adic integrable systems the situation is more complicated to describe in general, even for concrete examples (such as the Jaynes-Cummings model treated in our paper [**26**]). However, there is a common point:

PROPOSITION 9.19. *Let n be a positive integer. Let p be a prime number. Let (M, ω) be a $2n$-dimensional p-adic symplectic manifold. Let $F : (M, \omega) \to (\mathbb{Q}_p)^n$ be a p-adic analytic integrable system. Suppose that the components of F are either regular components ξ_i or given by one of the normal forms of Theorem 5.19. Then the fiber $F^{-1}(0)$ is a union of isotropic subspaces intersecting at the origin, and if all the components are regular, then the fiber is a Lagrangian subspace.*

PROOF. For the regular case, where the system is $(\xi_1, \ldots, \xi_n)$, clearly the fiber of 0 is a Lagrangian subspace. Otherwise, it is enough to prove the statement for the dimension 2 and 4 normal forms, and the conclusion follows by multiplying, because

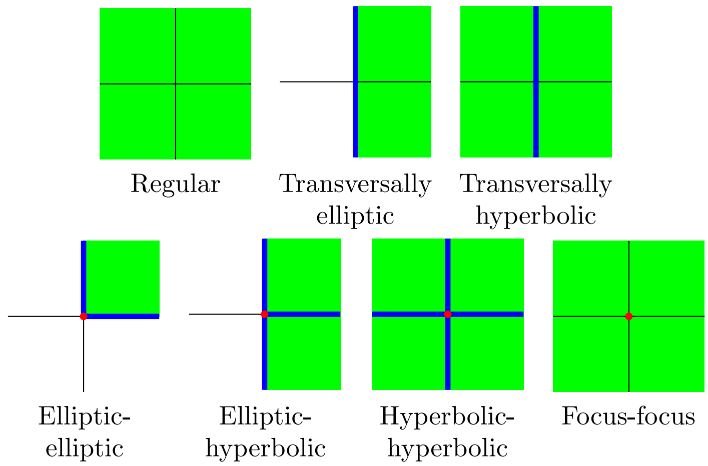


FIGURE 1. Images of some normal forms for the real case and the p-adic case with $p \not\equiv 1 \mod 4$. In the real case, the positive and negative sides of the axes represent, as usual, positive and negative numbers; if $p = 2$, the "positive" and "negative" sides represent numbers whose second digit is 0 and 1, respectively; finally, if $p \equiv 3 \mod 4$, the "positive" and "negative" sides represent even-order and odd-order numbers, respectively. (The points on the axes themselves have, as usual, a zero coordinate.) In each drawing, the green region represents regular values, the blue points are rank 1 critical values, and the red points are rank 0 critical values. An elliptic component has $c = 1$ in the notation of Corollary 9.5, a hyperbolic one has $c = -1$, and a focus-focus one has $c = -1$ in part (2) of Theorem 5.19.

the symplectic form ω_0 does not mix the coordinates of different components. For the normal forms, the result reduces to checking that

$$\omega_0((\pm d_1, 1, 0, 0), (0, 0, \pm d_2, 1)) = \omega_0((1, 0, 0, 0), (0, 0, 1, 0))$$
$$= \omega_0((\pm d, 1, 0, 0), (0, 0, 1, 0)) = 0. \qquad \square$$

From the point of view of integrable systems and symplectic singularity theory, the fibers and images of the local models in the p-adic case are very interesting and include for example the images displayed in Figures 1 and 2 and the fibers displayed in Figures 2 and 3.

We could define a p-adic version of the "Williamson type" defined for the real case at [12, p. 41]. In the real case, it consists of a tuple of integers $(k_{\mathrm{e}}, k_{\mathrm{h}}, k_{\mathrm{f}})$ that count the number of elliptic, hyperbolic and focus-focus components of the normal form. The problem with this approach is that the components of the normal forms are associated to blocks in the normal forms of matrices, which in the real case take only three possible forms. In the p-adic case, by Lemma 8.17, there can appear countably many different blocks, so the Williamson type will be a sequence (with a finite number of elements different from zero) instead of a tuple.

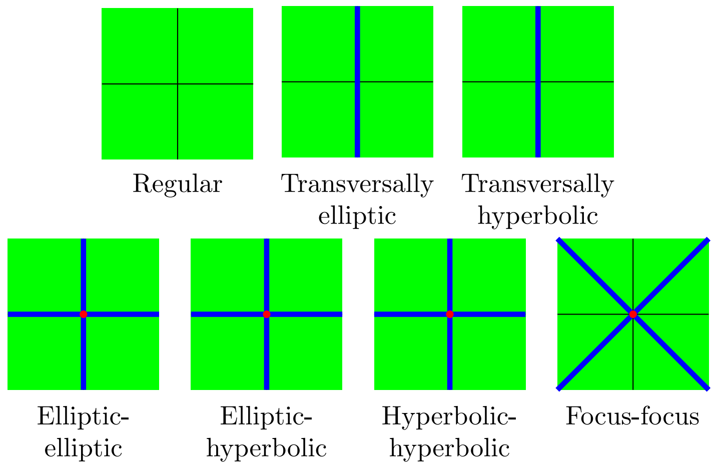


FIGURE 2. Images of the same normal forms for $p \equiv 1 \mod 4$. The slopes of the blue lines in the last image are i and $-$i. Note that the images of the systems with the same rank coincide, except perhaps for a coordinate change, because now the singular components of the systems belong to the same class: part (1) of Theorem 5.19, with $c_1 = c_2 = 1$.

For example, for $p = 2$, the first 11 elements of the sequence count the number of components with each possible c in Corollary 9.5 (associated to blocks of size two), the next 145 elements count the number of pairs of components with each possible form in parts (2) and (3) of Theorem 5.19 (associated to blocks of size four), after which come the counts of trios of components associated to blocks of size six, and so on.

We close this section with a mention of the Eliasson-Vey's linearization theorem [**49**, **50**, **116**, **132**, **134**], which in the real case states that any *smooth* integrable system can be brought to its Williamson normal form by a symplectomorphism. The *analytic* case of this theorem is due to Rüßmann [**116**] for two degrees of freedom and Vey [**132**] in arbitrary dimension. In the real case Eliasson's Theorem (assuming that there are no hyperbolic components) says that there is a local diffeomorphism φ and symplectic coordinates $\phi^{-1} = (x, \xi, y, \eta)$ such that $F \circ \phi = \varphi(g_1, g_2)$, where g_i is one of the elliptic, real or focus-focus models. The p-adic equivalent of this theorem is well beyond the scope of this paper, and we state it as a question.

QUESTION 9.20 (A p-adic Eliasson-Vey's theorem?). *Let n be a positive integer. Let p be a prime number. Given a $2n$-dimensional p-adic analytic symplectic manifold (M, ω), an integrable system $F : (M, \omega) \to (\mathbb{Q}_p)^n$ and a non-degenerate critical point m of F, determine under which conditions on the Williamson type of the critical point m there are open sets $U \subset M$ and $V \subset (\mathbb{Q}_p)^{2n}$, a p-adic analytic symplectomorphism $\phi : V \to U$ and a local diffeomorphism φ of $(\mathbb{Q}_p)^n$ such that $\phi(0) = m$ and*

$$(F - F(m)) \circ \phi = \varphi \circ (g_1, \ldots, g_n),$$

where $(g_1, \ldots, g_n)$ *is the Williamson normal form of* F *in* m*. (In the real case it is enough that there are no hyperbolic blocks.)*

REMARK 9.21. It is important to understand that the results of this section do provide a normal form of the functions involved in a neighborhood of the critical point, but do not give a normal form of the symplectic form in the entire neighborhood, only at the tangent space at the point. The question of whether such a normal form exists in the entire neighborhood and what shapes it may take is a completely different non-linear problem which we have not attempted to answer in this paper (this would be a Darboux type theorem for p-adic analytic symplectic manifolds, which would be a preliminary step to answer Question 9.20).

9.3. Application to classical mechanical systems

In this section we explain how Theorem 5.19 can be applied to further study the p-adic Jaynes-Cummings model introduced and studied in [26]. We recommend the books by Abraham-Marsden [1] and de León-Rodrigues [33] for an introduction to the mathematical study of mechanics and its connections to symplectic/differential geometry.

In our paper [26] we studied the Jaynes-Cummings model with p-adic coefficients (see Figure 3 for the fibers in the real case and Figure 4 for a fiber in the p-adic case). The system was defined therein, in analogy with the real case, as follows. For any number p, first we consider the product p-adic analytic manifold $\mathrm{S}_p^2 \times (\mathbb{Q}_p)^2$ with the p-adic symplectic form $\omega_{\mathrm{S}_p^2} + \mathrm{d}u \wedge \mathrm{d}v$. Here we recall that

$$\mathrm{S}_p^2 = \Big\{(x, y, z) \in \mathbb{Q}_p^2 : x^2 + y^2 + z^2 = 1\Big\}$$

and $\omega_{\mathrm{S}_p^2}$ is the area form in the sphere given by

$$\omega_{\mathrm{S}_p^2} = -\frac{1}{z}\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{y}\mathrm{d}x \wedge \mathrm{d}z = -\frac{1}{x}\mathrm{d}y \wedge \mathrm{d}z.$$

The *p-adic Jaynes-Cummings model* is given by the p-adic analytic map

$$F = (J, H) : \mathrm{S}_p^2 \times (\mathbb{Q}_p)^2 \to (\mathbb{Q}_p)^2,$$

where

$$\begin{cases} J(x, y, z, u, v) = \dfrac{u^2 + v^2}{2} + z; \\ H(x, y, z, u, v) = \dfrac{ux + vy}{2}, \end{cases}$$

where $(x, y, z) \in \mathrm{S}_p^2$ and $(u, v) \in (\mathbb{Q}_p)^2$.

By [26, Proposition 2.5], at $m_1 = (0, 0, -1, 0, 0)$, there is a p-adic linear symplectomorphism changing the local coordinates to (x, ξ, y, η) in which the p-adic symplectic form is given by $\omega = (\mathrm{d}x \wedge \mathrm{d}\xi + \mathrm{d}y \wedge \mathrm{d}\eta)/2$ and

$$F_1(x, \xi, y, \eta) = \frac{1}{2}(x^2 + \xi^2, y^2 + \eta^2) + \mathcal{O}((x, \xi, y, \eta)^3). \tag{35}$$

Here $F_1 = B \circ (F - F(0, 0, -1, 0, 0))$ with

$$B = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}.$$

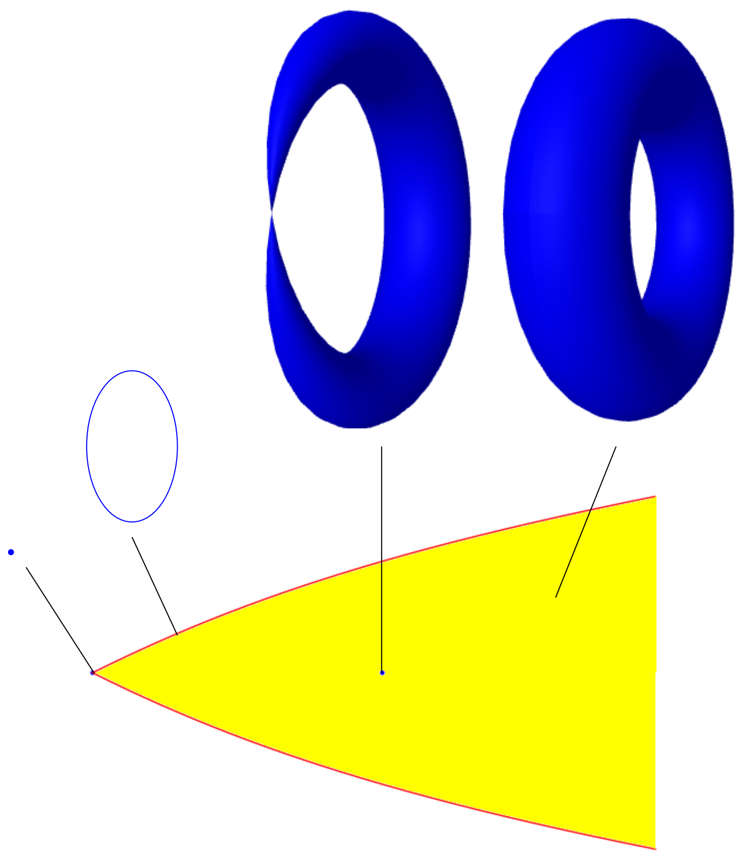


FIGURE 3. Image and fibers of the real Jaynes-Cummings model. The red curve consists of rank 1 critical points, and the two blue points are rank 0. The Jaynes-Cummings model is an example of a class of integrable systems called *semitoric systems*. The fibers of this system are a point, circles, 2-tori (generic fiber) and a pinched torus.

At $m_2 = (0, 0, 1, 0, 0)$, after the change, the p-adic symplectic form is also given by $\omega = (\mathrm{d}x \wedge \mathrm{d}\xi + \mathrm{d}y \wedge \mathrm{d}\eta)/2$ and

$$F_2(x, \xi, y, \eta) = (x\eta - y\xi, x\xi + y\eta) + \mathcal{O}((x, \xi, y, \eta)^3). \tag{36}$$

Here $F_2 = B \circ (F - F(0, 0, 1, 0, 0))$ with

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Hence we have the following consequence of Theorem 5.19.

COROLLARY 9.22. *Let p be a prime number. Then there exist open sets U_1 and U_2 such that $m_1 \in U_1$ and $m_2 \in U_2$ and a local linear symplectomorphism*

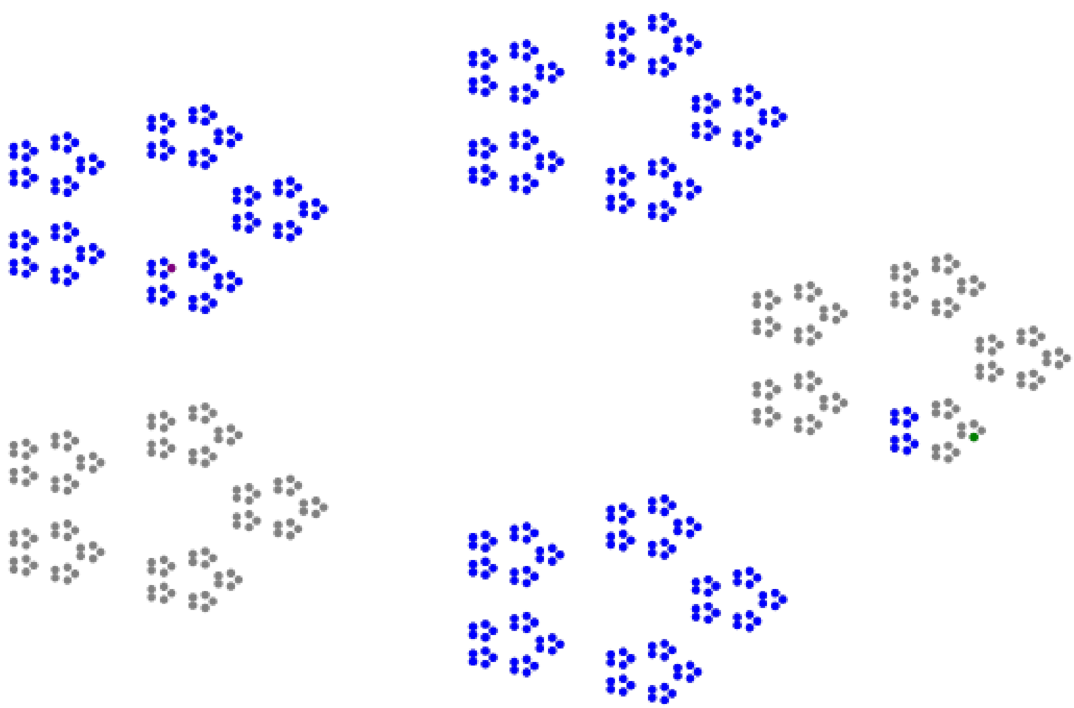


FIGURE 4. Fiber $F^{-1}(72,1)$ of the 5-adic Jaynes-Cummings model. The blue points are values of z for which the coordinates (x,y,u,v) form two p-adic circles, at the green points they only form one circle, at the purple point $z=j=72$ they have dimension 1 but they are not a circle, and the values of z which appear in grey are not in the fiber.

$\phi : U_1 \to U_2$ centered at m_1, such that

$$F_2(\phi(x,\xi,y,\eta)) = F_1(x,\xi,y,\eta) + \mathcal{O}((x,\xi,y,\eta)^3)$$

for $(x,\xi,y,\eta) \in U_1$, if and only if $p \equiv 1 \mod 4$, where F_1 and F_2 are as described in (35) and (36).

PROOF. By Theorem 5.19, F_1 and F_2 are linearly symplectomorphic to one of the possibilities listed in its statement, so it is enough to see that it is the same for $p \equiv 1 \mod 4$ and different otherwise. For the first normal form F_1, we have that

$$\Omega_0^{-1}\mathrm{d}^2(rJ_1 + sH_1) = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix} = \begin{pmatrix} 0 & -2r & 0 & 0 \\ 2r & 0 & 0 & 0 \\ 0 & 0 & 0 & -2s \\ 0 & 0 & 2s & 0 \end{pmatrix}$$

whose eigenvalues are $\pm 2\mathrm{i}r$ and $\pm 2\mathrm{i}s$. If $p \equiv 1 \mod 4$, we are in the situation of Proposition 6.17, so this is in case (1) of Theorem 5.19 linearly symplectomorphic to $x^2+\xi^2$. Otherwise, we are in the situation of Proposition 6.22, and for $\lambda = 2\mathrm{i}r$,

$$u^T\Omega_0\bar{u} = \begin{pmatrix}\mathrm{i} & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2} & 0 & 0\\ -\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}-\mathrm{i}\\ 1\\ 0\\ 0\end{pmatrix} = \mathrm{i}$$

and

$$\frac{2\lambda a}{u^T\Omega_0\bar{u}} = 4ra.$$

We need to find a and b such that $ab = 4r^2$ and $4ra \in \mathrm{DSq}(\mathbb{Q}_p, 1)$, or equivalently $r/a \in \mathrm{DSq}(\mathbb{Q}_p, 1)$. Taking $b = ac$, we have that $c = 4r^2/a^2$ is a square; moreover, if $p \equiv 3 \mod 4$, r/a has even order, hence $4 \mid \mathrm{ord}_p(c)$, and in the set

$$\left\{1, -1, p, -p, p^2\right\},$$

the c that we need is 1. If $p = 2$, the only c that is square is 1. The same reasoning holds for s instead of r, so this critical point is in case (1) with $c_1 = c_2 = 1$.

Respecting to F_2, we get

$$\Omega_0^{-1}\mathrm{d}^2(rJ_2+sH_2) = \begin{pmatrix}0 & -2 & 0 & 0\\ 2 & 0 & 0 & 0\\ 0 & 0 & 0 & -2\\ 0 & 0 & 2 & 0\end{pmatrix}\begin{pmatrix}0 & s & 0 & r\\ s & 0 & -r & 0\\ 0 & -r & 0 & s\\ r & 0 & s & 0\end{pmatrix} = \begin{pmatrix}-2s & 0 & 2r & 0\\ 0 & 2s & 0 & 2r\\ -2r & 0 & -2s & 0\\ 0 & -2r & 0 & 2s\end{pmatrix}$$

has as eigenvalues $\pm 2s \pm 2\mathrm{i}r$. If $p \equiv 1 \mod 4$, this is again in case (1) with $c_1 = c_2 = 1$, otherwise it is in case (2) with $c = -1$. □

More information about the Jaynes-Cummings model and other models of interest in physics can be found at [1, 33].

9.4. Examples

In this section we show examples which illustrate our theorems, so that they can be understood more concretely.

9.4.1. Examples with matrices.

EXAMPLE 9.23. This example follows the method given in the proof of Theorem 5.31 in order to find the normal form of a symmetric matrix. Let M be the following symmetric matrix:

$$M = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 5 & 6 & 7\\ 3 & 6 & 8 & 9\\ 4 & 7 & 9 & 10\end{pmatrix}.$$

The characteristic polynomial of $A = \Omega_0^{-1}M$ is $t^4 - 6t^2 - 2$. We start with $p = 2$. In order to classify M, we need to find two things: the family of the normal form, and the normal form itself. We first calculate

$$\lambda^2 = \frac{6 \pm \sqrt{36+8}}{2} = 3 \pm \sqrt{11}.$$

As $11 \equiv 3 \mod 8$, $\lambda^2 \notin \mathbb{Q}_2$ and we are in case (2) or (3) of Theorem 5.31 with $c = 3$. In order to find which one, we need to check whether $3 \pm \sqrt{11}$ is a square in $\mathbb{Q}_2[\sqrt{3}]$: it can be written as $3 \pm \sqrt{11/3}\sqrt{3}$, where $\sqrt{11/3} \in \mathbb{Q}_2$. Applying the criterion on Proposition 8.9(4), we see that

$$\operatorname{ord}(3) = \operatorname{ord}(\sqrt{11/3}) = 0,$$

so it is not a square and we are in case (3).

The next step is to find t_1 and t_2: this is the class of $3+\sqrt{11}$ in $\mathbb{Q}_2[\sqrt{3}]$ modulo squares. This can be found with the procedure in the proof of Corollary 8.10(2): we need to multiply by $1+\sqrt{3}$ to make $\operatorname{ord}(a) \neq \operatorname{ord}(b)$, which gives

$$(3+\sqrt{\frac{11}{3}}\sqrt{3})(1+\sqrt{3}) = 3+\sqrt{33}+\left(3+\sqrt{\frac{11}{3}}\right)\sqrt{3}.$$

Writing this again in the form $a+b\sqrt{3}$, it leads to $\operatorname{ord}(b)-\operatorname{ord}(a)=2$, $\operatorname{ord}(a)\equiv 1 \mod 2$ (so we have to multiply by 2) and $b/4a+\operatorname{digit}_2(a) \equiv 1 \mod 2$ (so we have to change sign). The class is $-2(1+\sqrt{3})$, that is, $t_1=t_2=-2$. This is not a class in the table, so we need to take its pair, $t_1=1$ and $t_2=1$.

It is only left to find a and b. The two possible classes have $b=0$, but one has $a=1$ and the other $a=-1$. To find the correct one, we go to the formula in Proposition 6.24.

$$\frac{a\alpha\gamma(b+\alpha)}{u^T\Omega_0\hat{u}} = \frac{a\alpha^2\gamma}{u^T\Omega_0\hat{u}} = \frac{3a\sqrt{-2(1+\sqrt{3})}}{u^T\Omega_0\hat{u}} \in \operatorname{DSq}(\mathbb{Q}_2[\sqrt{3}], 2(1+\sqrt{3})).$$

The class of this number, with $a=1$, is $-(1+\sqrt{3})$. As we see in Table 7, the position in row $\gamma^2=-2(1+\sqrt{3})$ and column $-(1+\sqrt{3})$ is unmarked, which means that the number is not in

$$\operatorname{DSq}(\mathbb{Q}_2[\sqrt{3}], 2(1+\sqrt{3})),$$

and we need to take $a=-1$. This finishes the classification of the family of the normal form:

$$r\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & -3 & 0 & 3 \end{pmatrix} + s\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & -9 \end{pmatrix}.$$

To find the concrete normal form, we need to put λ in the form $(r+s\alpha)\gamma$, and the result is

$$r = \frac{1}{2}\left(\sqrt{\frac{-3+\sqrt{11}+3\sqrt{3}-\sqrt{33}}{2}}+\sqrt{\frac{-3-\sqrt{11}-3\sqrt{3}-\sqrt{33}}{2}}\right)$$

$$s = \frac{1}{2\sqrt{3}}\left(\sqrt{\frac{-3+\sqrt{11}+3\sqrt{3}-\sqrt{33}}{2}}-\sqrt{\frac{-3-\sqrt{11}-3\sqrt{3}-\sqrt{33}}{2}}\right)$$

This is a common pattern in this example and the following two: the family of normal forms is very simple, but the concrete normal form is much more complicated.

Example 9.24. Now we classify the same matrix with $p=3$. 11 is still not a square in $\mathbb{Q}_3$, so we are in case (2) or (3) with $c=-1$. We write λ^2 as $3+\mathrm{i}\sqrt{-11}$, where $\sqrt{-11}\in\mathbb{Q}_3$, and we need to check whether this is a square in $\mathbb{Q}_3[\mathrm{i}]$. We use Proposition 8.6(1): $\min\{\operatorname{ord}(a),\operatorname{ord}(b)\} \equiv 0 \mod 2$ and $a^2+b^2=-2$ is a square in $\mathbb{Q}_3$, so λ^2 is a square in $\mathbb{Q}_3[\mathrm{i}]$ and we are in case (2). In this case $c = -1$ is the only parameter we need. This is the focus-focus family

$$\begin{pmatrix} 0 & s & 0 & r \\ s & 0 & -r & 0 \\ 0 & -r & 0 & s \\ r & 0 & s & 0 \end{pmatrix}$$

with

$$r = \frac{\sqrt{3+\sqrt{11}} - \sqrt{3-\sqrt{11}}}{2\mathrm{i}}, s = \frac{\sqrt{3+\sqrt{11}} + \sqrt{3-\sqrt{11}}}{2}$$

EXAMPLE 9.25. For the same matrix and $p = 5$, we see that $\lambda^2 = 3+\sqrt{11} \in \mathbb{Q}_5$, so we are in case (1). c_1 is given by the class of λ^2 modulo a square in $\mathbb{Q}_5$.

$$\lambda^2 = 3 + \sqrt{11} \equiv 3 + 1 = 4 \mod 5$$

is a square, so $c_1 = 1$ (actually, c_1 is in the class of $-\lambda^2$, but as $p \equiv 1 \mod 4$ it is equivalent to take λ^2). Analogously

$$\mu^2 = 3 - \sqrt{11} \equiv 3 - 1 = 2 \mod 5$$

so $c_2 = 2$ (which is c_0 for $p = 5$) or $c_2 = 2p^2 = 50$. To know which one, we go to Proposition 6.22:

$$\frac{2\mu a}{u^T \Omega_0 \bar{u}} = \ldots 203033a \in \mathrm{DSq}(\mathbb{Q}_5, 2)$$

which means, by Proposition 8.1(2), that a has even order, and since $\mathrm{ord}(ab) = 0$,

$$\mathrm{ord}(c_2) = \mathrm{ord}(b) - \mathrm{ord}(a) = -2\,\mathrm{ord}(a)$$

is multiple of 4, so this leaves $c_2 = 2$. The normal form is

$$\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 2s \end{pmatrix}$$

with

$$r = \sqrt{3+\sqrt{11}}, s = \sqrt{\frac{\sqrt{11}-3}{2}}.$$

EXAMPLE 9.26. Let M be the following matrix:

$$M = \begin{pmatrix} 2 & 6 & -2 & -3 \\ 6 & 11 & 1 & -5 \\ -2 & 1 & -6 & -2 \\ -3 & -5 & -2 & 3 \end{pmatrix}$$

The characteristic polynomial of $A = \Omega_0^{-1} M$ is $(t^2-5)^2$, so this matrix has repeated eigenvalues:

$$\{\sqrt{5}, \sqrt{5}, -\sqrt{5}, -\sqrt{5}\}.$$

For $p = 5$, $\sqrt{5} \notin \mathbb{Q}_5$, hence it is in case (3) of Theorem 5.33. We need to write $\lambda = r\sqrt{c}$ for $c \in Y_p$: this leads to $c = 5$ and $r = 1$. The value of a must be 1 or 2, and from (34) we obtain that $a = 1$ gives the result in $\mathrm{DSq}(\mathbb{Q}_5, -5)$. So the normal form is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 5 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

EXAMPLE 9.27. The same matrix with $p = 11$: now $\sqrt{5} \in \mathbb{Q}_{11}$, so it is in case (2) of Theorem 5.33, with $r = \lambda = \sqrt{5}$:

$$\begin{pmatrix} 0 & \sqrt{5} & 0 & 0 \\ \sqrt{5} & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{5} \\ 0 & 0 & \sqrt{5} & 0 \end{pmatrix}.$$

9.4.2. Examples with functions and integrable systems.

EXAMPLE 9.28. The matrices from Examples 9.23 and 9.26 can be translated to functions with a critical point in the origin (the first one is non-degenerate and the second one is degenerate):

$$f_1(x, \xi, y, \eta) = \frac{x^2}{2} + 2x\xi + 3xy + 4x\eta + \frac{5}{2}\xi^2 + 6\xi y + 7\xi\eta + 4y^2 + 9y\eta + 5\eta^2$$

$$f_2(x, \xi, y, \eta) = x^2 + 6x\xi - 2xy - 3x\eta + \frac{11}{2}\xi^2 + \xi y - 5\xi\eta - 3y^2 - 2y\eta + 3\eta^2$$

The class of the critical point of f_1 is that of Corollary 9.7(3), with $c = 3, t_1 = t_2 = -2, a = -1, b = 0$, for $p = 2$, that of part (2) with $c = -1$ for $p = 3$, and that of part (1) with $c_1 = 1$ and $c_2 = 2$ for $p = 5$.

The class of the critical point of f_2 is that of Corollary 9.8(3) with $c = 5$ for $p = 5$, and that of part (2) for $p = 11$.

EXAMPLE 9.29. Let p be a prime number such that $p \equiv 1 \mod 4$. Then the functions

$$F_{\mathrm{ee}}, F_{\mathrm{eh}}, F_{\mathrm{hh}}, F_{\mathrm{ff}} : ((\mathbb{Q}_p)^4, \omega_0) \to (\mathbb{Q}_p)^2$$

and the symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}\xi + \mathrm{d}y \wedge \mathrm{d}\eta$ on $\mathbb{Q}_p^4$:

- Elliptic-elliptic: $F_{\mathrm{ee}}(x, \xi, y, \eta) = (\frac{x^2+\xi^2}{2}, \frac{y^2+\eta^2}{2})$;
- Elliptic-hyperbolic: $F_{\mathrm{eh}}(x, \xi, y, \eta) = (\frac{x^2+\xi^2}{2}, y\eta)$;
- Hyperbolic-hyperbolic: $F_{\mathrm{hh}}(x, \xi, y, \eta) = (x\xi, y\eta)$;
- Focus-focus: $F_{\mathrm{ff}}(x, \xi, y, \eta) = (x\eta - y\xi, x\xi + y\eta)$,

are non-degenerate p-adic analytic integrable systems. Furthermore, all four systems are p-adically linearly symplectomorphic. This follows from Theorem 5.19.

EXAMPLE 9.30. Let p be a prime number such that $p \not\equiv 1 \mod 4$. Then any two distinct systems among those four are not linearly symplectomorphic. This follows from Theorem 5.19.

REMARK 9.31. In Corollary 9.5, the elliptic function corresponds to $c = 1$. The hyperbolic one, by Proposition 6.17, corresponds to $c = 1$ for $p \equiv 1 \mod 4$ and $c = -1$ otherwise. That is, six of the seven forms for $p \equiv 1 \mod 4$, three of the five for $p \equiv 3 \mod 4$, and nine of the eleven for $p = 2$, have no real equivalent.

In Theorem 5.19, the elliptic-elliptic model corresponds to (1) with $c_1 = c_2 = 1$. Changing elliptic components to hyperbolic results in changing the corresponding

c_i to -1, except if $p \equiv 1 \mod 4$, where there is no change. The focus-focus model is the same for $p \equiv 1 \mod 4$, and otherwise it is (2) with $c = -1$. *The vast majority of p-adic normal forms, including all those in point (3), have no real equivalent.*

We now comment on the S^1_p-actions on $\mathbb{Q}_p^2$ induced by those systems (we refer to [26, Appendix C] for a review of the concept of p-adic action), where we recall that S^1_p is defined by

$$\mathrm{S}^1_p = \Big\{(x,y) \in \mathbb{Q}_p^2 : x^2 + y^2 = 1\Big\}.$$

REMARK 9.32. Let p be a prime number.

(1) The elliptic function $f_{\mathrm{e}}(x,\xi) = \frac{x^2+\xi^2}{2}$ induces on $\mathbb{Q}_p^2$ an S^1_p-action given by
$$(u,v)\cdot(x,\xi) = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix},$$
for $(u,v) \in \mathrm{S}^1_p$ and $(x,\xi) \in \mathbb{Q}_p^2$.

(2) The first component $f_1(x,\xi,y,\eta) = x\eta - y\xi$ of the focus-focus system F_{ff} induces on $\mathbb{Q}_p^4$ an action with a similar formula to the previous one, simultaneously on the plane (x,y) and the plane (ξ,η):
$$(u,v)\cdot(x,\xi,y,\eta) = \begin{pmatrix} u & 0 & -v & 0 \\ 0 & u & 0 & -v \\ v & 0 & u & 0 \\ 0 & v & 0 & u \end{pmatrix}\begin{pmatrix} x \\ \xi \\ y \\ \eta \end{pmatrix},$$
for $(u,v) \in \mathrm{S}^1_p$ and $(x,\xi,y,\eta) \in \mathbb{Q}_p^4$.

Indeed, by [26, Corollary 4.5], there is a subgroup of S^1_p isomorphic to $p\mathbb{Z}_p$ by the correspondence $t \mapsto (\cos t, \sin t)$ that contains all elements near the origin. We need to prove that, if $\psi_t(x,\xi) = (\cos t, \sin t)\cdot(x,\xi)$, the vector field X_t of this flow (in the sense that $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x,\xi) = X_t(\psi_t(x,\xi))$) satisfies Hamilton's equations $\imath_{X_t}\omega_0 = \mathrm{d}f_{\mathrm{e}}$.

We have
$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x,\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix}$$
so $X_t(x,\xi) = (\xi,-x)$, and
$$\imath_{X_t}\omega_0 = x\mathrm{d}x + \xi\mathrm{d}\xi = \mathrm{d}f_{\mathrm{e}},$$
as we wanted.

For the focus-focus action, we have analogously that $X_t(x,\xi,y,\eta) = (-y,-\eta,x,\xi)$, and
$$\imath_{X_t}\omega_0 = \eta\mathrm{d}x - y\mathrm{d}\xi - \xi\mathrm{d}y + x\mathrm{d}\eta = \mathrm{d}f_1,$$
as we wanted.

9.5. Circular symmetries of the p-adic models

Here we generalize the content of Remark 9.32 and analyze the problem of existence of circle actions for arbitrary models. In the real case, for a fixed symplectic space (V,ω), most multiples of a Hamiltonian that admits a circle action do not admit it. This happens because a smooth circle action over V is defined by a smooth map $h : \mathrm{S}^1 \times V \to V$, satisfying $h(g_1, h(g_2,m)) = h(g_1g_2,m)$ and

$h(1,m)=m$, for $g_1,g_2\in\mathrm{S}^1$ and $m\in V$. Concretely, considering $g:\mathbb{R}\to\mathrm{S}^1$ given by $g(t)=(\cos t,\sin t)$, we must have

$$h(g(t+t'),m)=h(g(t)g(t'),m)=h(g(t),h(g(t'),m))$$

and $h(1,m)=m$, for $t,t'\in\mathbb{R}$ and $m\in V$. The induced vector field for this action, that is, $X_h(t)$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t}h(g(t),m)=X_h(t)(h(g(t),m)),$$

must coincide with X_f induced by the Hamiltonian f via the Hamilton equation $\imath_{X_f}\omega_0=\mathrm{d}f$. When f is multiplied by a constant k, that is, $f'=kf$, we also have $X_{f'}=kX_f$, so X_h must also be multiplied by k:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}h'(g(t),m)&=X_{h'}(t)(h'(g(t),m))\\&=kX_h(t)(h'(g(t),m))\\&=kX_h(kt)(h'(g(t),m))\end{aligned}$$

(the last equality happens because $X_h(t)=X_f$ is independent of t). This equation is solved by $h'(g(t),m)=h(g(kt),m)$, that is, the action is accelerated by a factor of k. As we must have $h'(g(2\pi),m)=h'(1,m)=m$, in general this is only possible if k is integer.

This will not happen in the p-adic case, because the p-adic circle is not closed (there is no $t\neq 0$ such that $g(t)=1$). So getting a circle action is much easier in this case, and actually any "small enough" multiple of a Hamiltonian admits a circle action.

PROPOSITION 9.33. *Let n be a positive integer and let p be a prime number. Let Ω_0 be the matrix of the standard symplectic form on $(\mathbb{Q}_p)^{2n}$. Given a p-adic analytic Hamiltonian $f:(\mathbb{Q}_p)^{2n}\to\mathbb{Q}_p$ such that $f(m)=m^TMm/2$, for a matrix $M\in\mathcal{M}_{2n}(\mathbb{Q}_p)$, f admits a p-adic analytic S^1_p-action (that is, there exists $h:\mathrm{S}^1_p\times(\mathbb{Q}_p)^{2n}\to(\mathbb{Q}_p)^{2n}$ analytic such that $h(g_1,h(g_2,m))=h(g_1g_2,m)$ and $h(1,m)=m$, for $g_1,g_2\in\mathrm{S}^1_p$ and $m\in(\mathbb{Q}_p)^{2n}$) if and only if $\mathrm{ord}(\lambda)\geq 0$ for all λ which is an eigenvalue of $\Omega_0^{-1}M$.*

PROOF. Suppose that f admits a circle action h and let $\psi(t,v)=h(g(t),v)$ and $A=\Omega_0^{-1}M$. Hamilton's equation $\imath_{X_f}\omega_0=\mathrm{d}f$ results in

$$X_f(m)^T\Omega_0v=\mathrm{d}f(m)(v)=m^TMv\Rightarrow X_f(m)=-\Omega_0^{-1}Mm=-Am.$$

Substituting in the flow equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,m)=-A\psi(t,m)$$

and $\psi(0,m)=m$, which solves as

$$\psi(t,m)=\exp(-tA)m$$

where exp denotes the matrix exponential. This must exist for all t in the domain of $g(t)=(\cos t,\sin t)$, that is, such that $|t|_p\leq k_p$, where $k_p=1/p$ for $p\neq 2$ and $k_2=1/4$. The exponential of $-tA$ exists if and only if the eigenvalues μ of $-tA$ satisfy $|\mu|_p\leq k_p$, which implies

$$|-t\lambda|_p=|t|_p|\lambda|_p\leq k_p.$$

As it must exist when $|t|_p = k_p$, we have $|\lambda|_p \leq 1$.

Conversely, suppose that $|\lambda|_p \leq 1$ for all eigenvalues of A. Then, for all t in the domain of g,

$$|-t\lambda|_p = |t|_p|\lambda|_p \leq k_p$$

and $\exp(-tA)$ exists. This means that $\psi(t,m)$ is well defined, and $h(g_0,m)$ exists for $g_0 \in \mathrm{Im}(g)$. By [26, Corollary 4.5], the quotient $\mathrm{S}^1_p/\mathrm{Im}(g)$ is a discrete group, so we can define its action arbitrarily without affecting the flow equation, and we have an action of S^1_p. □

REMARK 9.34. Often symplectic classifications of group actions in equivariant symplectic geometry include the assumption that the action is effective, as it is the case for example in Delzant's classification, Duistermaat-Pelayo [43] and Pelayo [96]. With this restriction (being effective), no multiple of a Hamiltonian which admits a circle action also admits an action. However, as Proposition 9.33 shows, in the p-adic case all small multiples of any Hamiltonian admit an effective action.

However, if we want the action to have the form $h(g,m) = h((u,v),m) = (uI + vB)m$, like the actions in Remark 9.32, the situation changes completely.

PROPOSITION 9.35. *Let n be a positive integer and let p be a prime number. Let Ω_0 be the matrix of the standard symplectic form on $(\mathbb{Q}_p)^{2n}$. Given a p-adic analytic Hamiltonian $f : (\mathbb{Q}_p)^{2n} \to \mathbb{Q}_p$ such that $f(m) = m^TMm/2$, for a matrix $M \in \mathcal{M}_{2n}(\mathbb{Q}_p)$, f admits a p-adic analytic S^1_p-action of the form $h((u,v),m) = (uI + vB)m$ if and only if the eigenvalues of $\Omega_0^{-1}M$ are i and $-\mathrm{i}$, both with multiplicity n. In that case, $B = -\Omega_0^{-1}M$.*

PROOF. The flow equation implies

$$\frac{\mathrm{d}}{\mathrm{d}t}h((\cos t,\sin t),m) = -Ah((\cos t,\sin t),m),$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cos tI + \sin tB)m = (-\sin tI + \cos tB)m = -A(\cos tI + \sin tB)m,$$

which implies $AB = I$ and $B = -A$. Hence, the action exists if and only if $A^2 = -I$. If this happens, the only possible eigenvalues are i and $-\mathrm{i}$, and since they must come in opposite pairs, each one appears n times. Conversely, if the eigenvalues of A are i and $-\mathrm{i}$, those of A^2 are -1, which implies that $A^2 = -I$. □

Of the normal forms in Theorem 5.19, those with a circle action of this form are the following:

- At point (1), only the elliptic component gives eigenvalues i and $-\mathrm{i}$. The remaining ones have different eigenvalues. Hence, we recover Remark 9.32(1).
- At point (2), only the focus-focus component, if $p \not\equiv 1 \mod 4$, has eigenvalues $(\mathrm{i},\mathrm{i},-\mathrm{i},-\mathrm{i})$. We recover Remark 9.32(2).
- At point (3), $t_1 + t_2\sqrt{c}$ is the square of an eigenvalue, so we must have $t_1 = -1$ and $t_2 = 0$ to have this kind of circle action.

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