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New Functional Inequalities with Applications to the *Arctan*-Fast Diffusion Equation

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Abstract. In this paper, we prove a couple of new nonlinear functional inequalities of Sobolev type akin to the logarithmic Sobolev inequality. In particular, one of the inequalities reads

$$\int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right)\partial_x u \,\mathrm{d}x \geqslant \arctan\left(\|u\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right)\|u\|_{\dot{W}^{1,1}(\mathbb{S}^1)}.$$

Then, these inequalities are used in the study of the nonlinear *arctan*-fast diffusion equation

$$\partial_t u - \partial_x \arctan\left(\frac{\partial_x u}{u}\right) = 0.$$

For this highly nonlinear PDE, we establish a number of well-posedness results and qualitative properties.

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Contents

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1. Introduction and Main Results

Functional inequalities are at the core of functional analysis and partial differential equations. In this paper, we prove the following nonlinear Sobolev inequality:

$$\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right) |\partial_x u| \, \mathrm{d}x \geqslant \arctan\left(\|u(t)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right) \|u(t)\|_{\dot{W}^{1,1}(\mathbb{S}^1)} .$$

Such a nonlinear Sobolev inequality has the same flavor as the logarithmic Sobolev inequality [16]

$$\int_{\mathbb{R}} u^2 \log(u^2) \, \mathrm{d}x \leqslant \frac{1}{2} \log\left(\frac{2}{\pi e} \int_{\mathbb{R}} (\partial_x u)^2 \, \mathrm{d}x\right).$$

We will make use of this inequality in the study of the following one-dimensional *arctan*-fast diffusion equation

$$\partial_t u - \partial_x \arctan\left(\frac{\partial_x u}{u}\right) = 0 \quad (x,t) \text{ on } \mathbb{S}^1 \times [0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x) \quad x \text{ on } \mathbb{S}^1. \tag{1.1b}$$

We will focus on the well-posedness for such equation and, in particular, we will establish the global existence of solutions for initial data satisfying certain properties.

There are several closely related problems in the literature. For instance, Equation (1.1) is a nonlinear diffusion somehow similar to the logarithmic fast diffusion equation

$$\partial_t u - \partial_x \left(\frac{\partial_x u}{u}\right) = 0 \qquad (x,t) \text{ on } \mathbb{S}^1 \times [0,T]. \tag{1.2}$$

In fact, using

$$\arctan(y) = y + \text{h.o.t.}$$

Equation (1.2) can be obtained from (1.1) when the effect of higher order nonlinearities is neglected. Remarkably, Eq. (1.2) is related to the Ricci flow and due to that, this equation has been extensively studied in the past years by many authors (see Ref. [23] and the references therein).

We could also quote the fast diffusion equation

$$\partial_t u = \partial_x \left(\frac{\partial_x u}{u^\sigma}\right) \quad \text{with} \quad \sigma > 1.$$

Equation (1.1) is also related to the following nonlocal fast diffusion equation:

$$\partial_t u - \partial_x \arctan\left(\frac{-Hu}{u}\right) = 0 \qquad (x,t) \text{ on } \mathbb{S}^1 \times [0,T], \tag{1.3}$$

where

$$Hu(x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{S}^1} \frac{u(y)}{\tan\left(\frac{x-y}{2}\right)}\,\mathrm{d}y,$$

is the Hilbert transform. Indeed, Eq. (1.3) is obtained from (1.1) by replacing the derivative inside the $\arctan(\cdot)$ by a term proportional to the Hilbert transform:

$$\partial_x \to -H.$$

A similar procedure establishes relations between the KdV and the Benjamin–Ono equation or between the Sine-Gordon and the Sine-Hilbert equation [13]. Equation (1.3) was derived by Steinerberger [22] when studying how the distribution of roots behaves under iterated differentiation (see also Refs. [1,20] for the mathematical study of some of its properties).

Actually, the principal reason that led us to study problem (1.1) is Eq. (1.3). Roughly speaking, our main aim is to understand how the nonlinear term driven by the $\arctan(\cdot)$ allows to prove the existence in the singular case, namely where u reaches zero. Hence, a first step is to analyze what happens in the local case (1.1).

Finally, since previous Eq. (1.1) can be written as

$$\partial_t u - \frac{u\partial_x^2 u - (\partial_x u)^2}{u^2 + (\partial_x u)^2} = 0, \tag{1.4}$$

we observe that there is a striking similarity with the one-dimensional relativistic heat equation

$$\partial_t u = \partial_x\left(\frac{u\partial_x u}{\sqrt{u^2+(\partial_x u)^2}}\right) = \frac{u\partial_x^2 u + (\partial_x u)^2}{\sqrt{u^2+(\partial_x u)^2}} - u(\partial_x u)^2 \frac{u+\partial_x^2 u}{\left(u^2+(\partial_x u)^2\right)^{3/2}},$$

see Refs. [2–9,11].

1.1. Main Results

As anticipated, one of the main results of this paper is

Theorem 1.1. *Let $0 < u \in W^{1,1}(\mathbb{S}^1)$ be a function such that*

$$\|u\|_{L^1(\mathbb{S}^1)} = 1.$$

Then, the following nonlinear Sobolev inequalities hold true:

$$\int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right)\partial_x u\,\mathrm{d}x \geqslant \arctan\left(\|u\|_{\dot W^{1,1}(\mathbb{S}^1)}\right)\|u\|_{\dot W^{1,1}(\mathbb{S}^1)},$$

$$\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right)\frac{|\partial_x u|}{u}\,\mathrm{d}x \geqslant \frac{1}{4\pi}\arctan\left(\frac{\|u\|^2_{\dot W^{1,1}(\mathbb{S}^1)}}{\||\partial_x u|u\|_{L^1(\mathbb{S}^1)}}\right)\frac{\|u\|^2_{\dot W^{1,1}(\mathbb{S}^1)}}{\||\partial_x u|u\|_{L^1(\mathbb{S}^1)}}.$$

The assumption $\|u\|_{L^1(\mathbb{S}^1)} = 1$ allows us to write the inequalities in Theorem 1.1 in a simpler form. However, this hypothesis is not necessary and an analogous result can be proved without this simplification.

We will make use of Theorem 1.1 in the study (1.1), proving the following results.

Theorem 1.2. *Let $0 < u_0 \in H^3(\mathbb{S}^1)$ be the initial data. Then, there exists a time $0 < T \leqslant \infty$, $T = T(\|u_0\|_{H^3}, \min_x u_0(x))$ and a unique positive solution to (1.1)*

$$0 < u \in C([0,T], H^3(\mathbb{S}^1)).$$

Theorem 1.3. *Let $0 < u_0 \in H^3(\mathbb{S}^1)$ the initial data for (1.1). Then, as long as the unique positive solution u to (1.1) exists, the following properties hold:*

- *Maximum principle:* $\|u(t)\|_{L^\infty(\mathbb{S}^1)} \leqslant \|u_0\|_{L^\infty(\mathbb{S}^1)}$,
- *Mass conservation:* $\|u(t)\|_{L^1(\mathbb{S}^1)} = \|u_0\|_{L^1(\mathbb{S}^1)}$,
- *Entropy balance:*

$$\mathcal{H}(t) + \int_0^t \mathcal{D}(s)\mathrm{d}s = \mathcal{H}(0),$$

where

$$\mathcal{H}(t) = \int_{\mathbb{S}^1} u(x,t)\log(u(x,t)) - u(x,t) + 1\,\mathrm{d}x$$

and

$$\mathcal{D}(t) = \int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u(x,t)}{u(x,t)}\right)\frac{\partial_x u(x,t)}{u(x,t)}\,\mathrm{d}x.$$

In particular, invoking Theorem 1.1,

$$\mathcal{H}(t) + \frac{1}{4\pi}\int_0^t \arctan\left(\frac{\|u(s)\|^2_{\dot{W}^{1,1}(\mathbb{S}^1)}}{\||\partial_x u(s)|u(s)\|_{L^1(\mathbb{S}^1)}}\right)\frac{\|u(s)\|^2_{\dot{W}^{1,1}(\mathbb{S}^1)}}{\||\partial_x u(s)|u(s)\|_{L^1(\mathbb{S}^1)}}\mathrm{d}s \leqslant \mathcal{H}(0).$$

- *Energy balance:*

$$\frac{1}{2}\|u(t) - \langle u_0\rangle\|^2_{L^2(\mathbb{S}^1)} + \int_0^t \mathcal{D}(s)\mathrm{d}s = \frac{1}{2}\|u_0 - \langle u_0\rangle\|^2_{L^2(\mathbb{S}^1)},$$

where

$$\mathcal{D}(t) = \int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right)\partial_x u\,\mathrm{d}x.$$

In particular, invoking Theorem 1.1,

$$\|u(t) - \langle u_0\rangle\|^2_{L^2(\mathbb{S}^1)} + \int_0^t \arctan\left(\|u(s)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right)\|u(t)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\,\mathrm{d}s \leqslant \|u_0 - \langle u_0\rangle\|^2_{L^2(\mathbb{S}^1)}.$$

- *Energy decay:*

$$\|u(t) - \langle u_0\rangle\|_{L^2(\mathbb{S}^1)} \leqslant \|u_0 - \langle u_0\rangle\|_{L^2(\mathbb{S}^1)} e^{-\frac{1}{2}\frac{\arctan(C\|u_0 - \langle u_0\rangle\|_{L^2(\mathbb{S}^1)})}{\|u_0 - \langle u_0\rangle\|_{L^2(\mathbb{S}^1)}}t}.$$

We study further properties of solutions to (1.1) passing to the following formulation of the problem.
We introduce the complex function $z(x,t) = u(x,t) + i\partial_x u(x,t)$ and, passing to polar coordinates, we rewrite (1.1) as

$$u(1+\tan^2(\theta))\partial_t\theta = \partial_x^2 \arctan\theta - \frac{\tan(\theta)}{1+\theta^2}\partial_x\theta = \frac{\partial_x^2\theta}{1+\theta^2} - \left(\frac{2\theta}{1+\theta^2} + \tan(\theta)\right)\frac{\partial_x\theta}{1+\theta^2},$$

being

$$\tan(\theta(x,t)) = \frac{\partial_x u(x,t)}{u(x,t)}.$$

Theorem 1.4. *Let $0 < u_0 \in H^3(\mathbb{S}^1)$ the initial data for (1.1). Then, as long as the unique positive solution u to (1.1) exists, the following properties hold:*

- *Boundedness of the slope:*

$$\|\theta(t)\|_{L^\infty(\mathbb{S}^1)} \leqslant \|\theta(0)\|_{L^\infty(\mathbb{S}^1)}.$$

  Furthermore,

$$\|\partial_x u(t)\|_{L^\infty(\mathbb{S}^1)} \leqslant \|\partial_x u(0)\|_{L^\infty(\mathbb{S}^1)} \frac{\max_x u_0(x)}{\min_x u_0(x)}.$$

- *Lyapunov functional:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} & u(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x \\ &= -\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x - \int_{\mathbb{S}^1}\frac{(\partial_x\theta)^2(2+\theta^2)}{1+\theta^2}\theta\tan(\theta)\,\mathrm{d}x \\ &\quad -\frac{1}{2}\int_{\mathbb{S}^1}\frac{(\partial_x\theta)^2\theta^2(2+\theta^2)}{1+\theta^2}(1+\tan^2(\theta))\,\mathrm{d}x \\ &\leqslant 0,\end{aligned}$$

  i.e.,

$$\begin{aligned}&\int_{\mathbb{S}^1} u(t)\left(1+\left(\frac{\partial_x u(t)}{u(t)}\right)^2\right)\left(\arctan^2\left(\frac{\partial_x u(t)}{u(t)}\right)+\arctan^4\left(\frac{\partial_x u(t)}{u(t)}\right)\right)\mathrm{d}x \\ &\quad \leqslant 2\int_{\mathbb{S}^1} u_0\left(1+\left(\frac{\partial_x u(0)}{u_0}\right)^2\right)\left(\arctan^2\left(\frac{\partial_x u(0)}{u_0}\right)+\arctan^4\left(\frac{\partial_x u(0)}{u_0}\right)\right)\mathrm{d}x.\end{aligned}$$

In particular, we deduce that

$$u \in L^\infty(0,T;W^{1,\infty}(\mathbb{S}^1)) \cap L^2(0,T;H^2(\mathbb{S}^1)).$$

We also prove two different existence results to problem (1.1) under lower regularity assumptions on the data.

In particular, we prove suitable uniform bounds to solutions of the approximating problem

$$\begin{aligned}\partial_t u_\kappa - \partial_x \arctan\left(\frac{\partial_x u_\kappa}{u_\kappa}\right) &= 0 \qquad && \text{in } \mathbb{S}^1\times(0,T_\kappa), \\ u_\kappa(x,0) &= \mathcal{J}_\kappa * u_0(x) && \text{in } \mathbb{S}^1,\end{aligned}$$

whose existence follows from Theorem 1.2.

We first exploit the Lyapunov functional found in Theorem 1.4 to obtain the following result.

Theorem 1.5. (Existence through Lyapunov functional)
Let $u_0 \in H^2(\mathbb{S}^1)$ *with* $u_0 > 0$, *and* $\left\|\arctan\left(\frac{\partial_x u_0}{u_0}\right)\right\|_{L^\infty(\mathbb{S}^1)}$ *suitably small. Then, solutions to problem* (1.1) *verify*

$$\int_{\mathbb{S}^1} \frac{\left(u(t)\partial_x^2 u(t) - (\partial_x u(t))^2\right)^2}{u(t)\left(u^2(t) + (\partial_x u(t))^2\right)} \mathrm{d}x$$
$$\leqslant \left(\int_{\mathbb{S}^1} \frac{\left(u_0 \partial_x^2 u(0) - (\partial_x u(0))^2\right)^2}{u_0\left(u_0^2 + (\partial_x u(0))^2\right)} \mathrm{d}x\right) e^{c\left(\int_0^t \left(\int_{\mathbb{S}^1} \frac{\left(u(t)\partial_x^2 u(t) - (\partial_x u(t))^2\right)^2}{\left(u^2(t) + (\partial_x u(t))^2\right)^2} \mathrm{d}x + 1\right) \mathrm{d}t\right)}.$$

In particular,

$$u \in L^\infty(0, T; H^2(\mathbb{S}^1)).$$

The next existence result we prove holds with Wiener data.
We recall that the definition of Wiener spaces is given by

$$A^\alpha(\mathbb{S}^1) = \left\{ u(x) \in L^1(\mathbb{S}^1) : \quad \|u\|_{A^\alpha(\mathbb{S}^1)} := \sum_{k \in \mathbb{Z}} |k|^\alpha |\widehat{u}(k)| < \infty \right\}.$$

We will make use of the interpolation inequality [10, Lemma 2.1]:

$$\|u\|_{A^p(\mathbb{S}^1)} \leqslant \|u\|^{1-\theta}_{A^0(\mathbb{S}^1)(\mathbb{S}^1)} \|u\|^\theta_{A^q(\mathbb{S}^1)} \quad \text{for} \quad 0 \leqslant p \leqslant q, \ \theta = \frac{p}{q}.$$

Theorem 1.6. (Existence in Wiener spaces) *Let*

$$w(x,t) = \frac{u(x,t) - \langle u_0 \rangle}{\langle u_0 \rangle},$$

with u *be the unique positive local solution to* (1.1). *Then, if* $w_0 = w(0) \in A^1(\mathbb{S}^1)$, *and*

$$\|w_0\|_{A^1(\mathbb{S}^1)} < \frac{1}{10},$$

we have that

$$w \in L^\infty(0, T; A^1(\mathbb{S}^1)) \cap L^1(0, T; A^3(\mathbb{S}^1)).$$

We observe that these two global existence results, although being stated in different spaces, have somehow the same flavor. Indeed, both have size restrictions in rather similar quantities, i.e.,

$$\frac{\partial_x u(0)}{u_0} \qquad \text{and} \qquad \frac{u(x,t) - \langle u_0 \rangle}{\langle u_0 \rangle}.$$

2. Proof of Theorem 1.1: Sobolev Inequalities

First, we prove the following inequality:

$$\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right) |\partial_x u| \,\mathrm{d}x \geqslant \arctan\left(\|u\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right) \|u\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\,.$$

We observe that

$$\int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right) \partial_x u \,\mathrm{d}x = \int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right) |\partial_x u| \,\mathrm{d}x.$$

We make use of the following inequality:

$$\arctan(z) \geqslant \begin{cases} \dfrac{\arctan(\xi)}{\xi} z & \text{if } z \leqslant \xi, \\ \arctan(\xi) & \text{if } z \geqslant \xi, \end{cases}$$

to deduce that

$$-\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right) |\partial_x u| \,\mathrm{d}x \leqslant -\begin{cases} \dfrac{\arctan \xi}{\xi} \displaystyle\int_{\mathbb{S}^1} \frac{|\partial_x u|^2}{u} \,\mathrm{d}x & \text{if } \dfrac{|\partial_x u|}{u} \leqslant \xi, \\ \arctan \xi \displaystyle\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x & \text{if } \dfrac{|\partial_x u|}{u} \geqslant \xi. \end{cases}$$

Since, by Hölder's inequality,

$$\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x \leqslant \left(\int_{\mathbb{S}^1} \frac{(\partial_x u)^2}{u} \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^1} u \,\mathrm{d}x\right)^{\frac{1}{2}},$$

we estimate

$$-\int_{\mathbb{S}^1} \frac{(\partial_x u)^2}{u} \,\mathrm{d}x \leqslant -\frac{\left(\int_{\mathbb{S}^1} |\partial_x u|\right)^2}{\int_{\mathbb{S}^1} u \,\mathrm{d}x},$$

and we improve the bound as

$$-\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right) |\partial_x u| \,\mathrm{d}x \leqslant -\begin{cases} \dfrac{\arctan \xi}{\xi} \dfrac{\left(\int_{\mathbb{S}^1} |\partial_x u|\right)^2}{\int_{\mathbb{S}^1} u \,\mathrm{d}x} & \text{if } \dfrac{|\partial_x u|}{u} \leqslant \xi, \\ \arctan \xi \displaystyle\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x & \text{if } \dfrac{|\partial_x u|}{u} \geqslant \xi. \end{cases} \tag{2.1}$$

Let

$$\xi = \int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x$$

in (2.1). This choice of ξ leads to

$$\begin{aligned} &-\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right) |\partial_x u| \,\mathrm{d}x \\ &\leqslant -\begin{cases} \dfrac{\arctan\left(\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x\right)}{\left(\int_{\mathbb{S}^1} u \,\mathrm{d}x\right) \int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x} \left(\displaystyle\int_{\mathbb{S}^1} |\partial_x u|\right)^2 & \text{if } \dfrac{|\partial_x u|}{u} \leqslant \displaystyle\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x, \\ \arctan\left(\displaystyle\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x\right) \left(\displaystyle\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x\right) & \text{if } \dfrac{|\partial_x u|}{u} \geqslant \displaystyle\int_{\mathbb{S}^1} |\partial_x u| \,\mathrm{d}x. \end{cases} \end{aligned}$$

Then, using the simplification

$$\int_{\mathbb{S}^1} u(x)\,\mathrm{d}x = 1,$$

$$-\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right)|\partial_x u|\ \mathrm{d}x \leqslant -\arctan\left(\|u\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right)\|u\|_{\dot{W}^{1,1}(\mathbb{S}^1)}.$$

Now, we prove the following inequality:

$$\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right)\frac{|\partial_x u|}{u}\,\mathrm{d}x \geqslant \frac{1}{4\pi}\arctan\left(\frac{\|u\|^2_{\dot{W}^{1,1}(\mathbb{S}^1)}}{\||\partial_x u|u\|_{L^1(\mathbb{S}^1)}}\right)\frac{\|u\|^2_{\dot{W}^{1,1}(\mathbb{S}^1)}}{\||\partial_x u|u\|_{L^1(\mathbb{S}^1)}}.$$

As before, we have that

$$\int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right)\frac{\partial_x u}{u}\,\mathrm{d}x \geqslant \begin{cases} \dfrac{\arctan\xi}{\xi}\displaystyle\int_{\mathbb{S}^1}\frac{(\partial_x u)^2}{u^2}\,\mathrm{d}x & \text{if } \dfrac{\partial_x u}{u} \leqslant \xi, \\ \arctan\xi\displaystyle\int_{\mathbb{S}^1}\frac{|\partial_x u|}{u}\,\mathrm{d}x & \text{if } \dfrac{\partial_x u}{u} \geqslant \xi. \end{cases}$$

Furthermore,

$$\int_{\mathbb{S}^1}\frac{|\partial_x u|}{u}\,\mathrm{d}x \leqslant \sqrt{2\pi}\left(\int_{\mathbb{S}^1}\frac{|\partial_x u|^2}{u^2}\,\mathrm{d}x\right)^{1/2},$$

We fix

$$\xi = \int_{\mathbb{S}^1}\frac{|\partial_x u|}{u}\,\mathrm{d}x,$$

and we obtain

$$\int_{\mathbb{S}^1} \arctan\left(\frac{|\partial_x u|}{u}\right)\frac{|\partial_x u|}{u}\,\mathrm{d}x \geqslant \frac{1}{4\pi}\arctan\left(\int_{\mathbb{S}^1}\frac{|\partial_x u|}{u}\,\mathrm{d}x\right)\int_{\mathbb{S}^1}\frac{|\partial_x u|}{u}\,\mathrm{d}x.$$

Using

$$\int_{\mathbb{S}^1}|\partial_x u|\ \mathrm{d}x \leqslant \left(\int_{\mathbb{S}^1}\frac{|\partial_x u|}{u}\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{S}^1}|\partial_x u|\,u\,\mathrm{d}x\right)^{1/2},$$

we conclude the desired inequality.

3. Proof of Theorem 1.2: The Existence Result with $u_0 \in H^3(\mathbb{S}^1)$

3.1. Well-Posedness

The well-posedness follows from the classical energy method [21], so we only sketch the proof.

We fix ε, κ and δ three positive parameters and define the approximate problems

$$\partial_t u^{(\varepsilon,\kappa,\delta)} - \partial_x \mathcal{J}_\kappa * \arctan\left(\frac{\partial_x \mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}}{\mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)} + \varepsilon}\right) = \varepsilon \mathcal{J}_\kappa * \partial_x^2 \mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)},$$

$$u^{(\varepsilon,\kappa,\delta)}(x,0) = \mathcal{J}_\kappa * u_0^{(\varepsilon,\kappa,\delta)}(x) + \delta,$$

where $\mathcal{J}_\kappa$ denotes the periodic heat kernel at time κ. The existence of a unique positive solution (up to a time $0 < T(\varepsilon,\kappa,\delta) \leqslant \infty$) $u^{(\varepsilon,\kappa,\delta)}$ follows from an application of Picard Theorem in H^3 [21]. Indeed, it is a tedious but straightforward computation to check that

$$F^{(\varepsilon,\kappa,\delta)}(u^{(\varepsilon,\kappa,\delta)}) = \varepsilon \mathcal{J}_\kappa * \partial_x^2 \mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)} + \partial_x \mathcal{J}_\kappa * \arctan\left(\frac{\partial_x \mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}}{\mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)} + \varepsilon}\right)$$

satisfies

$$F^{(\varepsilon,\kappa,\delta)} : H^3 \to H^3$$

and it is a Lipschitz operator

$$\|F^{(\varepsilon,\kappa,\delta)}(u^{(\varepsilon,\kappa,\delta)}) - F^{(\varepsilon,\kappa,\delta)}(v^{(\varepsilon,\kappa,\delta)})\|_{H^3} \leqslant C(\varepsilon,\kappa,\delta)\|u^{(\varepsilon,\kappa,\delta)} - v^{(\varepsilon,\kappa,\delta)}\|_{H^3}.$$

As a consequence, it exists a positive time of existence $0 < T(\varepsilon,\kappa,\delta)$ and a smooth solution

$$u^{(\varepsilon,\kappa,\delta)} \in C([0,T(\varepsilon,\kappa,\delta)], H^3(\mathbb{S}^1)).$$

The next step is to obtain uniform estimates for $0 < T^* < T(\varepsilon,\kappa,\delta)$.

It is easy to obtain κ-uniform bounds. Indeed, we test against

$$u^{(\varepsilon,\kappa,\delta)}$$

and, using the properties of the convolution, the symmetry of the heat kernel and Young's inequality, obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{L^2(\mathbb{S}^1)} + \varepsilon\|\mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{\dot{H}^1(\mathbb{S}^1)} \leqslant C(\varepsilon).$$

The higher order estimate can be obtained in a similar fashion. We test against

$$-\partial_x^6 u^{(\varepsilon,\kappa,\delta)}$$

and integrate by parts. We find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{\dot{H}^3(\mathbb{S}^1)} + \varepsilon\|\mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{\dot{H}^4(\mathbb{S}^1)}$$
$$= -\int_{\mathbb{S}^1} \partial_x^3 \arctan\left(\frac{\partial_x \mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}}{\mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)} + \varepsilon}\right) \partial_x^4 \mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}\,\mathrm{d}x.$$

The nonlinear term can be handled easily using the parabolicity and integration by parts. Then, we conclude

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{\dot{H}^3(\mathbb{S}^1)} \leqslant \mathcal{P}(\|\mathcal{J}_\kappa * u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{\dot{H}^3(\mathbb{S}^1)}) \leqslant \mathcal{P}(\|u^{(\varepsilon,\kappa,\delta)}(t)\|^2_{\dot{H}^3(\mathbb{S}^1)}),$$

which leads to the desired κ-uniform estimate. We can pass to the limit and obtain a solution

$$u^{(\varepsilon,\delta)} \in C([0,T(\varepsilon,\delta)], H^3(\mathbb{S}^1)).$$

In order to obtain ε-uniform estimates, we define

$$\mathcal{E}^{(\varepsilon,\delta)}(t) = \frac{1}{\min_x u^{(\varepsilon,\delta)}(x,t)} + \|u^{(\varepsilon,\delta)}(t)\|_{H^3(\mathbb{S}^1)}.$$

Now we use a pointwise argument (see Ref. [12] for more details). Being continuous and the domain a compact set, the solution has at least a minimum:

$$m^{(\varepsilon,\delta)}(t) = \min_x u^{(\varepsilon,\delta)}(x,t) = u^{(\varepsilon,\delta)}(\underline{x}_t, t).$$

Because of the positivity of the initial data, we have that $m^{(\varepsilon,\delta)}(0) > \delta > 0$. Following the argument in Ref. [12], we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} m^{(\varepsilon,\delta)}(t) = \partial_t u^{(\varepsilon,\delta)}(\underline{x}_t, t) = \frac{\partial_x^2 u^{(\varepsilon,\delta)}(\underline{x}_t)}{m^{(\varepsilon,\delta)}(t)} \quad \text{a.e..}$$

Indeed, due to the smoothness of $u^{(\varepsilon,\delta)}$ in space and time we have that $m^{(\varepsilon,\delta)}(t)$ is Lipschitz. To see that, we use the reverse triangle inequality and find that

$$|\min_x u^{(\varepsilon,\delta)}(t_1,x) - \min_x u^{(\varepsilon,\delta)}(t_2,x)| \leqslant \min_x(|u^{(\varepsilon,\delta)}(t_1,x) - u^{(\varepsilon,\delta)}(t_2,x)|) \leqslant C|t_1 - t_2|.$$

Using Rademacher's Theorem, we have that $\min_x u^{(\varepsilon,\delta)}(t,x)$ is differentiable almost everywhere. Thus, using that x_t is the point of minimum, we get that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m^{(\varepsilon,\delta)}(t) &= \lim_{h_j \to 0} \frac{m^{(\varepsilon,\delta)}(t+h_j) - m^{(\varepsilon,\delta)}(t)}{h_j} \\ &= \lim_{h_j \to 0} \frac{u^{(\varepsilon,\delta)}(x_{t+h_j}, t+h_j) - u^{(\varepsilon,\delta)}(x_t, t)}{h_j} \\ &\geqslant \partial_t u^{(\varepsilon,\delta)}(x_t, t). \end{aligned}$$

In the same way, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t} m^{(\varepsilon,\delta)}(t) \leqslant \partial_t u^{(\varepsilon,\delta)}(x_t, t).$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\min_x u(x,t)} = -\frac{\partial_t u(\underline{x}_t, t)}{m(t)^2} \leqslant C \frac{\|u(t)\|_{H^3(\mathbb{S}^1)}}{m(t)^3} \leqslant C(\mathcal{E}(t))^4.$$

We can estimate the evolution of the H^3 norm with the previous ideas together with the fact that

$$\frac{1}{u^{(\varepsilon,\delta)} + \varepsilon} \leqslant \frac{1}{u^{(\varepsilon,\delta)}} \leqslant \frac{1}{m^{(\varepsilon,\delta)}(t)}.$$

Thus, finally, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}^{(\varepsilon,\delta)}(t) \leqslant \mathcal{P}(\mathcal{E}^{(\varepsilon,\delta)}(t)).$$

Then, we obtain a ε-uniform bound and we can pass to the limit. We find

$$u^{(\delta)} \in C([0, T(\delta)], H^3(\mathbb{S}^1)).$$

With the previous ideas, we can also pass to the limit in δ and we conclude the local existence of classical solution

$$u \in C([0, T], H^3(\mathbb{S}^1)).$$

To obtain the uniqueness, we proceed with a standard contradiction argument.

4. Proof of Theorem 1.3: Properties of the Solution

4.1. Maximum Principle

Equation (1.1) can be written as

$$\partial_t u - \frac{u\partial_x^2 u - (\partial_x u)^2}{u^2 + (\partial_x u)^2} = 0.$$

Using the pointwise method [12], we find that

$$M(t) = \max_x u(x,t) = u(\overline{x}_t, t),$$

and

$$m(t) = \min_x u(x,t) = u(\underline{x}_t, t),$$

satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} M \leqslant 0, \text{ a.e.} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} m \geqslant 0, \text{ a.e..}$$

Integrating in time, we conclude this part. We observe that, in particular,

$$\|u(t)\|_{L^\infty(\mathbb{S}^1)} \leqslant \|u_0\|_{L^\infty(\mathbb{S}^1)}.$$

4.2. Conservation of Mass

The conservation of mass follows from the sign propagation and an integration in space.

4.3. Entropy Balance

The evolution of the entropy can be easily computed and we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) + \int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right)\frac{\partial_x u}{u}\,\mathrm{d}x = 0.$$

Using that

$$\arctan(z)z = \arctan(|z|)|z| \geqslant 0,$$

we conclude. Using Theorem 1.1, we conclude the desired estimate.

4.4. Energy Balance

The evolution of the L^2 energy can be computed similarly. We observe that the mean is preserved (see above). To estimate the decay, we compute the following:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - \langle u(t)\rangle\|_{L^2(\mathbb{S}^1)}^2 = \int_{\mathbb{S}^1} \partial_t u(u - \langle u_0\rangle)\,\mathrm{d}x = \int_{\mathbb{S}^1} \partial_t u u\,\mathrm{d}x.$$

Integrating by parts, we find

$$\mathcal{D}(t) = \int_{\mathbb{S}^1} \arctan\left(\frac{\partial_x u}{u}\right)\partial_x u\,\mathrm{d}x \geqslant 0.$$

In particular, invoking Theorem 1.1,

$$\|u(t) - \langle u_0\rangle\|_{L^2(\mathbb{S}^1)}^2 + \int_0^t \arctan\left(\|u(s)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right)\|u(s)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\,\mathrm{d}s$$
$$\leqslant \|u_0 - \langle u_0\rangle\|_{L^2(\mathbb{S}^1)}^2,$$

and we obtain that

$$\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}$$

decays.

4.5. Energy Decay

Using that, for certain point $\tilde{x}_t$,

$$u(x,t)-\langle u_0\rangle = u(x,t)-u(\tilde{x}_t,t)=\int_{\tilde{x}_t}^{x}\partial_x u(y,t)\,\mathrm{d}y,$$

we conclude the Poincaré inequality

$$C\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\leqslant \|\partial_x u(t)\|_{L^1(\mathbb{S}^1)}.$$

As a consequence,

$$\arctan\left(\|u(t)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\right)\|u(t)\|_{\dot{W}^{1,1}(\mathbb{S}^1)}\geqslant \arctan\left(C\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\right) C\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}.$$

Thus, integrating by parts and estimating using Theorem 1.1,

$$2\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}+C\arctan\left(C\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\right)\leqslant 0.$$

Using that

$$\arctan(z)\geqslant \frac{\arctan(C\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)})}{C\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}}z \qquad \text{if} \qquad z\leqslant C\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)},$$

the previous decay of the L^2 energy translates into the following inequality:

$$2\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}+\frac{\arctan\left(C\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\right)}{\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}}\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\leqslant 0,$$

from where we conclude the

$$\|u(t)-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\leqslant \|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}e^{-\frac{1}{2}\frac{\arctan\left(C\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}\right)}{\|u_0-\langle u_0\rangle\|_{L^2(\mathbb{S}^1)}}t}.$$

5. Proof of Theorem 1.4: Further Properties of the Solution with the θ Formulation

We consider the complex value $z(x,t)=u(x,t)+i\partial_x u(x,t)$. If we write this quantity in polar coordinates, we know that

$$\tan(\theta(x,t))=\frac{\partial_x u(x,t)}{u(x,t)}, \quad \text{hence} \quad \theta(x,t)=\arctan\left(\frac{\partial_x u(x,t)}{u(x,t)}\right). \tag{5.1}$$

Then, we use (1.1) to deduce the evolution equation of θ:

$$u(1+\tan^2(\theta))\partial_t\theta=\partial_x^2\arctan\theta-\frac{\tan(\theta)}{1+\theta^2}\partial_x\theta=\frac{\partial_x^2\theta}{1+\theta^2}-\left(\frac{2\theta}{1+\theta^2}+\tan(\theta)\right)\frac{\partial_x\theta}{1+\theta^2}. \tag{5.2}$$

Boundedness of the slope Taking in mind (5.1) and (5.2), let us take

$$\Psi(t)=\max_x\theta(x,t)=\theta(x_\Psi,t).$$

Then, a.e. $t \in [0,T]$, we have that

$$u(x_\Psi,t)(1+\tan^2\Psi(t))\partial_t\Psi(t) = \frac{\partial_x^2\theta(x_\Psi,t)}{1+\Psi^2(t)} \leqslant 0.$$

Observing also that $u(x_\Psi,t)(1+\tan^2\Psi(t)) > 0$, we obtain that

$$\max_x \theta(x,t) = \Psi(t) \leqslant \Psi(0) = \max_x \theta(x,0).$$

We can repeat the same argument for

$$\Phi(t) = \min_x \theta(x,t) = \theta(x_\Phi,t),$$

getting that

$$\min_x \theta(x,t) = \Phi(t) \geqslant \Phi(0) = \min_x \theta(x,0).$$

This implies that the function θ is bounded in the x variable a.e. t, i.e.,

$$\|\theta(t)\|_{L^\infty(\mathbb{S}^1)} \leqslant \|\theta(0)\|_{L^\infty(\mathbb{S}^1)} \qquad \text{a.e. } t \in [0,T].$$

We now recall the definition of θ to deduce that

$$\left\|\frac{\partial_x u(t)}{u(t)}\right\|_{L^\infty(\mathbb{S}^1)} = \|\tan(\theta(t))\|_{L^\infty(\mathbb{S}^1)} \leqslant \|\tan(\theta(0))\|_{L^\infty(\mathbb{S}^1)},$$

and the desired estimate follows from the boundedness of u:

$$\begin{aligned}\|\partial_x u(t)\|_{L^\infty(\mathbb{S}^1)} &= \left\|\frac{\partial_x u(t)}{u(t)}u(t)\right\|_{L^\infty(\mathbb{S}^1)} \leqslant \left\|\frac{\partial_x u(t)}{u(t)}\right\|_{L^\infty(\mathbb{S}^1)} \|u(t)\|_{L^\infty(\mathbb{S}^1)} \\ &\leqslant \|\tan(\theta(0))\|_{L^\infty(\mathbb{S}^1)} \|u_0\|_{L^\infty(\mathbb{S}^1)} \\ &\leqslant \|\partial_x u(0)\|_{L^\infty(\mathbb{S}^1)} \frac{\max_x u_0(x)}{\min_x u_0(x)}.\end{aligned}$$

5.1. Lyapunov Functional

We are going to prove that the functional

$$L(u) = \int_{\mathbb{S}^1} u(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x$$

is a Lyapunov functional. To this aim, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(u) &= \int_{\mathbb{S}^1} \partial_t u(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x + \int_{\mathbb{S}^1} u\partial_t(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} u(1+\tan^2(\theta))\partial_t\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x \\ &= \int_{\mathbb{S}^1} \partial_t u(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x + \int_{\mathbb{S}^1} u\tan(\theta)(1+\tan^2(\theta))\left(\theta^2+\frac{\theta^4}{2}\right)\partial_t\theta\,\mathrm{d}x \\ &\quad + \int_{\mathbb{S}^1} u(1+\tan^2(\theta))\left(\theta+\theta^3\right)\partial_t\theta\,\mathrm{d}x,\end{aligned} \tag{5.3}$$

and we use the equations

$$u(1+\tan^2(\theta))\partial_t\theta = \frac{\partial_x^2\theta}{1+\theta^2} - \left(\frac{2\theta}{1+\theta^2}+\tan(\theta)\right)\frac{\partial_x\theta}{1+\theta^2},$$

$$\partial_t u = \frac{u\partial_x^2 u - (\partial_x u)^2}{u^2+(\partial_x u)^2} = \partial_x\theta,$$

to rewrite (5.3) as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}L(u) &= \int_{\mathbb{S}^1} \partial_x\theta(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x + \int_{\mathbb{S}^1} \theta u(1+\tan^2(\theta)) \\
&\quad \left[1+\theta^2+\tan(\theta)\left(\theta+\frac{\theta^3}{2}\right)\right]\partial_t\theta\,\mathrm{d}x \\
&= \int_{\mathbb{S}^1} \partial_x\theta(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x \quad + \int_{\mathbb{S}^1} \theta\left[1+\theta^2+\tan(\theta)\left(\theta+\frac{\theta^3}{2}\right)\right] \\
&\quad \left(\frac{\partial_x^2\theta}{1+\theta^2} - \left(\frac{2\theta}{1+\theta^2}+\tan(\theta)\right)\frac{\partial_x\theta}{1+\theta^2}\right)\mathrm{d}x. \qquad (5.4)
\end{aligned}
$$

We claim that (5.4) is equivalent to

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} u(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x \\
&\quad = \int_{\mathbb{S}^1} \theta\left[1+\theta^2+\tan(\theta)\left(\theta+\frac{\theta^3}{2}\right)\right]\frac{\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x. \qquad (5.5)
\end{aligned}
$$

Indeed, the integrals

$$
\int_{\mathbb{S}^1} \partial_x\theta(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x
$$

and

$$
\int_{\mathbb{S}^1} \theta\left[1+\theta^2+\tan(\theta)\left(\theta+\frac{\theta^3}{2}\right)\right]\left(\frac{2\theta}{1+\theta^2}+\tan(\theta)\right)\frac{\partial_x\theta}{1+\theta^2}\,\mathrm{d}x,
$$

can be rewritten both in the form

$$
\int_{\mathbb{S}^1} \phi'(\theta)\partial_x\theta\,\mathrm{d}x,
$$

being

$$
\phi(\theta) = \int_0^\theta \psi(y)\,\mathrm{d}y,
$$

with $\psi(y)$ either

$$
\psi(y) = (1+\tan^2(y))\left(\frac{y^2}{2}+\frac{y^4}{4}\right),
$$

or

$$
\psi(y) = \left[1+y^2+\tan(y)\left(y+\frac{y^3}{2}\right)\right]\left(\frac{2y}{1+y^2}+\tan(y)\right)\frac{y}{1+y^2}.
$$

Then, by periodic boundary conditions, we obtain that

$$
\int_{\mathbb{S}^1} \phi'(\theta)\partial_x\theta\,\mathrm{d}x = \phi(\theta)\Big|_{-\pi}^{\pi} = 0.
$$

We come back to (5.5), and we rewrite the r.h.s. as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} u(1+\tan^2(\theta))\left(\frac{\theta^2}{2}+\frac{\theta^4}{4}\right)\mathrm{d}x &= \int_{\mathbb{S}^1} \theta\partial_x^2\theta\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}^1} \theta^2\tan(\theta)\partial_x^2\theta\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{S}^1} \tan(\theta)\partial_x^2\theta\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{S}^1} \frac{\tan(\theta)}{1+\theta^2}\partial_x^2\theta\,\mathrm{d}x.
\end{aligned}
$$

We compute the above integrals by integration by parts:

$$\int_{\mathbb{S}^1} \theta\partial_x^2\theta \,\mathrm{d}x = -\int_{\mathbb{S}^1} (\partial_x\theta)^2 \,\mathrm{d}x,$$

$$\frac{1}{2}\int_{\mathbb{S}^1} \theta^2 \tan(\theta)\partial_x^2\theta \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{S}^1} (\partial_x\theta)^2 \left(2\theta\tan(\theta) + \theta^2(1+\tan^2(\theta))\right) \mathrm{d}x,$$

$$\frac{1}{2}\int_{\mathbb{S}^1} \tan(\theta)\partial_x^2\theta \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{S}^1} (\partial_x\theta)^2(1+\tan^2(\theta)) \,\mathrm{d}x,$$

$$-\frac{1}{2}\int_{\mathbb{S}^1} \frac{\tan(\theta)}{1+\theta^2}\partial_x^2\theta \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{S}^1} \frac{(\partial_x\theta)^2}{1+\theta^2}(1+\tan^2(\theta)) \,\mathrm{d}x - \int_{\mathbb{S}^1} \frac{(\partial_x\theta)^2\theta\tan(\theta)}{1+\theta^2} \,\mathrm{d}x.$$

Summing up all the above integrals, we find that the r.h.s. of (5.5) simplifies as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} u(1+\tan^2(\theta)) \left(\frac{\theta^2}{2} + \frac{\theta^4}{4}\right) \mathrm{d}x \\ = -\int_{\mathbb{S}^1} (\partial_x\theta)^2 \,\mathrm{d}x - \int_{\mathbb{S}^1} \frac{(\partial_x\theta)^2(2+\theta^2)}{1+\theta^2}\theta\tan(\theta) \,\mathrm{d}x \\ -\frac{1}{2}\int_{\mathbb{S}^1} \frac{(\partial_x\theta)^2\theta^2(2+\theta^2)}{1+\theta^2}(1+\tan^2(\theta)) \,\mathrm{d}x.\end{aligned}$$

Using the properties of tan, we have that $x\tan(x) \geqslant 0$, and hence we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} u(1+\tan^2(\theta)) \left(\frac{\theta^2}{2} + \frac{\theta^4}{4}\right) \mathrm{d}x \leqslant 0,$$

and

$$\int_0^t \int_{\mathbb{S}^1} (\partial_x\theta)^2 \,\mathrm{d}x \leqslant L(u_0).$$

6. Proof of Theorem 1.5: Existence Through Lyapunov Functional

Let $-\partial_x^2\theta$ be the test function in (5.2). Then, the integral in space reads

$$\begin{aligned}-\int_{\mathbb{S}^1} u\left(1+\tan^2(\theta)\right)\partial_t\theta\partial_x^2\theta \,\mathrm{d}x = -\int_{\mathbb{S}^1} \frac{\left(\partial_x^2\theta\right)^2}{1+\theta^2} \mathrm{d}x + 2\int_{\mathbb{S}^1} \frac{\theta\partial_x\theta\partial_x^2\theta}{\left(1+\theta^2\right)^2} \mathrm{d}x \\ + \int_{\mathbb{S}^1} \tan(\theta)\frac{\partial_x\theta\partial_x^2\theta}{1+\theta^2} \mathrm{d}x. \qquad (6.1)\end{aligned}$$

We integrate by parts the integral in the l.h.s.:

$$\begin{aligned}&-\int_{\mathbb{S}^1} u\left(1+\tan^2(\theta)\right)\partial_t\theta\partial_x^2\theta \,\mathrm{d}x \\ &= \int_{\mathbb{S}^1} u\left(1+\tan^2(\theta)\right)\partial_x\theta\partial_{tx}\theta \,\mathrm{d}x + \int_{\mathbb{S}^1} \partial_x\left(u\left(1+\tan^2(\theta)\right)\right)\partial_x\theta\partial_t\theta \,\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} u\left(1+\tan^2(\theta)\right)\left(\partial_x\theta\right)^2 \,\mathrm{d}x - I_1 + I_2,\end{aligned}$$

with

$$I_1 = \frac{1}{2}\int_{\mathbb{S}^1} \partial_t\left(u\left(1+\tan^2(\theta)\right)\right)(\partial_x\theta)^2\,\mathrm{d}x,$$
$$I_2 = \int_{\mathbb{S}^1} \partial_x\left(u\left(1+\tan^2(\theta)\right)\right)\partial_x\theta\partial_t\theta\,\mathrm{d}x.$$

We use the equations

$$u(1+\tan^2(\theta))\partial_t\theta = \frac{\partial_x^2\theta}{1+\theta^2} - \left(\frac{2\theta}{1+\theta^2}+\tan(\theta)\right)\frac{\partial_x\theta}{1+\theta^2},$$
$$\partial_t u = \frac{u\partial_x^2 u - (\partial_x u)^2}{u^2+(\partial_x u)^2} = \partial_x\theta,$$
$$\frac{\partial_x u}{u} = \tan(\theta),$$

to rewrite I_1 and I_2 as follows:

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{\mathbb{S}^1}\partial_t u\left(1+\tan^2(\theta)\right)(\partial_x\theta)^2\,\mathrm{d}x + \int_{\mathbb{S}^1}\tan(\theta)u\left(1+\tan^2(\theta)\right)\partial_t\theta(\partial_x\theta)^2\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{S}^1}\left(1+\tan^2(\theta)\right)(\partial_x\theta)^3\,\mathrm{d}x + \int_{\mathbb{S}^1}\tan(\theta)(\partial_x\theta)^2\frac{\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x\\
&\quad - 2\int_{\mathbb{S}^1}\tan(\theta)\frac{\theta(\partial_x\theta)^3}{(1+\theta^2)^2}\,\mathrm{d}x - \int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^3}{1+\theta^2}\,\mathrm{d}x,\\
I_2 &= \int_{\mathbb{S}^1}\tan(\theta)u\left(1+\tan^2(\theta)\right)\partial_x\theta(1+2\partial_x\theta)\partial_t\theta\,\mathrm{d}x\\
&= \int_{\mathbb{S}^1}\tan(\theta)\frac{\partial_x\theta\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x - 2\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\theta}{(1+\theta^2)^2}\,\mathrm{d}x - \int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^2}{1+\theta^2}\,\mathrm{d}x\\
&\quad + 2\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x - 4\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^3\theta}{(1+\theta^2)^2}\,\mathrm{d}x - 2\int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^3}{1+\theta^2}\,\mathrm{d}x.
\end{aligned}$$

The difference among I_1 and I_2 reads

$$\begin{aligned}
I_1 - I_2 = &-\int_{\mathbb{S}^1}\tan(\theta)\frac{\partial_x\theta\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x + 2\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\theta}{(1+\theta^2)^2}\,\mathrm{d}x + \int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^2}{1+\theta^2}\,\mathrm{d}x\\
&-\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x + 2\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^3\theta}{(1+\theta^2)^2}\,\mathrm{d}x + \int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^3}{1+\theta^2}\,\mathrm{d}x\\
&+\frac{1}{2}\int_{\mathbb{S}^1}\left(1+\tan^2(\theta)\right)(\partial_x\theta)^3\,\mathrm{d}x.
\end{aligned}$$

Then, (6.1) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1}u\left(1+\tan^2(\theta)\right)(\partial_x\theta)^2\,\mathrm{d}x + \int_{\mathbb{S}^1}\frac{\left(\partial_x^2\theta\right)^2}{1+\theta^2}\,\mathrm{d}x\\
&\quad = I_1 - I_2 + 2\int_{\mathbb{S}^1}\frac{\theta\partial_x\theta\partial_x^2\theta}{(1+\theta^2)^2}\,\mathrm{d}x + \int_{\mathbb{S}^1}\tan(\theta)\frac{\partial_x\theta\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x\\
&\quad = 2\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\theta}{(1+\theta^2)^2}\,\mathrm{d}x + \int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^2}{1+\theta^2}\,\mathrm{d}x - \int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\partial_x^2\theta}{1+\theta^2}\,\mathrm{d}x\\
&\qquad + \int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^3}{1+\theta^2}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{S}^1}(1+\tan^2(\theta))(\partial_x\theta)^3\,\mathrm{d}x + 2\int_{\mathbb{S}^1}\frac{\theta\partial_x\theta\partial_x^2\theta}{(1+\theta^2)^2}\,\mathrm{d}x.
\end{aligned} \tag{6.2}$$

We easily estimate the sum of the first two terms in (6.2)

$$2\int_{\mathbb{S}^1}\tan(\theta)\frac{(\partial_x\theta)^2\theta}{(1+\theta^2)^2}\,\mathrm{d}x+\int_{\mathbb{S}^1}\tan^2(\theta)\frac{(\partial_x\theta)^2}{1+\theta^2}\,\mathrm{d}x$$

using that

$$\partial_x\theta\in L^2(0,T;L^2(\mathbb{S}^1)),$$

and

$$2\frac{\tan(\theta(t,x))\theta(t,x)}{1+\theta^2(t,x)}+\tan^2(\theta(t,x))\leqslant\left\|2\tan(\theta(0))\theta(0)+\tan^2(\theta(0))\right\|_{L^\infty(\mathbb{S}^1)}\leqslant c.$$

Furthermore, we can absorb the last term in (6.2) using Young's inequality and the boundedness in time and space of θ:

$$\begin{aligned}2\int_{\mathbb{S}^1}\frac{\theta\partial_x\theta\partial_x^2\theta}{\left(1+\theta^2\right)^2}\,\mathrm{d}x&\leqslant\varepsilon\int_{\mathbb{S}^1}\frac{\left(\partial_x^2\theta\right)^2}{1+\theta^2}\,\mathrm{d}x+c(\varepsilon)\int_{\mathbb{S}^1}\frac{\theta^2(\partial_x\theta)^2}{\left(1+\theta^2\right)^3}\,\mathrm{d}x\\&\leqslant\varepsilon\int_{\mathbb{S}^1}\frac{\left(\partial_x^2\theta\right)^2}{1+\theta^2}\,\mathrm{d}x+c(\varepsilon)\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x.\end{aligned}$$

So far, (6.2) can be bounded with

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1}u\left(1+\tan^2(\theta)\right)(\partial_x\theta)^2\,\mathrm{d}x+(1-\varepsilon)\int_{\mathbb{S}^1}\frac{\left(\partial_x^2\theta\right)^2}{1+\theta^2}\,\mathrm{d}x\\&\quad\leqslant c(\varepsilon)\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x+\int_{\mathbb{S}^1}|\tan(\theta)|\frac{(\partial_x\theta)^2\left|\partial_x^2\theta\right|}{1+\theta^2}\,\mathrm{d}x\\&\quad+\frac{1}{2}\int_{\mathbb{S}^1}\left(1+\frac{3+\theta^2}{1+\theta^2}\tan^2(\theta)\right)(\partial_x\theta)^3\,\mathrm{d}x.\end{aligned}\tag{6.3}$$

The second integral in the r.h.s. of (6.3) can be estimated as

$$\int_{\mathbb{S}^1}|\tan(\theta)|\frac{(\partial_x\theta)^2\left|\partial_x^2\theta\right|}{1+\theta^2}\,\mathrm{d}x\leqslant\varepsilon\left\|\frac{\partial_x^2\theta(t)}{\sqrt{1+\theta^2(t)}}\right\|_{L^2(\mathbb{S}^1)}^2+c(\varepsilon)\left\|\partial_x\theta(t)\right\|_{L^4(\mathbb{S}^1)}^4.$$

Since

$$\begin{aligned}\|\partial_x\theta(t)\|_{L^4(\mathbb{S}^1)}^4&\leqslant\left\|\partial_x^2\theta(t)\right\|_{L^2(\mathbb{S}^1)}^2\|\theta(t)\|_{L^\infty(\mathbb{S}^1)}^2\\&\leqslant\left\|\frac{\partial_x^2\theta(t)}{\sqrt{1+\theta^2(t)}}\right\|_{L^2(\mathbb{S}^1)}^2\|\theta(0)\|_{L^\infty(\mathbb{S}^1)}^2\left(1+\|\theta(0)\|_{L^\infty(\mathbb{S}^1)}^2\right),\end{aligned}$$

we can absorb

$$\begin{aligned}\int_{\mathbb{S}^1}|\tan(\theta)|\frac{(\partial_x\theta)^2\left|\partial_x^2\theta\right|}{1+\theta^2}\,\mathrm{d}x&\leqslant\left(\left(1+\|\theta(0)\|_{L^\infty(\mathbb{S}^1)}^2\right)\|\theta(0)\|_{L^\infty(\mathbb{S}^1)}^2+\varepsilon\right)\\&\quad\left\|\frac{\partial_x^2\theta(t)}{\sqrt{1+\theta^2(t)}}\right\|_{L^2(\mathbb{S}^1)}^2\end{aligned}$$

in the l.h.s. of (6.3) for sufficiently small ε and $\|\theta(0)\|_{L^\infty(\mathbb{S}^1)}$.

The last integral in the r.h.s. of (6.3) can be estimated by Sobolev's embedding as

$$\frac{1}{2}\int_{\mathbb{S}^1}\left(1+\frac{3+\theta^2}{1+\theta^2}\tan^2(\theta)\right)(\partial_x\theta)^3\,\mathrm{d}x \leqslant c\left\|\partial_x\theta(t)\right\|^3_{L^3(\mathbb{S}^1)} \leqslant c\left\|\partial_x\theta(t)\right\|^3_{H^{\frac{1}{6}}(\mathbb{S}^1)}.$$

We interpolate the last inequality and we use the Poincaré inequality to get

$$c\left\|\partial_x\theta(t)\right\|^3_{H^{\frac{1}{6}}(\mathbb{S}^1)} \leqslant c\left\|\partial_x\theta(t)\right\|^{\frac{5}{2}}_{L^2(\mathbb{S}^1)}\left\|\partial_x^2\theta(t)\right\|^{\frac{1}{2}}_{L^2(\mathbb{S}^1)} \leqslant c\left\|\partial_x\theta(t)\right\|^2_{L^2(\mathbb{S}^1)}\left\|\partial_x^2\theta(t)\right\|_{L^2(\mathbb{S}^1)}.$$

We apply the Young inequality to obtain the following estimate on this last integral:

$$\frac{1}{2}\int_{\mathbb{S}^1}\left(1+\frac{3+\theta^2}{1+\theta^2}\tan^2(\theta)\right)(\partial_x\theta)^3\,\mathrm{d}x \leqslant c(\varepsilon)\left\|\partial_x\theta(t)\right\|^4_{L^2(\mathbb{S}^1)}+\varepsilon\left\|\frac{\partial_x^2\theta(t)}{\sqrt{1+\theta^2(t)}}\right\|^2_{L^2(\mathbb{S}^1)}.$$

Finally, for ε and $\|\theta(0)\|_{L^\infty(\mathbb{S}^1)}$ suitably small, i.e., such that

$$\delta = 1-3\varepsilon-\left(1+\|\theta(0)\|^2_{L^\infty(\mathbb{S}^1)}\right)\|\theta(0)\|^2_{L^\infty(\mathbb{S}^1)}>0,$$

we obtain the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1}u\left(1+\tan^2(\theta)\right)(\partial_x\theta)^2\,\mathrm{d}x+\delta\int_{\mathbb{S}^1}\frac{\left(\partial_x^2\theta\right)^2}{1+\theta^2}\,\mathrm{d}x \leqslant c\left(\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x\right)\left(\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x+1\right).$$

To conclude our argument, we continue in the following way: we estimate

$$\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x = \int_{\mathbb{S}^1}\frac{u\left(1+\tan^2(\theta)\right)}{u\left(1+\tan^2(\theta)\right)}(\partial_x\theta)^2\,\mathrm{d}x \leqslant c\int_{\mathbb{S}^1}u\left(1+\tan^2(\theta)\right)(\partial_x\theta)^2\,\mathrm{d}x.$$

This means that we have a differential inequality of the type

$$y'(t)\leqslant c\left(1+\left(\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x\right)\right)y(t),$$

with

$$y(t)=\int_{\mathbb{S}^1}u\left(1+\tan^2(\theta)\right)(\partial_x\theta)^2\,\mathrm{d}x.$$

Then, we can apply a Grönwall type inequality obtaining that

$$\int_{\mathbb{S}^1}u(t)\left(1+\tan^2(\theta(t))\right)(\partial_x\theta(t))^2\,\mathrm{d}x \leqslant\left(\int_{\mathbb{S}^1}u_0\left(1+\tan^2(\theta(0))\right)(\partial_x\theta(0))^2\,\mathrm{d}x\right)e^{c\left(\int_0^t\left(\int_{\mathbb{S}^1}(\partial_x\theta(s))^2\,\mathrm{d}x+1\right)\mathrm{d}s\right)}.$$

Finally, we conclude using

$$\int_0^t\int_{\mathbb{S}^1}(\partial_x\theta)^2\,\mathrm{d}x\leqslant L(u_0).$$

7. Proof of Theorem 1.6: Existence in Wiener Spaces

This proof is done in the same spirit as Refs. [15, 17–19]. First we focus on the a priori estimates. We know that, for an H^3 initial data, the solution exists. Consequently, let us obtain the desired estimates under this extra assumption and, later on, we will generalize the argument to be able to drop it out.

Let

$$w(x,t) = \frac{u(x,t) - \langle u_0 \rangle}{\langle u_0 \rangle}.$$

Then, (1.4) becomes

$$\langle u_0 \rangle w_t = \frac{\partial_x^2 w + w\partial_x^2 w - (\partial_x w)^2}{1 + 2w + w^2 + (\partial_x w)^2},$$

with initial data

$$w_0(x) = w(x,0) = \frac{u_0(x) - \langle u_0 \rangle}{\langle u_0 \rangle}.$$

Since we assumed that $\|w_0\|_{(\mathbb{S}^1)} < 1/10$, we know that there exists a time $0 < T^*$, eventually smaller than the existence time T, such that

$$\|w(t)\|_{A^1(\mathbb{S}^1)} < 1/10 \qquad \forall t < T^*.$$

Hence,

$$\left|2w(x,t) + w^2(x,t) + (\partial_x w(x,t))^2\right| \leqslant 4\,\|w(t)\|_{A^1(\mathbb{S}^1)} < 1,$$

and we can develop in series

$$\langle u_0 \rangle w_t = \partial_x^2 w + w\partial_x^2 w - (\partial_x w)^2 + \left(\partial_x^2 w + w\partial_x^2 w - (\partial_x w)^2\right) \left(\sum_{n\geqslant 1} (-1)^n (2w + w^2 + (\partial_x w)^2)^n\right). \tag{7.1}$$

We want to write the Fourier coefficient of (7.1). Before getting into these computations, we rewrite

$$\sum_{n\geqslant 1} (-1)^n (2w + w^2 + (\partial_x w)^2)^n = \sum_{n\geqslant 1} (-1)^n \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r} 2^\ell w^{2j+\ell} (\partial_x w)^{2r}$$

for

$$\binom{n}{\ell,\, j,\, r} = \frac{n!}{\ell!\, j!\, r!}.$$

The Fourier coefficient of this series is given by

$$\begin{aligned}
&\widehat{\left(\partial_x^2 w + w\partial_x^2 w - (\partial_x w)^2\right) \left(\sum_{n\geqslant 1} (-1)^n (2w + w^2 + (\partial_x w)^2)^n\right)}(k,t) \\
&= \widehat{\left(\partial_x^2 w + w\partial_x^2 w - (\partial_x w)^2\right) \left(\sum_{n\geqslant 1} (-1)^n \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r} 2^\ell w^{2j+\ell} (\partial_x w)^{2r}\right)}(k,t) \\
&= \sum_{n\geqslant 1} (-1)^n \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r}
\end{aligned}$$

$$2^{\ell}\left(\widehat{w^{2j+\ell}(\partial_x w)^{2r}\partial_x^2 w}+\widehat{w^{2j+\ell+1}(\partial_x w)^{2r}\partial_x^2 w}-\widehat{w^{2j+\ell}(\partial_x w)^{2r+2}}\right)(k,t)$$
$$=\widehat{\mathcal{N}}_1(k,t)+\widehat{\mathcal{N}}_2(k,t)-\widehat{\mathcal{N}}_3(k,t),$$

for

$$\widehat{\mathcal{N}}_1(k,t)=-\sum_{n\geqslant 1}(-1)^n\sum_{\substack{\ell+j+r=n\\ \ell,j,r\geqslant 0}}\binom{n}{\ell,\,j,\,r}(-1)^r 2^{\ell}\sum_{m_0\in\mathbb{Z}}\cdots$$
$$\sum_{m_{2j+\ell+2r-3}\in\mathbb{Z}}(k-m_0)^2\widehat{w}(k-m_0,t)$$
$$\times(m_0-m_1)\widehat{w}(m_0-m_1,t)\left(\prod_{p=1}^{2r-1}\widehat{w}(m_p-m_{p+1},t)(m_p-m_{p+1})\right)$$
$$\times\left(\prod_{q=2r-1}^{2j+\ell+2r-3}\widehat{w}(m_q-m_{q+1},t)\right)\widehat{w}(m_{2j+\ell+2r-3},t),$$
$$\widehat{\mathcal{N}}_2(k,t)=-\sum_{n\geqslant 1}(-1)^n\sum_{\substack{\ell+j+r=n\\ \ell,j,r\geqslant 0}}\binom{n}{\ell,\,j,\,r}(-1)^r 2^{\ell}\sum_{m_0\in\mathbb{Z}}\cdots$$
$$\sum_{m_{2j+\ell+2r-2}\in\mathbb{Z}}(k-m_0)^2\widehat{w}(k-m_0,t)$$
$$\times(m_0-m_1)\widehat{w}(m_0-m_1,t)\left(\prod_{p=1}^{2r-1}\widehat{w}(m_p-m_{p+1},t)(m_p-m_{p+1})\right)$$
$$\times\left(\prod_{q=2r-1}^{2j+\ell+2r-2}\widehat{w}(m_q-m_{q+1},t)\right)\widehat{w}(m_{2j+\ell+2r-2},t),$$
$$\widehat{\mathcal{N}}_3(k,t)=\sum_{n\geqslant 1}(-1)^n\sum_{\substack{\ell+j+r=n\\ \ell,j,r\geqslant 0}}\binom{n}{\ell,\,j,\,r}(-1)^{r+1}2^{\ell}\sum_{m_0\in\mathbb{Z}}\cdots$$
$$\sum_{m_{2j+\ell+2r-2}\in\mathbb{Z}}$$
$$\times(k-m_0)\widehat{w}(k-m_0,t)\left(\prod_{p=0}^{2r}\widehat{w}(m_p-m_{p+1},t)(m_p-m_{p+1})\right)$$
$$\times\left(\prod_{q=2r-1}^{2j+\ell+2r-2}\widehat{w}(m_q-m_{q+1},t)\right)\widehat{w}(m_{2j+\ell+2r-2},t),$$

and the Fourier coefficient of (7.1) is given by

$$\langle u_0\rangle\widehat{w}_t(k,t)=-k^2\widehat{w}(k,t)-\sum_{m\in\mathbb{Z}}m^2\widehat{w}(m,t)\widehat{w}(k-m,t)$$
$$-\sum_{m\in\mathbb{Z}}m(k-m)\widehat{w}(m,t)\widehat{w}(k-m,t)$$
$$+\mathcal{N}_1(k,t)+\mathcal{N}_2(k,t)-\mathcal{N}_3(k,t).$$

We now want to take the sum in $k \in \mathbb{Z}$ and to estimate the $A^1(\mathbb{S}^1)$ semi-norm of the time derivative.
Since

$$\partial_t |\widehat{w}(t,k)| = Re\left(\overline{\widehat{w}}(t,k)\partial_t \widehat{w}(t,k)\right)/|\widehat{w}(t,k)|,$$

we have that

$$\sum_{k\in\mathbb{Z}} |k|\partial_t|\widehat{w}(t,k)| = \frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{A^1(\mathbb{S}^1)} \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} |k|^3 |\widehat{w}(k,t)| = \|w(t)\|_{A^3(\mathbb{S}^1)}.$$

Using the Tonelli's Theorem and interpolation in Wiener spaces, we estimate the left terms as

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |k| \left|\sum_{m\in\mathbb{Z}} m^2 \widehat{w}(m,t)\widehat{w}(k-m,t)\right| \leqslant \|w(t)\|_{A^1(\mathbb{S}^1)} \|w(t)\|_{A^2(\mathbb{S}^1)} \\
&\quad \leqslant \|w(t)\|_{A^0(\mathbb{S}^1)} \|w(t)\|_{A^3(\mathbb{S}^1)}, \\
&\sum_{k\in\mathbb{Z}} |k| \left|\sum_{m\in\mathbb{Z}} m(k-m) \widehat{w}(m,t)\widehat{w}(k-m,t)\right| \leqslant \|w(t)\|_{A^1(\mathbb{S}^1)} \|w(t)\|_{A^2(\mathbb{S}^1)} \\
&\quad \leqslant \|w(t)\|_{A^0(\mathbb{S}^1)} \|w(t)\|_{A^3(\mathbb{S}^1)},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |k| \left|\widehat{\mathcal{N}}_1(k,t)\right| \leqslant \|w(t)\|_{A^3(\mathbb{S}^1)} \sum_{n\geqslant 1} \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r} 2^\ell \|w(t)\|^{2r}_{A^1(\mathbb{S}^1)} \|w(t)\|^{2j+\ell}_{A^0(\mathbb{S}^1)} \\
&\quad = \|w(t)\|_{A^3(\mathbb{S}^1)} \sum_{n\geqslant 1} \left(2\|w(t)\|_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^1(\mathbb{S}^1)}\right)^n, \\
&\sum_{k\in\mathbb{Z}} |k| \left|\widehat{\mathcal{N}}_2(k,t)\right| \leqslant \|w(t)\|_{A^3(\mathbb{S}^1)} \sum_{n\geqslant 1} \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r} 2^\ell \|w(t)\|^{2r}_{A^1(\mathbb{S}^1)} \|w(t)\|^{2j+\ell+1}_{A^0(\mathbb{S}^1)} \\
&\quad = \|w(t)\|_{A^3(\mathbb{S}^1)} \|w(t)\|_{A^0(\mathbb{S}^1)} \sum_{n\geqslant 1} \left(2\|w(t)\|_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^1(\mathbb{S}^1)}\right)^n, \\
&\sum_{k\in\mathbb{Z}} |k| \left|\widehat{\mathcal{N}}_3(k,t)\right| \leqslant \|w(t)\|_{A^2(\mathbb{S}^1)} \|w(t)\|_{A^1(\mathbb{S}^1)} \sum_{n\geqslant 1} \\
&\qquad \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r} 2^\ell \|w(t)\|^{2r}_{A^1(\mathbb{S}^1)} \|w(t)\|^{2j+\ell}_{A^0(\mathbb{S}^1)} \\
&\quad \leqslant \|w(t)\|_{A^3(\mathbb{S}^1)} \|w(t)\|_{A^0(\mathbb{S}^1)} \sum_{n\geqslant 1} \sum_{\substack{\ell+j+r=n \\ \ell,\, j,\, r\geqslant 0}} \binom{n}{\ell,\, j,\, r} 2^\ell \|w(t)\|^{2r}_{A^1(\mathbb{S}^1)} \|w(t)\|^{2j+\ell}_{A^0(\mathbb{S}^1)} \\
&\quad = \|w(t)\|_{A^3(\mathbb{S}^1)} \|w(t)\|_{A^0(\mathbb{S}^1)} \sum_{n\geqslant 1} \left(2\|w(t)\|_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^1(\mathbb{S}^1)}\right)^n.
\end{aligned}$$

Since, for every $t < T^*$,

$$\begin{aligned}
&\sum_{n\geqslant 1} \left(2\|w(t)\|_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^1(\mathbb{S}^1)}\right)^n \\
&\quad = \frac{1}{1-(2\|w(t)\|_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^0(\mathbb{S}^1)} + \|w(t)\|^2_{A^1(\mathbb{S}^1)})} - 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\left\|w(t)\right\|_{A^0(\mathbb{S}^1)} + \left\|w(t)\right\|^2_{A^0(\mathbb{S}^1)} + \left\|w(t)\right\|^2_{A^1(\mathbb{S}^1)}}{1-(2\left\|w(t)\right\|_{A^0(\mathbb{S}^1)} + \left\|w(t)\right\|^2_{A^0(\mathbb{S}^1)} + \left\|w(t)\right\|^2_{A^1(\mathbb{S}^1)})}\\
&\leqslant \frac{4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}},
\end{aligned}
$$

we improve the bounds on the $\mathcal{N}_i$ terms as

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} |k| \left|\widehat{\mathcal{N}}_1(k,t)\right| + \sum_{k\in\mathbb{Z}} |k| \left|\widehat{\mathcal{N}}_2(k,t)\right| + \sum_{k\in\mathbb{Z}} |k| \left|\widehat{\mathcal{N}}_3(k,t)\right|\\
&\quad\leqslant \left\|w(t)\right\|_{A^3(\mathbb{S}^1)} \left(1+2\left\|w(t)\right\|_{A^0(\mathbb{S}^1)}\right) \frac{4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\langle u_0\rangle \frac{\mathrm{d}}{\mathrm{d}t}\left\|w(t)\right\|_{A^1(\mathbb{S}^1)} + \left\|w(t)\right\|_{A^3(\mathbb{S}^1)}\\
&\quad\leqslant \left(2\left\|w(t)\right\|_{A^0(\mathbb{S}^1)} + \left(2\left\|w(t)\right\|_{A^0(\mathbb{S}^1)}+1\right)\frac{4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}\right)\left\|w(t)\right\|_{A^3(\mathbb{S}^1)}\\
&\quad= \left(\frac{2\left\|w(t)\right\|_{A^0(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|^2_{A^1(\mathbb{S}^1)}} + \frac{4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}\right)\left\|w(t)\right\|_{A^3(\mathbb{S}^1)}\\
&\quad\leqslant \frac{6\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}\left\|w(t)\right\|_{A^3(\mathbb{S}^1)}.
\end{aligned}
$$

The smallness assumptions on $\left\|w_0\right\|_{A^0(\mathbb{S}^1)}$ and on the $t<T^*$ imply that

$$
\frac{6\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}}{1-4\left\|w(t)\right\|_{A^1(\mathbb{S}^1)}} \leqslant c < 1,
$$

so we get that

$$
\langle u_0\rangle \frac{\mathrm{d}}{\mathrm{d}t}\left\|w(t)\right\|_{A^1(\mathbb{S}^1)} + (1-c)\left\|w(t)\right\|_{A^3(\mathbb{S}^1)} \leqslant 0,
$$

and the thesis follows integrating in time.

Let us explain now how to get the extra assumption on the initial data. We consider the problem

$$
\begin{aligned}
\langle u_0\rangle w^N_t &= \partial_x^2 w^N + w^N\partial_x^2 w^N - \left(\partial_x w^N\right)^2\\
&\quad + \left(\partial_x^2 w^N + w^N\partial_x^2 w^N - \left(\partial_x w^N\right)^2\right)\left(\sum_{n=1}^{N}(-1)^n(2w^N+(w^N)^2+(\partial_x w^N)^2)^n\right).
\end{aligned}
$$

For each fixed N we can obtain a local in time solution using Galerkin method. Repeating the previous estimates in the Wiener space A^1, we find that

$$
\begin{aligned}
w^N &\overset{*}{\rightharpoonup} w \quad \text{in} \quad L^\infty(0,T;L^\infty),\\
\partial^3 w^N &\overset{*}{\rightharpoonup} \partial^3 w \quad \text{in} \quad \mathcal{M}(0,T;L^\infty).
\end{aligned}
$$

Using the ideas in Ref. [14] we can further get the desired space

$$
L^\infty(0,T;A^1)\cap\mathcal{L}^\infty(0,T;A^3).
$$

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Declarations

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