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On the Three-Space Property for Subprojective and Superprojective Banach Spaces

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Abstract. We introduce the notion of subprojective and superprojective operators and we use them to prove a variation of the three-space property for subprojective and superprojective spaces. As an application, we show that some spaces considered by Johnson and Lindenstrauss are both subprojective and superprojective.

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1. Introduction

A Banach space X is called *subprojective* if every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X, and X is called *superprojective* if every closed infinite-codimensional subspace of X is contained in an infinite-codimensional subspace complemented in X; note that finite-dimensional spaces are trivially both subprojective and superprojective. These two classes of Banach spaces were introduced by Whitley [20] to find conditions for the conjugate of an operator to be strictly singular or strictly cosingular. More recently, they have been used to obtain some positive solutions to the perturbation classes problem for semi-Fredholm operators. This problem has a negative solution in general [9], but there are some positive answers when one of the spaces is subprojective or superprojective [11,14].

Subprojectivity passes on to subspaces and superprojectivity passes on to quotients, and both are stable under direct sums [13,19], but neither of them is a three-space property: for every $1 < p < \infty$ there exists a non-subprojective, non-superprojective space X with a subspace $M \subseteq X$ such that $M \simeq X/M \simeq \ell_p$, which is both subprojective and superprojective [19,

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Proposition 2.8] [13, Proposition 3.2]. However, slightly stronger hypotheses on M and X/M do imply the subprojectivity or superprojectivity of X, as seen below. For the subprojectivity, it is enough that X/M is subprojective and that every closed infinite-dimensional subspace of M contains an infinite-dimensional subspace complemented in X (which is stronger than being complemented in M, as the definition of subprojectivity of M would require), and there is an equivalent condition for superprojectivity (see Theorem 2.7). We introduce two classes of operators, namely subprojective and superprojective operators, which allow to show that a Banach space X is subprojective or superprojective when certain conditions such as these are met by a closed subspace M of X and its induced quotient X/M.

As an application, these sufficient conditions will be used to prove that two examples of Banach spaces introduced by Johnson and Lindenstrauss to study the properties of weakly compactly generated spaces are both subprojective and superprojective.

We will use standard notation. X, Y and Z will be Banach spaces. Given a closed subspace M of X, we will denote the inclusion of M into X by J_M, and Q_M will be the quotient map of X onto X/M. A (bounded, linear) operator $T \in \mathcal{L}(X, Y)$ is said to be strictly singular if there is no closed infinite-dimensional subspace M of X such that the restriction TJ_M is an isomorphism; it is said to be strictly cosingular if there is no closed infinite-codimensional subspace M of Y such that $Q_M T$ is surjective.

2. Subprojective and Superprojective Operators

Definition. An operator $T \in \mathcal{L}(X, Y)$ is subprojective if every closed infinite-dimensional subspace M of X such that TJ_M is an isomorphism contains a closed infinite-dimensional subspace N such that $T(N)$ is complemented in Y.

An operator $T \in \mathcal{L}(X, Y)$ is superprojective if every closed infinite-codimensional subspace M of Y such that $Q_M T$ is surjective is contained in a closed infinite-codimensional subspace N such that $T^{-1}(N)$ is complemented in X.

Note that a Banach space X is subprojective (resp., superprojective) if and only if the identity I_X is subprojective (resp., superprojective). Also, strictly singular operators are trivially subprojective, and strictly cosingular operators are trivially superprojective.

The following result, which is a consequence of [2, Lemma 2.2], will be useful at several places.

Lemma 2.1. *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be an operator.*

- (i) *If M is a closed subspace of X such that TJ_M is an isomorphism and $T(M)$ is complemented in Y, then M is complemented in X.*
- (ii) *If N is a closed subspace of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is complemented in X, then N is complemented in Y.*

Proof. (i) Let $N(T)$ be the kernel of T. If $M \cap N(T) = 0$ and N is a closed subspace of Y such that $Y = T(M) \oplus N$, then $X = M \oplus T^{-1}(N)$.

(ii) Let $R(T)$ be the range of T. If $R(T) + N = Y$ and M is a closed subspace of X such that $X = M \oplus T^{-1}(N)$, then $T(M)$ is closed and $Y = T(M) \oplus N$.

□

Subprojective (resp., superprojective) operators are stable under left (resp., right) composition.

Proposition 2.2. *Let X, Y and Z be Banach spaces and let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ be operators.*

(i) *If S is subprojective, then ST is subprojective.*
(ii) *If T is superprojective, then ST is superprojective.*

Proof. (i) Let M be a closed infinite-dimensional subspace of X such that STJ_M is an isomorphism. Then $T(M)$ is a closed infinite-dimensional subspace of Y and $SJ_{T(M)}$ is an isomorphism. Since S is subprojective, there exists a closed infinite-dimensional subspace N of $T(M)$ such that $S(N)$ is complemented in Z, and then $(TJ_M)^{-1}(N)$ is a closed infinite-dimensional subspace of M whose image $(ST)\big((TJ_M)^{-1}(N)\big) = S(N)$ is complemented in Z.

(ii) Let M be a closed infinite-codimensional subspace of Z such that $Q_M ST$ is surjective. Then $S^{-1}(M)$ is a closed infinite-codimensional subspace of Y and $Q_{S^{-1}(M)}T$ is surjective. Since T is superprojective, there exists a closed infinite-codimensional subspace N of Y containing $S^{-1}(M)$ such that $T^{-1}(N)$ is complemented in X, and then $S(N)$ is a closed infinite-codimensional subspace of Z containing M where $(ST)^{-1}(S(N)) = T^{-1}(N)$ is complemented in X.

□

Applying this result to the identity of a subprojective or superprojective space yields the following.

Corollary 2.3. *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be an operator.*

(i) *If Y is subprojective, then T is subprojective.*
(ii) *If X is superprojective, then T is superprojective.*

Question. Let S, $T \in \mathcal{L}(X, Y)$.

1. If S and T are subprojective, is $S + T$ subprojective?
2. If S and T are superprojective, is $S + T$ superprojective?

The subprojectivity of an embedding and the superprojectivity of a quotient map appear often enough that it is worth noting the following characterisation for them.

Proposition 2.4. *Let X be a Banach space and let Z be a closed subspace of X.*

(i) *J_Z is subprojective if and only if every closed infinite-dimensional subspace of Z contains an infinite-dimensional subspace complemented in X, in which case Z is subprojective.*
(ii) *Q_Z is superprojective if and only if every closed infinite-codimensional subspace of X containing Z is contained in an infinite-codimensional subspace complemented in X, in which case X/Z is superprojective.*

Proof. (i) This is a direct consequence of the definition of subprojective operator and Lemma 2.1(i).

(ii) Assume that Q_Z is superprojective and let M be a closed infinite-codimensional subspace of X containing Z. Then $Q_Z(M)$ is a closed infinite-codimensional subspace of X/Z and, by hypothesis, there exists an infinite-codimensional subspace N of X/Z containing $Q_Z(M)$ and such that $Q_Z^{-1}(N)$ is complemented in X, where $Q_Z^{-1}(N)$ contains M and is still infinite-codimensional.

Conversely, let M be a closed infinite-codimensional subspace of X/Z. Then $Q_Z^{-1}(M)$ is a closed infinite-codimensional subspace of X containing Z, so there exists an infinite-codimensional subspace N containing $Q_Z^{-1}(M)$ and complemented in X, and then $Q_Z(N)$ is infinite-codimensional in X/Z and contains M, and $Q_Z^{-1}(Q_Z(N)) = N$ is complemented in X.

Finally, let M be a closed infinite-codimensional subspace M of X/Z. Then $Q_Z^{-1}(M)$ is contained in an infinite-codimensional subspace N complemented in X, so M is contained in $Q_Z(N)$, which is closed, hence complemented in X/Z by Lemma 2.1(ii).

□

The following lemmata will be used in Theorem 2.7 to handle subspaces of a space depending on its relative position with respect to another subspace.

Lemma 2.5. *Let X be a Banach space, let M and N be closed subspaces of X such that $M \cap N = 0$ and $M + N$ is not closed. Then there exists an automorphism $U : X \longrightarrow X$ such that $U(M) \cap N$ is infinite-dimensional.*

Proof. Take normalised sequences $(x_n)_{n\in\mathbb{N}}$ in M and $(y_n)_{n\in\mathbb{N}}$ in N such that $\|x_n - y_n\| < 2^{-n}$ for every $n \in \mathbb{N}$. Since any weak cluster point of $(x_n)_{n\in\mathbb{N}}$ must be in $M \cap N = 0$, by passing to a subsequence [3, Theorem 1.5.6] we can assume that $(x_n)_{n\in\mathbb{N}}$ is a basic sequence and that there exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* such that $\langle x_i^*, x_j\rangle = \delta_{ij}$ for every i, $j \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|x_n^*\| \, \|x_n - y_n\| < 1$. Then $K(x) = \sum_{n=1}^{\infty} \langle x_n^*, x\rangle (x_n - y_n)$ defines an operator $K : X \longrightarrow X$ with $\|K\| < 1$ and $U = I - K$ is an automorphism on X that maps $U(x_n) = y_n$ for every $n \in \mathbb{N}$, so $U(M) \cap N$ is infinite-dimensional.

□

Lemma 2.6. *Let X be a Banach space, let M and N be closed subspaces of X such that $M + N$ is dense in X but not closed. Then there exists an automorphism $U : X \longrightarrow X$ such that $\overline{U^{-1}(M) + N}$ is infinite-codimensional in X.*

Proof. $M+N$ is dense in X but not closed, so $M^\perp \cap N^\perp = 0$ and $M^\perp + N^\perp$ is not closed either [18, Theorem IV.4.8]. Take a normalised sequence $(x_n^*)_{n\in\mathbb{N}}$ in $M^\perp$ and another sequence $(y_n^*)_{n\in\mathbb{N}}$ in $N^\perp$ such that $\|x_n^* - y_n^*\| < 2^{-n}$ for every $n \in \mathbb{N}$. Since any weak* cluster point of $(x_n^*)_{n\in\mathbb{N}}$ must be in $M^\perp \cap N^\perp = 0$, by passing to a subsequence [10, Lemma 3.1.19] we can assume that $(x_n^*)_{n\in\mathbb{N}}$ is a basic sequence and find a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\langle x_i^*, x_j\rangle = \delta_{ij}$ for every i, $j \in \mathbb{N}$ and $\sum_{n=1}^\infty \|x_n\| \, \|x_n^* - y_n^*\| < 1$. Then $K(x) = \sum_{n=1}^\infty \langle x_n^* - y_n^*, x\rangle x_n$ defines an operator $K\colon X \longrightarrow X$ with $\|K\| < 1$ and $U = I - K$ is an automorphism on X whose conjugate maps $U(x_n^*) = y_n^*$ for every $n \in \mathbb{N}$, so $U^*(M^\perp) \cap N^\perp = (U^{-1}(M) + N)^\perp$ is infinite-dimensional and $\overline{U^{-1}(M) + N}$ is infinite-codimensional. □

Theorem 2.7. *Let X be a Banach space and let Z be a closed subspace of X.*

(i) *If J_Z and Q_Z are both subprojective, then X is subprojective.*
(ii) *If J_Z and Q_Z are both superprojective, then X is superprojective.*

Proof. (i) Let M be a closed infinite-dimensional subspace of X. If $M \cap Z$ is infinite-dimensional, then it contains another infinite-dimensional subspace complemented in X by the hypothesis on J_Z and Proposition 2.4(i). Otherwise, if $M \cap Z$ is finite-dimensional, we can assume that $M \cap Z = 0$ by passing to a further subspace if necessary. If $M + Z$ is closed, then $Q_Z J_M$ is an isomorphism and M contains an infinite-dimensional subspace complemented in X by the hypothesis on Q_Z. We are left with the case where $M \cap Z = 0$ and $M + Z$ is not closed. By Lemma 2.5, there exists an automorphism $U\colon X \longrightarrow X$ such that $U(M) \cap Z$ is infinite-dimensional. Let N be an infinite-dimensional subspace of $U(M) \cap Z$ complemented in X, again by Proposition 2.4(i); then $U^{-1}(N) \subseteq M$ and is still complemented in X.

(ii) Let M be a closed infinite-codimensional subspace of X. If $\overline{M+Z}$ is infinite-codimensional, then it is contained in another infinite-codimensional subspace complemented in X by the hypothesis on Q_Z and Proposition 2.4(ii). Otherwise, if $\overline{M+Z}$ is finite-codimensional, we can assume that $\overline{M+Z} = X$ by enlarging M with a finite-dimensional subspace if necessary. If $M + Z$ is closed, so $M + Z = X$, then $Q_M J_Z$ is surjective and M is contained in an infinite-codimensional subspace complemented in X by the hypothesis on J_Z. We are left with the case where $M + Z$ is dense in X but not closed. By Lemma 2.6, there exists an automorphism $U\colon X \longrightarrow X$ such that $\overline{U^{-1}(M) + Z}$ is infinite-codimensional in X. Let N be an infinite-codimensional subspace complemented in X such that $\overline{U^{-1}(M) + Z} \subseteq N$, again by Proposition 2.4(ii); then $M \subseteq U(N)$, which is still infinite-codimensional and complemented in X.

□

Theorem 2.7 implies a variation of the 3-space property for subprojectivity and superprojectivity. Given a Banach space X and a closed subspace Z of X, the inclusion J_Z is subprojective if and only if every closed infinite-dimensional subspace of Z contains an infinite-dimensional subspace complemented in X by Proposition 2.4(i), and this is stronger than being subprojective; on the other hand, for Q_Z to be subprojective, it is sufficient (but not necessary) that X/Z be subprojective, by Corollary 2.3. So, if X/Z is subprojective and J_Z is subprojective, then X is subprojective. Similarly, if Z is superprojective and Q_Z is superprojective, then X is superprojective.

Also, Theorem 2.7 is not a characterisation. While J_Z must be subprojective if X is subprojective by Corollary 2.3, Q_Z need not be. For instance, take any surjection $T\colon \ell_1 \longrightarrow C([0,1])$ and define the operator $Q\colon \ell_1 \oplus \ell_1 \longrightarrow C([0,1])$ as $Q(x,y) = x + Ty$. Then Q is also clearly surjective and an isomorphism on its first component, but ℓ_1 cannot contain any subspace whose image by Q is complemented in $C([0,1])$.

Related to Theorem 2.7, as mentioned in the introduction, there exists a non-subprojective, non-superprojective space X with a subspace $M \subseteq X$ such that $M \simeq X/M \simeq \ell_p$, which is both subprojective and superprojective [19, Proposition 2.8] [13, Proposition 3.2], where Q_M is strictly singular [17, Theorem 6.4], hence subprojective, while X is not, and J_M is strictly cosingular [17, Theorem 6.4], hence superprojective, while X is not. Thus, J_M cannot be subprojective although M is, and Q_M cannot be superprojective although X/M is.

$$J_Z, Q_Z \text{ subprojective} \Rightarrow X \text{ subprojective} \Rightarrow$$
$$\Rightarrow J_Z \text{ subprojective} \Rightarrow Z \text{ subprojective}$$

A similar situation holds for superprojectivity.

$$J_Z, Q_Z \text{ superprojective} \Rightarrow X \text{ superprojective} \Rightarrow$$
$$\Rightarrow Q_Z \text{ superprojective} \Rightarrow X/Z \text{ superprojective}$$

As a particular case of Theorem 2.7, we have the following.

Corollary 2.8. *Let X be a Banach space such that*

(i) *every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X^{**}; and*
(ii) *X^{**}/X is subprojective.*

Then X^{**} is subprojective.

In [12], it is proved that, under certain conditions, the J-sum $J(\Phi)$ of Banach spaces, as defined by Bellenot [6], and its bidual $J(\Phi)^{**}$ are subprojective. As a particular case, for every separable subprojective space X there exists a separable Banach space $J(\Phi)$ such that $J(\Phi)^{**}$ is subprojective and $J(\Phi)^{**}/J(\Phi)$ is isomorphic to X. An essential part of the proof is to show that any closed infinite-dimensional subspace of $J(\Phi)$ contains a further infinite-dimensional subspace complemented in $J(\Phi)^{**}$, which is to say that the inclusion of $J(\Phi)$ in $J(\Phi)^{**}$ is subprojective [12, Theorem 5.2(i)]. That result is, in fact, a particular case of Corollary 2.8.

A related result was proved by Argyros and Raikoftsalis [4]. Let Y be a separable reflexive space. Then:

- For every $1 < p < \infty$, there is a separable reflexive space $X_p(Y)$ that is hereditarily complemented ℓ_p, hence subprojective, and Y is a quotient of $X_p(Y)$.

- Y is a quotient of a separable hereditarily-c_0 space $X_0(Y)$.

Of course, in both cases the kernel of the quotient map is subprojective.

Remark 2.9. In Corollary 2.8, condition (i) can be replaced by

(i') *every closed infinite-dimensional subspace of X contains an infinite-dimensional reflexive subspace complemented in X.*

3. Subprojectivity and Superprojectivity of the Johnson–Lindenstrauss Space

Here we apply the results in the previous section to study the Banach spaces introduced by Johnson and Lindenstrauss in [16, Examples 1 and 2].

Let Γ be a set with the cardinality of the continuum and let $\{ N_\gamma : \gamma \in \Gamma \}$ be a family of infinite subsets of $\mathbb{N}$ such that $N_\gamma \cap N_{\gamma'}$ is finite if $\gamma \neq \gamma'$. For each $\gamma \in \Gamma$, let $\phi_\gamma \in \ell_\infty$ be the characteristic function of N_γ.

Let $V = c_0 \subset \ell_\infty$ and let JL_0 be the linear span of $V \cup \{ \phi_\gamma : \gamma \in \Gamma \}$ in ℓ_∞ endowed with the norm

$$\left\| y + \sum_{i=1}^{k} a_i \phi_{\gamma_i} \right\|_{JL} = \max\left\{ \left\| y + \sum_{i=1}^{k} a_i \phi_{\gamma_i} \right\|_\infty, \left\| (a_i)_{i=1}^k \right\|_2 \right\}$$

for every $y \in V$ and $(\gamma_i)_{i=1}^k$ with $\gamma_i \neq \gamma_j$ if $i \neq j$.

The space JL is defined as the completion of $(JL_0, \| \cdot \|_{JL})$. These are some of its properties.

Theorem 3.1. [16, Example 1]

(i) *V is a subspace of JL isometric to c_0 and JL/V is isometric to $\ell_2(\Gamma)$.*
(ii) *Weakly compact subsets of JL are separable. Hence every reflexive subspace of JL is separable and V is not complemented.*
(iii) *JL^* is isomorphic to $\ell_1 \oplus \ell_2(\Gamma)$.*

A Banach space X is weakly compactly generated (WCG, in short) if there exists a weakly compact subset of X that generates a subspace that is dense in X. Clearly, separable spaces and reflexive spaces are WCG, but JL is not by property (ii), as it is not separable. Hence being WCG is not a three-space property [16].

Note that, while we treat JL as a unique space, there are different JL spaces depending on the choice of the family $\{ N_\gamma : \gamma \in \Gamma \}$, and that the resulting spaces may not be isomorphic [5]. However, the properties in Theorem 3.1, and those proved below, are common to all possible JL spaces obtained this way.

Corollary 3.2. *Every infinite-dimensional reflexive subspace of JL is a complemented copy of ℓ_2.*

Proof. Let M be an infinite-dimensional reflexive subspace of JL. Then $M \cap V$ is finite-dimensional and, by Lemma 2.5, $M + V$ is closed. Passing to a finite-codimensional subspace N of M, the restriction $Q_V J_N$ is an isomorphism. Since $Q_V(N)$ is complemented in $\ell_2(\Gamma)$, N is complemented in JL by Lemma 2.1(i), and then so is M. □

Using subprojective and superprojective operators, it is possible to prove that the space JL is both subprojective and superprojective.

Lemma 3.3. *Let* $\{\gamma_i : 1 \leq i \leq n\}$ *be a finite subset of* Γ *and let* $M_k = N_{\gamma_k} \setminus \bigcup_{i=1}^{k-1} N_{\gamma_i}$ *for every* $1 \leq k \leq n$. *Then* $(\chi_{M_k})_{i=1}^n$ *is equivalent to the unit vector basis of* ℓ_2^n.

Proof. For every $1 \leq k \leq n$, define $F_k = \bigcup_{i=1}^{k-1}(N_{\gamma_k} \cap N_{\gamma_i})$, which is a finite set, and note that N_{γ_k} is the disjoint union of M_k and F_k, so $\chi_{M_k} = \chi_{N_{\gamma_k}} - \chi_{F_k}$, which is well defined in JL. Since the sets $(M_k)_{k=1}^n$ are pairwise disjoint, $\left\|\sum_{k=1}^n a_k\chi_{M_k}\right\|_\infty = \|(a_k)_{k=1}^n\|_\infty$ and $\left\|\sum_{k=1}^n a_k\chi_{M_k}\right\|_{JL} = \|(a_k)_{k=1}^n\|_2$ for every $(a_k)_{k=1}^n$. □

As a consequence, the same applies to any countable subset of Γ with respect to ℓ_2.

We will need the following result, which was essentially proved in [8, Theorem 2.2]. We include a proof here because our statement is different and also encompasses complex spaces.

Proposition 3.4. *Let* X *be a Banach space that does not contain any copies of* ℓ_1. *Then every copy of* c_0 *in* X *contains another copy of* c_0 *complemented in* X.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X equivalent to the unit vector basis of c_0 and take a bounded sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* such that $\langle x_i^*, x_j\rangle = \delta_{ij}$ for every i, $j \in \mathbb{N}$. It is easy to check that $(x_n^*)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_1.

Since X does not contain any copies of ℓ_1, there exists a normalised weak* null block sequence of $(x_n^*)_{n\in\mathbb{N}}$ ([15, Theorem 1(a)] for the real case, [1, Appendix A] for the complex case). Write $y_k^* = \sum_{i=m_k}^{m_{k+1}-1} a_i x_i^*$ for such a sequence and let $\varepsilon_i = a_i/|a_i|$, or $\varepsilon_i = 1$ if $a_i = 0$, for every $i \in \mathbb{N}$. Then $y_k = \sum_{i=m_k}^{m_{k+1}-1} \varepsilon_i x_i$ defines a sequence $(y_k)_{k\in\mathbb{N}}$ in $[x_n : n \in \mathbb{N}]$ equivalent to the unit vector basis of c_0 such that $P(x) = \sum_{k=1}^\infty \langle y_k^*, x\rangle y_k$ is a projection in X onto $[y_k : k \in \mathbb{N}]$. □

Proposition 3.5.

(i) J_V *is subprojective.*
(ii) Q_V *is superprojective.*
(iii) JL *is both subprojective and superprojective.*
(iv) JL^* *is subprojective but not superprojective.*
(v) JL^{**} *is neither subprojective nor superprojective.*

Proof. (i) JL does not contain any copies of ℓ_1, as neither c_0 nor $\ell_2(\Gamma)$ contain copies of ℓ_1, and not containing any copies of ℓ_1 is a three-space property [7, Theorem 3.2.d]. Then every closed infinite-dimensional subspace of $V \simeq c_0$ contains another copy of c_0 that is complemented in JL by Proposition 3.4 and J_V is subprojective by Proposition 2.4.

(ii) Let M be a closed infinite-codimensional subspace of JL containing V. Then JL/M is a quotient of $JL/V \equiv \ell_2(\Gamma)$, so, taking a bigger M

if necessary, we can assume that JL/M is separable. Since $Q_V(M)$ is closed, we can consider the decomposition $\ell_2(\Gamma) = Q_V(M) \oplus Q_V(M)^\perp$.

Let $\{ e_\gamma : \gamma \in \Gamma \}$ be the basis of $\ell_2(\Gamma)$. Since $Q_V(M)^\perp$ is separable, there exists a sequence of different points $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $Q_V(M)^\perp \subseteq [e_{\gamma_n} : n \in \mathbb{N}]$. Consider an orthonormal basis of $Q_V(M)^\perp$; then that basis is a normalised weakly null sequence, so it has a subsequence $(f_k)_{k\in\mathbb{N}}$ that is equivalent to a block basis of $(e_{\gamma_n})_{n\in\mathbb{N}}$, i.e., there exist $1 = m_1 < m_2 < m_3 < \cdots$ in $\mathbb{N}$ and a sequence of scalars $(b_n)_{n\in\mathbb{N}}$ such that $y_k = \sum_{i=m_k}^{m_{k+1}-1} b_i e_{\gamma_i}$ satisfies $\|y_k - f_k\| < 2^{-2k}$ for every $k \in \mathbb{N}$.

Now, for every $n \in \mathbb{N}$, define $F_n = \bigcup_{i=1}^{n-1}(N_{\gamma_n} \cap N_{\gamma_i})$ and $M_n = N_{\gamma_n} \backslash F_n$, so that $(\chi_{M_n})_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_2 by Lemma 3.3 and $Q_V(\chi_{M_n}) = Q_V(\chi_{N_{\gamma_n}}) = e_{\gamma_n}$ for every $n \in \mathbb{N}$, which means that the restriction of Q_V to $[\chi_{M_k} : k \in \mathbb{N}]$ is an isomorphism onto $[e_{\gamma_n} : n \in \mathbb{N}]$. Define $x_k = \sum_{i=m_k}^{m_{k+1}-1} b_i \chi_{M_i} \in JL$ for every $k \in \mathbb{N}$, so that $Q_V(x_k) = y_k$ for every $k \in \mathbb{N}$. Then the restriction of Q_V to $[x_k : k \in \mathbb{N}]$ is an isomorphism onto $[y_k : k \in \mathbb{N}]$ and $\ell_2(\Gamma) = [y_k : k \in \mathbb{N}] \oplus N$ with N a closed subspace containing $Q_V(M)$, and $JL = [x_k : k \in \mathbb{N}] \oplus Q_V^{-1}(N)$. Hence $Q_V^{-1}(N)$ is a complemented infinite-codimensional subspace containing M.

(iii) Q_V is subprojective and J_V is superprojective by Corollary 2.3, as $JL/V \equiv \ell_2(\Gamma)$ is subprojective and $V = c_0$ is superprojective, so JL is both subprojective and superprojective by (i), (ii) and Theorem 2.7.

(iv) This follows from $JL^* \simeq \ell_1 \oplus \ell_2(\Gamma)$ [19, Proposition 2.2] [13, Proposition 4.1], as ℓ_1 and $\ell_2(\Gamma)$ are subprojective, but ℓ_1 is not superprojective.

(v) JL^{**} contains a (complemented) copy of ℓ_∞.

□

There is a second example in [16], which is the closed subspace X_{JL} generated by $V \cup \{ \phi_\gamma : \gamma \in \Gamma \} \cup \{\chi_\mathbb{N}\}$ in ℓ_∞. Since X_{JL} is a commutative Banach algebra with the pointwise multiplication, it is isometric to some $C(K)$ space by Gelfand's representation theorem [16, Example 2]. It is easy to check that X_{JL}/V is isometric to $c_0(\Gamma)$, which is WCG because the natural inclusion $\ell_2(\Gamma) \longrightarrow c_0(\Gamma)$ has dense range. However, since weakly compact subsets of ℓ_∞ are separable, X_{JL} is not WCG, so V is not complemented in X_{JL}.

Recall that a Banach space X is said to have property (V) if every non-weakly compact operator $T\colon X \longrightarrow Y$ is an isomorphism on a subspace of X isomorphic to c_0. It is well known that $C(K)$ spaces have property (V), so X_{JL} has property (V).

Proposition 3.6. *X_{JL} is subprojective and superprojective.*

Proof. X_{JL} is hereditarily c_0 because being hereditarily c_0 is a three-space property [7, Theorem 3.2.e]. As such, any closed infinite-dimensional subspace of $V = c_0$ contains a further subspace isomorphic to c_0 and complemented in X_{JL} by Proposition 3.4, which means that J_V is subprojective. As in the case of JL, Q_V is subprojective by Corollary 2.3, as $X_{JL}/V \equiv c_0(\Gamma)$ is subprojective, so X_{JL} is subprojective by Theorem 2.7.

For the superprojectivity, we will prove that Q_V is superprojective. First of all, $\ell_1(\Gamma)$ does not have any reflexive subspaces, so $X_{JL}/V \equiv c_0(\Gamma)$ does not have any reflexive quotients. Let W be a closed infinite-codimensional subspace of X_{JL} containing V; then Q_W is not weakly compact, and X_{JL} has property (V), so there exists a subspace W of X_{JL} isomorphic to c_0 such that Q_W is an isomorphism on M. Now again X_{JL}/W does not have any reflexive quotients, so it does not contain ℓ_1 and we can assume that $Q_W(M)$ is complemented by Proposition 3.4. Write $X_{JL}/W = Q_W(M) \oplus N$; then $X_{JL} = M \oplus Q_W^{-1}(N)$, where $Q_W^{-1}(N)$ contains W.

Finally, again J_V is superprojective by Corollary 2.3, so X_{JL} is superprojective by Theorem 2.7. □

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