Facultad de Ciencias

THE RICCATI EQUATION

(LA ECUACIÓN DE RICCATI)

Trabajo de Fin de Grado
para acceder al
GRADO EN MATEMÁTICAS

Autor: Diego Fernández Porres

Director: Luis Alberto Fernández

Febrero - 2025

Abstract

The Riccati equation is a non-linear first order ordinary differential equation that was studied by Jacopo Francesco Riccati, among other illustrious mathematicians, in the 18th century. The Riccati equation plays a very important role in the theoretical study of ordinary differential equations and has numerous practical applications in fields such as physics, chemistry and economics, among others.

This project covers the main properties of the Riccati equation, its known simplifications, its relation with other differential equations and different cases where its general solution is known or easy to calculate.

This document ends by showing how the Riccati equation appears in the resolution of current problems such as the Schrödinger equation or in the resolution of optimal control problems.

Resumen

La ecuación de Riccati es una ecuación diferencial ordinaria de primer orden no lineal que fue estudiada por Jacopo Francesco Riccati, entre otros matemáticos ilustres, en el siglo XVIII. La ecuación de Riccati juega un papel muy importante en el estudio teórico de las ecuaciones diferenciales ordinarias y tiene numerosas aplicaciones prácticas en campos como la física, la química o la economía entre otros.

Este trabajo de fin de grado recoge las principales propiedades de la ecuación de Riccati, sus simplificaciones conocidas, su relación con otras ecuaciones diferenciales y distintos casos donde se conoce o es sencillo calcular su solución general.

La memoria finaliza mostrando cómo aparece la ecuación de Riccati en la resolución de problemas actuales como la ecuación de Schrödinger o en la resolución de problemas de control óptimo.

Contents

1 Introduction

There are two types of subjects in the mathematics degree. Subjects that can be called "core subjects" and those that can be called "specific subjects". Examples of core subjects can be Calculus or Algebra. I call them core subjects because the concepts studied in them are commonly used as tools in the rest of the subjects. In this sense, in Calculus we can find the study of limits or the concept of derivative and in Algebra the understanding of vector space among many other things.

Within the specific subjects we have the example of the subject ''Ecuaciones Diferenciales ordinarias'' . It is understandable to dedicate a subject to ODEs because their study is not only of theoretical interest, but also plays a fundamental role in disciplines such as physics, economics, optimal control and dynamical systems theory among others. One of the multiple equations studied in this subject is the Riccati equation, but it is presented as just another differential equation.

The Riccati equation is a non-linear non-homogeneous first-order ordinary differential equation developed in the 18th century by the mathematician Jacopo Francesco Riccati which expresses itself as:

$$y'(x) + r(x)y^2(x) + q(x)y(x) = p(x) \tag{1}$$

where r, q and p are continuous functions and $r \neq 0$.

Figure 1: Jacopo Francesco Riccati

The aim of this undergraduate final project is to collect the main known properties of the Riccati Equation and to show that it is not an ordinary differential equation like any other. To this end, in the first section we will reflect on its theoretical and practical relevance. In the next section we will study the existence of simplified forms of the Riccati equation, as well as compiling different lists of cases in which a solution to the Riccati equation is known. In the fourth section of this paper we will study cases of the Riccati equation with constant coefficients and then apply them to separable cases of the Riccati equation. And in the last and fifth section, we will give a glimpse of some of the many applications of this equation.

2 The importance of the Riccati equation

One of the main objectives of this final project is to show that the Riccati equation is not just another equation studied in the course of Ordinary Differential Equations.

In this section, we will explore how the Riccati equation represents a significant step forward, not only in the theoretical study of ordinary differential equations but also in practical applications in other fields such as physics, biology, or engineering.

It has been known for a long time that we have a general formula for the solution of a first-order linear ODE. If we wanted to have a general formula for all ODEs, regardless of order and linearity, the logical thing to do would be to continue with the homogeneous second order or with the systems of linear equations of first order. This is where the Riccati equation comes in, since a second-order linear homogeneous ODE or a system of first-order ODEs will have a solution if only if it has its associated Riccati equation. In the following subsection we will prove this statement by looking at the changes of variable that transforms the second-order linear ODE and the system into a Riccati equation and vice versa.

As for the practical part, we will see in the section ''Aplications of the Riccati equation'' many examples of the usefulness of this equation. In this section we will focus on what for the author is the most important application. The quadratic approximation of a non-linear ordinary differential equation.

2.1 The influence of the Riccati equation on the theoretical study of ODEs

Let us prove the statements made above, the existence of a general formula for the solution of a first-order linear ODE, and the relationship between homogeneous second-order linear ODEs and systems of two first-order linear equations with the Riccati equation.

2.1.1 The existence of solution for first-order linear ODEs

Proposition 1

For every first- order linear ordinary differential equation defined as:

$$z'(x) = a(x)z(x) + b(x) \tag{2}$$

it exist a general solution expressed as:

$$z(x) = e^{\int a(x)\,dx}(C + \int b(x)e^{-\int a(x)\,dx}\,dx)$$

Proof

The general solution is obtained from the following method: First we multiply by the exponential of minus a primitive of $a(x)$, so we get:

$$z'(x) \cdot e^{-\int a(x)\,dx} = a(x)z(x)e^{-\int a(x)\,dx} + b(x)e^{-\int a(x)\,dx} \Longrightarrow$$

$$\Longrightarrow z'(x)e^{-\int a(x)\,dx} - a(x)z(x)e^{-\int a(x)\,dx} = b(x)e^{-\int a(x)\,dx}$$

Integrating each term of the equation, we obtain:

$$z(x)e^{-\int a(x)\,dx} = C + \int b(x)e^{-\int a(x)\,dx}\,dx$$ [1]

Now we arrive at the previously mentioned formula.

$$z(x) = e^{\int a(x)\,dx}(C + \int b(x)e^{-\int a(x)\,dx}\,dx) \tag{3}$$

[1]Throughout this document C will be an arbitrary constant that absorbs other constants if it is needed.

Let us apply this formula to an example:

Example 1

We will get the solution for:

$$z'(x) = z(x) - x \tag{4}$$

In this case $a(x) = 1$ and $b(x) = -x$ so the solution for 4 must be:

$$z(x) = e^{\int dx}(C - \int xe^{-\int dx}\, dx) = e^x(C - \int xe^{-x}\, dx) = e^x(C + (1+x)e^{-x}) = Ce^x + x + 1$$

Let's check that our solution verifies (4):

$$z'(x) = Ce^x + 1; \qquad z'(x) = z(x) - x = Ce^x + x + 1 - x = Ce^x + 1$$

2.1.2 Relationship between second-order linear ODE and simplified Riccati equation

We define simplified Riccati equation as:

$$y'(x) + r(x)y^2(x) = p(x) \tag{5}$$

and we define a second-order linear homogeneous ordinary differential equation as:

$$u''(x) + a(x)u'(x) + b(x)u(x) = 0 \tag{6}$$

Later we will show that any Riccati equation can be transformed into a simplified Riccati equation. In 1796 Euler proved the relationship between (1) and (6).On this section we will study the relationship between (6) and (5) so indeed, after proving the relationship between (1) and (6), we will be proving what Euler proved.

Proposition 2

Every Riccati simplified equation defined as:

$$y'(x) + r(x)y^2(x) = p(x)$$

can be expressed as second-order linear homogeneous ODE as:

$$u''(x) - \frac{r'(x)}{r(x)}u'(x) - p(x)r(x)u(x) = 0$$

Proof

Let us start with a change of unknown function:

$$y(x) = \frac{u'(x)}{r(x)u(x)} \Longrightarrow y'(x) = \frac{u''(x)r(x)u(x) - r'(x)u(x)u'(x) - (u'(x))^2r(x)}{r^2(x)u^2(x)}$$

Substituting in (5) we get:

$$\frac{u''(x)r(x)u(x) - r'(x)u(x)u'(x) - (u'(x))^2r(x)}{r^2(x)u^2(x)} + \frac{(u'(x))^2r(x)}{r(x)^2u(x)^2} = p(x) \Rightarrow$$

$$\Rightarrow \frac{u''(x)r(x)u(x) - r'(x)u(x)u'(x)}{r^2(x)u^2(x)} = p(x) \Rightarrow u''(x) - \frac{r'(x)}{r(x)}u'(x) = p(x)r(x)u(x)$$

$$\Rightarrow u''(x) - \frac{r'(x)}{r(x)}u'(x) - p(x)r(x)u(x) = 0$$

Let us apply this transformation to an example:

Example 2

We will transform the following simplified Riccati equation:

$$y'(x) - xe^x y^2(x) = \frac{1}{x^2 e^x} \tag{7}$$

Starting with the following change of unknown function:

$$y(x) = \frac{-u'(x)}{xe^x u(x)}; \qquad y'(x) = \frac{-u''(x)u(x)xe^x + (1+x)e^x u(x)u'(x) + (u'(x))^2 xe^x}{x^2 e^{2x} u^2(x)}$$

Substituting in (7):

$$\frac{-u''(x)u(x)xe^x + (1+x)e^x u(x)u'(x) + (u'(x))^2 xe^x}{x^2 e^{2x} u^2(x)} - \frac{(u'(x))^2 xe^x}{x^2 e^{2x} u^2(x)} = \frac{1}{x^2 e^x}$$

Simplifying:

$$\frac{-u''(x)x + (1+x)u'(x)}{x^2 e^x u(x)} = \frac{1}{x^2 e^x} \Rightarrow -u''(x)x + (1+x)u'(x) = u(x) \Rightarrow$$

$$\Rightarrow u''(x) - \frac{(1+x)}{x}u'(x) + \frac{1}{x}u(x) = 0 \tag{8}$$

Let's check that the coefficients are as mentioned in the proposition:

$$\frac{-r'(x)}{r(x)} = -\frac{(-xe^x)'}{-xe^x} = -\frac{-e^x - xe^x}{-xe^x} = -\frac{x+1}{x}$$

$$-p(x)r(x) = -\frac{1}{x^2 e^x}(-xe^x) = \frac{1}{x}$$

Proposition 3

Every second-order linear homogeneous ODE defined as:

$$u''(x) + a(x)u'(x) + b(x)u(x) = 0$$

can be expressed as the following simplified Riccati equation:

$$y'(x) + y^2(x) = \frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x)$$

Proof

We start with the change of unknown function:

$$u(x) = w(x)e^{-\int v(x)dx} \Rightarrow u'(x) = (w'(x) - w(x)v(x))e^{-\int v(x)dx} \Rightarrow$$

$$\Rightarrow u''(x) = (w''(x) - 2w'(x)v(x) - w(x)v'(x) + w(x)v^2(x))e^{-\int v(x)dx}$$

substituting in (6), multiplying by $e^{\int v(x)dx}$ and grouping we get:

$$w''(x) + w'(x)(a(x) - 2v(x)) + w(x)(v^2(x) - v'(x) - v(x)a(x) + b(x)) = 0$$

where, if we take $v(x) = \frac{a(x)}{2}$, in order to simplify the equation we get:

$$w''(x) + w(x)\left(\frac{a^2(x)}{4} - \frac{a'(x)}{2} - \frac{a^2(x)}{2} + b(x)\right) = 0 \Rightarrow$$

$$\Rightarrow w''(x) = w(x)\left(\frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x)\right) \Rightarrow \frac{w''(x)}{w(x)} = \left(\frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x)\right)$$

Now we continue with another change of unknown function:

$$y(x) = \frac{w'(x)}{w(x)} \Rightarrow y'(x) = \frac{w''(x)w(x) - (w'(x))^2}{w^2(x)}$$

Analyzing it carefully we should find that:

$$y'(x) + y^2(x) = \frac{w''(x)}{w(x)}$$

So substituting in the previous equation we get:

$$y'(x) + y^2(x) = \frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x)$$

Let's apply it to an example.

Example 3

Now we will apply the previous transformation to (8):

$$u(x) = w(x)e^{\int \frac{x+1}{2x}} = w(x)e^{\frac{x}{2}+\frac{ln(x)}{2}} = w(x)e^{\frac{x}{2}}\sqrt{x}$$

$$u'(x) = \left(w'(x)\sqrt{x} + w(x)\frac{x+1}{2\sqrt{x}}\right)e^{\frac{x}{2}}$$

$$u''(x) = \left(w''(x)\sqrt{x} + w'(x)\frac{x+1}{\sqrt{x}} + w(x)\frac{x^2+2x-1}{4x\sqrt{x}}\right)e^{\frac{x}{2}}$$

Substituting in (8) and multiplying by $e^{\frac{-x}{2}}$:

$$\left(w''(x)\sqrt{x} + w'(x)\frac{x+1}{\sqrt{x}} + w(x)\frac{x^2+2x-1}{4x\sqrt{x}}\right) - \frac{1+x}{x}\left(w'(x)\sqrt{x} + w(x)\frac{x+1}{2\sqrt{x}}\right) +$$

$$+\frac{1}{x}w(x)\sqrt{x} = 0$$

Simplifying:

$$w''(x)\sqrt{x} + w(x)\frac{-x^2+2x-3}{4x\sqrt{x}} = 0 \Rightarrow w''(x)\sqrt{x} = w(x)\frac{x^2-2x+3}{4x\sqrt{x}} \Rightarrow \frac{w''(x)}{w(x)} = \frac{x^2-2x+3}{4x^2}$$

Applying the change of unknown function $y(x) = \frac{w'(x)}{w(x)}$ we get that $y'(x) + y^2(x) = \frac{w''(x)}{w(x)}$, as we have seen before:

$$y'(x) + y^2(x) = \frac{x^2-2x+3}{4x^2}$$

We just need to confirm that $\frac{x^2-2x+3}{4x^2} = \frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x)$

$$\frac{a^2(x)}{4} + \frac{a'(x)}{2} - b(x) = \frac{\left(\frac{-x-1}{x}\right)^2}{4} + \frac{\frac{1}{x^2}}{2} - \frac{1}{x} = \frac{x^2+2x+1}{4x^2} + \frac{2}{4x^2} - \frac{4x}{4x^2} = \frac{x^2-2x+3}{4x^2}$$

2.1.3 Relationship between first-order homogeneous linear ODE system and Riccati equation

Proposition 4

Every first-order homogeneous linear ODE system defined as:

$$\begin{cases} x'(t) = a(t)x(t) + b(t)y(t) \\ y'(t) = c(t)x(t) + d(t)y(t) \end{cases} \tag{9}$$

can be expressed as the following Riccati equation:

$$v'(t) + b(t)v^2(t) + (a(t) - d(t))v(t) = c(t)$$

Proof

First we start with the next change of unknown function:

$$v(t) = \frac{y(t)}{x(t)} \Rightarrow v'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t)}$$

Substituting $x'(t)$ and $y'(t)$ by their value in (9) we get:

$$v'(t) = \frac{(c(t)x(t) + d(t)y(t))x(t) - (a(t)x(t) + b(t)y(t))y(t)}{x^2(t)}$$

Applying now $y(t) = v(t)x(t)$:

$$v'(t) = \frac{(c(t)x(t) + d(t)v(t)x(t))x(t) - (a(t)x(t) + b(t)v(t)x(t))v(t)x(t)}{x^2(t)} \Rightarrow$$

$$\Rightarrow v'(t) = c(t) + d(t)v(t) - (a(t) + b(t)v(t))v(t) = -b(t)v^2(t) + (d(t) - a(t))v(t) + c(t) \Rightarrow$$
$$\Rightarrow v'(t) + b(t)v^2(t) + (a(t) - d(t))v(t) = c(t)$$

Let us apply this transformations to an example

Example 4

We will transform the following system into his related Riccati equation:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -\frac{1}{t}x(t) + \frac{t+1}{t}y(t) \end{cases} \tag{10}$$

As we have done in the previous proof, let us consider $v(t) = \frac{y(t)}{x(t)}; v'(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x^2(t)}$. Substituting $x'(t)$ and $y'(t)$ by their values in (10):

$$v'(t) = \frac{(-\frac{1}{t}x(t) + \frac{t+1}{t}y(t))x(t) - y(t)y(t)}{x^2(t)} = \frac{-\frac{1}{t}x^2(t) + \frac{t+1}{t}y(t)x(t) - y^2(t)}{x^2(t)}$$

Now, using that $y(t) = v(t)x(t)$:

$$v'(t) = \frac{-\frac{1}{t}x^2(t) + \frac{t+1}{t}x^2(t)v(t) - x^2(t)v^2(t)}{x^2(t)} = -\frac{1}{t} + \frac{t+1}{t}v(t) - v^2(t)$$

Arriving to his associated Riccati equation:

$$v'(t) + v^2(t) - \frac{t+1}{t}v(t) = -\frac{1}{t} \tag{11}$$

As in the previous examples, we can check the coefficients.

Proposition 5

Every Riccati equation defined as:

$$y'(t) + r(t)y^2(t) + q(t)y(t) = p(t)$$

can be expressed as the following first-order homogeneous linear ODE system equation:

$$\begin{cases} u'(t) = p(t)w(t) \\ w'(t) = r(t)u(t) + q(t)w(t) \end{cases}$$

Proof

First we introduce a change of unknown function:

$$y(t) = \frac{u(t)}{w(t)} \Rightarrow y'(t) = \frac{u'(t)w(t) - u(t)w'(t)}{w^2(t)}$$

Where $w(t)$ and $u(t)$ are two unknown functions. Substituting in (1):

$$\frac{u'(t)w(t) - u(t)w'(t)}{w^2(t)} + r(t)\frac{u^2(t)}{w^2(t)} + q(t)\frac{u(t)}{w(t)} = p(t)$$

Multiplying this expresion by $w^2(t)$:

$$u'(t)w(t) - u(t)w'(t) + r(t)u^2(t) + q(t)u(t)w(t) = p(t)w^2(t) \Rightarrow$$

$$u'(t)w(t) - u(t)w'(t) = -r(t)u^2(t) - q(t)u(t)w(t) + p(t)w^2(t)$$

We will take as the first equation of the system:

$$u'(t) = p(t)w(t)$$

Substituting in the previous expression u'(x) by its value on the first equation of the system:

$$p(t)w^2(t) - u(t)w'(t) = -r(t)u^2(t) - q(t)u(t)w(t) + p(t)w^2(t) \Rightarrow$$

$$-u(t)w'(t) = -r(t)u^2(t) - q(t)u(t)w(t)$$

Multiplying each term of the equation by $\frac{-1}{u(t)}$ we get the second equation of the system as we wanted to show:

$$w'(t) = r(t)u(t) + q(t)w(t)$$

Let us transform (11) into a system as an example:

Example 5

We start introducing the functions: $v(t) = \frac{u(t)}{w(t)}$ such as we did in the proof:

$$u'(t)w(t) - u(t)w'(t) = -u^2(t) + \frac{t+1}{t}u(t)w(t) + \frac{w^2(t)}{t}$$

Taking as the first equation:

$$u'(t) = \frac{w(t)}{t}$$

we arrive to the second equation of the system:$w'(t) = u(t) - \frac{t+1}{t}w(t)$. It is easy to verify that the coefficients coincide with the expected coefficients by applying the proposition

2.2 A very important application of the Riccati equation

As mentioned in the Introduction of this Section, we consider the main practical application of the Riccati equation to be the quadratic approximation of any regular first-order ordinary differential equation. In physics, chemistry, economics, and all other sciences, there are phenomena that can be described by first-order ODEs.When the ODE can not be solved explicitly one of the best ways to approach the solution is to approximate it by means of Taylor formula to functions of two variables. With this, we not only manage to reduce the difficulty of the equation, but also to deepen the study of one of the two variables. As examples, in physics we would have the equation that describes the velocity of an object in free fall with air resistance: $v'(t) = g - kv^2(t)$. In chemistry, we would have the equation that describes a second-order reaction in which the reactants are consumed in a non-linear process: $A'(t) = -kA^2(t)$. In economics we would have the function modeling economic growth with competition depending on total output: $y'(t) = ay(t) - by^2(t)$. Where a is an economic growth coefficient and b is a parameter that measures market competition or saturation, which depends on the total quantity produced. If we suppose that $f(x, y)$ can be developed in powers of y , then applying Taylor formula as we mentioned above:

$$y'(x) = f(x, y(x)) \approx f(x, y_0) + f_y(x, y_0)(y(x) - y_0) + \frac{1}{2} f_{yy}(x, y_0)(y(x) - y_0)^2 + \frac{1}{6} f_{yyy}(x, y_0)(y(x) - y_0)^3 + \ldots.$$

If we approach $f(x, y)$ with just two terms of the Taylor series we get a firs-order linear ODE and taking $y_0 = 0$:

$$y'(x) = f(x, 0) + f_y(x, 0)y(x)$$

But, in order to go for a better approach, using three terms of the series, we get a Riccati equation:

$$y'(x) = f(x, 0) + f_y(x, 0)y(x) + \frac{1}{2} f_{yy}(x, 0)y^2(x)$$

So if we have a solution for all Riccati equations we will always have at least a quadratic approximation for any regular first-order ODE

3 Attempting to arrive at the general solution of the Riccati equation

In the previous section we have seen that if we get a general formula to solve Riccati equation we will also have a formula solving second-order homogeneous linear ODE and a formula solving the system previously seen. So let us try to simplify the Riccati equation to come closer to a general solution.

3.1 The easiest case: The Bernouilli equation

When $p(x) \equiv 0$, the Riccati equation becomes a Bernouilli equation which is easier to solve:

$$y'(x) + q(x)y(x) = -r(x)y^2(x) \tag{12}$$

In order to solve it we just need to apply a change of unknown function:

$$y(x) = \frac{-1}{u(x)}; \qquad y'(x) = \frac{u'(x)}{u^2(x)}$$

Substituting in (12):

$$\frac{u'(x)}{u^2(x)} - \frac{q(x)}{u(x)} = \frac{-r(x)}{u^2(x)} \Rightarrow u'(x) = q(x)u(x) - r(x)$$

Using now (3), the solution of this first-order linear ODE is:

$$u(x) = e^{\int q(x)\,dx}(C + \int(-r(x))e^{-\int q(x)\,dx}\,dx)$$

Undoing the change of unknown function, we get that the general solution for (12) is

$$y(x) = \frac{-1}{e^{\int q(x)\,dx}(C + \int(-r(x))e^{-\int q(x)\,dx}\,dx)}$$

Let's put it into practice with an example:

Example 6

We are going to apply our formula to:

$$y'(x) + \frac{1}{x}y(x) = -y^2(x)$$

The solution according to our formula is:

$$y(x) = \frac{-1}{e^{\int \frac{1}{x}\,dx}(C + \int(-1)e^{-\int \frac{1}{x}\,dx}\,dx)} = \frac{-1}{e^{ln(x)+C_1}(C - \int e^{-(ln(x)+C_1)}\,dx)} \Rightarrow$$

$$\Rightarrow y(x) = \frac{-1}{xC_1(C - \frac{1}{C_1}\int \frac{1}{x}\,dx)} \Rightarrow$$

$$\Rightarrow y(x) = \frac{-1}{x(C - ln(x) + C_2)} = \frac{1}{xln(x) - Cx}$$

3.2 Simplified Riccati equation

The next simplification for (1) is to get $q(x) = 0$, reaching (5).

Proposition 6

For every Riccati equation :

$$y'(x) + r(x)y^2(x) + q(x)y(x) = p(x)$$

can be expressed as his associated simplified Riccati equation as:

$$z'(x) + r(x)e^{-\int q(x)dx}z^2(x) = p(x)e^{\int q(x)dx}$$

Proof

We start by the following change of unknown function:

$$y(x) = z(x)e^{-\int q(x)dx} \Rightarrow y'(x) = (z'(x) - z(x)q(x))e^{-\int q(x)dx}$$

Substituting in (1):

$$(z'(x) - q(x)z(x))e^{-\int q(x)dx} + r(x)e^{-2\int q(x)dx}z^2(x) + q(x)e^{-\int q(x)dx}z(x) = p(x)$$

Multiplying each term by $e^{\int q(x)dx}$ and simplifying we get the expression that we were searching. Let us illustrate this with the following example:

Example 7

Let's calculate the simplified form of the following equation:

$$y'(x) - xy^2(x) - y(x) = \frac{1}{x^2} \tag{13}$$

First we apply the following change of unknown function:

$$y(x) = z(x)e^{\int dx} = z(x)e^x \Rightarrow y'(x) = (z'(x) + z(x))e^x$$

Substituting in (13):

$$(z'(x) + z(x))e^x - xz^2(x)e^{2x} - z(x)e^x = \frac{1}{x^2} \Rightarrow z'(x)e^x - xz^2(x)e^{2x} = \frac{1}{x^2} \Rightarrow$$

$$\Rightarrow z'(x) - xz^2(x)e^x = \frac{1}{x^2e^x}$$

And we have reached (7) which was the simplified Riccati equation that we transformed into his second-order homogeneous linear ODE in Example 2

3.2.1 Arriving at an even more simplified Riccati equation

In this secction we are going to explain how is possible to simplify even more (5), by getting $r(x) \equiv 1$ or getting $p(x) \equiv 1$.

Proposition 7

Let us assume that $r(x)$ is twice derivable and non zero and $q(x)$ is derivable. Then, every Riccati equation defined as:

$$y'(x) + r(x)y^2(x) + q(x)y(x) = p(x)$$

can be expressed as his associated simplified Riccati equation expressed as:

$$u'(x) + u^2(x) = p_2(x)$$

or expressed as:

$$u'(x) + p_2(x)u^2(x) = 1$$

where $p_2(x) = \frac{4r^2(x)p(x)+2q'(x)r(x)-2r''(x)+3\frac{(r'(x))^2}{r(x)}-2r'(x)q(x)+q^2(x)r(x)}{-4r(x)}$

Proof

As usual we start with a change of unknown function:

$$y(x) = \frac{u(x)}{r(x)} - \frac{1}{2}\frac{q(x) - \frac{r'(x)}{r(x)}}{r(x)}$$

$$y'(x) = \frac{u'(x)r(x) - r'(x)u(x)}{r^2(x)} - \frac{1}{2}\frac{q'(x)r(x) - r'(x)q(x) - r''(x) + \frac{2(r'(x))^2}{r(x)}}{r^2(x)} \stackrel{def}{\equiv} \frac{u'(x)r(x) - r'(x)u(x)}{r^2(x)} + f(x)$$

Substituting in (1) and simplifying:

$$\frac{u'(x)r(x) - r'(x)u(x)}{r^2(x)} + f(x) + \frac{1}{4r(x)}\left(2u(x) - q(x) + \frac{r'(x)}{r(x)}\right)^2 + \frac{q(x)}{2r(x)}\left(2u(x) - q(x) + \frac{r'(x)}{r(x)}\right) = p(x)$$

Operating this expression and letting on the left side terms of the unknown function, and grouping the non-dependent terms of $u(x)$ in $p_1(x)$:

$$\frac{u'(x)r(x) - r'(x)u(x)}{r^2(x)} + \frac{1}{4r(x)}\left(4u^2(x) - 4u(x)q(x) + 4u(x)\frac{r'(x)}{r(x)}\right) + \frac{q(x)}{r(x)}u(x) = p_1(x) \Rightarrow$$

$$\frac{u'(x)r(x) - r'(x)u(x)}{r^2(x)} + \frac{u^2(x)}{r(x)} - \frac{q(x)}{r(x)}u(x) + \frac{r'(x)}{r^2(x)}u(x) + \frac{q(x)}{r(x)}u(x) = p_1(x)$$

with $p_1(x) = p(x) - f(x) - \frac{q^2(x) + \frac{(r'(x))^2}{r^2} - \frac{2q(x)r'(x)}{r(x)}}{4r(x)} + \frac{q^2(x) - \frac{r'(x)q(x)}{r(x)}}{2r(x)}$ Operating we reach the simplified Riccati equation with $r(x) = 1$

$$u'(x) + u^2(x) = p_1(x)r(x) = p_2(x) = \frac{4r^2(x)p(x) + 2q'(x)r(x) - 2r''(x) + 3\frac{(r'(x))^2}{r(x)} - 2r'(x)q(x) + q^2(x)r(x)}{4r(x)} \tag{14}$$

Now, in order to get $p(x) \equiv 1$ we add a new change of unknown function:

$$u(x) = \frac{1}{v(x)} \Rightarrow u'(x) = \frac{-v'(x)}{v^2(x)}$$

Applying it in (14):

$$\frac{-v'(x)}{v^2(x)} + \frac{1}{v^2(x)} = p_2(x) \Rightarrow -v'(x) + 1 = p_2(x)v^2(x) \Rightarrow -v'(x) - p_2(x)v^2(x) = -1$$

getting the reduced form that we were looking for:

$$v'(x) + p_2(x)v^2(x) = 1$$

Example 8

Let us put it into practice using (13) as an example. We start with the following change of unknown function:

$$y(x) = \frac{-u(x)}{x} - \frac{1+\frac{1}{x}}{2x} = \frac{-1}{2x}\left(2u(x) + 1 + \frac{1}{x}\right)$$

$$y'(x) = \frac{-u'(x)x + u(x)}{x^2} + \frac{1}{2}\frac{x^2+2x}{x^4}$$

Substituting in (13):

$$\frac{-u'(x)x + u(x)}{x^2} + \frac{1}{2}\frac{x^2+2x}{x^4} - \frac{1}{4x}\left(2u(x) + 1 + \frac{1}{x}\right)^2 + \frac{1}{2x}\left(2u(x) + 1 + \frac{1}{x}\right) = \frac{1}{x^2} \Rightarrow$$

$$\Rightarrow \frac{-u'(x)x + u(x)}{x^2} - \frac{u^2(x)}{x} - \frac{1}{4x} - \frac{1}{4x^3} - \frac{u(x)}{x} - \frac{u(x)}{x^2} - \frac{1}{2x^2} + \frac{u(x)}{x} + \frac{1}{2x} + \frac{1}{2x^2} = \frac{1}{x^2} - \frac{1}{2}\frac{x^2+2x}{x^4} \Rightarrow$$

Grouping and simplifying we get:

$$\Rightarrow \frac{-u'(x)}{x} - \frac{u^2(x)}{x} = \frac{1}{x^2} - \frac{1}{2}\frac{x^2+2x}{x^4} + \frac{1}{4x} + \frac{1}{4x^3} + \frac{1}{2x^2} - \frac{1}{2x} - \frac{1}{2x^2} = \frac{-x^2+2x-3}{4x^3} \Rightarrow$$

Multiplying each term by $-x$ we arrive at the desired equation:

$$u'(x) + u^2(x) = \frac{x^2-2x+3}{4x^2} \tag{15}$$

Now, let's introduce the following change of variable to obtain $p(x) = 1$

$$u(x) = \frac{1}{v(x)} \Rightarrow v'(x) + \frac{x^2-2x+3}{4x^2}v^2(x) = 1$$

Let us confirm that:

$$p_2(x) = \frac{4r^2(x)p(x) + 2q'(x)r(x) - 2r''(x) + 3\frac{(r'(x))^2}{r(x)} - 2r'(x)q(x) + q^2(x)r(x)}{4r(x)} = \frac{4 - \frac{3}{x} - 2 - x}{-4x} =$$

$$= \frac{x^2-2x+3}{4x^2}$$

3.2.2 Solving the Riccati equation knowing a particular solution

Proposition 10

For any Riccati equation expressed as:

$$y'(x) + r(x)y^2(x) + q(x) = p(x)$$

for which a particular solution $y_1(x)$is found, there is a general solution expressed as:

$$y(x) = y_1(x) + \frac{1}{e^{\int 2y_1(x)r(x)+q(x)\,dx}(C + \int r(x)e^{-\int 2y_1(x)r(x)+q(x)\,dx}\,dx)}$$

Proof

We start with a change of unknown function:

$$y(x) = y_1(x) + \frac{1}{u(x)} \Rightarrow y'(x) = y_1'(x) - \frac{u'(x)}{u^2(x)}$$

Substituting in (1):

$$y_1'(x) - \frac{u'(x)}{u^2(x)} + r(x)\left(y_1(x) + \frac{1}{u(x)}\right)^2 + q(x)\left(y_1(x) + \frac{1}{u(x)}\right) = p(x) \Rightarrow$$

$$\Rightarrow y_1'(x) - \frac{u'(x)}{u^2(x)} + r(x)\left(y_1^2(x) + \frac{1}{u^2(x)} + \frac{2y_1(x)}{u(x)}\right) + q(x)\left(y_1(x) + \frac{1}{u(x)}\right) = p(x) \Rightarrow$$

As $y_1(x)$ is a particular solution:

$$-\frac{u'(x)}{u^2(x)} + r(x)\left(\frac{1}{u^2(x)} + \frac{2y_1(x)}{u(x)}\right) + q(x)\left(\frac{1}{u(x)}\right) = 0 \Rightarrow$$

$$\Rightarrow -u'(x) + r(x)(1 + 2y_1(x)u(x)) + q(x)u(x) = 0$$

$$\Rightarrow u'(x) = (2y_1(x)r(x) + q(x))u(x) + r(x)$$

Which is a first-order linear ODE that we have resolved in section 2.1:

$$u(x) = e^{\int 2y_1(x)r(x)+q(x)\,dx}(C + \int r(x)e^{-\int 2y_1(x)r(x)+q(x)\,dx}\,dx) \Rightarrow$$

$$\Rightarrow y(x) = y_1(x) + \frac{1}{e^{\int 2y_1(x)r(x)+q(x)\,dx}(C + \int r(x)e^{-\int 2y_1(x)r(x)+q(x)\,dx}\,dx)}$$

Let's put it into practice by using as an example (13):

Example 10

First we need to find a particular solution for $y'(x) - xy^2(x) - y(x) = \frac{1}{x^2}$, which we find by inspection to be $y_0 = \frac{-1}{x}$. We will now use the change of unknown function seen above:

$$y(x) = \frac{-1}{x} + \frac{1}{u(x)}; y'(x) = \frac{1}{x^2} - \frac{u'(x)}{u^2(x)}$$

Substituting in (13):

$$\frac{1}{x^2} - \frac{u'(x)}{u^2(x)} - x\left(\frac{1}{x^2} - \frac{2}{xu(x)} + \frac{1}{u^2(x)}\right) - \left(\frac{-1}{x} + \frac{1}{u(x)}\right) = \frac{1}{x^2}$$

Simplifying:

$$-\frac{u'(x)}{u^2(x)} + \frac{2}{u(x)} - \frac{x}{u^2(x)} - \frac{1}{u(x)} = 0 \Rightarrow -u'(x) + u(x) - x = 0 \Rightarrow$$

$$\Rightarrow u'(x) = u(x) - x$$

We have seen in Example 1 that $u(x) = Ce^x + x + 1$ is the general solution of this equation. Undoing the change of unknown function we get to the solution for (13)

$$y(x) = \frac{-1}{x} + \frac{1}{Ce^x + x + 1}$$

In the website [2] we can find 16 different solved cases whose solution is obtained from knowing a particular solution. That is why we are going to study how a Riccati equation must be to have $g(x)$ as a particular solution.

Proposition 11

A given differentiable function $g(x)$ is a particular solution of a Riccati equation expressed as:

$$y'(x) + r(x)y^2(x) + q(x) = p(x)$$

when $p(x) = g'(x)$ and $q(x) = -r(x)g(x)$

Proof

If $g(x)$ is a particular solution, then it must satisfy (1):

$$g'(x) + r(x)g^2(x) + q(x)g(x) = p(x)$$

If we suppose $p(x) = g'(x)$ then we get:

$$r(x)g^2(x) + q(x)g(x) = 0 \Rightarrow q(x) = -r(x)g(x)$$

So the Riccati equation must be:

$$y'(x) + r(x)y^2(x) - r(x)g(x)y(x) = g'(x)$$

We have seen that this condition is sufficient. In order to see that it is not necessary, we are going to show an example where $p(x) \neq g'(x)$:

$$y'(x) - y^2(x) + xy(x) = ax - a^2$$

This is a particular example of the special case 1 seen in [2]. It has $g(x) \equiv a$ as particular solution. $g'(x) \equiv 0 \neq p(x) = ax - a^2 \iff a \neq 0$.

3.3 Solutions to the Riccati equation if more particular solutions are available

In the previous section we have seen that having one particular solution makes the problem of solving the Riccati equation much easier but, what if we have two, three, four or even more particular solutions? That question was answered by Euler in 1762 presenting the following proposition:

Proposition 12

The general solution of the Riccati equation expressed as:

$$y'(x) + r(x)y^2(x) + q(x) = p(x)$$

when two particular solutions, $y_1(x)$ and $y_2(x)$ are known has the following form:

$$y(x) = \frac{y_2(x) - y_1(x)Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}}{1 - Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}}$$

Proof

We start doing the same change of unknown function, but this time twice:

$$y(x) = y_1(x) + \frac{1}{u(x)} \qquad y(x) = y_2(x) + \frac{1}{v(x)}$$

So repeating the process of subsection 4.2.1 we will get:

$$u'(x) - (2y_1(x)r(x) + q(x))u(x) - r(x) = 0$$

And

$$v'(x) - (2y_2(x)r(x) + q(x))v(x) - r(x) = 0$$

Now we multiply by $v(x)$ the first one and by $u(x)$ the second one, getting:

$$v(x)u'(x) - (2y_1(x)r(x) + q(x))u(x)v(x) - v(x)r(x) = 0$$

And

$$u(x)v'(x) - (2y_2(x)r(x) + q(x))v(x)u(x) - u(x)r(x) = 0$$

The next step is to subtract the second from the first:

$$v(x)u'(x) - u(x)v'(x) + (2r(x)(y_2(x) - y_1(x))u(x)v(x) + r(x)(u(x) - v(x)) = 0 \Rightarrow$$

$$\frac{1}{u(x)}u'(x) - \frac{1}{v(x)}v'(x) + 2r(x)(y_2(x) - y_1(x)) + r(x)\left(\frac{1}{v(x)} - \frac{1}{u(x)}\right) = 0 \Rightarrow$$

$$\frac{1}{u(x)}u'(x) - \frac{1}{v(x)}v'(x) + 2r(x)(y_2(x) - y_1(x)) + r(x)(y(x) - y_2(x) - y(x) + y_1(x)) = 0 \Rightarrow$$

$$\frac{1}{u(x)}u'(x) - \frac{1}{v(x)}v'(x) + r(x)(y_1(x) - y_2(x)) = 0 \Rightarrow$$

$$\frac{1}{v(x)}u'(x) - \frac{u(x)}{v^2(x)}v'(x) + r(x)(y_1(x) - y_2(x))\frac{u(x)}{v(x)} = 0 \Rightarrow$$

$$\Rightarrow \left(\frac{u(x)}{v(x)}\right)' + r(x)(y_1(x) - y_2(x))\frac{u(x)}{v(x)} = 0$$

Taking a new unknown function and substituting:

$$w(x) = \frac{u(x)}{v(x)} \Rightarrow$$

$$w'(x) = r(x)(y_1(x) - y_2(x))w(x)$$

which is a simpler first-order linear ODE that the one obtained in subsection 4.2.1.

$$w(x) = Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}$$

After solving this ODE we will undo the change of unknown function:

$$y(x) = y_1(x) + \frac{1}{u(x)} \Rightarrow u(x) = \frac{1}{y(x) - y_1(x)}$$

Using the same argument:

$$v(x) = \frac{1}{y(x) - y_2(x)} \Rightarrow w(x) = \frac{y(x) - y_2(x)}{y(x) - y_1(x)} \Rightarrow y(x) - y_2(x) = w(x)(y(x) - y_1(x)) \Rightarrow$$

$$\Rightarrow y(x) = \frac{y_2(x) - y_1(x)w(x)}{1 - w(x)} = \frac{y_2(x) - y_1(x)Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}}{1 - Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}} \tag{16}$$

Let us illustrate this case with an example with $A = -1$ and $B = -2$

Example 11

We are going to solve:

$$xy'(x) - y^2(x) - y(x) = -2 \Rightarrow xy'(x) = y^2(x) + y(x) - 2 \tag{17}$$

Since 1 and -2 are the roots of the polynomial $\lambda^2 + \lambda - 2$, $y_1(x) \equiv 1$ and $y_2(x) \equiv -2$ are particular solutions of (17). As in the theoretical proof, by the next changes of unknown function $y(x) = 1 + \frac{1}{u(x)}$ and $y(x) = -2 + \frac{1}{v(x)}$ and, after operating, $w(x) = \frac{u(x)}{v(x)}$:

$$w'(x) = \frac{-3}{x} w(x)$$

Using (3):

$$w(x) = Ce^{\int \frac{-3}{x} dx} = Ce^{ln(x^{-3})} = Cx^{-3}$$

Utilizing (16):

$$y(x) = \frac{2 + Cx^{-3}}{Cx^{-3} - 1}$$

We can observe how the expression of the solution with two particular solutions is ‘’simpler‘’ than the solution with only one particular solution known. Will this ‘’simplicity‘’ increase if we have one more particular solution? This was studied by E.Weyer in 1875 and C.E. Picard in 1877 reaching the follwing proposition

Proposition 13

The general solution of the Riccati equation expressed as:

$$y'(x) + r(x)y^2(x) + q(x) = p(x)$$

when three particular solutions, $y_1(x)$, $y_2(x)$ and $y_3(x)$ are known has the following form:

$$y(x) = \frac{y_2(x) - y_1(x)C^{\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}}}{1 - C^{\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}}}$$

Proof

In subsection 4.2.1 we have seen that applying the following change of unknown function to the Riccati equation:

$$y = y_1(x) + \frac{1}{u_1(x)} \Rightarrow u_1(x) = \frac{1}{y(x) - y_1(x)}$$

We get that $u_1(x)$ is solution for a first-order linear ODE, so $u_1(x)$ has the form of (3)

$$u_1(x) = C_1 f(x) + g(x)$$

Now we define two new unknown functions which are also solutions of (1) and , as $u_1(x)$, they have te form of (2)

$$u_2(x) = \frac{1}{y_2(x) - y_1(x)} = C_2 f(x) + g(x)$$

$$u_3(x) = \frac{1}{y_3(x) - y_1(x)} = C_3 f(x) + g(x)$$

Now, it is easy to see that:

$$\frac{u_1(x) - u_2(x)}{u_3(x) - u_2(x)} = C; \quad \frac{C_1 f(x) + g(x) - (C_2 f(x) + g(x))}{C_3 f(x) + g(x) - (C_2 f(x) + g(x))} = \frac{(C_1 - C_2)f(x)}{(C_3 - C_2)f(x)} = \frac{(C_1 - C_2)}{(C_3 - C_2)} = C$$

Substituting with their values seen above:

$$C = \frac{\frac{1}{y(x)-y_1(x)} - \frac{1}{y_2(x)-y_1(x)}}{\frac{1}{y_3(x)-y_1(x)} - \frac{1}{y_2(x)-y_1(x)}} = \frac{\frac{y_2(x)-y_1(x)-y(x)+y_1(x)}{(y(x)-y_1(x))(y_2(x)-y_1(x))}}{\frac{y_2(x)-y_1(x)-y_3(x)+y_1(x)}{(y_3(x)-y_1(x))(y_2(x)-y_1(x))}} = \frac{\frac{y_2(x)-y(x)}{y(x)-y_1(x)}}{\frac{y_2(x)-y_3(x)}{y_3(x)-y_1(x)}} \Rightarrow$$

$$\Rightarrow \frac{y(x)-y_2(x)}{y(x)-y_1(x)} = C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)} \Rightarrow y(x)-y_2(x) = (y(x)-y_1(x))C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)} \Rightarrow$$

$$\Rightarrow y(x)\left(1 - C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}\right) = y_2(x) - y_1(x)C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)} \Rightarrow$$

$$y(x) = \frac{y_2(x) - y_1(x)C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}}{1 - C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}} \tag{18}$$

Let's take it to a practical case with the following example:

Example 12

We are going to use the same equation that we used in Example 10:

$$xy'(x) - y^2(x) - y(x) = -2$$

In the previous example, we have seen that the general solution for this equation is:

$$y(x) = \frac{2 + Cx^{-3}}{Cx^{-3} - 1}$$

We already know that $y_1(x) \equiv 1$ and $y_2(x) \equiv -2$ are particular solutions of our equation. In order to obtain any particular solution, we just have to give values to the constant C. For example, $y_2(x)$ is obtained by taking $C = 0$ and $y_1(x)$ is obtained by taking $C = \pm\infty$ Let's define then a third particular solution:

$$y_3(x) = \frac{2 + C_3x^{-3}}{C_3x^{-3} - 1}$$

Now, let's verify (18)

$$\frac{y_2(x) - y_1(x)C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}}{1 - C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}} = \frac{-2 - 1 \cdot C\frac{\frac{2+C_3x^{-3}}{C_3x^{-3}-1}+2}{\frac{2+C_3x^{-3}}{C_3x^{-3}-1}-1}}{1 - C\frac{\frac{2+C_3x^{-3}}{C_3x^{-3}-1}+2}{\frac{2+C_3x^{-3}}{C_3x^{-3}-1}-1}} \Rightarrow$$

$$\Rightarrow \frac{-2 - C\frac{\frac{3C_3x^{-3}}{C_3x^{-3}-1}}{\frac{3}{C_3x^{-3}-1}}}{1 - C\frac{\frac{3C_3x^{-3}}{C_3x^{-3}-1}}{\frac{3}{C_3x^{-3}-1}}} = \frac{-2 - C(C_3x^{-3})}{1 - C(C_3x^{-3})}$$

Grouping the constants together and multiplying the numerator and denominator by -1 we see that the equation is verified

Now we ask ourselves: Is there an end to this procedure? If we continue to increase the number of available particular solutions, will the simplicity of the solution of the Riccati equation obtained continue to increase?. The answer we give to these questions is that following the method used so far leads to the following property:

Proposition 14

For all four particular solutions of a Riccati equation $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ it is satisfied that:

$$\frac{(y_4(x)-y_2(x))(y_3(x)-y_1(x))}{(y_4(x)-y_1(x))(y_3(x)-y_2(x))}=C$$

Proof

Using the same arguments as in subsection 4.2.3 , but now using:

$$u_1(x)=\frac{1}{y_4(x)-y_1(x)}$$

$$u_2(x)=\frac{1}{y_2(x)-y_1(x)}$$

$$u_3(x)=\frac{1}{y_3(x)-y_1(x)}$$

We get:

$$\frac{y_4(x)-y_2(x)}{y_4(x)-y_1(x)}=C\frac{y_3(x)-y_2(x)}{y_3(x)-y_1(x)}\Rightarrow\frac{(y_4(x)-y_2(x))(y_3(x)-y_1(x))}{(y_4(x)-y_1(x))(y_3(x)-y_2(x))}=C$$

Let us apply it in the following example, to the equation seen in Example 10 of which we know that $y_1(x)\equiv 1$, $y_2(x)\equiv -2$, $y_3(x)=\frac{2+C_3x^{-3}}{C_3x^{-3}-1}$ and $y_4(x)=\frac{2+C_4x^{-3}}{C_4x^{-3}-1}$ are particular solutions.

Example 13

We are going to prove that

$$\frac{(y_4(x)-y_2(x))(y_3(x)-y_1(x))}{(y_4(x)-y_1(x))(y_3(x)-y_2(x))}=C$$

Where C is any constant and $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$ are particular solutions of a Riccati equation. As we mentioned before they are solutions for $xy'(x)-y^2(x)-y(x)=-2$

$$\frac{(y_4(x)-y_2(x))(y_3(x)-y_1(x))}{(y_4(x)-y_1(x))(y_3(x)-y_2(x))}=\frac{(\frac{2+C_4x^{-3}}{C_4x^{-3}-1}+2)(\frac{2+C_3x^{-3}}{C_3x^{-3}-1}-1)}{(\frac{2+C_4x^{-3}}{C_4x^{-3}-1}-1)(\frac{2+C_3x^{-3}}{C_3x^{-3}-1}+2)}=\frac{(\frac{3C_4x^{-3}}{C_4x^{-3}-1})(\frac{3}{C_3x^{-3}-1})}{(\frac{3}{C_4x^{-3}-1})(\frac{3C_3x^{-3}}{C_3x^{-3}-1})}=\frac{C_4}{C_3}$$

And obviously de division of two constants is constant

3.4 The original Riccati equation

Let us present a simple Riccati equation whose solution involves Bessel functions which was studied by Riccati.

$$y'(x)+Ay^2(x)=Bx^n \tag{19}$$

We define Bessel function of the first kind as:

$$J_\nu(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(\nu+j+1)}\left(\frac{x}{2}\right)^{\nu+2j}\qquad\qquad \nu\in\mathbb{R}$$

If $\nu\notin\mathbb{Z}$, then the following equation:

$$u(x)=C_1J_\nu(x)+C_2J_{-\nu}(x) \tag{20}$$

is the general solution for the Bessel equation:

$$L(x;\nu)\equiv x^2u''(x)+xu'(x)+(x^2-\nu^2)u(x)=0 \tag{21}$$

In this section we will transform equations (19) and (21) by variable and unknown function changes in order to reach the same equation, proving that the solution of (19) is related to the solution of (21).

Proposition 8

Let us assume that the Bessel equation defined as:

$$x^2u''(x) + xu'(x) + (x^2 - \nu^2)u(x) = 0$$

has the general solution:

$$u(x) = C_1J_\nu(x) + C_2J_{-\nu}(x)$$

Then, the general solution of the ODE:

$$sw''(s) + (1+\nu)w'(s) + w(s) = 0$$

is given by:

$$w(s) = s^{\frac{-\nu}{2}}(C_1J_\nu(2\sqrt{s}) + C_2J_{-\nu}(2\sqrt{s}))$$

Proof

We start with the following variable change:

$$s = \frac{x^2}{4} \Rightarrow \frac{ds}{dx} = \frac{x}{2} = \sqrt{s} \quad ; \qquad \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} = \frac{du}{ds}\sqrt{s} \quad ; \qquad \frac{d^2u}{d^2x} = \frac{d^2u}{d^2s}s + \frac{1}{2}\frac{du}{ds}$$

Substituting in (21) we get:

$$4s(su''(s) + \frac{1}{2}u'(s)) + 2\sqrt{s}u'(s)\sqrt{s} + (4s - \nu^2)u(s) = 0 \Rightarrow$$

$$\Rightarrow 4s^2u''(s) + 4su'(s) + (4s - \nu^2)u(s) = 0$$

Now we will introduce a change of unknown function:

$$u(s) = s^{\frac{\nu}{2}}w(s) \quad ; \qquad u'(s) = \frac{\nu}{2}w(s)s^{\frac{\nu}{2}-1} + s^{\frac{\nu}{2}}w'(s);$$

$$u''(s) = \frac{\nu}{2}\left(w(s)\left(\frac{\nu}{2}-1\right)s^{\frac{\nu}{2}-2} + w'(s)s^{\frac{\nu}{2}-1}\right) + \frac{\nu}{2}w'(s)s^{\frac{\nu}{2}-1} + w''(s)s^{\frac{\nu}{2}}$$

Substituting:

$$(4s^2\left(\frac{\nu}{2}\left(w(s)\left(\frac{\nu}{2}-1\right)s^{\frac{\nu}{2}-2} + w'(s)s^{\frac{\nu}{2}-1}\right) + \frac{\nu}{2}w'(s)s^{\frac{\nu}{2}-1} + w''(s)s^{\frac{\nu}{2}}\right) +$$

$$+4s\left(\frac{\nu}{2}w(s)s^{\frac{\nu}{2}-1} + s^{\frac{\nu}{2}}w'(s)\right) + (4s-\nu^2)s^{\frac{\nu}{2}}w(s) = 0 \Rightarrow$$

$$\Rightarrow 4s^{\frac{\nu}{2}+2}w''(s) + w'(s)\left(\frac{\nu}{2}4s^{\frac{\nu}{2}+1} + \frac{\nu}{2}4s^{\frac{\nu}{2}+1} + 4s^{\frac{\nu}{2}+1}\right) +$$

$$+w(s)\left(\frac{\nu}{2}\left(\frac{\nu}{2}-1\right)4s^{\frac{\nu}{2}} + \frac{\nu}{2}4s^{\frac{\nu}{2}} + (4s-\nu^2)s^{\frac{\nu}{2}}\right) = 0 \Rightarrow$$

$$\Rightarrow 4s^{\frac{\nu}{2}+2}w''(s) + w'(s)\left((\nu+1)4s^{\frac{\nu}{2}+1}\right) + 4s^{\frac{\nu}{2}+1}w(s) = 0$$

Dividing by $4s^{\frac{\nu}{2}+1}$ we arrive at the following equation:

$$sw''(s) + (1+\nu)w'(s) + w(s) = 0 \tag{22}$$

Applying the same changes of unknown function and variables that we have applied to (21), to his solution (20) we get that the solution for (22) is:

$$w(s) = s^{\frac{-\nu}{2}}(C_1J_\nu(2\sqrt{s}) + C_2J_{-\nu}(2\sqrt{s})) \tag{23}$$

On the other side we have (19).

Proposition 9

Every Riccati equation given by:

$$y'(x) + Ay^2(x) = Bx^n$$

with A and B being constants and $n \in \mathbb{N} \setminus \{-2\}$, can be transformed:

$$sw''(s) + \left(1 - \frac{1}{n+2}\right) w'(s) + w(s) = 0 \tag{24}$$

Proof

We will first apply the following change of unknown function.

$$y(x) = \frac{u(x)}{x} \quad ; \qquad y'(x) = \frac{u'(x)x - u(x)}{x^2}$$

Substituting in (19):

$$\frac{u'(x)x - u(x)}{x^2} + A\frac{u^2(x)}{x^2} = Bx^n \Rightarrow$$
$$u'(x)x + Au^2(x) - u(x) = Bx^{n+2} \tag{25}$$

From there we will apply a variable change:

$$s = \frac{-AB}{(n+2)^2}x^{n+2} \quad ; \qquad \frac{ds}{dx} = \frac{-AB}{n+2}x^{n+1}$$

We need to assume that $n \neq -2$ for obvious reasons. In the next section we will solve the case $n = -2$. We will also add a change of unknown function:

$$u(s) = \frac{n+2}{A}s\frac{w'(s)}{w(s)} \quad ; \qquad u'(s) = \frac{n+2}{A}\left(\frac{w'(s)}{w(s)} + s\frac{w''(s)w(s) - (w'(s))^2}{w^2(s)}\right)$$

Knowing that $\frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx}$, we will calculate each term of (25) separately:

$$u'(x)x = u'(s)\frac{ds}{dx}x = \frac{n+2}{A}\left(\frac{w'(s)}{w(s)} + s\frac{w''(s)w(s) - (w'(s))^2}{w^2(s)}\right)\frac{-AB}{n+2}x^{n+2} \Rightarrow$$

$$\Rightarrow u'(x)x = -B\left(\frac{w'(s)}{w(s)} + s\frac{w''(s)w(s) - (w'(s))^2}{w^2(s)}\right)s\frac{(n+2)^2}{-AB} = \frac{(n+2)^2}{A}s\left(\frac{w'(s)}{w(s)} + s\frac{w''(s)w(s) - (w'(s))^2}{w^2(s)}\right)$$

$$Au^2(x) = Au^2(s) = \frac{(n+2)^2}{A}s^2\left(\frac{w'(s)}{w(s)}\right)^2$$

$$-u(x) = -u(s) = \frac{-(n+2)}{A}s\frac{w'(s)}{w(s)} \quad ; \qquad Bx^{n+2} = s\frac{(n+2)^2}{-A}$$

Now substituting in (25):

$$\frac{(n+2)^2}{A}s\left(\frac{w'(s)}{w(s)} + s\frac{w''(s)w(s) - (w'(s))^2}{w^2(s)}\right) + \frac{(n+2)^2}{A}s^2\left(\frac{w'(s)}{w(s)}\right)^2 - \frac{(n+2)}{A}s\frac{w'(s)}{w(s)} = s\frac{(n+2)^2}{-A}$$

Multiplying each term by $\frac{A}{s(n+2)^2}$:

$$\left(\frac{w'(s)}{w(s)} + s\frac{w''(s)w(s) - (w'(s))^2}{w^2(s)}\right) + s\left(\frac{w'(s)}{w(s)}\right)^2 - \frac{1}{n+2}\frac{w'(s)}{w(s)} = -1 \Rightarrow$$

$$\Rightarrow \frac{w'(s)}{w(s)}\left(1 - \frac{1}{n+2}\right) + s\frac{w''(s)}{w(s)} = -1 \Rightarrow sw''(s) + \left(1 - \frac{1}{n+2}\right)w'(s) = -w(s) \Rightarrow$$

$$\Rightarrow sw''(s) + \left(1 - \frac{1}{n+2}\right)w'(s) + w(s) = 0$$

Achieving the general solution

Using (23) the solution for (24) is:

$$w(s) = s^{\frac{1}{2(n+2)}}(C_1 J_{\frac{-1}{n+2}}(2\sqrt{s}) + C_2 J_{\frac{1}{n+2}}(2\sqrt{s})) \tag{26}$$

Undoing the changes of variable and unknown function and changing we get the solution for (19). We will first change $n+2$ by $2k$ in order to simplify $w(s)$:

$$w(s) = (\sqrt{s})^{\frac{1}{2k}}(C_1 J_{\frac{-1}{2k}}(2\sqrt{s}) + C_2 J_{\frac{1}{2k}}(2\sqrt{s}))$$

First we will undo the change of variable:

$$w(x) = \left(\frac{\sqrt{-AB}}{2k}\right)^{\frac{1}{2k}} \sqrt{x}\left(C_1 J_{\frac{-1}{2k}}\left(\frac{1}{k}\sqrt{-AB}x^k\right) + C_2 J_{\frac{1}{2k}}\left(\frac{1}{k}\sqrt{-AB}x^k\right)\right) \Rightarrow$$

Transforming the constants C_1 and C_2

$$\Rightarrow w(x) = \sqrt{x}\left(C_1 J_{\frac{-1}{2k}}\left(\frac{1}{k}\sqrt{-AB}x^k\right) + C_2 J_{\frac{1}{2k}}\left(\frac{1}{k}\sqrt{-AB}x^k\right)\right)$$

Secondly we will undo the change of unknown function:

$$u(s) = \frac{2k}{A}s\frac{w'(s)}{w(s)} \Rightarrow u(x) = \frac{2k}{A}\frac{(-AB)}{(2k)^2}x^{2k}\frac{w'(x)}{w(x)}\frac{2k}{(-AB)}x^{2k-1} = \frac{1}{A}\frac{w'(x)}{w(x)}x$$

And finally we get the solution using that: $y(x) = \frac{u(x)}{x}$:

$$y(x) = \frac{1}{A}\frac{w'(x)}{w(x)}; w(x) = \sqrt{x}\left(C_1 J_{\frac{-1}{2k}}\left(\frac{1}{k}\sqrt{-AB}x^k\right) + C_2 J_{\frac{1}{2k}}\left(\frac{1}{k}\sqrt{-AB}x^k\right)\right)$$

3.4.1 Solving (19) with $n = -2$

As in the general case, we start with the following change of unknown function: $y(x) = \frac{u(x)}{x}$ Substituting in (19):

$$\frac{u'(x)x - u(x)}{x^2} + A\frac{u^2(x)}{x^2} = Bx^{-2} \Rightarrow u'(x)x - u(x) + Au^2(x) = B$$

If we write the previous function in the following form:

$$u'(x)x = -Au^2(x) + u(x) + B \tag{27}$$

It is easy to see that $u(x) \equiv \lambda$,where λ is a root of the quadratic equation $-Ax^2 + x + B$, is a particular solution for (27). We will see in section 3.2.3 how the solution of a Riccati equation looks like when we have one or two particular solutions. In that section we will use a concrete case of (27) as an example of solving a Riccati equation with two particular solutions. For the moment, we will show an example of the general case of (19).

Example 9

We will take the parachutist equation as an example. This equation is of the form of (19) with $A = 1$, $B = 1$ and $n = 0$. Applying the same changes of variable and unknown function:

$$y(x) = \frac{u(x)}{x}; \qquad s = \frac{-x^2}{4}; \qquad u(s) = 2s\frac{w'(s)}{w(s)}$$

We know, using (26) that:

$$w(s) = s^{\frac{1}{4}}(C_1 J_{\frac{-1}{2}}(2\sqrt{s}) + C_2 J_{\frac{1}{2}}(2\sqrt{s})) \Rightarrow$$

Since $J_{\frac{-1}{2}}(2\sqrt{s}) = \sqrt{\frac{1}{\pi\sqrt{s}}}cos(2\sqrt{s})$ and $J_{\frac{1}{2}}(2\sqrt{s}) = \sqrt{\frac{1}{\pi\sqrt{s}}}sen(2\sqrt{s})$, we arrive at the following expression for $w(s)$:

$$\Rightarrow w(s) = \sqrt{\frac{1}{\pi}}(C_1 cos(2\sqrt{s}) + C_2 sen(2\sqrt{s}))$$

Applying now that $u(s) = 2s\frac{w'(s)}{w(s)}$:

$$u(s) = 2s\frac{(-C_1\frac{1}{\sqrt{s}}sen(2\sqrt{s}) + C_2\frac{1}{\sqrt{s}}cos(2\sqrt{s}))}{C_1 cos(2\sqrt{s}) + C_2 sen(2\sqrt{s})} = 2\sqrt{s}\frac{(-C_1 sen(2\sqrt{s}) + C_2 cos(2\sqrt{s}))}{C_1 cos(2\sqrt{s}) + C_2 sen(2\sqrt{s})}$$

As $s = \frac{-x^2}{4}$, $\sqrt{s} = \frac{ix}{2} \Rightarrow 2\sqrt{s} = ix$:

$$u(x) = ix\frac{(-C_1 sen(ix) + C_2 cos(ix))}{(C_1 cos(ix) + C_2 sen(ix))}$$

Knowing that $sen(ix) = i \cdot senh(x)$ and $cos(ix) = cosh(x)$:

$$y(x) = \frac{u(x)}{x} = \frac{ix}{x}\frac{(-iC_1 senh(x) + C_2 cosh(x))}{(C_1 cosh(x) + iC_2 senh(x))} = \frac{C_1(e^x - e^{-x}) + iC_2(e^x + e^{-x})}{C_1(e^x + e^{-x}) + iC_2(e^x - e^{-x})} \Rightarrow$$

$$\Rightarrow \frac{(C_1 + iC_2)e^x - (C_1 - iC_2)e^{-x}}{(C_1 + iC_2)e^x + (C_1 - iC_2)e^{-x}}$$

Dividing the numerator and denominator by $(C_1 + iC_2)$ and defining a new constant $K = \frac{(C_1 - iC_2)}{(C_1 + iC_2)}$ to simplify the equation, we get:

$$y(x) = \frac{e^x - Ke^{-x}}{e^x + Ke^{-x}}$$

4 Separable cases of the Riccati equation

Now we will focus on another simpler cases of the Riccati equation, separable cases. We define a first-order separable ODE when we it has the form$y'(x) = F_1(y(x))F_2(x)$

We will first analyze the case when $p(x), q(x)$ and $r(x)$ are constants, which will be useful for solving the cases that we will present later on:

4.1 Riccati equation with constant coefficients

As $r(x) \equiv R, q(x) \equiv Q$ and $p(x) \equiv P$ we have (1) being transformed into:

$$y'(x) + Ry^2(x) + Qy(x) = P \Rightarrow y'(x) = -Ry^2(x) - Qy(x) + P \tag{28}$$

Solving this case is closely related to the roots of the polynomial $-R\lambda^2 - Q\lambda + P$ which are $\frac{Q\pm\sqrt{Q^2+4RP}}{-2R}$. We will study three different scenarios. The first one, when both roots are real and different, which means $Q^2 + 4RP > 0$, the second one when both are the same real root, $Q^2 + 4RP = 0$ and the third, when both are complex, $Q^2 + 4RP < 0$.

Proposition 15

If the polynomial $-R\lambda^2 - Q\lambda + P$ has two different real roots $\lambda_1 = \alpha$ and $\lambda_2 = \beta$, then the general solution of the Riccati equation:

$$y'(x) + Ry^2(x) + Qy(x) = P$$

is given by:

$$y(x) = \frac{\beta - \alpha C e^{R(\alpha-\beta)x}}{1 - Ce^{R(\alpha-\beta)x}}$$

Proof

In this case we can rewrite (28) as:

$$y'(x) = -R(y(x) - \alpha)(y(x) - \beta)$$

It is easy to see that $y_1(x) \equiv \alpha$ and $y_2(x) \equiv \beta$ are two particular solutions for (28). Applying what we have seen in subsection 4.2.2:

$$y(x) = \frac{y_2(x) - y_1(x)Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}}{1 - Ce^{\int r(x)(y_1(x)-y_2(x))\,dx}} = \frac{\beta - \alpha Ce^{\int R(\alpha-\beta)\,dx}}{1 - Ce^{\int R(\alpha-\beta)\,dx}} = \frac{\beta - \alpha Ce^{R(\alpha-\beta)x+C_1}}{1 - Ce^{R(\alpha-\beta)x+C_1}} \Rightarrow$$

$$\Rightarrow y(x) = \frac{\beta - \alpha Ce^{R(\alpha-\beta)x}}{1 - Ce^{R(\alpha-\beta)x}} \tag{29}$$

Let us illustrate this case with an example taking $R = -1$, $Q = -4$ and $P = 3$:

Example 14

We are going to solve:

$$y'(x) - y^2(x) - 4y(x) = 3 \Rightarrow y'(x) = y^2(x) + 4y(x) + 3$$

The roots of $\lambda^2 + 4\lambda + 3$ are $\alpha = -1$ and $\beta = -3$. Applying (29) the general solution is:

$$y(x) = \frac{-3 - (-1)Ce^{-1(-1-(-3))x}}{1 - Ce^{-1(-1-(-3))x}} = \frac{-3 + Ce^{-2x}}{1 - Ce^{-2x}}$$

Proposition 16

If the polynomial $-R\lambda^2 - Q\lambda + P$ has two equal real roots $\lambda_1 = \lambda_2 = \alpha$, then the solution of the Riccati equation expressed as:

$$y'(x) + Ry^2(x) + Qy(x) = P$$

is given by:

$$y(x) = \frac{-Q}{2R} + \frac{1}{C + Rx}$$

Proof

Now we rewrite (28) as:

$$y'(x) = -R(y(x) - \alpha)^2 \Rightarrow \frac{y'(x)}{(y(x) - \alpha)^2} = -R$$

We can easily calculate $y(x)$ by integrating each term of the equation or, we can apply what we have seen on subsection 4.2.1 because it is easy to see that $y(x) \equiv \alpha = \frac{-Q}{2R}$ is a particular solution. Integrating each term we get:

$$\int \frac{dy}{(y-\alpha)^2} = \int -Rdx \Rightarrow \frac{-1}{y-\alpha} = -Rx + C \Rightarrow y - \alpha = \frac{1}{Rx + C} \Rightarrow$$

$$y(x) = \alpha + \frac{1}{Rx + C} = \frac{-Q}{2R} + \frac{1}{Rx + C}$$

Now, applying subsection 4.2.1. :

$$y(x) = y_1(x) + \frac{1}{e^{\int 2y_1(x)r(x)+q(x)\,dx}(C + \int r(x)e^{-\int 2y_1(x)r(x)+q(x)\,dx}\,dx)} \Rightarrow$$

$$\Rightarrow y(x) = \frac{-Q}{2R} + \frac{1}{e^{\int 2\frac{-Q}{2R}R+Q\,dx}(C + \int Re^{-\int 2\frac{-Q}{2R}R+Q\,dx}\,dx)} = \frac{-Q}{2R} + \frac{1}{e^{\int 0\,dx}(C + \int Re^{-\int 0\,dx}\,dx)} \Rightarrow$$

$$\Rightarrow y(x) = \frac{-Q}{2R} + \frac{1}{C + Rx} \tag{30}$$

Let's apply it to a specific case where $R = -1$;$Q = 6$ and $P = 9$

Example 15

We are going to solve:

$$y'(x) - y^2(x) + 6y(x) = 9 \Rightarrow y'(x) = y^2(x) - 6y(x) + 9$$

The roots of $x^2 - 6x + 9$ are $\alpha = \beta = 3$. Applying now (30) the solution must be:

$$y(x) = \frac{-6}{-2} + \frac{1}{-x + C} = 3 - \frac{1}{x - C}$$

Proposition 17

If the polynomial $-R\lambda^2 - Q\lambda + P$ has two complex roots $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$, then the general solution of the Riccati equation expressed as:

$$y'(x) + Ry^2(x) + Qy(x) = P$$

is given by:

$$y(x) = \frac{\sqrt{-(Q^2+4RP)} \cdot tan\left(-\frac{\sqrt{-(Q^2+4RP)}}{2}x + C_1\right) - Q}{2R}$$

Proof

In this case both roots are complex so it is not easy to find particular solutions such as in the previous cases. We can rewrite the equation as:

$$y'(x) = -R(y(x) - (a+bi))(y(x) - (a-bi)) = -R((y(x) - a) - bi)((y(x) - a) + bi) \Rightarrow$$

$$\Rightarrow \frac{y'(x)}{(y(x) - a)^2 + b^2} = -R$$

Where $a = \frac{-Q}{2R}$ and $b = \frac{\sqrt{-(Q^2+4RP)}}{2R}$. Integrating each term we have:

$$\int \frac{dy}{(y-a)^2 + b^2} = \int -Rdx \Rightarrow \frac{1}{b} arctg\left(\frac{y-a}{b}\right) = -Rx + C \Rightarrow arctg\left(\frac{y-a}{b}\right) = -Rbx + C \Rightarrow$$

$$\Rightarrow \frac{y-a}{b} = tan\left(-Rbx + C\right) \Rightarrow y(x) = b \cdot tan\left(-Rbx + C\right) + a$$

Substituting the values of a and b:

$$y(x) = \frac{\sqrt{-(Q^2+4RP)}}{2R} tan\left(-R\frac{\sqrt{-(Q^2+4RP)}}{2R}x + C\right) - \frac{Q}{2R} \Rightarrow$$

$$\Rightarrow y(x) = \frac{\sqrt{-(Q^2+4RP)} \cdot tan\left(-\frac{\sqrt{-(Q^2+4RP)}}{2}x + C\right) - Q}{2R} \tag{31}$$

As in the two previous cases, let's check the solution in an example by taking $R = -1$, $Q = 2$ and $P = 2$

Example 16

We are going to solve:

$$y'(x) - y^2(x) + 2y(x) = 2 \Rightarrow y'(x) = y^2(x) - 2y(x) + 2$$

The roots of $\lambda^2 - 2\lambda + 2$ are $\alpha = 1 + i$ and $\beta = 1 - i$. Applying (31) the solution must be:

$$y(x) = \frac{\sqrt{-(2^2+4(-1)2)} \cdot tan\left(-\frac{\sqrt{-(2^2+4(-1)2)}}{2}x + C\right) - 2}{2R} = \frac{\sqrt{4} \cdot tan\left(-\frac{\sqrt{4}}{2}x + C\right) - 2}{-2} \Rightarrow$$

$$y(x) = \frac{2 \cdot tan(-x + C) - 2}{-2} = -tan(-x + C_1) + 1$$

4.2 Separable case seen on Rao's article

Solving the previous case will be useful for other separable cases when the part of the equation related with the unknown function has constant terms. As an example, the one that appears on [3]: We can assume $r(x) \geq 0$ in this section, if it is necessary we can apply the change of unknown function $y(x) = -y(x)$ We will start applying a change of unknown function on (4):

$$y(x) = u(x)v(x) - \frac{q(x)}{r(x)}; y^2(x) = u^2(x)v^2(x) + \frac{q^2(x)}{r^2(x)} - \frac{2u(x)v(x)q(x)}{r(x)};$$

$$y'(x) = u'(x)v(x) + u(x)v'(x) - \frac{q'(x)r(x) - r'(x)q(x)}{r^2(x)}$$

Substituting in (4):

$$u'(x)v(x) + u(x)v'(x) - \frac{q'(x)r(x) - r'(x)q(x)}{r^2(x)} + r(x)\left(u^2(x)v^2(x) + \frac{q^2(x)}{r^2(x)} - \frac{2u(x)v(x)q(x)}{r(x)}\right) +$$

$$+q(x)\left(u(x)v(x) - \frac{q(x)}{r(x)}\right) = p(x) \Rightarrow$$

$$\Rightarrow u'(x)v(x) + u(x)v'(x) - \frac{q'(x)r(x) - r'(x)q(x)}{r^2(x)} + u^2(x)v^2(x)r(x) + \frac{q^2(x)}{r(x)} - 2u(x)v(x)q(x)+$$

$$+u(x)v(x)q(x) - \frac{q^2(x)}{r(x)} = p(x) \Rightarrow$$

$$\Rightarrow u'(x)v(x) + u(x)v'(x) - \frac{q'(x)r(x) - r'(x)q(x)}{r^2(x)} + u^2(x)v^2(x)r(x) - u(x)v(x)q(x) = p(x) \Rightarrow$$

$$\Rightarrow u'(x)v(x)r^2(x)+u(x)v'(x)r^2(x)-q'(x)r(x)+r'(x)q(x)+u^2(x)v^2(x)r^3(x)-u(x)v(x)q(x)r^2(x) = p(x)r^2(x) \Rightarrow$$
$$\Rightarrow u'(x)v(x)r^2(x) = -u(x)v'(x)r^2(x)+q'(x)r(x)-r'(x)q(x)-u^2(x)v^2(x)r^3(x)+u(x)v(x)q(x)r^2(x)+p(x)r^2(x) \Rightarrow$$
$$\Rightarrow u'(x)v(x)r^2(x) = (-v^2(x)r^3(x))u^2(x)+r^2(x)(v(x)q(x)-v'(x))u(x)+q'(x)r(x)-r'(x)q(x)+p(x)r^2(x)$$

Now, taking $w(x) = q'(x)r(x) - r'(x)q(x) + p(x)r^2(x)$, we get:

$$u'(x)v(x)r^2(x) = (-v^2(x)r^3(x))u^2(x) + r^2(x)(v(x)q(x) - v'(x))u(x) + w(x) \tag{32}$$

Now we have three different cases, the first when we work on an interval in which $w(x) \equiv 0$, the second when $w(x) \geq 0$, and the third when $w(x) \leq 0$

4.2.1 $w \equiv 0$

In this case we can also choose $v(x)$ so that $v(x)q(x) - v'(x) = 0$, so (13) gets transformed into:

$$u'(x)v(x)r^2(x) = (-v^2(x)r^3(x))u^2(x) \Rightarrow \frac{u'(x)}{u^2(x)} = -v(x)r(x)$$

It is easy to see, applying the formula for first-order linear ODE, that $v = Ce^{\int q(x)dx}$. We will choose C=1. The equation (14) is the separable case we were searching for. It is easy to solve it just integrating:

$$\int \frac{du}{u^2} = \int -e^{\int q(x)dx} r(x)dx \Rightarrow -\frac{1}{u} = -\int e^{\int q(x)dx} r(x)dx + C \Rightarrow u(x) = \frac{1}{\int e^{\int q(x)dx} r(x)dx + C}$$

So undoing the change of unknown function:

$$y(x) = \frac{e^{\int q(x)dx}}{\int e^{\int q(x)dx} r(x)dx + C} - \frac{q(x)}{r(x)}$$

4.2.2 $w > 0$

In this case we apply the following change of unknown function:

$$v(x) = \sqrt{\frac{w(x)}{r^3(x)}}; \quad v'(x) = \frac{\sqrt{r(x)}}{2\sqrt{w(x)}} \frac{w'(x)r(x) - 3r'(x)w(x)}{r^3(x)}$$

Substituting in (32):

$$u'(x)\sqrt{\frac{w(x)}{r^3(x)}}r^2(x) = -w(x)u^2(x)+r^2(x)\left(\sqrt{\frac{w(x)}{r^3(x)}}q(x) - \frac{\sqrt{r(x)}}{2\sqrt{w(x)}}\frac{w'(x)r(x) - 3r'(x)w(x)}{r^3(x)}\right)u(x)+w(x) \Rightarrow$$

$$\Rightarrow u'(x)\sqrt{w(x)r(x)} = -w(x)u^2(x)+\left(\sqrt{w(x)r(x)}q(x) - \frac{\sqrt{r(x)}}{2\sqrt{w(x)}}\frac{w'(x)r(x) - 3r'(x)w(x)}{r(x)}\right)u(x)+w(x) \Rightarrow$$

$$\Rightarrow u'(x) = -\sqrt{\frac{w(x)}{r(x)}}u^2(x) + \left(q(x) - \frac{w'(x)r(x) - 3r'(x)w(x)}{2w(x)r(x)}\right)u(x) + \sqrt{\frac{w(x)}{r(x)}} \Rightarrow$$

$$\Rightarrow u'(x) = -\sqrt{\frac{w(x)}{r(x)}}\left(u^2(x) - \frac{2q(x)r(x)w(x) - w'(x)r(x) + 3r'(x)w(x)}{2w(x)\sqrt{w(x)r(x)}}u(x) - 1\right)$$

Assuming that:

$$\frac{2q(x)r(x)w(x) - w'(x)r(x) + 3r'(x)w(x)}{2w(x)\sqrt{w(x)r(x)}} = K \tag{33}$$

being K a constant value we get the separable equation that we were searching for:

$$u'(x) = -\sqrt{\frac{w(x)}{r(x)}}(u^2(x) - Ku(x) - 1)$$

As we have seen, the solution depends on the roots of $\lambda^2 - K\lambda - 1$ that are $\alpha = \frac{K+\sqrt{K^2+4}}{2}$ and $\beta = \frac{K-\sqrt{K^2+4}}{2}$. As $K^2+4 > 4 > 0$ we will always have two different real roots. Applying Proposition 12 we get that:

$$y(x) = \frac{\beta - \alpha C e^{\int \sqrt{\frac{w(x)}{r(x)}}(\sqrt{K^2+4})\,dx}}{1 - Ce^{\int \sqrt{\frac{w(x)}{r(x)}}(\sqrt{K^2+4})\,dx}}$$

Let us apply it to an example:

Example 17

We are going to solve:

$$y'(x) + e^{2x}y^2(x) - 2y(x) = e^{-2x}$$

Let us first calculate the value of $w(x)$ in order to verify that $w(x) > 0$:

$$w(x) = q'(x)r(x) - r'(x)q(x) + p(x)r^2(x) = 5e^{2x} > 0$$

Let us check the condition (33):

$$\frac{2q(x)r(x)w(x) - w'(x)r(x) + 3r'(x)w(x)}{2w(x)\sqrt{w(x)r(x)}} = \frac{-20e^{4x} - 10e^{4x} + 30e^{4x}}{10e^{4x}\sqrt{5}} = 0 = K$$

So we get:

$$u'(x) = -\sqrt{5}(u^2(x) - 1)$$

Now that we know that the conditions are met, let's look for the root of $\lambda^2 - 1$, which are: $\alpha = 1$ and $\beta = -1$. Applying our formula we find that the solution is:

$$u(x) = \frac{1 + Ce^{\int \sqrt{5}(-2)\,dx}}{1 - Ce^{\int \sqrt{5}(-2)\,dx}} = \frac{1 + Ce^{-2\sqrt{5}x}}{1 - Ce^{-2\sqrt{5}x}}$$

Undoing the change of unknown function:

$$y(x) = u(x)\sqrt{5}e^{-2x} + 2e^{-2x}$$

4.2.3 $w < 0$

In this case we choose v so that:

$$v(x) = \sqrt{\frac{w(x)}{-r^3(x)}}$$

Substituting in (32), and applying the same reasoning that we have applied in the previous case we get that: :

$$u'(x) = \sqrt{\frac{w(x)}{-r(x)}}(u^2(x) - Ku(x) + 1)$$

Where K is a constant defined as:

$$K = \frac{3r'(x)w(x) + 2q(x)r(x)w(x) - r(x)w'(x)}{2\sqrt{r(x)(-w(x))^3}} \tag{34}$$

Depending on the roots of $\lambda^2 - K\lambda + 1$. We will reach the solution applying the same reasoning that we have applied when $w > 0$. Let us apply it to an example:

Example 18

We are going to solve:

$$y'(x) - e^{-2x}y^2(x) - 2y(x) = e^{2x}$$

As $r < 0$ we will apply the mentioned change of unknown function $\hat{y} = -y$ leading to:

$$\hat{y}'(x) + e^{-2x}\hat{y}^2(x) - 2\hat{y}(x) = -e^{2x}$$

Let us first calculate the value of $w(x)$:

$$w(x) = q'(x)r(x) - r'(x)q(x) + p(x)r^2(x) = -5e^{-2x} < 0$$

Let us check that (34)is met:

$$\frac{3r'(x)w(x) + 2q(x)r(x)w(x) - r(x)w'(x)}{2\sqrt{r(x)(-w(x))^3}} = \frac{30e^{-4x} + 20e^{-4x} - 10e^{-4x}}{2\sqrt{(5e^{-2x})^3e^{-2x}}} = \frac{4}{\sqrt{5}} = K$$

So we get:

$$u'(x) = \sqrt{5}(u^2(x) - Ku(x) + 1)$$

Now that we know that the conditions are met, let's look for the solution of our equation. We need to calculate the roots of the above polynomial, which are: $\alpha = \frac{2+i}{\sqrt{5}}$ and $\beta = \frac{2-i}{\sqrt{5}}$. Applying Proposition 17 we find that the solution is:

$$u(x) = \frac{2 + tan(x + C)}{\sqrt{5}}$$

So undoing the unknown function changes:

$$y(x) = e^{2x}tan(x + C)$$

4.3 Separable case by Allen and Stein

In [4] it is showed a simpler change of unknown function that also ends in a separable ODE case. Assuming that $\frac{p}{-r} > 0$. This change of unknown function is:

$$y(x) = \sqrt{\frac{p(x)}{-r(x)}}u(x); \qquad y^2(x) = \frac{p(x)}{-r(x)}u^2(x)$$

$$y'(x) = \frac{1}{2}\sqrt{\frac{-r(x)}{p(x)}}\frac{r'(x)p(x) - p'(x)r(x)}{r^2(x)}u(x) + u'(x)\sqrt{\frac{p(x)}{-r(x)}}$$

Substituting in (4):

$$\frac{1}{2}\sqrt{\frac{-r(x)}{p(x)}}\frac{r'(x)p(x) - p'(x)r(x)}{r^2(x)}u(x) + u'(x)\sqrt{\frac{p(x)}{-r(x)}} + r(x)\left(\frac{p(x)}{-r(x)}u^2(x)\right) + q(x)\left(\sqrt{\frac{p(x)}{-r(x)}}u(x)\right) = p(x) \Rightarrow$$

$$\Rightarrow u'(x)\sqrt{\frac{p(x)}{-r(x)}} = p(x)u^2(x) - \left(\sqrt{\frac{p(x)}{-r(x)}}q(x) + \frac{1}{2}\sqrt{\frac{-r(x)}{p(x)}}\frac{r'(x)p(x) - p'(x)r(x)}{r^2(x)}\right)u(x) + p(x) \Rightarrow$$

$$\Rightarrow u'(x) = \sqrt{-r(x)p(x)}u^2(x) - \left(q(x) + \frac{p'(x)r(x) - r'(x)p(x)}{-2p(x)r(x)}\right)u(x) + \sqrt{-r(x)p(x)} \Rightarrow$$

$$\Rightarrow u'(x) = \sqrt{-r(x)p(x)}\left(u^2(x) - \left(\frac{2q(x)p(x)r(x) + p'(x)r(x) - r'(x)p(x)}{2p(x)r(x)\sqrt{-p(x)r(x)}}\right)u(x) + 1\right) \Rightarrow$$

If we suppose that

$$\frac{2q(x)p(x)r(x)+p'(x)r(x)-r'(x)p(x)}{2p(x)r(x)\sqrt{-p(x)r(x)}}=K \tag{35}$$

being K constant we get:

$$u'(x)=\sqrt{-r(x)p(x)}(u^2(x)-Ku(x)+1)$$

Now applying the same reasoning as in Rao's case we will find that the roots of the polynomial $\lambda^2-K\lambda+1$ which are $\frac{K\pm\sqrt{K^2-4}}{2}$.As in the case of Rao, we can find ourselves in three different scenarios, two real roots, two complex roots or two equal roots. Let us apply it to an example:

Example 19

We are going to solve:

$$y'(x)-e^{-x^{\frac{4}{3}}}y^2(x)-\left(1+\frac{4}{3}x^{\frac{1}{3}}\right)y(x)=e^{x^{\frac{4}{3}}}$$

First we need to verify that $\frac{p(x)}{-r(x)}\geq 0$

$$\frac{p(x)}{-r(x)}=\frac{e^{x^{\frac{4}{3}}}}{e^{-x^{\frac{4}{3}}}}=e^{2x^{\frac{4}{3}}}>0 \quad \forall x\in\mathbb{R}$$

In [4] it is said that $\frac{2q(x)p(x)r(x)+p'(x)r(x)-r'(x)p(x)}{2p(x)r(x)\sqrt{-p(x)r(x)}}=-1$ .Now that we know that the conditions are met, let's look for the solution of our equation as we did in the Rao case. We need to calculate the roots of the polynomial $\lambda^2+\lambda+1$, which are:$\alpha=\frac{-1+i\sqrt{3}}{2}$ and $\beta=\frac{-1-i\sqrt{3}}{2}$. Applying Proposition 17 we get that:

$$u(x)=\frac{\sqrt{3}\cdot tan\left(\frac{\sqrt{3}}{2}x+C\right)-1}{2}$$

Undoing the unknown function change:

$$y(x)=e^{x^{\frac{4}{3}}}u(x)$$

5 Applications of the Riccati equation

We have already seen some applications for the Riccati equation, such as solving second-order homogeneous linear ODE, or solving homogeneous system of ODE or solving the parachutist equation. In this section, we will see more important applications of the Riccati equation.

5.1 The Schrödinger equation

The Schrödinger equation is the most important PDE in quantum mechanics, and it is expressed as follows in the one-dimensional space case:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t) \tag{36}$$

Deduced by the Austrian physicist Erwin Schrödinger in the 1920s, it represents particles in sling functions and seeks the conservation of energy.

The Riccati equation appears in the earch of solution for the Schrödinger equation in some particular cases. In the study of the time-independent Schrödinger equation and in the search for a solution to the general Schrodinger equation.

We are going to try to find a solution for it as we did for the heat equation in the subject "Introduction to Partial Differential Equations", by separation of variables. We are going to look for a solution of the form:

$$\psi(x,t) = G(t)F(x) \Rightarrow \frac{\partial\psi(x,t)}{\partial t} = F(x)G'(t)\ ; \quad \frac{\partial^2\psi(x,t)}{\partial x^2} = F''(x)G(t)$$

Substituting in (36):

$$i\hbar F(x)G'(t) = -\frac{\hbar^2}{2m}F''(x)G(t) + V(x,t)F(x)G(t)$$

Assuming that $F(x) \neq 0$ and $G(x) \neq 0$, and dividing by $F(x)G(t)$ each term we get:

$$i\hbar\frac{G'(t)}{G(t)} = -\frac{\hbar^2}{2m}\frac{F''(x)}{F(x)} + V(x,t)$$

Now in order to apply the following reasoning we need $V(x,t)$ to be either a function of t or a function of x.So let's solve the concrete case where the potential is the harmonic oscillator, where $V(x,t)$ is defined as a function of x: $V(x,t) = \frac{mw^2}{2}x^2$, where w is 2π times the classical oscillation frequency. Substituting into our equation:

$$i\hbar\frac{G'(t)}{G(t)} = -\frac{\hbar^2}{2m}\frac{F''(x)}{F(x)} + \frac{mw^2}{2}x^2$$

Now we have an equation where the term on the left side depends only on the variable t and the term on the right side depends only on the variable x. So, for equality to occur, both terms must be equal to a constant which we will call $-\lambda$. And we get two "new" equations:

$$i\hbar\frac{G'(t)}{G(t)} = E \Rightarrow G'(t) + \frac{-iE}{\hbar}G(t) = 0 \tag{37}$$

$$-\frac{\hbar^2}{2m}\frac{F''(x)}{F(x)} + \frac{mw^2}{2}x^2 = E \Rightarrow F''(x) - \left(\frac{m^2w^2}{\hbar^2}x^2 - \frac{2m}{\hbar^2}E\right)F(x) = 0 \tag{38}$$

(37) is a first-order linear equation whose solution can be calculated by applying (3):

$$G(t) = Ce^{\int\frac{-iE}{\hbar}dt} = Ce^{\frac{-iE}{\hbar}t}$$

(38) is an homogeneous second-order linear equation that is easier to solve if we transform it into his associated Riccati simplified equation using the transformation: $y(x) = \frac{F'(x)}{F(x)}$

$$y'(x) + y^2(x) = \frac{m^2w^2}{\hbar^2}x^2 - \frac{2m}{\hbar^2}E$$

Let us now look for particular solutions of the form: $F(x) = \alpha x + \beta$. Substituting in our equation we have:

$$\alpha + \alpha^2 x^2 + 2\alpha\beta x + \beta^2 = \frac{m^2 w^2}{\hbar^2} x^2 - \frac{2m}{\hbar^2} E \Rightarrow \alpha = \frac{-mw}{\hbar}; \;\; \beta = 0$$

this equation is only verified if $E = \frac{w\hbar}{2}$ Applying the transformation $y(x) = -\frac{mw}{\hbar}x + \frac{1}{u(x)}$ we get to the following first-order linear equation:

$$u'(x) = -\frac{2mw}{\hbar} x u(x) + 1$$

we calculate $u(x)$ using (3)

$$u(x) = e^{-\frac{mw}{\hbar}x^2} \left(C + \int e^{\frac{mw}{\hbar}x^2} dx \right)$$

undoing the transformation we get:

$$y(x) = -\frac{mw}{\hbar}x + \frac{e^{\frac{mw}{\hbar}x^2}}{C + \int e^{\frac{mw}{\hbar}x^2} dx}$$

and undoing the first transformation:

$$F(x) = Ae^{\int \left(-\frac{mw}{\hbar}x + \frac{e^{\frac{mw}{\hbar}x^2}}{C + \int e^{\frac{mw}{\hbar}x^2} dx} \right) dx} = Ae^{-\frac{mw}{2\hbar}x^2} e^{log(C + \int e^{\frac{mw}{\hbar}x^2} dx)} = Ae^{-\frac{mw}{2\hbar}x^2} (C + \int e^{\frac{mw}{\hbar}x^2} dx) \Rightarrow$$

$$F(x) = e^{-\frac{mw}{2\hbar}x^2} (C_1 + C_2 \int e^{\frac{mw}{\hbar}x^2} dx)$$

where A is another arbitrary constant. In order for our solution to make physical sense, they must be bounded when $|x| \to \pm\infty$.That is why we have to take $C_2 = 0$. So for $E_0 = \frac{w\hbar}{2}$:

$$F_0(x) = Ce^{-\frac{mw}{2\hbar}x^2}$$

To determine the remaining energy levels we will look for solutions of the form:

$$F(x) = e^{-\frac{mw}{2\hbar}x^2} H \left(\sqrt{\frac{mw}{\hbar}} x \right) \Rightarrow$$

$$\Rightarrow F''(x) = \left(\frac{mw}{\hbar} H'' \left(\sqrt{\frac{mw}{\hbar}} x \right) - 2x \left(\frac{mw}{\hbar} \right)^{\frac{3}{2}} H' \left(\sqrt{\frac{mw}{\hbar}} x \right) + \left(\frac{m^2 w^2}{\hbar^2} x^2 - \frac{mw}{\hbar} \right) H \left(\sqrt{\frac{mw}{\hbar}} x \right) \right) e^{\frac{-mw}{2\hbar}x^2}$$

Substituting in (38) and simplifying we get:

$$\frac{\hbar^2}{2m} \left(\frac{mw}{\hbar} H'' \left(\sqrt{\frac{mw}{\hbar}} x \right) - 2x \left(\frac{mw}{\hbar} \right)^{\frac{3}{2}} H' \left(\sqrt{\frac{mw}{\hbar}} x \right) + \left(\frac{m^2 w^2}{\hbar^2} x^2 - \frac{mw}{\hbar} \right) H \left(\sqrt{\frac{mw}{\hbar}} x \right) \right) = \left(\frac{mw^2}{2} x^2 - E \right) \Longleftrightarrow$$

$$\Longleftrightarrow \frac{hw}{2} H'' \left(\sqrt{\frac{mw}{\hbar}} x \right) - x\sqrt{w^3 \hbar m} H' \left(\sqrt{\frac{mw}{\hbar}} x \right) + \left(E - \frac{\hbar w}{2} \right) H \left(\sqrt{\frac{mw}{\hbar}} x \right) = 0 \Longleftrightarrow$$

taking $z = \sqrt{\frac{mw}{\hbar}} x$

$$\Longleftrightarrow H''(z) - 2zH'(z) + \left(\frac{2E}{\hbar w} - 1 \right) H(z) = 0$$

Which, if $\frac{2E_n}{\hbar w} - 1 = 2n$ with $n \in \mathbb{N}$, is the Hermite ODE studied in the degree.

This ODE has a solution only at these values, so $E_n = \left(n + \frac{1}{2}\right) \frac{w\hbar}{2}$ and it results that H(z) is a multiple of the Hermite polynomial of degree n. So:

$$F(x) = Ce^{-\frac{mw}{2\hbar}x^2} \mathrm{H}_n \left(\sqrt{\frac{mw}{\hbar}} x \right)$$

where H_n is the Hermite polynomial of degree n. And finally:

$$\psi(x,t) = \sum_{n=0}^{\infty} C_n e^{\frac{-i\left(n+\frac{1}{2}\right)}{\hbar}t} e^{-\frac{mw}{2\hbar}x^2} \mathrm{H}_n \left(\sqrt{\frac{mw}{\hbar}} x \right)$$

5.2 Optimal control problem

An optimal control problem is a type of mathematical problem that tries to find the best way to control a dynamic system in order to optimize a given objective function. We define the linear status equation , which determines the dynamical system, as:

$$(SE) \qquad x'(t) = a(t)x(t) + b(t)u(t); \;\; x(0) = x_0$$

With $a(t)$ and $b(t)$ continuous known functions defined on an interval [0,T]. The typical function to minimise in this type of problem is:

$$(CP) \qquad \underset{u(t)\in L^2[0,T]}{Minimize} \;\; J(u) = \frac{\alpha}{2}\int_0^T (x_u(t))^2 dt \; + \; \frac{\beta}{2}\int_0^T u^2(t)dt \; + \; \frac{\gamma}{2}x_u^2(T)$$

Where α, β and γ are known positive constants and $x_u(t)$ represents the only continuous solution in [0,T] of the equation of state for each $u \in L^2[0,T]$.

It can be shown that there is a unique solution $\overline{u}$ for the optimal control problem (CP) if $\beta > 0$, but we will focus on determine it by using the necessary first-order optimality conditions, assuming its existence.

First, let us obtain the necessary first-order optimality conditions. We will denote $\overline{x} = x_{\overline{u}}$ and $x_\lambda = x_{\lambda u+(1-\lambda)\overline{u}}$ with $u \in L^2[0,T]$ and $\lambda \in (0,1)$. We will call $\overline{x}$ the optimum state and $\overline{u}$ the optimum control. As $\overline{u}$ is the solution for (CP) then it verifies:

$$J(\overline{u}) \le J(\lambda u + (1-\lambda)\overline{u}) \qquad \forall\lambda \in [0,1] \quad \forall u \in L^2[0,T]$$

$$\iff \frac{\alpha}{2}\int_0^T (\overline{x}(t))^2 dt \; + \; \frac{\beta}{2}\int_0^T u^2(t)dt \; + \; \frac{\gamma}{2}\overline{x}^2(T) \le$$

$$\le \frac{\alpha}{2}\int_0^T (x_\lambda(t))^2 dt \; + \; \frac{\beta}{2}\int_0^T (\lambda u(t) + (1-\lambda)\overline{u}(t))^2(t)dt \; + \; \frac{\gamma}{2}(x_\lambda(T))^2 \iff$$

$$\iff \frac{\alpha}{2}\int_0^T (x_\lambda^2(t) - \overline{x}^2(t))dt + \frac{\beta}{2}\int_0^T (\lambda^2(u(t)-\overline{u}(t))^2 + 2\lambda(u(t)-\overline{u}(t))\overline{u}(t))dt + \frac{\gamma}{2}(x_\lambda^2(T) - \overline{x}^2(T)) \ge 0 \tag{39}$$

Now we define a new unknown function:

$$z(t) = \frac{x_\lambda(t) - \overline{x}(t)}{\lambda} \qquad \forall\lambda \in (0,1)$$

We know that:

$$z'(t) = a(t)z(t) + b(t)(u(t) - \overline{u}(t)); \quad z(0) = 0$$

And we can divide (39) by $\lambda \in (0,1)$ getting:

$$\frac{\alpha}{2}\int_0^T z(t)(x_\lambda(t)+\overline{x}(t))dt + \frac{\beta}{2}\int_0^T (\lambda(u(t)-\overline{u}(t))^2 + 2(u(t)-\overline{u}(t))\overline{u}(t))dt + \frac{\gamma}{2}(x_\lambda(T)+\overline{x}(T))z(T) \ge 0$$

taking the limit of this expression when $\lambda \to 0$ we arrive at:

$$\alpha\int_0^T z(t)(\overline{x}(t))dt \; + \; \beta\int_0^T (u(t) - \overline{u}(t))\overline{u}(t))dt \; + \; \gamma(\overline{x}(T))z(T) \ge 0 \tag{40}$$

Introducing now the adjoint state $p(t)$ as the unique solution of:

$$(ASE) \qquad -p'(t) = a(t)p(t) + \alpha\overline{x}(t) \; ; \quad p(T) = \gamma\overline{x}(T)$$

integrating by parts we have that:

$$\int_0^T z(t)p'(t)dt = z(t)p(t)|_{t=0}^{t=T} - \int_0^T p(t)z'(t)dt \underset{z(0)=0}{\iff} \int_0^T z(t)p'(t)dt + \int_0^T p(t)z'(t)dt = z(T)\gamma\overline{x}(T) \iff$$

$$\iff \int_0^T z(t)(-a(t)p(t) - \alpha\overline{x}(t))dt + \int_0^T p(t)(a(t)z(t) + b(t)(u(t) - \overline{u}(t)))dt = z(T)\gamma\overline{x}(T) \iff$$

$$\iff \alpha \int_0^T z(t)(\overline{x}(t))dt + z(T)\gamma\overline{x}(T) = \int_0^T p(t)(b(t)(u(t) - \overline{u}(t)))dt$$

Then (40) is equivalent to:

$$\int_0^T (p(t)b(t) + \beta\overline{u}(t))(u(t) - \overline{u}(t))dt \geq 0 \qquad \forall u \in L^2[0,T]$$

In particular, taking $u(t) = \overline{u}(t) + p(t)b(t) + \beta\overline{u}(t)$ we obtain:

$$\int_0^T (p(t)b(t) + \beta\overline{u}(t))^2 dt = 0 \Rightarrow$$

$$\Rightarrow \overline{u}(t) = \frac{-p(t)b(t)}{\beta} \tag{41}$$

Let us suppose now that the adjoint state has the form $p(t) = s(t)\overline{x}(t)$, with $s(t)$ an unknown function. This is known in the literature as feedback control.

Differentiating $p(t)$:

$$p'(t) = s'(t)\overline{x}(t) + s(t)\overline{x}'(t) \underset{(SE)}{=} s'(t)\overline{x}(t) + s(t)(a(t)x(t) + b(t)u(t))$$

Using now the (ASE):

$$-a(t)p(t) - \alpha\overline{x}(t) = s'(t)\overline{x}(t) + s(t)(a(t)x(t) + b(t)u(t))$$

Using (**??**):

$$s'(t)\overline{x}(t) + s(t)(a(t)x(t) + b(t)\frac{-p(t)b(t)}{\beta}) + a(t)p(t) + \alpha\overline{x}(t) = 0 \Rightarrow$$

Substituting again $p(t)$ by $s(t)\overline{x}(t)$ we get to a Riccati equation.

$$s'(t) - \frac{b^2(t)}{\beta}s^2(t) + 2a(t)s(t) = -\alpha$$

Let us apply to an example:

Example

In this example we will take $\alpha = \beta = 1$, $\gamma = 0$, $a(t) \equiv 0$ and $b(t) \equiv 1$ giving us the folling Riccati equation:

$$s'(t) - s^2(t) = -1$$

It is easy to see that $s_1(t) \equiv 1$ and$s_2(t) = -1$ are particular solutions of our Riccati equation because they are the roots of the polynomial $\lambda^2 - 1$. Applying proposition 12 we get that:

$$s(t) = \frac{Ce^{-2t} + 1}{Ce^{-2t} - 1}$$

Adding the final condition s(T)=0

$$s(t) = \frac{1 - e^{2(t-T)}}{1 + e^{2(t-T})}$$

Using (SE) with the optimal control $\overline{u}$(t):

$$\overline{x}(t) = x_0 \frac{e^t + e^{2T-t}}{1 + e^{2T}}$$

Using the the expression of the feedback control we get:

$$\overline{u}(t) = x_0 \frac{e^t - e^{2T-t}}{1 + e^{2T}}$$

References

[1] Amigó, J. M. (2008). La ecuación de Riccati. *SeMA Journal: Boletín de la Sociedad Española de Matemática Aplicada, No. 45*, pp. 147-170.

[2] Polyanin, A. EqWorld, the world of mathematical equations. *General Riccati equation.* https://eqworld.ipmnet.ru/en/solutions/ode/ode0123.pdf.

[3] Rao, P. R. P. (1962). The Riccati differential equation. The American Mathematical Monthly, Vol. 69, No. 10, p. 995.

[4] Allen, J.L and Stein, F.M (1964).On solutions of certain Riccati Differential Equations. The American Mathematical Monthly, Vol. 71, No. 10 , pp. 1113-1115.

[5] Haley, S. B. (1997). An underrated entanglement: Riccati and Schrödinger equations. American Journal of Physics, Vol. 65, No. 3, pp. 237–243.

[6] Hasegawa, K. (2001). The Riccati equation and its applications in physics (PhD thesis). Division of Mathematics and Natural Sciences, Reed College.

[7] Murphy, G. M. (1960). Ordinary differential equations and their solutions. D. Van Nostrand Company.