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Full Length Article

# Measure-preserving mappings from the unit cube to some symmetric spaces[☆]

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## Abstract

We construct measure-preserving mappings from the $d$-dimensional unit cube to the $d$-dimensional unit ball and the compact rank one symmetric spaces, namely the $d$-dimensional sphere, the real, complex, and quaternionic projective spaces, and the Cayley plane. We also give a procedure to generate measure-preserving mappings from the $d$-dimensional unit cube to product spaces and fiber bundles under certain conditions.


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## 1. Introduction and main results

Given two measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, a bijective mapping $\varphi\colon \Omega_1 \to \Omega_2$ is said to be *measure preserving* if both $\varphi$ and $\varphi^{-1}$ are measurable mappings and moreover

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$\mu_2(A) = \mu_1(\varphi^{-1}(A))$ for every $A \in \Sigma_2$; or, equivalently, if $\mu_1(A) = \mu_2(\varphi(A))$ for every $A \in \Sigma_1$. In this work, we look for measure-preserving smooth diffeomorphisms between Riemannian manifolds. With a slight abuse of notation, we sometimes remove a null set from $\Omega_1$ or $\Omega_2$.

The problem of finding measure-preserving mappings from one manifold to another has applications in cartography, computer graphics, medical imaging, signal processing, or, more generally, in any area that requires good discretizations of a certain space. Thus, when looking for uniform collections of points or uniform grids (that is, grids all of whose cells have the same volume) on a manifold $\mathcal{M}$, a frequent approach consists in generating collections or grids with that property on a simpler, easily discretizable space such as the unit cube, and then transporting them to $\mathcal{M}$ through a measure-preserving mapping. In this sense, most of the research has been carried out for two-dimensional and three-dimensional manifolds (see [12–15,18–20] and references therein). In [18], the authors also obtained a mapping from the $n$-dimensional sphere of radius $r$ in $\mathbb{R}^{n+1}$ to the $n$-dimensional ball of radius $R$ in $\mathbb{R}^n$ that generalizes the equal-area Lambert mapping.

Measure-preserving mappings are also relevant in the theory of partial differential equations on Lipschitz domains (see [11]), in the generation of low-discrepancy points (see, for example, [1,5,7,9,10]), and, more recently, they have been used to generate projective constellations for noncoherent communications over single-input-multiple-output (SIMO) channels; see [17], where the authors constructed a measure-preserving mapping from the unit square to the complex projective line $\mathbb{CP}^1$, or [6] for the higher dimensional case. However, to the best of our knowledge, there are no constructive procedures to generate measure-preserving mappings from the $d$-dimensional unit cube to the $d$-sphere and to the remaining projective spaces.

### 1.1. Notation

In this paper, $\lambda$ denotes the Lebesgue measure in $\mathbb{R}^d$, and $B^d(0, R)$ denotes the open ball of radius $R \in (0, \infty]$ in $\mathbb{R}^d$ (if $R = \infty$, this means just $\mathbb{R}^d$). When $R = 1$, we denote it by $\mathbb{B}^d$. We will call $(0, 1)^d$ the (open) unit cube.

We denote the measure associated to the normal distribution $\mathcal{N}(0, c)$ in $\mathbb{R}^d$ by $\mu_c$, that is,

$$d\mu_c(x) = \frac{1}{(2\pi c)^{d/2}} e^{-\|x\|^2/(2c)}\, d\lambda(x).$$

It is well known that the mapping

$$\begin{aligned} \Phi_{\mathbb{R}^d}: \quad (0,1)^d \quad &\longrightarrow \quad \mathbb{R}^d, \\ \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \quad &\longmapsto \quad \begin{pmatrix} \sqrt{2}\,\mathrm{erf}^{-1}(2x_1 - 1) \\ \vdots \\ \sqrt{2}\,\mathrm{erf}^{-1}(2x_d - 1) \end{pmatrix}, \end{aligned} \tag{1}$$

is measure preserving from $((0, 1)^d, \lambda)$ to $(\mathbb{R}^d, \mu_{c=1})$, where $\mathrm{erf}^{-1}$ is the inverse of the error function $\mathrm{erf}\colon \mathbb{R} \to (-1, 1)$ given by

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\, ds.$$

Given any continuous function $\omega\colon (0, R) \to (0, \infty)$, we consider the associated measure in $B^d(0, R)$ given by the weight function $\omega(\|x\|)$ and denote it by $\mu_\omega$, that is,

$$d\mu_\omega(x) = \omega(\|x\|)\, d\lambda(x).$$

We will always assume that $\mu_\omega$ is a probability measure, i.e.,

$$1 = \int_{x \in B^d(0,R)} d\mu_\omega(x) = \int_{x \in B^d(0,R)} \omega(\|x\|)\, d\lambda(x) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^R \omega(s) s^{d-1}\, ds. \tag{2}$$

Finally, if we have a Riemannian manifold $\mathcal{M}$ (including the case that $\mathcal{M}$ is the unit cube with the standard structure, the usual $d$-sphere, or any open set of a compact Riemannian manifold), we denote by unif the uniform measure in $\mathcal{M}$ according to its volume form. For example, $(\mathbb{B}^d, \text{unif})$ is the unit ball endowed with the Lebesgue measure normalized to have volume 1, which can be denoted in our previous notation by $\mu_{\omega=1/\operatorname{vol}(\mathbb{B}^d)}$.

### 1.2. The compact rank one symmetric spaces

The compact rank one symmetric spaces (CROSSes) are the $n$-sphere $\mathbb{S}^n$ and the real, complex, quaternionic, and octonionic projective spaces $\mathbb{RP}^n, \mathbb{CP}^n$, $\mathbb{HP}^n$, and $\mathbb{OP}^2$. These spaces, which were classified by É. Cartan, are examples of locally harmonic Blaschke manifolds; in fact, Lichnerowicz's conjecture claims that the CROSSes are the only Riemannian manifolds of this kind. They are also the only compact connected two-point homogeneous Riemannian manifolds. See [4] for more information about these spaces.

Let $\mathcal{M}$ be a CROSS and let $d$, $D$, and $V$ be, respectively, its real dimension, its diameter (that is, the maximum Riemannian distance between two points in $\mathcal{M}$), and its volume. The exponential map based on the north pole

$$\begin{array}{rccc} \exp_{\mathcal{M}}: & B^d(0,D) & \longrightarrow & \mathcal{M}, \\ & v & \longmapsto & \exp_{(0,\ldots,0,1)}(v), \end{array}$$

is a diffeomorphism onto $\mathcal{M} \setminus \mathcal{X}$, where $\mathcal{X}$ is a measure zero set (just a point in the case $\mathcal{M} = \mathbb{S}^n$ and a hyperplane for the projective spaces). Moreover, the absolute value of the Jacobian of $\exp_{\mathcal{M}}$, which we simply denote by $\operatorname{Jac}\exp_{\mathcal{M}}$, is known as the volume density and has the form $\Omega(\|v\|)$ for a certain function $\Omega$. As a consequence, we have the following lemma:

**Lemma 1.1.** *Let $\mathcal{M}$ be a* CROSS. *Then, the exponential map $\exp_{\mathcal{M}}$ is a measure-preserving mapping from $(B^d(0,D), \mu_{\omega=\Omega/V})$ to $(\mathcal{M}, \text{unif})$.*

**Proof.** Since $\exp_{\mathcal{M}}$ is a smooth diffeomorphism, both $\exp_{\mathcal{M}}$ and its inverse are measurable mappings. Now, let $\mathcal{A} \subseteq B^d(0,D)$ be a measurable set. Applying the change of variables theorem to $\exp_{\mathcal{M}}$, we have

$$\begin{aligned} \mu_{\omega=\Omega/V}(\mathcal{A}) &= \int_{x \in B^d(0,D)} \chi_{\mathcal{A}}(x)\, d\mu_{\omega=\Omega/V}(x) \\ &= \int_{x \in B^d(0,D)} \chi_{\mathcal{A}}(x) \frac{\Omega(\|x\|)}{\operatorname{vol}(\mathcal{M})}\, d\lambda(x) \\ &= \int_{x \in B^d(0,D)} \chi_{\mathcal{A}}(x) \frac{\operatorname{Jac}\exp_{\mathcal{M}}(x)}{\operatorname{vol}(\mathcal{M})}\, d\lambda(x) \\ &= \frac{1}{\operatorname{vol}(\mathcal{M})} \int_{y \in \mathcal{M}} \chi_{\mathcal{A}}(\exp_{\mathcal{M}}^{-1}(y))\, dy \\ &= \frac{1}{\operatorname{vol}(\mathcal{M})} \int_{y \in \mathcal{M}} \chi_{\exp_{\mathcal{M}}(\mathcal{A})}(y)\, dy \\ &= \operatorname{unif}(\exp_{\mathcal{M}}(\mathcal{A})). \quad \square \end{aligned}$$

**Table 1**
The volume density in the CROSSes is $\omega_p(q) = \Omega(r)$, where $r = d_R(p, q)$ is the Riemannian distance. In this table, we show $r^{d-1}\Omega(r)$, where $d = \dim_{\mathbb{R}}(\mathcal{M})$, for the CROSSes. We also include the diameter $D = \mathrm{diam}(\mathcal{M})$, the volume $V = \mathrm{vol}(\mathcal{M})$, and the exponential map $\exp_{\mathcal{M}}$.
*Source:* Table taken from [3].

| $\mathcal{M}$ | $d$ | $D$ | $V$ | $\exp_{\mathcal{M}}(v)$ | $r^{d-1}\Omega(r)$ |
|---|---|---|---|---|---|
| $\mathbb{S}^n$ | $n$ | $\pi$ | $\frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$ | $\begin{pmatrix} \frac{v}{\|v\|} \sin \|v\| \\ \cos \|v\| \end{pmatrix}$ | $\sin^{n-1} r$ |
| $\mathbb{RP}^n$ | $n$ | $\pi/2$ | $\frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$ | $\begin{pmatrix} \frac{v}{\|v\|} \tan \|v\| \\ 1 \end{pmatrix}$ | $\sin^{n-1} r$ |
| $\mathbb{CP}^n$ | $2n$ | $\pi/2$ | $\frac{\pi^n}{n!}$ | $\begin{pmatrix} \frac{v}{\|v\|} \tan \|v\| \\ 1 \end{pmatrix}$ | $\sin^{2n-1} r \cos r$ |
| $\mathbb{HP}^n$ | $4n$ | $\pi/2$ | $\frac{\pi^{2n}}{(2n+1)!}$ | $\begin{pmatrix} \frac{v}{\|v\|} \tan \|v\| \\ 1 \end{pmatrix}$ | $\sin^{4n-1} r \cos^3 r$ |
| $\mathbb{OP}^2$ | 16 | $\pi/2$ | $\frac{\pi^8}{1320\,\Gamma(8)}$ | $\begin{pmatrix} \frac{v}{\|v\|} \tan \|v\| \\ 1 \end{pmatrix}$ | $\sin^{15} r \cos^7 r$ |

Table 1 summarizes the dimension, the diameter, the volume, the exponential map, and the volume density of the CROSSes.

### 1.3. Main results

Let $\gamma$ denote the incomplete gamma function:

$$\gamma(t, x) = \int_0^x s^{t-1} e^{-s}\, ds.$$

Our first main result is the following proposition, which yields a measure-preserving mapping from the unit cube to the unit ball:

**Proposition 1.2.** *Let $\varphi_{\mathbb{B}^d} : (\mathbb{R}^d, \mu_{c=1}) \to (\mathbb{B}^d, \mathrm{unif})$ be the mapping given by*

$$\varphi_{\mathbb{B}^d}(x) = \frac{x}{\|x\|} \left( \frac{\gamma\left(\frac{d}{2}, \frac{\|x\|^2}{2}\right)}{\Gamma(\frac{d}{2})} \right)^{1/d}.$$

*Then, the mapping $\Phi_{\mathbb{B}^d} = \varphi_{\mathbb{B}^d} \circ \Phi_{\mathbb{R}^d} : ((0, 1)^d, \mathrm{unif}) \to (\mathbb{B}^d, \mathrm{unif})$, where $\Phi_{\mathbb{R}^d}$ is as in Eq. (1), is measure preserving.*

The next main result provides a procedure to generate measure-preserving mappings from the unit cube to each CROSS.

**Theorem 1.3.** *Let $\mathcal{M}$ be a CROSS and let $\varphi_{\mathcal{M}} : (\mathbb{R}^d, \mu_{c=1}) \to (B^d(0, D), \mu_{\omega=\Omega/V})$ be the mapping given by $\varphi_{\mathcal{M}}(x) = x\rho(\|x\|)/\|x\|$, where $\rho = \rho(r)$, $\rho : [0, \infty) \to [0, D)$, is the unique solution to*

$$\int_0^\rho \omega(s) s^{d-1}\, ds = \frac{1}{2\pi^{d/2}} \gamma\left(\frac{d}{2}, \frac{r^2}{2}\right).$$

*Then, the mapping* $\Phi_{\mathcal{M}} = \exp_{\mathcal{M}} \circ \varphi_{\mathcal{M}} \circ \Phi_{\mathbb{R}^d} : ((0,1)^d, \mathrm{unif}) \to (\mathcal{M}, \mathrm{unif})$ *is measure preserving.*

The following commutative diagram illustrates the construction described in Theorem 1.3:

$$\begin{array}{ccc}
((0,1)^d, \mathrm{unif}) & \xrightarrow{\Phi_{\mathbb{R}^d}} & (\mathbb{R}^d, \mu_{c=1}) \\
\Big\downarrow{\scriptstyle \Phi_{\mathcal{M}}} & & \Big\downarrow{\scriptstyle \varphi_{\mathcal{M}}} \\
(\mathcal{M}, \mathrm{unif}) & \xleftarrow[\exp_{\mathcal{M}}]{} & (B^d(0, D), \mu_{\omega = \Omega/V})
\end{array}$$

It follows straightforwardly from the definition of measure-preserving mapping that, given two Riemannian manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, and two measure-preserving mappings

$$\Phi_{\mathcal{M}_1} : ((0,1)^{\dim(\mathcal{M}_1)}, \mathrm{unif}) \to (\mathcal{M}_1, \mathrm{unif}), \qquad \Phi_{\mathcal{M}_2} : ((0,1)^{\dim(\mathcal{M}_2)}, \mathrm{unif}) \to (\mathcal{M}_2, \mathrm{unif}),$$

the mapping

$$\begin{array}{rccc}
\Phi_{\mathcal{M}_1 \times \mathcal{M}_2} : & ((0,1)^{\dim(\mathcal{M}_1)+\dim(\mathcal{M}_2)}, \mathrm{unif}) & \longrightarrow & (\mathcal{M}_1 \times \mathcal{M}_2, \mathrm{unif}), \\
& (x, y) & \longmapsto & (\Phi_{\mathcal{M}_1}(x), \Phi_{\mathcal{M}_2}(y)),
\end{array}$$

where $x \in (0,1)^{\dim(\mathcal{M}_1)}$ and $y \in (0,1)^{\dim(\mathcal{M}_2)}$, is also measure preserving. As a consequence, since by Theorem 1.3 we have measure-preserving mappings from the unit cube to any CROSS, we also have a constructive procedure to generate measure-preserving mappings from the unit cube to any finite product of CROSSes. In this work we generalize this property to the case of fiber bundles making use of the Normal Jacobian NJac (see Appendix A for details):

**Theorem 1.4.** *Let $E$, $B$, and $F$ be Riemannian manifolds, where we assume that the measures in $E$, $B$, and $F$ are normalized to have unit volume, and let $F \hookrightarrow E \xrightarrow{\pi} B$ be a smooth fiber bundle such that* $\mathrm{NJac}\,\pi(x)$ *is constant for every $x \in E$. Let* $\Phi_B : ((0,1)^{\dim(B)}, \mathrm{unif}) \to (B, \mathrm{unif})$ *and* $\Phi_F : ((0,1)^{\dim(F)}, \mathrm{unif}) \to (F, \mathrm{unif})$ *be measure-preserving mappings. Let* $\Psi_y : (F, \mathrm{unif}) \to (\pi^{-1}(y), \mathrm{unif})$ *be a measure-preserving mapping for every $y \in B$ such that the mapping* $\xi : (B \times F, \mathrm{unif}) \to (E, \mathrm{unif})$ *given by* $\xi(y, z) = \Psi_y(z)$ *is measurable. Then, $\xi$ is measure preserving and hence the mapping*

$$\begin{array}{rccc}
\Phi_E : & ((0,1)^{\dim(E)}, \mathrm{unif}) & \longrightarrow & (E, \mathrm{unif}), \\
& (y, z) & \longmapsto & \Psi_{\Phi_B(y)}(\Phi_F(z)),
\end{array}$$

*where* $y \in (0,1)^{\dim(B)}$ *and* $z \in (0,1)^{\dim(F)}$*, is measure preserving.*

### *1.4. Structure of the paper*

In Section 2, we prove our main technical result, which yields a procedure to generate measure-preserving mappings from $(\mathbb{R}^d, \mu_c)$ to $(B^d(0, R), \mu_\omega)$, and we prove Proposition 1.2. In Section 3, we prove Theorem 1.3 and we construct measure-preserving mappings from the unit cube to each CROSS. In Section 4, we prove Theorem 1.4 and we show an alternative procedure to construct measure-preserving mappings from the unit cube to odd-dimensional spheres using the Hopf fibration. Appendix A is devoted to the smooth coarea formula, a technical tool. Finally, in Appendix B we present some auxiliary computations.

## 2. A technical result

Recall that $\omega\colon(0,R)\to(0,\infty)$ is any continuous function satisfying (2).

**Theorem 2.1.** *Let $G(\rho)=\int_0^\rho \omega(s)s^{d-1}\,ds$, and let $\rho=\rho(r)$ be the unique solution to*

$$G(\rho)=\frac{1}{2\pi^{d/2}}\gamma\left(\frac{d}{2},\frac{r^2}{2c}\right). \tag{3}$$

*Then, the mapping $\varphi\colon(\mathbb{R}^d,\mu_c)\to(B^d(0,R),\mu_\omega)$ given by $\varphi(x)=x\rho(\|x\|)/\|x\|$ is measure preserving.*

**Proof.** First, note that $\omega(r)>0$ implies that $G$ is an increasing function with $G(0)=0$. Moreover, Eq. (2) implies that $G(R)=\Gamma(d/2)/2\pi^{d/2}$, which means that $\rho$ is a well-defined bijection with $\lim_{r\to\infty}\rho(r)=R$. The inverse of $\varphi$ is easily computed: $\varphi^{-1}(y)=y\rho^{-1}(\|y\|)/\|y\|$.

Computing the derivative with respect to $r$ at both sides of (3), we get

$$\omega(\rho)\rho^{d-1}\rho'(r)=\frac{r^{d-1}}{(2\pi c)^{d/2}}e^{-r^2/(2c)}. \tag{4}$$

Now, let $f(r)=\rho(r)/r$ and compute the Jacobian of $\varphi(x)=xf(\|x\|)$ by choosing an orthonormal basis $v_1^x,\dots,v_d^x$ at $x\in\mathbb{R}^d$, with $v_1^x=x/\|x\|$. A straightforward computation shows that $D\varphi(x)$ preserves the orthogonality of that basis and yields

$$\begin{aligned}
\operatorname{Jac}\varphi(x) &= f(\|x\|)^{d-1}(f(\|x\|)+\|x\|f'(\|x\|))\\
&= \frac{\rho(\|x\|)^{d-1}}{\|x\|^{d-1}}\rho'(\|x\|)\\
&\overset{(4)}{=} \frac{1}{(2\pi c)^{d/2}\omega(\rho(\|x\|))}e^{-\|x\|^2/(2c)}.
\end{aligned}$$

Then, given any measurable set $\mathcal{A}\subseteq\mathbb{R}^d$, we can check that the measure of $\mathcal{A}$ in $(\mathbb{R}^d,\mu_c)$ equals that of $\varphi(\mathcal{A})$ in $(B^d(0,R),\mu_\omega)$ using the change of variables theorem: if we denote by $\chi_{\mathcal{A}}$ the characteristic function of $\mathcal{A}$, then

$$\begin{aligned}
\mu_c(\mathcal{A}) &= \int_{x\in\mathbb{R}^d}\chi_{\mathcal{A}}(x)\,d\mu_c(x)\\
&= \int_{x\in\mathbb{R}^d}\chi_{\mathcal{A}}(x)\frac{1}{(2\pi c)^{d/2}}e^{-\|x\|^2/(2c)}\,d\lambda(x)\\
&= \int_{x\in\mathbb{R}^d}\chi_{\mathcal{A}}(x)\omega(\rho(\|x\|))\operatorname{Jac}\varphi(x)\,d\lambda(x)\\
&= \int_{y\in B^d(0,R)}\chi_{\mathcal{A}}(\varphi^{-1}(y))\omega(\rho(\|\varphi^{-1}(y)\|))\,d\lambda(y)\\
&= \int_{y\in B^d(0,R)}\chi_{\varphi(\mathcal{A})}(y)\omega(\|y\|)\,d\lambda(y)\\
&= \int_{y\in B^d(0,R)}\chi_{\varphi(\mathcal{A})}(y)\,d\mu_\omega(y)\\
&= \mu_\omega(\varphi(\mathcal{A})),
\end{aligned}$$

which proves the theorem. □

**Example 2.2** (*A Measure-Preserving Mapping from* $(\mathbb{R}^d, \mu_c)$ *to* $(\mathbb{R}^d, \mu_b)$)**.** In this case, we have $R = \infty$ and

$$\omega(r) = \frac{e^{-r^2/(2b)}}{(2\pi b)^{d/2}}.$$

Following Theorem 2.1, let

$$G(\rho) = \int_0^\rho \omega(s)s^{d-1}\,ds = \frac{1}{(2\pi b)^{d/2}}\int_0^\rho s^{d-1}e^{-s^2/(2b)}\,ds = \frac{1}{2\pi^{d/2}}\gamma\left(\frac{d}{2}, \frac{\rho^2}{2b}\right).$$

We have to obtain $\rho$ from

$$\frac{1}{2\pi^{d/2}}\gamma\left(\frac{d}{2}, \frac{\rho^2}{2b}\right) = \frac{1}{2\pi^{d/2}}\gamma\left(\frac{d}{2}, \frac{r^2}{2c}\right),$$

which is obviously solved by $\rho(r) = r\sqrt{b/c}$. Thus, we conclude that

$$\varphi(x) = x\sqrt{\frac{b}{c}}$$

defines a measure-preserving mapping from $(\mathbb{R}^d, \mu_c)$ to $(\mathbb{R}^d, \mu_b)$.

**Example 2.3** (*A Measure-Preserving Mapping from* $(\mathbb{R}^d, \mu_{c=1})$ *to* $(\mathbb{R}^d, \mu_{\text{stereo}})$)**.** Let $\mu_{\text{stereo}}$ be the measure that makes the stereographic projection a measure-preserving mapping, that is,

$$d\mu_{\text{stereo}}(x) = \frac{1}{\text{vol}(\mathbb{S}^d)}\frac{2^d}{(1+\|x\|^2)^d}\,d\lambda(x).$$

In this case, we have $c = 1$, $R = \infty$, and

$$\omega(r) = \frac{1}{\text{vol}(\mathbb{S}^d)}\frac{2^d}{(1+r^2)^d} = \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}}\frac{2^d}{(1+r^2)^d}.$$

We compute

$$G(\rho) = \int_0^\rho \omega(s)s^{d-1}\,ds = \frac{2^{d-1}\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}\int_0^\rho \frac{s^{d-1}}{(1+s^2)^d}\,ds.$$

If $d = 2$,

$$G(\rho) = \frac{2\Gamma(\frac{3}{2})}{\pi^{3/2}}\int_0^\rho \frac{s}{(1+s^2)^2}\,ds = \frac{\rho^2}{2\pi(1+\rho^2)}.$$

Therefore, we have to obtain $\rho$ from

$$\frac{\rho^2}{2\pi(1+\rho^2)} = \frac{1}{2\pi}\gamma\left(1, \frac{r^2}{2}\right),$$

that is,

$$\rho^2 = \frac{1}{e^{-r^2/2}} - 1 = e^{r^2/2} - 1.$$

Hence, we conclude that

$$\varphi(x) = \frac{x}{\|x\|}\sqrt{e^{\|x\|^2/2} - 1}$$

defines a measure-preserving mapping from $(\mathbb{R}^2, \mu_{c=1})$ to $(\mathbb{R}^2, \mu_{\text{stereo}})$.

**Example 2.4** (*A Measure-Preserving Mapping from* $(\mathbb{R}^d, \mu_{c=1})$ *to* $(\mathbb{B}^d, \text{unif})$)**.** In this case, we have $c = 1$, $R = 1$, and

$$\omega(r) = \frac{1}{\text{vol}(\mathbb{B}^d)} = \frac{\Gamma(\frac{d}{2}+1)}{\pi^{d/2}} = \frac{d\Gamma(\frac{d}{2})}{2\pi^{d/2}}.$$

We easily compute

$$G(\rho) = \int_0^\rho \omega(s)s^{d-1}\,ds = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}\rho^d,$$

and we have to obtain $\rho$ from

$$\frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}\rho^d = \frac{1}{2\pi^{d/2}}\gamma\left(\frac{d}{2}, \frac{r^2}{2}\right),$$

concluding that, following the notation of Proposition 1.2,

$$\varphi_{\mathbb{B}^d}(x) = \frac{x}{\|x\|}\left(\frac{\gamma\left(\frac{d}{2}, \frac{\|x\|^2}{2}\right)}{\Gamma(\frac{d}{2})}\right)^{1/d}$$

defines a measure-preserving mapping from $(\mathbb{R}^d, \mu_{c=1})$ to $(\mathbb{B}^d, \text{unif})$.

**Proof of Proposition 1.2.** Immediate from Example 2.4 and the fact that $\Phi_{\mathbb{R}^d}$ is measure preserving. □

## 3. Measure-preserving mappings from the unit cube to the compact rank one symmetric spaces

After Theorem 2.1, the proof of Theorem 1.3 is now straightforward:

**Proof of Theorem 1.3.** Immediate from Theorem 2.1, Lemma 1.1, and the fact that $\Phi_{\mathbb{R}^d}$ is measure preserving. □

We can now generate measure-preserving mappings from the unit cube to all the CROSSes following Theorem 2.1: it suffices to consider $\exp_{\mathcal{M}} \circ \varphi_{\mathcal{M}} \circ \Phi_{\mathbb{R}^d}$, where, according to Theorem 2.1, the mapping $\varphi_{\mathcal{M}} : (\mathbb{R}^d, \mu_{c=1}) \to (B^d(0, D), \mu_{\omega=\Omega/V})$ can be computed, to some extent, explicitly. We do the computations for the different choices of $\mathcal{M}$ in the next few subsections. Recall that, for each CROSS $\mathcal{M}$, we denote its real dimension by $d$, its diameter by $D$, and its volume by $V$ (see Table 1).

### 3.1. The unit sphere $\mathbb{S}^n$

In this case, we have $d = n$, $D = \pi$, and

$$\omega(r) = \frac{\Omega(r)}{V} = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}}\frac{\sin^{n-1} r}{r^{n-1}}.$$

**Corollary 3.1.** *The mapping* $\varphi_{\mathbb{S}^n} : (\mathbb{R}^n, \mu_{c=1}) \to (B^n(0,\pi), \mu_{\omega=\Omega/V})$ *given by* $\varphi_{\mathbb{S}^n}(x) = x\rho(\|x\|)/\|x\|$ *is measure preserving if* $\rho = \rho(r)$, $\rho : [0,\infty) \to [0,\pi)$, *satisfies*

$$\int_0^\rho \sin^{n-1} s\, ds = \frac{\sqrt{\pi}}{\Gamma(\frac{n+1}{2})}\gamma\left(\frac{n}{2}, \frac{r^2}{2}\right).$$

*As a consequence, the mapping* $\Phi_{\mathbb{S}^n} = \exp_{\mathbb{S}^n} \circ \varphi_{\mathbb{S}^n} \circ \Phi_{\mathbb{R}^n} : ((0,1)^n, \mathrm{unif}) \to (\mathbb{S}^n, \mathrm{unif})$ *is measure preserving. For* $n = 1$ *we have*

$$\rho(r) = \sqrt{\pi}\gamma\left(\frac{1}{2}, \frac{r^2}{2}\right) = \pi\,\mathrm{erf}\left(\frac{r}{\sqrt{2}}\right),$$

*and so*

$$\Phi_{\mathbb{S}^1}(x) = (-\sin 2\pi x, -\cos 2\pi x) \cong -ie^{-i2\pi x}.$$

*For* $n = 2$ *we can compute* $\rho(r)$ *explicitly:*

$$\rho(r) = 2\arccos e^{-r^2/4},$$

*and hence*

$$\Phi_{\mathbb{S}^2}(x) = \left(\frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|}2e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/4}\sqrt{1-e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/2}}, 2e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/2} - 1\right).$$

**Proof.** From Theorem 2.1 we just need to check that

$$\int_0^\rho \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}}\sin^{n-1} s\, ds = \frac{1}{2\pi^{n/2}}\gamma\left(\frac{n}{2}, \frac{r^2}{2}\right),$$

which is equivalent to the formula in the corollary. The case $n = 2$ reads

$$\sin^2\frac{\rho}{2} = \frac{1-\cos\rho}{2} = \gamma\left(1, \frac{r^2}{2}\right) = 1 - e^{-r^2/2},$$

which is equivalent to the last claim in the corollary. □

Note that the integral on the left-hand side of the expression in Corollary 3.1 is an incomplete beta function:

$$\int_0^\rho \sin^{n-1} s\, ds = 2^{n-1}B_{\sin^2(\rho/2)}\left(\frac{n}{2}, \frac{n}{2}\right).$$

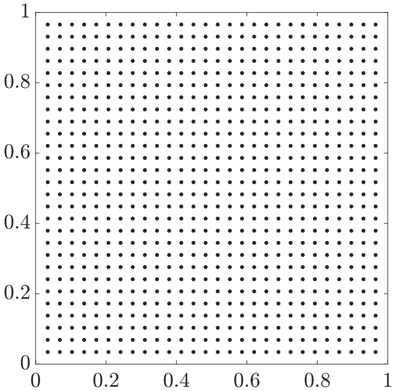
Hence, it is not possible to obtain a closed expression for $\Phi_{\mathbb{S}^n}$ when $n > 2$. In Section 4 we consider a different approach that provides measure-preserving mappings with closed expressions for odd-dimensional spheres.

Figs. 1 and 2 illustrate the measure-preserving mapping obtained in Corollary 3.1 for the particular case of the two-dimensional sphere $\mathbb{S}^2$.

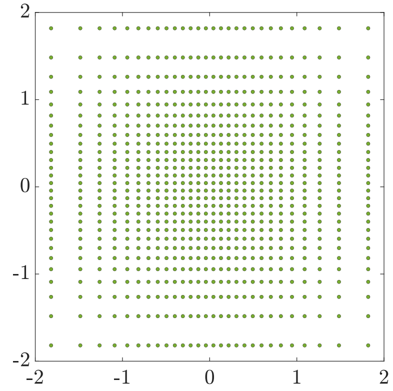
### *3.2. The real projective space* $\mathbb{RP}^n$

In this case, we have $d = n$, $D = \pi/2$, and

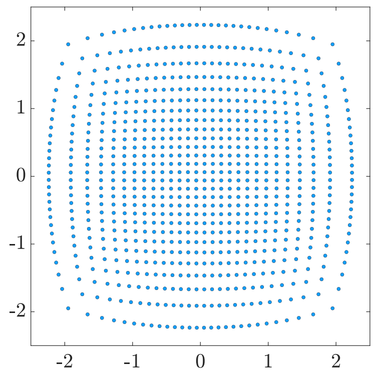
$$\omega(r) = \frac{\Omega(r)}{V} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}\frac{\sin^{n-1} r}{r^{n-1}}.$$



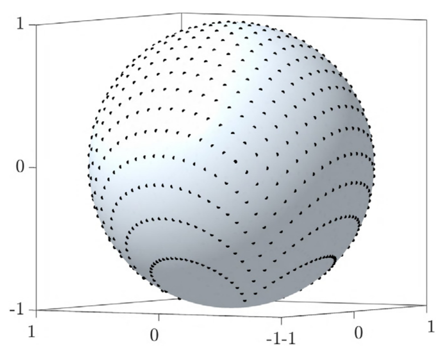

(A) Square mesh in $(0, 1)^2$



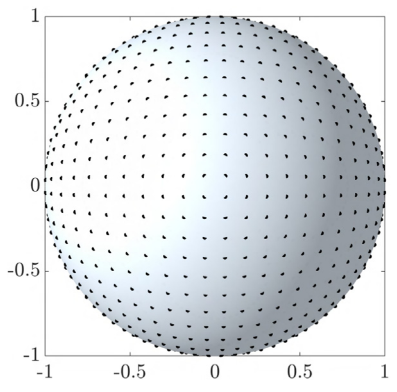

(B) Image of the mesh under $\Phi_{\mathbb{R}^2}$



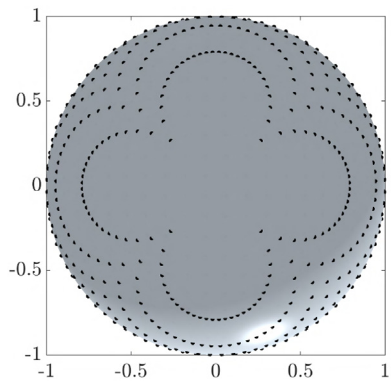

(C) Image of the mesh under $\varphi_{\mathbb{S}^2} \circ \Phi_{\mathbb{R}^2}$




(D) Image of the mesh under $\exp_{\mathbb{S}^2} \circ \varphi_{\mathbb{S}^2} \circ \Phi_{\mathbb{R}^2}$, lateral view

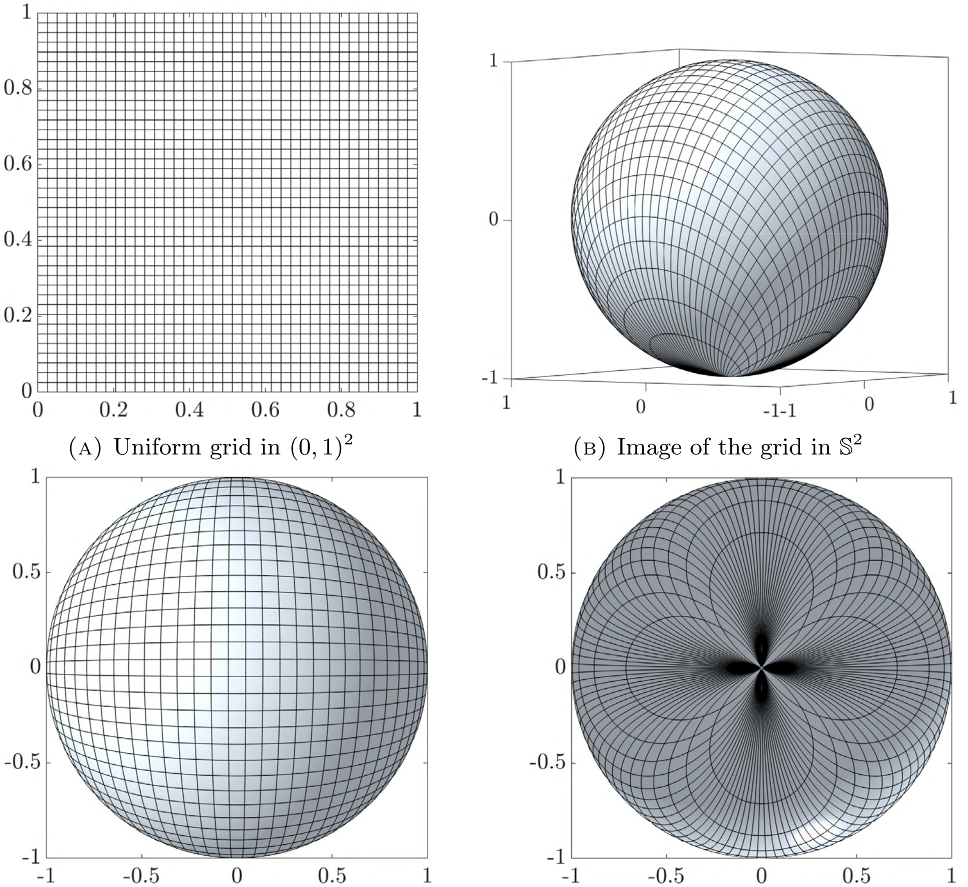



(E) Image of the mesh under $\exp_{\mathbb{S}^2} \circ \varphi_{\mathbb{S}^2} \circ \Phi_{\mathbb{R}^2}$, north pole view




(F) Image of the mesh under $\exp_{\mathbb{S}^2} \circ \varphi_{\mathbb{S}^2} \circ \Phi_{\mathbb{R}^2}$, south pole view

**Fig. 1.** The measure-preserving mapping $\Phi_{\mathbb{S}^2} = \exp_{\mathbb{S}^2} \circ \varphi_{\mathbb{S}^2} \circ \Phi_{\mathbb{R}^2} : ((0, 1)^2, \text{unif}) \to (\mathbb{S}^2, \text{unif})$ transforms points on $(0, 1)^2$ into points on $\mathbb{S}^2$. For an initial collection of 784 mesh points on $(0, 1)^2$, we show the different steps from the unit square to the sphere.



(A) Uniform grid in $(0,1)^2$

(B) Image of the grid in $\mathbb{S}^2$

(C) Image of the grid in $\mathbb{S}^2$, north pole view

(D) Image of the grid in $\mathbb{S}^2$, south pole view

**Fig. 2.** The measure-preserving mapping $\Phi_{\mathbb{S}^2} = \exp_{\mathbb{S}^2} \circ \varphi_{\mathbb{S}^2} \circ \Phi_{\mathbb{R}^2} : ((0,1)^2, \mathrm{unif}) \to (\mathbb{S}^2, \mathrm{unif})$ transforms uniform grids in $(0,1)^2$ into uniform grids in $\mathbb{S}^2$. We show the image of a grid in $(0,1)^2$ formed by 1521 cells.

**Corollary 3.2.** *The mapping* $\varphi_{\mathbb{RP}^n} : (\mathbb{R}^n, \mu_{c=1}) \to (B^n(0,\pi/2), \mu_{\omega=\Omega/V})$ *given by* $\varphi_{\mathbb{RP}^n}(x) = x\rho(\|x\|)/\|x\|$ *is measure preserving if* $\rho = \rho(r)$, $\rho : [0,\infty) \to [0,\pi/2)$, *satisfies*

$$\int_0^\rho \sin^{n-1} s \, ds = \frac{\sqrt{\pi}}{2\Gamma(\frac{n+1}{2})} \gamma\left(\frac{n}{2}, \frac{r^2}{2}\right).$$

*As a consequence, the mapping* $\Phi_{\mathbb{RP}^n} = \exp_{\mathbb{RP}^n} \circ \varphi_{\mathbb{RP}^n} \circ \Phi_{\mathbb{R}^n} : ((0,1)^n, \mathrm{unif}) \to (\mathbb{RP}^n, \mathrm{unif})$ *is measure preserving. For* $n = 1$ *we have*

$$\rho(r) = \frac{\sqrt{\pi}}{2} \gamma\left(\frac{1}{2}, \frac{r^2}{2}\right) = \frac{\pi}{2} \operatorname{erf}\left(\frac{r}{\sqrt{2}}\right),$$

*and so*

$$\Phi_{\mathbb{RP}^1}(x) = (-\cot \pi x, 1).$$

*For $n = 2$ we can compute $\rho(r)$ explicitly:*

$$\rho(r) = \arccos e^{-r^2/2},$$

*and hence*

$$\Phi_{\mathbb{RP}^2}(x) = \left( \frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|} \sqrt{e^{\|\Phi_{\mathbb{R}^2}(x)\|^2} - 1}, 1 \right).$$

**Proof.** From Theorem 2.1 we just need to check that

$$\int_0^\rho \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \sin^{n-1} s \, ds = \frac{1}{2\pi^{n/2}} \gamma\left(\frac{n}{2}, \frac{r^2}{2}\right),$$

which is equivalent to the formula in the corollary. The case $n = 2$ reads

$$1 - \cos\rho = \gamma\left(1, \frac{r^2}{2}\right) = 1 - e^{-r^2/2},$$

which is equivalent to the last claim in the corollary. □

### 3.3. *The complex projective space* $\mathbb{CP}^n$

In this case, we have $d = 2n$, $D = \pi/2$, and

$$\omega(r) = \frac{\Omega(r)}{V} = \frac{n!}{\pi^n} \frac{\sin^{2n-1} r \cos r}{r^{2n-1}}.$$

**Corollary 3.3.** *The mapping $\varphi_{\mathbb{CP}^n} : (\mathbb{R}^{2n}, \mu_{c=1}) \to (B^{2n}(0, \pi/2), \mu_{\omega=\Omega/V})$ given by $\varphi_{\mathbb{CP}^n}(x) = x\rho(\|x\|)/\|x\|$ is measure preserving if $\rho = \rho(r)$, $\rho : [0, \infty) \to [0, \pi/2)$, satisfies*

$$\rho(r) = \arcsin\left( \left( \frac{1}{(n-1)!} \gamma\left(n, \frac{r^2}{2}\right) \right)^{1/(2n)} \right).$$

*As a consequence, the mapping $\Phi_{\mathbb{CP}^n} = \exp_{\mathbb{CP}^n} \circ \varphi_{\mathbb{CP}^n} \circ \Phi_{\mathbb{R}^{2n}} : ((0, 1)^{2n}, \mathrm{unif}) \to (\mathbb{CP}^n, \mathrm{unif})$ is measure preserving.*

**Proof.** From Theorem 2.1 we just need to check that

$$\int_0^\rho \frac{n!}{\pi^n} \sin^{2n-1} s \cos s \, ds = \frac{1}{2\pi^n} \gamma\left(n, \frac{r^2}{2}\right),$$

which is equivalent to

$$\sin^{2n}\rho = \frac{1}{(n-1)!} \gamma\left(n, \frac{r^2}{2}\right),$$

and the corollary follows. □

### 3.4. *The quaternionic projective space* $\mathbb{HP}^n$

In this case, we have $d = 4n$, $D = \pi/2$, and

$$\omega(r) = \frac{\Omega(r)}{V} = \frac{(2n+1)!}{\pi^{2n}} \frac{\sin^{4n-1} r \cos^3 r}{r^{4n-1}}.$$

**Corollary 3.4.** *The mapping $\varphi_{\mathbb{HP}^n} : (\mathbb{R}^{4n}, \mu_{c=1}) \to (B^{4n}(0, \pi/2), \mu_{\omega=\Omega/V})$ given by $\varphi_{\mathbb{HP}^n}(x) = x\rho(\|x\|)/\|x\|$ is measure preserving if $\rho = \rho(r)$, $\rho : [0, \infty) \to [0, \pi/2)$, satisfies*

$$\int_0^\rho \sin^{4n-1} s \cos^3 s \, ds = \frac{1}{2(2n+1)!} \gamma\left(2n, \frac{r^2}{2}\right).$$

*As a consequence, the mapping $\Phi_{\mathbb{HP}^n} = \exp_{\mathbb{HP}^n} \circ \varphi_{\mathbb{HP}^n} \circ \Phi_{\mathbb{R}^{4n}} : ((0,1)^{4n}, \mathrm{unif}) \to (\mathbb{HP}^n, \mathrm{unif})$ is measure preserving.*

**Proof.** From Theorem 2.1 we just need to check that

$$\int_0^\rho \frac{(2n+1)!}{\pi^{2n}} \sin^{4n-1} s \cos^3 s \, ds = \frac{1}{2\pi^{2n}} \gamma\left(2n, \frac{r^2}{2}\right),$$

which is equivalent to the formula in the corollary. □

### 3.5. The Cayley plane $\mathbb{OP}^2$

In this case, we have $d = 16$, $D = \pi/2$, and

$$\omega(r) = \frac{\Omega(r)}{V} = \frac{1320\Gamma(8)}{\pi^8} \frac{\sin^{15} r \cos^7 r}{r^{15}}.$$

**Corollary 3.5.** *The mapping $\varphi_{\mathbb{OP}^2} : (\mathbb{R}^{16}, \mu_{c=1}) \to (B^{16}(0, \pi/2), \mu_{\omega=\Omega/V})$ given by $\varphi_{\mathbb{OP}^2}(x) = x\rho(\|x\|)/\|x\|$ is measure preserving if $\rho = \rho(r)$, $\rho : [0, \infty) \to [0, \pi/2)$, satisfies*

$$\int_0^\rho \sin^{15} s \cos^7 s \, ds = \frac{1}{2640\Gamma(8)} \gamma\left(8, \frac{r^2}{2}\right).$$

*As a consequence, the mapping $\Phi_{\mathbb{OP}^2} = \exp_{\mathbb{OP}^2} \circ \varphi_{\mathbb{OP}^2} \circ \Phi_{\mathbb{R}^{16}} : ((0,1)^{16}, \mathrm{unif}) \to (\mathbb{OP}^2, \mathrm{unif})$ is measure preserving.*

**Proof.** From Theorem 2.1 we just need to check that

$$\int_0^\rho \frac{1320\Gamma(8)}{\pi^8} \sin^{15} s \cos^7 s \, ds = \frac{1}{2\pi^8} \gamma\left(8, \frac{r^2}{2}\right),$$

which is equivalent to the formula in the corollary. □

In Table 2 we show the cases for which we have a closed expression for the measure-preserving mapping $\Phi_{\mathcal{M}}$. In addition, in the next section we present an approach that will allow us to obtain measure-preserving mappings with explicit expressions for any odd-dimensional sphere.

## 4. Measure-preserving mappings from the unit cube to fiber bundles

In this section we show how to construct measure-preserving mappings from the unit cube to the total space $E$ of the smooth fiber bundle $F \hookrightarrow E \xrightarrow{\pi} B$, where the total space $E$, the base space $B$, and the fiber $F$ are Riemannian manifolds, assuming that we have measure-preserving mappings from the corresponding unit cubes to $B$ and $F$.

To prove Theorem 1.4 we need the following lemma. The main technical tool used in its proof is the smooth coarea formula (see Appendix A for more details).

**Table 2**
Summary of the manifolds for which we have a closed formula for the measure-preserving mapping $\Phi_{\mathcal{M}}$, where $\Phi_{\mathbb{R}^d}$ is as in Eq. (1). The computations are straightforward from our main results; see Appendix B.

| $\mathcal{M}$ | $\Phi_{\mathcal{M}} = \exp_{\mathcal{M}} \circ \varphi_{\mathcal{M}} \circ \Phi_{\mathbb{R}^d} : ((0,1)^d, \text{unif}) \to (\mathcal{M}, \text{unif})$ |
|---|---|
| $\mathbb{B}^n$ | $\frac{\Phi_{\mathbb{R}^n}(x)}{\|\Phi_{\mathbb{R}^n}(x)\|}\left(\frac{\gamma\left(\frac{n}{2}, \frac{\|\Phi_{\mathbb{R}^n}(x)\|^2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{1/n}$ |
| $\mathbb{S}^1$ | $(-\sin 2\pi x, -\cos 2\pi x)$ |
| $\mathbb{S}^2$ | $\left(\frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|} 2e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/4}\sqrt{1-e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/2}}, 2e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/2}-1\right)$ |
| $\mathbb{RP}^1$ | $(-\cot \pi x, 1)$ |
| $\mathbb{RP}^2$ | $\left(\frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|}\sqrt{e^{\|\Phi_{\mathbb{R}^2}(x)\|^2}-1}, 1\right)$ |
| $\mathbb{CP}^1$ | $\left(\frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|}\sqrt{e^{\|\Phi_{\mathbb{R}^2}(x)\|^2/2}-1}, 1\right)$ |
| $\mathbb{CP}^n$ | $\left(\frac{\Phi_{\mathbb{R}^{2n}}(x)}{\|\Phi_{\mathbb{R}^{2n}}(x)\|}\left(-1+\frac{1}{1-\left(\frac{1}{(n-1)!}\gamma\left(n, \frac{\|\Phi_{\mathbb{R}^{2n}}(x)\|^2}{2}\right)\right)^{1/n}}\right)^{1/2}, 1\right)$ |

**Lemma 4.1.** *Let $E$, $B$, and $F$ be finite-volume Riemannian manifolds, and let $F \hookrightarrow E \xrightarrow{\pi} B$ be a smooth fiber bundle such that* $\mathrm{NJac}\,\pi(x)$ *is constant for every $x \in E$. Let $\Psi_y : (F, \text{unif}) \to (\pi^{-1}(y), \text{unif})$ be a measure-preserving mapping for every $y \in B$ and consider the mapping*

$$\begin{array}{llll} \xi : & (B \times F, \text{unif}) & \longrightarrow & (E, \text{unif}), \\ & (y, z) & \longmapsto & \Psi_y(z). \end{array}$$

*If $\xi$ is measurable, then it is measure preserving. Moreover, if the measures in $E$, $B$, and $F$ are normalized to have unit volume, then* $\mathrm{NJac}\,\pi(x) = 1$ *for every $x \in E$.*

**Proof.** Without loss of generality, assume that the measures in $E$, $B$, and $F$ are normalized. We first check that $\mathrm{NJac}\,\pi(x) = 1$ for every $x \in E$. Since $F \hookrightarrow E \xrightarrow{\pi} B$ is a smooth fiber bundle, we know that $\pi$ is a submersion and hence we can apply the smooth coarea formula. Therefore,

$$1 = \mathrm{vol}(E) = \int_{x \in E} dx = \int_{y \in B} \int_{z \in \pi^{-1}(y)} \frac{1}{C}\, dz dy = \frac{\mathrm{vol}(\pi^{-1}(y))\,\mathrm{vol}(B)}{C}$$
$$= \frac{\mathrm{vol}(F)\,\mathrm{vol}(B)}{C} = \frac{1}{C},$$

and so $C = 1$. Now we prove that $\xi$ is measure preserving. Let $\mathcal{A} \subseteq E$ be a measurable set. We have to prove that

$$\mathrm{vol}(\mathcal{A}) = \mathrm{vol}(\xi^{-1}(\mathcal{A})).$$

Using again the smooth coarea formula together with the fact that $\mathrm{NJac}\,\pi(x) = 1$ for all $x \in E$, we have

$$\begin{aligned}\operatorname{vol}(\mathcal{A}) &= \int_{x\in E} \chi_{\mathcal{A}}(x)\,dx = \int_{y\in B}\int_{z\in\pi^{-1}(y)} \chi_{\mathcal{A}}(z)\frac{1}{\operatorname{NJac}\pi(z)}\,dzdy \\ &= \int_{y\in B}\int_{z\in\pi^{-1}(y)} \chi_{\mathcal{A}}(z)\,dzdy = \int_{y\in B} \operatorname{vol}(\mathcal{A}\cap\pi^{-1}(y))\,dy \\ &= \int_{y\in B} \operatorname{vol}(\Psi_y^{-1}(\mathcal{A}\cap\pi^{-1}(y)))\,dy \\ &= \int_{y\in B}\int_{z\in F} \chi_{\Psi_y^{-1}(\mathcal{A}\cap\pi^{-1}(y))}(z)\,dzdy.\end{aligned} \tag{5}$$

Note that

$$\begin{aligned}\chi_{\Psi_y^{-1}(\mathcal{A}\cap\pi^{-1}(y))}(z) = 1 &\iff z\in\Psi_y^{-1}(\mathcal{A}\cap\pi^{-1}(y)) \iff \Psi_y(z)\in\mathcal{A}\cap\pi^{-1}(y) \\ &\iff \Psi_y(z)\in\mathcal{A} \iff \xi(y,z)\in\mathcal{A} \\ &\iff (y,z)\in\xi^{-1}(\mathcal{A}) \iff \chi_{\xi^{-1}(\mathcal{A})}(y,z) = 1.\end{aligned}$$

Therefore, since

$$\begin{aligned}\operatorname{vol}(\xi^{-1}(\mathcal{A})) &= \int_{(y,z)\in B\times F} \chi_{\xi^{-1}(\mathcal{A})}(y,z)\,d(y,z) = \int_{y\in B}\int_{z\in F} \chi_{\xi^{-1}(\mathcal{A})}(y,z)\,dzdy \\ &= \int_{y\in B}\int_{z\in F} \chi_{\Psi_y^{-1}(A\cap\pi^{-1}(y))}(z)\,dzdy \overset{(5)}{=} \operatorname{vol}(\mathcal{A}),\end{aligned}$$

the lemma follows. □

**Proof of Theorem 1.4.** From Lemma 4.1 we have that the mapping $\xi : B\times F\to E$ given by $\xi(y,z) = \Psi_y(z)$ is measure preserving. Since $\Phi_E = \xi\circ\Phi_{B\times F}$, and both mappings are measure preserving, the theorem follows. □

**Example 4.2** (*The Hopf Fibration*)**.** Consider $\mathbb{S}^1 \subset \mathbb{C}$ and $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Recall that the (complex) Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{h} \mathbb{CP}^n$ is given by

$$\begin{array}{llcl} h: & \mathbb{S}^{2n+1} & \longrightarrow & \mathbb{CP}^n, \\ & (y_1,\ldots,y_{n+1}) & \longmapsto & [y_1:\cdots:y_{n+1}]. \end{array}$$

The fiber of each $[y] = [y_1:\cdots:y_{n+1}] \in \mathbb{CP}^n$ is a unit circle in $\mathbb{S}^{2n+1}$ given by

$$h^{-1}([y]) = \{w\in\mathbb{S}^{2n+1} : [w] = [y]\}.$$

For each $[y]\in\mathbb{CP}^n$, we choose a unit norm representative $y$ smoothly out of a lower-dimensional set, and, thinking of the elements of $\mathbb{S}^1$ as unimodular complex numbers, we consider the mapping

$$\begin{array}{llcl} \Psi_{[y]}: & (\mathbb{S}^1,\text{unif}) & \longrightarrow & (h^{-1}([y]),\text{unif}), \\ & \zeta & \longmapsto & \zeta y, \end{array}$$

which is an isometry and hence it is measure preserving. Therefore, by Theorem 1.4, the mapping

$$\begin{array}{llcl} \Phi^h_{\mathbb{S}^{2n+1}}: & ((0,1)^{2n+1},\text{unif}) & \longrightarrow & (\mathbb{S}^{2n+1},\text{unif}), \\ & (y,t) & \longmapsto & \Psi_{\Phi_{\mathbb{CP}^n}(y)}(\Phi_{\mathbb{S}^1}(t)) = \Phi_{\mathbb{S}^1}(t)\Phi_{\mathbb{CP}^n}(y), \end{array}$$

where $y \in \mathbb{C}^n \cong \mathbb{R}^{2n}$ and $t \in \mathbb{R}$ (and recall that we are assuming that the representative of $\Phi_{\mathbb{CP}^n}(y)$ has unit norm), is measure preserving. Note that we have explicit expressions for both $\Phi_{\mathbb{S}^1}$ and $\Phi_{\mathbb{CP}^n}$, and hence for $\Phi^h_{\mathbb{S}^{2n+1}}$:

$$\Phi^h_{\mathbb{S}^{2n+1}}(y,t) = \Phi_{\mathbb{S}^1}(t)\Phi_{\mathbb{CP}^n}(y)$$
$$= -ie^{-i2\pi t}\frac{\left(\frac{\Phi_{\mathbb{R}^{2n}}(y)}{\|\Phi_{\mathbb{R}^{2n}}(y)\|}\left(-1+\frac{1}{1-\left(\frac{1}{(n-1)!}\gamma\left(n,\frac{\|\Phi_{\mathbb{R}^{2n}}(y)\|^2}{2}\right)\right)^{1/n}}\right)^{1/2},1\right)}{\left\|\left(\frac{\Phi_{\mathbb{R}^{2n}}(y)}{\|\Phi_{\mathbb{R}^{2n}}(y)\|}\left(-1+\frac{1}{1-\left(\frac{1}{(n-1)!}\gamma\left(n,\frac{\|\Phi_{\mathbb{R}^{2n}}(y)\|^2}{2}\right)\right)^{1/n}}\right)^{1/2},1\right)\right\|}.$$

For the particular case of $\mathbb{S}^3$ we have

$$\Phi^h_{\mathbb{S}^3}(y,t) = \Phi_{\mathbb{S}^1}(t)\Phi_{\mathbb{CP}^1}(y) = -ie^{-i2\pi t}e^{-\|\Phi_{\mathbb{R}^2}(y)\|^2/4}\left(\frac{\Phi_{\mathbb{R}^2}(y)}{\|\Phi_{\mathbb{R}^2}(y)\|}\sqrt{e^{\|\Phi_{\mathbb{R}^2}(y)\|^2/2}-1},1\right)$$
$$= \left(-ie^{-i2\pi t}\frac{\Phi_{\mathbb{R}^2}(y)}{\|\Phi_{\mathbb{R}^2}(y)\|}\sqrt{1-e^{-\|\Phi_{\mathbb{R}^2}(y)\|^2/2}}, -ie^{-i2\pi t}e^{-\|\Phi_{\mathbb{R}^2}(y)\|^2/4}\right),$$

since $\|\Phi_{\mathbb{CP}^1}(y)\| = e^{\|\Phi_{\mathbb{R}^2}(y)\|^2/4}$. Note that we are considering $\Phi_{\mathbb{R}^2}(y) \in \mathbb{C}$ through the canonical isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ given by $(a,b) \mapsto a+bi$.


## Acknowledgments

We would like to thank two anonymous referees for their helpful suggestions and comments.


## Appendix A. The smooth coarea formula

We devote this appendix to the smooth coarea formula, an integral formula due to Federer [8] and Howard [16] that generalizes the change of variables formula and Fubini's theorem. We refer the interested reader to [2, Section 2] for some examples of use.

Let $\mathcal{M}, \mathcal{N}$ be Riemannian manifolds. Given a smooth mapping $\varphi : \mathcal{M} \to \mathcal{N}$, let $D\varphi(x) : T_x\mathcal{M} \to T_{\varphi(x)}\mathcal{N}$ denote the differential mapping, where $T_x\mathcal{M}$ is the tangent space to $\mathcal{M}$ at $x \in \mathcal{M}$ and $T_{\varphi(x)}\mathcal{N}$ is the tangent space to $\mathcal{N}$ at $\varphi(x) \in \mathcal{N}$.

**Definition A.1** (*Normal Jacobian*)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds and let $\varphi : \mathcal{M} \to \mathcal{N}$ be a $C^1$ surjective map. Let $n = \dim(\mathcal{N})$ be the real dimension of $\mathcal{N}$. For every point $x \in \mathcal{M}$ such that the differential mapping $D\varphi(x)$ is surjective, let $v_1^x, \dots, v_n^x$ be an orthogonal basis of $(\ker(D\varphi(x)))^\perp$. Then we define the normal Jacobian of $\varphi$ at $x$, written as $\mathrm{NJac}\,\varphi(x)$, as the volume in the tangent space $T_{\varphi(x)}\mathcal{N}$ of the parallelepiped spanned by $D\varphi(x)(v_1^x), \dots, D\varphi(x)(v_n^x)$. In the case that $D\varphi(x)$ is not surjective, we define $\mathrm{NJac}\,\varphi(x) = 0$.

**Theorem A.2** (*Smooth Coarea Formula*)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two Riemannian manifolds of dimension $m$ and $n$, respectively, where $m \ge n$. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth surjective mapping such that the differential mapping $D\varphi(x)$ is surjective for almost every $x \in \mathcal{M}$. Let $\psi : \mathcal{M} \to \mathbb{R}$ be an integrable mapping. Then, the following equalities hold:*

$$\int_{x\in\mathcal{M}} \psi(x)\,dx = \int_{y\in\mathcal{N}}\int_{x\in\varphi^{-1}(y)} \psi(x)\frac{1}{\mathrm{NJac}\,\varphi(x)}\,dx dy,$$
$$\int_{x\in\mathcal{M}} \psi(x)\,\mathrm{NJac}\,\varphi(x)\,dx = \int_{y\in\mathcal{N}}\int_{x\in\varphi^{-1}(y)} \psi(x)\,dx dy.$$

Note that if $m = n$ and $\varphi$ is a diffeomorphism we recover the classical change of variables theorem.

## Appendix B. Auxiliary computations

In this appendix, we show the explicit computations leading to the formulas in Table 2.

### B.1. Explicit expression of $\Phi_{\mathbb{B}^n}$

Recall from Proposition 1.2 that we have

$$\varphi_{\mathbb{B}^n}(x) = \frac{x}{\|x\|}\left(\frac{\gamma\left(\frac{n}{2}, \frac{\|x\|^2}{2}\right)}{\Gamma(\frac{n}{2})}\right)^{1/n}.$$

Hence,

$$\Phi_{\mathbb{B}^n}(x) = \varphi_{\mathbb{B}^n}(\Phi_{\mathbb{R}^n}(x)) = \frac{\Phi_{\mathbb{R}^n}(x)}{\|\Phi_{\mathbb{R}^n}(x)\|}\left(\frac{\gamma\left(\frac{n}{2}, \frac{\|\Phi_{\mathbb{R}^n}(x)\|^2}{2}\right)}{\Gamma(\frac{n}{2})}\right)^{1/n}.$$

### B.2. Explicit expression of $\Phi_{\mathbb{S}^1}$

Although we could simply define $\Phi_{\mathbb{S}^1}(x) = e^{i2\pi x}$, let us find the expression of this mapping using the general procedure. In this case, we have

$$\Phi_{\mathbb{R}}(x) = \sqrt{2}\,\mathrm{erf}^{-1}(2x-1).$$

Following Corollary 3.1, to find $\varphi_{\mathbb{S}^1}$ we have to obtain $\rho$ from

$$\int_0^\rho \sin^{n-1} s\, ds = \frac{\sqrt{\pi}}{\Gamma(\frac{n+1}{2})}\gamma\left(\frac{n}{2}, \frac{r^2}{2}\right).$$

Since in this case $n = 1$, we have

$$\rho(r) = \sqrt{\pi}\gamma\left(\frac{1}{2}, \frac{r^2}{2}\right) = \pi\,\mathrm{erf}\left(\frac{r}{\sqrt{2}}\right).$$

Hence,

$$\varphi_{\mathbb{S}^1}(x) = \frac{\pi x}{|x|}\,\mathrm{erf}\left(\frac{|x|}{\sqrt{2}}\right).$$

Therefore,

$$\begin{aligned}\varphi_{\mathbb{S}^1}(\Phi_{\mathbb{R}}(x)) &= \frac{\pi\sqrt{2}\,\mathrm{erf}^{-1}(2x-1)}{|\sqrt{2}\,\mathrm{erf}^{-1}(2x-1)|}\,\mathrm{erf}\left(\frac{|\sqrt{2}\,\mathrm{erf}^{-1}(2x-1)|}{\sqrt{2}}\right)\\ &= \frac{\pi\,\mathrm{erf}^{-1}(2x-1)}{|\mathrm{erf}^{-1}(2x-1)|}\,\mathrm{erf}(|\mathrm{erf}^{-1}(2x-1)|).\end{aligned}$$

Since both erf and $\mathrm{erf}^{-1}$ are odd functions, the absolute values cancel each other and so

$$\varphi_{\mathbb{S}^1}(\Phi_{\mathbb{R}}(x)) = \pi\,\mathrm{erf}(\mathrm{erf}^{-1}(2x-1)) = \pi(2x-1).$$

Recall from Table 1 that the exponential map $\exp_{\mathbb{S}^1} : (-\pi, \pi) \to \mathbb{S}^1$ is given by

$$\exp_{\mathbb{S}^1}(v) = \left(\frac{v}{|v|}\sin|v|, \cos|v|\right).$$

Hence,

$$\Phi_{\mathbb{S}^1}(x) = \exp_{\mathbb{S}^1}(\pi(2x-1)) = \left(\frac{\pi(2x-1)}{|\pi(2x-1)|}\sin|\pi(2x-1)|, \cos|\pi(2x-1)|\right)$$
$$= (\sin(2\pi x-\pi), \cos(2\pi x-\pi)) = (-\sin 2\pi x, -\cos 2\pi x)$$
$$\cong -ie^{-i2\pi x}.$$

### B.3. Explicit expression of $\Phi_{\mathbb{S}^2}$

Recall from Corollary 3.1 that we have

$$\varphi_{\mathbb{S}^2}(x) = \frac{x}{\|x\|}\cdot 2\arccos e^{-\|x\|^2/4}.$$

Let us compute first $\exp_{\mathbb{S}^2}\circ\varphi_{\mathbb{S}^2}$. Recall from Table 1 that

$$\exp_{\mathbb{S}^2}(v) = \left(\frac{v}{\|v\|}\sin\|v\|, \cos\|v\|\right).$$

Hence,

$$\exp_{\mathbb{S}^2}(\varphi_{\mathbb{S}^2}(x)) = \exp_{\mathbb{S}^2}\left(\frac{x}{\|x\|}\cdot 2\arccos e^{-\|x\|^2/4}\right)$$
$$= \left(\frac{\frac{x}{\|x\|}\cdot 2\arccos e^{-\|x\|^2/4}}{\|\frac{x}{\|x\|}\cdot 2\arccos e^{-\|x\|^2/4}\|}\sin\left\|\frac{x}{\|x\|}\cdot 2\arccos e^{-\|x\|^2/4}\right\|, \cos\left\|\frac{x}{\|x\|}\cdot 2\arccos e^{-\|x\|^2/4}\right\|\right).$$

Due to the parity of the sine and the cosine, we can rewrite the previous expression as

$$\exp_{\mathbb{S}^2}(\varphi_{\mathbb{S}^2}(x)) = \left(\frac{x}{\|x\|}\sin(2\arccos e^{-\|x\|^2/4}), \cos(2\arccos e^{-\|x\|^2/4})\right).$$

To further simplify these expressions, note that, for $-1 < x < 1$,

$$\sin(2\arccos(x)) = 2\sin(\arccos(x))\cos(\arccos(x)) = 2x\sqrt{1-\cos^2(\arccos(x))}$$
$$= 2x\sqrt{1-x^2},$$

and

$$\cos(2\arccos(x)) = \cos^2(\arccos(x)) - \sin^2(\arccos(x)) = x^2-(1-x^2)$$
$$= 2x^2-1.$$

Therefore,

$$\exp_{\mathbb{S}^2}(\varphi_{\mathbb{S}^2}(x)) = \left(\frac{x}{\|x\|}2e^{-\|x\|^2/4}\sqrt{1-e^{-\|x\|^2/2}}, 2e^{-\|x\|^2/2}-1\right).$$

Hence, we conclude that

$$\Phi_{\mathbb{S}^2}(x) = \left(\frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|}2e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/4}\sqrt{1-e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/2}}, 2e^{-\|\Phi_{\mathbb{R}^2}(x)\|^2/2}-1\right).$$

### B.4. Explicit expression of $\Phi_{\mathbb{RP}^1}$

In this case, we have

$$\Phi_{\mathbb{R}}(x) = \sqrt{2}\,\mathrm{erf}^{-1}(2x-1).$$

Following Corollary 3.2, to find $\varphi_{\mathbb{RP}^1}$ we have to obtain $\rho$ from

$$\int_0^\rho \sin^{n-1} s\, ds = \frac{\sqrt{\pi}}{2\Gamma(\frac{n+1}{2})}\gamma\left(\frac{n}{2}, \frac{r^2}{2}\right).$$

Since in this case $n = 1$, we have

$$\rho(r) = \frac{\sqrt{\pi}}{2}\gamma\left(\frac{1}{2}, \frac{r^2}{2}\right) = \frac{\pi}{2}\,\mathrm{erf}\left(\frac{r}{\sqrt{2}}\right).$$

Hence,

$$\varphi_{\mathbb{RP}^1}(x) = \frac{\pi x}{2|x|}\,\mathrm{erf}\left(\frac{|x|}{\sqrt{2}}\right).$$

Therefore,

$$\begin{aligned}\varphi_{\mathbb{RP}^1}(\Phi_{\mathbb{R}}(x)) &= \frac{\pi\sqrt{2}\,\mathrm{erf}^{-1}(2x-1)}{2|\sqrt{2}\,\mathrm{erf}^{-1}(2x-1)|}\,\mathrm{erf}\left(\frac{|\sqrt{2}\,\mathrm{erf}^{-1}(2x-1)|}{\sqrt{2}}\right)\\ &= \frac{\pi\,\mathrm{erf}^{-1}(2x-1)}{2|\mathrm{erf}^{-1}(2x-1)|}\,\mathrm{erf}(|\mathrm{erf}^{-1}(2x-1)|).\end{aligned}$$

Since both erf and $\mathrm{erf}^{-1}$ are odd functions, the absolute values cancel each other and so

$$\varphi_{\mathbb{RP}^1}(\Phi_{\mathbb{R}}(x)) = \frac{\pi}{2}\,\mathrm{erf}(\mathrm{erf}^{-1}(2x-1)) = \frac{\pi}{2}(2x-1).$$

Recall from Table 1 that the exponential map $\exp_{\mathbb{RP}^1} : (-\pi/2, \pi/2) \to \mathbb{RP}^1$ is given by

$$\exp_{\mathbb{RP}^1}(v) = \left(\frac{v}{|v|}\tan|v|, 1\right).$$

Hence,

$$\Phi_{\mathbb{RP}^1}(x) = \exp_{\mathbb{RP}^1}\left(\frac{\pi}{2}(2x-1)\right) = \left(\frac{\pi(2x-1)}{|\pi(2x-1)|}\tan\left|\frac{\pi}{2}(2x-1)\right|, 1\right).$$

Since the tangent function is odd, we can simplify the previous expression as follows:

$$\Phi_{\mathbb{RP}^1}(x) = \left(\tan\left(\frac{\pi}{2}(2x-1)\right), 1\right) = \left(\tan\left(\pi x - \frac{\pi}{2}\right), 1\right) = (-\cot \pi x, 1).$$

### B.5. Explicit expression of $\Phi_{\mathbb{RP}^2}$

Recall from Corollary 3.2 that we have

$$\varphi_{\mathbb{RP}^2}(x) = \frac{x}{\|x\|}\arccos e^{-\|x\|^2/2}.$$

Let us compute $\exp_{\mathbb{RP}^2} \circ \varphi_{\mathbb{RP}^2}$. Recall from Table 1 that

$$\exp_{\mathbb{RP}^2}(v) = \left(\frac{v}{\|v\|}\tan\|v\|, 1\right).$$

Hence,

$$\exp_{\mathbb{RP}^2}(\varphi_{\mathbb{RP}^2}(x)) = \left( \frac{\frac{x}{\|x\|} \arccos e^{-\|x\|^2/2}}{\|\frac{x}{\|x\|} \arccos e^{-\|x\|^2/2}\|} \tan \left\| \frac{x}{\|x\|} \arccos e^{-\|x\|^2/2} \right\|, 1 \right).$$

Since the tangent function is odd, we can simplify the previous expression as follows:

$$\exp_{\mathbb{RP}^2}(\varphi_{\mathbb{RP}^2}(x)) = \left( \frac{x}{\|x\|} \tan \arccos e^{-\|x\|^2/2}, 1 \right).$$

Note that

$$\tan \arccos(x) = \frac{\sin \arccos(x)}{\cos \arccos(x)} = \frac{\sqrt{1-x^2}}{x} = \sqrt{\frac{1}{x^2} - 1}.$$

Hence,

$$\exp_{\mathbb{RP}^2}(\varphi_{\mathbb{RP}^2}(x)) = \left( \frac{x}{\|x\|} \sqrt{e^{\|x\|^2} - 1}, 1 \right),$$

and we conclude that

$$\Phi_{\mathbb{RP}^2}(x) = \left( \frac{\Phi_{\mathbb{R}^2}(x)}{\|\Phi_{\mathbb{R}^2}(x)\|} \sqrt{e^{\|\Phi_{\mathbb{R}^2}(x)\|^2} - 1}, 1 \right).$$

### *B.6. Explicit expression of $\Phi_{\mathbb{CP}^n}$*

Recall from Corollary 3.3 that we have

$$\varphi_{\mathbb{CP}^n}(x) = \frac{x}{\|x\|} \arcsin\left( \left( \frac{1}{(n-1)!} \gamma\left(n, \frac{\|x\|^2}{2}\right) \right)^{1/(2n)} \right).$$

Recall from Table 1 that

$$\exp_{\mathbb{CP}^n}(v) = \left( \frac{v}{\|v\|} \tan \|v\|, 1 \right).$$

Let us compute $\exp_{\mathbb{CP}^n} \circ \varphi_{\mathbb{CP}^n}$. As for the case of $\mathbb{RP}^2$, the parity of the tangent function implies that

$$\exp_{\mathbb{CP}^n}(\varphi_{\mathbb{CP}^n}(x)) = \left( \frac{x}{\|x\|} \tan \arcsin\left( \left( \frac{1}{(n-1)!} \gamma\left(n, \frac{\|x\|^2}{2}\right) \right)^{1/(2n)} \right), 1 \right).$$

Note that, in our range,

$$\tan \arcsin(x) = \frac{\sin \arcsin(x)}{\cos \arcsin(x)} = \frac{x}{\sqrt{1-x^2}} = \sqrt{-1 + \frac{1}{1-x^2}}.$$

Hence,

$$\exp_{\mathbb{CP}^n}(\varphi_{\mathbb{CP}^n}(x)) = \left( \frac{x}{\|x\|} \sqrt{-1 + \frac{1}{1 - \left(\frac{1}{(n-1)!}\gamma(n, \frac{\|x\|^2}{2})\right)^{1/n}}}, 1 \right),$$

and we conclude that

$$\Phi_{\mathbb{CP}^n}(x) = \left( \frac{\Phi_{\mathbb{R}^{2n}}(x)}{\|\Phi_{\mathbb{R}^{2n}}(x)\|} \sqrt{-1 + \frac{1}{1 - \left(\frac{1}{(n-1)!}\gamma(n, \frac{\|\Phi_{\mathbb{R}^{2n}}(x)\|^2}{2})\right)^{1/n}}}, 1 \right).$$

## Data availability

No data was used for the research described in the article.

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