On the global well-posedness of interface dynamics for gravity Stokes flow

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Abstract

In this paper, we establish the global-in-time well-posedness for an arbitrary $C^{1+\gamma}$, $0 < \gamma < 1$, initial internal periodic wave for the free boundary gravity Stokes system in two dimensions. This classical well-posedness result is complemented by a weak solvability result in the case of C^{γ} or Lipschitz interfaces. In particular, we show new cancellations that prevent the so-called two-dimensional Stokes paradox, despite the polynomial growth of the Stokeslet in this horizontally periodic setting. The bounds obtained in this work are exponential in time, which are in strong agreement with the growth of the solutions obtained in [22]. Additionally, these new cancellations are used to establish global-in-time well-posedness for the Stokes-transport system with initial densities in L^p for $2 < p < \infty$. Furthermore, we also propose and analyze several one-dimensional models that capture different aspects of the full internal wave problem for the gravity Stokes system, showing that all of these models exhibit finite-time singularities. This fact evidences the fine structure of the nonlinearity in the full system, which allows the free boundary problem to be globally well-posed, while simplified versions blow-up in finite time.

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1 Introduction and main results

The study of the dynamics of free boundary problems dates back, at least, to the works of Laplace (1776) and Lagrange (1781 and 1786) on the dynamics of water waves. In this problem, there is one fluid filling the time-dependent domain $\Omega(t)$ bounded by the moving interface $\Gamma(t)$. This fluid is assumed to be surrounded by vacuum.

Since these pioneer works, many celebrated researchers have studied the case of an inviscid fluid with a free boundary. In particular, today is well-known that the one-phase Euler equations with moving free-surface are locally well-posed in Sobolev spaces if the pressure function satisfies the so-called Rayleigh-Taylor (RT) stability condition, that requires the sign of the normal derivative of the pressure to be negative, namely,

$$\left.\frac{\partial p}{\partial n(t)}\right|_{t=0} < 0$$

on the initial surface $\Gamma(0)$ (see [12] and the references therein).

When two inviscid fluids are considered, the situation changes drastically as the dynamics naturally develop instabilities ([32] and the references therein). In fact, the two-phase Euler equations are used as the basic model for the Rayleigh-Taylor and Kelvin-Helmholtz instabilities for fluids with different densities under gravity forces and fluids with the same density but discontinuous velocities at the interface $\Gamma(t)$, respectively. Perturbations of the flat interface in the linear regime grow; the nonlinearity is unable of stabilizing such a growth. As a result, this nonlinear two-fluid problem where the two fluids have constant (different) densities ρ^+ and ρ^-

$$\rho^\pm(u^\pm_t + (u^\pm \cdot \nabla)u^\pm) = -\nabla p^\pm - (0, \rho^\pm)^T \quad \text{in} \quad \Omega^\pm(t), \tag{1a}$$
$$\nabla \cdot u^\pm = 0 \quad \text{in} \quad \Omega^\pm(t), \tag{1b}$$
$$[\![p]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{1c}$$
$$z_t = u(z,t) \quad \text{on} \quad \Gamma(t), \tag{1d}$$

where $[\![f]\!] = f^+ - f^-$ on $\Gamma(t)$, is ill-posed in Sobolev spaces when surface tension effects are neglected [32].

A similar problem arises when the fluids fill a porous medium. In this case, the problem is known in the literature as the Muskat/interface Hele-Shaw problem [1, 6, 7, 8, 13]. There, the system under study is given by Darcy's law

$$u^\pm = -\nabla p^\pm - (0, \rho^\pm)^T \quad \text{in} \quad \Omega^\pm(t), \tag{2a}$$
$$\nabla \cdot u^\pm = 0 \quad \text{in} \quad \Omega^\pm(t), \tag{2b}$$
$$[\![p]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{2c}$$
$$z_t = u(z,t) \quad \text{on} \quad \Gamma(t). \tag{2d}$$

The two-phase Muskat problem can also be ill-posed if the RT condition is not satisfied [13] while the one-phase Muskat problem (in an unbounded domain, at least) satisfies the RT condition automatically. Furthermore, besides its own mathematical interest, the Muskat problem has been a successful benchmark problem for the free boundary Euler equations and conversely. In particular, the ideas leading to turning waves for the two-phase Muskat were also implemented in the case of water waves [8] and the splash singularities in water waves [5] were later extended to the one-phase Muskat problem [7].

Even if for many applications, the inviscid approximation is enough in practice, a better, more careful study requires the inclusion of viscosity effects (see [25] and the references therein). In this case, one has to study the free boundary problem for the Navier-Stokes equations

$$\rho^\pm(u^\pm_t + (u^\pm \cdot \nabla)u^\pm) - \Delta u^\pm = -\nabla p^\pm - (0, \rho^\pm)^T \quad \text{in} \quad \Omega^\pm(t), \tag{3a}$$
$$\nabla \cdot u^\pm = 0 \quad \text{in} \quad \Omega^\pm(t), \tag{3b}$$
$$[\![(\nabla u + \nabla u^T - p\,\mathrm{Id})n]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{3c}$$
$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{3d}$$
$$z_t = u(z,t) \quad \text{on} \quad \Gamma(t). \tag{3e}$$

One of the classical references on such a system is [17]. However, in the last year there has been some works studying the free boundary problem for such a system both in the two-phase and one-phase case [21, 27, 28].

After going to dimensionless variables and the Boussinesq approximation, system (3) reads

$$\frac{1}{Pr}(u_t^\pm + (u^\pm \cdot \nabla)u^\pm) - \Delta u^\pm = -\nabla p^\pm - (0, Ra\,\rho^\pm)^T \quad \text{in} \quad \Omega^\pm(t), \tag{4a}$$

$$\nabla \cdot u^\pm = 0 \quad \text{in} \quad \Omega^\pm(t), \tag{4b}$$

$$[\![(\nabla u + \nabla u^T - p\,\mathrm{Id})n]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{4c}$$

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{4d}$$

$$z_t = u(z,t) \quad \text{on} \quad \Gamma(t), \tag{4e}$$

where Pr and Ra are the Prandtl and Rayleigh dimensionless numbers [26, 35]. See [14, 20] for recent works on this problem. As a consequence of the model, in the asymptotic regime $1 \ll Pr$, a good approximation is the two-phase Stokes system

$$-\Delta u^\pm = -\nabla p^\pm - (0, \rho^\pm)^T \quad \text{in} \quad \Omega^\pm(t), \tag{5a}$$

$$\nabla \cdot u^\pm = 0 \quad \text{in} \quad \Omega^\pm(t), \tag{5b}$$

$$[\![(\nabla u + (\nabla u)^T - p\,\mathrm{Id})n]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{5c}$$

$$[\![u]\!] = 0 \quad \text{on} \quad \Gamma(t), \tag{5d}$$

$$z_t = u(z,t) \quad \text{on} \quad \Gamma(t). \tag{5e}$$

Furthermore, the previous system has been also derived from a microscopic formulation of sedimenting particles in a fluid [29] and from a Vlasov-Stokes kinetic system [30]. This system and close variants (as the surface tension case) have been studied by many different researchers in the previous years [2, 11, 26, 34, 36, 37, 38, 39, 40, 41, 44].

In this paper we study the case of two fluids with different constant densities evolving by (5) in absence of surface tension in two spatial dimensions. We observe that in our paper [22] we proved that system (5) can be equivalently written using the following contour dynamics formulation

$$z_t(\alpha,t) = (\rho^- - \rho^+)\int_{\mathbb{T}} \mathcal{S}(z(\alpha,t) - z(\beta,t)) \cdot \partial_\beta z^\perp(\beta,t) z_2(\beta,t)d\beta, \tag{6}$$

where the so-called x_1-periodic Stokeslet reads

$$\mathcal{S}(y) = \frac{1}{8\pi}\log\left(2(\cosh(y_2) - \cos(y_1))\right) \cdot I - \frac{y_2}{8\pi(\cosh(y_2) - \cos(y_1))}\begin{pmatrix} -\sinh(y_2) & \sin(y_1) \\ \sin(y_1) & \sinh(y_2) \end{pmatrix}.$$

We note that the Stokeslet in this horizontally periodic setting grows polynomially. This growth is faster that the analog in the whole plane where the Stokeslet behaves as a logarithm. Indeed, in our setting

$$|\mathcal{S}(y)| \sim \frac{1}{4\pi}|y_2| \text{ as } |y_2| \to \infty,$$

while without the periodicity condition the analog kernel [24] behaves as

$$\frac{1}{4\pi}\log|y| \text{ as } |y| \to \infty.$$

Equation (6) resembles the vortex patch equation [4, 16, 43]

$$z_t(\alpha,t) = -\omega_0 \int_{\mathbb{T}} \mathcal{V}(z(\alpha,t) - z(\beta,t)) \cdot \partial_\beta z(\beta,t)d\beta, \tag{7}$$

where the kernel reads

$$\mathcal{V}(y) = \frac{1}{4\pi}\log\left(y_1^2 + y_2^2\right) \cdot I.$$

Even if (part of) the kernels involved are similar, at the linear level both equations behave very differently. Indeed, the presence of the z_2 term in (6) coming from the gravity, imposes a crucial anisotropy in the problem (5). As a consequence of such a term, the linear operator in (6) has order -1 while the linear operator in (7) has order 0.

A number of well-posedness results for the two-dimensional case without surface tension are available in the literature. On the one hand, for (5) in a planar domain bounded in the vertical direction, Leblond established the global existence and uniqueness for bounded initial density [34]. This result is refined to three-dimensional bounded domains in [35] . In particular, such results cover the case of a density patch described above. The same case but in a bounded planar domain was also proved by Antontsev, Yurinsky and Meirmanov [2] for the case of a C^2 initial interface. However, the inherent anisotropy of the system, where the vertical direction plays a crucial role compared to the horizontal one, makes the case with an unbounded domain out of reach of the previous references. In the case of the whole plane, the only available well-posedness result is the one by Grayer II, that addresses the global existence of solutions for an initial density that is in $L^1 \cap L^\infty$ with compact support [26]. The hypothesis on the support is crucial as it serves to control it over time due to the transport character of the conservation of mass equation for the density . In fact, such a compact support is propagated by the transport structure of the problem and allows the author to handle the logarithmic growth of the Stokeslet kernel. Part of our analysis relies on a new cancellation found in the Stokeslet that ensures that the velocity u in our case has finite L^p energy instead of linear or logarithmic growth. Also, notice the recent result [33], where the authors extend [26], proving persistance of $C^{k,\gamma}$ regularity of the patch solution for for any $k \geq 1$, $0 < \gamma < 1$.

One of our main results in this paper is the following:

Theorem 1 (Global well-posedness of two-phase Stokes)**.** *Let* $0 < \gamma < 1$ *and* $z_0(\alpha) \in C^{1+\gamma}(\mathbb{T})$ *be the initial data for the two-phase Stokes problem* (5) *satisfying the arc-chord condition. Then, there exists a solution*

$$z(\alpha, t) \in C([0, T]; C^{1+\gamma})$$

for any $T > 0$*. Furthermore, this solution is unique and lacks self-intersections due to the boundedness of the arc-chord condition*

$$\sup_{\alpha \neq \beta} \frac{|\alpha - \beta|}{|z(\alpha, t) - z(\beta, t)|} < \infty, \quad t \in [0, T].$$

We observe that the arc-chord condition cannot blow up in finite time and that the initial data has arbitrary size. The proof of this result is based on new bounds for the velocity field for the case of density patches that exploits the structure of the kernels involved [22]. Once this regularity is controlled, using the contour dynamics equation for a general curve $z(\alpha, t)$ given in [22], any higher order regularity can be controlled ($C^{k+\gamma}$ with $k \geq 2$).

The main importance of our contribution is three-fold. First, our result is the first one handling the case where the fluids fill an unbounded-in-x_2 domain without imposing at the same time some integrability or compact support condition on the densities. This is a main difficulty in any rigorous proof of the result due to the growth of the Stokeslet (as $|x_2|$ in this direction) and the anisotropy of the system where the x_2 variable plays a crucial role. In particular, we prove that the *Stokes paradox* [23] does not hold besides the polynomial growth of the Stokeslet in this horizontally periodic setting. Second, our result also supersedes [2] in the sense that it requires barely C^1 initial data (any $C^{1+\gamma}$ is in fact enough) instead of C^2. In particular, our result covers initial interfaces with unbounded curvature and large slopes. This level of regularity is the same as the one in our previous local-in-time result in [22]. Finally, our result applies regardless of the stratification of the fluids, *i.e.* it proves that there is not Rayleigh-Taylor sign condition required to define global solutions and also regardless of the size and geometry of the initial data. In fact, in our previous work [22] we establish an instability result in the Rayleigh-Taylor unstable regime and prove that there are solutions with exponential growth in certain norm. In this paper we establish a new exponential bound for the Lagrangian trajectories that strongly agrees with the growth in [22]. Furthermore, as observed in [35], the $C^{1+\gamma}$ regularity of the velocity field can be used to propagate the C^γ or $W^{1,\infty}$ norm of the interface and, as a consequence, to define *Lagrangian solutions* of low regularity. Due to the new estimates that we prove for the velocity such Lagrangian solutions are also weak solutions of the system. We refer to [31] for a study between the different concepts of solutions.

In this work we focus on patch type configurations for the density. However, our global-in-time well-posedness result can be extended to a different class of initial densities. In particular we have the following result:

Theorem 2 (Global well-posedness of Stokes-transport)**.** *Let $\rho_0 \in L^p(\mathbb{T}\times\mathbb{R})$ with $2 < p < \infty$ be the initial data for the Stokes-transport system. Then, there exists a unique global-in-time solution. Furthermore, the Lagrangian trajectories $X(a,t)$ lie in the space $C^1([0,T];C^{1,\gamma})$, $\gamma = 1-2/p$.*

Now, we turn our attention to the study of reduced-order models for the contour dynamics equation (6). In order to do that, let us recall that, at the linear level and in the stable stratification of the densities (we set $\rho^- - \rho^+ = 1$ for simplicity), we find the linear dynamics

$$h_t = -\Lambda^{-1}(h), \quad h(\alpha,0) = h_0(\alpha), \tag{8}$$

where h denotes the graph of the function that characterizes the free boundary at the linear level: $z(\alpha,t) = (\alpha, h(\alpha,t))$. We will use the following shorthand notation for the solutions of the previous linear equation:

$$h(\alpha,t) = e^{-\Lambda^{-1}t}h_0(\alpha).$$

The operator $\Lambda^{-1}(h)$ in the periodic one-dimensional torus is defined as

$$\Lambda^{-1}(h)\,(\alpha) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(4\sin^2\left(\frac{\alpha-\beta}{2}\right)\right)h(\beta)d\beta. \tag{9}$$

This operator was studied in more detail in the authors' previous work [22]. For completeness, we recall the following result from [22] stating decay estimates in L^2 for the linear dynamics:

Proposition 1 (Decay in L^2 with Sobolev data)**.** *Let us consider equation (8) with zero mean initial data h_0. Then we have that the solution verifies*

$$\|h\|_{L^2} \le C\|h_0\|_{H^{s_0}}(1+t)^{-s}, \tag{10}$$

where

$$0 < s \le s_0.$$

With these results in mind, we propose the following reduced-order models of nonlocal and non-linear partial differential equations:

$$h_t + hh_\alpha = -\Lambda^{-1}(h) \quad \alpha\in\mathbb{T},\, t\in[0,T], \tag{11a}$$
$$h_0(\alpha,0) = h_0(\alpha) \quad \alpha\in\mathbb{T}, \tag{11b}$$

$$h_t + h^2h_\alpha = -\Lambda^{-1}(h) \quad \alpha\in\mathbb{T},\, t\in[0,T], \tag{12a}$$
$$h_0(\alpha,0) = h_0(\alpha) \quad \alpha\in\mathbb{T}, \tag{12b}$$

$$h_t + \frac{1}{2}\mathcal{H}\left(h^2\right)h_\alpha = -\Lambda^{-1}(h) \quad \alpha\in\mathbb{T},\, t\in[0,T], \tag{13a}$$
$$h_0(\alpha,0) = h_0(\alpha) \quad \alpha\in\mathbb{T}. \tag{13b}$$

The Hilbert transform in the periodic one-dimensional torus is defined as

$$\mathcal{H}(h)(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cot\left(\frac{\alpha-\beta}{2}\right)h(\beta)d\beta. \tag{14}$$

All three previous models share the linear dynamics with the full non-linear system (6) in the stable regime of the densities. Additionally, we have considered different non-linearities of quadratic and cubic order that correspond to various phenomena and provide an interesting link between the gravity-Stokes dynamics and other systems arising in fluid dynamics. In particular, the system (11) appears as an asymptotic model of Euler alignment for a specific kernel. Indeed, the Euler alignment system [42] reads

$$\rho_t + (v\rho)_\alpha = 0,$$
$$v_t + vv_\alpha = \int \phi(\alpha-\beta)(v(\beta)-v(\alpha))\rho(\beta)d\beta.$$

If we now make the ansatz

$$\rho = 1+\varepsilon^2 h, \quad v = \varepsilon\tilde{v},$$

we find

$$\varepsilon^2 h_t + \varepsilon(\tilde{v}(1+\varepsilon^2 h))_\alpha = 0,$$
$$\varepsilon\tilde{v}_t + \varepsilon^2\tilde{v}\tilde{v}_\alpha = \varepsilon\int \phi(\alpha-\beta)(\tilde{v}(\beta)-\tilde{v}(\alpha))d\beta + \varepsilon^3\int \phi(\alpha-\beta)(\tilde{v}(\beta)-\tilde{v}(\alpha))h(\beta)d\beta.$$

Neglecting terms of order ε^2 we find that the previous system decouples

$$\varepsilon h_t + \tilde{v}_\alpha = 0,$$
$$\tilde{v}_t + \varepsilon\tilde{v}\tilde{v}_\alpha = \int \phi(\alpha-\beta)(\tilde{v}(\beta)-\tilde{v}(\alpha))d\beta.$$

If we now take ϕ the kernel associated to Λ^{-1} we derive (11). This model is closely related with the so-called fractal Burgers equation, considered for example in [18].

Systems (12) and (13) are reduced-order models for the gravity Stokes system. If we formally write the contour dynamics equation (6) (setting $\rho^- - \rho^+ = 1$) as the sum of linear and non-linear contributions

$$h_t = \mathcal{L}(h) + \mathcal{N}(h),$$

we find by a formal expansion of (6) that the specific form of the non-linear part is

$$\mathcal{N}(h) = \mathcal{C}(h) + \text{ higher order terms},$$

where $\mathcal{C}$ is of cubic order (see Section 3 for a detailed derivation of $\mathcal{C}$). Consequently, we propose system (12) as a model with a cubic non-linearity which is of local transport type. This model can also be classified as a cubic fractal Burger's model. Finally, system (13) arises from the truncation of the contour dynamics form of the gravity Stokes system up to some cubic remainder. We notice that in this case, the cubic term is of non-local transport type, and it is closely related to reduced-order models of 3D Euler considered in [9, 10, 15].

For all these one-dimensional systems we prove the following (roughly stated) result [1]:

Theorem 3 (Global well-posedness vs. finite time singularities for 1D models)**.** *Let $h_0 \in H^n$ for n high enough be the initial data for* (11)*,* (12) *or* (13)*. Then, if h_0 is small enough in appropriate Sobolev spaces the solution emanating from it is globally defined. At the contrary, if the initial data h_0 is large enough, the solution blows up in finite time.*

The finite-time blow-up for the 1D models is attained following the ideas in [3, 9, 10, 18]. This result shows that the fine structure of the nonlinearity is required to have globally defined smooth solutions for the free boundary. Indeed, we show that when only some of the nonlinear interactions are considered, the linear damping is unable to dominate the dynamics and large solutions blow up.

[1] See Section 3 for a more precise statement.

2 Global well-posedness for the gravity Stokes system

As in our previous work [22], we consider two incompressible, viscous and immiscible fluids filling the domain $\mathbb{T}\times\mathbb{R}$. Both fluids are separated by the curve

$$\Gamma(t)=\{(z_1(\alpha,t),z_2(\alpha,t));\quad \alpha\in[-\pi,\pi],\quad z(\alpha+2\pi k,t)=(2\pi k,0)+z(\alpha,t)\}.$$

Then the upper fluid fills the domain $\Omega^+(t)$, while the lower fluid lies in $\Omega^-(t)$. Thus, the problem that we consider reads

$$-\Delta u^\pm=-\nabla p^\pm-(0,\rho^\pm)^T \qquad x\in\Omega^\pm(t),t\in[0,T], \tag{15a}$$
$$\nabla\cdot u^\pm=0 \qquad x\in\Omega^\pm(t),t\in[0,T], \tag{15b}$$
$$[\![\nabla u+(\nabla u)^T-p\,\mathrm{Id}]\!]\cdot(\partial_\alpha z)^\perp=0 \qquad x\in\Gamma(t),t\in[0,T], \tag{15c}$$
$$[\![u]\!]=0 \qquad x\in\Gamma(t),t\in[0,T], \tag{15d}$$
$$z_t=u(z,t) \qquad t\in[0,T], \tag{15e}$$
$$z=z_0 \qquad t=0. \tag{15f}$$

We observe that there exists a constant M big enough such that

$$\mathbb{T}\times[M,+\infty)\subset\Omega^+(t)\quad\text{and}\quad\mathbb{T}\times(-\infty,-M]\subset\Omega^-(t).$$

We recall that the velocity field can be expressed as a convolution of the density function with the so-called x_1-periodic Stokeslet

$$u=(\nabla^\perp\Delta^{-2}\nabla^\perp)(0,\rho)^T=\mathcal{S}*(0,\rho)^T \tag{16}$$

where

$$\mathcal{S}(y)=\frac{1}{8\pi}\log\left(2(\cosh(y_2)-\cos(y_1))\right)\cdot I-\frac{y_2}{8\pi(\cosh(y_2)-\cos(y_1))}\begin{pmatrix}-\sinh(y_2) & \sin(y_1)\\ \sin(y_1) & \sinh(y_2)\end{pmatrix}. \tag{17}$$

See our previous work [22] for a rigorous derivation of $\mathcal{S}$. A direct calculation shows that the component $\mathcal{S}_{1,1}(y)$ growths as $|y_2|$ if $y_2\to\infty$. However, from the explicit expression of $\mathcal{S}$ in (17) and the structure of the gravity force in (16), the term $\mathcal{S}_{1,1}$ is canceled in the convolution and therefore

$$u_1(x,t)=\frac{1}{8\pi}\int_{\mathbb{T}\times\mathbb{R}}\frac{-y_2\sin(y_1)}{\cosh(y_2)-\cos(y_1)}\rho(x-y,t)dy,$$
$$u_2(x,t)=\frac{1}{8\pi}\int_{\mathbb{T}\times\mathbb{R}}\left(\log\left(2(\cosh(y_2)-\cos(y_1))\right)-\frac{y_2\sinh(y_2)}{\cosh(y_2)-\cos(y_1)}\right)\rho(x-y,t)dy.$$

In compact notation,

$$u=\tilde{\mathcal{S}}*\rho=(\tilde{\mathcal{S}}_1*\rho,\tilde{\mathcal{S}}_2*\rho),$$

where

$$\tilde{\mathcal{S}}=\frac{1}{8\pi}\left(\frac{-y_2\sin(y_1)}{\cosh(y_2)-\cos(y_1)},\log(2(\cosh(y_2)-\cos(y_1)))-\frac{y_2\sinh(y_2)}{\cosh(y_2)-\cos(y_1)}.\right)$$

In the new kernels above we have found extra cancellation, obtaining better integrability properties as shown in the next Lemma.

Lemma 1. *The Stokeslet applied to the gravity force satisfies*

$$\tilde{\mathcal{S}}\in L^p(\mathbb{T}\times\mathbb{R})\ \ \textit{for}\ \ 1\le p<\infty,\quad\textit{and}\ \ \nabla\tilde{\mathcal{S}}\in L^q(\mathbb{T}\times\mathbb{R})\ \ \textit{for}\ \ 1\le q<2.$$

Proof. We will study the behavior of these kernels at 0 and ∞, where they can develop singularities.

In a neighborhood B_0 of $y=0$, the x_1-periodic Stokeslet behaves like the classical Stokeslet in the 2D plane and it is easy to check that it is integrable (it behaves as $\log|y|$), so that

$$\|\tilde{\mathcal{S}}\|_{L^p(B_0)}<\infty,\ \text{for}\ p<\infty.$$

At $|y_2| \to \infty$, it is clear that
$$|\tilde{\mathcal{S}}_1(y)| \lesssim |y_2| e^{-|y_2|}.$$

Furthermore, we prove that $\tilde{\mathcal{S}}_2$ also decays exponentially when $|y_2| \to +\infty$. On the one hand, let us fix y_1 such that $\cos(y_1) \neq 0$. We compute

$$\begin{aligned}\partial_{y_2} \tilde{\mathcal{S}}_2(y) &= -\frac{y_2 \cosh(y_2)}{\cosh(y_2) - \cos(y_1)} + \frac{y_2 \sinh^2(y_2)}{(\cosh(y_2) - \cos(y_1))^2} \\ &= -\frac{y_2(1 + \cosh(y_2)\cos(y_1))}{(\cosh(y_2) - \cos(y_1))^2}.\end{aligned}$$

Then,
$$\lim_{y_2 \to +\infty} \frac{\tilde{\mathcal{S}}_2(y)}{-\cos(y_1)\int_{y_2}^{\infty} \frac{s}{\cosh(s)} ds} = \lim_{y_2 \to +\infty} \frac{\partial_{y_2} \tilde{\mathcal{S}}_2(y)}{\cos(y_1)\frac{y_2}{\cosh(y_2)}} = 1.$$

On the other hand, when $\cos(y_1) = 0$,

$$\lim_{y_2 \to +\infty} \frac{\tilde{\mathcal{S}}_2(y_2)}{\int_{y_2}^{\infty} \frac{s}{\cosh(s)} ds} = 0.$$

This proves that
$$\tilde{\mathcal{S}}_2(y) \sim -\cos(y_1) \int_{y_2}^{\infty} \frac{s}{\cosh(s)} ds$$
for $y_2 \to \infty$, which translates into exponential decay in y_2 of $\tilde{\mathcal{S}}_2(y)$. Namely,

$$|\tilde{\mathcal{S}}_2(y)| \lesssim e^{-\frac{y_2}{2}}.$$

The same analysis holds for the case $y_2 \to -\infty$.

Therefore, we have proved that $\tilde{\mathcal{S}} \in L^p(\mathbb{T} \times \mathbb{R})$ for $p < \infty$. As $\nabla \tilde{\mathcal{S}}$ behaves as $|y|^{-1}$ close to zero and has a similar exponential decay at infinity, a direct approach provides $\nabla \tilde{\mathcal{S}} \in L^q(\mathbb{T} \times \mathbb{R})$ for $1 \leq q < 2$. □

This particular behavior of the kernel for the gravity forcing will have consequences in the boundedness of the velocity field and its derivatives, which at the same time will imply regularity of the free interface. With this in mind, we prove the following result:

Lemma 2. *The velocity u solving (5) is integrable in the vertical strip $\mathbb{T} \times \mathbb{R}$.*

Proof. We split the density as
$$\rho = \rho^c(x_1, x_2, t) + \rho^\infty(x_2, t),$$
where
$$\rho^c = \begin{cases} \rho^+, & z_2(\alpha, t) \leq x_2 \leq 2\|z_2(t)\|_{L^\infty}, \\ \rho^-, & -2\|z_2(t)\|_{L^\infty} \leq x_2 \leq z_2(\alpha, t), \\ 0 & \text{otherwise} \end{cases}$$
and
$$\rho^\infty = \begin{cases} \rho^+, & x_2 > 2\|z_2(t)\|_{L^\infty}, \\ \rho^-, & x_2 < -2\|z_2(t)\|_{L^\infty}, \\ 0 & \text{otherwise.} \end{cases}$$
Then, at the level of the stream function,

$$\Delta^2 \psi = \partial_{x_1} \rho^c + \partial_{x_1} \rho^\infty.$$

Note that $\partial_{x_1} \rho^\infty = 0$ in the weak sense, as ρ_∞ only depends on x_2 and t. Hence, ρ^∞ does not contribute to ψ, and consequently neither does to the velocity field u. Therefore, the velocity becomes
$$u(x, t) = \tilde{\mathcal{S}} * \rho^c(x, t). \tag{18}$$

This trick can also be seen by including ρ^∞ into the pressure as $\nabla q(x,t) = \rho^\infty(x_2,t)$ for some continuous function q. As a consequence,

$$\|u\|_{L^p} \leq \|\rho_0\|_{L^\infty} \|\tilde{\mathcal{S}}\|_{L^1} (4\|z_2(t)\|_{L^\infty})^{1/p}.$$

□

The structure of the gravity force allows to deal with the growth of the two-dimensional Stokeslet. Equipped with these estimates, let us proceed to prove the main result of this paper.

Proof of Theorem 1. Let us consider a Lagrangian approach: let $X(a,t)$ be the trajectory with initial data $X(a,0) = a$ driven by the velocity field $u(x,t)$ as in (16). Then, the free boundary is transformed as

$$X(z_0(\alpha),t) = z(\alpha,t).$$

The velocity of the trajectories is

$$\frac{d}{dt} X(a,t) = u(X(a,t),t)$$

and the gradient evolves as

$$\frac{d}{dt} \nabla_a X(a,t) = \nabla u(X(a,t),t) \nabla_a X(a,t).$$

Consequently,

$$\|X - \mathrm{Id}\,\|_{L^\infty} \leq \int_0^t \|u(s)\|_{L^\infty} ds, \tag{19}$$

$$\exp\left(-\int_0^t \|\nabla u(s)\|_{L^\infty} ds\right) \leq \|\nabla X\|_{L^\infty} \leq \exp\left(\int_0^t \|\nabla u(s)\|_{L^\infty} ds\right) \tag{20}$$

and similarly

$$\begin{aligned} \frac{d}{dt} |\nabla X|_\gamma &\leq \|\nabla u\|_{L^\infty} |\nabla X|_\gamma + |\nabla u|_\gamma \|\nabla X\|_{L^\infty}^{1+\gamma} \\ &\leq \|\nabla u\|_{L^\infty} |\nabla X|_\gamma + |\nabla u|_\gamma \left(\exp\left(\int_0^t \|\nabla u(s)\|_{L^\infty} ds\right)\right)^{1+\gamma}. \end{aligned} \tag{21}$$

Control of the supremum of the free boundary is given by control of $\|u\|_{L^\infty}$, due to (19). Indeed, we only need to control the second component $X(z_0(\alpha),t)_2 = z_2(\alpha,t)$ in L^∞, which is the only one that could be unbounded, namely

$$\frac{d}{dt} z_2(\alpha,t) = u_2(z(\alpha,t),t),$$

so that

$$\|z_2(t)\|_{L^\infty} \leq \|z_2(0)\|_{L^\infty} + \int_0^t \|u(s)\|_{L^\infty} ds.$$

We observe that Lemma 1 gives the uniform control

$$\|u\|_{L^\infty} \lesssim \|\tilde{\mathcal{S}}\|_{L^1} \|\rho_0\|_{L^\infty}, \tag{22}$$

thus

$$\|z_2\|_{L^\infty} \leq \|z_0\|_{L^\infty} + C\|\rho_0\|_{L^\infty} t.$$

Now, we note from (20) and (21), that control of $\|\nabla u\|_{L^\infty}$ and $|\nabla u|_\gamma$ translates into control of $|X|_{C^{1+\gamma}}$. For the Lipschitz norm of the velocity field u, it is possible to bound as follows

$$\|\nabla u\|_{L^\infty} \leq \|\nabla \tilde{\mathcal{S}}\|_{L^1} \|\rho_0\|_{L^\infty},$$

as a consequence of Lemma 1 . Additionally, by Sobolev embedding

$$|\nabla u|_\gamma \leq \|\nabla^2 u\|_{L^p},$$

for $2 < p < \infty$ and $\gamma = 1 - \frac{2}{p}$. Moreover,

$$\|\nabla^2 u\|_{L^p} = \|\nabla^2 \partial_{x_1} \nabla^\perp \Delta^{-2}(\rho^c)\|_{L^p} \leq C\|\rho^c\|_{L^p} \leq C(\|\rho_0\|_{L^\infty})\|z_2\|_{L^\infty}^{1/p} \leq C(1+t)^{1/p}$$

by Calderon-Zygmund theory, since $\nabla^2\nabla^\perp\Delta^{-2}\nabla^\perp$ is a 0-th order kernel. Consequently, the control of $\|\nabla u\|_{L^\infty}$ and $|\nabla u|_\gamma$ holds for every positive time $t > 0$.

Furthermore, the bound of the Lipschitz norm of the velocity field ensures that the arc-chord condition does not collapse at any finite time. Indeed, for $a \neq b$

$$\frac{d}{dt}\frac{X(b,t) - X(a,t)}{|b-a|} = \frac{u(X(b,t),t) - u(X(a,t))}{|b-a|},$$

and in particular

$$\frac{d}{dt}\frac{|X(b,t) - X(a,t)|}{|b-a|} \geq -\|\nabla u\|_{L^\infty}\frac{|X(b,t) - X(a,t)|}{|b-a|}.$$

Therefore, using Gronwall's Lemma it is possible to obtain

$$\exp\left(-\int_0^t \|\nabla u\|_{L^\infty} ds\right) \leq \frac{|X(a,t) - X(b,t)|}{|a-b|},$$

so the arc-chord condition does not collapse.

We can conclude then that

$$X(a,t) \in C([0,T]; C^{1+\gamma})$$

and the associated free boundary

$$z(\alpha,t) \in C([0,T]; C^{1+\gamma})$$

for $0 < \gamma < 1$ and any $T > 0$, which provides the existence of global regular solutions and in particular of the nonlocal contour dynamics system given in (6), where the free boundary lacks self-intersections.

In particular, collecting the previous bounds, we have that

$$\|\nabla X(t)\|_{L^\infty} \leq e^{Ct}.$$

This is in strong agreement with the exponential growth obtained in [22].

□

Corollary 1 (Existence of weak solutions)**.** *Let $z_0(\alpha)$ be the initial data for the two-phase Stokes problem (5) satisfying the arc-chord condition. Assume that $z_0(\alpha) \in C^\gamma(\mathbb{T})$, then, there exists a weak solution*

$$z(\alpha,t) \in C([0,T]; C^\gamma)$$

for any $T > 0$. Similarly, assume that $z_0(\alpha) \in W^{1,\infty}(\mathbb{T})$, then, there exists a weak solution

$$z(\alpha,t) \in C([0,T]; W^{1,\infty})$$

for any $T > 0$.

Proof. As noted in [31], any Lagrangian density patch solution with integrable velocity is also a weak (distributional) solution. Thus, to obtain the previous result it is enough mollify the initial data z_0 and invoke Theorem 1 and Lemma 2. □

Corollary 2 (Uniqueness of Lagrangian solutions)**.** *We consider two Lagrangian solutions $X^1(a,t)$ and $X^2(a,t)$ emanating from the initial data $X^1(a,0) = X^2(a,0) = a$, where $a \in \mathbb{T}\times\mathbb{R}$. Then, for every $T > 0$,*

$$X^1(a,t) = X^2(a,t), \quad t \in [0,T].$$

In particular, this shows uniqueness for stronger classes of solutions: weak and regular solutions.

Proof. We can write the velocity field as

$$u(x,t) = \int_{\mathbb{T}\times\mathbb{R}} \tilde{\mathcal{S}}\left(x-y\right)\right) \rho(y,t)dy = \int_{\mathbb{T}\times\mathbb{R}} \tilde{\mathcal{S}}\left(x - X^j(b,t)\right)\big) \rho_0(b)db$$

via the change of variables $y = X^j(b,t)$ (and $dy = db$ as a consequence of incompressibility). We also note that the density is transported through the trajectory, so that $\rho(X^j(b,t),t) = \rho_0(b)$. In particular,

$$X^j(a,t) = a + \int_0^t \int_{\mathbb{T}\times\mathbb{R}} \tilde{\mathcal{S}}\left(X^j(a,s) - X^j(b,s)\right)\big) \rho_0(b)db.$$

Then, the difference of two Lagrangian solutions emanating from the same initial data reads

$$\begin{aligned} X^2(a,t) - X^1(a,t) = \int_0^t \int_{\mathbb{T}\times\mathbb{R}} & \nabla\tilde{\mathcal{S}}\left((1-\lambda)X^2(a,s) + \lambda X^1(a,s) - (1-\lambda)X^2(b,s) - \lambda X^1(b,s)\right) \\ & \times \left(X^2(a,s) - X^1(a,s) - X^2(b,s) + X^1(b,s)\right) \rho_0(b)db, \end{aligned}$$

where we used the mean value theorem and $\lambda \in (0,1)$. Hence, since

$$b \mapsto (1-\lambda)X^2(a,s) + \lambda X^1(a,s) - (1-\lambda)X^2(b,s) - \lambda X^1(b,s)$$

is a local isomorphism at least for some time interval $[0,T_1]$ with $T_1 > 0$ (to be proved at the end), we can estimate

$$\|X^2(a,t) - X^1(a,t)\|_{L^\infty} \lesssim 2t\|\nabla\tilde{\mathcal{S}}\|_{L^1}\|\rho_0\|_{L^\infty} \sup_{s\in[0,t]} \|X^2(a,s) - X^1(a,s))\|_{L^\infty}.$$

Choosing

$$0 < t < \frac{1}{4\|\nabla\tilde{\mathcal{S}}\|_{L^1}\|\rho_0\|_{L^\infty}} = T_2, \quad t \le T_1$$

and taking supremum in t with this restriction,

$$\begin{aligned} \sup_{t\in(0,\min\{T_1,T_2\})} \|X^2(a,t) - X^1(a,t)\|_{L^\infty} &\lesssim \frac{1}{2} \sup_{t\in(0,\min\{T_1,T_2\})} \sup_{s\in[0,t]} \|X^2(a,s) - X^1(a,s))\|_{L^\infty} \\ &\lesssim \frac{1}{2} \sup_{t\in(0,\min\{T_1,T_2\}]} \|X^2(a,t) - X^1(a,t))\|_{L^\infty}. \end{aligned}$$

Therefore,

$$X^2(a,t) - X^1(a,t) = 0 \text{ for } t \in (0, \min\{T_1,T_2\}).$$

We can iterate this argument to get uniqueness for any size $T > 0$ of the time interval.

Finally, let us prove that

$$b \mapsto (1-\lambda)X^2(a,s) + \lambda X^1(a,s) - (1-\lambda)X^2(b,s) - \lambda X^1(b,s)$$

is a local isomorphism, which reduces to prove that

$$b \mapsto (1-\lambda)X^2(b,s) + \lambda X^1(b,s)$$

is a local isomorphism. We have that

$$\frac{d}{dt}\left(\nabla X^j(a,t)\right) = \nabla u(X^j(a,t),t)\nabla X^j(a,t).$$

Consequently,

$$\frac{d}{dt}\left(\nabla X^j(a,t) - \mathrm{Id}\right) = \nabla u(X^j(a,t),t)\left(\nabla X^j(a,t) - \mathrm{Id}\right) + \nabla u(X^j(a,t),t),$$

so that, via a Gronwall estimate,

$$\|\nabla X^j(\cdot,t) - \mathrm{Id}\,\|_{L^\infty} \le t\|\nabla u\|_{L^\infty} \exp\left(t\|\nabla u\|_{L^\infty}\right).$$

and in particular

$$\|\nabla\left((1-\lambda)X^2(\cdot,t) + \lambda X^1(\cdot,t)\right) - \mathrm{Id}\,\|_{L^\infty} \le t\|\nabla u\|_{L^\infty} \exp\left(t\|\nabla u\|_{L^\infty}\right).$$

This shows the desired result for $0 < t \sim 0$. □

In fact, the previous ideas can be used to consider also different type of solutions. Indeed, if we consider now densities ρ that, instead of piecewise constant step functions, enjoy certain integrability at infinity, we can prove Theorem 2.

Proof of Theorem 2. We recall Lemma 1. Using the conservation of the L^p norm of the density, it is possible to get

$$\|u\|_{L^p} \le \|\tilde{\mathcal{S}}\|_{L^1}\|\rho_0\|_{L^p} \text{ for } 1 \le p \le \infty.$$

Additionally, since $\nabla\tilde{\mathcal{S}} \in L^q$ for $1 \le q < 2$, we get that

$$\|\nabla u\|_{L^\infty} \le \|\nabla\tilde{\mathcal{S}}\|_{L^q}\|\rho_0\|_{L^p} \text{ with } \frac{1}{q} + \frac{1}{p} = 1,$$

for p in the range $2 < p \le \infty$. Finally, we have that, for $1 < p < \infty$,

$$\|\nabla^2 u\|_{L^p} \le \|\rho_0\|_{L^p},$$

due to Calderon-Zygmund theory. Thus, for the common range $2 < p < \infty$, we can consider the Lagrangian trajectories

$$\frac{d}{dt}X(a,t) = u(X(a,t),t),$$

and find that

$$X(a,t) \in C^1([0,T];C^{1+\gamma})$$

for $0 < \gamma < 1$, analogously as in Theorem 1. The uniqueness follows as in Corollary 2. This concludes the result. □

3 Global well-posedness vs. finite time singularities in several one-dimensional models of the internal waves for the gravity Stokes system

3.1 Derivation of the reduced order model (13)

Recall the gravity Stokes contour dynamics equation for an initial interface given by a graph in the stable regime of the densities (see [22] for more details):

$$\begin{aligned}
h_t(\alpha,t) &= \frac{\bar{\rho}}{2\pi}\int_{\mathbb{T}} \log\left(4\sin^2\left(\frac{\alpha-\beta}{2}\right)\right) h(\beta)[1+h_\alpha(\alpha)h_\alpha(\beta)]d\beta \\
&+ \frac{\bar{\rho}}{2\pi}\int_{\mathbb{T}} \log\left(\frac{\sinh^2\left(\frac{h(\alpha,t)-h(\beta,t)}{2}\right)}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} + 1\right) h(\beta)[1+h_\alpha(\alpha)h_\alpha(\beta)]d\beta \\
&+ \frac{\bar{\rho}}{2\pi}\int_{\mathbb{T}} \frac{h(\beta)(h(\alpha)-h(\beta))}{2\left(\sinh^2\left(\frac{h(\alpha,t)-h(\beta,t)}{2}\right)+\sin^2\left(\frac{\alpha-\beta}{2}\right)\right)}\left[(h_\alpha(\alpha)h_\alpha(\beta)-1)\sinh(h(\alpha)-h(\beta))\right]d\beta \\
&+ \frac{\bar{\rho}}{2\pi}\int_{\mathbb{T}} \frac{h(\beta)(h(\alpha)-h(\beta))}{2\left(\sinh^2\left(\frac{h(\alpha,t)-h(\beta,t)}{2}\right)+\sin^2\left(\frac{\alpha-\beta}{2}\right)\right)}\left[(h_\alpha(\alpha)+h_\alpha(\beta))\sin(\alpha-\beta)\right]d\beta.
\end{aligned}$$

Here, $\bar{\rho} = \frac{\rho^- - \rho^+}{4}$, and in particular, in the stable case $\bar{\rho} > 0$. For simplicity, in the following we will assume $\bar{\rho} = 1$.

Let us consider a formal decomposition of the contour dynamics equation above in its linear and non-linear parts:

$$h_t = \mathcal{L}(h) + \mathcal{N}(h).$$

The linear contribution for this equation is only coming from the first term, namely,

$$\mathcal{L}(h) = -\Lambda^{-1}(h)$$

The following contributions are of cubic order (there are no quadratic terms) and they come from all the four terms. Thus,

$$\mathcal{N}(h) = \mathcal{C}(h) + \text{ higher order terms},$$

where $\mathcal{C}$ is the cubic contribution. By performing a formal Taylor expansion of the nonlocal equation, we find explicitly the following cubic terms:

$$\mathcal{C}_1(h) = \frac{1}{2} h_\alpha(\alpha) \mathcal{H}(h^2).$$

$$\mathcal{C}_2(h) = \frac{1}{4}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(h(\alpha)-h(\beta))^2}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} h(\beta) d\beta.$$

$$\mathcal{C}_3(h) = -\frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(h(\alpha)-h(\beta))^2}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} h(\beta) d\beta.$$

We note that

$$\begin{aligned} \mathcal{C}_2(h)+\mathcal{C}_3(h) &= \frac{1}{4}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(h(\alpha)-h(\beta))^3}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta - h(\alpha)\frac{1}{4}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{2h(\alpha)(h(\alpha)-h(\beta))-(h(\alpha)^2-h(\beta)^2)}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta \\ &= \frac{1}{4}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(h(\alpha)-h(\beta))^3}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta + h^2\Lambda(h) - \frac{1}{2} h\Lambda(h^2). \end{aligned}$$

We also find the cubic term

$$\begin{aligned} \mathcal{C}_4(h) &= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{h(\alpha)-h(\beta)}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} h(\beta)(h_\alpha(\alpha)+h_\alpha(\beta))\sin(\alpha-\beta) d\beta \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \cot\left(\frac{\alpha-\beta}{2}\right) h(\beta)(h(\alpha)-h(\beta))(h_\alpha(\alpha)+h_\alpha(\beta)) d\beta \\ &= \frac{1}{2}(h^2)_\alpha \mathcal{H}(h) - h_\alpha \mathcal{H}(h^2) + \frac{1}{2} h\Lambda(h^2) - \frac{1}{3}\Lambda(h^3). \end{aligned}$$

All together, we have that

$$\mathcal{C}(h) = -\frac{1}{2} h_\alpha \mathcal{H}(h^2) + h^2\Lambda(h) + \frac{1}{2}(h^2)_\alpha \mathcal{H}(h) - \frac{1}{3}\Lambda(h^3) + \frac{1}{4}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(h(\alpha)-h(\beta))^3}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta.$$

We will select the linear term and the fist cubic term in $\mathcal{C}(h)$ as our model. Motivation for this type of model can be seen at [3, ?, 9, 10, 15] .

Above, the fractional Laplacian $\Lambda(h) = (-\Delta)^{1/2}(h)$ is defined as

$$\Lambda(h)(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(\alpha)-h(\beta)}{2\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta. \tag{23}$$

Note that the following relations hold:

$$\partial_\alpha \Lambda^{-1}(h) = -\mathcal{H}(h), \quad \partial_\alpha \mathcal{H}(h) = \Lambda(h).$$

For this reduced model of the Stokes gravity contour dynamics equation (13) and related systems (11) and (12), we prove global in time well-posedness for small initial data and the formation of finite time singularities under certain conditions on the initial data. In the following sections, since we focus on one-dimensional models and there is no ambiguity, we will use the standard notation $x \in \mathbb{T}$ (or y as auxiliary variable) to denote the physical one-dimensional variable, instead of $\alpha \in \mathbb{T}$ (or β as auxiliary variable) .

3.2 Global well-posedness

In this section we prove the global existence of solutions for systems (11), (12) and (13), given that the initial data is small enough in Sobolev spaces of good enough regularity. We recall that the three systems share the same linear dynamics, and this is crucial for the proof of global existence. In fact, in order to control the non-linear dynamics we will exploit Proposition 1, which shows decay estimates of the linear operator Λ^{-1}, which will be transferred to the non-linear part via Duhamel principle.

Theorem 4 (Existence of solutions for small data)**.** *Let $h_0 \in \dot{H}^4(\mathbb{T})$ be a zero mean initial data for the Cauchy problem (11). Then, if $\|h_0\|_{\dot{H}^4(\mathbb{T})}$ is small enough, there exists a solution*

$$h \in C([0,T], H^4), \forall T > 0.$$

Furthermore, the solution verifies

$$\|h(t)\|_{L^2} \leq C(1+t)^{-2.25}.$$

Proof. We will omit the time dependence of h when it is clear from the context. The proof is based on finding a polynomial estimate for the quantity

$$|||h||| = \sup_{t\in[0,T]} \left((1+t)^{2.25}\|h\|_{L^2} + \|h\|_{\dot{H}^4}\right). \tag{24}$$

Using the Duhamel formula, we can write the solution as

$$h(x,t) = e^{-\Lambda^{-1}t}h_0(x) + \int_0^t e^{-\Lambda^{-1}(t-s)}\left[h(x,s)h_x(x,s)\right] ds. \tag{25}$$

Taking the L^2 norm and using Proposition 1, we get

$$\begin{aligned}
\|h(t)\|_{L^2} &\leq \|e^{-\Lambda^{-1}t}h_0\|_{L^2} + \int_0^t \|e^{-\Lambda^{-1}(t-s)}\left[h(s)h_x(s)\right]\|_{L^2} ds, \\
&\leq C\|h_0\|_{\dot{H}^{2.5}}(1+t)^{-2.25} + C\int_0^t (1+t-s)^{-2.25}\|h(s)h_x(s)\|_{\dot{H}^{2.5}} ds, \\
&\lesssim \|h_0\|_{\dot{H}^{2.5}}(1+t)^{-2.25} + \int_0^t (1+t-s)^{-2.25}\|h^2(s)\|_{\dot{H}^{3.5}} ds, \\
&\lesssim \|h_0\|_{\dot{H}^{2.5}}(1+t)^{-2.25} + \int_0^t (1+t-s)^{-2.25}\|h(s)\|_{L^\infty}\|h(s)\|_{\dot{H}^{3.5}} ds.
\end{aligned}$$

Using interpolation, we find that

$$\begin{aligned}\|h(t)\|_{L^2} &\lesssim \|h_0\|_{\dot H^{2.5}}(1+t)^{-2.25}\\ &\quad+\int_0^t (1+t-s)^{-2.25}\|h(s)\|_{L^2}^{7/8}\|h(s)\|_{\dot H^4}^{1/8}\|h(s)\|_{L^2}^{1/8}\|h(s)\|_{\dot H^4}^{7/8}ds,\\ &\lesssim \|h_0\|_{\dot H^{2.5}}(1+t)^{-2.25}\\ &\quad+\int_0^t (1+t-s)^{-2.25}(1+s)^{-2.25}(1+s)^{2.25}\|h(s)\|_{L^2}\|h(s)\|_{\dot H^4}ds,\\ &\lesssim \|h_0\|_{\dot H^{2.5}}(1+t)^{-2.25}+|||h|||^2\int_0^t(1+t-s)^{-2.25}(1+s)^{-2.25}ds,\\ &\lesssim \|h_0\|_{\dot H^{2.5}}(1+t)^{-2.25}+|||h|||^2\,(1+t)^{-2.25},\end{aligned}$$

where we have used Moser inequality , Gagliardo-Nirenberg interpolation inequality and Lemma 2.4 in [19]. Then,

$$(1+t)^{2.25}\|h(t)\|_{L^2}\lesssim \|h_0\|_{\dot H^{2.5}}+|||h|||^2\,. \tag{26}$$

We have to bound $\|h\|_{\dot H^4}$ using energy estimates.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\partial_x^4 h|^2dx &= \int \partial_x^4 h\partial_x^4\left(-hh_x-\Lambda^{-1}h\right)dx,\\ &= -\int|\Lambda^{1/2}\partial_x^3 h|^2-\int\partial_x^4 h\partial_x^4(hh_x)dx,\\ &\le -5.5\int(\partial_x^4 h)^2h_xdx-10\int\partial_x^4 h\partial_x^3 h\partial_x^2 hdx,\\ &=: -5.5I_1-10I_2.\end{aligned}$$

We can bound the first contribution as follows

$$\begin{aligned}|I_1| &\le \|h_x\|_{L^\infty}\|h\|_{\dot H^4}^2,\\ &\lesssim \|h\|_{\dot H^4}^{3/8}\|h\|_{L^2}^{5/8}\|h\|_{\dot H^4}^2,\\ &\lesssim (1+t)^{-2.25\frac{5}{8}}\,|||h|||^3\,.\end{aligned}$$

The second term can be handled similarly

$$\begin{aligned}I_2 &= \int\partial_x^4 h\partial_x^3 h\partial_x^2 hdx,\\ &= -\int\partial_x^3 h(\partial_x^4 h\partial_x^2 h+\partial_x^3 h\partial_x^3 h)dx,\\ &= -I_2-\int(\partial_x^3 h)^3dx,\end{aligned}$$

which leads to

$$\begin{aligned}|I_2| &\le \int|\partial_x^3 h|^3dx,\\ &\le \|\partial_x^3 h\|_{L^\infty}\|\partial_x^3 h\|_{L^2}^2,\\ &\lesssim \|h\|_{\dot H^4}^{7/8}\|h\|_{L^2}^{1/8}\|h\|_{\dot H^4}^{2\cdot 3/4}\|h\|_{L^2}^{2/4},\\ &\lesssim (1+t)^{-2.25\frac{5}{8}}\,|||h|||^3\,.\end{aligned}$$

Collecting all the estimates

$$\frac{d}{dt}\|h(t)\|_{\dot H^4}^2 \lesssim (1+t)^{-2.25\frac{5}{8}} \,|||h|||^3\,,$$

$$\begin{aligned}\|h(t)\|_{\dot H^4}^2 &\lesssim \|h_0\|_{\dot H^4}^2 + |||h|||^3 \int_0^t (1+s)^{-2.25\frac{5}{8}},\\ &\lesssim \|h_0\|_{\dot H^4}^2 + |||h|||^3 \int_0^\infty (1+s)^{-2.25\frac{5}{8}},\\ &\lesssim \|h_0\|_{\dot H^4}^2 + |||h|||^3\,.\end{aligned}$$

Thus,

$$\|h(t)\|_{\dot H^4} \lesssim \|h_0\|_{\dot H^4} + |||h|||^{\frac{3}{2}}\,. \tag{27}$$

As a conclusion, by (26) and (27), we find

$$|||h||| \lesssim 2\|h_0\|_{\dot H^4} + |||h|||^2 + |||h|||^{\frac{3}{2}}\,. \tag{28}$$

If the initial data $\|h_0\|_{\dot H^4}$ is small enough, we get that $|||h|||$ is bounded independently of the choice of T. As a consequence, global existence for small initial data is proved. □

Theorem 5 (Existence of solutions for small data)**.** *Let $h_0 \in \dot H^3(\mathbb{T})$ be a zero mean initial data for the Cauchy problem (12). Then, if $\|h_0\|_{\dot H^3(\mathbb{T})}$ is small enough, there exists a solution*

$$h \in C([0,T], H^3), \forall\, T>0.$$

Furthermore, the solution verifies

$$\|h(t)\|_{L^2} \le C(1+t)^{-1.25}.$$

Proof. The proof is based on a polynomial estimate on the energy

$$|||h||| = \sup_{t\in[0,T]} \left((1+t)^{1.25}\|h\|_{L^2} + \|h\|_{\dot H^3}\right). \tag{29}$$

Using the Duhamel formula, we can write the solution as

$$h(x,t) = e^{-\Lambda^{-1}t}h_0(x) + \int_0^t e^{-\Lambda^{-1}(t-s)}\left[h^2(x,s)h_x(x,s)\right]ds. \tag{30}$$

Taking the L^2 norm and using Proposition 1, we get that

$$\begin{aligned}\|h(t)\|_{L^2} &\le \|e^{-\Lambda^{-1}t}h_0\|_{L^2} + \int_0^t \|e^{-\Lambda^{-1}(t-s)}\left[h^2(s)h_x(s)\right]\|_{L^2}ds,\\ &\le C\|h_0\|_{\dot H^{1.5}}(1+t)^{-1.25} + C\int_0^t (1+t-s)^{-1.25}\|h^2(s)h_x(s)\|_{\dot H^{1.5}}ds,\\ &\lesssim \|h_0\|_{\dot H^{1.5}}(1+t)^{-1.25}\\ &\quad + \int_0^t (1+t-s)^{-1.25}\left(\|h(s)\|_{L^\infty}^2\|h(s)\|_{\dot H^{2.5}} + \|h^2(s)\|_{\dot H^{1.5}}\|h_x(s)\|_{L^\infty}\right)ds.\end{aligned}$$

Using interpolation, we find that

$$\begin{aligned}\|h(t)\|_{L^2} &\lesssim \|h_0\|_{\dot H^{1.5}}(1+t)^{-1.25}\\ &\quad + \int_0^t (1+t-s)^{-1.25}\left(\|h(s)\|_{L^2}^{1/6+5/3}\|h(s)\|_{\dot H^3}^{5/6+1/3} + \|h(s)\|_{L^2}^{1/2+5/6}\|h(s)\|_{\dot H^3}^{1/2+1/6+1}\right)ds,\\ &\lesssim \|h_0\|_{\dot H^{1.5}}(1+t)^{-1.25}\\ &\quad + \int_0^t (1+t-s)^{-1.25}\left((1+s)^{-1.25\cdot 11/6} + (1+s)^{-1.25\cdot 4/3}\right)|||h|||^3\,ds,\\ &\lesssim \|h_0\|_{\dot H^{1.5}}(1+t)^{-1.25} + |||h|||^3\,(1+t)^{-1.25},\end{aligned}$$

Then,

$$(1+t)^{1.25}\|h(t)\|_{L^2} \lesssim \|h_0\|_{\dot{H}^{1.5}} + |||h|||^3\,. \tag{31}$$

Now we estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\partial_x^3 h|^2 dx &= \int \partial_x^3 h \partial_x^3\left(-h^2h_x - \Lambda^{-1}h\right)dx,\\ &= -\int|\Lambda^{1/2}\partial_x^2 h|^2 - \int \partial_x^3 h\partial_x^3(h^2h_x)dx,\\ &\leq -\int \partial_x^3 h\partial_x^3(h^2h_x)dx,\\ &= C_1\int(\partial_x^3 h)^2 h_x h dx + C_2\int(\partial_x^2 h)^2\partial_x^3 h h dx + C_3\int(h_x)^2\partial_x^3 h\partial_x^2 h dx,\\ &=: C_1I_1 + C_2I_2 + C_3I_3.\end{aligned}$$

We now compute that

$$\begin{aligned}|I_1| &\lesssim \|h\|_{L^\infty}\|h_x\|_{L^\infty}\|h\|^2_{\dot{H}^3},\\ &\lesssim \|h\|_{L^2}^{5/6+1/2}\|h\|_{\dot{H}^3}^{2+1/6+1/2},\\ &\lesssim (1+t)^{-1.25\cdot 4/3}\,|||h|||^4\,.\end{aligned}$$

The second integral can be estimated as

$$\begin{aligned}|I_2| &\leq \|h\|_{L^\infty}\|\partial_x^3 h\|_{L^2}\|(\partial_x^2 h)^2\|_{L^2},\\ &\lesssim \|h\|_{L^\infty}\|h\|_{\dot{H}^3}\|\partial_x^2 h\|^2_{L^4},\\ &\lesssim \|h\|_{L^2}^{5/6+1/6}\|h\|_{\dot{H}^3}^{1+1/6+11/6},\\ &\lesssim (1+t)^{-1.25}\,|||h|||^4\,.\end{aligned}$$

Finally, the third integral is bounded as follows

$$\begin{aligned}|I_3| &\leq \|h_x\|^2_{L^\infty}\|\partial_x^3 h\|_{L^2}\|\partial_x^2 h\|_{L^2},\\ &\lesssim \|h\|_{L^2}^{1+1/3}\|h\|_{\dot{H}^3}^{2+2/3},\\ &\lesssim (1+t)^{-1.25\cdot 4/3}\,|||h|||^4\,.\end{aligned}$$

Putting every estimate together we find that

$$\begin{aligned}\frac{d}{dt}\|h(t)\|^2_{\dot{H}^3} &\lesssim (1+t)^{-1.25}\,|||h|||^4\,,\\ &\lesssim \|h_0\|^2_{\dot{H}^3} + |||h|||^4\int_0^\infty(1+s)^{-1.25}.\end{aligned}$$

Thus,

$$\|h(t)\|_{\dot{H}^3} \lesssim \|h_0\|_{\dot{H}^3} + |||h|||^2\,. \tag{32}$$

Finally, by (31) and (32),

$$|||h||| \lesssim \|h_0\|_{\dot{H}^3} + |||h|||^2 + |||h|||^3\,. \tag{33}$$

If the initial data $\|h_0\|_{\dot{H}^3}$ is small enough, we get that $|||h|||$ is bounded independently of the choice of T. As a consequence, global existence for small initial data is proved. □

Theorem 6 (Existence of solutions for small data)**.** *Let $h_0 \in H^3(\mathbb{T})$ be the initial data for the Cauchy problem* (13)*. Then, if $\|h_0\|_{H^3(\mathbb{T})}$ is small enough, there exists a solution*

$$h \in C([0,T],H^3),\ \forall T>0.$$

Furthermore, the solution verifies

$$\|h(t)\|_{L^2} \leq C(1+t)^{-1.25}.$$

Proof. As in the previous case, we estimate the energy

$$|||h||| = \sup_{t\in[0,T]} \left((1+t)^{1.25}\|h\|_{L^2} + \|h\|_{\dot{H}^3}\right). \tag{34}$$

We start with the low regularity norm. Using the Duhamel formula, we can write the solution as

$$h(x,t) = e^{-\Lambda^{-1}t}h_0(x) - \frac{1}{2}\int_0^t e^{-\Lambda^{-1}(t-s)}\left[\mathcal{H}(h^2)(x,s)h_x(x,s)\right] ds. \tag{35}$$

Taking the L^2 norm and using Proposition 1, we get

$$\begin{aligned}
\|h(t)\|_{L^2} &\le \|e^{-\Lambda^{-1}t}h_0\|_{L^2} + \frac{1}{2}\int_0^t \|e^{-\Lambda^{-1}(t-s)}\left[\mathcal{H}(h^2)(s)h_x(s)\right]\|_{L^2} ds,\\
&\le C\|h_0\|_{\dot{H}^{1.5}}(1+t)^{-1.25} + C\int_0^t (1+t-s)^{-1.25}\|\mathcal{H}(h^2)(s)h_x(s)\|_{\dot{H}^{1.5}} ds,\\
&\lesssim \|h_0\|_{\dot{H}^{1.5}}(1+t)^{-1.25} + |||h|||^3\,(1+t)^{-1.25}.
\end{aligned}$$

Then,

$$(1+t)^{2.25}\|h(t)\|_{L^2} \lesssim \|h_0\|_{\dot{H}^{1.5}} + |||h|||^3. \tag{36}$$

Now we continue with the high regularity term:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\partial_x^3 h|^2 dx &= \int \partial_x^3 h \partial_x^3\left(-\mathcal{H}(h^2)h_x - \Lambda^{-1}h\right) dx,\\
&= -\int |\Lambda^{1/2}\partial_x^2 h|^2 - \int \partial_x^3 h\partial_x^3(\mathcal{H}(h^2)h_x)dx,\\
&\le C_1\int \partial_x^3 h\partial_x^2\Lambda(h^2)h_x dx + C_2\int \partial_x(\partial_x^2 h)^2\partial_x\Lambda(h^2)dx + C_3\int \partial_x^3 h\Lambda(h^2)\partial_x^3 h dx,\\
&=: C_1 I_1 + C_2 I_2 + C_3 I_3.
\end{aligned}$$

As before we can estimate

$$\begin{aligned}
|I_1| &\lesssim \|h_x\|_{L^\infty}\|h^2\|_{\dot{H}^3}\|h\|_{\dot{H}^3},\\
&\lesssim \|h_x\|_{L^\infty}\|h\|_{L^\infty}\|h\|_{\dot{H}^3}^2,\\
&\lesssim \|h\|_{L^2}\|h\|_{\dot{H}^3}^3,\\
&\lesssim (1+t)^{-1.25}\,|||h|||^4.
\end{aligned}$$

The second term can be bounded as

$$\begin{aligned}
|I_2| &\lesssim \|(\partial_x^2 h)^2\|_{\dot{H}^1}\|h^2\|_{\dot{H}^2},\\
&\lesssim \|\partial_x^2 h\|_{L^\infty}\|h\|_{\dot{H}^3}\|h\|_{L^\infty}\|h\|_{\dot{H}^2},\\
&\lesssim \|h\|_{L^2}^{1+1/3}\|h\|_{\dot{H}^3}^{2+2/3},\\
&\lesssim (1+t)^{-1.25\cdot 4/3}\,|||h|||^4.
\end{aligned}$$

Finally, the third term can be estimated as

$$\begin{aligned}
|I_3| &\lesssim \|h\|_{\dot{H}^3}^2\|\Lambda h^2\|_{L^\infty},\\
&\lesssim \|h\|_{\dot{H}^3}^2\|h^2\|_{W^{1,\infty}},\\
&\lesssim \|h\|_{\dot{H}^3}^2\|h^2\|_{L^2}^{1/2}\|h^2\|_{\dot{H}^3}^{1/2},\\
&\lesssim \|h\|_{\dot{H}^3}^{2+1/2}\|h\|_{L^4}\|h\|_{L^\infty}^{1/2},\\
&\lesssim \|h\|_{\dot{H}^3}^{2+1/2+1/12+1/12}\|h\|_{L^2}^{11/12+5/12},\\
&\lesssim (1+t)^{-1.25\cdot 4/3}\,|||h|||^4.
\end{aligned}$$

Collecting every estimate we conclude that

$$\frac{d}{dt}\|h(t)\|_{\dot{H}^3}^2 \lesssim (1+t)^{-1.25} \, |||h|||^4 ,$$
$$\lesssim \|h_0\|_{\dot{H}^3}^2 + |||h|||^4 \int_0^\infty (1+s)^{-1.25}.$$

Then,

$$\|h(t)\|_{\dot{H}^3} \lesssim \|h_0\|_{\dot{H}^3} + |||h|||^2 . \tag{37}$$

Analogously as in the previous cubic model, these estimates prove global existence for small initial data now in the inhomogeneous H^3 due to the lack of hypothesis on the mean of the initial data. □

3.3 Finite-time singularities

Theorem 7 (Blow-up for the quadratic 1D model)**.** *Let $h(x,t)$ be a smooth solution of the Cauchy problem* (11)*. Then, there exists C such that if the initial data $h_0(x)$ satisfies*

$$\min_{x\in\mathbb{T}} h_x(x,0) \le -C,$$

$h_x(x,t)$ blows up in finite time.

Proof. We consider the evolution of

$$m(t) = \min_{x\in\mathbb{T}} h_x(x,t) = h_x(x_{\mathrm{m}}(t),t),$$

where $x_{\mathrm{m}}(t)$ denotes the point where the minimum is achieved. Using a pointwise argument, we have that

$$\frac{d}{dt}m(t) = -m^2(t) - \Lambda^{-1}h_x(x_{\mathrm{m}}(t),t).$$

We compute

$$\begin{aligned}
\Lambda^{-1}h_x(x_{\mathrm{m}}(t)) &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(4\sin^2\left(\frac{y}{2}\right)\right) h_x(x_{\mathrm{m}}(t)-y)dy,\\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sin^2\left(\frac{y}{2}\right)\right) h_x(x_{\mathrm{m}}(t)-y)dy,\\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sin^2\left(\frac{y}{2}\right)\right) (h_x(x_{\mathrm{m}}(t)-y)-h_x(x_{\mathrm{m}}(t)))dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sin^2\left(\frac{y}{2}\right)\right) h_x(x_{\mathrm{m}}(t))dy,\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sin^2\left(\frac{y}{2}\right)\right) (h_x(x_{\mathrm{m}}(t))-h_x(x_{\mathrm{m}}(t)-y))dy + \log(4)m(t),\\
&\ge \log(4)m(t).
\end{aligned}$$

From the previous computation we find that

$$\frac{d}{dt}m(t) \le -(m(t))^2 - \log(4)m(t).$$

This ODE blows up if $m(0)$ is large enough compared with $\log(4)$, which concludes the result.

□

Theorem 8 (Blow-up for the cubic local 1D model)**.** *Let $h(x,t)$ be a smooth solution of the Cauchy problem* (12)*. Then, given $0<\delta<1/2$, if*

$$\int_{-1}^{1} \left(-h_0^2(x) + h_0^2(0)\right)\left(|x|^{-\delta} - 1\right)\mathrm{sign}(x)dx \ge \|h_0\|_{L^2},$$

the solution $h(x,t)$ develops a finite-time singularity.

Remark 1. *The hypothesis of the previous theorem is needed to find a contradiction assuming that the solution does not develop a singularity for all times. By considering the smooth evolution of this non-local quantity in time (39), we find that it should blow up in finite-time. Moreover, it holds that $L(t) \lesssim \|h(t)\|^2_{L^\infty}$. However, according to the smooth evolution, $\|h\|_{L^\infty}$ is bounded for all finite times, and this is a contradiction.*

Proof. This proof is based on arguments shown in [18]. Firstly, note that we have uniform control of the L^2 norm by

$$\|h\|^2_{L^2} + 2\int_0^t \|\Lambda^{-1/2}h\|^2_{L^2} ds \leq \|h_0\|^2_{L^2}.$$

The L^∞ norm is also controlled via control of the maximum and minimum values of h. We define

$$M(t) = \max_{x\in\mathbb{T}} h(x,t), \quad m(t) = \min_{x\in\mathbb{T}} h(x,t).$$

If we assume that the supremum at time $t > 0$ is attained at the maximum point, i.e.,

$$\|h(\cdot,t)\|_{L^\infty} = M(t),$$

then $M(t) \geq 0$ and it holds

$$\frac{d}{dt}M(t) \leq \|\Lambda^{-1}h\|_{L^\infty} \leq C\|h_0\|_{L^2}.$$

Similarly, if the supremum is attained at the minimum point, then $m(t) \leq 0$ and a similar bound holds. Consequently,

$$\|h(\cdot,t)\|_{L^\infty} \leq \|h_0\|_{L^\infty} + C\|h_0\|_{L^2} t. \tag{38}$$

Let us define

$$W(x,t) = h^2(x + y(t),t), \quad y'(t) = h^2(y(t),t), \quad y(0) = 0.$$

Then, by multiplication of equation (12a) by h, one gets

$$\begin{aligned}\frac{d}{dt}W(x,t) &= \partial_t h^2(x+y(t),t) + \partial_x h^2(x+y(t),t)h^2(y(t),t),\\ &= \partial_x h^2(x+y(t),t)\left(h^2(y(t),t) - h^2(x+y(t),t)\right) - 2h(x+y(t),t)\Lambda^{-1}h(x+y(t),t),\\ &= \partial_x W(x,t)\left(W(0,t) - W(x,t)\right) - 2h(x+y(t),t)\Lambda^{-1}h(x+y(t),t).\end{aligned}$$

Now, given $\delta < 1/2$, we define the functional

$$L(t) = \int_{-1}^1 \left(-W(x,t) + W(0,t)\right)\left(|x|^{-\delta} - 1\right)\mathrm{sign}(x)dx. \tag{39}$$

Then, the evolution in time of $L(t)$ is given by

$$\begin{aligned}\frac{d}{dt}L(t) &= \int_{-1}^1 \left(-\partial_t W(x,t) + \partial_t W(0,t)\right)\left(|x|^{-\delta} - 1\right)\mathrm{sign}(x)dx,\\ &= \int_{-1}^1 \partial_x W(x,t)\left(W(x,t) - W(0,t)\right)\left(|x|^{-\delta} - 1\right)\mathrm{sign}(x)dx,\\ &\quad + 2\int_{-1}^1 \left(h(x+y(t),t)\Lambda^{-1}h(x+y(t),t) - h(y(t),t)\Lambda^{-1}h(y(t),t)\right)\left(|x|^{-\delta} - 1\right)\mathrm{sign}(x)dx,\\ &= I_1 + I_2.\end{aligned}$$

The first term can be estimated as

$$
\begin{aligned}
I_1 &= \frac{1}{2}\int_{-1}^{1} \partial_x \left(W(x,t)-W(0,t)\right)^2 \left(|x|^{-\delta}-1\right)\text{sign}(x)dx,\\
&= \frac{1}{2}\left\{\left(W(x,t)-W(0,t)\right)^2 \left(|x|^{-\delta}-1\right)\text{sign}(x)\right\}_{-1}^{1},\\
&\quad -\frac{1}{2}\int_{-1}^{1}\left(W(x,t)-W(0,t)\right)^2 \partial_x\left[\left(|x|^{-\delta}-1\right)\text{sign}(x)\right]dx.
\end{aligned}
$$

In fact, we compute

$$
\begin{aligned}
I_1 &= \delta\frac{1}{2}\int_{-1}^{1}\left(W(x,t)-W(0,t)\right)^2 |x|^{-\delta-1}dx,\\
&\geq \delta\frac{1}{4}\left(\int_{-1}^{1}|-W(x,t)+W(0,t)|\,|x|^{-\frac{\delta+1}{2}}dx\right)^2,\\
&\geq \delta\frac{1}{4}\left(\int_{-1}^{1}|-W(x,t)+W(0,t)|\left(|x|^{-\delta}-1\right)dx\right)^2,\\
&\geq C(\delta)L(t)^2,
\end{aligned}
$$

where we have used Hölder inequality and the fact that $\delta < 1/2$. The second term is bounded by

$$
\begin{aligned}
I_2 &= 2\int_{-1}^{1}\left(-h(x+y(t),t)\Lambda^{-1}h(x+y(t),t)+u(y(t),t)\Lambda^{-1}h(y(t),t)\right)\left(|x|^{-\delta}-1\right)\text{sign}(x)dx,\\
&\leq \|h\|_{L^2}\|\Lambda^{-1}h\|_{L^\infty}\||x|^{-\delta}-1\|_{L^2(-1,1)},\\
&\leq C(\delta)\|h_0\|_{L^2}^2,
\end{aligned}
$$

as long as $\delta < 1/2$.

Both estimates provide

$$
\frac{d}{dt}L(t) \geq C(\delta)\left(L(t)^2 - \|h_0\|_{L^2}^2\right). \tag{40}
$$

Assuming that

$$
L(0) - \|h_0\|_{L^2} > 0,
$$

the inequality (40) implies finite-time blow up of $L(t)$. This is a contradiction due to (38) and

$$
L(t) \lesssim \|h(t)\|_{L^\infty}^2.
$$

□

Theorem 9 (Blow-up for the cubic non-local 1D model)**.** *Let $h(x,t)$ be a smooth solution of the Cauchy problem (13). Then, if the initial data $h_0(x)$ has odd symmetry and we fix $0 < \delta < 1/2$, $h_x(x,t)$ blows up in finite time.*

Proof. If we repeat the previous pointwise estimate, we find that

$$
\frac{d}{dt}M(t) \leq \|\Lambda^{-1}h\|_{L^\infty} \leq CM(t).
$$

Using the same idea for $m(t)$, we conclude

$$
\|h(\cdot,t)\|_{L^\infty} \leq \|h_0\|_{L^\infty}e^{Ct}. \tag{41}
$$

As the initial data is odd and such symmetry is preserved by the evolution, the resulting solution is also odd. As a consequence, it conserves the mean as in the original problem (5). Let us define

$$
J(t) = \int_0^\infty \frac{h^2}{|x|^{1+\delta}}dx.
$$

for $\delta < \frac{1}{2}$. Then, by Cordoba-Cordoba-Fontelos inequality [9]

$$\begin{aligned}\frac{d}{dt}J(t) &= \int_0^\infty \frac{\partial_t h^2}{|x|^{1+\delta}}dx,\\ &= -\frac{1}{2}\int_0^\infty \frac{\mathcal{H}(h^2)\partial_x(h^2)}{|x|^{1+\delta}}dx - 2\int_0^\infty \frac{h\Lambda^{-1}h}{|x|^{1+\delta}}dx,\\ &\geq \frac{C(\delta)}{2}\int_0^\infty \frac{h^4}{|x|^{2+\delta}}dx - 2\int_0^\infty \frac{h\left(\Lambda^{-1}h(x)-\Lambda^{-1}h(0)\right)}{|x|^{1+\delta}}dx.\end{aligned}$$

Note that

$$\begin{aligned}\left|\Lambda^{-1}h(x)-\Lambda^{-1}h(x+h)\right| &\leq |h|^{1/2}\left|\Lambda^{-1}h\right|_{\dot{C}^{1/2}},\\ &\lesssim |h|^{1/2}\|h_0\|_{L^\infty}e^{Ct}.\end{aligned}$$

Furthermore,

$$\begin{aligned}-\int_0^\infty \frac{h\left(\Lambda^{-1}h(x)-\Lambda^{-1}h(0)\right)}{|x|^{1+\delta}}dx &= -\int_0^L \frac{h\left(\Lambda^{-1}h(x)-\Lambda^{-1}h(0)\right)}{|x|^{1+\delta}}dx - \int_L^\infty \frac{h\left(\Lambda^{-1}h(x)-\Lambda^{-1}h(0)\right)}{|x|^{1+\delta}}dx,\\ &\geq -C_1\|h_0\|_{L^\infty}^2e^{2C_2t}\int_0^L \frac{1}{|x|^{1/2+\delta}}dx - C_1\|h_0\|_{L^\infty}^2e^{2C_2t}\int_L^\infty \frac{1}{x^{1+\delta}}dx,\\ &\geq -C\|h_0\|_{L^\infty}^2e^{Ct}.\end{aligned}$$

Now, by Hölder inequality,

$$\int_0^1 \frac{h^4}{x^{2+\delta}}dx \geq C\left(\int_0^1 \frac{h^2}{x^{1+\delta}}dx\right)^2.$$

Owing to the control of h in L^∞ and the integrability of $\int_1^\infty x^{-1-\delta}$ and $\int_1^\infty x^{-2-\delta}$, we conclude

$$\int_0^\infty \frac{h^4}{x^{2+\delta}}dx \geq C(\delta,\|h_0\|_{L^\infty})\left(J^2(t)-J(t)e^{Ct}-e^{Ct}\right).$$

All together, we have that

$$\frac{d}{dt}J(t) \geq C(\delta,\|u_0\|_{L^\infty})\left(J^2(t)-J(t)e^{Ct}-e^{Ct}\right).$$

Then, $J(t)$ blows-up in finite time for certain large enough initial data $\|u_0\|_{L^\infty}$.

In particular, since

$$J(t) \leq \int_0^1 \frac{h^2}{|x|^{1+\delta}}dx + \int_1^\infty \frac{h^2}{|x|^{1+\delta}}dx \lesssim \|h(.,t)\|_{L^\infty}\|h_x(.,t)\|_{L^\infty} + \|h(.,t)\|_{L^\infty}^2$$

and we have control of $\|h(\cdot,t)\|_{L^\infty}$ by (41), it implies the finite-time blow-up of $h_x(x,t)$.

□

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