# Multitriangulations and Tropical Pfaffians\*

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Abstract. The k-associahedron  $\mathcal{A}ss_k(n)$  is the simplicial complex of (k+1)-crossing-free subgraphs of the complete graph with vertices on a circle. Its facets are called k-triangulations. We explore the connection of  $\mathcal{A}ss_k(n)$  with the *Pfaffian variety*  $\mathcal{P}f_k(n)$  of antisymmetric matrices of rank  $\leq 2k$ . First, we characterize the Gröbner cone  $Grob_k(n)$  for which the initial ideal of  $I(\mathcal{P}f_k(n))$  equals the Stanley–Reisner ideal of  $\mathcal{A}ss_k(n)$  (that is, the monomial ideal generated by (k+1)-crossings). We then look at the tropicalization of  $\mathcal{P}f_k(n)$  and show that  $\mathcal{A}ss_k(n)$  embeds naturally as the intersection of  $trop(\mathcal{P}f_k(n))$  and  $Grob_k(n)$ , and that it is contained in the totally positive part  $trop^+(\mathcal{P}f_k(n))$  of it. We show that for k = 1 and for each triangulation T of the n-gon, the projection of this embedding of  $\mathcal{A}ss_k(n)$  to the n-3 coordinates corresponding to diagonals in T gives a complete polytopal fan, realizing the associahedron. This fan is linearly isomorphic to the g-vector fan of the cluster algebra of type A, shown to be polytopal by Hohlweg, Pilaud, and Stella in 2018.

Key words. multitriangulations, tropical geometry, Pfaffians, associahedron

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**1. Introduction.** Throughout the paper we consider  $[n] = \{1, 2, ..., n\}$  as the vertex set of a complete graph  $K_n$ , thus calling *edges* its size-two subsets. We think of the *n* vertices as drawn in order along a circle (or any other convex closed curve in the plane) so that two edges  $\{i, j\}, \{i', j'\} \in {[n] \choose 2}$  with i < j and i' < j' are said to *cross* each other if either i < i' < j < j' or i' < i < j' < j. A *k*-crossing is a set of *k* edges that mutually cross each other.

Our object of study is the simplicial complex with vertex set  $\binom{[n]}{2}$  and with faces consisting of subsets containing no (k + 1)-crossing. For k = 1 this complex is (the polar of) the face complex of the associahedron. In particular, it is a polytopal sphere, and from the algebraic geometric perspective it is related to the Grassmannian  $\mathcal{G}r_2(n)$ . For higher k the complex is still known to be a topological sphere, but its polytopality is open. We study this complex from the perspective of algebraic geometry, for its relation to the Pfaffian variety.

#### Multitriangulations.

Definition 1.1. A subset  $T \subseteq {\binom{[n]}{2}}$  is called (k+1)-free if it contains no (k+1)-crossings. The maximal (k+1)-free graphs are called k-triangulations, or multitriangulations if k is not specified.

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Many nice combinatorial properties of k-triangulations are known [35, 36, 48]. For example, all k-triangulations of the n-gon have the same cardinality, equal to k(2n - 2k - 1) [33, 17]. We are interested in the abstract simplicial complex  $\mathcal{A}ss_k(n)$  on the vertex set  $\binom{[n]}{2}$  whose faces are (k + 1)-free graphs. Hence, facets are k-triangulations.

If an edge  $\{i, j\}$  has  $|i-j| \leq k$  (where indices are taken modulo n, and distance is measured cyclically), then it lies in every k-triangulation since it cannot be part of any (k+1)-crossing. We call these edges *irrelevant* and call *irrelevant face* the face of  $\mathcal{A}ss_k(n)$  they span. We can thus define the reduced complex,  $\overline{\mathcal{A}ss_k}(n)$ , the faces of which are the (k+1)-free sets of *relevant* edges. The exact relation between  $\mathcal{A}ss_k(n)$  and  $\overline{\mathcal{A}ss_k}(n)$  is that the former is the join of the latter with the irrelevant face, and hence the latter is the link of the former at the irrelevant face. Based on the fact that  $\overline{\mathcal{A}ss_1}(n)$  is the polar complex to the face poset of the standard associahedron we define the following.

Definition 1.2. We call  $\overline{Ass}_k(n)$  the k-associahedron, or multiassociahedron of parameters n, k. We refer to  $Ass_k(n)$  as the extended multiassociahedron.

Jonsson [26] and Dress et al. [15] proved that  $\overline{Ass}_k(n)$  is a shellable simplicial sphere, and conjectured it to be polytopal. This conjecture is one of our motivations.

Conjecture 1.3 (Jonsson). For every  $k \ge 1$  and  $n \ge 2k + 1$ ,  $\overline{Ass}_k(n)$  is isomorphic to the face lattice of a simplicial polytope of dimension k(n-2k-1).

Besides the case k = 1, the conjecture is known to hold for the following cases:  $n \le 2k + 3$ and for n = 2 and k = 8 there is an ad hoc construction of the polytope [4]. In a forthcoming paper [14] we show polytopality for the remaining cases with  $n \le 10$ , namely,  $(k,n) \in$  $\{(2,9), (2,10), (3,10)\}$ . For some additional cases there are constructions that realize  $\overline{Ass}_k(n)$ as a complete simplicial fan. This happens for n = 2k + 4 [2] and for k = 2 and  $n \le 13$  [32].

Polytopality of  $\overline{Ass}_k(n)$  is also relevant from the perspective of Coxeter combinatorics. Let (W, S) be a Coxeter system. Let  $w \in W$  be an element in the group and Q a word of a certain length N and containing a reduced expression for w as a subword. The subword complex of Q and w is the simplicial complex with vertex set  $\{1, \ldots, N\}$  consisting of subsets of positions that can be deleted from Q and still contain a reduced expression for w. Knutson and Miller [30, Theorem 3.7 and Question 6.4] proved that every subword complex is either a shellable ball or sphere, and they asked whether all spherical subword complexes are polytopal. It turns out that  $\overline{Ass}_k(n)$  is a spherical subword complex for the Coxeter system of type  $A_{n-2k-1}$  [48, Theorem 2.1] and, moreover, it is universal: every other spherical subword complex of type A appears as a link in some  $\overline{Ass}_k(n)$  [37, Proposition 5.6]. In particular, Conjecture 1.3 would provide, in type A, a positive answer to the question of Knutson and Miller. (Versions of multiassociahedra for the rest of finite Coxeter groups exist, with the same implications [11]).

**Pfaffians and tropical varieties.** In the case k = 1, one way of realizing the associahedron is as the positive part of the space of "tree metrics," which coincides with the tropicalization  $\operatorname{trop}(\mathcal{G}r_2(n))$  of the Grassmannian  $\mathcal{G}r_2(n)$  (see [44, 45, 46] or Remark 3.12). More precisely, we have the following.

Theorem 1.4 ([45, section 5]). The totally positive tropical Grassmannian trop<sup>+</sup>( $\mathcal{G}r_2(n)$ ) is a simplicial fan isomorphic to (a cone over) the extended associahedron  $\overline{\mathcal{A}ss}_1(n)$ .

Let us briefly recall what the tropicalization of a variety, and its positive part, are (see also [5]). Let  $I \subset \mathbb{K}[x_1, \ldots, x_N]$  be a polynomial ideal and let  $V = V(I) \subset \mathbb{K}^N$  be its corresponding variety. Each vector  $v \in \mathbb{R}^N$ , considered as giving weights to the variables, defines an initial ideal  $\operatorname{in}_v(I)$ , consisting of the initial forms  $\operatorname{in}_v(f)$  of the polynomials in f. For the purposes of this paper we take the following definitions. (These are not the standard definitions, but are equivalent to them as shown for example in [45, Propositions 2.1 and 2.2].)

**Definition 1.5.** The tropical variety  $\operatorname{trop}(V)$  of V equals the set of  $v \in \mathbb{R}^N$  for which  $\operatorname{in}_v(I)$  does not contain any monomial. If  $\mathbb{K} = \mathbb{C}$ , the totally positive part of  $\operatorname{trop}(V)$ , denoted  $\operatorname{trop}^+(V)$ , equals the set of  $v \in \mathbb{R}^N$  for which  $\operatorname{in}_v(I)$  does not contain any polynomial with all coefficients real and positive.

Pachter and Sturmfels [34, p. 107] hint at the fact that the relation between the associated on and  $\mathcal{G}r_2(n)$  extends to a relation between the multiassociated on  $\mathcal{A}ss_k(n)$  and the tropical variety of Pfaffians of degree k+1. Recall that a *Pfaffian of degree* k is the square root of the determinant of an antisymmetric matrix M of size 2k. Considering the entries of M as indeterminates (over a certain field  $\mathbb{K}$ ), the Pfaffian is a homogeneous polynomial of degree k in  $\mathbb{K}[x_{i,j}, \{i, j\} \in {\binom{[2k]}{2}}]$  with one monomial for each of the (2k-1)!! perfect matchings in [2k] (see section 2.1).

For each  $n \geq 2k + 2$ , let  $I_k(n)$  be the ideal in  $\mathbb{K}[x_{i,j}, \{i, j\} \in {\binom{[n]}{2}}]$  generated by all the Pfaffians of degree k + 1. Let  $\mathcal{P}f_k(n) \subset \mathbb{K}^{\binom{n}{2}}$  be the corresponding algebraic variety. That is, points in  $\mathcal{P}f_k(n)$  are antisymmetric  $n \times n$  matrices with coefficients in  $\mathbb{K}$  and of rank at most 2k. It is well known and easy to see that  $\mathcal{P}f_1(n)$  equals the Grassmannian  $\mathcal{G}r_2(n)$  in its Plücker embedding and, as pointed out in [34],  $\mathcal{P}f_k(n)$  equals the kth secant variety of it.

For k = 1, Pfaffians are a universal Gröbner basis of  $I_k(n)$  [34, 44]. For k > 1 they are not (see Example 2.11), but it is known that they are a Gröbner basis for certain choices of monomial orders: in [20] it is proved that this happens for a v that selects as initial monomial in each Pfaffian the (k + 1)-nesting and in [27] for one that selects the (k + 1)-crossing.

**This paper.** We explore relations between k-triangulations and the algebraic variety  $\mathcal{P}f_k(n)$ . Our starting point is restricting Gröbner bases and tropicalization to weight vectors satisfying the following "four-point positivity" conditions.

**Definition 1.6.** We say that a weight vector  $v \in \mathbb{R}^{\binom{[n]}{2}}$  is four-point positive (abbreviated fp-positive) if for all  $1 \le a < a' < b < b' \le n$  we have that

(1.1) 
$$v_{a,b} + v_{a',b'} \ge \max\{v_{a,a'} + v_{b,b'}, v_{a,b'} + v_{a',b}\}.$$

We denote by  $\operatorname{FP}_n$  the subset of  $\mathbb{R}^{\binom{[n]}{2}}$  consisting of fp-positive vectors. That is to say,  $v \in \operatorname{FP}_n$  if the maximum weight given by v to the three matchings among four points is attained always for the matching that forms a 2-crossing.

Although the polyhedron  $\operatorname{FP}_n \subset \mathbb{R}^{\binom{[n]}{2}}$  of fp-positive vectors (the solution set of (1.1)) is defined by  $2\binom{n}{4}$  inequalities, the following  $\binom{n}{2} - n$  alone are an irredundant description of it, with indices considered cyclically:

(1.2) 
$$v_{a,b} + v_{a+1,b+1} - v_{a,b+1} - v_{a+1,b} \ge 0 \quad \forall \{a,b\} \in \binom{[n]}{2} \text{ with } |a-b| > 1.$$

The left-hand side coefficient vectors (that is, the facet normals of  $\operatorname{FP}_n$ ) are linearly independent, so that  $\operatorname{FP}_n$  is linearly isomorphic to an orthant plus a lineality space of dimension n. We like to think of  $\operatorname{FP}_n$  as the "positive orthant" of  $\mathbb{R}^{\binom{[n]}{2}}$  regarding Pfaffians. It can be interpreted as the space of weights that represent *separation vectors* among sides of the *n*-gon, or as the weights that are monotone with respect to crossings among perfect matchings of each fixed even set  $U \subset [n]$ . See Proposition 2.5 and Corollary 2.6 for details.

Algebraically, fp-positive vectors are the monomial weight vectors for which the leading form of every 3-term Plücker relation

$$x_{a,b}x_{a',b'} - x_{a,a'}x_{b,b'} - x_{a,b'}x_{a',b}, \qquad 1 \le a < a' < b < b' \le n,$$

contains the crossing monomial. These relations generate the ideal of the Grassmannian  $\mathcal{G}r_2(n)$ . In particular, fp-positive vectors are the (closed) Gröbner cone of  $\mathcal{G}r_2(n)$  producing as initial ideal the one generated by 2-crossings  $x_{a,b}x_{a',b'}$ .

Extending this, we denote by  $\operatorname{Grob}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$  the Gröbner cone consisting of weights that select the (k+1)-crossing as the leading monomial (or as one of them) in every Pfaffian of degree k+1. What we say above can then be stated as  $\operatorname{Grob}_1(n) = \operatorname{FP}_n$ , and the result of [27] says that  $\operatorname{Grob}_k(n)$  has a nonempty interior. In section 2 we show that  $\operatorname{FP}_n \subset \operatorname{Grob}_k(n)$ (Theorem 2.8) and give an explicit description of  $\operatorname{Grob}_k(n)$ , both by inequalities and by generators (Theorem 2.9).

Theorem 1.7 (Theorem 2.9). For any k > 2n + 2,  $\operatorname{Grob}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$  is a simplicial cone with a lineality space of dimension n.

- 1. It is generated by the following:
  - (lineality space) For each i ∈ [n], the line generated by the indicator vector of the set {{i, j} : j ∈ [n] \ i}.
  - ("short" generators) For each  $\{i, j\} \in [n]$  with  $1 \le |i j| \le k$ , the negative basis vector corresponding to  $\{i, j\}$ .
  - ("long" generators) For each  $\{i, j\} \in [n]$  with  $|i j| \ge k + 2$ , the ray of  $FP_n$  corresponding to  $\{i, j\}$ .
- 2. An irredundant facet description of it is given by the following  $\binom{[n]}{2} n$  inequalities:
  - ("long" inequalities) For each  $\{i, j\} \in [n]$  with  $|i j| \ge k + 1$ , the inequality (1.2) corresponding to  $\{i, j\}$ .
    - ("short" inequalities) For each  $\{i, j\} \in [n]$  with  $2 \leq |i j| \leq k$ , the sum of the inequalities (1.2) corresponding to all the  $\{i', j'\}$  with  $|i' j'| \leq k + 1$  and with  $\{i, j\}$  contained in the short side of  $\{i', j'\}$ .

In particular,  $\operatorname{Grob}_k(n)$  contains  $\operatorname{FP}_n$  for every k and n.

This description has the following combinatorial interpretation: modulo its lineality space (of dimension n, equal to that of  $FP_n$ ),  $\operatorname{Grob}_k(n)$  is a simplicial cone with one facet and generator corresponding to each of the  $\binom{[n]}{2} - n$  edges of length at least two. The long facet-inequalities (those corresponding to relevant edges) are the same as the corresponding ones in  $FP_n$ , and the short ones are looser in  $\operatorname{Grob}_k(n)$  than in  $FP_n$ .

Moreover, we show that the monomial initial ideal of  $I_k(n)$  produced by any generic weight vector  $v \in \operatorname{Grob}_k(n)$  equals the Stanley–Reisner ideal of  $\mathcal{A}ss_k(n)$ . That is, to say, the ideal in  $\mathbb{K}[x_{i,j},\{i,j\}\in \binom{[2k]}{2}]$  generated by (k+1)-crossings. This, in turn, implies that k-triangulations

are bases of the algebraic matroid of  $\mathcal{P}f_k(n)$  (Corollary 2.15). We find this of interest for two reasons (see section 2.4 for details).

On the one hand, the algebraic matroid  $\mathcal{M}(\mathcal{P}f_k(n))$  of  $\mathcal{P}f_k(n)$  is closely related to lowrank completion of antisymmetric matrices [1, 29]: given a subset  $T \subset {[n] \choose 2}$  of positions for entries in an antisymmetric matrix M of size  $n \times n$ , a generic choice of values for those entries can be extended to an antisymmetric matrix of rank < 2k if and only if T is independent in  $\mathcal{M}(\mathcal{P}f_k(n))$ . Thus, we have the following.

Theorem 1.8. Let  $T \subset {\binom{[n]}{2}}$ .

- 1. If T is (k+1)-free and K is algebraically closed, then for any generic choice of values  $v \in \mathbb{K}^T$  there is at least one skew-symmetric matrix of rank  $\leq 2k$  with the entries prescribed by v.
- 2. If T contains a k-triangulation then for any choice of values  $v \in \mathbb{K}^T$  there is only a finite number (maybe zero) of skew-symmetric matrices of rank  $\leq 2k$  with those prescribed entries.

On the other hand, the algebraic matroid of  $\mathcal{P}f_k(n)$  coincides with the generic hyperconnectivity matroid in dimension 2k defined by Kalai [28]. The fact that k-triangulations are bases in it is closely related to the conjecture by Pilaud and Santos [36] that they are bases in the generic bar-and-joint rigidity matroid in dimension 2k (Conjecture 2.18).

In section 3 we turn our attention to the tropicalization of  $\mathcal{P}f_k(n)$ . More precisely, we denote by  $\mathrm{Pf}_k(n) \subset \mathbb{R}^{\binom{[n]}{2}}$  the intersection of the tropical hypersurfaces corresponding to Pfaffians of degree k. This is by definition a tropical *prevariety*. It contains the tropical variety trop( $\mathcal{P}f_k(n)$ ) but it does not, in general, coincide with it, as we show in Theorem 3.7.

In the light of Theorem 1.7, it makes sense to look at the part of  $Pf_k(n)$  contained in the Gröbner cone  $\operatorname{Grob}_k(n)$ . That is, we define

$$\operatorname{Pf}_{k}^{+}(n) := \operatorname{Pf}_{k}(n) \cap \operatorname{Grob}_{k}(n).$$

Since the crossing monomial is the only positive monomial in each 3-term Plücker relation, for k = 1 we have

$$\operatorname{trop}^+(\mathcal{P}f_1(n)) = \operatorname{trop}(\mathcal{P}f_1(n)) \cap \operatorname{FP}_n = \operatorname{Pf}_1^+(n).$$

One of our main results partially generalizes this to higher k.

Theorem 1.9 (See Theorem 3.9 and Corollary 3.11).

1.  $\operatorname{Pf}_{k}^{+}(n) = \operatorname{Grob}_{k}(n) \cap \operatorname{trop}(\mathcal{P}f_{k}(n)) \subset \operatorname{trop}^{+}(\mathcal{P}f_{k}(n)).$ 2.  $\operatorname{Pf}_{k}^{+}(n)$  is the union of the faces of  $\operatorname{Grob}_{k}(n)$  corresponding to (k+1)-free graphs.

This theorem says that for a  $v \in \operatorname{Grob}_k(n)$ , being in  $\operatorname{Pf}_k(n)$  is equivalent to the fact that the "long inequalities" of Theorem 1.7 (that is, the inequalities (1.2) for  $|a-b| \ge k+1$ ) are satisfied with equality except in a (k+1)-free set. Moreover, when this happens v can be proved to be in trop( $\mathcal{P}f_k(n)$ ) and, in fact, in trop<sup>+</sup>( $\mathcal{P}f_k(n)$ ).

In part (2), by the face corresponding to a certain graph  $G \subset {[n] \choose 2}$  we mean the intersection of the facets of  $\operatorname{Grob}_k(n)$  corresponding to  $\binom{[n]}{2} \setminus G$  in the description of Theorem 1.7. That is, we consider  $\operatorname{Grob}_k(n)$  as (a cone over) the simplex with vertex set  $\binom{[n]}{2}$ , so that every simplicial complex on  $\binom{[n]}{2}$  is a subcomplex of its face complex. Hence, Theorem 1.9 has the following interpretation.

Corollary 1.10. As a simplicial fan and modulo its lineality space,  $\mathrm{Pf}_k^+(n) = \mathrm{Grob}_k(n) \cap \mathrm{trop}(\mathcal{P}f_k(n))$  is isomorphic to (the cone over) the extended multiassociahedron  $\mathcal{A}ss_k(n)$ .

*Remark* 1.11.  $Pf_k^+(n)$  is not equal to  $trop^+(\mathcal{P}f_k(n))$ . Put differently, four point positivity implies but is not the same as positivity in the sense of Definition 1.5. See Example 3.13.

Theorem 1.9 suggests that one way to realize the multiassociahedron as a polytope would be to find a projection  $\mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{k(2n-2k-1)}$  that is injective on  $\mathrm{Pf}_k^+(n)$ . This would embed  $\mathcal{A}ss_k(n)$  as a full-dimensional simplicial fan in  $\mathbb{R}^{k(2n-2k-1)}$  whose link at the irrelevant face would necessarily realize the multiassociahedron  $\overline{\mathcal{A}ss}_k(n)$  as a complete fan in  $\mathbb{R}^{k(n-2k-1)}$ . A second step is needed in order to show polytopality: to find appropriate right-hand sides showing that the complete fan is polytopal.

We have achieved both steps for k = 1. We show that, for any seed triangulation T, the projection  $\mathbb{R}^{\binom{[n]}{2}} \to \mathbb{R}^{2n-3}$  that keeps only the coordinates corresponding to edges in T is injective on  $\mathrm{Pf}_1^+(n)$  (Corollary 4.2). That is, we have an explicit projection sending  $\mathrm{Pf}_1^+(n)$  to (the normal fan of) the associahedron. It was pointed out to us by Pilaud that the embedding that we obtain is exactly the so-called **g**-vector fan associated with the seed triangulation. **g**-vector fans can be defined in an arbitrary cluster algebra of finite type and starting with any seed cluster, and they were shown to be polytopal by Hohlweg, Pilaud, and Stella [23]. See section 4 for details.

Theorem 1.12 (Corollary 4.7). For each seed triangulation T of the n-gon, projection of  $Pf_1^+(n)$  to the n-3 coordinates of the diagonals in T gives a realization of the (n-3)-associahedron in  $\mathbb{R}^{n-3}$  isomorphic to the **g**-vector fan of T.

This would seem to open up the possibility of using these same ideas to find polytopal realizations of  $\overline{Ass}_k(n)$  for higher k, by adapting to k-triangulations the (quite simple) procedure used to define the **g**-vectors from a seed triangulation. Unfortunately, our final result Corollary 4.10 says that this approach is doomed to fail, under certain natural assumptions.

# 2. The variety of antisymmetric matrices of bounded rank.

**2.1.** Matchings and the Pfaffian of an antisymmetric matrix. The complete graph on a set of vertices  $U \subset [n]$  of size 2k it has (2k - 1)!! matchings (by which we always mean a *perfect* matching), one of which is the unique k-crossing with vertex set U. The *parity* of a matching E is the parity of the number of pairwise crossings among the edges in E. This parity coincides with the parity as a permutation, when the pairs of matched vertices are written one after another, in increasing order within each pair. By *swapping* two pairs  $\{a, b\}$  and  $\{c, d\}$  in a matching E we mean removing them and inserting one of the other two matchings of  $\{a, b, c, d\}$  instead. Observe that one of the three matchings of  $\{a, b, c, d\}$  has a crossing (that is, it is odd) and the other two are crossing-free (hence even).

Lemma 2.1. A swap changes parity if and only if one of the two pairs of edges in the swap (the pair removed or the pair inserted) is a crossing.

*Proof.* Let  $\{a, b\}$  and  $\{c, d\}$  be the initial pairs and  $\{a, d\}$  and  $\{b, c\}$  the new pairs. Any other edge from the matching crosses the cycle *abcda* an even number of times. Hence, the only change in the number of crossings comes from whether the edges in the swap cross.

Recall that an antisymmetric matrix of odd size n has zero determinant because

$$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M).$$

For even size there is the following classical result.

Theorem 2.2 (Cayley 1852 [7]). Let M be a size 2k antisymmetric matrix. Then

(2.1) 
$$\det M = \left(\sum_{E \ matching} s(E) \prod_{(i,j) \in E, i < j} m_{ij}\right)^2,$$

where the sum is taken over the matchings of [2k] and  $s(E) = \pm 1$  according to the parity of E.

The expression inside the parenthesis in this theorem, that is, the square root of the determinant of an antisymmetric matrix, is called the Pfaffian of M.

**2.2. Four-point positive weight vectors.** Let  $I_k(n) \subset \mathbb{K}[x_{i,j}, \{i, j\} \in {\binom{[n]}{2}}]$  be the ideal generated by all Pfaffians of degree k of an antisymmetric matrix of size  $n \times n$  (with indeterminate coefficients) and let  $\mathcal{P}f_k(n)$  be the corresponding algebraic variety, whose points are the antisymmetric matrices of rank at most 2k.

We now introduce certain term orders for the variables that produce as the initial ideal of  $I_k(n)$  the monomial ideal generated by (k+1)-crossings. For this, we need to introduce a change of basis in  $\mathbb{R}^{\binom{[n]}{2}}$ , and a change of point of view on the *n*-gon.

Let us call ath side of the *n*-gon the edge  $\{a-1,a\}$  (with indices taken modulo *n*). Then, any choice of real numbers  $w_{i,j}$  (with  $\{i,j\} \in {[n] \choose 2}$ ) for the edges connecting vertices of the *n*-gon induces a "separation" distance between each pair of sides, as the sum of w's of the edges separating those sides. That is, we have the following.

Definition 2.3. Given a vector  $w \in \mathbb{R}^{\binom{[n]}{2}}$ , the separation vector  $d(w) \in \mathbb{R}^{\binom{[n]}{2}}$  induced by w is defined as

(2.2) 
$$d_{a,b}(w) = \sum_{\substack{(i,j) \in \binom{[n]}{2} \\ a \le i < b \le j < a}} w_{ij} \qquad \forall \{a,b\} \in \binom{[n]}{2}.$$

Here the order symbols "<" and " $\leq$ " for indices are considered cyclically. E.g., a < b < c < a means that a, b, c are different and they appear in that cyclic order along the n-gon.

Figure 2.1 shows an example of this transformation. To compute  $d_{26}(w)$ , where a = 2 and b = 6 denote two sides of the octagon, we have to sum the  $w_{ij}$ s in the complete bipartite graph on the two subsets of vertices separated by a and b.

The entries of d(w) are going to be used as weights for variables in our monomial orders, but we want to have in mind the weight vector w from which they come. This is well defined thanks to the following result, which implies that the transformation from w to d(w) is a linear isomorphism in  $\mathbb{R}^{\binom{[n]}{2}}$ .

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Figure 2.1. The transformation from w to d.

Proposition 2.4. For any  $w \in \mathbb{R}^{\binom{[n]}{2}}$ , and every  $\{a, b\} \in \binom{[n]}{2}$ , we have

(2.3) 
$$2w_{a,b} = d_{a,b}(w) + d_{a+1,b+1}(w) - d_{a,b+1}(w) - d_{a+1,b}(w),$$

where  $d_{a,a}(w) = 0$  by convention.

Hence, each  $v \in \mathbb{R}^{\binom{[n]}{2}}$  can be expressed uniquely as d(w) for a certain  $w \in \mathbb{R}^{\binom{[n]}{2}}$ .

*Proof.* It is enough to check that the rest of the  $w_{ij}$ 's cancel out when  $d_{a,b}(w) + d_{a+1,b+1}(w) - d_{a,b+1}(w) - d_{a+1,b}(w)$  is computed via (2.2).

That is, we can think of d(w) and w as different choices of linear coordinates for  $\mathbb{R}^{\binom{[n]}{2}}$ .

Proposition 2.5. Let  $v \in \mathbb{R}^{\binom{[n]}{2}}$  be a weight vector. The following conditions are equivalent:

- 1.  $v \in FP_n$ . That is, it satisfies the positive four-point conditions (1.1) in Definition 1.6. 2. v satisfies the  $\binom{n}{2} - n$  inequalities (1.2).
- 3. v = d(w) in the sense of Definition 2.3 for a w with  $w_{a,b} \ge 0$  for all  $\{a,b\} \in {[n] \choose 2}$  with |a-b| > 1.
- 4. For every  $k \ge 1$  and every  $U \in {\binom{[n]}{2k}}$  the weights given by v to matchings in U are monotone with respect to swaps that create crossings.
- 5. For every  $k \ge 1$  and every  $U \in {\binom{[n]}{2k}}$  the maximum weight given by v to matchings in U is attained by the k-crossing.

*Proof.* The equivalence of parts 1 and 4 is obvious and the equivalence of 2 and 3 follows from Proposition 2.4. The implications  $5 \Rightarrow 1 \Rightarrow 2$  are also trivial because the inequalities in condition 1 are nothing but the case k = 2 of condition 5, and they contain the inequalities in condition 2 as a subset.

The implication  $4 \Rightarrow 5$  follows from the fact that every matching can monotonically be converted into a full crossing by swaps that create crossings.

Finally, the implication  $3 \Rightarrow 4$  follows from the fact that if  $1 \le a < a' < b < b' \le n$ , then (2.2) gives

$$\begin{split} v_{a,b} + v_{a',b'} &= W_1 + W_2 + W_3, \\ v_{a,a'} + v_{b,b'} &= W_1 + W_2, \\ v_{a,b'} + v_{a',b} &= W_1 + W_3, \end{split}$$

where

$$\begin{split} W_{1} &= \sum_{\substack{a \leq i < a' \leq j < b}} w_{ij} + \sum_{\substack{a' \leq i < b \leq j < b'}} w_{ij} + \sum_{\substack{b \leq i < b' \leq j < a}} w_{ij} + \sum_{\substack{b' \leq i < a \leq j < a'}} w_{ij} \\ W_{2} &= \sum_{\substack{a \leq i < a', \\ b \leq j < b'}} w_{ij}, \\ W_{3} &= \sum_{\substack{a' \leq i < b, \\ b' \leq j < a}} w_{ij}. \end{split}$$

Since w is nonnegative (except perhaps for consecutive indices) we have that  $W_2, W_3 \ge 0$  and hence  $v_{a,b} + v_{a',b'}$  is greater than or equal to both of  $v_{a,a'} + v_{b,b'}$  and  $v_{a,b'} + v_{a',b}$ .

That is to say,  $FP_n$  is essentially the positive orthant in the *w* coordinates, except for one detail. Proposition 2.4 implies that the inequalities (1.2) from the introduction are equivalent to

$$w_{a,b} \ge 0 \qquad \forall \{a,b\} \in \binom{[n]}{2} \text{ with } |a-b| > 1;$$

but the inequalities  $w_{a,a+1} \ge 0$  are not valid in FP<sub>n</sub>. The *n*-dimensional subspace generated by the vectors with  $w_{a,b} = 0$  if |a - b| > 1 and  $w_{a,a+1}$  arbitrary can be thought of as the "irrelevant" part of the *w* coordinates; in fact, it is the lineality space of FP<sub>n</sub>. This suggests we give it a name. We denote

$$L_{n} := \left\{ d(w) : w \in \mathbb{R}^{\binom{[n]}{2}} \text{ and } w_{i,j} = 0 \text{ if } |i-j| > 1 \right\} \cong \mathbb{R}^{n},$$
  
FP\_{n}^{+} :=  $\left\{ d(w) : w \in \mathbb{R}^{\binom{[n]}{2}}_{\geq 0} \right\} \cong \mathbb{R}^{\binom{[n]}{2}}_{\geq 0}.$ 

Corollary 2.6.  $\operatorname{FP}_n = L_n + \operatorname{FP}_n^+$ , and it is linearly isomorphic to  $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^{\binom{[n]}{2}-n}$ .

*Proof.* By Proposition 2.4 the map  $w \to d(w)$  is a linear automorphism in  $\mathbb{R}^{\binom{[n]}{2}}$ ; by Proposition 2.5, FP<sub>n</sub> is the image FP<sub>n</sub><sup>+</sup> of the positive orthant plus the linear subspace  $L_n$ .

**2.3.** Pfaffians as a Gröbner basis for four-point positive weights. The following is the main result of Jonsson and Welker [27], although it is also stated without proof in [34, p. 107]).

Theorem 2.7 ([27]). There is a (lexicographical) term order for which  $in_v(I_k(n))$  is the monomial ideal generated by all (k+1)-crossings.

The term order of Jonsson and Welker necessarily selects in each Pfaffian the monomial corresponding to the (k+1)-crossing (in fact, it is designed to have that property), and Pfaffins are a Gröbner basis for it since each Pfaffian contains one and only one of the generators in the initial ideal. Once we know this, any term order that selects this same monomial in each Pfaffian will produce the same initial ideal by, for example, Exercise 8.4 in [12, p. 435]. Proposition 2.5(5) says that this includes the order induced by any (generic) fp-positive vector  $v \in FP_n$ . Hence, we have the following statement, a bit more general than Theorem 2.7.

**Theorem 2.8.** Pfaffians of degree 2k + 2 are a Gröbner basis for  $I_k(n)$  with respect to any monomial order that selects the k-crossing in each Pfaffian; in particular, for the ordering of any fp-positive vector  $v \in FP_n$ .

If the fp-positive vector is sufficiently generic then  $in_v(I_k(n))$  is the monomial ideal generated by all (k+1)-crossings.

The case k = 1 of this theorem is classical, via the equality  $\mathcal{P}f_1(n) = \mathcal{G}r_2(n)$ ; see [34, Theorem 3.20] and Remark 3.12 below). In fact, in this case the last sentence in the theorem is an "if and only if." Indeed,  $FP_n$  is, by definition, the closed Gröbner cone of  $I_1(n)$  corresponding to the initial ideal generated by 2-crossings.

In general, let  $\operatorname{Grob}_k(n)$  be the Gröbner cone of  $I_k(n)$  corresponding to the ideal of (k+1)crossings. For higher k it is no longer true that  $FP_n = \operatorname{Grob}_k(n)$ , we only have the containement  $FP_n \subset \operatorname{Grob}_k(n)$  which follows from the previous theorem. Our next result explicitly describes  $\operatorname{Grob}_k(n)$ .

For arbitrary k, the Gröbner cone is the intersection of the normal cones of each (k+1)crossing in the Newton polytope of the corresponding Pfaffian. A priori, this intersection is described by the following family of linear inequalities, running over all the even cycles  $(i_0, i_1, \ldots, i_{2l-1}, i_0)$  of length 2l that contain an l-crossing contained in a (k+1)-crossing, for  $l \leq k+1$ :

(2.4) 
$$v_{i_0i_1} - v_{i_1i_2} + \dots - v_{i_{2l-1}i_0} \ge 0.$$

But most of these inequalities are redundant. For example, for k = 1,  $\operatorname{Grob}_1(n) = \operatorname{FP}_n$ which is defined by  $2\binom{n}{4}$  four-point conditions, but only the  $\binom{n}{2} - n$  in (1.2) are irredundant. In fact, it turns out that for every k and every  $n \ge 2k+3$ , the Gröbner cone is simplicial.

**Theorem 2.9.** For  $n \ge 2k+3$ ,  $\operatorname{Grob}_k(n)$  is, modulo the lineality space  $L_n$ , a simplicial cone given by the following inequalities, one for each  $\{i, j\}$  with  $|j - i| \ge 2$ :

(2.5) $w_{ij} \ge 0 \quad if |j-i| > k$ (long inequalities),  $\sum_{\substack{i' \le i < j \le j' \le i' + k + 1}} w_{i'j'} \ge 0 \quad if \ 2 \le |j - i| \le k$ (short inequalities). (2.6)

The ray opposite to the facet indexed by  $\{i, j\}$  is generated by

- the basis vector indexed by  $\{i, j\}$  in the w coordinates if  $|j i| \ge k + 2$ , and
- the negative basis vector indexed by  $\{i+1,j\}$  in the v coordinates if |j-i| < k+2.

Observe that the long inequalities are also facet-defining for  $FP_n$  and the short ones are sums of facet-defining short inequalities in  $FP_n$ .

*Proof.* First let us see that the inequalities are valid in the cone. The first group (2.5) is obvious, because the (k+1)-crossing has higher weight than any swap. For the second group, let  $\{i, i+\ell\}$  be an edge with  $\ell \leq k$ . For each set U of 2k+2 sides of the n-gon and each edge  $e \in T$  we call the *length of e with respect to U* and denote it  $\ell_U(e)$  the smallest size of the two parts of U separated by e. (Equivalently, it is the usual length of the edge as a diagonal of the *n*-gon when all the sides not in U are contacted.) For a matching M of U and an edge ewe denote by  $c_M(e)$  the number of edges of M that cross e.

Consider the matching

$$M = \{\{i+1, i+\ell\}, \{i+2, i+k+3\}, \{i+3, i+k+4\}, \dots, \{i+\ell-1, i+k+\ell\}, \{i-k+\ell-1, i+\ell+1\}, \{i-k+\ell, i+\ell+2\}, \dots, \{i, i+k+2\}\}.$$

This is a k-crossing plus the edge  $\{i+1, i+\ell\}$ . The coefficient of a w will be the same in this matching than in the (k+1)-crossing, that is,  $\ell_U(e) = c_M(e)$ , except for the edges  $\{i', j'\}$  with  $i' \leq i < i+l \leq j' \leq i'+k+1$ , for which  $c_M(e) = \ell_U(e) - 2$ . Hence the left-hand side of (2.6) is half the difference between the weights, which proves the inequalities.

Once we know that the inequalities are valid, let  $G_{ij}$  be the ray defined in the statement. We only need to show that at each  $G_{ij}$  all inequalities are equalities, except for the one of index ij, and that the  $G_{ij}$  indeed lie in  $\operatorname{Grob}_k(n)$ .

Indeed, if |i - j| > k + 1, then  $G_{ij}$  has all w coordinates equal to zero except  $w_{ij} > 0$ . it is clear that all inequalities of the form (2.6) are equalities (since they only involve w's of length  $\leq k + 1$ ) and all of type (2.5) except the one for ij are equalities (by construction). If  $|i - j| \leq k + 1$ , in  $G_{ij}$  we have that the only nonzero v coordinate is  $v_{i+1,j}$ , which is negative. We take it equal to -1. Proposition 2.4 implies that in the w coordinates the only nonzero ones are

$$w_{i+1,j} = w_{i,j-1} = -\frac{1}{2}, \qquad w_{i+1,j-1} = w_{i,j} = \frac{1}{2}.$$

Now, if  $j - i \leq k$ , (2.5) always gives 0 and (2.6) gives 1/2 exactly for one sum, the one corresponding to  $\{i, j\}$ , and 0 for the rest. If j - i = k + 1, (2.6) always gives 0 and (2.5) gives 1 only for  $w_{i,j}$ .

It remains to see that these rays are in  $\operatorname{Grob}_k(n)$ :

- For the w basis vectors this follows from the fact that they are in  $FP_n$ .
- For the negative v basis vectors, we are giving weight -1 to an irrelevant edge and 0 to all the other edges; it is clear that every (k + 1)-crossing gets weight zero, and every other matching gets nonpositive weight.

Remark 2.10. Theorem 2.9 fails for n = 2k + 2, but in this case it is easy to describe  $\operatorname{Grob}_k(n)$ . Since we have a single Pfaffian, the Gröbner fan is simply the normal fan of its Newton polytope. In particular, none of the equalities (2.4) are redundant and  $\operatorname{Grob}_k(n)$  has as many facets as there are matchings of [2k + 2] whose symmetric difference with the k + 1-crossing is a single cycle. For example,

- for k = 2, n = 6, all matchings differ from the 3-crossing in a single cycle. Thus, the  $\operatorname{Grob}_2(6)$  has (modulo its lineality space) dimension  $\binom{6}{2} 6 = 9$  and 14 facets;
- for k = 3, n = 8, there are matchings differing from the 4-crossing in two cycles of length four. There are exactly 12 of them, coming from the three ways of partitioning the 4-crossing into two pairs of edges and the two ways of completing each pair of edges into a four-cycle. Hence,  $\operatorname{Grob}_3(8)$  has dimension  $\binom{8}{2} - 8 = 20$  and it has 105 - 1 - 12 = 92 facets.

One difference between k = 1 and k > 1 is that for k = 1 Pfaffians are a universal Gröbner basis for the ideal  $I_1(n)$  (one proof is that every other Gröbner cone of  $I_1(n)$  can be sent to  $FP_n$  by a permutation of [n]; see [44, Theorem 4.3]). The same is known to fail for higher Grassmannians (see, e.g., [44, section 7] or [31, Example 4.3.10]), and it also fails for higher Pfaffians.

*Example* 2.11 (Pfaffians are not a universal Gröbner basis). Let n = 9 and k = 2. Consider the vector with

$$v_{12} = v_{34} = v_{56} = v_{47} = v_{89} = 2,$$
  
 $v_{58} = v_{69} = 1,$   
 $v_{17} = v_{28} = v_{39} = 10,$ 

and the rest of entries equal to zero. We are going to show that, regardless of the field  $\mathbb{K}$ , Pfaffians are not a Gröbner basis for this choice of v (or any small perturbation of it).

Call f and g the Pfaffians on the sets  $U = \{1, 2, 3, 4, 5, 6\}$  and  $V = \{4, 5, 6, 7, 8, 9\}$ , which have as matchings of highest weight  $\{12, 34, 56\}$  and  $\{56, 47, 89\}$ , both of weight 6. That is,

$$in(f) = x_{12}x_{34}x_{56}, \quad in(g) = x_{56}x_{47}x_{89}.$$

The following polynomial, which is nothing but the S-polynomial of f and g that arises in Buchberger's algorithm, lies in  $I_3(9)$ :

$$h := x_{12}x_{34}g - x_{47}x_{89}f$$

The only monomials of weight > 6 in h are the initial terms of the two parts  $x_{12}x_{34}g$ and  $x_{47}x_{89}f$ , which cancel out, and  $x_{12}x_{34}x_{47}x_{58}x_{69}$ , of weight 8. Hence, we have that  $in(h) = x_{12}x_{34}x_{47}x_{58}x_{69}$ .

In particular, if Pfaffians were a universal Gröbner basis, there should be a Pfaffian whose leading monomial divides in(h). That is, there should be a set  $W \subset [9]$  of six elements whose matching M of maximum weight is contained in  $\{12, 34, 47, 58, 69\}$ . This W does not exist. Indeed, W cannot contain any of the pairs  $\{1,7\}$ ,  $\{2,8\}$ , or  $\{3,9\}$ , because then its highest matching would have weight  $\geq 10$ . And every set of three edges among  $\{12, 34, 47, 58, 69\}$  not containing any of those pairs of vertices contains the edges  $\{58, 69\}$ , which cannot be in the leading term of any Pfaffian since they produce a smaller weight than their swap  $\{56, 89\}$ .

*Remark* 2.12. That Pfaffians are a Gröbner basis for the ideal  $I_k(n)$  they generate was known before [27]. The earliest proof we are aware of is by Herzog and Trung [20], who construct a lexicographic order for which the initial ideal  $in_{\leq}(I_k(n))$  is generated by the (k+1)-nestings. Here  $\{a,d\}$  and  $\{b,c\}$  are nested if  $1 \leq a < b < c < d \leq n$ .

This result was recovered by Sturmfels and Sullivant [49] as a special case of a more general behavior; Sturmfels and Sullivant study the relation between the Gröbner bases of an ideal I and those of its secant ideals  $I^{\{k\}}$ , and call a monomial order "delightful" if the initial ideal of  $I^{\{k\}}$  can be obtained from that of I by the following simple combinatorial rule: the standard monomials in  $in_{<}(I^{\{k\}})$  are the products of k standard monomials of  $in_{<}(I)$ . They then consider  $I_k(n) = I_1(n)^{\{k\}}$  as an example [49, Example 4.13], and show that the lexicographic order of Herzog and Trung [20] is delightful.

It is worth noticing that fp-positive orders are not delightful in the sense of [49]. Indeed, the maximal square-free standard monomials in our initial ideal are the k-triangulations of the

*n*-gon, and not every k-triangulation is the union of k triangulations. For a trivial example observe that the complete graph on 5 vertices is a 2-triangulation but it is not the union of two triangulations of the pentagon. Related to this, see [36, section 6].

Theorem 2.8 has a natural interpretation via (k + 1)-free sets and multitriangulations. Observe that k(2n - 2k - 1), the dimension of  $\mathcal{A}ss_k(n)$ , coincides with that of  $\mathcal{P}f_k(n)$ .

Corollary 2.13. If the weight vector v for the variables in  $\mathbb{K}[x_{i,j}, \{i, j\} \in {\binom{[n]}{2}}]$  lies in  $\operatorname{Grob}_k(n)$  (for example, if it is fp-positive) and generic then the initial ideal of  $I_k(n)$  equals the Stanley-Reisner ideal of the extended k-associahedron  $\mathcal{A}ss_k(n)$ . That is, it is the radical monomial ideal whose square-free standard monomials form, as a simplicial complex,  $\mathcal{A}ss_k(n)$ .

2.4. The algebraic matroid of  $\mathcal{P}f_k(n)$  and low-rank matrix completion. Let  $I \subset \mathbb{K}[x_1, \ldots, x_N]$  be a prime ideal; the algebraic matroid of I, which we denote as  $\mathcal{M}(I)$ , has the variables  $E := \{x_1, \ldots, x_N\}$  as elements and a subset  $S \subset E$  is independent if I does not contain any nonzero polynomial in  $\mathbb{K}[S]$ . If  $\mathbb{K}$  is algebraically closed and V = V(I) is the irreducible variety of V, then dependence and independence of a subset S of variables can be told via the natural projection map  $\pi_S : V \subset \mathbb{K}^N \to \mathbb{K}^S$ , as follows. A set is independent in  $\mathcal{M}(I)$  if and only if the corresponding projection map  $\pi_S : V \to K^S$  is dominant; that is, its image is (Zariski) dense. We use [39, 40, 29] as our main sources for algebraic matroids.

**Theorem 2.14.** Let  $\mathbb{K}$  be an algebraically closed field,  $I \subset \mathbb{K}[x_1, \ldots, x_N]$  a prime ideal, and V its algebraic variety. For each  $S \subset [N]$  denote by  $\pi_S : \mathbb{K}^{[N]} \to \mathbb{K}^S$  the coordinate projection to S. Then

- 1. S is independent in  $\mathcal{M}(I)$  if and only if  $\pi_S(V)$  is Zariski dense in  $\mathbb{K}^S$ .
- 2. the rank of S is equal to the dimension of  $\pi_S(V)$ ;
- 3. S is spanning if and only if  $\pi_S$  is finite-to-one: for every  $x \in \mathbb{K}^S$  the fiber  $\pi_S^{-1}(x)$  is finite (perhaps empty).

**Proof.** The first part is Theorem 15 in [40]. For the second, the rank of S is the maximum size among independent subsets of S, which are the subsets T for which  $\pi_T(V) = \pi_T(\pi_S(V))$  has dimension |T|. The maximal ones are those which have the same size as the dimension of  $\pi_S(V)$ , so this is the rank.

The third part is a consequence of the second, because a projection has the same dimension as the variety if and only if the fiber has dimension zero, and a fiber has dimension zero if and only if it is finite.

This statement has as a consequence that, over an algebraically closed field, we can speak of the algebraic matroid of the irreducible variety V, and denote it  $\mathcal{M}(V)$ , instead of looking at the ideal.

We now turn our attention to the case of  $\mathcal{P}f_k(n)$ .

Corollary 2.15. (k + 1)-free subsets of edges are independent in the algebraic matroid of  $\mathcal{P}f_k(n)$  and k-triangulations are bases.

After Proposition 2.16 we show examples of nonplanar graphs that are independent in  $\mathcal{P}f_1(n)$ . This implies that the converse of Corollary 2.15 is false; not every basis of  $\mathcal{P}f_1(n)$  is, after relabeling vertices, a triangulation.

**Proof.** Let S be a dependent set in the matroid. Then there is a polynomial f in  $I_k(n)$  using only the variables in S and the initial monomial of f according to any fp-positive weight also uses only variables in S. By Corollary 2.13  $I_k(n)$  has an initial ideal consisting only of (k + 1)-crossing monomials, hence f has a monomial with a (k + 1)-crossing, and S is not (k + 1)-free.

For the second part, it is enough to see that the rank of the matroid equals  $2nk - \binom{2k+1}{2}$ . This is because points in  $\mathcal{P}f_k(n)$  are antisymmetric matrices of rank  $\leq 2k$ . In order to construct one such matrix M we can choose generic elements in the first 2k rows above the diagonal and every other element  $M_{i,j}$  (i, j > 2k) is uniquely determined by them. Indeed, the Pfaffian of the rows and columns indexed by  $[2k] \cup \{i, j\}$  has the form  $AM_{i,j} + B$ , where A is the Pfaffian of [2k]. Since our choice was generic,  $A \neq 0$ .

This proof already shows the relation between independence in the algebraic matroid of  $\mathcal{P}f_k(n)$  and low-rank completion of partially known antisymmetric matrices. Suppose that we are given a matrix  $M \in \mathbb{K}^{n \times n}$ , of which we only know a subset T of entries, we want to deduce the rest of entries with the restriction that M needs to be antisymmetric and have at most range 2k. Corollary 2.15 then immediately allows us to prove Theorem 1.8.

**Proof of Theorem** 1.8. Consider the projection  $\pi_T : \mathbb{K}^{\binom{[n]}{2}} \to \mathbb{K}^T$  that keeps only the coordinates of T. In part (1) we are saying that  $\pi_T$  is almost surjective (any element has a preimage except for a zero measure set) and in part (2) that it is finite-to-one (every point  $x \in \mathbb{K}^T$  has a finite fiber  $\pi^{-1}(x)$ ). Both parts follow from Corollary 2.15, via the characterization of algebraic matroids in Theorem 2.14.

It is worth mentioning that the algebraic matroid of  $\mathcal{P}f_k(n)$  coincides with the generic hyperconnectivity matroid in dimension 2k introduced by Kalai [28]. Let us review this relation. The hyperconnectivity matrix of a configuration  $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{R}^d$  is defined to be

$$H(\mathbf{p}) := \begin{pmatrix} \mathbf{p}_2 & -\mathbf{p}_1 & 0 & \dots & 0 & 0 \\ \mathbf{p}_3 & 0 & -\mathbf{p}_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{p}_n & 0 & 0 & \dots & 0 & -\mathbf{p}_1 \\ 0 & \mathbf{p}_3 & -\mathbf{p}_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{p}_n & -\mathbf{p}_{n-1} \end{pmatrix}$$

We call the hyperconnectivity matroid of  $\mathbf{p}$  the linear matroid  $\mathcal{H}_d(\mathbf{p})$  of rows of  $H(\mathbf{p})$ . There clearly exists an open dense subset of configurations where the matroid is the most free; we call that matroid the generic hyperconnectivity matroid of dimension d and denote it  $\mathcal{H}_d$ .

On the other hand, if an algebraic variety V is parametrized by a polynomial map  $T : \mathbb{R}^M \to V \subset \mathbb{R}^N$ , then the algebraic matroid of V equals the linear matroid of rows of the Jacobian of T at a sufficiently generic point of  $\mathbb{R}^M$  [39, Proposition 2.5]. In our case,  $\mathcal{P}f_k(n)$  is parametrized by the following linear map:

(2.7) 
$$T: (\mathbb{R}^n)^{2k} \to \mathcal{P}f_k(n) \subset \mathbb{R}^{\binom{n}{2}},$$
$$(\mathbf{a}_1, \mathbf{b}_1, \dots, \mathbf{a}_k, \mathbf{b}_k) \mapsto \sum_{l=1}^k \left(a_{l,i}b_{l,j} - a_{l,j}b_{l,i}\right)_{1 \le i < j \le n}$$

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where  $\mathbf{a}_l = (a_{l,1}, \ldots, a_{l,n})$  and  $\mathbf{b}_l = (b_{l,1}, \ldots, b_{l,n})$ . The Jacobian of T at a point  $(\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_k, \mathbf{b}_k)$  then coincides with the hyperconnectivity matrix of the configuration  $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ , where

$$\mathbf{p}_i = (b_{1,i}, -a_{1,i}, \dots, b_{k,i}, -a_{k,i}).$$

As a consequence we get the following (known) result, which is implicit for example in [34, Theorem 3.23].

**Proposition 2.16.** The algebraic matroid of  $\mathcal{P}f_k(n)$  coincides with the generic hyperconnectivity matroid in dimension 2k.

With this result it is easy to construct nonplanar graphs that are bases in  $\mathcal{P}f_1(n) = \mathcal{H}_2(n)$ . Start with any nonplanar graph and subdivide every edge into two parts. The graph G obtained is independent in every two-dimensional rigidity matroid, in particular in  $\mathcal{H}_2(n)$ , because iteratively removing the new vertices, which all have degree two, we get the empty graph. Hence, G can be extended to a nonplanar basis of  $\mathcal{H}_2(n)$ . As a consequence, not every basis of  $\mathcal{P}f_1(n)$  is a triangulation.

Corollary 2.17. k-triangulations are bases in the generic hyperconnectivity matroid of dimension 2k.

This statement is related to the following conjecture of Pilaud and Santos.

Conjecture 2.18 ([36, Conjecture 8.6]). k-triangulations are bases in the generic bar-andjoint rigidity matroid of dimension 2k.

Let us denote by  $\mathcal{R}_d$  the generic bar-and-joint rigidity matroid. It is known that hyperconnectivity falls under the framework of rigidity theory in the sense that both  $\mathcal{R}_d$  and  $\mathcal{H}_d$ are abstract rigidity matroids as defined by Edmonds: matroids of rank  $dn - \binom{d+1}{2}$  on the ground set  $\binom{[n]}{2}$  with the property that every complete graph on d+1 elements is independent. It is conjectured that  $\mathcal{R}_d$  is freer than  $\mathcal{H}_d$ , which would make Proposition 2.17 imply Conjecture 2.18, but the conjecture is open starting at dimension 3. (For d = 1 both matroids coincide with the usual graphical matroid of the complete graph; for d = 2 there are combinatorial characterizations of independent graphs in both of them: Laman graphs in  $\mathcal{R}_2$ , and the graphs described in [1] in  $\mathcal{H}_2$ .)

It is known, however, that for points chosen along the moment curve the two matroids coincide.

Theorem 2.19 ([13]). Let d be a positive integer and let  $t_1, \ldots, t_n \in \mathbb{R}$  be (distinct) real numbers. Let  $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \subset \mathbb{R}^d$  be the corresponding configuration of points along the moment curve, so that  $\mathbf{p}_i = (t_i, \ldots, t_i^d)$ ,  $i = 1, \ldots, n$ . Then,  $\mathcal{H}_d(\mathbf{p}) = \mathcal{R}_d(\mathbf{p})$ .

In particular, a statement that would imply both Proposition 2.17 and Conjecture 2.18 is that k-triangulations are bases for the matroid  $\mathcal{H}_d(\mathbf{p}) = \mathcal{R}_d(\mathbf{p})$  when  $\mathbf{p}$  is a generic collection of points along the moment curve. We refer to [13] and the references in there for an up-to-date account of the relation between  $\mathcal{H}_d$  and  $\mathcal{R}_d$ .

# 3. The tropicalization of $\mathcal{P}f_k(n)$ .

**3.1. The tropical Pfaffian variety and prevariety.** Recall that the *tropical hypersurface* trop(f) of a polynomial  $f \in \mathbb{K}[x_1, \ldots, x_N]$  is the collection of weight vectors  $v \in \mathbb{R}^N$  for

which  $in_v(f)$  is not a monomial. Put differently, the weight vectors for which the maximum weight among monomials in f is attained at least twice. It is a polyhedral fan, namely, the codimension one skeleton of the normal fan of the Newton polytope of f.

If V is the algebraic variety of an ideal I, the *tropicalization* of V equals

$$\operatorname{trop}(V) := \bigcap_{f \in I} \operatorname{trop}(f).$$

A finite subset  $B \subset I$  such that  $\operatorname{trop}(V) := \bigcap_{f \in B} \operatorname{trop}(f)$ , which always exists, is called a *tropical basis of I*. Not every generating set of I (not even a universal Gröbner basis of I; see [3, Example 10] or [31, Example 2.6.1]) is a tropical basis. In general, a finite intersection of tropical hypersurfaces is called a *tropical prevariety*, while the tropicalization of a variety is a *tropical variety* [31, Definitions 3.1.1 and 3.2.1]. The tropical prevariety defined by a finite set of polynomials  $\{f_1, \ldots, f_n\}$  contains, but is sometimes not equal to, the tropical variety of the ideal  $(f_1, \ldots, f_n)$  generated by them.

Looking at the case of Pfaffians, for each subset U of [n] of size 2k + 2 we have as tropical hypersurface the set of vectors  $v \in \mathbb{R}^{\binom{[n]}{2}}$  for which the maximum

$$\left\{\sum_{\{i,j\}\in E} v_{ij}: E \text{ matching in } U\right\},\$$

is attained at least twice. We denote by  $Pf_k(n)$  the intersection of all these tropical hypersurfaces for the different  $U \in {[n] \choose 2k}$ . We call it the *tropical Pfaffian prevariety*. It contains the tropicalization  $trop(\mathcal{P}f_k(n))$  of  $\mathcal{P}f_k(n)$  and it is known to coincide with it in the following cases:

- If n = 2k + 2, since then we have a single Pfaffian defining trop( $\mathcal{P}f_k(n)$ ).
- If k = 1, by the results in [44] and the fact that  $\mathcal{P}f_1(n)$  coincides with the Grassmannian  $\mathcal{G}r_2(n)$  (see Remark 3.12 below).

The following example looks at the first open case.

**Example 3.1.** For k = 2 and n = 7, using **Gfan** [25] we have computed Pf<sub>2</sub>(7) as the intersection of the seven hypersurfaces corresponding to Pfaffians. The result is a nonsimplicial fan of pure dimension 18 with 77 rays and a lineality space of dimension 7 (as expected). It has 73395 maximal cones, all of them with multiplicity 1. These cones correspond to 33 classes of symmetry via permutations of variables. The 77 rays are the following:

- The 21 vectors in the standard basis of the coordinates v, and their 21 opposites. That is, for each  $\{i, j\} \in \binom{7}{2}$ , the two vectors with  $v_{ij} = \pm 1$  and  $v_{i'j'} = 0$  otherwise.
- The 35 vectors obtained as follows: for each  $\{i, j, k\} \in \binom{7}{3}$ , the vector with  $v_{ij} = v_{ik} = v_{jk} = 1$  and  $v_{i'j'} = 0$  otherwise.

7 of the 14 extremal rays of FP<sub>7</sub> are among these vectors. In the *w* coordinates these are the vectors with  $w_{ij} = 1$  and all other entries equal to zero, for the fourteen choices of nonconsecutive *i* and *j*. The seven with i = j - 2 coincide (modulo the lineality space) with the *v*-basis vectors with  $v_{j-1,j} = -1$ , which are rays, and the seven with i = j - 3 are the vectors with  $v_{j-2,j} = v_{j-1,j} = v_{j-2,j-1} = -1$ , that is, the opposites to some rays, but they are not rays themselves. None of the other 77 rays computed by **Gfan** lie in FP<sub>7</sub>.

The cone corresponding to a given 2-triangulation cannot be in this prevariety, because its rays are not among those rays. But it can be the result of intersecting a cone from the prevariety with FP<sub>7</sub>, because, by Remark 2.10, the Gröbner cone in which it is contained is a bit greater than FP<sub>7</sub>. In fact, a v coming from a 2-triangulation is in the cone spanned by the rays  $v_{j-1,j} = -1$  for all j and  $v_{j-2,j} = -1$  for  $\{j-3, j\}$  in the 2-triangulation. The intersection of this cone with FP<sub>7</sub> is the cone in the 2-associahedron.

In this case, we want to check whether the tropical prevariety  $Pf_2(7)$  coincides with the variety trop( $\mathcal{P}f_2(7)$ ). To do that, we need to compute the tropical variety as a subfan of the Gröbner fan. However, it is not enough to check that the cones in both fans are the same, because the tropical prevariety may not be a subfan of the Gröbner fan.

 $\operatorname{trop}(\mathcal{P}f_2(7))$  is a simplicial fan with 84420 cones, that belong to 35 equivalence classes. The equality as sets for the two fans can now be checked by showing that all the simplicial cones in  $\operatorname{trop}(\mathcal{P}f_2(7))$  are contained in a cone of  $\operatorname{Pf}_2(7)$  and the union of the cones contained in the same cone gives the whole cone.

The prevariety contains 71820 simplicial and 1575 nonsimplicial cones. The simplicial ones are also cones of the variety, so that part is correct. Now there are 12600 remaining cones in the variety, that correspond to the nonsimplicial part. The nonsimplicial cones can be triangulated in two ways: in 8 cones and in 3 cones. The triangulation in 8 cones of all them can be shown to match exactly the cones of the variety, and we are done.

To better understand the difference between  $\operatorname{Pf}_k(n)$  and  $\operatorname{trop}(\mathcal{P}f_k(n))$  we are now going to relate them to two different notions of rank for a tropical matrix. For this, it is convenient to extend  $\mathbb{R}$  to the *tropical semiring*  $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$  with the operations max as "addition" and + as "multiplication". By a tropical  $n_1 \times n_2$ -matrix we mean an  $n_1 \times n_2$ -matrix with entries in  $\overline{\mathbb{R}}$ . To distinguish between tropical (pre)varieties in  $\mathbb{R}^n$  and  $\overline{\mathbb{R}}^n$  we denote as  $\overline{V}$  the extension to  $\overline{\mathbb{R}}^n$  of a tropical variety or prevariety  $V \in \mathbb{R}^n$ .

Clearly, for every family F of polynomials, the prevariety of F in  $\mathbb{R}^n$  is topologically closed, so it contains the closure of the prevariety in  $\mathbb{R}^n$ , and the same holds for varieties. The converse is not always true, as the following example shows.

*Example* 3.2. Let  $I = (x_1x_3 + x_2, x_2x_3 + x_1)$ . The tropical variety it defines in  $\mathbb{R}^3$  equals  $\{(a, a, 0) : a \in \mathbb{R}\}$ , while the variety it defines in  $\overline{\mathbb{R}}^3$  contains that plus the points  $\{(-\infty, -\infty, b) : b \in \overline{\mathbb{R}}\}$ .

Observe that this ideal is not prime, since it contains  $x_1(x_3^2 - 1)$  but it does not contain any of its factors  $x_1, x_3 + 1$ , or  $x_3 - 1$ . We do not know whether for prime ideals it is always true that the closure of V equals  $\overline{V}$ .

The following two notions of rank were introduced in [16].

Definition 3.3 (tropical rank [31, Def. 5.3.3]). A square matrix  $M \in \mathbb{R}^{r \times r}$  is tropically singular if the maximum in the tropical determinant

trop det(M) := 
$$\max_{\sigma \in S_r} \sum_{i=1}^r m_{i\sigma(i)}$$

is attained at least twice, and tropically regular otherwise.

The tropical rank of a tropical matrix is the largest size of a tropically regular minor in it.

Stated differently, the tropical rank of M is the largest r such that M is not in the tropical prevariety of the  $r \times r$  minors or, equivalently, the smallest r such that M is in the tropical prevariety of the  $(r + 1) \times (r + 1)$  minors.

Definition 3.4 (Kapranov rank [31, Def. 5.3.2]). Let  $M \in \mathbb{R}^{n_1 \times n_2}$  be a tropical matrix. The Kapranov rank of M over a valuated field  $\mathbb{K}$  is the smallest rank of a lift of the matrix, that is, a matrix  $\widetilde{M} \in \mathbb{K}$  such that the degree of  $\widetilde{M}_{ij}$  is  $m_{ij}$ .

The tropical variety of the  $(r + 1) \times (r + 1)$  minors is the tropicalization of the (classical) variety of the matrices with rank at most r. Hence, the Kapranov rank is the smallest r such that M is in the tropical variety of the  $(r + 1) \times (r + 1)$  minors, or the largest r such that M is not in the tropical variety of the  $r \times r$  minors.

Observe that the Kapranov rank of M depends on the field  $\mathbb{K}$  under consideration, while the tropical rank does not. The relation of the two notions of rank to the tropical variety and prevariety of minors readily shows that the Kapranov rank is greater than or equal to the tropical rank [16, Theorem 1.4]. Two small examples where the two notions do not coincide appear in [16, section 7] (a  $7 \times 7$  matrix of tropical rank three and Kapranov rank four) and [41] (a  $6 \times 6$  matrix of tropical rank four and Kapranov rank five). The two examples are reproduced in [42, section 4] where Shitov, completing the work of Develin, Santos, and Sturmfels [16], Chan, Jensen, and Rubei [9], and himself [43] shows that these two examples are the smallest possible.

Lemma 3.5 ([42]). For given positive integers  $r, n_1, n_2$  the following are equivalent:

- 1. The  $(r+1) \times (r+1)$  minors are a tropical basis for the variety of  $n_1 \times n_2$  matrices of rank r (over any of the complex, real, or rational fields).
- 2.  $r \leq 2$ , or  $r = \min\{n_1, n_2\}$ , or r = 3 and  $\min\{n_1, n_2\} \leq 6$ .

Since these notions of rank distinguish between the variety and prevariety of minors, antisymmetric versions of them will distinguish between the variety and prevariety of Pfaffians. (The same idea for the *symmetric* case is explored in [50].)

Let  $M \in \mathbb{R}^{n_1 \times n_2}$  be a tropical matrix and let  $n = n_1 + n_2$ . Let  $K \in \mathbb{R}$  be a sufficiently big constant. From M and K we construct the following  $n \times n$  matrix:

$$\operatorname{Sym}(M,K) := \begin{pmatrix} N_1 & M \\ M^t & N_2 \end{pmatrix} \in \overline{\mathbb{R}}^{n \times n},$$

where  $(N_1)_{ij} = m_{i1} + m_{j1} - K$  and  $(N_2)_{ij} = m_{1i} + m_{1j} - K$  for  $i \neq j$ , and  $(N_1)_{ii} = (N_2)_{ii} = -\infty$ . We have a corresponding vector  $v(M, K) \in \mathbb{R}^{\binom{[n]}{2}}$  of entries of Sym(M, K):

$$v_{ij} := \begin{cases} m_{i,j-n_1} & \text{if } 1 \le i \le n_1 < j \\ m_{i1} + m_{j1} - K & \text{if } 1 \le i, j \le n_1, \\ m_{1,i-n_1} + m_{1,j-n_1} - K & \text{if } i, j > n_1. \end{cases}$$

For example, for the  $2 \times 3$  matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

we have

$$\operatorname{Sym}(M, 10) = \begin{pmatrix} -\infty & -5 & 1 & 2 & 3 \\ -5 & -\infty & 4 & 5 & 6 \\ 1 & 4 & -\infty & -7 & -6 \\ 2 & 5 & -7 & -\infty & -5 \\ 3 & 6 & -6 & -5 & -\infty \end{pmatrix}$$

and

$$v(M, 10) = (-5, 1, 2, 3, 4, 5, 6, -7, -6, -5)$$

where the negative entries are obtained subtracting 10 from the sum of the two corresponding elements from the first row or from the first column of M.

We also consider the matrix and vector  $\operatorname{Sym}(M, \infty)$  and  $v(M, \infty) \in \overline{\mathbb{R}}^{\binom{[n]}{2}}$  obtained using  $\infty$  instead of K. That is

$$\operatorname{Sym}(M,\infty) := \begin{pmatrix} -\infty & M \\ M^t & -\infty \end{pmatrix} \in \overline{\mathbb{R}}^{n \times n}$$

Lemma 3.6. Let  $M \in \overline{\mathbb{R}}^{n_1 \times n_2}$  be a tropical matrix and  $K \in \overline{\mathbb{R}}$ . For the vector  $v(M, K) \in \overline{\mathbb{R}}^{\binom{[n]}{2}}$  defined above we have

- 1. for K sufficiently large,  $v(M, K) \in Pf_k(n)$  if and only if the tropical rank of M is at most k;
- 2.  $v(M, \infty) \in \operatorname{trop}(\mathcal{P}f_k(n))$  if and only if the Kapranov rank of M is at most k.

**Proof.** For part (1), assume first that  $v(M, K) \in Pf_k(n)$ , and consider a  $(k+1) \times (k+1)$  minor of M. This corresponds to a set  $U \in \binom{[n]}{2k+2}$  with half of the elements in  $[1, \ldots, n_1]$  and the other half in  $[n_1 + 1, \ldots, n]$ . Since  $v(M, K) \in Pf_k(n)$ , there are at least two perfect matchings in U of maximum weight. Since we chose K very big, none of these matchings come from the  $N_1$  or  $N_2$  parts of Sym(M, K). This implies that the minor of M that we started with is tropically singular.

Conversely, assume that trop rank  $M \leq k$ . Let  $U \in {[n] \choose 2k+2}$  and consider a perfect matching E in U with maximal weight, which is a term in the Pfaffian of U. We have three cases:

- If all the edges in E are between  $[n_1]$  and  $[n_1 + 1, n]$ , E corresponds to a permutation in M attaining the tropical determinant. As trop rank  $M \leq k$ , there must be another permutation with the same weight.
- If all the edges in E except one are between  $[n_1]$  and  $[n_1 + 1, n]$ , suppose  $E = \{\{i_1, j_1\}, \dots, \{i_{k+1}, j_{k+1}\}\}$  and  $i_1, \dots, i_{k+1}, j_1 \leq n_1 < j_2, \dots, j_{k+1}$  (the other case is symmetric). Then

$$w(E) = v_{i_1j_1} + \dots + v_{i_{k+1}j_{k+1}}$$
  
=  $m_{i_11} + m_{j_11} + m_{i_2,j_2-n_1} + \dots + m_{i_{k+1},j_{k+1}-n_1} - K$ 

We have now two cases:

- If  $j_l = n_1 + 1$  for some l, for example  $j_2 = n_1 + 1$ , then

$$w(E) = v_{i_1i_2} + v_{j_1,n_1+1} + v_{i_3j_3} + \dots + v_{i_{k+1}j_{k+1}}.$$

- If  $j_l > n_1 + 1$  for all  $l, w(E) - m_{j_1 1} + K$  is the weight of the permutation  $\{i_1, 1\}, \{i_2, j_2 - n_1\}, \ldots, \{i_{k+1}, j_{k+1} - n_1\}$  in M. Since the tropical rank of M is smaller than k + 1, there is another permutation with weight greater than or equal to  $w(E) - m_{j_1 1} + K$ . That is,

$$w(E) - m_{j_11} + K \le m_{i'_11} + m_{i'_2, j_2 - n_1} + \dots + m_{i'_{k+1}, j_{k+1} - n_1},$$
  
where  $(i'_1, \dots, i'_{k+1})$  is a permutation of  $(i_1, \dots, i_{k+1})$ . Equivalently  
 $w(E) \le (m_{i'_11} + m_{j_11} - K) + m_{i'_2, j_2 - n_1} + \dots + m_{i'_{k+1}, j_{k+1} - n_1}$ 

$$= v_{i_1'j_1} + v_{i_2'j_2} + \dots + v_{i_{k+1}'j_{k+1}}.$$

As E is maximal, this is an equality, and we have another matching in U with the same weight.

• If there is more than one edge inside  $[n_1]$  or inside  $[n_1+1,n]$ , suppose for example we have the edges  $\{a,b\}$  and  $\{c,d\}$  with  $a,b,c,d \le n_1$ . Then any of the two swaps among these four elements preserves the weight; indeed,

$$v_{a,b} + v_{c,d} = m_{a1} + m_{b1} + m_{c1} + m_{d1} - 2K = v_{a,c} + v_{b,d} = v_{a,d} + v_{b,c}$$

In any case, there is another matching with the same weight as E, and this finishes part (1). For part (2), if M has Kapranov rank at most k then there is a lift  $\widetilde{M}$  of M of rank k. Thus,

$$\begin{pmatrix} 0 & \widetilde{M} \\ \widetilde{M}^t & 0 \end{pmatrix}$$

is an antisymmetric lift of  $\text{Sym}(M, \infty)$  of rank 2k.

Conversely, if  $v(M, \infty) \in \operatorname{trop}(\mathcal{P}f_k(n))$ , consider an antisymmetric matrix in  $\mathcal{P}f_k(n)$  projecting to it, hence of rank 2k. This matrix necessarily has zero entries in the places where  $v(M, \infty)$  has  $-\infty$ , so it is of the form

$$\begin{pmatrix} 0 & \widetilde{M} \\ \widetilde{M}^t & 0 \end{pmatrix},$$

where M is a matrix of rank at most k and projecting to M.

**Theorem 3.7.** If there is a matrix  $M \in \mathbb{R}^{n_1 \times n_2}$  of tropical rank  $\leq k$  and Kapranov rank > k then  $\operatorname{Pf}_k(n) \neq \operatorname{trop}(\mathcal{P}f_k(n))$ , where  $n = n_1 + n_2$ .

This happens, for example, for k = 3 and any  $n \ge 14$  and for any  $k \ge 4$  and  $n \ge 2k + 4$ .

*Proof.* Let  $M \in \mathbb{R}^{n_1 \times n_2}$  be a matrix of tropical rank  $\leq k$  and Kapranov rank > k. By part (1) of Lemma 3.6 we have that  $v(M, K) \in Pf_k(n)$  for every sufficiently big K.

Also, by part (2) of the lemma,  $v(M, \infty) \notin \operatorname{trop}(\mathcal{P}f_k(n))$ . In particular,  $v(M, \infty)$  is not in the closure of  $\operatorname{trop}(\mathcal{P}f_k(n))$ , which implies it is not true that  $v(M, K) \in \operatorname{trop}(\mathcal{P}f_k(n))$  for all sufficiently big K.

Thus,  $\operatorname{Pf}_k(n) \neq \operatorname{trop}(\mathcal{P}f_k(n))$ .

Summing up, the cases where we do not know whether  $Pf_k(n) = trop(\mathcal{P}f_k(n))$  are

• k = 2 and  $n \ge 8$ ;

- k = 3 and  $n \in \{9, 10, 11, 12, 13\};$
- $k \ge 4$  and n = 2k + 3.

**3.2.** The *k*-associahedron as the fp-positive part of the tropical Pfaffian variety. We are interested in the part of  $Pf_k(n)$  contained in  $Grob_k(n)$ .

Definition 3.8. We define

$$\operatorname{Pf}_{k}^{+}(n) := \operatorname{Pf}_{k}(n) \cap \operatorname{Grob}_{k}(n).$$

We call it the (k + 1)-free part of the tropical Pfaffian variety of parameters n and k for two reasons. On the one hand, the initial ideal corresponding to  $\operatorname{Grob}_k(n)$  is the Stanley–Reisner ideal of the complex of (k + 1)-free sets. But, more significantly, our results in this section say that  $\operatorname{Pf}_k^+(n)$  coincides with the points of  $\operatorname{Grob}_k(n)$  which, expressed in the w-coordinates, have (k + 1)-free support.

**Theorem 3.9.** Let  $v = d(w) \in \operatorname{Grob}_k(n)$  be a vector in the Gröbner cone. This includes the case where w is nonnegative (or, equivalently,  $v \in \operatorname{FP}_n$ ). Then,

- 1.  $v \in Pf_k^+(n)$  if and only if the support of w is (k+1)-free;
- 2. if the above holds, then for every subset  $U \subset {\binom{[n]}{2}}$  of size 2k+2 one of the maximal matchings of U for v is the one producing a (k+1)-crossing, and a second one is obtained from it by a swap of two consecutive edges in the (k+1)-crossing.

*Proof.* Let  $U = \{a_0, a_1, \ldots, a_{2k+1}\}$  written in cyclic order, and let  $E_0$  be the (k+1)-crossing in it, that is, the matching that pairs  $a_i$  with  $a_{k+1+i}$ . As we already know, the maximum weight given by v to matchings of U is attained at  $E_0$ .

If the support of w is (k+1)-free, there must be an l such that no edge in the support of w has an end between sides  $a_l$  and  $a_{l+1}$  and the other between  $a_{l+k+1}$  and  $a_{l+k+2}$ . Then,

$$E_1 := E_0 \setminus \{\{a_l, a_{l+k+1}\}, \{a_{l+1}, a_{l+k+2}\}\} \cup \{\{a_l, a_{l+k+2}\}, \{a_{l+1}, a_{l+k+1}\}\}$$

has the same weight as  $E_0$ , so that  $v \in Pf_k^+(n)$  and part (2) holds.

Conversely, if the support of w contains a (k+1)-crossing then there is a  $U = \{a_0, a_1, \ldots, a_{2k+1}\}$  such that each  $a_i$  lies in one of the 2k+2 regions defined by that crossing, and then the matching  $E_0$  of U has weight strictly larger than any other matching. In particular,  $v \notin Pf_k(n)$ .

We now want to show that  $Pf_k^+(n)$  is contained in  $trop(\mathcal{P}f_k(n))$ . That is to say, even if the tropical Pfaffian variety and prevariety may not coincide, their "(k+1)-free parts" coincide. We need the following lemma, the proof of which we postpone to section 3.3.

Lemma 3.10. Let  $v = d(w) \in \operatorname{Grob}_k(n)$  be sufficiently generic. Then, for every subset  $U \in \binom{[n]}{2k+2}$  we have that U has the same number of positive and negative matchings of maximum weight with respect to v.

Corollary 3.11.  $\operatorname{Pf}_k^+(n) \subset \operatorname{trop}(\mathcal{P}f_k(n))$ . Moreover,  $\operatorname{Pf}_k^+(n) \subset \operatorname{trop}^+(\mathcal{P}f_k(n))$ .

Let us point out that  $Pf_k(n)$  and  $Pf_k^+(n)$  are independent of the field  $\mathbb{K}$ , while trop( $\mathcal{P}f_k(n)$ ) and trop<sup>+</sup>( $\mathcal{P}f_k(n)$ ) are (probably) not. The first statement is over an arbitrary field. The second statement is stronger, but it makes sense only over fields of characteristic zero.

*Proof.* Let  $v \in Pf_k^+(n)$ . We want to show that  $v \in trop(\mathcal{P}f_k(n))$ . In fact, it is enough to show this under the assumption that v is sufficiently generic (within  $Pf_k^+(n)$ ), since  $trop(\mathcal{P}f_k(n))$ 

is closed. By Theorem 3.9, being generic in  $Pf_k^+(n)$  implies that v = d(w) for a w with support equal to a k-triangulation. By Lemma 3.10 the latter implies that the initial form of every Pfaffian for the weight vector v vanishes at the point  $(1, \ldots, 1)$ . Since Pfaffians are a Gröbner basis for v by Theorem 2.8, we have that

$$(1,\ldots,1) \in V(\operatorname{in}_v(I_k(n))).$$

This clearly implies that  $in_v(I_k(n))$  contains no monomials (over an arbitrary field) and that it does not contain polynomials with all coefficients real and of the same sign (over fields of characteristic zero).

Putting together Theorem 3.9 and Corollary 3.11 we conclude Theorem 1.9.

*Remark* 3.12. Since Pfaffians of degree two coincide with the 3-term Plücker relations that generate the Grassmannian  $\mathcal{G}r_2(n)$ , we have that  $\mathcal{P}f_1(n) = \mathcal{G}r_2(n)$  and that  $\mathrm{Pf}_1(n)$  equals the *Dressian*  $\mathcal{D}r_2(n)$  (the tropical prevariety defined by quadratic Plücker relations [31, section 4.4]).

It was proven in [44] that  $\mathcal{D}r_2(n) = \operatorname{trop}(\mathcal{G}r_2(n))$  (equivalently, that  $\operatorname{Pf}_1(n) = \operatorname{trop}(\mathcal{P}f_1(n))$ , by showing that  $\operatorname{trop}(\mathcal{G}r_2(n))$  also coincides with the space  $\mathcal{T}ree_n$  of *tree metrics* for trees with n leaves. The proof is reproduced in [31, Theorem 4.3.3] and the idea of it is the following: the tropical hypersurface corresponding to the Pfaffian of degree two (or the 3-term Plücker relation) of a certain  $U \subset {n \choose 4}$  equals the solution set of

$$v_{i,j} + v_{k,l} \le \max\{v_{i,k} + v_{j,l}, v_{i,l} + v_{j,k}\} \quad \forall \{i, j\} \in \binom{U}{2}.$$

These relations (taken for all U) are exactly the *four-point conditions* that characterize tree metrics [6]. Hence,  $\operatorname{trop}(\mathcal{P}f_1(n)) \subset \operatorname{Pf}_1(n) = \mathcal{T}ree_n$ . For the converse, for any given (generic)  $v \in \mathcal{T}ree_n = \operatorname{Pf}_1(n)$  there is a ternary tree T with nonnegative weights w on its edges and realizing v as a tree metric. By relabeling its leaves, we can assume that T is the dual tree of a certain triangulation of the *n*-gon. Hence, v coincides (after relabeling, but this does not change  $\operatorname{trop}(\mathcal{P}f_1(n))$ ) with the d(w) of Definition 2.3 for this choice of weights. Theorem 3.9 and Corollary 3.11 then imply that  $v \in \operatorname{Pf}_1^+(n) \subset \operatorname{trop}(\mathcal{P}f_1(n))$ .

We do not have a concrete example showing that  $Pf_2(n) \neq trop(\mathcal{P}f_2(n))$  for any n, nor  $Pf_k(2k+3) \neq trop(\mathcal{P}f_k(2k+3))$  for any k, but the above proof cannot work for  $k \geq 2$  since not every cone in  $Pf_k(n)$  can be sent to  $Pf_k^+(n)$  by a relabeling of the vertices. This is illustrated in the following example.

*Example* 3.13. Let n = 6 and k = 2. Observe that  $Pf_2(6) = trop(\mathcal{P}f_2(6))$  since it is a hypersurface.

Consider the  $v \in \mathbb{R}^{\binom{[6]}{2}}$  defined by

$$v_{1,3} = v_{2,3} = v_{2,4} = v_{4,5} = v_{5,6} = v_{1,6} = 1,$$

and  $v_{i,j} = 0$  for every other i, j. This v lies in Pf<sub>2</sub>(6) since it gives maximum weight to (exactly) two matchings, namely, {13, 24, 56} and {23, 45, 16}.

Since the first matching is negative and the second one is positive, we have that  $v \in$  trop<sup>+</sup>( $\mathcal{P}f_2(6)$ ). Since the two matchings do not differ by a single swap, part (2) of Theorem 3.9 implies that no relabeling sends v to  $\mathrm{Pf}_2^+(6)$ .

The example also shows that  $\operatorname{trop}^+(\mathcal{P}f_k(n))$  is not contained in the Gröbner cone of k+1crossings, but that is also easy to achieve with the following simpler example: let  $v_{13} = 1$  and every other  $v_{ij} = 0$ . For any  $k \ge 2$  and every  $n \ge 6$  this gives a point in  $\operatorname{trop}^+(\mathcal{P}f_k(n))$  (in every maximum matching of size 3 we can swap the two edges of weight zero to get a maximum matching of the opposite sign) that is not in the Gröbner cone (in any U containing  $\{1,3\}$  the matching using  $\{1,3\}$  has weight larger than the 3-crossing).

**3.3. Proof of Lemma 3.10.** In the following result we call an *accordion* any sequence  $e_1, \ldots, e_m$  of edges from  $\binom{[n]}{2}$  such that (a) for every  $i = 1, \ldots, n-1$ ,  $e_i$  and  $e_{i+1}$  share a vertex; (b) for every  $i = 2, \ldots, n-1$ , the endpoints of  $e_{i-1}$  and  $e_{i+1}$  that are not in  $e_i$  lie on opposite sides of the line containing  $e_i$ .

The only property of k-triangulations that we need in what follows (apart from the fact that they are (k + 1)-free) is the following.

Lemma 3.14. Let T be a k-triangulation of the n-gon, for some k. Then, every two edges of T that do not cross are part of an accordion contained in T.

*Proof.* Let  $e = \{a, b\}$  and  $e' = \{a', b'\}$  be the two edges of T; we assume without loss of generality that  $1 \le a \le a' < b' \le b \le n$ . We will use induction on  $\min\{a' - a, b - b'\}$ , taking as base cases those with a = a' or b = b', which are trivial. Hence, for the inductive step we suppose that e and e' have no endpoints in common.

If  $\{a,b'\} \in T$ , we are done, so we assume that  $\{a,b'\} \notin T$ . Then there is a k-crossing K in T that crosses that edge. That is,  $K \cup \{a,b'\}$  is a (k+1)-crossing contained in  $T \cup \{a,b'\}$ . Let e'' be the edge next to  $\{a,b'\}$  in the positive direction in this (k+1)-crossing. If e'' crossed e (resp., e'), then every edge in K would cross e (resp., e'), which would imply that T contains the (k+1)-crossing  $K \cup \{e\}$  (resp.,  $K \cup \{e'\}$ ). Thus, e'' does not cross any of e or e'. Inductive hypothesis implies that T contains an accordion from e to e'' and an accordion from e'' to e', and the union of these two accordions is an accordion from e to e'.

We now consider a subset  $U \in {[n] \choose 2k+2}$  (as a set of sides, not vertices, of the *n*-gon) and  $v = d(w) \in \operatorname{Grob}_k(n)$  sufficiently generic. Genericity implies, by Theorem 3.9, that the support of w is a certain k-triangulation T. For each edge  $e \in T$  we call the *length of e with respect to* U, and denote as  $\ell_U(e)$  the smallest size of the two parts of U separated by e. If both parts are equal (that is, if  $\ell_U(e) = k + 1$ ) we say that e is a diameter of U.

For a matching M of U and an edge e of T we denote by  $c_M(e)$  the number of edges of M that cross e. Remember that, v being in the Gröbner cone, the maximum weight among matchings of U is the weight of the (k + 1)-crossing.

Lemma 3.15. Let M be a matching of U. Then, M is of maximum weight with respect to v if and only if for every  $e \in T$  we have that  $\ell_U(e) = c_M(e)$ .

*Proof.* Observe that the equality  $\ell_U(e) = c_M(e)$  holds for the case when M is the (k+1)-crossing, and that, for arbitrary M, knowing which edges of T cross each edge of M is enough to compute the weight of M. This shows the sufficiency of  $\ell_U(e) = c_M(e)$ .

Now suppose that  $\ell_U(e) > c_M(e)$  for some edge  $e \in T$ . Take a vector w' obtained setting  $w_e$  to its minimum possible value while staying in  $\operatorname{Grob}_k(n)$ . For v' = d(w'), the (k+1)-crossing is still the maximum weight matching, so

$$\sum_{e \in T} w'_e c_M(e) \le \sum_{e \in T} w'_e l_U(e) \Rightarrow \sum_{e \in T} w'_e (l_U(e) - c_M(e)) \ge 0.$$

Our condition in w implies that  $w_e > w'_e$ , so

$$\sum_{e \in T} w_e(l_U(e) - c_M(e)) > 0 \Rightarrow \sum_{e \in T} w_e c_M(e) < \sum_{e \in T} w_e l_U(e) < \sum_{e \in$$

Hence, M is not of maximum weight.

For the rest of this section, we collapse the *n*-gon to a (2k + 2)-gon by leaving only the sides labeled by U; that is, by contracting all edges e with  $\ell_U(e) = 0$ . We denote as  $T_U$  the subgraph of  $K_{2k+2}$  obtained from T after this collapse. We introduce the following partial order among edges of  $T_U$  (or, in fact, among edges of  $K_{2k+2}$ ): e and f are incomparable if they either cross or are separated by a diameter of U, and if they are comparable then they are ordered according to their  $\ell_U$ .

Observe that both  $\ell_U(e)$  and  $c_M(e)$  depend only on the class of e in  $T_U$ . Thus, Lemma 3.15 needs only to be checked in  $T_U$  and not in T. (That is, only one representative edge of T for each class in  $T_U$  needs to be checked.) But even more is true. Let  $T_U^{\text{max}}$  be the set of edges of  $T_U$  that are maximal (within  $T_U$ ) for this order.

Lemma 3.16. Let M be a matching of U. If  $\ell_U(e) = c_M(e)$  holds for the edges in  $T_U^{\max}$  then it holds for all edges in  $T_U$ , hence in T.

**Proof.** Let e < e' be two edges of  $T_U$  and suppose that  $\ell_U(e') = c_M(e')$ . Then, the edges of M that cross e' match the  $\ell_U(e')$  edges of the (2k+2)-gon on the shorter side of e' to the same number of edges on the longer side (if e' is a diameter it does not matter which side we call short). By the definition of e < e', the smaller side of e is contained in the smaller side of e', so the same holds for e and  $\ell_U(e) = c_M(e)$ .

This last lemma suggests we should look at the properties of  $T_{II}^{\text{max}}$ .

### Lemma 3.17.

- 1. Every two edges in  $T_{II}^{\text{max}}$  either cross each other or share a vertex.
- 2. There is a vertex of the (2k+2)-gon not used in  $T_U^{\max}$ .

*Proof.* For part (1) we use Lemma 3.14 and the observation that the passage from T to  $T_U$  preserves accordions. In particular, every two edges of  $T_U$  that do not cross are part of an accordion in  $T_U$ . Only two of the edges of an accordion contained in  $T_U$  can be in  $T_U^{\text{max}}$ , and they share a vertex; hence, every two edges in  $T_U^{\text{max}}$  that do not cross share a vertex.

This finishes the proof of part (1) and gives us two possibilities:

- If all the edges in  $T_U^{\max}$  mutually cross, then  $T_U^{\max}$  is a *j*-crossing for some j < k + 1. Hence, at least one (in fact at least two) of the 2k + 2 vertices of the (2k + 2)-gon are not used.
- If two edges e and e' of  $T_U^{\max}$  share a vertex p, then none of them is a diameter and, in fact, they are on opposite sides of the diameter using p. Then the opposite vertex q of that diameter is not used in  $T_U^{\max}$  because it is impossible for an edge with an

endpoint in q other than the diameter itself to cross or share a vertex with both of e and e'.

In both cases we have a proof of part (2).

Lemma 3.18. Let p be a vertex of the (2k+2)-gon not used in  $T_U^{\text{max}}$ . Let a and b be the elements of U next to p. Then, no maximal matching of U matches a to b.

**Proof.** To seek a contradiction, suppose that M is a maximal matching and that  $\{a, b\} \in M$ . We claim that, for any other edge  $\{c, d\} \in M$ , no edge of  $T_U^{\max}$  has a and b on one side and c and d on the other side. Suppose that there is such an edge e. Then, by Lemmas 3.15 and 3.16, we have  $c_M(e) = l_U(e)$ , and the swaps  $\{a, c\}, \{b, d\}$  and  $\{a, d\}, \{b, c\}$  cross  $T_U^{\max}$  more often than the original pair of edges  $\{a, b\}, \{c, d\}$ ; that is, more often than the single edge  $\{c, d\}$  (since  $\{a, b\}$  does not cross  $T_U^{\max}$ ). This implies that, after swapping, we have  $c_M(e) > l_U(e)$ , which is not possible. This proves the claim.

Now, since all edges of  $T_U^{\max}$  have a and b on the same side, we conclude that this side must contain one of c or d for every  $\{c,d\} \in M$  other than  $\{a,b\}$ . In particular, for every  $e \in T_U^{\max}$  the side of e containing a and b has length at least k + 2 (it contains a, b, and one vertex of each of the other k edges in M). This gives the following contradiction: let p' be one of the vertices of the (2k + 2)-gon next to p. The edge  $\{p, p'\}$  is in  $T_U$ , since every boundary edge of the 2k+2-gon is. Hence, there must be an edge in  $T_U$  that is greater than  $\{p, p'\}$  in the partial order, and that edge can have length at most k + 1 on the side containing a and b.

We are now ready to prove Lemma 3.10.

*Proof of Lemma* 3.10. Let p be a vertex of the (2k+2)-gon not used in  $T_U^{\max}$ , which exists by Lemma 3.17. Let a and b be the first elements of U on both directions starting at p.

Let us denote by  $\mathcal{M}$  the set of matchings of U not using the edge  $\{a, b\}$ . This contains all matchings of maximum weight by Lemma 3.18. Consider the map  $\phi : \mathcal{M} \to \mathcal{M}$  that takes each matching  $M \in \mathcal{M}$  and swaps in it the edges that contain a and b in the way that does not produce the pair  $\{a, b\}$ . This map is well-defined since there are three possible matchings among four vertices and we are excluding one of them. We have the following:

- The map  $\phi$  is obviously an involution.
- The map  $\phi$  sends matchings of maximum weight to matchings of maximum weight by Lemmas 3.15 and 3.16, since every edge of  $T_U^{\text{max}}$  leaves a and b on the same side.
- If a' and b' are the elements of U matched to a and b in a certain matching M then the matching of a, b, a', b' that has a crossing is involved in the swap from M to  $\phi(M)$ (because the matching that is not involved in the swap is  $\{a, b\}, \{a', b'\}$ , which does not have a crossing). Hence, M and  $\phi(M)$  have opposite parity, by Lemma 2.1.

Putting these facts together we conclude that  $\phi$  restricts to a bijection between the odd and the even matchings of U of maximum weight.

4. Recovering the g-vector fan for k = 1. In this section we look at the case k = 1 and show how to project  $Pf_1^+(n)$  isomorphically to the associahedron  $\overline{Ass}_1(n)$ . In doing so we recover the so-called g-vector fan of the associahedron defined in the context of cluster algebras. Throughout the section let  $T \subset {\binom{[n]}{2}}$  be an arbitrary triangulation of the *n*-gon, that we call the *seed triangulation*. Then we have the following.

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Lemma 4.1. For every  $(v_{i,j})_{i,j} \in \operatorname{Pf}_1^+(n)$ , knowing the entries of v corresponding to T, we can recover all other entries. That is, the projection  $\pi : \operatorname{Pf}_1^+(n) \to \mathbb{R}^T \cong \mathbb{R}^{2n-3}$  that restricts each vector  $(v_{i,j})_{i,j}$  to the entries with  $\{i, j\} \in T$  is injective.

*Proof.* Let  $v \in Pf_1^+(n)$  and let us see that we can recover the entry  $v_{i,j}$  for any  $\{i, j\} \in {\binom{[n]}{2}}$ , knowing the entries of v corresponding to edges of T.

The proof is by induction on the number of triangles of T crossed by  $\{i, j\}$ . If only two triangles are crossed, then  $\{i, j\}$  is the only unknown entry from the quadruple  $U = \{i, j, k, l\}$ consisting of those two triangles, and the edges  $\{i, j\}$  and  $\{k, l\}$  cross. Since  $d \in Pf_1^+(n)$ , we have that the maximum weight among the three matchings in U is attained by  $\{ij, kl\}$  and at least one of the other two matchings, so we can write

$$v_{i,j} = \max\{v_{i,k} + v_{j,l}, v_{i,l} + v_{j,k}\} - v_{k,l}$$

If  $\{i, j\}$  crosses more than two triangles, let  $\{k, i, l\}$  be the triangle incident to i and crossed by  $\{i, j\}$ . By inductive hypothesis, all the entries among the 4-tuple  $\{i, j, k, l\}$  are known except for the entry  $\{i, j\}$ , so we can recover  $v_{i,j}$  with the same formula as above.

That is,  $\pi$  embeds  $\operatorname{Pf}_1^+(n)$  as a full-dimensional fan  $\pi(\operatorname{Pf}_1^+(n)) \subset \mathbb{R}^T \cong \mathbb{R}^{2n-3}$ . If we now compose it with a second projection

$$\phi: \mathbb{R}^T \to \mathbb{R}^{\overline{T}} \cong \mathbb{R}^{n-3}$$

that sends the irrelevant face of  $\pi(\operatorname{Pf}_1^+(n))$  to zero we will automatically have that  $\phi(\pi(\operatorname{Pf}_1^+(n)))$ is a fan isomorphic to the link of the irrelevant face in  $\pi(\operatorname{Pf}_1^+(n))$ , that is, isomorphic to  $\overline{Ass}_1(n)$ , the normal fan of the associahedron. Here,  $\overline{T}$  denotes the relevant part (the n-3 diagonals) of T.

Corollary 4.2. The projection

$$\phi \circ \pi : \mathrm{Pf}_1^+(n) \to \mathbb{R}^T \cong \mathbb{R}^{n-3}$$

gives a realization of the associahedron  $\overline{Ass}_1(n)$  as a complete fan.

*Proof.* This projection is conewise linear (linear in each cone). After normalizing, it becomes a continuous map from the (n-4)-dimensional sphere  $\overline{\mathcal{A}ss}_1(n)$  to the unit sphere in  $\mathbb{R}^{n-3}$  and, by Lemma 4.1, it is injective. Since every injective continuous map from a sphere to itself is a homeomorphism,  $\phi(\pi(\mathrm{Pf}_1^+(n)))$  is complete.

*Remark* 4.3. Lemma 4.1 and its Corollary 4.2 do not hold for  $k \ge 2$ . In fact, suppose we take T to be any k-triangulation containing all the edges of the form (1,i) and (2,i), which exists since  $k \ge 2$ . Consider now the cone corresponding to a k-triangulation T' that does not use a certain edge (1,i). In this cone we have  $w_{1,i} = 0$  and hence

$$v_{1,i} + v_{2,i+1} = v_{1,i+1} + v_{2,i}.$$

Thus, the projection  $\pi$  is not injective; it collapses the cone of T' to a lower dimension.

We now want to give a more explicit description of the fan in Corollary 4.2; that is, explicit coordinates in  $\mathbb{R}^{n-3}$  for the ray corresponding to each diagonal  $\{i, j\} \in {[n] \choose 2}$ . For this,

remember that T is embedded as a true triangulation using the vertices of our *n*-gon, while the diagonals  $\{a, b\} \in {[n] \choose 2}$  corresponding to coordinates in our ambient space correspond to pairs of sides. For any given diagonal  $\delta$  we define the following *crossing sign* of  $\{a, b\}$  with respect to  $\delta$  and the **g**-vector of  $\{a, b\}$  with respect to T as follows.

Definition 4.4 (See [23, Proposition 33] or [24, Definition 1.1]). Let  $\delta$  be a diagonal in  $\overline{T}$  and  $\{a,b\} \in {[n] \choose 2}$ . Let  $q(\delta)$  be the quadrilateral in T consisting of  $\delta$  and its two adjacent triangles. We define the crossing sign of  $\{i, j\}$  with respect to  $\delta$  in T:

$$\varepsilon(\delta \in T, \{a, b\}) := \begin{cases} +1 & \text{if } \{a, b\} \text{ crosses } q(\delta) \text{ as a } \mathsf{Z} \text{ ("zig")}, \\ -1 & \text{if } \{a, b\} \text{ crosses } q(\delta) \text{ as a } \mathsf{\Sigma} \text{ ("zag")}, \\ 0 & \text{otherwise.} \end{cases}$$

We define the **g**-vector of  $\{a, b\}$  with respect to T as

$$\boldsymbol{g}(T,\{a,b\}) := (\varepsilon(\delta \in T,\{a,b\}))_{\delta \in \overline{T}}$$

*Remark* 4.5.  $\mathbf{g}(T, \{a, b\})$  has the following interpretation: the edges of T crossed by  $\{i, j\}$  form an accordion in the sense of section 3.3. The signs in the vector  $\mathbf{g}(T, \{a, b\})$  record at which edges the accordion turns left or right. In particular, the **g**-vector is zero for edges of T that are not in the accordion, but also for those in which the accordion "does not turn."

This definition of **g**-vectors, which we have taken from Hohlweg, Pilaud, and Stella [23], is a specialization to the disc of the *shear coordinates* described for arbitrary surfaces by Fomin and Thurston in [18]. They consider the **g**-vector fan obtained considering as cones all the possible clusters (which, in type A are the triangulations) and taking as generators the **g**-vectors for a fixed but arbitrary seed triangulation T. The main result of [23] is that these fans are polytopal. It turns out that these fans are linearly isomorphic to the ones of Corollary 4.2.

**Theorem 4.6.** In the basis of  $\mathbb{R}^{n-3}$  consisting of the rays corresponding to the diagonals of  $\overline{T}$  we have that for every  $\{a,b\} \in {[n] \choose 2}$ , the vector  $\mathbf{g}(T,\{a,b\})$  spans the ray of  $im(\phi \circ \pi)$  corresponding to  $\{a,b\}$ .

**Proof.** For each  $(i,j) \in {\binom{[n]}{2}}$  let  $W_{i,j}$  be the generator of  $\operatorname{FP}_n$  corresponding to a certain  $\{i,j\}$ . That is,  $W_{i,j} = d(w)$  for the vector w with  $w_{i,j} = 1$  and  $w_{i',j'} = 0$  if  $\{i',j'\} \neq \{i,j\}$ . We think of  $W_{i,j}$  as the standard basis vector in the coordinates  $w_{i,j}$ , and let  $V_{i,j}$  be the standard basis vector in the coordinates  $w_{i,j}$ , and let  $V_{i,j}$  are also the generators for the fan structure in  $\operatorname{Pf}_k^+(n)$ , so that  $\phi \circ \pi(W_{i,j})$  is the corresponding generator of  $\phi \circ \pi(\operatorname{Pf}_1^+(n))$ .

The relations in Definition 2.3, which express the coordinates v in terms of the coordinates w, get transposed to the following relations among the vectors  $W_{i,j}$  and  $V_{a,b}$ :

(4.1) 
$$W_{i,j} = \sum_{\substack{\{a,b\} \in \binom{[n]}{2}\\ i < b \le j < a \le i}} V_{a,b}.$$

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Observe that the projections  $\pi$  and  $\phi$  are defined by their images at the vectors V and W, respectively.  $\pi$  sends  $V_{i,j}$  to zero if  $\{i, j\} \notin T$ , and  $\phi$  sends  $\pi(W_{i,i+1})$  to zero for every i. For simplicity, for each vector  $V \in \mathbb{R}^{\binom{[n]}{2}}$  we will denote  $\overline{V} := \phi(\pi(V)) \in \mathbb{R}^{n-3}$ , and the same for  $\overline{W}$ .

Let  $\{i, j\}$  be a diagonal of T. We then have

$$\overline{W}_{i,j} + \overline{W}_{i+1,j+1} = \overline{W}_{i,i+1} + \overline{W}_{j,j+1} = 0,$$

where the first equality comes from (4.1) taking into account that the only edges of T crossing  $\{i, j\}$  or  $\{i + 1, j + 1\}$  are those with an endpoint in i or j, and each of them crosses  $\{i, j\}$  and  $\{i + 1, j + 1\}$  the same number of times as it crosses  $\{i, i + 1\}$  or  $\{j, j + 1\}$ . (Namely, they all cross once except for the edge  $\{i, j\}$  which crosses twice). The second equality comes from the fact that  $\phi(\pi(W_{i,i+1}) = 0$  for every i. Thus we have

$$\overline{W}_{i,j} = -\overline{W}_{i+1,j+1}$$

for each diagonal  $\{i, j\}$  of T.

Now, let a and b be two sides of the n-gon and consider the accordion in T between a and b. Let  $\{i_1, j_1\}, \ldots, \{i_\ell, j_\ell\}$  be the diagonals of T at which the accordion has an "inflection point" (it changes from turning left to turning right, or vice versa, that is,  $\{a, b\}$  crosses  $\{i_m, j_m\}$  as a Z or a  $\Sigma$ , alternatively). The statement we want to prove is that

(4.2) 
$$\overline{W}_{a,b} = \sum_{\delta \in \overline{T}} \varepsilon(\delta \in T, \{a,b\}) \overline{W}_{\delta} = \sum_{m} \varepsilon(\{i_m, j_m\} \in T, \{a,b\}) \overline{W}_{i_m, j_m}.$$

Note that  $-\overline{W}_{i_m,j_m}$  equals  $\overline{W}_{i_m+1,j_m+1}$ , so we are taking the sum of the edges in the zigzag turned in the direction of the path. Indeed, the sum in the right-hand side includes three times the diagonals  $\{i_m, j_m\}$ , twice the rest of the diagonals in the accordion, and once the rest of the edges with an endpoint in vertices where an  $\{i_m, j_m\}$  meets the next one. Subtracting the irrelevant  $\overline{W}$ 's for these vertices, we get exactly once the diagonals separating a and b, and only them.

Corollary 4.7. Let T be any triangulation of the n-gon. The associahedral fan  $im(\phi \circ \pi)$  in  $\mathbb{R}^{n-3}$  of Corollary 4.2 equals the **g**-vector fan of T. Hence, it is polytopal.

Remark 4.8. From the perspective of cluster algebras, associahedra are the type A case of the generalized associahedra that Fomin and Zelevinsky [19] defined as simplicial spheres and Chapoton, Fomin, and Zelevinsky [10] constructed as polytopes, using the so-called **d**-vector fans for certain seed clusters. In type A, this construction was generalized by Ceballos, Santos, and Ziegler [8, section 5] to obtain Catalan-many associahedra by showing that any triangulation works as the seed triangulation in the **d**-vector construction.

The construction of generalized associahedra via **g**-vectors instead of **d**-vectors was first achieved in various special cases by, among others, Hohlweg-Lange [21], Hohlweg, Lange, and Thomas [22], Pilaud and Stump [38], and Stella [47], before the general case was settled by Hohlweg, Pilaud, and Stella in [23].

The associahedral fans obtained by Santos via **d**-vector fans and by Hohlweg, Pilaud, and Stella via **g**-vector fans from a seed triangulation T have certain similarities:

- 1. For each of the n-3 diagonals  $\{i, j\} \in \overline{T}$ , the ray corresponding to  $\{i, j\}$  is opposite to another ray. That is, the corresponding facets in the associahedron are parallel.
- 2. Every other ray can be expressed as a  $\{+1, 0, -1\}$  combination in the basis given by those n-3 rays.

However, they are not the same. In the **g**-vector fan the ray opposite to a diagonal  $\{i, j\}$  of T is  $\{i+1, j+1\}$  while in the **d** construction it is the diagonal inserted in T by the flip of  $\{i, j\}$ .

One could think that there is a variant of **g**-vectors for k > 1. For example, for k = 2 it is known that multitriangulations are complexes of 5-sided stars [36], and a **g**-vector can be defined assigning different values for  $\varepsilon(\{i, j\} \in T, \{a, b\})$  depending on the position of  $\{a, b\}$ with respect to the two stars incident to  $\{i, j\}$ . A priori, the problem would be how to define these  $\varepsilon(\{i, j\} \in T, \{a, b\})$  so that they work. If the two edges cross, there are 4 possible positions for a and the same number for b, giving 16 different positions, and the idea would be to use different coefficients as  $\varepsilon$  depending on which of the 16 possibilities (or 10, if we mod out symmetry) we are in.

However, this idea can not work for n big enough.

**Theorem 4.9.** For a k-triangulation T with k > 1, if there is an edge not contained in any pair of adjacent stars of T, it is impossible to realize the k-associahedron as a **g**-vector fan with seed T, independently of the values chosen for  $\varepsilon$ .

*Proof.* Let  $\{a, b\}$  be the edge. We will show that  $\mathbf{g}(a, b) + \mathbf{g}(a + 1, b + 1) = \mathbf{g}(a, b + 1) + \mathbf{g}(a + 1, b)$ . Then, we can choose a k-triangulation that contains these edges (for k > 1 it will exist), and its cone will not have full dimension.

This equality can be checked one coordinate at a time. For an edge  $\{i, j\} \in T$ , either a is not in the two stars delimited by  $\{i, j\}$  or b is not. In the first case,  $\varepsilon(\{i, j\} \in T, \{a, c\}) = \varepsilon(\{i, j\} \in T, \{a + 1, c\})$  for any c, concretely for c = b and c = b + 1, and the equality holds for this component. The same happens if b is not in the two stars.

Corollary 4.10. It is impossible to realize the k-associahedron as a g-vector fan, independently of the values chosen for  $\varepsilon$  for k > 1 and n big enough.

*Proof.* Suppose it is possible. Then all edges must be contained in a pair of adjacent stars. There are as many pairs of adjacent stars as relevant edges in T, that is, k(n-2k-1). Each pair contains at most 4k vertices that form  $\binom{4k}{2}$  edges, so we get

$$\binom{n}{2} \le k(n-2k-1)\binom{4k}{2}$$

which is false for n big enough.

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