TOWARDS AN ECOSYSTEM FOR COMPUTER-SUPPORTED GEOMETRIC REASONING

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In this study, we explore automated reasoning tools (ART) in geometry education and we argue that these tools are part of a wider, nascent ecosystem for computer-supported geometric reasoning. To provide some context, we set out to summarize the capabilities of ART in GeoGebra (GGb), and we discuss the first research proposals of its use in the classroom. While the design and development of ART have been embraced already by several teams of mathematics researchers and developers, the educational community, which is an essential actor in this ecosystem, has not provided sufficient feedback yet on this new technology. We therefore propose a concrete path for incorporating ART in the classroom. We outline a set of necessary procedures towards this goal, and we include a discussion on the benefits and concerns arising from the use of these automated tools in the mathematical learning process.

Keywords: automated reasoning tools, GeoGebra, computer-supported reasoning, elementary geometry

Subject classification codes: 97G40, 68T15

INTRODUCTION

Dynamic Geometry Systems (DGS), such as the popular GeoGebra1 software, are useful tools that can improve the teaching and learning of reasoning and proof (Sinclair, Bartolini Bussi, de Villiers, Jones, Kortenkamp and Owens, 2016). While, initially, these software environments were meant mainly for geometric visualization and experimentation, modern DGS have started to include features for automated reasoning that allow for the automatic and mathematically rigorous verification and discovery of geometric theorems (Kovács, Recio and Vélez, 2018).

Recently, the computer algebra system Giac was embedded in GeoGebra (Kovács and Parisse, 2015), allowing for the implementation of automated proving algorithms based on the algebraic approach described in Recio and Vélez (1999). The result is a collection of GeoGebra features and commands that allow to conjecture, discover and prove statements on a given geometric construction. Though these tools clearly have considerable potential in the mathematics classroom, it is not clear yet how they should be used effectively.

The aim of the present paper is to establish a concrete path towards accommodating GeoGebra ART in a computer-supported geometric reasoning ecosystem. We first review and bring new light on tasks and experiments that have been performed in previous papers. Then, we discuss the advantages and

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[1] https://www.geogebra.org/

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disadvantages that this novelty could contribute to the learning and teaching of geometry. Finally, we establish this issue in a theoretical framework, that of the "mathematical working space" (Richard, Oller & Meavilla, 2016; Richard, Venant, Gagnon, 2019). As we will point out, the proposed approach requires to be embedded in a larger ecosystem that should be developed by the scientific and teaching community, and cover all aspects of computer-supported geometric reasoning.

A SHORT INTRODUCTION TO GEOGEBRA AUTOMATED REASONING TOOLS (ART)

GeoGebra Automated Reasoning Tools (GGb-ART) currently include several commands: The *Relation* command, that can be used for the automatic finding of geometric conjectures and the verification or denial of these conjectures; the *LocusEquation* command, which calculates the implicit equation of a free point such that a given property holds; the *Prove* and *ProveDetails* commands, which decide if a statement is true in general and, eventually, give some additional conditions for its truth, avoiding degenerate cases; and the *Envelope* command, which computes the equation of a curve which is tangent to a family of objects while a certain parent of the family moves on a path. A complete tutorial can be found in Kovács, Recio, and Vélez (2017).

A typical example of GGb-ART is shown in Figure 1. In this example, a triangle ABC and two of its medians are drawn. Then, the *Prove* command is used to verify that the barycenter F divides the median $\overline{AD}$ into two segments, one of which ($h = \overline{AF}$) measures twice as much as the other ($i = \overline{FD}$). The *Prove* command for the statement $h = 2i$ returns the Boolean value "*true*".

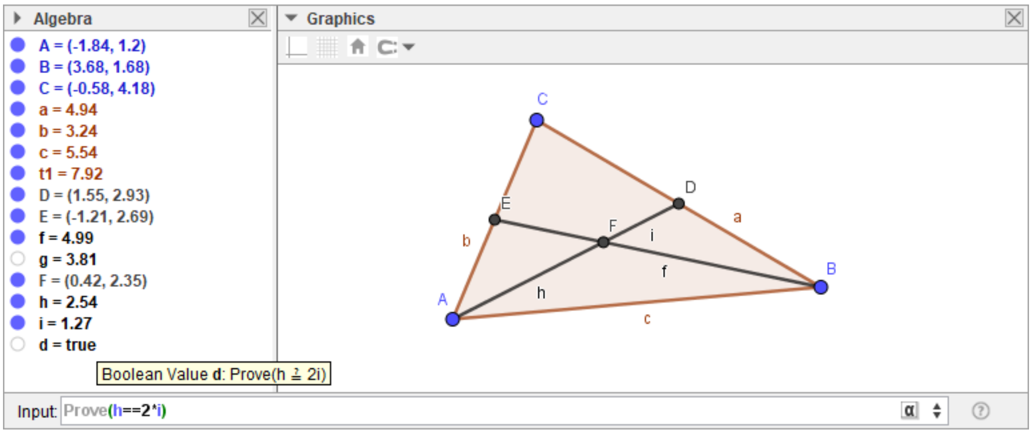

Figure 1: Example of the *Prove* command in GGb-ART (GeoGebra 5), asking in the input bar to prove that segment $h = AF$ is twice segment $i = FD$, where D is the midpoint of BC and F is the intersection of the medians. The output is just d = True, a well-known result in elementary geometry.

Out of these tools, the *Relation Tool* command may be the most interesting in the educational context, in our opinion. This command was already available in the first versions of GeoGebra (in 2002), though it only provided a *numerical* verification at that time. It allows to select two geometrical objects in a construction, and then it automatically looks for typical relations among them, including[2] *perpendicularity, parallelism, equality* and *incidence*. Finally, it shows a message box with the obtained information (*yes/no, the relation holds*, from a numerical or approximate point of view). Starting with GeoGebra version 5 this message box now displays an extra button with the caption "*More...*" which displays the results of some *symbolic* computations when pressed. In other words, by pressing the "*More...*" button, GeoGebra's ART subsystem starts and selects (by some heuristics) an appropriate prover method[3] to decide if the numerically obtained property is indeed absolutely and rigorously true in general.

Moreover, if the conjectured relation does not (mathematically speaking) hold in all generality, the command can determine some geometric extra-conditions, which need to hold true in order to make the given statement generally correct. These are the so-called *non-degeneracy conditions,* which usually prescribe that some of the input objects (for example, a freely defined triangle) should not be degenerate for the relation to hold true.

The *Relation Tool* is already available not only on the classic desktop platform of GeoGebra, but also in the web version, running, for instance, on smartphones, as shown in Figure 2 and Figures 3a and 3b.

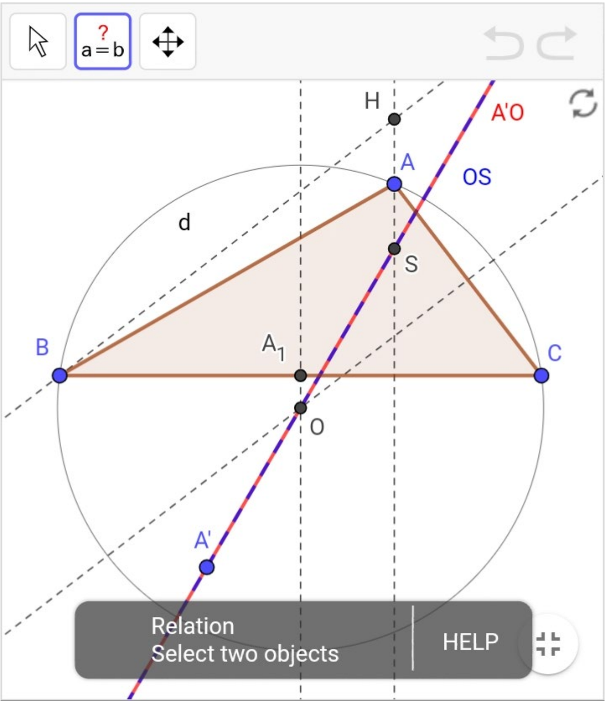

Figure 2: Screenshot of GeoGebra ART web version on a smartphone, concerning Chou's Example 230 (Chou, 1987). The user builds lines *OS* and *A'O* (blue and red in the Figure) and calls the *Relation Tool* to verify if they are equal (cf. Fig. 3a and 3b below).

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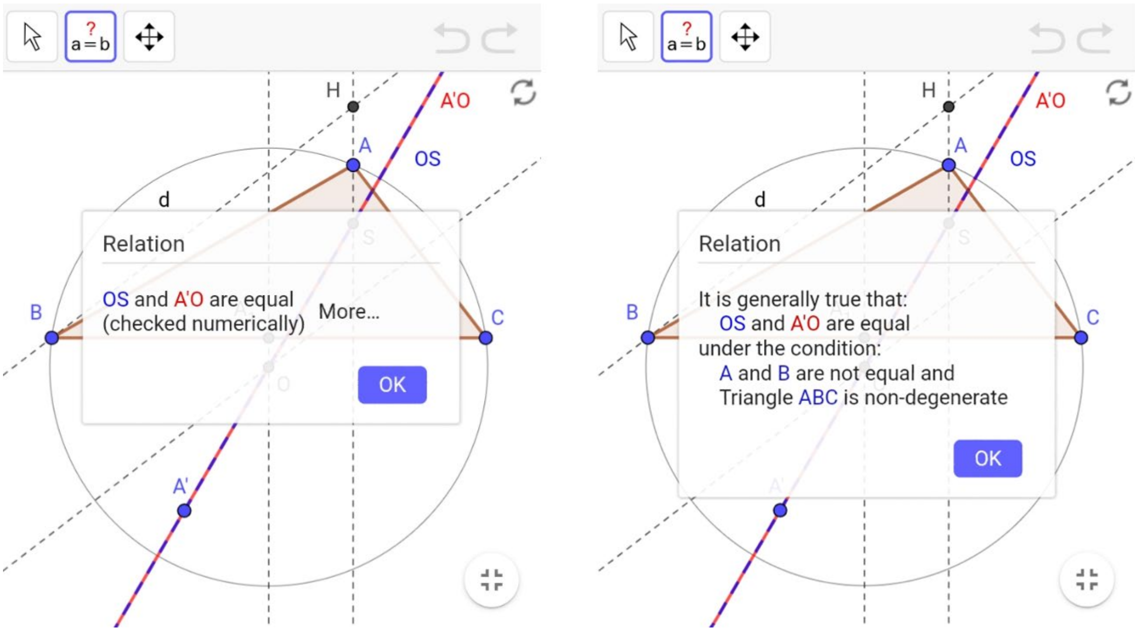
[2] See https://wiki.geogebra.org/en/Relation_Command for a full list.

[3] Such as a*) the Gröbner basis method*, b) *Wu's characteristic method*, c) *the area method*, or d) *Recio's exact check method* as the underlying automated theorem proving technique.

Figure 2 concerns Chou's Example 230 (Chou, 1987): "*Show that the symmetric (S) of the orthocentre (H) of a triangle (ABC) with respect to a vertex (A), and the symmetric (A') of that vertex with respect to the midpoint of the opposite side (A1), are collinear with the circumcentre (O) of the triangle*". The user builds lines OS and A'O (blue and red in Figure 2) and calls the *Relation Tool* to verify if they are equal. Then, in Fig. 3a, *Relation* shows that both lines OS and A'O are numerically coincident. And in Figure 3b, after clicking the "*More...*" button, it is shown that both lines are rigorously equal, meaning that the alignment of SOA' is a theorem that always holds, except for some degenerate cases (triangle collapsing to a line or a point).



Figures 3a and 3b: Chou's *Example 230* solved by means of the *Relation Tool*. Left, *Relation* shows that both lines *OS* and *A'O* are numerically coincident. Right, after clicking the "*More...*" button, it is rigorously stated that both lines are always equal.

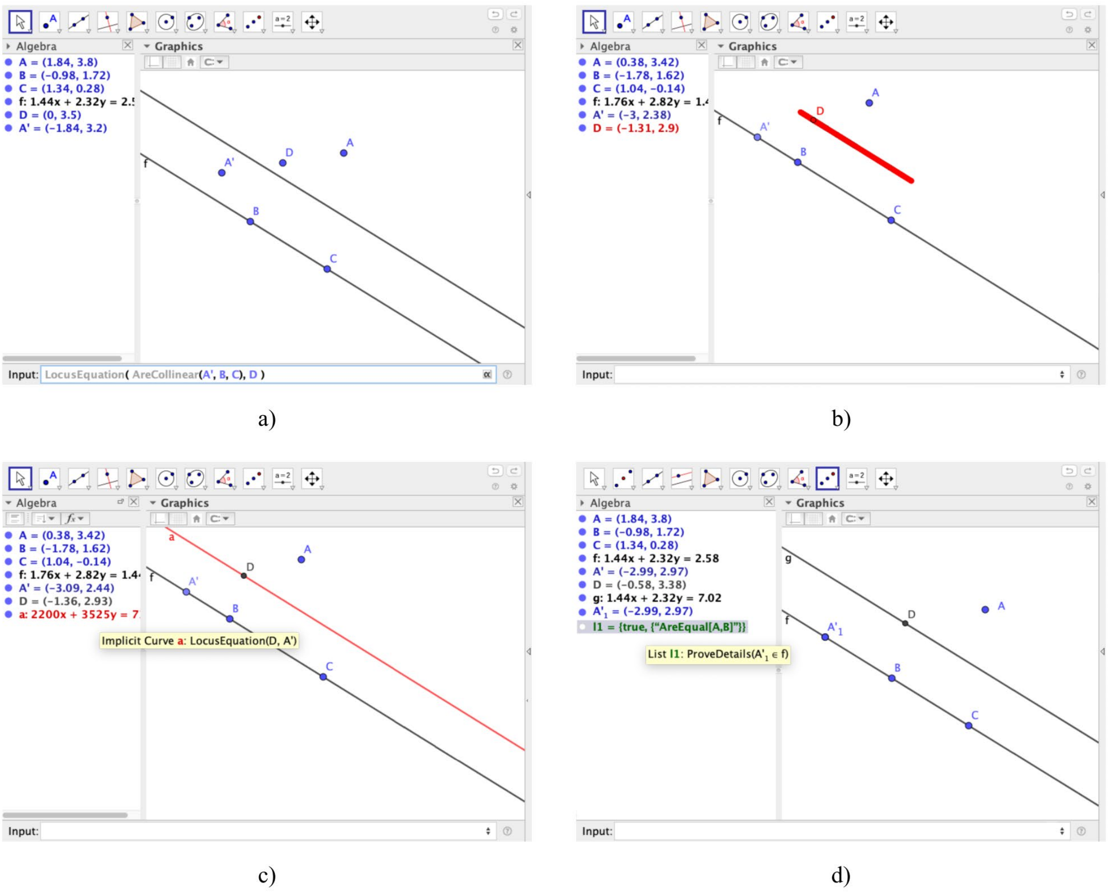
Finally, let us remark that, in the current version of *GeoGebra*, the user cannot obtain a visible or readable proof of the proven theorem by merely using the *Relation, Prove* or *Locus Equation Tool*. Nevertheless, the reported result is based on a mathematically correct proof, which, philosophically, is a completely different, higher-level result than a collection of instances obtained by dragging, as is the classic DGS way.

LEARNING AND TEACHING GEOMETRY USING GGB-ART

While GGb-ART provides students with a powerful tool to explore, conjecture, discover and prove properties in a geometric construction draw with GeoGebra, it is not clear yet how they should be used effectively in the classroom. A pioneering research study on the inclusion of DGS-ART in the classroom is presented in Hohenwarter, Kovács and Recio (2019). There, the authors propose a workflow for the design of geometry learning through ART as follows:

- Before the exercise, the teacher shows a demonstration of ART in DGS and he or she poses open-ended question such as “find all points *P* in the plane that have a certain property” (an “implicit locus problem”).
- Step 1 of the exercise requires the student to make a construction and to perform some calculations with the DGS.
- In Step 2, the student makes a conjecture, which, in the case of the locus problem describes the characteristics of the output curve.
- In Step 3, the student applies the ART tools to verify the conjecture. The result is accepted as a theorem and can be verified mathematically using pen and paper.
- Finally, the teacher explains to the student that the obtained theorem now represents a piece that can be used to achieve more involved statements, and to assemble new algorithms.

To illustrate this workflow, we now describe a detailed example. Let us suppose the teacher draws three non-aligned points *A*, *B*, *C*, line *f* through *BC*, a new point *D* and the symmetric of *A* with respect to *D*, *A'* (see Figure 4a). Then he/she asks the students to figure out where to place *D* so that *A'* lies on line *f*.



a) b) c) d)

Figure 4: A detailed example of the proposed workflow to incorporate ART in the mathematics classroom.

The students know that, by construction, *D* is the midpoint of segment *AA'*, so they might try to explore the locus of this midpoint when *A'* is placed on line *f*. Thus, they just take a point *A'* on line *f* and build, with the DGS, the midpoint *D* of *AA'*. Next, the traditional way of approaching this problem with a DGS would be to move *A'* along *f* and keep track of the position of *D* (see Figure 4b). But with GeoGebra ART, the students have a simpler approach: they just ask GeoGebra about the locus of *D* when *A'* moves along the line *f* (see Figure 4c). The (not only visual, but also mathematically rigorous) solution is that *D* must lie on line *a*.

But GeoGebra does not provide any information about what are the geometric characteristics of such line *a*. Graphically, *a* looks like a parallel line to *BC*, though GeoGebra cannot verify such conjecture, since line *a* is not a geometric object built by the user. Thus, the student is obliged to construct a line that he/she thinks might coincide with line *a*. He/she recalls that point *D* is the midpoint of *AA'*, so that line *a* passes through any midpoint of *A* and a point on line *BC*. Indeed, he/she takes whatever point *A'* on line *f*, and builds the midpoint *D* of *AA'* and the line *g* going through *D* and parallel to *f*.

Now it is time to apply GeoGebra ART to this line *g* (see Figure 4d). A free point *D* is chosen on this line, and the symmetric A'_1 of *A* with respect to *D* is constructed. Then, the student asks whether it is true that A'_1 lies on line *f*. The answer is (see Figure 4d) that it is always true, except in the degenerate case when $A=B$.

In Recio, Richard and Vélez (2019), a recent experience is described that analyses the inclusion of certain tasks based on GGb-ART in different learning sequences. The experience was developed in the framework of different mathematics teacher initial training studies, taking place in the Universities of Montreal (Canada), Cantabria and Nebrija (Spain). In that paper, the authors argued that the execution of the proposed tasks, of an open nature, would enhance mathematical reasoning while taking advantage of the potential of ART.

In fact, the paper specifies a set of tasks calling the students to visually conjecture or discover a theorem on a given geometric construction, and then requiring them to use GGb-ART to instrumentally verify this conjecture. Thus, regarding these tasks, GGb-ART is assumed to play the role of an "oracle" (cf. Sutherland and Balacheff, 1999), both guiding the development of mathematical reasoning and, finally, helping the students to formulate the corresponding traditional mathematical proof. More in detail, the activities proposed in Recio, Richard and Vélez (2019) are of two types. The first ones begin with a geometric statement whose conclusion should be reached by the student in a guided way through questions that lead him to observe, verify, ask the machine, reflect or go back and reformulate with the help of GeoGebra. The second kind of activities include open statements which the student must approach by using different *milieus* (Brousseau, 1998) and types of inference and then reflecting and arguing on the presumed accuracy of the provided answers.

As pointed out above, the final objective of the work described in Recio, Richard and Vélez (2019) was to study how to promote, with the help of ART, new and more effective ways of developing mathematical reasoning skills, enhancing the development by the student of different types of inference: deduction, induction and abduction (or the process of forming explanatory hypotheses). In particular, the authors addressed the following specific research question: "*How tasks, based on ART, should be designed so that they will lead to a new, more effective, path in the development of* *mathematical reasoning skills?*" (Recio, Richard and Vélez, 2019, p. 82).

In this context, the experiments and didactic initiatives carried out by these authors conclude that students were genuinely surprised by the existence and capabilities of such tools. Indeed, if it is usual for the students to undertake instrumented mathematical work (in the sense of Richard, Venant, and Gagnon, 2019), and while it is true that using ART helped them to detect certain limitations of visual and instrumented reasoning, they still feel more comfortable when reasoning with the sole help of standard GeoGebra features, showing in this way the overwhelming weight of tradition. Thus, the authors conclude: "*We think this limited experience shows already it is worth to continue working towards establishing the more convenient conditions for implementing a different approach to the teaching of Euclidean geometry*" (Recio, Richard & Vélez, 2019, p. 88).

AN ECOSYSTEM FOR COMPUTER-SUPPORTED GEOMETRIC REASONING

Although some Automated Reasoning Tools are already present in a few Dynamic Geometry Systems (DGS), many of their features are experimental and an important amount of work remains to be done to mature this technology and to enhance its use in the educational context.

In the communities of mathematics research and computer science, several research efforts by different groups are being implemented as components of a larger ecosystem for computer-supported geometric reasoning. Among these, we highlight: i) automated theorem provers such as GGb-ART, ii) repositories of geometric theorems for benchmarking provers such as TGTP (Quaresma, 2010), and iii) standards for exchanging information between DGS, automated provers and theorem repositories (Quaresma and Baeta, 2014; Chen, 2014).

However, in an education context, the route towards using this new technology in geometry teaching and learning remains largely unexplored. In Figure 5, we outline the stages required to introduce ART in the classroom. The first stage requires educational researchers to investigate what it means to teach and learn in this extended framework, and its implications for elementary geometry (Hohenwarter, Kovács and Recio, 2019). Their research requires to incorporate feedback from experimental use in the classroom. The second stage involves the training of teachers, based on the research findings, extending their previous training in mathematics, didactics and computer science (Recio, Richard and Vélez, 2019). In the final stage, ART can be used in the classroom, where students are instructed in their use and presented task-based sequences, as described in the previous section.

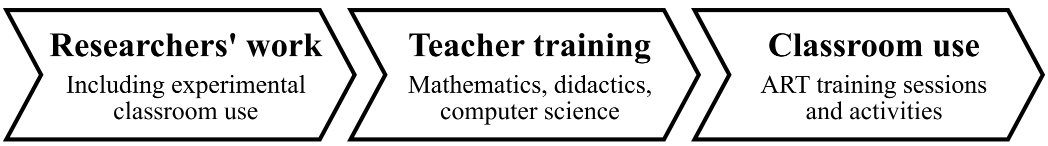


Figure 5: Necessary stages for introducing ART in education contexts.

In what follows, we will sketch some reflections and describe some basic notions and elements that we consider are necessary for the constitution of a sound procedure for dealing with this novel situation. Indeed, when talking about mathematics, we are tempted to believe that there is one and only one logic for everyone, confusing the activity and mathematical reasoning of the students with their cultural product: the standard method of communication (Brousseau, 2004). Moreover, in a classroom, each student reasons according to his or her own modalities: what is true for one is not necessarily true for the other, i.e., not at that moment or not according to the same degree of conviction. Following the example of modal logics (e.g. temporal or epistemic logics), which allow for differentiation in the expressiveness of reasoning according to the one who states it, the didactic system can be seen as a set of reasoning in interactions, traditional or instrumental, which participate in the construction of new knowledge.

The idea of building an ecosystem for carrying out mathematical work and, *a fortiori*, for computer-assisted geometrical reasoning, highlights a community of beings that perform their mathematical work in relation to their environment (the classroom) and during which reference knowledge, artefacts and sign systems that allow expressiveness, interact. Reasoning or proving would be a type of activity finalized in a given system, so that the old-fashioned mathematician, i.e., those who make little use of artefacts, would be in a different ecosystem than those who work with them (e.g., instrumented reasoning or instrumental proofs).

The set of the potential interactions related to a type of epistemic necessity is named the space of *epistemic necessity* (Richard, Oller & Meavilla, 2016). This allows us to test the ability of those who do their mathematical work to tolerate variations by modifying the semiotic, discursive and instrumental conditions in the accomplishment of a task. This ability is called *the valence of the mathematical work* and we can show the effects, for example, for a student, teacher or expert to prove in a given space of epistemic necessity. As in Richard, Oller and Meavilla (2016), the concept of *tolerance* is based on the *engineering tolerance*, which focuses on the permissible limit(s) of the potential interactions in a mathematical working space. Thus, in our context, the tolerance analysis is the study of the operating domain of these interactions. With a specific reference knowledge, artefacts and sign systems, acting into a mathematical working space related to a given space of epistemic necessity, we use the generic term of *variation of the mathematical work*. Then, the use or absence of automated reasoning tools constitutes such a variation, but it may have deeper implications for the valence of the mathematical work or the consideration of the ecosystem itself.

DISCUSSION

While GGb-ART have proven to be very useful tools for exploring geometric statements and conjectures, the idea of using them in the classroom raises several concerns. First, as pointed out by Lin, Yang, Lee, Taback and Stylianides (2012), the powerful visualization features of DGS with ART may discourage students from engaging in actual proving. In addition, these visualization features could blur the boundaries between conjecturing and proving, as perceived by students. Second, DGS with ART can be considered "geometry calculators", which raises the concern that students could lose their ability to develop a local "deductive theory" (de Villiers, 1990). GGb-ART also present challenges, as they require a new way of designing task, including open statements, and conjecturing or discovering before proving.

The goal of using GGb-ART in the classroom is to help our students to do mathematics better or faster, in a more creative way by exploring, discovery and conjecturing, which fosters their curiosity and critical spirit, as well as gives them a way of reasoning focused on competencies for the digital age.

While the classroom use of DGS-ART is still very much in an incipient and experimental phase, one should bear in mind that these tools are now readily available to the more than 100 million GeoGebra users worldwide. Hohenwarter, Kovács and Recio (2019) note that, "*as with pocket calculators, people will probably start using ART for checking geometric facts without the consensus of the pedagogical community on its role.*"

It is our wish and our goal to work helping the community towards achieving a different scenario.

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