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On relativistic hydrodynamics: Bernoulli process equations

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Abstract

A relativistic covariant description of a Bernoulli process is presented in terms of a set of two fundamental equations: Newton's second law and the first law of thermodynamics. The set is first obtained in the rest frame of the process S, then in a frame $\bar{\text{S}}$ moving at a constant velocity relative to S. It is shown that the set is covariant under Lorentz transformation, and reduces to the classical equations at the low-speed limit.

1 Introduction

Fluid dynamics is a difficult physics subject [1]. It is a many-body, or continuous media [2], problem, with system's internal [3] and external [4] forces and interactions, difficult to reduce to a one-body problem. As a representative of fluid dynamics, we will use a generalised version, including thermal effects, of Bernoulli's (after Daniel Bernoulli) process equation.

1.1 Classical Bernoulli equation

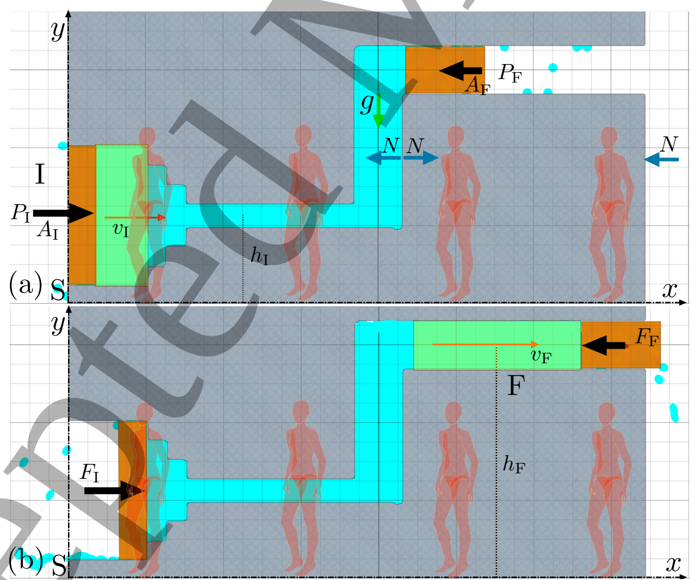

Figure 1: Bernoulli process. Observer-lab frame S, incompresible fluid flowing thorough a resting pipe. (a) Initial state (I), with P_I, $\mathbf{F}_\text{I} = P_\text{I}\mathbf{A}_\text{I}$, v_I and h_I. (b) Final state (F), with P_F, $\mathbf{F}_\text{F} = P_\text{F}\mathbf{A}_\text{F}$, v_F and h_F. By continuity equation $A_\text{I}v_\text{I}\Delta t = A_\text{F}v_\text{F}\Delta t$ (greenish areas).

The Bernoulli process, described by the Bernoulli equation [5], can be found in almost any university physics textbook.

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Classical physics Bernoulli's equation [6] describes the energy density equation [7] of a process in which an incompressible fluid, i.e., constant density ρ ($\rho = M/\mathcal{V}$), mass M and volume $\mathcal{V}$, flows (with zero viscosity), under pressure gradient and varying velocity, through a rigid walls pipe, with varying cross-section and height, relating pressures $P_{\rm k}$, speeds $v_{\rm k}$ and fluid heights $h_{\rm k}$ at two different, initial (I) and final (F) (k : I, F) points (see Fig. 1), with:

$$P_{\rm I} + \frac{1}{2}\rho v_{\rm I}^2 + \rho g h_{\rm I} = P_{\rm F} + \frac{1}{2}\rho v_{\rm F}^2 + \rho g h_{\rm F} \,. \tag{1}$$

Equation (1) works well, properly describing a process, when no dissipative forces are exerted on the fluid [8]. Many ways have been proposed to obtain Eq. (1) [9]. Despite several centuries of existence [10], equation terms physical meaning, particularly the pressure terms appearing in it [11, pp. 80-4], continues producing literature.

Equation (1) applies the first law of thermodynamics (FLT) (in terms of density and pressure instead of mass and work) [12]:

$$\Delta K_{\rm cm} + \Delta U = W + Q \,. \tag{2}$$

In Eq. (1), there are (implicitly): (i) hydrostatic system ($P\mathcal{V}T$), configuration work terms, $\delta W_{\rm k} = -P_{\rm k} {\rm d}\mathcal{V}$ (k: I, F), pressure described by fluid thermal equation of state $P = P(\mathcal{V}, T)$, (ii) work $W_{\rm g}$ performed by gravitational force $\mathbf{G} = (0, -Mg, 0)$, with $W_{\rm g} = Mg(h_{\rm F} - h_{\rm I})$, and (iii) no internal mechanical energy (rotational kinetic energy, for instance) variation, with thermal internal energy variation $\Delta U_T = Mc_\xi(T_{\rm F} - T_{\rm I})$, being c_ξ fluid ξ specific heat, and heat Q,

$$\frac{1}{2}M\hat{v}_{\rm F}^2 - \frac{1}{2}Mv_{\rm I}^2 + \Delta U = P_{\rm F}\mathcal{V}_{\rm F} - P_{\rm I}\mathcal{V}_{\rm I} - Mg(h_{\rm F} - h_{\rm I}) + Q \,, \tag{3}$$

when dissipative forces are involved.

1.2 Covariant equations

An interesting way to clarify the physical meaning of an equation, and the terms that integrate it, is to obtain covariant equations for the processes to which it is applied. For the case of the classical Bernoulli theorem, its relativistic generalization was first established in Ref. [13, 14], where an hydrodynamical theory in general relativity was developed. The scope of the present work is that of the special theory of relativity. To fully understand the proceses analyzed here, all external forces must be considered, including the gravitational force exerted by the Earth (see Fig. 1). According to the General Theory of Relativity, gravity does not actually exert a force but instead curves space-time. In this work, the gravitational interaction will be considered from a phenomenological point of view as a conservative force, in order to avoid unnecessary complexity. Obtaining covariant equations is an important issue in any branch of physics and is of great conceptual interest in physics.

According to the principle of relativity, any covariant equation, e.g., the first law of thermodynamics in frame reference $\bar{\rm S}$ ($\overline{\rm FLT}$), must be linear combination of the Newton's second law (NSL) and the first law of thermodynamics ($\overline{\rm FLT}$) in frame S; and vice-versa [15, pp. 1-10].

On the one hand, when thermal effects are involved in an actual Bernoulli process, it is difficult to know how Eq. (1) generalizes [16, pp. 40-6–40-9]. On the other hand, confusion between (i) Newton's second law complementary dynamical relationship [17] (NSL-CDR), and (ii) the energy equation (FLT) for the process, also contributes to physics for Bernoulli's process misunderstanding.

An extended classical description of Bernoulli's process must include: the linear momentum equation (NSL), and the energy balance equation (FLT). Since the NSL for this process is usually not considered, it is challenging to obtain the classical covariant description of Bernoulli's equation [18].

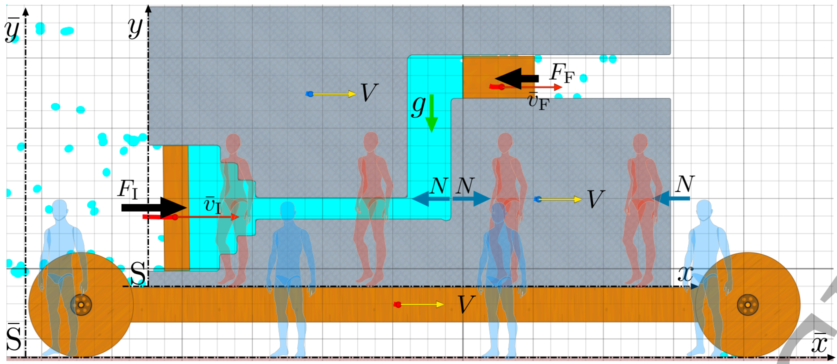

Figure 2: Bernoulli process on a moving wagon-lab. Frame $\bar{\mathrm{S}}(\bar{x}, \bar{y})$. Bar magnitudes, e.g., $\bar{v}$, as mesured in frame $\bar{\mathrm{S}}$, with the train station ground at rest, and moving pipe (lab).

1.3 Performing experiments

Providing a covariant equation for a process involves a description of an experiment in two inertial frames (asynchronous formulation of relativity [19]). Figure 2 shows a sketch of the process already seen in Fig. 1, described in frame $\bar{\mathrm{S}}$, in relative motion with respect to frame S.

Carrying out an experiment means: (i) designing the experiment, ensuring that conditions under which the theory has been developed – i.e., theoretical equations have been obtained – (e.g., the fluid is incompressible) are fulfilled, (ii) collecting the space-time coordinates of certain events, (iii) obtaining the numerical values of different physical magnitudes, based on the events space-time coordinates, and (iv) comparing these results with those anticipated by the theory.

For a given experiment, mass (M) and gravity acceleration (g) are known parameter. Sensors measure intensive magnitudes, e.g., pressure P, and temperature T. Sensor send the collected information to computers in frames S and $\bar{\mathrm{S}}$. Time t_0, and time intervals $[0, \Delta t]$ are measured by synchronized clocks in S.

Magnitudes like forces or velocities must be obtained in every inertial frame by the lattice of rules and synchronized clocks that conform it. The set of observers in, e. g., frame $\mathrm{S}(x, y, z)$, measure displacements, velocities and, where appropriate, accelerations through experimental space-time observations of events coordinates. Position $(x_{\mathrm{j}}, y_{\mathrm{j}}, z_{\mathrm{j}})$ for observer (j) is known in frame S and time recorded when an event (E) occurs in its position; for example, the piston-(I) event, in which the (centre of the) piston (I) is in observer (j) position at time t_{j}. When the experiment is over, this set of observers share the recorded information. If the observer (a)-$(x_{\mathrm{a}}, y_{\mathrm{a}}, z_{\mathrm{a}})$ observes the piston (I) – event-(I) – at instant $t_0 = 0$ and the observer at (b)-$(x_{\mathrm{b}}, y_{\mathrm{a}}, z_{\mathrm{a}})$ observes the event-(I) at $t_0 + \Delta t$, they know that piston displacement has been $x_{\mathrm{I}} = x_{\mathrm{b}} - x_{\mathrm{a}}$; if the observer at (c)-$(x_{\mathrm{c}}, y_{\mathrm{a}}, z_{\mathrm{a}})$ detects the event piston-(I) at time t_0 and the observer at (d)-$(x_{\mathrm{c}} + \mathrm{d}x, y_{\mathrm{a}}, z_{\mathrm{a}})$ at time $t_0 + \Delta t$, they can assign speed $v(t_0) \approx \Delta x / \Delta t$ to piston (I).

When necessary displacements, heights, and velocities have been obtained in proper frame, every proper magnitude is substituted into the corresponding equation, checking for consistent results. When inertial observers, for example, one in frame S and another in frame $\bar{\mathrm{S}}$, compare their results, their conclusions must be *equivalent*: each equation in $\bar{\mathrm{S}}$, must be linear combination of equations in S, and vice-versa.

1.4 Special relativity

When working with the concepts and methods of Einstein's special theory of relativity (STR), one naturally obtains covariant equations under the Lorentz transformation between inertial frames. The aim in this work is to express the Bernoulli equation in relativistic covariant form [20], and then to obtain its classical physics covariant description in the low-speed $v/c \to 0$ limit. The search for a covariant equation for the Bernouli process is a good example of a useful application of the methods and concepts of the STR to processes occurring at speeds much slower than the speed of light (with some similarity to the relativistic treatment of the magnetic field associated with electric charges moving at speeds much slower than the speed of light).

The article is organized as follows. In Sec. 2 the main four-vectors for the process are defined in proper frame S: linear-momentum–energy four-vector for the state of the system, linear-impulse–work four-vector for the interactions of the system with its environment through forces, and the heat four-vector. These four-vectors enter into the four-vector fundamental equation (FVFE) for the process. Next, the NSL and the FLT equations for the Bernoulli process are deduced, first relativistic and then in classical physics, discussing the conditions when the process evolves with mechanical energy conservation and recovering Eq. (1). In Sec. 3 the equations of the process in a moving frame $\bar{\mathrm{S}}$ are obtained, applying the Lorentz transformation to the FVFE in S. The relativistic effects are discussed, and the relativistic equations are obtained. In the low-speed limit, the classical Bernoulli equation in frame $\bar{\mathrm{S}}$ is obtained. In Sec. 4, some conclusions are drawn, highlighting the didactic and pedagogical advantages of using the concepts and methods of STR. In the Appendix the application of the relativistic principle of locality is implemented; the system is divided into elements, the NSL and the FLT equations are obtained for each fluid element, then the FVFE is obtained for each element, and adding over all of them the FVFE for the Bernoulli process is obtained.

2 Relativistic Bernoulli's process equations

The classical energy equation, the FLT, usually needs to be complemented with their mechanical NSL equations. Description of forces arriving from mechanical potentials, conservative forces exerting linear impulse and performing work on the system, do not combine easily with non-conservative forces, exerting linear impulse on the system but performing no work. Since the system is at rest in frame S, it is often overlooked that in frame $\bar{\mathrm{S}}$, the process evolves with work terms $\bar{W}_{\mathrm{cb}} \neq 0$ in $\bar{\mathrm{S}}$ related to forces not performing work $W_{\mathrm{cb}} = 0$ in frame S.

In classical physics, there is no an algorithmic procedure allowing to change between inertial frames. This is because in classical physics it is not recognized: (i) that there is not a single (four-vector matrix) equation (ansatz), from which NSL (first ansatz) and FLT (second ansatz) can be obtained for the process, and (ii) that an equation's transformation operator between inertial frames could be applied to a four-vector matrix equation of that type.

An interesting way to analyze an equation, is to consider how it is described in different inertial frames [21], using the concepts and methods of Einstein's STR, a naturaly covariant theory with mathematical rigour and versatility.

It is usually considered that Einstein's STR should just be applied to processes in which speeds comparable to the speed of light c are involved; therefore, at first sight, it would seem that a relativistic description of the Bernoulli equation would not be justified [18].

On the one hand, we understand relativistic hydrodynamics as the description of a fluid dynamics process (e.g., Bernoulli process) in inertial frames in relative motion – frame S (Fig. 1) is a lab inside a moving wagon or conveyor belt (Fig. 2) –, speed $V \ll c$, fulfilling the STR postulates, considerations and results, even at low speed [22]. On the other hand, when thermal effects are present – e.g., the energy exchange between a system and its environment by heat, described as electromagnetic radiation or photons –, physical effects taking place at the speed of light [23]. So, in this paper we want to obtain a relativistic (covariant) four-vector fundamental equation describing the Bernoulli process.

Four-vector fundamental equation. When a relativistic equation describes the Bernoulli process, the linear impulse–linear momentum variation equation (NSL) and the energy equation (FLT) are simultaneously obtained in proper frame S. Then, by applying the Lorentz transformation to the four-vector fundamental equation in S, the four-vector fundamental equation in $\bar{\mathrm{S}}$ is obtained, giving the $\overline{\mathrm{NSL}}$ and the $\overline{\mathrm{FLT}}$ equations in $\bar{\mathrm{S}}$. Equations obtained in $\bar{\mathrm{S}}$ are equivalent to equations in S, i.e., fulfill the principle of relativity.

Given the system and the process, chosen the reference frame (S), the four-vector fundamental equation formalism [19]:

$$E_{\rm f}^{\mu} - E_{\rm I}^{\mu} = \Sigma_{\rm k} W_{\rm k}^{\mu} + Q^{\mu} \,, \tag{4}$$

allows to obtain mechanics and thermodynamics equations for a process.

The Lorentz transformation, $\mathcal{L}_{\nu}^{\mu}(V)$, applied on the four-vector fundamental equation in frame S, transforms Eq. (4) into that in frame $\bar{\rm S}$:

$$\mathcal{L}_{\nu}^{\mu}(V)[E_{\rm f}^{\nu} - E_{\rm I}^{\nu} = \Sigma_{\rm k} W_{\rm k}^{\nu} + Q^{\nu}] \rightarrow \bar{E}_{\rm f}^{\mu} - \bar{E}_{\rm I}^{\mu} = \Sigma_{\rm k} \bar{W}_{\rm k}^{\mu} + \bar{Q}^{\mu} \,, \tag{5}$$

$$\mathcal{L}_{\mu}^{\nu}(-V)[\bar{E}_{\rm f}^{\mu} - \bar{E}_{\rm I}^{\mu} = \Sigma_{\rm k} \bar{W}_{\rm k}^{\mu} + \bar{Q}^{\mu}] \rightarrow E_{\rm f}^{\nu} - E_{\rm I}^{\nu} = \Sigma_{\rm k} W_{\rm k}^{\nu} + Q^{\nu} \,, \tag{6}$$

and vice-versa from $\bar{\rm S}$ to S, by Lorentz transformation $\mathcal{L}_{\nu}^{\mu}(-V)$.

2.1 Fluid dynamics four-vectors

An inclined angle θ pipe (see Fig. 3) is going to be considered. By application of the principle of locality (see Appendix), the system is divided into elements, each with same inertia $\mathcal{M}$ and volume $\mathcal{V}$, and constant density ρ. The system is taken as the set of elements: (I), 1, 2, ..., k-1, k, k+1, ..., r-1, r, (F). External forces due to external pressures $\mathbf{F}_{\rm I} = P_{\rm I}\mathbf{A}_{\rm I}$ and $\mathbf{F}_{\rm F} = P_{\rm F}\mathbf{A}_{\rm F}$ are applied to this system. There are external gravitational forces $\mathbf{F}_{\rm g|k} = (0, -Mg, 0)$, applied to each element of the system. There are non-conservative forces on the fluid due to the pipe. When thermal effects take place, they are represented in terms of a heat four-vector.

2.1.1 Internal energy and inertia.

The Bernoulli equation involves the fluid internal energy E_0 and the NSL equation involves its inertia $\mathcal{M}$. Einstein's inertia of energy principle [23] allows the two concepts to be related as $\mathcal{M} = c^{-2} E_0$.

The fluid volume $\mathcal{V}$ is made up of $m_{\rm j}$ elementary particles (protons, neutrons, electrons, in H and O atoms, for instance), assembled as N fluid molecules – H_2O for instance–, with bound energy $\tilde{U}_{\xi} = \tilde{U}_{\rm N} + \tilde{U}_{\rm A} + \tilde{U}_{\rm M} < 0$, where $\tilde{U}_{\rm N}$, $\tilde{U}_{\rm A}$ and $\tilde{U}_{\rm M}$ are (bonding) energies for nuclei, atoms and molecules [23].

The concept of rigid solid is not allowed in relativity, i.e., zero adiabatic compressibility coefficient $\kappa_S = -(1/\mathcal{V})(\partial\mathcal{V}/\partial P)_S \approx 0$, and sound speed, $v_{\rm s} = (1/\rho\kappa_S)^{1/2} = (\gamma P/\rho)^{1/2}$ above the light-speed.

Van der Waals thermal equation of state. The Van der Waals thermal equation of state is a good equation for describing dense but compressible fluids ($\kappa_S \approx 10^{-11}$ Pa^{-1}). Let be n moles of fluid of chemical composition ξ (H_2O for instance) described by the Van der Waals equation:

$$P(\xi, \mathcal{V}, T) = \frac{nRT}{(\mathcal{V} - nb)} - \frac{an^2}{\mathcal{V}^2} \,, \tag{7}$$

being a and b fluid ξ characteristic constants. Its internal energy $E_0(\xi, T, \mathcal{V})$ is given by:

$$E_0(\xi, T, \mathcal{V}) = \Sigma_{\rm j} m_{\rm j} c^2 - \left|\tilde{U}_{\xi}\right| + n\tilde{c}_{\xi} T - \frac{an^2}{\mathcal{V}} \,; U_0 = \Sigma_{\rm j} m_{\rm j} c^2 - \left|\tilde{U}_{\xi}\right| , \tag{8}$$

with $n = N/N_{\rm A}$ (N, number of molecules and $N_{\rm A}$, Avogadro's number).

Four-vectors are obtained in frame S in which: (i) pipe walls, through which the fluid flows, are at rest – the relative velocity pipe-observer is zero; (ii) forces, conservative and non-conservative, apply simultaneously during time interval $[0, \Delta t]$; (iii) energy exchanged by heat with thermal reservoir surrounding the system are described, if any, by a set of thermal photons.

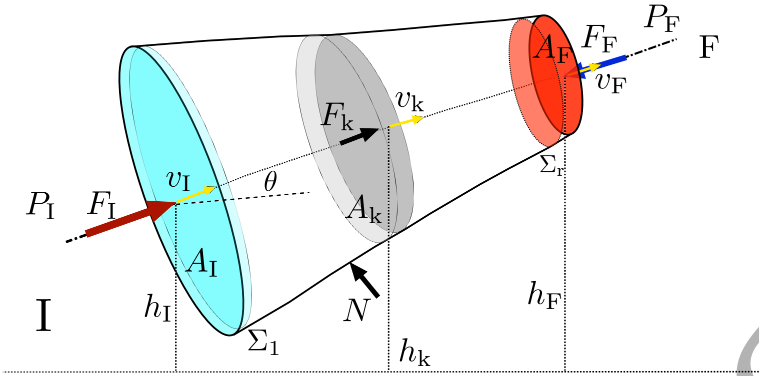

Figure 3: In time interval $[0, \Delta t]$, fluid inertia $\mathcal{M} = \rho A_{\rm I} v_{\rm I} \Delta t$ (fluid volume $\mathcal{V} = A_{\rm I} v_{\rm I} \Delta t = A_{\rm I} L_{\rm I}$), velocity $\mathbf{v}_{\rm I}$, located at height $h_{\rm I}$ is translated from region (I) to intermediate region (1), under force $\mathbf{F}_{\rm I} = P_{\rm I} \mathbf{A}_{\rm I}$. During the same time interval, fluid inertia $\mathcal{M} = \rho A_{\rm F} v_{\rm F} \Delta t$, is translated from the intermediate region (r) to region (F), moving with velocity $\mathbf{v}_{\rm F}$, located at height $h_{\rm F}$, against force $\mathbf{F}_{\rm F} = P_{\rm F} \mathbf{A}_{\rm F}$. Pipe walls could exert force on the fluid. There could be frictional forces, due to fluid viscosity, or to a porous plug, and thermal effects (not represented). Distance between surface $\Sigma_{\rm k}$ and surface $\Sigma_{\rm k+1}$ is $L_{\rm k}$ (see Fig. 4 in Appendix).

2.1.2 Linear-momentum–energy four-vectors

A finite process, time interval $[0, \Delta t]$ will be considered. Inertia $\mathcal{M}$ transits from region (I) to region (F), the mechanical state of the fluid inside the pipe does not change (Fig. 3) during that time interval.

For fluid element (k), velocity $\mathbf{v}_{\rm k} = (v_{x\rm k}, v_{y\rm k}, 0)$, modulus $v_{\rm k} = (v_{x\rm k}^2 + v_{y\rm k}^2)^{1/2}$, linear momentum $\mathbf{p}_{\rm k} = (\gamma_v \mathcal{M} v_{x\rm k}, \gamma_v \mathcal{M} v_{y\rm k}, 0)$ and energy $E_{\rm k} = \gamma_{v_{\rm k}} \mathcal{M} c^2$ (total energy $E = E_0 + K$), the linear-momentum–energy four-vector $E_{\rm k}^\mu$ is given by:

$$E_{\rm k}^\mu = \begin{pmatrix} c\gamma_{v_{\rm k}} \mathcal{M} v_{x\rm k} \\ c\gamma_{v_{\rm k}} \mathcal{M} v_{y\rm k} \\ 0 \\ \gamma_{v_{\rm k}} \mathcal{M} c^2 \end{pmatrix} \equiv \begin{pmatrix} c\gamma_{v_{\rm k}} \mathcal{M} \mathbf{v}_{\rm k} \\ \gamma_{v_{\rm k}} \mathcal{M} c^2 \end{pmatrix}, \tag{9}$$

with *spatial* – $c\mathbf{p}_{\rm k} \equiv c\gamma_{v_{\rm k}} \mathcal{M} \mathbf{v}_{\rm k}$ – and *temporal* – $E_{\rm k} \equiv \gamma_{v_{\rm k}} \mathcal{M} c^2$ –, four-vector components, where $\gamma_{v_{\rm k}} = (1 - \beta_{\rm k}^2)^{-1/2}$ is the Lorentz factor and $\beta_{\rm k} = v_{\rm k}/c$.

Element (I) has an initial thermal state at temperature T and volume $\mathcal{V}$, moving its centre-of-mass, located at height $h_{\rm I}$, with velocity $\mathbf{v}_{\rm I}$. In its final thermal state at temperature T and volume $\mathcal{V}$, its centre-of-mass, at height $h_{\rm F}$, the fluid moves with velocity $\mathbf{v}_{\rm F}$. Initial and final fluid states are given, respectively, by the following four-vectors:

$$E_{\rm I}^\mu = \begin{pmatrix} c\gamma_{v_{\rm I}} \mathcal{M} \mathbf{v}_{\rm I} \\ \gamma_{v_{\rm I}} \mathcal{M} c^2 \end{pmatrix}; \; E_{\rm F}^\mu = \begin{pmatrix} c\gamma_{v_{\rm F}} \mathcal{M} \mathbf{v}_{\rm F} \\ \gamma_{v_{\rm F}} \mathcal{M} c^2 \end{pmatrix}. \tag{10}$$

Element (k) linear-momentum–energy variation four-vector. During the process, element (I) disappears and is transformed, due to the forces exerted on it, into element (1); element (k) is transformed into element (k+1) and so on; element (r) is transformed into element (F). By adding over fluid elements, and since there exists a stationary state inside the pipe, the mechanical state of the set of elements inside the pipe does not vary during the process. However, the linear impulse exerted and work performed on the fluid elements inside the pipe do not cancel each other out.

By the *continuity equation*: $A_{\rm I} L_{\rm I} = A_{\rm k} L_{\rm k} = A_{\rm F} L_{\rm F} = \mathcal{V}$. For fluid element inside the pipe (k), which becomes the element (k+1), the following linear-momentum–energy variation four-vector is found (see Appendix):

$$\Delta E^{\mu}_{\rm k} \equiv \left(\begin{array}{c} c\mathcal{M}\Delta(\gamma_{v_{\rm k}}\mathbf{v}_{\rm k}) \\ \mathcal{M}\Delta(\gamma_{v_{\rm k}}c^2) \end{array} \right) \equiv \left(\begin{array}{c} c\mathcal{M}(\gamma_{v_{\rm k+1}}\mathbf{v}_{\rm k+1}) \\ \mathcal{M}(\gamma_{v_{\rm k+1}}c^2) \end{array} \right) - \left(\begin{array}{c} c\mathcal{M}(\gamma_{v_{\rm k}}\mathbf{v}_{\rm k}) \\ \mathcal{M}(\gamma_{v_{\rm k}}c^2) \end{array} \right). \tag{11}$$

By adding over $\Delta E^{\mu}_{\rm k}$ four-vectors:

$$\Sigma_{\rm k=I}^{\rm k=F} \Delta E^{\mu}_{\rm k} = E^{\mu}_{\rm F} - E^{\mu}_{\rm I}\,. \tag{12}$$

This important result is obtained by means of the relativistic principle of locality (see Appendix).

2.1.3 Linear-impulse–work four-vectors

Each force exerted on the fluid must be characterised by its linear impulse and its work.

1. *Pressure impulse–work four-vectors* $W^{\mu}_{\rm I}$, $W^{\mu}_{\rm F}$.

Inclined pipe angle θ (see Appendix Fig. 4). Constant pressure $P_{\rm I}$ is exerted, during time interval $[0, \Delta t]$, on piston (I), cross-section $\mathbf{A}_{\rm I} = (A_{\rm I}\cos\theta, A_{\rm I}\sin\theta, 0)$ with, piston (I) and force $\mathbf{F}_{\rm I} = P_{\rm I}\mathbf{A}_{\rm I}$, displacement $\mathbf{L}_{\rm I} = (L_{\rm I}\cos\theta, L_{\rm I}\sin\theta, 0)$, with $\mathbf{v}_{\rm I} = \Delta\mathbf{L}_{\rm I}/\Delta t = (v_{\rm I}\cos\theta, v_{\rm I}\sin\theta, 0)$.

Constant pressure $P_{\rm F}$ is exerted, during time interval $[0, \Delta t]$, on piston (F), cross-section $\mathbf{A}_{\rm F} = (A_{\rm F}\cos\theta, A_{\rm F}\sin\theta, 0)$ with, piston (F) and force $\mathbf{F}_{\rm F} = P_{\rm F}\mathbf{A}_{\rm F}$, displacement $\mathbf{L}_{\rm F} = (L_{\rm F}\cos\theta, L_{\rm F}\sin\theta, 0)$, with $\mathbf{v}_{\rm F} = \Delta\mathbf{L}_{\rm F}/\Delta t = (v_{\rm F}\cos\theta, v_{\rm F}\sin\theta, 0)$.

For these $\mathbf{F}_{\rm I} = P_{\rm I}\mathbf{A}_{\rm I}$ and $\mathbf{F}_{\rm F} = -P_{\rm F}\mathbf{A}_{\rm F}$ external pressure origin forces, impulse–work four-vectors $W^{\mu}_{\rm I}$ and $W^{\mu}_{\rm F}$, are given by:

$$W^{\mu}_{\rm I} = \left(\begin{array}{c} cP_{\rm I}\mathbf{A}_{\rm I}\Delta t \\ P_{\rm I}A_{\rm I}L_{\rm I} \end{array} \right), W^{\mu}_{\rm F} = \left(\begin{array}{c} -cP_{\rm F}\mathbf{A}_{\rm F}\Delta t \\ -P_{\rm F}A_{\rm F}L_{\rm F} \end{array} \right), \tag{13}$$

with $L_{\rm I} = v_{\rm I}\Delta t$, and $L_{\rm F} = v_{\rm F}\Delta t$, respectively, with volume $\mathcal{V} = A_{\rm I}L_{\rm I}$ and $\mathcal{V} = A_{\rm F}L_{\rm F}$, respectively.

2. *Gravitational impulse–work four-vector* $W^{\mu}_{\rm rg|k}$.

Gravitational force $\mathbf{F}_{\rm g|k} = (0, -\mathcal{M}g, 0)$, displacement $\mathbf{L}_{\rm k} = (L_{\rm k}\cos\theta, L_{\rm k}\sin\theta, 0)$, velocity $\mathbf{v}_{\rm k} = (v_{\rm k}\cos\theta, v_{\rm k}\sin\theta, 0)$, with $(\mathbf{A}_{\rm k}\cdot\mathbf{v}_{\rm k})\Delta t = \mathbf{A}_{\rm k}\cdot\mathbf{L}_{\rm k} = \mathcal{V}$, is exerted on each element (1,2, …, k-1, k, k+1, …, r-1, r) inside the pipe, performing work $W_{\rm g|k} \equiv \mathbf{F}_{\rm g|k}\cdot\mathbf{L}_{\rm k} = -\mathcal{M}gL_{\rm k}\sin\theta$ during the process. The momentum–energy four-vector associated to the gravitational force is given by:

$$W^{\mu}_{\rm g|k} = \left(\begin{array}{c} c\mathcal{M}\mathbf{g} \\ \mathcal{M}(\mathbf{g}\cdot\mathbf{v}_{\rm k}) \end{array} \right)\Delta t = \left(\begin{array}{c} c\mathcal{M}\mathbf{g}\Delta t \\ -\mathcal{M}gL_{\rm k}\sin\theta \end{array} \right). \tag{14}$$

The total gravitational force linear-impulse–work four vector, $W^{\mu}_{\rm rg}$, is obtained by adding over the (r) elements:

$$W^{\mu}_{\rm rg} \equiv \Sigma_{\rm k=1}^{\rm k=r} W^{\mu}_{\rm g|k} = \left(\begin{array}{c} cr\mathcal{M}\mathbf{g}\Delta t \\ -\mathcal{M}gL_{\rm p}\sin\theta \end{array} \right), \tag{15}$$

where $L_{\rm p} = \Sigma_{\rm k=1}^{\rm k=r} L_{\rm k}$ is the length of the pipe and $h_{\rm F} - h_{\rm I} = (\Sigma_{\rm k}L_{\rm k})\sin\theta = L_{\rm p}\sin\theta$ the difference in heights between the centres-of-mass of elements (F) and (I).

3. *Pipe restriction force* $W^{\mu}_{\rm N}$.

Even in an idealised process, the pipe walls will exert forces $\mathbf{n}_{\rm k}$ on the moving fluid (see Fig. 4). For force $\mathbf{n}_{\rm k}$ exerted on fluid element (k), the impulse–work four-vector is given by:

$$W^{\mu}_{\rm n|k} = \left(\begin{array}{c} c\mathbf{n}_{\rm k}\Delta t \\ 0 \end{array} \right), W^{\mu}_{\rm N} \equiv \Sigma_{\rm k=1}^{\rm k=r} W^{\mu}_{\rm n|k} = \left(\begin{array}{c} c\mathbf{N}\Delta t \\ 0 \end{array} \right).$$

From force $\mathbf{N}$ exerted by pipe walls on the fluid, the linear-impulse–energy four-vector $W^{\mu}_{\rm N}$ can be obtained by adding over fluid elements four-vectors $W^{\mu}_{\rm n|k}$, with $\mathbf{N} = \Sigma_{\rm k=1}^{\rm k=r}\mathbf{n}_{\rm k} = (N_x, N_y, 0)$.

4. *External agent force* $F^{\rm ext}$ *on the pipe.*

When the fluid flows inside the pipe, the tube exerts net force $\mathbf{N}$ on the fluid, and the fluid exerts force $\mathbf{F}_{\mathrm{f/p}} = -\mathbf{N}$ on the pipe. To write equations for the pipe is not interesting, as the problem would become more complicated. Assuming the pipe linear momentum variation is zero in S, it must be admitted that an external agent must exert force $\mathbf{F}^{\mathrm{ext}} = \mathbf{N}$ on the pipe in such a way that the net resultant fluid-external agent force applied to the pipe is zero. An external agent forces that the velocity of the pipe is zero in lab frame S [24].

Since force $\mathbf{F}^{\mathrm{ext}}$ is not directly exerted on the fluid, its role is not usually taken into consideration. In the lab frame S, external force F^{ext} on the pipe exerts linear impulse and performs no work, so it will not appear into the energetic equations describing the process in this frame. However, in frame $\bar{\mathrm{S}}$, force $\mathbf{N}$ exerts linear impulse and performs work $\bar{W}^{\mathrm{ext}}_{\mathrm{cb}} = (\mathbf{N}\cdot\mathbf{V})\Delta t$, as well as force $\mathbf{N}$ exerted on the fluid, work that will appear in the equations describing the process in frame $\bar{\mathrm{S}}$. Work $\bar{W}^{\mathrm{ext}}_{\mathrm{cb}}$ must be performed by an external agent, in order to maintain constant the lab S speed [25].

2.1.4 Relativistic heat and entropy variation

In a kind of Bernoulli process involving dissipative forces, with pseudo-work $\mathrm{p}W < 0$, mechanical energy is dissipated. For an adiabatic process, pipe does not allow to exchange energy as heat with its surroundings, variation in fluid temperature will occur. Considering diathermic pipe walls, energy exchanged by heat with surroundings is allowed. In frame S this energy exchange is emitted as electromagnetic radiation. During time interval $[t_0, t_0 + \Delta t]$, $N_{\mathrm{ph}} = aT^3\Delta t$ photons are emitted [26, pp. 159-161], with frequency $\nu = bT$ (Wien's law, in the monochromatic approximation [27]), and power $\dot{E} = \sigma A T^4$ (Stefan-Boltzmann law), with energy exchanged as heat $Q = N_{\mathrm{ph}} h\nu$.

Relativistic heat definition. In classical thermodynamics, the Born-Caratheódory definition of heat [28], $Q = W - W_{\mathrm{ad}}$, exchanged by a system in a given process, is the difference between the adiabatic work (univocally determined, $\Delta U = U_{\mathrm{F}} - U_{\mathrm{I}} = W_{\mathrm{ad}}$) and the actual work performed in the process, connecting the same two, initial (I) and final (F), states of the system; in classical thermodynamics no linear momentum is attached to energy as heat. For a given process, according to the second law of thermodynamics (SLT), heat must be a relativistic invariant, the same in any inertial frame. In general, energy Q emitted as heat in a process in frame $\hat{\mathrm{S}}$ is defined as the norm $\|\hat{Q}^\mu\|$ of the four-vector $\hat{Q}^\mu$ [24],

$$\hat{Q}^\mu \equiv \begin{pmatrix} c\hat{\mathbf{p}}_Q \\ \hat{E}_Q \end{pmatrix}, \|\hat{Q}^\mu\| \equiv (\hat{E}_Q^2 - c^2\hat{p}_Q^2)^{1/2} = Q\,.$$

With $\hat{E}_Q^2 = c^2\hat{p}_Q^2 + \mathcal{M}_Q^2 c^4$, this definition of heat retains from energy $\hat{E}_Q$ the part $E_Q = \mathcal{M}_Q c^2$, not endowed with linear momentum, energy that really contributes to the system, environment and universe, entropy variations. This heat definition ensures that heat involved in a process as well as entropy variation, are frame invariants.

Heat four-vector. To ensure that heat is energy emitted with net zero linear momentum (i.e., maximum entropy of the universe variation [29]), thermal photons are emitted as opposite pairs ($\mathrm{c}\tilde{\mathrm{c}}$), with directions $\mathbf{u}_{\mathrm{c}} = (\cos\theta_{\mathrm{c}}, \sin\theta_{\mathrm{c}}, 0)$, and $\mathbf{u}_{\tilde{\mathrm{c}}} = (-\cos\theta_{\mathrm{c}}, -\sin\theta_{\mathrm{c}}, 0)$, and with linear-momentum–energy four-vectors in frame S:

$$E^\mu_{\mathrm{ph|c}} = \begin{pmatrix} h\nu\mathbf{u}_{\mathrm{c}} \\ h\nu \end{pmatrix}, E^\mu_{\mathrm{ph|\tilde{c}}} = \begin{pmatrix} h\nu\mathbf{u}_{\tilde{\mathrm{c}}} \\ h\nu \end{pmatrix}. \tag{16}$$

For heat four-vector, Q^μ, adding on $N_{\mathrm{ph}}/2$ opposite photon pairs ($\mathrm{c}\tilde{\mathrm{c}}$), one has:

$$Q^\mu = \Sigma_{N_{\mathrm{ph}}/2}(E^\mu_{\mathrm{ph|c}} + E^\mu_{\mathrm{ph|\tilde{c}}}) = \begin{pmatrix} 0 \\ Q \end{pmatrix}, \tag{17}$$

with $Q = \dot{N}_{\mathrm{ph}} h\nu\Delta t$ the energy emitted as heat during time interval $[0, \Delta t]$, where $\dot{N}_{\mathrm{ph}} = \mathrm{d}N_{\mathrm{ph}}/\mathrm{d}t$.

2.2 Four-vector equation: frame S

In frame S, the pipe is at rest, forces are simultaneously applied during time interval $[0, \Delta t]$, and emitted photons have zero net linear momentum. The following circumstances apply to the process:

1. The system consists of elements (I), 1, 2, ..., $\mathrm{k}-1, \mathrm{k}, \mathrm{k}+1$, ..., r-1, r, (F).
2. Only elements (I) and (F) change their mechanical state during the process.
3. External forces $\mathbf{F}_{\mathrm{I}} = P_{\mathrm{I}}\mathbf{A}_{\mathrm{I}}$ and $\mathbf{F}_{\mathrm{F}} = -P_{\mathrm{F}}\mathbf{A}_{\mathrm{F}}$ perform works $W_{\mathrm{I}} = P_{\mathrm{I}}\mathcal{V}$ and $W_{\mathrm{F}} = -P_{\mathrm{F}}\mathcal{V}$ respectively.
4. Gravitational forces, with $\mathbf{g} = (0, -g, 0)$, exert linear impulse and perform work on all (r) elements inside the pipe, with associated four-vectors $W^{\mu}_{\mathrm{rg}} = \Sigma^{\mathrm{k=r}}_{\mathrm{k=1}} W^{\mu}_{\mathrm{g|k}}$.
5. Forces $\mathbf{F}_{\mathrm{k}} = P_{\mathrm{k}}\mathbf{A}_{\mathrm{k}}$ are internal forces cancelling each other out and performing zero net internal work.
6. External pipe-on-fluid forces, $\mathbf{n}_k$, exert linear impulse and perform no work (but pseudo-work). It can be considered $\mathbf{N} = (N_x, N_y, 0)$, with $\mathbf{N} = \Sigma_{\mathrm{k}}\mathbf{n}_{\mathrm{k}}$.
7. Heat, described as a set of thermal photons, has zero net linear momentum (and zero net linear impulse on the system) in frame S, with $Q = \Sigma_{\mathrm{k}} Q_{\mathrm{k}}$.

For a Bernoulli process in time interval $[0, \Delta t]$, one has the four-vector fundamental equation:

$$E^{\mu}_{\mathrm{I}} - E^{\mu}_{\mathrm{F}} = W^{\mu}_{\mathrm{I}} + W^{\mu}_{\mathrm{F}} + W^{\mu}_{\mathrm{rg}} + W^{\mu}_{\mathrm{N}} + Q^{\mu} \,. \tag{18}$$

In its matrix form, Eq. (18) is given by:

$$\begin{pmatrix} c\gamma_{v_{\mathrm{F}}}\mathcal{M}v_{\mathrm{F}}\cos\theta \\ c\gamma_{v_{\mathrm{F}}}\mathcal{M}v_{\mathrm{F}}\sin\theta \\ 0 \\ \gamma_{v_{\mathrm{F}}}\mathcal{M}c^2 \end{pmatrix} - \begin{pmatrix} c\gamma_{v_{\mathrm{I}}}\mathcal{M}v_{\mathrm{I}}\cos\theta \\ c\gamma_{v_{\mathrm{I}}}\mathcal{M}v_{\mathrm{I}}\sin\theta \\ 0 \\ \gamma_{v_{\mathrm{I}}}\mathcal{M}c^2 \end{pmatrix} = \begin{pmatrix} cP_{\mathrm{I}}A_{\mathrm{I}}\cos\theta\Delta t \\ cP_{\mathrm{I}}A_{\mathrm{I}}\sin\theta\Delta t \\ 0 \\ P_{\mathrm{I}}\mathcal{V} \end{pmatrix} - \begin{pmatrix} cP_{\mathrm{F}}A_{\mathrm{F}}\cos\theta\Delta t \\ cP_{\mathrm{F}}A_{\mathrm{F}}\sin\theta\Delta t \\ 0 \\ P_{\mathrm{F}}\mathcal{V} \end{pmatrix} + \\ + \begin{pmatrix} 0 \\ -cr\mathcal{M}g\Delta t \\ 0 \\ -\mathcal{M}gL_{\mathrm{p}}\sin\theta \end{pmatrix} + \begin{pmatrix} cN_x\Delta t \\ cN_y\Delta t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q \end{pmatrix} . \tag{19}$$

Height difference between (I) and (F) is $h_{\mathrm{F}} - h_{\mathrm{I}} = L_{\mathrm{p}}\sin\theta$. Initial and final volumes are identical, $\mathcal{V} = A_{\mathrm{I}}L_{\mathrm{I}} = A_{\mathrm{F}}L_{\mathrm{F}}$. The velocity in this equation is the velocity of the centre-of-mass of the system in each region (I), or (F).

The derivation of Eq. (19) is equivalent to considering as the system a fluid, with inertia $\mathcal{M}$, moving during time interval $[0, \Delta t]$ from point (I) to point (F) subjected to various external forces, some of which are conservative, performing work, and some performing pseudo-work, dissipating mechanical energy; the fluid inside the pipe play no role in this process, remaining in the same steady state (see Fig. 3). If any, thermal effects are considered into heat four-vector Q^{μ}.

Equation (19) can be obtained if a FVFE is posed for all fluid elements (Fig. 3), (I), 1, 2, ... k-1, k, k+1, ..., r-1, r, (F), and adding over them. This methodology has been chosen, by application of the locality principle, in the Appendix, to obtain the Eq. (19) for the process.

2.3 Bernoulli relativistic equations. Frame S

Equation (19) provides, by components, the relativistic NSL and FLT equations, allowing a complete mechanical-thermodynamical description of the process. From this FVFE, the following relativistic NSL and FLT equations are obtained:

Newton's second law. The NSL equation for the process is obtained considering the FVFE four-vectors linear impulse and linear momentum variation components. For a finite process, with forces applied during the interval $[0, \Delta t]$, one has:

$$\begin{aligned} \mathcal{M}(\gamma_{v_{\mathrm{F}}}v_{\mathrm{F}}\cos\theta - \gamma_{v_{\mathrm{I}}}v_{\mathrm{I}}\cos\theta) &= (P_{\mathrm{I}}A_{\mathrm{I}}\cos\theta - P_{\mathrm{F}}A_{\mathrm{F}}\cos\theta)\Delta t + N_x\Delta t \,, \\ \mathcal{M}(\gamma_{v_{\mathrm{F}}}v_{\mathrm{F}}\sin\theta - \gamma_{v_{\mathrm{I}}}v_{\mathrm{I}}\sin\theta) &= (P_{\mathrm{I}}A_{\mathrm{I}}\sin\theta - P_{\mathrm{F}}A_{\mathrm{F}}\sin\theta)\Delta t - r\mathcal{M}g\Delta t + N_y\Delta t \end{aligned} \tag{20}$$

which constitutes the linear-impulse–linear momentum variation equation for the process.

Newton's second law complementary dynamical relationship. For the pseudo-work–kinetic energy variation equation (NSL-CDR) one has:

$$\gamma_{v_\mathrm{F}}\mathcal{M}c^2 - \gamma_{v_\mathrm{I}}\mathcal{M}c^2 = (P_\mathrm{I} - P_\mathrm{F})\mathcal{V} - \mathcal{M}g(h_\mathrm{F} - h_\mathrm{I}) + [\Sigma_{\mathrm{k}=1}^{\mathrm{k}=\mathrm{r}}(\mathbf{v}_\mathrm{k}\cdot\mathbf{n}_\mathrm{k})]\Delta t\,, \tag{21}$$

with pseudo-work:

$$\mathrm{p}W = \Sigma_{\mathrm{k}=1}^{\mathrm{k}=\mathrm{r}}(\mathbf{v}_\mathrm{k}\cdot\mathbf{n}_\mathrm{k})\Delta t = \Sigma_{\mathrm{k}=1}^{\mathrm{k}=\mathrm{r}}(\mathbf{L}_\mathrm{k}\cdot\mathbf{n}_\mathrm{k})\,. \tag{22}$$

First law of thermodynamics. The FLT equation for the process is obtained considering the FVFE energy and work four-vectors, temporal components. For the FLT for the process one has:

$$(\gamma_{v_\mathrm{F}} - \gamma_{v_\mathrm{I}})\mathcal{M}c^2 = (P_\mathrm{I} - P_\mathrm{F})\mathcal{V} - \mathcal{M}g(h_\mathrm{F} - h_\mathrm{I}) + Q\,. \tag{23}$$

Forces $\mathbf{F}_\mathrm{I}$ and $\mathbf{F}_\mathrm{F}$ perform work, $W_\mathrm{I} = P_\mathrm{I}A_\mathrm{I}L_\mathrm{I} = P_\mathrm{I}\mathcal{V}$ and $W_\mathrm{F} = P_\mathrm{F}A_\mathrm{F}L_\mathrm{F} = P_\mathrm{F}\mathcal{V}$; also the gravitational forces perform work $W_\mathrm{rg} = \mathcal{M}g(h_\mathrm{F} - h_\mathrm{I}) = \mathcal{M}gL_\mathrm{p}\sin\theta$; heat Q takes into account possible thermal effects during the process. With $\rho\mathcal{V} = \mathcal{M}$, one achieves relationship:

$$\gamma_{v_\mathrm{F}}\rho c^2 - \gamma_{v_\mathrm{I}}\rho c^2 = -\rho g(h_\mathrm{F} - h_\mathrm{I}) + P_\mathrm{I} - P_\mathrm{F} + q\,, \tag{24}$$

where $q = Q/\mathcal{V}$ is heat per unit volume. In general, comparing the NSL-CDR Eq. (21) and the FLT Eq. (23), one obtains:

$$Q = -\mathrm{p}W\,. \tag{25}$$

Pseudo-work quantifies heat. Equation (25) quantifies dissipative effects. For the entropy of the universe increment during the process, $\Delta S_\mathrm{U} = |Q|/T > 0$ is obtained, with $Q = \|Q^\mu\|$, showing the irreversibility of the process when $\mathrm{p}W < 0$.

Mechanical energy conservation. Whether the process is carried out with $\mathrm{p}W = 0$ [i.e., with $\Sigma_{\mathrm{k}=1}^{\mathrm{k}=\mathrm{r}}(\mathbf{n}_\mathrm{k}\cdot\mathbf{v}_\mathrm{k}) = 0$], no thermal effects will take place and the relativistic Bernoulli's equation is given by [6]:

$$P_\mathrm{F} + \rho(\gamma_{v_\mathrm{F}} - 1)c^2 + \rho g h_\mathrm{F} = P_\mathrm{I} + \rho(\gamma_{v_\mathrm{I}} - 1)c^2 + \rho g h_\mathrm{I}\,. \tag{26}$$

2.4 Bernoulli's classical equations. Frame S

The classical description of the Bernoulli process is carried out exclusively with his mechanical energy equation. Relativity tells us that we must work simultaneously with the NSL and the FLT for the process. In turn, from the NSL it will be possible to obtain the NSL-CDR equation, based on the pseudo-work, and by comparison between the NSL-CDR and the FLT, the heat equation and the entropy of the universe variation (SLT). This description allows a more complete physics analysis of the process.

In the low-speed limit, with $\lim_{v/c\to 0}\gamma_v = 1$ and $\lim_{v/c\to 0}(\gamma_v - 1)c^2 = \frac{1}{2}v^2$, the well-known Bernoulli equation, for an incompressible fluid flowing without friction or viscosity inside a variable section and height pipe, is obtained. In this limit, the energy equation (FLT) and the NSL-CDR equation are the same. In the absence of dissipative forces, Bernoulli's equation can be obtained from NSL as its NSL-CDR, for a pure mechanical process [30].

Classical NSL **equation.** For the classical linear-impulse–linear-momentum variation equation for the Bernoulli equation one has ($M \equiv \mathcal{M}$):

$$M(\mathbf{v}_\mathrm{F} - \mathbf{v}_\mathrm{I}) = [(P_\mathrm{I}\mathbf{A}_\mathrm{I} - P_\mathrm{F}\mathbf{A}_\mathrm{F}) + rM\mathbf{g} + \mathbf{N}]\Delta t\,. \tag{27}$$

An external force $\mathbf{F}^\mathrm{ext} \equiv \mathbf{N}$ is needed in order to maintain pipe walls at rest.

Classical NSL-CDR **equation.** From Eq. (21), in the low-speed limit, the following pseudo-work–kinetic energy variation classical equation is obtained:

$$\frac{1}{2}Mv_\mathrm{F}^2 - \frac{1}{2}Mv_\mathrm{I}^2 = (P_\mathrm{I} - P_\mathrm{F})\mathcal{V} - Mg(h_\mathrm{F} - h_\mathrm{I}) + \mathrm{p}W\,. \tag{28}$$

Classical FLT **equation.** From Eq. (23), in the low speed-limit, the following classical FLT equation for the Bernoulli process is obtained:

$$\frac{1}{2}M(v_{\rm F}^2-v_{\rm I}^2)=(P_{\rm I}-P_{\rm F})\mathcal{V}-Mg(h_{\rm F}-h_{\rm I})+Q \Longrightarrow P_{\rm I}+\frac{1}{2}\rho v_{\rm I}^2+\rho g h_{\rm I}=P_{\rm F}+\frac{1}{2}\rho v_{\rm F}^2+\rho g h_{\rm F}+q. \quad (29)$$

Heat equation. Comparing the NSL-CDR equation and the FLT, the heat-pseudo-work equation, Eq. (25), is obtained.

Mechanical energy conservation. In the absence of mechanical energy dissipation, the classical equations for the Bernouilli process are obtained:

$$\begin{aligned} M(\mathbf{v}_{\rm F}-\mathbf{v}_{\rm I}) =& [(P_{\rm I}\mathbf{A}_{\rm I}-P_{\rm F}\mathbf{A}_{\rm F})+rM\mathbf{g}+\mathbf{N}]\Delta t\,, \\ \begin{pmatrix} Mv_{\rm F}\cos\theta \\ Mv_{\rm F}\sin\theta \\ 0\end{pmatrix}-\begin{pmatrix} Mv_{\rm I}\cos\theta \\ Mv_{\rm I}\sin\theta \\ 0\end{pmatrix} =& \left[\begin{pmatrix} P_{\rm I}A_{\rm I}\cos\theta \\ P_{\rm I}A_{\rm I}\sin\theta \\ 0\end{pmatrix}-\begin{pmatrix} P_{\rm F}A_{\rm F}\cos\theta \\ P_{\rm F}A_{\rm F}\sin\theta \\ 0\end{pmatrix}+\begin{pmatrix} 0 \\ -rMg \\ 0\end{pmatrix}+\begin{pmatrix} N_x \\ N_y \\ 0\end{pmatrix}\right]\Delta t\,, \\ \frac{1}{2}M(v_{\rm F}^2-v_{\rm I}^2) =& (P_{\rm I}-P_{\rm F})\mathcal{V}-MgL_{\rm p}\sin\theta\,, \\ \frac{1}{2}M(v_{\rm F}^2-v_{\rm I}^2) =& (P_{\rm I}-P_{\rm F})\mathcal{V}-MgL_{\rm p}\sin\theta\,. \end{aligned} \quad (30)$$

In this limit, the classical FLT equation is given by Eq. (1), which is not a general equation for the speed variation of a fluid flowing through a varying-cross pipe, subject to a gradient of pressures and heights. It is only valid under the rather restrictive conditions stated above.

3 Four-vector equation. Frame $\bar{\rm S}$

Figure 2 shows a diagram of how the experiment is described from the point of view of observers in frame $\bar{\rm S}$. Frame $\bar{\rm S}(\bar{x},\bar{y})$ is integrated by various observers, each one located at a node, equipped with measuring devices, in particular, a clock synchronized with the rest of the observers in $\bar{\rm S}$ (see Sec. 1.3). Each $\bar{\rm S}$ observer (j) is characterized by its coordinates $\bar{\mathbf{x}}_{\rm j}\equiv(\bar{x}_{\rm j},\bar{y}_{\rm j},\bar{z}_{\rm j})$ and it stores spatio-temporal coordinates for events taking place at its location.

Fluid elements displacements $\bar{\mathbf{x}}_{\rm k}$ and velocities $\bar{\mathbf{v}}_{\rm k}$ are measured by the set of observers in $\bar{\rm S}$, posing a series of equations: $\overline{\rm NSL}$, $\overline{\rm NSL\text{-}CDR}$, $\overline{\rm FLT}$. Then, it will be checked if their experimental observations comply with these proposed equations.

Lorentz transformation. As previously indicated, an advantage of the STR four-vector formulation, and the description of processes by a FVFE, is that the process can be described in a generic frame $\bar{\rm S}$ moving with velocity $\mathbf{V}$ relative to a proper frame S, by applying the Lorentz transformation. For the standard configuration, with velocity $\mathbf{V}=(-V,0,0)$ of frame $\bar{\rm S}$ relative to frame S, the corresponding Lorentz transformation, $\mathcal{L}_\nu^\mu(-V)$, in its matrix form, is given by:

$$\mathcal{L}_\nu^\mu(-V)\equiv\begin{pmatrix} \gamma_V & 0 & 0 & \beta_V\gamma_V \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_V\gamma_V & 0 & 0 & \gamma_V \end{pmatrix}, \quad (31)$$

where $\gamma_V=(1-\beta_V^2)^{-1/2}$, with $\beta_V=V/c$. It is easy to check that an equation in frame $\bar{\rm S}$, $\overline{\rm NSL}$ or $\overline{\rm FLT}$, will be linear combination of NSL and FLT equations in frame S, and vice versa, with V dependent coeficients γ_V and $\beta_V\gamma_V$. Thus, four-vector fundamental equation descriptions in S and $\bar{\rm S}$ are *equivalent* (as demanded by the principle of relativity).

As stated above, formalism coherence demands that frame S observer can anticipate the FVFE for the observers in frame $\bar{\rm S}$ by means of:

$$\begin{aligned} \mathcal{L}_\nu^\mu(-V)[E_{\rm F}^\nu-E_{\rm I}^\nu &= W_{\rm I}^\nu+W_{\rm F}^\nu+W_{\rm rg}^\nu+W_{\rm N}^\nu+Q^\nu]\rightarrow \\ \bar{E}_{\rm F}^\mu-\bar{E}_{\rm I}^\mu &= \bar{W}_{\rm I}^\mu+\bar{W}_{\rm F}^\mu+\bar{W}_{\rm rg}^\mu+\bar{W}_{\rm N}^\mu+\bar{Q}^\mu\,. \end{aligned} \quad (32)$$

In its matrix form, the FVFE in frame $\bar{\rm S}$ is given by:

$$\begin{pmatrix} c\gamma_{\bar v_{\rm F}}\mathcal{M}\bar v_{\rm F}\cos\bar\theta \\ c\gamma_{\bar v_{\rm F}}\mathcal{M}\bar v_{\rm F}\sin\bar\theta \\ 0 \\ \gamma_{\bar v_{\rm F}}\mathcal{M}c^2 \end{pmatrix} - \begin{pmatrix} c\gamma_{\bar v_{\rm I}}\mathcal{M}\bar v_{\rm I}\cos\bar\theta \\ c\gamma_{\bar v_{\rm I}}\mathcal{M}\bar v_{\rm I}\sin\bar\theta \\ 0 \\ \gamma_{\bar v_{\rm I}}\mathcal{M}c^2 \end{pmatrix} = \begin{pmatrix} cP_{\rm I}A_{\rm I}\Delta\bar t_{\rm I} \\ cP_{\rm I}A_{\rm I}\sin\theta\Delta t \\ 0 \\ P_{\rm I}A_{\rm I}\bar L_{\rm I} \end{pmatrix} - \begin{pmatrix} cP_{\rm F}A_{\rm F}\Delta\bar t_{\rm F} \\ cP_{\rm F}A_{\rm F}\sin\theta\Delta t \\ 0 \\ P_{\rm F}A_{\rm F}\bar L_{\rm F} \end{pmatrix} +$$
$$+ \begin{pmatrix} -c\gamma_V[c^{-2}\mathcal{M}gL_{\rm p}\sin\theta]V \\ -cr\mathcal{M}g\Delta t \\ 0 \\ -c\gamma_V\mathcal{M}gL_{\rm p}\sin\theta \end{pmatrix} + \begin{pmatrix} cN_x\Delta\hat t \\ cN_y\Delta t \\ 0 \\ N_xV\Delta\hat t \end{pmatrix} + \begin{pmatrix} c\gamma_V(c^{-2}Q)V \\ 0 \\ 0 \\ \gamma_V Q \end{pmatrix}, \tag{33}$$

with relativistic transformations (A:I, F):

$$\begin{aligned} \gamma_{\bar v_{\rm A}}\bar v_{\rm A}\cos\bar\theta &= \gamma_{v_{\rm A}}\gamma_V(v_{\rm A}\cos\theta + V)\,, \\ \gamma_{\bar v_{\rm A}}\bar v_{\rm A}\sin\bar\theta &= \gamma_{v_{\rm F}}v_{\rm F}\sin\theta\,, \\ \gamma_{\bar v_{\rm F}} &= \gamma_{v_{\rm F}}\gamma_V(1 + v_{\rm F}\cos\theta V/c^2)\,, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \bar v_{x{\rm A}} \equiv \bar v_{\rm A}\cos\bar\theta &= (v_{x{\rm A}} + V)/(1 + v_{x{\rm A}}V/c^2) = (v_{\rm A}\cos\theta + V)/(1 + v_{\rm A}\cos\theta V/c^2)\,, \\ \bar v_{y{\rm A}} \equiv \bar v_{\rm A}\sin\bar\theta &= \gamma_V^{-1}v_{y{\rm A}}/(1 + v_{x{\rm A}}V/c^2) = \gamma_V^{-1}v_{\rm A}\sin\theta/(1 + v_{\rm A}\cos\theta V/c^2)\,, \\ \bar L_{\rm I} &= \gamma_V(L_{\rm I} + V\cos\theta\Delta t)\,, \bar L_{\rm F} = \gamma_V(L_{\rm F} + V\cos\theta\Delta t)\,, \\ \Delta\bar t_{\rm I} &= \gamma_V(\cos\theta\Delta t + Vc^{-2}L_{\rm I})\,, \Delta\bar t_{\rm F} = \gamma_V(\cos\theta\Delta t + Vc^{-2}L_{\rm F})\,, \\ \Delta\hat t &= \gamma_V\Delta t\,. \end{aligned} \tag{35}$$

Relativistic effects. The description of the process in frame $\bar{\rm S}$ presents several relativistic effects:

1. *Non simultaneity.* Forces, $\mathbf{F}_{\rm I}$, $\mathbf{F}_{\rm F}$, and $\mathbf{N}$, that were simultaneously exerted during time interval $[0,\Delta t]$ in frame S, are not simultaneously applied in frame $\bar{\rm S}$ (according to the asyncrhonous formulation of relativity [19]), with time intervals $[0,\Delta\bar t_{\rm I}]$, $[0,\Delta\bar t_{\rm F}]$, and $[0,\Delta\hat t]$, respectively. This effect is an order c^{-2} relativistic effect, which will be neglected in the low-speed limit (absolute time interval $[0,\Delta t]$).

2. *Velocity transformations.* Fluid relativistic velocities transformations guarantee that no speed in $\bar{\rm S}$ reaches light-speed c, even when $V \approx c$. In the low-speed limit the Huygens-Galileo velocity transformations,
$$\bar v_{x{\rm A}} = v_{x{\rm A}} + V\,, \bar v_{y{\rm A}} = v_{y{\rm A}}\,, \bar{\mathbf{v}}_{\rm A} = \mathbf{v}_{\rm A} + \mathbf{V}\,, \tag{36}$$
are recovered.

3. *Conveyor belt effect.* In frame $\bar{\rm S}$, work $\bar W_{\rm cb} = \gamma_V(\mathbf{N}\cdot\mathbf{V})\Delta t$, temporal component in four-vector $\bar W^\mu_{\rm N}$ in Eq. (33) is performed by the external force $\mathbf{F}^{\rm ext} \equiv \mathbf{N}$: i.e., force $\mathbf{N}$ component N_x moves with speed V during time interval $\Delta\hat t = \gamma_V\Delta t$; this effect has no relation to the speed of the fluid, just with frame $\bar{\rm S}$ relative speed to frame S. It is $\bar W_{\rm cb}$ the work performed by the external agent that keeps the wagon-lab, or the belt-lab, moving at constant speed V when the flowing fluid exerts force $\mathbf{N}$ on the pipe [25].

4. *Angle transformation.* Angle θ, tg$\theta = v_y/v_x$, transforms to angle $\bar\theta$, with tg$\bar\theta = \bar v_y/\bar v_x$, tg$\bar\theta = \gamma_V^{-1}v\sin\theta/(v\cos\theta + V)$ a kind of velocity aberration effect.

5. *Inertia of work effect.* Work related inertia $\bar{\mathcal{M}}_g = c^{-2}\mathcal{M}gL_{\rm p}\sin\theta$ contributes to the linear impulse in $\bar{\rm S}$, according to the inertia of energy principle.

6. *Heat inertia and entropy of the universe increment.* According to Einstein's IEP, heat inertia $\mathcal{M}_Q \equiv c^{-2}Q$ contributes to the linear momentum in $\bar{\mathrm{S}}$. When heat exchanged is described in S as a set of thermal photons, relativistic Doppler and aberration effects are obtained [24], with the relativistic transformations in frequency and direction (e.g., $\bar{E}^{\mu}_{\mathrm{ph|c}} = \mathcal{L}^{\mu}_{\nu}(-V)E^{\nu}_{\mathrm{ph|c}}$, etc),

$$\begin{aligned}
\bar{\nu}_{\mathrm{c}} &= \nu\gamma_V(1+\beta_V\cos\theta_{\mathrm{c}})\,, & \bar{\nu}_{\bar{\mathrm{c}}} &= \nu\gamma_V(1-\beta_V\cos\theta_{\mathrm{c}})\,,\\
\cos\bar{\theta}_{\mathrm{c}} &= (\cos\theta_{\mathrm{c}}+\beta_V)/(1+\beta_V\cos\theta_{\mathrm{c}})\,, & \cos\bar{\theta}_{\bar{\mathrm{c}}} &= (-\cos\theta_{\mathrm{c}}+\beta_V)/(1-\beta_V\cos\theta_{\mathrm{c}})\,,\\
\sin\bar{\theta}_{\mathrm{c}} &= \gamma_V^{-1}\sin\theta_{\mathrm{c}}/(1+\beta_V\cos\theta_{\mathrm{c}})\,, & \sin\bar{\theta}_{\bar{\mathrm{c}}} &= -\gamma_V^{-1}\sin\theta_{\mathrm{c}}/(1-\beta_V\cos\theta_{\mathrm{c}})\,,\\
\mathrm{tg}\bar{\theta}_{\mathrm{c}} &= \gamma_V^{-1}\sin\theta_{\mathrm{c}}/(\cos\theta_{\mathrm{c}}+\beta_V)\,, & \mathrm{tg}\bar{\theta}_{\bar{\mathrm{c}}} &= -\gamma_V^{-1}\sin\theta_{\mathrm{c}}/(-\cos\theta_{\mathrm{c}}+\beta_V)\,.
\end{aligned} \tag{37}$$

Then, by adding over photons linear-momentum–energy four-vectors in $\bar{\mathrm{S}}$ one obtains:

$$\begin{aligned}
h\Sigma_{\mathrm{c}\bar{\mathrm{c}}}(\bar{\nu}_{\mathrm{c}}+\bar{\nu}_{\bar{\mathrm{c}}}) &= \gamma_V(N_{\mathrm{ph}}h\nu)\,,\\
c^{-1}h\Sigma_{\mathrm{c}\bar{\mathrm{c}}}(\bar{\nu}_{\mathrm{c}}\cos\bar{\theta}_{\mathrm{c}}+\bar{\nu}_{\bar{\mathrm{c}}}\cos\bar{\theta}_{\bar{\mathrm{c}}}) &= \gamma_V(c^{-2}N_{\mathrm{ph}}h\nu)V\,,\\
c^{-1}h\Sigma_{\mathrm{c}\bar{\mathrm{c}}}(\bar{\nu}_{\mathrm{c}}\sin\bar{\theta}_{\mathrm{c}}+\bar{\nu}_{\bar{\mathrm{c}}}\sin\bar{\theta}_{\bar{\mathrm{c}}}) &= 0\,.
\end{aligned} \tag{38}$$

These transformations allows to obtain the four-vector $\bar{Q}^{\mu}$ in the same way that it has been obtained by applying on four-vector Q^{μ} the Lorentz transformation. In frame $\bar{\mathrm{S}}$, a nonzero linear momentum is attached to heat (unlike in S, where heat linear momentum is zero). Therefore, an associated linear impulse is present due to emitted photons with different frequencies in different directions (with respect to how they are emitted in S): this effect is a relativistic order c^{-2} effect and will be neglected in the low-speed classical limit.

Since the physical meaning of the entropy of the universe variation is related to lost work, W_{ls}, or dissipated mechanical energy, $T\Delta S_{\mathrm{U}} = W_{\mathrm{ls}}$, the entropy change of the universe for the process must be a relativistic invariant, the same in every inertial frame. According to its definition, heat in $\bar{\mathrm{S}}$ is the norm of $\bar{Q}^{\mu}$ four-vector, with $\bar{\mathbf{p}}_Q = \gamma_V\mathcal{M}_Q\mathbf{V}$, and $\bar{E}_Q = \gamma_V\mathcal{M}_Qc^2$,

$$\bar{Q}^{\mu} = \begin{pmatrix} c\gamma_V(c^{-2}Q)\mathbf{V} \\ \gamma_V Q \end{pmatrix}\,, \tag{39}$$

with $\|\bar{Q}^{\mu}\| = [E_Q^2 - c^2\mathbf{p}_{\mathrm{Q}}\cdot\mathbf{p}_{\mathrm{Q}}]^{1/2} = Q$, and $T\Delta\bar{S}_{\mathrm{U}} = \|\bar{Q}^{\mu}\| = \|Q^{\mu}\|$. Since $\|Q^{\mu}\| = [Q^2-0]^{1/2} = Q$, then $\Delta\bar{S}_{\mathrm{U}} = \Delta S_{\mathrm{U}}$.

3.1 Relativistic equations. Frame $\bar{\mathrm{S}}$.

Newton's second law in $\bar{\mathrm{S}}$. From Eq. (33), the linear-impulse–linear-momentum variation ($\overline{\mathrm{NSL}}$) in frame $\bar{\mathrm{S}}$ is obtained:

$$\begin{aligned}
&\begin{pmatrix} \gamma_{\bar{v}_{\mathrm{F}}}\mathcal{M}\bar{v}_{\mathrm{F}}\cos\bar{\theta} \\ \gamma_{\bar{v}_{\mathrm{F}}}\mathcal{M}\bar{v}_{\mathrm{F}}\sin\bar{\theta} \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma_{\bar{v}_{\mathrm{I}}}\mathcal{M}\bar{v}_{\mathrm{I}}\cos\bar{\theta} \\ \gamma_{\bar{v}_{\mathrm{I}}}\mathcal{M}\bar{v}_{\mathrm{I}}\sin\bar{\theta} \\ 0 \end{pmatrix} = \begin{pmatrix} P_{\mathrm{I}}A_{\mathrm{I}}\Delta\bar{t}_{\mathrm{I}} \\ P_{\mathrm{I}}A_{\mathrm{I}}\sin\theta\Delta t \\ 0 \end{pmatrix} - \begin{pmatrix} P_{\mathrm{F}}A_{\mathrm{F}}\Delta\bar{t}_{\mathrm{F}} \\ P_{\mathrm{F}}A_{\mathrm{F}}\sin\theta\Delta t \\ 0 \end{pmatrix} + \\
&\quad + \begin{pmatrix} -\gamma_V[c^{-2}\mathcal{M}gL_{\mathrm{p}}\sin\theta]V \\ -r\mathcal{M}g\Delta t \\ 0 \end{pmatrix} + \begin{pmatrix} N_x\Delta\hat{t} \\ N_y\Delta t \\ 0 \end{pmatrix} + \begin{pmatrix} \gamma_V(c^{-2}Q)V \\ 0 \\ 0 \end{pmatrix}\,.
\end{aligned} \tag{40}$$

Forces that were simultaneously applied in S, are not applied simultaneously in $\bar{\mathrm{S}}$. Inertia $\mathcal{M}_{\mathrm{w}} \equiv c^{-2}\mathcal{M}g(h_{\mathrm{F}}-h_{\mathrm{I}})$ attached to energy exchanged as work contributes to the fluid linear momentum variation (order c^{-2} relativistic effect). Inertia $\mathcal{M}_Q \equiv c^{-2}Q$ attached to energy interchanged as heat contributes to the fluid linear momentum variation (order c^{-2} relativistic effect).

First law of thermodynamics in $\bar{\mathrm{S}}$. For the energy ($\overline{\mathrm{FLT}}$) equation in $\bar{\mathrm{S}}$ one has:

$$\mathcal{M}(\gamma_{\bar{v}_{\mathrm{F}}}-\gamma_{\bar{v}_{\mathrm{I}}})c^2 = -\gamma_V\mathcal{M}g(h_{\mathrm{F}}-h_{\mathrm{I}}) + P_{\mathrm{I}}A_{\mathrm{I}}\bar{L}_{\mathrm{I}} - P_{\mathrm{F}}A_{\mathrm{F}}\bar{L}_{\mathrm{F}} + \bar{W}_{\mathrm{cb}} + \gamma_V Q\,, \tag{41}$$

with the conveyor belt effect work: $\bar{W}_{\rm cb} = \mathbf{N} \cdot \mathbf{V}\Delta\hat{t}$, as the work performed on the wagon, or conveyor belt, to keep it moving with constant speed V.

Newton's second law complementary dynamical relationship in $\bar{\rm S}$. For the $\overline{\rm NSL} - \overline{\rm CDR}$ equation in $\bar{\rm S}$ one has:

$$\mathcal{M}(\gamma_{\bar{v}_{\rm F}} - \gamma_{\bar{v}_{\rm I}})c^2 = -\gamma_V \mathcal{M}g(h_{\rm F} - h_{\rm I}) + P_{\rm I}A_{\rm I}\bar{L}_{\rm I} - P_{\rm F}A_{\rm F}\bar{L}_{\rm F} + \bar{W}_{\rm cb} + \gamma_V[\Sigma_{\rm k}(\mathbf{n}_{\rm k} \cdot \mathbf{v}_{\rm k})]\Delta t\,, \tag{42}$$

with pseudo-work ${\rm p}\bar{W} = \gamma_V({\rm p}W)$.

Principle of relativity. The FVFE formalism has been developed in order to demonstrate that transformations of NSL and FLT equations comply with the principle of relativity: a relativistic equation, $\overline{\rm NSL}$ or $\overline{\rm FLT}$, in frame $\bar{\rm S}$ can be expressed as a linear combination of NSL and FLT equations in frame S, and vice-versa, with transformational symmetry between the relativistic NSL and the FLT equations.

There is a symmetry between the relativistic NSL and FLT equations transformations from S to $\bar{\rm S}$ and vice-versa.

$\overline{\rm NSL}$ equation. By applying the Lorentz transformation to the FVFE, the component (x) of the equation $\overline{\rm NSL}$, can be expressed as:

$$c\overline{\rm NSL}_x = \gamma_V(c{\rm NSL}_x - \beta_V{\rm FLT})\,. \tag{43}$$

The FLT in S contributes to $\overline{\rm NSL}_x$ in $\bar{\rm S}$: it is an order c^{-2} relativistic effect.

$\overline{\rm FLT}$ equation. The $\overline{\rm FLT}$ equation can be expressed as:

$$\overline{\rm FLT} = \gamma_V({\rm FLT} - \beta_V c{\rm NSL}_x)\,. \tag{44}$$

The NSL in S contributes to $\overline{\rm FLT}$ in $\bar{\rm S}$: it is a c^0 relativistic effect, and then, also classical.

3.2 Bernoulli classical equations in $\bar{\rm S}$

Classical Newton's second law in $\bar{\rm S}$. In the low-speed limit, from Eq. (40), the classical $\overline{\rm NSL}$ equation is obtained:

$$M(\bar{\mathbf{v}}_{\rm F} - \bar{\mathbf{v}}_{\rm I}) = [(P_{\rm I}\mathbf{A}_{\rm I} - P_{\rm F}\mathbf{A}_{\rm F}) + rM\mathbf{g} + \mathbf{N}]\Delta t\,. \tag{45}$$

In its matrix form:

$$\begin{pmatrix} M\bar{v}_{\rm F}\cos\bar{\theta} \\ M\bar{v}_{\rm F}\sin\bar{\theta} \\ 0 \end{pmatrix} - \begin{pmatrix} M\bar{v}_{\rm I}\cos\bar{\theta} \\ M\bar{v}_{\rm I}\sin\bar{\theta} \\ 0 \end{pmatrix} = \left[\begin{pmatrix} P_{\rm I}A_{\rm I}\cos\theta \\ P_{\rm I}A_{\rm I}\sin\theta \\ 0 \end{pmatrix} - \begin{pmatrix} P_{\rm F}A_{\rm F}\cos\theta \\ P_{\rm F}A_{\rm F}\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -rMg \\ 0 \end{pmatrix} + \begin{pmatrix} N_x \\ N_y \\ 0 \end{pmatrix}\right]\Delta t\,. \tag{46}$$

This equation is not operational. Even when the pseudo-work for the process is zero $\Sigma_{\rm k}(\mathbf{n}_{\rm k} \cdot \mathbf{v}_{\rm k}) = 0$, forces $\mathbf{n}_{\rm k}$ will still exist and will have to be obtained from the geometry of the pipe and the flow of the fluid, which will not always be easy. Multiplying the equation in x by $\cos\theta$ and the equation in y by $\sin\theta$, we have:

$$M\bar{v}_{\rm F} - M\bar{v}_{\rm I} = (P_{\rm I}A_{\rm I} - P_{\rm F}A_{\rm F} - rMg\sin\theta)\Delta t + \hat{N}\Delta t\,, \tag{47}$$

which is the equation of the NSL along the symmetry axis of the pipe. Multiplying the equation in x by $\sin\theta$ and the equation in y by $\cos\theta$ and subtracting,

$$N_x\sin\theta - N_y\cos\theta + rMg\cos\theta = 0\,, \tag{48}$$

which gives the condition for the components of the force N so that the flow only has component along the symmetry axis of the pipe.

The FLT equation contribution to Newton's second law in $\bar{\mathrm{S}}$ is an order c^{-2} effect, so it disappears (inertia of energy effect) in the classical limit. With the Huygens-Galileo velocity transformation, $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{V}$, one has:

$$M(\mathbf{v}_{\mathrm{F}} - \mathbf{v}_{\mathrm{I}}) = [(P_{\mathrm{I}}\mathbf{A}_{\mathrm{I}} + P_{\mathrm{F}}\mathbf{A}_{\mathrm{F}}) + rM\mathbf{g} + \mathbf{N}]\Delta t \,. \tag{49}$$

The classical NSL and $\overline{\mathrm{NSL}}$ equations, in S and in $\bar{\mathrm{S}}$ respectively, are identical. The classical NSL description of the Bernoulli process does not comply with the principle of relativity.

Classical first law of thermodynamics in $\bar{\mathrm{S}}$. In the low-speed limit, from Eq. (41), the classical $\overline{\mathrm{FLT}}$ is obtained in $\bar{\mathrm{S}}$:

$$\frac{1}{2}M(\bar{v}_{\mathrm{F}}^2 - \bar{v}_{\mathrm{I}}^2) = (P_{\mathrm{I}}A_{\mathrm{I}}\bar{L}_{\mathrm{I}} + P_{\mathrm{F}}A_{\mathrm{F}}\bar{L}_{\mathrm{F}}) + MgL_{\mathrm{p}}\sin\theta + (\mathbf{N}\cdot\mathbf{V})\Delta t + Q \,, \tag{50}$$

with $\mathbf{N}\cdot\mathbf{V} = N_x V$. Time interval is absolute. The following classical equations for the velocity and length transformations are obtained:

$$\begin{aligned} \bar{v}_{x\mathrm{A}} &\equiv \bar{v}_{\mathrm{A}}\cos\bar{\theta} = (v_{\mathrm{A}}\cos\theta + V)\,, \\ \bar{v}_{y\mathrm{A}} &\equiv \bar{v}_{\mathrm{A}}\sin\bar{\theta} = v_{\mathrm{A}}\sin\theta \\ \bar{L}_{\mathrm{I}} &= L_{\mathrm{I}} + V\cos\theta\Delta t\,, \bar{L}_{\mathrm{F}} = L_{\mathrm{F}} + V\cos\theta\Delta t\,. \\ \bar{Q} &= Q\,, \end{aligned} \tag{51}$$

with work $\bar{W}_{\mathrm{cb}} = (\mathbf{N}\cdot\mathbf{V})\Delta t$ (conveyor belt effect). This $\overline{\mathrm{FLT}}$ equation in $\bar{\mathrm{S}}$ can be expressed as linear combination of NSL_x and the FLT equations in S.

The effect of the NSL equation, in S, on the $\overline{\mathrm{FLT}}$ equation, in $\bar{\mathrm{S}}$, is a genuine effect, also appearing in the classical limit. Heat in $\bar{\mathrm{S}}$ is given by $\bar{Q} = Q$.

3.3 Classical covariant equations for the Bernoulli process

The mass M is the same for all observers. Forces n_k, pressures P_k, sections A_k, are the same for all observers, and also g. The time intervals Δt are identical. The distances between two planes, L_{k} are identical. Physical varying magnitudes between observers are displacements, velocities of the elements and displacements and velocities of the application point of the forces. The linear momentum variations are identical, but kinetic energy variations and work are different. Heat, pseudo-work and temperature are identical.

For generic observers S and $\bar{\mathrm{S}}$ for which the lab is moving at speed $\mathbf{V} = (-V, 0, 0)$, the following classical equations related to pseudo-work are found respectively:

$$\begin{aligned} \mathrm{p}W &= \Sigma_{\mathrm{k}}(\mathbf{n}_{\mathrm{k}}\cdot\mathbf{v}_{\mathrm{k}})\Delta t\,, \\ \mathrm{p}\bar{W} &= \Sigma_{\mathrm{k}}(\mathbf{n}_{\mathrm{k}}\cdot\bar{\mathbf{v}}_{\mathrm{k}})\Delta t = \Sigma_{\mathrm{k}}(\mathbf{n}_{\mathrm{k}}\cdot\mathbf{v}_{\mathrm{k}})\Delta t + \mathbf{V}\cdot(\Sigma_{\mathrm{k}}\mathbf{n}_{\mathrm{k}})\Delta t = \mathrm{p}W + (\mathbf{N}\cdot\mathbf{V})\Delta t\,. \end{aligned} \tag{52}$$

Covariant equations $\overline{\mathrm{NSL}}$, $\overline{\mathrm{NSL}}$-$\overline{\mathrm{CDR}}$ and $\overline{\mathrm{FLT}}$ are given, respectively, by:

$$\begin{aligned} M(\bar{\mathbf{v}}_{\mathrm{F}} - \bar{\mathbf{v}}_{\mathrm{I}}) &= [(P_{\mathrm{I}}\mathbf{A}_{\mathrm{I}} - P_{\mathrm{F}}\mathbf{A}_{\mathrm{F}}) + rM\mathbf{g} + \mathbf{N}]\Delta t\,, \\ \begin{pmatrix} M\bar{v}_{\mathrm{F}}\cos\theta \\ M\bar{v}_{\mathrm{F}}\sin\theta \\ 0 \end{pmatrix} - \begin{pmatrix} M\bar{v}_{\mathrm{I}}\cos\theta \\ M\bar{v}_{\mathrm{I}}\sin\theta \\ 0 \end{pmatrix} &= \left[\begin{pmatrix} P_{\mathrm{I}}A_{\mathrm{I}}\cos\theta \\ P_{\mathrm{I}}A_{\mathrm{I}}\sin\theta \\ 0 \end{pmatrix} - \begin{pmatrix} P_{\mathrm{F}}A_{\mathrm{F}}\cos\theta \\ P_{\mathrm{F}}A_{\mathrm{F}}\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -rMg \\ 0 \end{pmatrix} + \begin{pmatrix} N_x \\ N_y \\ 0 \end{pmatrix}\right]\Delta t\,, \\ \frac{1}{2}M(\bar{v}_{\mathrm{F}}^2 - \bar{v}_{\mathrm{I}}^2) &\approx (P_{\mathrm{I}}A_{\mathrm{I}}\bar{L}_{\mathrm{I}} - P_{\mathrm{F}}A_{\mathrm{F}}\bar{L}_{\mathrm{F}}) - Mg(h_{\mathrm{F}} - h_{\mathrm{I}}) + \Sigma_{\mathrm{k}}(\mathbf{n}_{\mathrm{k}}\cdot\bar{\mathbf{v}}_{\mathrm{k}})\Delta t\,, \\ \frac{1}{2}M(\bar{v}_{\mathrm{F}}^2 - \bar{v}_{\mathrm{I}}^2) &= (P_{\mathrm{I}}A_{\mathrm{I}}\bar{L}_{\mathrm{I}} - P_{\mathrm{F}}A_{\mathrm{F}}\bar{L}_{\mathrm{F}}) - Mg(h_{\mathrm{F}} - h_{\mathrm{I}}) + (\mathbf{N}\cdot\mathbf{V})\Delta t + Q\,, \\ Q &= \mathrm{p}W\,. \end{aligned} \tag{53}$$

Every observer measures the same heat, equal to the pseudo-work of forces exerted on the fluid by the walls of the pipe in frame S where the pipe remains at rest.

When the fluid flows in such a way that $\Sigma_{\rm k}(\mathbf{n}_{\rm k}\cdot\mathbf{v}_{\rm k})\Delta t = 0$, no thermal effects are present and $Q = 0$.

This set of equations constitute the classical covariant form of Bernoulli equation: each of these equations can be expressed as linear combination of the corresponding equations in frame S and vice-versa.

4 Conclusions

Covariant equations can be obtained for a Bernoulli process, by applying the concepts and methods of Einstein's special theory of relativity. This approach overcomes the classical physics transformational asymmetry between Newton's second law and the first law of thermodynamics equations. The solution lies in using a single hypothesis, the four-vector fundamental equation, instead of the two independent equations, Newton's second law and the first law of thermodynamics. This equation is based on the principle of inertia of energy, which relates internal energy and inertia, and is implemented by the Lorentz transformation.

Obtaining a covariant description for the Bernoulli process is a good example of the application of the concepts and methods of the special theory of relativity to a process in which speeds are not comparable to light-speed. Since the relativistic description of a process is *per se* covariant, the classical covariant description for Bernoulli's process is obtained from its relativistic description in the low-speed limit.

To apply the special theory of relativity to a Bernoulli process, the following steps must be carried out:

1. First, the corresponding four-vectors associated to physical magnitudes must be obtained, in order to consider both the Newton's second law and the first law of thermodynamics. Then, a covariant relativistic four-vector equation for the process will be posed.

2. By application of the Lorentz transformation, the four-vector fundamental equation for the moving-lab frame, $\bar{\rm S}$, will be obtained from the four-vector fundamental equation in the proper frame S and the corresponding relativistic transformations.

3. After the relativistic description, the classical covariant description of the process in the low-speed limit can be obtained.

Proceeding with the Lorentz transformation on the four-vector fundamental equation in frame S, the following is guaranteed:

1. The principle of relativity is fulfilled and four-vector fundamental equations in S and $\bar{\rm S}$ have the same functional form.

2. Predictions can be made and can be contrasted experimentally about how an observer in frame $\bar{\rm S}$ describes the process in S and vice-versa.

3. The components of the four-vector equation in $\bar{\rm S}$ are linear combinations of the equations in frame S and vice-versa, which can be checked by application of the Lorentz transformation onto the corresponding four-vector fundamental equation in $\bar{\rm S}$.

On the one hand, by solving problems using formalisms complying with the special theory of relativity postulates and requirements, we are forced to think about physical issues that are usually omitted when solving the same problem under the classical approach. On the other hand, when changing between inertial frames to describe the same process, covariant equations are needed, which, in turn, requires consideration of relativistic transformations of velocities, displacements, time intervals, forces, etc. Since the Lorentz transformation is used in relativity, new relativistic effects can be discovered when dealing with new problems.

The validity of the classical Bernoulli equation has been discussed. The relativistic four-vector fundamental equation formalism allows the consideration of a general process and simultaneously obtain the linear momentum and energy equations of the process. The classical Bernoulli equation is valid without mechanical energy dissipation. In that case, the process is purely mechanical, and mechanical energy is conserved throughout the process: Newton's second law complementary dynamic relationship match with the energy equation.

The classical physics covariant form for the Bernoulli process energy equation requires considering Newton's second law for the process and taking into account the force exerted by an external agent, imposing the pipe moving with constant speed (zero speed in proper frame S). This force exerts linear impulse and performs work in every frame (conveyor belt effect), except in proper frame S. The relativistic description of the process implicitly considers these two circumstances.

Appendix. Principle of locality

The special theory of relativity is a *local* theory [23]: equations must relate causes (e.g., force, lineal impulse, work) and effects (e.g., linear momentum variation, energy variation) for the same spatiotemporal event; it must be ensured that a force is applied at the point where it produces effects; contact forces cannot act at a distance; even forces coming from the interaction of the system with a field, must be considered applied to the different elements that conform the system.

Since there is no incompressible fluid (nor rigid solid), and forces cannot act at a distance, according to the principle of locality, the system must be divided into elements such that forces are applied to the elements for which the equations are written, such that forces can be identified as applied to the elements whose mechanical-thermodynamical state they modify. Thus, when dealing with the relativistic description of a process in which a large body is involved, the equation for the whole system is obtained by adding over the system elements equations.

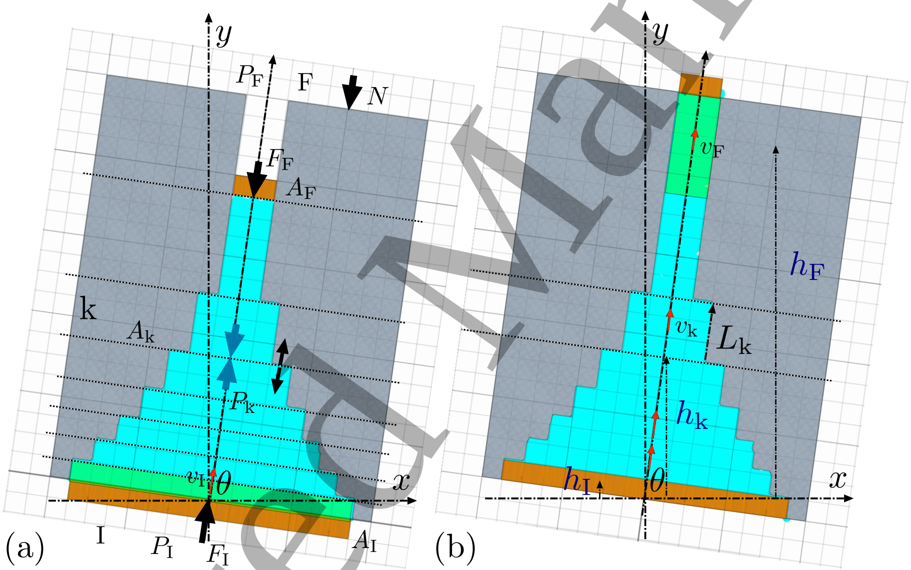

Figure 4: Inclined angle θ pipe. (a) Initial state. (b) Final state. Element (k) surface A_k, width L_k, distance to surface A_F, ξ_k. For $\theta = 0$, horizontal pipe, no gravitational effects; for $\theta = \pi/2$, vertical pipe.

Fig. 4 schematizes the locality principle application to a Bernoulli process [10]. The space between the volume element at point (I) and the same volume element at point (F) is sliced into 1, 2, . . . , k-1, k, k+1,. . . r-1, r, elements. A four-vector fundamental equation will be proposed for each fluid element, including (I) and (F). The outgoing flow of element (k-1) must be the incoming flow of element (k), etc. Intermediate, pipe inside, forces, $\mathbf{F}_k = P_k \mathbf{A}_k$, are system internal forces, so they will do not appear into the final equations. Through elements edges, the pipe surface exerts normal forces $\mathbf{n}_k$ on the fluid inside the element, with net force $\mathbf{N} = \Sigma_k \mathbf{n}_k$ for the whole pipe. An external agent must exert resultant force $\mathbf{F}^{\text{ext}} = \mathbf{N}$ on the pipe to ensure that it moves with constant speed.

The process of continuously transporting volume $\mathcal{V} = \rho \mathcal{M}$, and inertia $\mathcal{M}$, from point (I), up to point (F), implies that a steady state remains inside the pipe.

On mathematical surface Σ_k, with section A_k, pressure P_k is exerted on its bottom face, with $\mathbf{F}_k = P_k\mathbf{A}_k$ and $\tilde{P}_k = P_F + \rho g\xi_k$, where ξ_k is the distance between surfaces Σ_k and Σ_F. Fluid column pressure $\Delta\tilde{P}_k = \rho g\xi_k$ is obtained. Fluid inside element (k), becomes the fluid in element (k+1), etc.

During time interval $[0, \Delta t]$, the volume of the fluid is conserved on every pipe element, with $(\mathbf{A}_k \cdot \mathbf{v}_k)\Delta t = (\mathbf{A}_{k+1} \cdot \mathbf{v}_{k+1})\Delta t$, or $\mathbf{A}_k \cdot \mathbf{L}_k = \mathbf{A}_{k+1} \cdot \mathbf{L}_{k+1} = \mathcal{V}$ (continuity equation), where L_k is the width of element (k): $L_k = \xi_k - \xi_{k+1} = h_{k+1} - h_k$.

Figure 4 sketches a Bernoulli process with an inclined angle θ pipe. Applying the locality principle, three equations will be obtained for the process: (i) Newton's second law equation, (ii) Newton's second law complementary dynamical relationship, and (iii) the first law of thermodynamics. From these equations, the four-vector fundamental equation for a generalised Bernoulli process is obtained.

Newton's second law. Fluid in element (k) occupies the volume of element $(k+1)$, varying its speed from v_k to v_{k+1} varying its linear momentum as:

$$\Delta\mathbf{p}_k \equiv \mathcal{M}\Delta(\gamma_{v_k}\mathbf{v}_k) = \mathcal{M}\gamma_{v_{k+1}}\mathbf{v}_{k+1} - \mathcal{M}\gamma_{v_k}\mathbf{v}_k\,. \tag{54}$$

This variation in linear momentum is due to the net linear impulse exerted on element (k): force P_k exerts pressure on the bottom face of surface Σ_k, where pressure $P_f + \rho g\xi_k$ is exerted from its top face. The linear impulse on element (k) is given by:

$$\mathbf{I}_k = [(P_k\mathbf{A}_k - P_F\mathbf{A}_{k+1}) + \rho\mathbf{g}A_k\xi_{k+1} + \mathbf{n}_k]\Delta t\,. \tag{55}$$

The NSL is applied to each fluid element, and the following set of equations is obtained:

$$\begin{aligned}
\mathcal{M}\Delta(\gamma_{v_I}\mathbf{v}_I) &\equiv \mathcal{M}\gamma_{v_1}\mathbf{v}_1 - \mathcal{M}\gamma_{v_I}\mathbf{v}_I = [(P_I\mathbf{A}_I - P_F\mathbf{A}_I) + \rho\mathbf{g}A_I\xi_I]\Delta t + \mathbf{n}_I\Delta t\,,\\
\mathcal{M}\Delta(\gamma_{v_1}\mathbf{v}_1) &\equiv \mathcal{M}\gamma_{v_2}\mathbf{v}_2 - \mathcal{M}\gamma_{v_1}\mathbf{v}_1 = [(P_1\mathbf{A}_1 - P_F\mathbf{A}_1) + \rho\mathbf{g}A_1\xi_1]]\Delta t + \mathbf{n}_1\Delta t\,,\\
&\dots\\
\mathcal{M}\Delta(\gamma_{v_{k-1}}\mathbf{v}_{k-1}) &\equiv \mathcal{M}\gamma_{v_k}\mathbf{v}_k - \mathcal{M}\gamma_{v_{k-1}}\mathbf{v}_{k-1} = [(P_{k-1}\mathbf{A}_{k-1} - P_F\mathbf{A}_{k-1}) + \rho\mathbf{g}A_{k-1}\xi_{k-1}]\Delta t + \mathbf{n}_{k-1}\Delta t\,,\\
\mathcal{M}\Delta(\gamma_{v_k}\mathbf{v}_k) &\equiv \mathcal{M}\gamma_{v_{k+1}}\mathbf{v}_{k+1} - \mathcal{M}\gamma_{v_k}\mathbf{v}_k = [(P_k\mathbf{A}_k - P_F\mathbf{A}_k) + \rho\mathbf{g}A_k\xi_k]\Delta t + \mathbf{n}_k\Delta t\,,\\
&\dots\\
\mathcal{M}\Delta(\gamma_{v_{r-1}}\mathbf{v}_{r-1}) &\equiv \mathcal{M}\gamma_{v_r}\mathbf{v}_r - \mathcal{M}\gamma_{v_{r-1}}\mathbf{v}_{r-1} = [(P_{r-1}\mathbf{A}_{r-1} - P_F\mathbf{A}_{r-1}) + \rho\mathbf{g}A_{r-1}\xi_{r-1})]\Delta t + \mathbf{n}_{r-1}\Delta t\,,\\
\mathcal{M}\Delta(\gamma_{v_r}\mathbf{v}_r) &\equiv \mathcal{M}\gamma_{v_F}\mathbf{v}_F - \mathcal{M}\gamma_{v_r}\mathbf{v}_r = [(P_r\mathbf{A}_r - P_F\mathbf{A}_r) + \rho\mathbf{g}A_r\xi_r]\Delta t + \mathbf{n}_r\Delta t\,.
\end{aligned} \tag{56}$$

Pressure $P_F + \rho g\xi_k \sin\theta$ on the top face of surface Σ_k, is equal to the pressure on top face of surface Σ_{k+1}, $P_F + \rho g\xi_{k+1}$, plus the liquid column pressure in (k), $\Delta P_k = \rho g A_k L_k \sin\theta$, where $L_k = v_k\Delta t$ is the liquid width in element (k), $A_k L_k = \mathcal{V}$. Thus, impulse $\mathbf{I}_k$ exerted on element (k) is:

$$\mathbf{I}_k = [(P_k\mathbf{A}_k - P_{k+1}\mathbf{A}_{k+1}) + \rho\mathbf{g}A_k L_k + \mathbf{n}_r]\Delta t\,. \tag{57}$$

Then, the NSL equation for element (k) is:

$$\begin{pmatrix} I_k\cos\theta \\ I_k\sin\theta \\ 0 \end{pmatrix} = \left[\begin{pmatrix} P_kA_k\cos\theta \\ P_kA_k\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} -P_{k+1}A_{k+1}\cos\theta \\ -P_{k+1}A_{k+1}\sin\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\mathcal{M}g \\ 0 \end{pmatrix} + \begin{pmatrix} n_k\cos\theta \\ n_k\sin\theta \\ 0 \end{pmatrix}\right]\Delta t\,. \tag{58}$$

We have the set of local NSL equations, one per element:

$$\begin{aligned}
\mathcal{M}\gamma_{v_1}\mathbf{v}_1 - \mathcal{M}\gamma_{v_\mathrm{I}}\mathbf{v}_\mathrm{I} &= (P_\mathrm{I}\mathbf{A}_\mathrm{I} - P_1\mathbf{A}_1)\Delta t - \rho\mathbf{g}A_\mathrm{I}L_\mathrm{I}\Delta t + \mathbf{n}_\mathrm{I}\Delta t\,,\\
\mathcal{M}\gamma_{v_2}\mathbf{v}_2 - \mathcal{M}\gamma_{v_1}\mathbf{v}_1 &= (P_1\mathbf{A}_1 - P_2\mathbf{A}_2)\Delta t - \rho\mathbf{g}A_1L_1\Delta t + \mathbf{n}_1\Delta t\,,\\
&\dots\\
\mathcal{M}\gamma_{v_\mathrm{k}}\mathbf{v}_\mathrm{k} - \mathcal{M}\gamma_{v_{\mathrm{k}-1}}\mathbf{v}_{\mathrm{k}-1} &= (P_{\mathrm{k}-1}\mathbf{A}_{\mathrm{k}-1} - P_\mathrm{k}\mathbf{A}_\mathrm{k})\Delta t - \rho\mathbf{g}A_{\mathrm{k}-1}L_{\mathrm{k}-1}\Delta t + \mathbf{n}_{\mathrm{k}-1}\Delta t\,,\\
\mathcal{M}\gamma_{v_{\mathrm{k}+1}}\mathbf{v}_{\mathrm{k}+1} - \mathcal{M}\gamma_{v_\mathrm{k}}\mathbf{v}_\mathrm{k} &= (P_\mathrm{k}\mathbf{A}_\mathrm{k} - P_{\mathrm{k}+1}\mathbf{A}_{\mathrm{k}+1})\Delta t - \rho\mathbf{g}A_\mathrm{k}L_\mathrm{k}\Delta t + \mathbf{n}_\mathrm{k}\Delta t\,,\\
&\dots\\
\mathcal{M}\gamma_{v_\mathrm{r}}\mathbf{v}_\mathrm{r} - \mathcal{M}\gamma_{v_{\mathrm{r}-1}}\mathbf{v}_{\mathrm{r}-1} &= (P_{\mathrm{r}-1}\mathbf{A}_{\mathrm{r}-1} - P_\mathrm{r}\mathbf{A}_\mathrm{r})\Delta t - \rho\mathbf{g}A_{\mathrm{r}-1}L_{\mathrm{r}-1}\Delta t + \mathbf{n}_{\mathrm{r}-1}\Delta t\\
\mathcal{M}\gamma_{v_\mathrm{F}}\mathbf{v}_\mathrm{F} - \mathcal{M}\gamma_{v_\mathrm{r}}\mathbf{v}_\mathrm{r} &= (P_\mathrm{r}\mathbf{A}_\mathrm{r} - P_\mathrm{F}\mathbf{A}_\mathrm{F})\Delta t - \rho\mathbf{g}A_\mathrm{r}L_\mathrm{r}\Delta t - \mathbf{n}_\mathrm{r}\Delta t\,.
\end{aligned} \tag{59}$$

Adding over the elements equations, pressure forces inside the pipe cancel each other, $\mathbf{n}_\mathrm{k}$ forces are added, the process global equation is obtained, involving just elements (I) and (F):

$$\mathcal{M}(\gamma_{v_\mathrm{F}}\mathbf{v}_\mathrm{F}) - \mathcal{M}(\gamma_{v_\mathrm{I}}\mathbf{v}_\mathrm{I}) = [(P_\mathrm{I}\mathbf{A}_\mathrm{I} - P_\mathrm{F}\mathbf{A}_\mathrm{F}) + r\mathcal{M}\mathbf{g} + \mathbf{N}]\Delta t\,. \tag{60}$$

In matrix form:

$$\begin{pmatrix}\mathcal{M}\gamma_{v_\mathrm{F}}v_\mathrm{F}\sin\theta\\ \mathcal{M}\gamma_{v_\mathrm{F}}v_\mathrm{F}\cos\theta\\ 0\end{pmatrix} - \begin{pmatrix}\mathcal{M}\gamma_{v_\mathrm{I}}v_\mathrm{I}\sin\theta\\ \mathcal{M}\gamma_{v_\mathrm{I}}v_\mathrm{I}\cos\theta\\ 0\end{pmatrix} = \left[\begin{pmatrix}P_\mathrm{I}A_\mathrm{I}\cos\theta\\ P_\mathrm{I}A_\mathrm{I}\sin\theta\\ 0\end{pmatrix} - \begin{pmatrix}P_\mathrm{F}A_\mathrm{F}\cos\theta\\ P_\mathrm{F}A_\mathrm{F}\sin\theta\\ 0\end{pmatrix} - \begin{pmatrix}0\\ r\mathcal{M}g\\ 0\end{pmatrix} + \begin{pmatrix}N_x\\ N_y\\ 0\end{pmatrix}\right]\Delta t\,. \tag{61}$$

For element (k):

$$\mathcal{M}(\gamma_{v_\mathrm{k}}\mathbf{v}_\mathrm{k}) - \mathcal{M}(\gamma_{v_\mathrm{I}}\mathbf{v}_\mathrm{I}) = [(P_\mathrm{I}\mathbf{A}_\mathrm{I} - P_\mathrm{k}\mathbf{A}_\mathrm{k}) + k\mathcal{M}\mathbf{g} + \mathbf{N}_\mathrm{k}]\Delta t\,, \tag{62}$$

with $\mathbf{N}_\mathrm{k} = \Sigma_{\mathrm{r}=1}^{\mathrm{r}=\mathrm{k}}\mathbf{n}_\mathrm{r}$,

$$\begin{pmatrix}\mathcal{M}\gamma_{v_\mathrm{k}}v_\mathrm{k}\sin\theta\\ \mathcal{M}\gamma_{v_\mathrm{k}}v_\mathrm{k}\cos\theta\\ 0\end{pmatrix} - \begin{pmatrix}\mathcal{M}\gamma_{v_\mathrm{I}}v_\mathrm{I}\sin\theta\\ \mathcal{M}\gamma_{v_\mathrm{I}}v_\mathrm{I}\cos\theta\\ 0\end{pmatrix} = \left[\begin{pmatrix}P_\mathrm{I}A_\mathrm{I}\cos\theta\\ P_\mathrm{I}A_\mathrm{I}\sin\theta\\ 0\end{pmatrix} - \begin{pmatrix}P_\mathrm{k}A_\mathrm{k}\cos\theta\\ P_\mathrm{k}A_\mathrm{k}\sin\theta\\ 0\end{pmatrix} - \begin{pmatrix}0\\ k\mathcal{M}g\\ 0\end{pmatrix} + \begin{pmatrix}N_{x|\mathrm{k}}\\ N_{y|\mathrm{k}}\\ 0\end{pmatrix}\right]\Delta t\,. \tag{63}$$

For a fluid vertical cylinder ($\theta = \pi/2$), with $\mathbf{n}_\mathrm{k} = 0$, $A_\mathrm{k} = A$ and $L_\mathrm{k} = L$, with $\Sigma_\mathrm{k}L_\mathrm{k} = rL = h_\mathrm{t} - h_\mathrm{b}$ [(t) $\equiv$ top, (b) $\equiv$ bottom], the hydrostatic equation $P_\mathrm{b}A - (P_\mathrm{t}A + r\mathcal{M}g) = 0$, with $L = (h_\mathrm{t} - h_\mathrm{b})/r$, and $AL = \mathcal{V}$, $\rho = \mathcal{M}/\mathcal{V}$, transforms into $P_\mathrm{b} = P_\mathrm{t} + \rho g(h_\mathrm{t} - h_\mathrm{b})$.

Newton's second law complementary dynamical relationship. Multiplying each element (k) NSL equation by $\mathbf{v}_\mathrm{k}$ and from relationship [20]:

$$\mathbf{v}_\mathrm{k}\cdot\Delta(\gamma_{v_\mathrm{k}}\mathbf{v}_\mathrm{k}) = \Delta(\gamma_{v_\mathrm{k}}c^2)\,, \tag{64}$$

one has:

$$\begin{aligned}
\mathcal{M}\mathbf{v}_\mathrm{k}\cdot\Delta(\gamma_{v_\mathrm{k}}\mathbf{v}_\mathrm{k}) &= \{(P_\mathrm{k}\mathbf{A}_\mathrm{k} - P_{\mathrm{k}+1}\mathbf{A}_{\mathrm{k}+1}) - \rho\mathbf{g}A_\mathrm{k}L_\mathrm{k}\sin\theta + \mathbf{n}_\mathrm{k}\}\cdot\mathbf{v}_\mathrm{k}\Delta t \;\rightarrow\\
\mathcal{M}\Delta(\gamma_{v_\mathrm{k}}c^2) &\approx [P_\mathrm{k}(\mathbf{A}_\mathrm{k}\cdot\mathbf{L}_\mathrm{k}) - P_{\mathrm{k}+1}(\mathbf{A}_{\mathrm{k}+1}\cdot\mathbf{L}_{\mathrm{k}+1}) + (\mathbf{v}_\mathrm{k}\cdot\mathbf{n}_\mathrm{k})]\Delta t\,.
\end{aligned} \tag{65}$$

Then, with $(\mathbf{g}\cdot\mathbf{v}_\mathrm{k})A_\mathrm{k}\Delta t = gA_\mathrm{k}L_\mathrm{k}\sin\theta = g\mathcal{V}\sin\theta$, $A_\mathrm{k}L_\mathrm{k} = \mathcal{V}$ and assuming $A_\mathrm{k} \approx A_{\mathrm{k}\pm1}$ and $L_\mathrm{k} \approx L_{\mathrm{k}\pm1}$, the set of NSL-CDR equations is obtained:

$$\begin{aligned}
\mathcal{M}\Delta(\gamma_{v_\mathrm{I}}c^2) &\approx [P_\mathrm{I} - P_1]\mathcal{V} - \mathcal{M}\sin\theta v_\mathrm{I}\Delta t + (\mathbf{v}_\mathrm{I}\cdot\mathbf{n}_\mathrm{I})\Delta t\,,\\
\mathcal{M}\Delta(\gamma_{v_1}c^2) &\approx [P_1 - P_2]\mathcal{V} - \mathcal{M}\sin\theta v_1\Delta t + (\mathbf{v}_1\cdot\mathbf{n}_1)\Delta t\,,\\
&\dots\\
\mathcal{M}\Delta(\gamma_{v_\mathrm{k}}c^2) &\approx [P_\mathrm{k} - P_{\mathrm{k}+1}]\mathcal{V} - \mathcal{M}\sin\theta v_\mathrm{k}\Delta t + (\mathbf{v}_\mathrm{k}\cdot\mathbf{n}_\mathrm{k})\Delta t\,,\\
\mathcal{M}\Delta(\gamma_{v_{\mathrm{k}+1}}c^2) &\approx [P_{\mathrm{k}+1} - P_\mathrm{k}]\mathcal{V} - \mathcal{M}\sin\theta v_{\mathrm{k}+1}\Delta t + (\mathbf{v}_{\mathrm{k}+1}\cdot\mathbf{n}_{\mathrm{k}+1})\Delta t\,,\\
&\dots\\
\mathcal{M}\Delta(\gamma_{v_{\mathrm{r}-1}}c^2) &\approx [P_{\mathrm{r}-1} - P_\mathrm{r}]\mathcal{V} - \mathcal{M}\sin\theta v_{\mathrm{r}-1}\Delta t + (\mathbf{v}_{\mathrm{r}-1}\cdot\mathbf{n}_{\mathrm{r}-1})\Delta t\,,\\
\mathcal{M}\Delta(\gamma_{v_\mathrm{r}}c^2) &\approx [P_\mathrm{r} - P_\mathrm{F}]\mathcal{V} - \mathcal{M}\sin\theta v_\mathrm{r}\Delta t + (\mathbf{v}_\mathrm{r}\cdot\mathbf{n}_\mathrm{r})\Delta t\,.
\end{aligned} \tag{66}$$

Finally, assuming:

$$\mathcal{M}\Delta(\gamma_{v_{\rm k}}c^2) \equiv \mathcal{M}\gamma_{v_{\rm k+1}}c^2 - \mathcal{M}\gamma_{v_{\rm k}}c^2\,, \tag{67}$$

and adding over elements, the whole NSL-CDR is obtained:

$$\gamma_{v_{\rm F}}\mathcal{M}c^2 - \gamma_{v_{\rm I}}\mathcal{M}c^2 = (P_{\rm I} - P_{\rm F})\mathcal{V} - \mathcal{M}g(h_{\rm F} - h_{\rm I}) + [\Sigma_{\rm k}(\mathbf{v}_{\rm k}\cdot\mathbf{n}_{\rm k})]\Delta t\,. \tag{68}$$

The process pseudo-work is thus:

$${\rm p}W = [\Sigma_{\rm k}(\mathbf{v}_{\rm k}\cdot\mathbf{n}_{\rm k})]\Delta t\,. \tag{69}$$

First law of thermodynamics. The fluid in element(k) occupies the volume of element $(\mathrm{k}+1)$, varying its kinetic energy as:

$$\Delta K_{\rm k} \equiv \mathcal{M}\Delta(\gamma_{v_{\rm k}}c^2) = \mathcal{M}\gamma_{v_{\rm k+1}}c^2 - \mathcal{M}\gamma_{v_{\rm k}}c^2\,, \tag{70}$$

due to work performed on it:

$$W_{\rm k} = \{[P_{\rm k}\mathbf{A}_{\rm k} - P_{\rm k+1}\mathbf{A}_{\rm k+1}] - \rho\mathbf{g}A_{\rm k}L_{\rm k}\}\cdot\mathbf{v}_{\rm k}\Delta t\,, \tag{71}$$

being $W_{\rm k} = \rho g A_{\rm k}L_{\rm k}v_{\rm k}\Delta t = \mathcal{M}gv_{\rm k}\Delta t = \mathcal{M}gL_{\rm k}\sin\theta$ the work to raise the liquid of the element (k) itself by distance $L_{\rm k}\sin\theta$.

The local set of equations for the FLT of the process is then:

$$\begin{aligned}
\mathcal{M}\gamma_{v_1}c^2 - \mathcal{M}\gamma_{v_{\rm I}}c^2 &= [P_{\rm I}(\mathbf{A}_{\rm I}\cdot\mathbf{v}_{\rm I}) - P_1(\mathbf{A}_1\cdot\mathbf{v}_1)]\Delta t - \rho g A_{\rm I}L_{\rm I}v_{\rm I}\sin\theta\Delta t + Q_{\rm I}\,,\\
\mathcal{M}\gamma_{v_2}c^2 - \mathcal{M}\gamma_{v_1}c^2 &= [P_1(\mathbf{A}_1\cdot\mathbf{v}_1) - P_2(\mathbf{A}_2\cdot\mathbf{v}_2)]\Delta t - \rho g A_1L_1v_1\sin\theta\Delta t + Q_1\,,\\
&\dots\\
\mathcal{M}\gamma_{v_{\rm k+1}}c^2 - \mathcal{M}\gamma_{v_{\rm k}}c^2 &= [P_{\rm k}(\mathbf{A}_{\rm k}\cdot\mathbf{v}_{\rm k}) - P_{\rm k+1}(\mathbf{A}_{\rm k+1}\cdot\mathbf{v}_{\rm k}+1)]\Delta t - \rho g A_{\rm k}L_{\rm k}v_{\rm k}\sin\theta\Delta t + Q_{\rm k}\,,\\
\mathcal{M}\gamma_{v_{\rm k+2}}c^2 - \mathcal{M}\gamma_{v_{\rm k+1}}c^2 &= [P_{\rm k+1}(\mathbf{A}_{\rm k+1}\cdot\mathbf{v}_{\rm k+1}) - P_{\rm k+2}(\mathbf{A}_{\rm k+2}\cdot\mathbf{v}_{\rm k}+2)]\Delta t - \rho g A_{\rm k+1}L_{\rm k+1}v_{\rm k+1}\sin\theta\Delta t + Q_{\rm k+1}\,,\\
&\dots\\
\mathcal{M}\gamma_{v_{\rm r}}c^2 - \mathcal{M}\gamma_{v_{\rm r-1}}c^2 &= [P_{\rm r-1}(\mathbf{A}_{\rm r-1}\cdot\mathbf{v}_{\rm r-1}) - P_{\rm r}(\mathbf{A}_{\rm r}\cdot\mathbf{v}_{\rm r})]\Delta t - \rho g A_{\rm r-1}L_{\rm r-1}v_{\rm r-1}\sin\theta\Delta t + Q_{\rm r-1}\\
\mathcal{M}\gamma_{v_{\rm F}}c^2 - \mathcal{M}\gamma_{v_{\rm r}}c^2 &= [P_{\rm r}(\mathbf{A}_{\rm r}\cdot\mathbf{v}_{\rm r}) - P_{\rm F}(\mathbf{A}_{\rm F}\cdot\mathbf{v}_{\rm F})]\Delta t - \rho g A_{\rm r}L_{\rm r}v_{\rm r}\sin\theta\Delta t + Q_{\rm r}\,.
\end{aligned} \tag{72}$$

Adding over the FLT equations of the fluid elements, the FLT for the process is obtained:

$$\begin{aligned}
&(\gamma_{v_{\rm F}} - \gamma_{v_{\rm I}})\mathcal{M}c^2 = (P_{\rm I} - P_{\rm F})\mathcal{V} - \mathcal{M}g(h_{\rm F} - h_{\rm I}) + Q\ \rightarrow\\
&\rho(\gamma_{v_{\rm F}} - \rho\gamma_{v_{\rm I}})c^2 = (P_{\rm I} - P_{\rm F}) - \rho g(h_{\rm F} - h_{\rm I}) + q\,,
\end{aligned} \tag{73}$$

where $Q = \Sigma_{\rm k}Q_{\rm k} = q\mathcal{V}$.

By comparing NSL-CDR Eq. (68) with the FLT Eq. (73), one has:

$$Q = {\rm p}W\,.$$

The process pseudo-work pW quantifies the mechanical energy dissipated and emitted to the thermal reservoir by heat, Q.

Four-vector fundamental equation for fluid element (k). The FVFE for element (k) is given by:

$$\begin{aligned}
&\begin{pmatrix} c\gamma_{v_{\rm k+1}}\mathcal{M}v_{\rm k+1}\cos\theta\\ c\gamma_{v_{\rm k+1}}\mathcal{M}v_{\rm k+1}\sin\theta\\ 0\\ \gamma_{v_{\rm k+1}}\mathcal{M}c^2\end{pmatrix} - \begin{pmatrix} c\gamma_{v_{\rm k}}\mathcal{M}v_{\rm k}\cos\theta\\ c\gamma_{v_{\rm k}}\mathcal{M}v_{\rm k}\sin\theta\\ 0\\ \gamma_{v_{\rm k}}\mathcal{M}c^2\end{pmatrix} = \begin{pmatrix} c(P_{\rm k}A_{\rm k} - P_{\rm k+1}A_{\rm k+1})\cos\theta\Delta t\\ c(P_{\rm k}A_{\rm k} - P_{\rm k+1}A_{\rm k+1})\sin\theta\Delta t\\ 0\\ (P_{\rm k}\mathbf{A}_{\rm k} - P_{\rm k+1}\mathbf{A}_{\rm k+1})\cdot\mathbf{v}_{\rm k}\Delta t\end{pmatrix} +\\
&\qquad \begin{pmatrix} 0\\ -c\mathcal{M}g\Delta t\\ 0\\ -\mathcal{M}gL_{\rm k}\sin\theta\end{pmatrix} + \begin{pmatrix} cn_{\rm k}\cos\theta\Delta t\\ cn_{\rm k}\sin\theta\Delta t\\ 0\\ 0\end{pmatrix} + \begin{pmatrix} 0\\ 0\\ 0\\ Q_{\rm k}\end{pmatrix}\,,
\end{aligned} \tag{74}$$

with volume $\mathcal{V} = (\mathbf{A}_\mathrm{k} \cdot \mathbf{v}_\mathrm{k})\Delta t = \mathbf{A}_\mathrm{k} \cdot \mathbf{L}_\mathrm{k} = A_\mathrm{k} L_\mathrm{k} = \mathcal{V}$, and inertia $\mathcal{M} = \rho\mathcal{V}$.

Four-vector fundamental equation for the Bernoulli process. Adding over all elements, the four-vector fundamental equation for a generalised Bernoulli process is obtained:

$$\begin{pmatrix} c\gamma_{v_\mathrm{F}}\mathcal{M}v_\mathrm{F}\cos\theta \\ c\gamma_{v_\mathrm{F}}\mathcal{M}v_\mathrm{F}\sin\theta \\ 0 \\ \gamma_{v_\mathrm{F}}\mathcal{M}c^2 \end{pmatrix} - \begin{pmatrix} c\gamma_{v_\mathrm{I}}\mathcal{M}v_\mathrm{I}\cos\theta \\ c\gamma_{v_\mathrm{I}}\mathcal{M}v_\mathrm{I}\sin\theta \\ 0 \\ \gamma_{v_\mathrm{I}}\mathcal{M}c^2 \end{pmatrix} = \begin{pmatrix} cP_\mathrm{I}A_\mathrm{I}\cos\theta\Delta t \\ cP_\mathrm{I}A_\mathrm{I}\sin\theta\Delta t \\ 0 \\ P_\mathrm{I}\mathcal{V} \end{pmatrix} - \begin{pmatrix} cP_\mathrm{F}A_\mathrm{F}\cos\theta\Delta t \\ cP_\mathrm{F}A_\mathrm{F}\sin\theta\Delta t \\ 0 \\ P_\mathrm{F}\mathcal{V} \end{pmatrix} + \begin{pmatrix} 0 \\ -cr\mathcal{M}g\Delta t \\ 0 \\ -\mathcal{M}gL_\mathrm{p}\sin\theta \end{pmatrix} + \begin{pmatrix} cN_x\Delta t \\ cN_y\Delta t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q \end{pmatrix}, \quad (75)$$

where $\mathbf{N} = \Sigma_\mathrm{k}\mathbf{n}_\mathrm{k}$.

The description of the process is assumed as if fluid inertia were taken from point (I) to point (F) without changing the state of the fluid inside the pipe. By applying the principle of locality, a many-body problem with (r+2) elements is reduced to a one-body problem.

The role of the tube in the process is considered through force **N**, which determine the thermal, dissipative effects. The result depends on the geometry of the model.

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