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GLOBAL EXISTENCE FOR CERTAIN FOURTH ORDER EVOLUTION EQUATIONS

Rafael Granero-Belinchón*[1] and Martina Magliocca[2]

[1]Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Spain

[2]Departamento de Análisis Matemático, Universidad de Sevilla, Spain

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Abstract. In this paper we established three global in time results for two fourth order nonlinear parabolic equations. The first of such equations involved the Hessian and appeared in epitaxial growth. For such an equation, we gave conditions ensuring the global existence of the solution. For certain regime of the parameters, our size condition involved the norm in a critical space with respect to the scaling of the equation and improved previous existing results in the literature for this equation. The second of the equations under study was a thin film equation with a porous medium nonlinearity. For this equation, we established conditions leading to the global existence of the solution.

1. Introduction and main results.

High order partial differential equations (PDEs) are a very interesting topic due to their many applications such as beam dynamics, thin films [17, 18, 19, 20], crystal dynamics [14, 16], and many others. From a mathematical viewpoint, its study is more challenging that standard second order PDEs, for instance, due to its lack of maximum principles and some other features.

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*Corresponding author: Rafael Granero-Belinchón.

Our main goal is proving existence and decay results in Wiener spaces to the following problems:

$$\begin{cases} \partial_t u = K_0 \Delta u + 2K_1 \det D^2 u - K_2 \Delta^2 u - \dfrac{K_3}{2} \Delta \left(\Delta u\right)^2 & \text{in} \quad (0,T) \times \mathbb{T}^2, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{T}^2, \end{cases} \tag{1.1}$$

with

$$K_0 \geq 0, \quad K_1 \geq 0, \quad K_2 > 0, \quad K_3 \geq 0,$$

and

$$\begin{cases} \partial_t u = -\mathrm{div}\left(u \nabla \Delta u\right) - \chi \Delta u^p & \text{in} \quad (0,T) \times \mathbb{T}^2, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{T}^2, \end{cases} \tag{1.2}$$

with

$$\chi > 0.$$

Equation in (1.1) models epitaxial growth, and its geometrical derivation can be found in [9, 10] and [11, Section 2] (where the meaning of the constants K_i is explained). Roughly speaking, this growth process consists in the superposition of layers due to deposition of new material, all under high vacuum conditions. As pointed out in [10], this phenomenon has several applications such as crystal growth. Crystal surfaces are made up of terraces separated by steps of atomic height. These steps contain straight parts separated by kinks. On the terraces, there are surface vacancies resulting from missing surface atoms. Under ultra-high vacuum conditions, atoms are sent onto the surface, and they diffuse until they are incorporated.
The authors of this work considered the case $K_0 = K_3 = 0$, and they proved the existence of solutions to

$$\partial_t u = \det D^2 u - \Delta^2 u + f(t,x) \qquad \text{in} \quad (0,T) \times \Omega, \qquad \Omega \subsetneq \mathbb{R}^2,$$

which are global in time under smallness assumptions on the data, or local in time with arbitrary data. They assumed H^2 initial data, and they studied both the homogeneous case $f \equiv 0$ and the case $f \in L^2(0,T;L^2(\Omega))$ with Dirichlet or Navier boundary conditions. When $f \equiv 0$ and $\|u_0\|_{H^2}$ is large enough, they also proved that the $W^{1,4}$ norm of the solution blows up in finite time. Later, Escudero [9] improved the blow up result previously obtained in [10].

Another interesting model of which is linked to (1.2) is contained in [21]. Here, the authors studied a problem modeling the effect of odd viscosity on the instability of liquid film along a wavy inclined bottom with linear temperature variation, and they found that the free boundary evolution equation verifies the following asymptotic equation:

$$\partial_t u = -A(u)\partial_x u - \alpha \partial_x \left(B(u)\partial_x u + C(u)\partial_x^3 u\right),$$

which, for appropriate choice of A, B, and C, is equivalent to (1.2).

There exists a huge literature concerning equations as in (1.2). Equation

$$\partial_t u = -\mathrm{div}\left(u^n \nabla \Delta u\right) - \chi \Delta u^p, \qquad u > 0, \quad \chi \in \mathbb{R}, \tag{1.3}$$

describes the evolution of the thin-film liquid height u spreading on a solid surface. The fourth order term takes into account the surface tension, while the porous medium one is related to the Van der Waals forces.

The case $\chi = 0$, $n = 1$ has been deeply studied in [17]. The author also provides the derivation of the one-dimensional model, and a detailed description of the physical experiment motivating the interest of the equation itself. The cases with $\chi = 0$ and $n \in (0, 2)$ and $n \in [2, 3)$ can be found in [6, 15], respectively.

The case $\chi = -1$ in (1.3) has been dealt with in [7], and the relation among the positive parameters p and n has been investigated.

The blow up of solutions has been proved in the one-dimensional case and for $\chi = 1$ in [3] when $p \geq n + 3$, and this result has been refined in [20] in the critical case $p = n + 3$.
Always in the case $\chi = 1$, the existence of self-similar solutions to (1.3), as well as blow up results, are contained in [19] with $p = n + 3$, [18] with $0 < n < 3$, $n \leq p$, and [12] for the case of the first critical exponent $p = n + 1 + 2/N$, $0 < n < 3/2$, $N \geq 1$.

Finally, equations as (1.2) are related also to approximations of nonlocal aggregation-diffusion models [2, 8] and tumor growth [5, 13].

In this paper, we are going to establish the global existence of weak solutions and decay assuming Wiener initial data. The main advantage of these spaces when compared to classical estimates on L^2 based Sobolev spaces is that Wiener spaces usually allow to reach the critical functional space with respect to the scaling invariance of the equation. The technique we are going to apply is contained in [14] (see also [4, 16, 1]). We present the notions of definitions of weak solutions we consider below.

Definition 1.1 (Weak solution to (1.1))**.** We say that a function u is a weak solution of (1.1) if

$$u \in L^\infty((0, T) \times \mathbb{T}^2) \cap L^2(0, T; W^{2,4}(\mathbb{T}^2))$$

and verifies the following weak formulation:

$$\int_{\mathbb{T}^2} u_0 \varphi(0)\, dx + \iint_{\mathbb{T}^2 \times (0,T)} u \partial_t \varphi + \left(K_0 u + \frac{K_3}{2} (\Delta u)^2 \right) \Delta \varphi + 2K_1 \varphi \det D^2 u - K_2 u \Delta^2 \varphi \, dx\, dt = 0$$

for every

$$\varphi \in W^{1,1}(0, T; L^1(\mathbb{T}^2)) \cap L^1(0, T; W^{4,1}(\mathbb{T}^2)) \cap L^2(0, T; H^2(\mathbb{T}^2)).$$

Taking advantage of the fact that the equation (1.2) conserves the mean, we define the new variable

$$v(t, x) = u(t, x) - \frac{1}{4\pi^2} \int u_0(x)\, dx.$$

Without loss of generality from this point onward we assume that

$$\frac{1}{4\pi^2} \int u_0(x) = 1.$$

Hence, (1.2) becomes

$$\begin{cases} \partial_t v = -\Delta^2 v - \operatorname{div}(v \nabla \Delta v) - \chi \Delta(1 + v)^p & \text{in} \quad (0, T) \times \mathbb{T}^2, \\ v(0, x) = v_0(x) = u_0(x) - 1 & \text{in} \quad \mathbb{T}^2. \end{cases} \tag{1.4}$$

Definition 1.2 (Weak solution to (1.4))**.** We say that a function v is a weak solution of (1.4) if

$$v \in L^\infty((0,T)\times\mathbb{T}^2)\cap L^2(0,T;H^2(\mathbb{T}^2)),$$

and verifies the following weak formulation:

$$-\int_{\mathbb{T}^2} v_0\varphi(0)\,dx + \iint_{\mathbb{T}^2\times(0,T)} -v\partial_t\varphi - v\nabla\Delta v\cdot\nabla\varphi + \chi(v+1)^{p-1}\Delta\varphi\,dx\,dt = 0$$

for every

$$\varphi \in W^{1,1}(0,T;L^1(\mathbb{T}^2))\cap L^{q'}(0,T;H^2(\mathbb{T}^2)) \quad \text{with} \quad 1\le q<2.$$

The k-th Fourier coefficients of a 2π-periodic function on $\mathbb{T}^d$ are

$$\widehat{u}(k) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} u(x)e^{-ik\cdot x}dx,$$

and the Fourier series expansion of u is given by

$$u(x) = \sum_{k\in\mathbb{Z}^d}\widehat{u}(k)e^{ik\cdot x}.$$

Using this, we define the Wiener spaces for $s\ge 0$:

$$A^s = \left\{ u\in L^1(\mathbb{T}^d) : \|u\|_{\dot{A}^s} = \sum_{k\in\mathbb{Z}^d}|k|^s\,|\widehat{u}(k)| < \infty \right\}.$$

We note that A^0 is a Banach algebra and, furthermore,

$$A^s \subset C^s \subset H^s.$$

We will largely make use of the interpolation inequality [4, Lemma 2.1]:

$$\|u\|_{A^p} \le \|u\|_{A^0}^{1-\theta}\|u\|_{A^q}^{\theta} \quad \text{for} \quad 0\le p\le q,\ \theta=\frac{p}{q}.$$

Theorems 1.3 and 1.4 contain two existence results for problem (1.1), and the main difference concerns the regularity of the initial data.

Theorem 1.3 (Existence result to (1.1) for $K_3>0$)**.** *Let $K_3>0$ in (1.1) and consider $u_0\in A^2$ is a zero mean initial data such that*

$$K_2 - 2(K_3+K_1)\|u_0\|_{A^2} > 0 \qquad \text{if } K_0=0, \tag{1.5}$$

and

$$\min\left\{K_2 - 2K_3\|u_0\|_{A^2},\, K_0 - 2K_1\|u_0\|_{A^2}\right\} > 0 \qquad \text{if } K_0>0.$$

Then, there exists at least one global weak solution of (1.1) in the sense of Definition 1.1

$$u\in L^\infty(0,T;A^2)\cap L^1(0,T;A^6) \quad \forall T.$$

Furthermore, the solution satisfies

$$\|u(t)\|_{A^2} \le e^{-(K_2-2(K_3+K_1)\|u_0\|_{A^2})t}\,\|u_0\|_{A^2} \qquad \text{if } K_0=0,$$

and

$$\|u(t)\|_{A^2} \le e^{-\min\left\{K_2-2K_3\|u_0\|_{A^2},K_0-2K_1\|u_0\|_{A^2}\right\}t}\,\|u_0\|_{A^2} \qquad \text{if } K_0>0.$$

Similarly, in the case $K_3=0$, we have that:

Theorem 1.4 (Existence result to (1.1) for $K_3 = 0$)**.** *Let $K_3 = 0$ in (1.1) and consider $u_0 \in A^0$ is a zero mean initial data such that*

$$K_2 - 2K_1 \|u_0\|_{A^0} > 0.$$

Then, there exists at least one global weak solution of (1.1) in the sense of Definition 1.1

$$u \in L^\infty(0,T;A^0) \cap L^1(0,T;A^4) \quad \forall T.$$

Furthermore, the solution satisfies

$$\|u(t)\|_{A^0} \leq e^{-(K_2-2K_1\|u_0\|_{A^0})t} \|u_0\|_{A^0}.$$

This particular result improves the previous global in time result contained in [10] due to the fact that our size condition is given in A^0 instead of H^2. In fact, the space A^0 is a critical space with respect to the scaling of the equation

$$u_\lambda(x,t) = u(\lambda x, \lambda^4 t).$$

We now present our existence result concerning problem (1.4).

Theorem 1.5 (Existence result to (1.4))**.** *Let $0 \leq u_0 \in A^0$ be a an initial data satisfying*

$$\frac{1}{4\pi^2}\int u_0 dx = 1$$

together with the smallness condition

$$1 - 2\|v_0\|_{A^0} - \frac{c\chi p!}{2}\left(\|v_0\|_{A^0} + 2\sum_{q=1}^{p-1}\|v_0\|_{A^0}^q\right) > 0.$$

Then, there exists at least one global weak solution of (1.4) in the sense of Definition 1.2:

$$v \in L^\infty(0,T;A^0) \cap L^1(0,T;A^4) \quad \forall T.$$

Furthermore, the solution satisfies

$$\|v(t)\|_{A^0} \leq \exp\left(-\left(1 - 2\|v_0\|_{A^0} - c\chi p!\left(\|v_0\|_{A^0} + 2\sum_{q=1}^{p-1}\|v_0\|_{A^0}^q\right)\right)t\right)\|v_0\|_{A^0}.$$

The above results could be extended to whole $\mathbb{R}^2$ considering the Wiener spaces $\mathcal{A}^s$ defined through the Fourier transform:

$$\mathcal{A}^s = \left\{u \in L^1(\mathbb{T}^2) : \|u\|_{\dot{\mathcal{A}}^s} = \int_{\mathbb{R}^2} |\xi|^s |\widehat{u}(\xi)| < \infty\right\}. \tag{1.6}$$

For other boundary conditions, a similar approach could possibly be implemented using Wiener spaces defined via eigenfunctions of the bilaplacian with such boundary conditions.

In the following, we write

$$f_{,j} = \partial_{x_j} f$$

for the space derivative in the $j-$th direction.

2. **Proof of Theorems 1.3 and 1.4. Approximate problem:** We explicitly compute each term of (1.1):

$$\Delta u = u_{,ii}\,,$$
$$2\det D^2 u = u_{,ii}\,u_{,jj} - u_{,ij}\,u_{,ij}\,,$$
$$\Delta^2 u = u_{,iijj}\,,$$
$$\frac{1}{2}\Delta\left(\Delta u\right)^2 = u_{,ii}\,u_{,jjkk}\,.$$

Hence, (1.1) is equivalent to

$$\begin{cases} \partial_t u = K_0 u_{,ii} + K_1\left(u_{,ii}\,u_{,jj} - u_{,ij}\,u_{,ij}\right) - K_2 u_{,iijj} - K_3 u_{,ii}\,u_{,jjkk} & \text{in} \quad (0,T)\times\mathbb{T}^2, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{T}^2. \end{cases}$$

We consider the following approximating problem:

$$\begin{cases} \partial_t u^{(n)} = K_0 u^{(n)}{}_{,ii} + K_1 P_n\left(u^{(n)}{}_{,ii}\,u^{(n)}{}_{,jj} - u^{(n)}{}_{,ij}\,u^{(n)}{}_{,ij}\right) & \\ \qquad - K_2 u^{(n)}{}_{,iijj} - K_3 P_n\left(u^{(n)}{}_{,ii}\,u^{(n)}{}_{,jjkk}\right) & \text{in} \quad (0,T)\times\mathbb{T}^2, \\ u^{(n)}(0,x) = P_n u_0(x) & \text{in} \quad \mathbb{T}^2, \end{cases} \tag{2.1}$$

where P_n is the projector on the Fourier modes satisfying

$$|k| \leq n.$$

Using this Galerkin projection approximation, we obtain a (finite dimensional) nonlinear system of ordinary differential equations (ODEs). Indeed, it is enough to solve for the $2n+1$ Fourier modes of $u^{(n)}$. As this ODE system has a Lipschitz nonlinearity (it's merely a number of multiplications), the classical Picard theorem leads to the local existence of solution up to time T_n. So far, the solution to the approximate problems may exist locally in time. Furthermore, we have to discard that $\liminf T_n = 0$. However, we will show below that there exists a common and positive time interval of existence.

A priori estimates in Wiener spaces: Let us omit the superscript (n) in the following computations. We rewrite each term of the right hand side of (2.1) in Fourier:

$$\widehat{u_{,ii}\,u_{,jj} - u_{,ij}\,u_{,ij}}(t,k) = \sum_{m\in\mathbb{Z}^2}\Big(|m|^2\,|k-m|^2 - m_i m_j (k_i - m_i)(k_j - m_j)\Big)\times$$
$$\times\,\hat{u}(t,m)\hat{u}(t,k-m),$$
$$\widehat{u_{,iijj}}(t,k) = |k|^4\,\hat{u}(t,k),$$
$$\widehat{u_{,ii}}(t,k) = |k|^2\,\hat{u}(t,k),$$
$$\widehat{u_{,ii}\,u_{,jjkk}}(t,k) = -\sum_{m\in\mathbb{Z}^2}|m|^4|k-m|^2\hat{u}(t,m)\hat{u}(t,k-m).$$

Note that the contribution of the term

$$K_0\Delta u$$

is always negative in Wiener spaces.

Assume that $K_0 = 0$. Then, the Fourier coefficient of (2.1) is given by

$$\partial_t \widehat{u}(k,t) = K_1 \sum_{m\in\mathbb{Z}^2} \left(|m|^2 \, |k-m|^2 - m_i m_j (k_i - m_i)(k_j - m_j) \right) \widehat{u}(t,m)\widehat{u}(t,k-m)$$
$$- K_2 |k|^4 \widehat{u}(t,k) + K_3 \sum_{m\in\mathbb{Z}^2} |m|^4 |k-m|^2 \widehat{u}(t,k-m)\widehat{u}(t,m). \quad (2.2)$$

In order to estimate the A^2 semi-norm of $\partial_t u$, we use the fact that

$$\partial_t |\widehat{u}(t,k)| = \frac{Re\left(\overline{\widehat{u}}(t,k)\partial_t \widehat{u}(t,k)\right)}{|\widehat{u}(t,k)|}, \quad (2.3)$$

to obtain the estimates

$$\sum_{k\in\mathbb{Z}^2} |k|^2 \partial_t \, |\widehat{u}(k,t)| = \frac{d}{dt} \|u(t)\|_{A^2}, \qquad \sum_{k\in\mathbb{Z}^2} |k|^6 \, |\widehat{u}(t,k)| = \|u(t)\|_{A^6},$$

$$\sum_{k\in\mathbb{Z}^2} |k|^2 \left| \sum_{m\in\mathbb{Z}^2} \left(|m|^2 \, |k-m|^2 - m_i m_j (k_i - m_i)(k_j - m_j) \right) \widehat{u}(t,m)\widehat{u}(t,k-m) \right|$$
$$\leq 2 \|u(t)\|_{A^2} \|u(t)\|_{A^4},$$

and also, by Tonelli's Theorem,

$$\sum_{k\in\mathbb{Z}^2} |k|^2 \left| \sum_{m\in\mathbb{Z}^2} |m|^4 |k-m|^2 \widehat{u}(t,m)\widehat{u}(t,k-m) \right|$$
$$\leq \sum_{m\in\mathbb{Z}^2} |m|^4 \, |\widehat{u}(t,m)| \sum_{k\in\mathbb{Z}^2} |k|^2 |k-m|^2 \, |\widehat{u}(t,k-m)|$$
$$\leq \sum_{m\in\mathbb{Z}^2} |m|^4 \, |\widehat{u}(t,m)| \sum_{k\in\mathbb{Z}^2} |k-m|^4 \, |\widehat{u}(t,k-m)|$$
$$+ \sum_{m\in\mathbb{Z}^2} |m|^6 \, |\widehat{u}(t,m)| \sum_{k\in\mathbb{Z}^2} |k-m|^2 \, |\widehat{u}(t,k-m)|$$
$$\leq \|u(t)\|_{A^4}^2 + \|u(t)\|_{A^2} \|u(t)\|_{A^6}$$
$$\leq 2 \|u(t)\|_{A^2} \|u(t)\|_{A^6},$$

where the last inequality follows interpolating.

We gather the previous estimates, obtaining

$$\frac{d}{dt} \|u(t)\|_{A^2} \leq -\left(K_2 - 2K_3 \|u(t)\|_{A^2}\right) \|u(t)\|_{A^6} + 2K_1 \|u(t)\|_{A^2} \|u(t)\|_{A^4}. \quad (2.4)$$

We now estimate the last term in the r.h.s. of (2.4) as below,

$$2K_1 \|u(t)\|_{A^2} \|u(t)\|_{A^4} \leq 2K_1 \|u(t)\|_{A^2} \|u(t)\|_{A^6},$$

so that

$$\frac{d}{dt} \|u(t)\|_{A^2} \leq -\left(K_2 - 2(K_3 + K_1) \|u(t)\|_{A^2}\right) \|u(t)\|_{A^6}.$$

Thus, if $u_0 \in A^2(\mathbb{T}^2)$ is such that

$$K_2 - 2(K_3 + K_1) \|u_0\|_{A^2} > 0$$

and using a contradiction argument in time, we obtain that u is uniformly bounded in

$$W^{1,1}(0,T; A^2(\mathbb{T}^2)) \cap L^1(0,T; A^6(\mathbb{T}^2)), \quad (2.5)$$

and, furthermore, it decays

$$\|u(t)\|_{A^2} \leq e^{-(K_2-2(K_3+K_1)\|u_0\|_{A^2})t} \|u_0\|_{A^2}.$$

If $K_0 > 0$, we can improve the smallness condition (1.5) in the following way.

The term $-K_0 |k|^2 \widehat{u}(t,k)$ appears in the r.h.s. of (2.2). Then, reasoning as in the case $K_0 = 0$, the inequality in (2.4) takes the following form:

$$\frac{d}{dt}\|u(t)\|_{A^2} \leq -\left(K_2 - 2K_3 \|u(t)\|_{A^2}\right)\|u(t)\|_{A^6} - \left(K_0 - 2K_1 \|u(t)\|_{A^2}\right)\|u(t)\|_{A^4}.$$

We can thus avoid to estimate $\|u(t)\|_{A^4}$ with $\|u(t)\|_{A^6}$, requiring $u_0 \in A^2$ such that

$$\min\left\{K_2 - K_3 \|u_0\|_{A^2}, K_0 - 2K_1 \|u_0\|_{A^2}\right\} > 0.$$

Moreover, in the case where $K_0 = K_3 = 0$, we can improve the previous result and find that

$$\frac{d}{dt}\|u(t)\|_{A^0} \leq +2K_1 \|u(t)\|_{A^2}^2 - K_2 \|u(t)\|_{A^4}.$$

Using interpolation, we obtain that

$$\frac{d}{dt}\|u(t)\|_{A^0} \leq \left(-K_2 + 2K_1 \|u(t)\|_{A^0}\right)\|u(t)\|_{A^4},$$

from where we can conclude the desired estimates as before.

An analogous estimate holds in the case $K_0 > 0$.

Estimates for the approximate problem: Using that

$$\sum_k |P_n f| \leq \|f\|_{A^0},$$

we observe that the previous a priori estimates are valid for the approximate problem

$$\begin{cases} \partial_t u^{(n)} = K_0 u^{(n)}{}_{,ii} + K_1 P_n\left(u^{(n)}{}_{,ii}\, u^{(n)}{}_{,jj} - u^{(n)}{}_{,ij}\, u^{(n)}{}_{,ij}\right) & \\ \qquad -K_2 u^{(n)}{}_{,iijj} - K_3 P_n(u^{(n)}{}_{,ii}\, u^{(n)}{}_{,jjkk}) & \text{in} \quad (0,T)\times\mathbb{T}^2, \\ u^{(n)}(0,x) = P_n u_0(x) & \text{in} \quad \mathbb{T}^2. \end{cases}$$

Compactness results: Up to subsequences, we have that

$$u^{(n)} \to u \qquad \text{a.e. } (0,T)\times\mathbb{T}^2, \tag{2.6}$$

$$u^{(n)} \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(0,T;W^{2,\infty}(\mathbb{T}^2)), \tag{2.7}$$

$$u^{(n)} \overset{*}{\rightharpoonup} u \qquad \text{in } \mathcal{M}(0,T;W^{6,\infty}(\mathbb{T}^2)), \tag{2.8}$$

$$u^{(n)} \overset{*}{\rightharpoonup} u \qquad \text{in } L^{\frac{2p}{3p-6}}(0,T;W^{p,\infty}(\mathbb{T}^2)) \quad \text{with} \quad 2 < p < 6, \tag{2.9}$$

$$u^{(n)} \to u \qquad \text{in } L^2(0,T;H^2(\mathbb{T}^2)). \tag{2.10}$$

Indeed, the weak-$*$ convergences (2.7) and (2.8) follow from the uniform bound in (2.5) and the Banach-Alaoglu Theorem. We use the interpolation inequality to say that,

$$\left\|u^{(n)}(t) - u(t)\right\|_{A^p} \leq \left\|u^{(n)}(t) - u(t)\right\|_{A^2}^{\frac{6-p}{2p}} \left\|u^{(n)}(t) - u(t)\right\|_{A^6}^{\frac{3p-6}{2p}},$$

and we integrate in time

$$\int_0^T \left\|u^{(n)}(t) - u(t)\right\|_{A^p}^{\frac{2p}{3p-6}} dt \leq \left\|u^{(n)} - u\right\|_{L^\infty(A^2)}^{\frac{6-p}{3p-6}} \int_0^T \left\|u^{(n)}(t) - u(t)\right\|_{A^6} dt.$$

Then, (2.9) follows recalling (2.5) and observing that $\|f\|_{W^{p,\infty}} \leq \|f\|_{A^p}$. Similarly, the a.e. convergence in (2.6) follows from the previous ones.

We now focus on the strong convergence in (2.10). Interpolation inequality implies that

$$\left\|u^{(n)} - u\right\|_{L^2(H^2)} \leq \left\|u^{(n)} - u\right\|_{L^2(L^2)}^{\frac{1}{3}} \left\|u^{(n)} - u\right\|_{L^2(H^3)}^{\frac{2}{3}} \leq c \left\|u^{(n)} - u\right\|_{L^2(L^2)}^{\frac{1}{3}}$$

by (2.9) with $p = 3$. Then, we just have to prove the strong convergence

$$u^{(n)} \to u \qquad \text{in } L^2(0,T;L^2(\mathbb{T}^2)) \tag{2.11}$$

to deduce (2.10). We want to apply classical compactness results in order to get (2.11). Then, we need some spaces X and Y such that

$$u^{(n)} \text{ uniformly bounded } L^2(0,T;X),$$
$$\partial_t u^{(n)} \text{ uniformly bounded } L^1(0,T;Y),$$

verifying

$$X \overset{compact}{\hookrightarrow} L^2(\mathbb{T}^2) \hookrightarrow Y.$$

We set $X = H^2(\mathbb{T}^2)$, so the boundedness in $L^2(0,T;H^2(\mathbb{T}^2))$ follows from the one in $L^2(0,T;H^3(\mathbb{T}^2))$ and the finiteness of the domain. We choose $Y = H^{-2}(\mathbb{T}^2)$ and we prove the uniform boundedness of the time derivative using that

$$\|\partial_t u^{(n)}(t)\|_{H^{-2}} = \sup_{\substack{\varphi \in H^2(\mathbb{T}) \\ \|\varphi\|_{H^2} \leq 1}} \left|\langle \partial_t u_n(t), \varphi\rangle\right|.$$

Then, we estimate as

$$\begin{aligned}\left|\int_{\mathbb{T}^2} \partial_t u^{(n)} \varphi \, dx\right| &\leq c \int_{\mathbb{T}^2} \left(\left|u^{(n)}\right| + \left|\Delta u^{(n)}\right| + (\Delta u^{(n)})^2\right) |\Delta\varphi| \, dx \\ &\quad + c \int_{\mathbb{T}^2} \left|\det D^2 u^{(n)}\right| \varphi \, dx \\ &\leq c \left(1 + \left\|u^{(n)}\right\|_{W^{2,\infty}}\right) \left\|u^{(n)}\right\|_{H^2} \|\varphi\|_{H^2}.\end{aligned}$$

Then,

$$\left\|\partial_t u^{(n)}(t)\right\|_{H^{-2}} \leq c \left(1 + \left\|u^{(n)}\right\|_{W^{2,\infty}}\right) \left\|u^{(n)}\right\|_{H^2},$$

and

$$\left\|\partial_t u^{(n)}(t)\right\|_{L^2(H^{-2})} \leq c \left(1 + \left\|u^{(n)}\right\|_{L^\infty(W^{2,\infty})}\right) \left\|u^{(n)}\right\|_{L^2(H^2)} < c.$$

Passing to the limit: We want to take the limit in n in

$$\begin{aligned}&\int_{\mathbb{T}^2} P_n u_0 \varphi(0) \, dx + \iint_{\mathbb{T}^2 \times (0,T)} u^{(n)} \partial_t \varphi + \left(K_0 u^{(n)} + \frac{K_3}{2} (\Delta u^{(n)})^2\right) \Delta\varphi \, dx \, dt \\ &\quad + \iint_{\mathbb{T}^2 \times (0,T)} 2K_1 \varphi \det D^2 u^{(n)} - K_2 u^{(n)} \Delta^2 \varphi \, dx \, dt = 0,\end{aligned}$$

being $\varphi \in W^{1,1}(0,T;L^1(\mathbb{T}^2)) \cap L^1(0,T;W^{4,1}(\mathbb{T}^2)) \cap L^2(0,T;H^2(\mathbb{T}^2))$.

We only detail the convergence of

$$\iint_{\mathbb{T}^2 \times (0,T)} (\Delta u^{(n)})^2 \Delta\varphi \, dx \, dt$$

because the one of

$$\iint_{\mathbb{T}^2\times(0,T)} \varphi \det D^2 u^{(n)}\, dx\, dt$$

is similar to the first one, and the others follow from the assumptions on φ and the weak-$*$ convergence (2.7). We have

$$\begin{aligned}&\iint_{\mathbb{T}^2\times(0,T)} \left((\Delta u^{(n)})^2 - (\Delta u)^2\right) \Delta\varphi\, dx\, dt\\ &= \iint_{\mathbb{T}^2\times(0,T)} \Delta(u^{(n)}-u)\Delta(u^{(n)}+u)\Delta\varphi\, dx\, dt\\ &\le \left\|u^{(n)}+u\right\|_{L^\infty(W^{2,\infty})} \iint_{\mathbb{T}^2\times(0,T)} \left|\Delta(u^{(n)}-u)\right| |\Delta\varphi|\, dx\, dt\end{aligned}$$

thanks to (2.7). We now apply Hölder‘s inequality, obtaining that

$$\begin{aligned}&\iint_{\mathbb{T}^2\times(0,T)} \left((\Delta u_n)^2 - (\Delta u)^2\right) \Delta\varphi\, dx\, dt\\ &\le \|u_n+u\|_{L^\infty(W^{2,\infty})} \|u_n-u\|_{L^2(H^2)} \|\varphi\|_{L^2(H^2)},\end{aligned}$$

which converges to zero as $n\to\infty$ by (2.10).

3. **Proof of Theorem 1.5. Approximate problem:** We observe that in the new variable v, problem (1.4) is equivalent to

$$\begin{cases} \partial_t v = -v_{,iijj} - v_{,i}\, v_{,jji} - v v_{,iijj} - \chi p(p-1)(1+v)^{p-2} v_{,i}\, v_{,i} - \chi p(1+v)^{p-1} v_{,ii} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad (0,T)\times\mathbb{T}^2, \\ v^{(n)}(0,x) = u_0(x) - 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \mathbb{T}^2. \end{cases}$$

We define the following approximate problem

$$\begin{cases} \partial_t v^{(n)} = -v^{(n)}{}_{,iijj} - v^{(n)}{}_{,i}\, v^{(n)}{}_{,jji} - v^{(n)} v^{(n)}{}_{,iijj} \\ \qquad -\chi p(p-1)(1+v^{(n)})^{p-2} v^{(n)}{}_{,i}\, v^{(n)}{}_{,i} - \chi p(1+v^{(n)})^{p-1} v^{(n)}{}_{,ii} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad (0,T)\times\mathbb{T}^2, \\ v(0,x) = P_n(u_0(x)-1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \mathbb{T}^2. \end{cases} \tag{3.1}$$

Similar comments to (2.1) hold for (3.1).

A priori estimates in Wiener spaces: Let us omit the superscript (n) in the following computations. We use that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

and we rewrite each term of the r.h.s. of (3.1) in Fourier:

$$\begin{aligned}\widehat{v_{,iijj}}(t,k) &= |k|^4 \hat{u}(t,k),\\ \widehat{v_{,i}\, v_{,jji}}(t,k) &= \sum_{m\in\mathbb{Z}^2} m_i(k_i-m_i)|k-m|^2 \hat{v}(t,m)\hat{v}(t,k-m),\\ \widehat{v v_{,iijj}}(t,k) &= \sum_{m\in\mathbb{Z}^2} |k-m|^4 \hat{v}(t,m)\hat{v}(t,k-m),\end{aligned}$$

$$
\begin{aligned}
&\widehat{(1+v)^{p-2}v_{,i}\,v_{,i}}(t,k)\\
&=\sum_{q=0}^{p-2}\binom{p-2}{q}\widehat{v^q v_{,i}\,v_{,i}}(t,k)\\
&=-\sum_{q=0}^{p-2}\binom{p-2}{q}\sum_{m^1\in\mathbb{Z}^2}\cdots\sum_{m^{q+1}\in\mathbb{Z}^2}(k_i-m_i^1)(m_i^1-m_i^2)\widehat{v}(t,k-m^1)\widehat{v}(t,m^1-m^2)\\
&\qquad\times\prod_{\ell=2}^{q}\widehat{v}(t,m^\ell-m^{\ell+1})\widehat{v}(t,m^{q+1})\\
&=-\sum_{q=0}^{p-2}\frac{(p-2)!}{q!(p-2-q)!}\mathcal{N}_1^{(q)}(t,k),\\
&\widehat{(1+v)^{p-1}v_{,ii}}(k,t)\\
&=\sum_{q=0}^{p-1}\binom{p-1}{q}\widehat{v^q v_{,ii}}(t,k)\\
&=-\sum_{q=0}^{p-1}\binom{p-1}{q}\sum_{m^1\in\mathbb{Z}^2}\cdots\sum_{m^q\in\mathbb{Z}^2}\left|k-m^1\right|^2\widehat{v}(t,k-m^1)\prod_{\ell=1}^{q-1}\widehat{v}(t,m^\ell-m^{\ell+1})\widehat{v}(t,m^q)\\
&=-\sum_{q=0}^{p-1}\frac{(p-1)!}{q!(p-1-q)!}\mathcal{N}_2^{(q)}(t,k).
\end{aligned}
$$

Then, the Fourier coefficients of (3.1) satisfy

$$
\begin{aligned}
\partial_t\widehat{v}(t,k)=-|k|^4\widehat{v}(t,k)&-\sum_{m\in\mathbb{Z}^2}m_i(k_i-m_i)|k-m|^2\widehat{v}(t,m)\widehat{v}(t,k-m)\\
&-\sum_{m\in\mathbb{Z}^2}|k-m|^4\widehat{v}(t,m)\widehat{v}(t,k-m)+\chi p!\sum_{q=0}^{p-2}\frac{\mathcal{N}_1^{(q)}(t,k)}{q!(p-2-q)!}\\
&+\chi p!\sum_{q=1}^{p-1}\frac{\mathcal{N}_2^{(q)}(t,k)}{q!(p-1-q)!}.
\end{aligned}
$$

We recall (2.3) to deduce

$$
\sum_{k\in\mathbb{Z}^2}\partial_t\left|\widehat{v}(k,t)\right|=\frac{d}{dt}\left\|v(t)\right\|_{A^0},\quad \sum_{k\in\mathbb{Z}^2}|k|^4\left|\widehat{v}(k,t)\right|=\left\|v(t)\right\|_{A^4},
$$

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}^2}\left|\sum_{m\in\mathbb{Z}^2}m_i(k_i-m_i)|k-m|^2\widehat{v}(t,m)\widehat{v}(t,k-m)\right|&\le\left\|v(t)\right\|_{A^1}\left\|v(t)\right\|_{A^3}\\
&\le\left\|v(t)\right\|_{A^0}\left\|v(t)\right\|_{A^4},\\
\sum_{k\in\mathbb{Z}^2}\left|\sum_{m\in\mathbb{Z}^2}|k-m|^4\widehat{v}(t,m)\widehat{v}(t,k-m)\right|&\le\left\|v(t)\right\|_{A^0}\left\|v(t)\right\|_{A^4},
\end{aligned}
$$

$$
\sum_{q=0}^{p-2}\frac{1}{q!(p-2-q)!}\sum_{k\in\mathbb{Z}^2}\left|\mathcal{N}_1^{(q)}(t,k)\right|\le c\sum_{q=0}^{p-2}\left\|v(t)\right\|_{A^0}^q\left\|v(t)\right\|_{A^1}^2
$$

$$\leq c\sum_{q=0}^{p-2}\|v(t)\|_{A^0}^{q+1}\|v(t)\|_{A^2}\,,$$

$$\sum_{q=0}^{p-1}\frac{1}{q!(p-1-q)!}\sum_{k\in\mathbb{Z}^2}\left|\mathcal{N}_2^{(q)}(t,k)\right|\leq c\sum_{q=0}^{p-1}\|v(t)\|_{A^0}^{q}\|v(t)\|_{A^2}\,,$$

and we estimate the A^0 semi-norm of v as

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_{A^0}&\\ \leq-(1-2\|v(t)\|_{A^0})\|v(t)\|_{A^4}+c\chi p!\left(\sum_{q=0}^{p-2}\|v(t)\|_{A^0}^{q+1}+\sum_{q=0}^{p-1}\|v(t)\|_{A^0}^{q}\right)\|v(t)\|_{A^2}&\\ =-(1-2\|v(t)\|_{A^0})\|v(t)\|_{A^4}+c\chi p!\left(\|v(t)\|_{A^0}+2\sum_{q=1}^{p-1}\|v(t)\|_{A^0}^{q}\right)\|v(t)\|_{A^2}\,.\end{aligned}$$

We estimate the $\|v(t)\|_{A^2}$ as

$$\|v(t)\|_{A^2}\leq\|v(t)\|_{A^0}^{\frac{1}{2}}\|v(t)\|_{A^4}^{\frac{1}{2}}\leq\frac{1}{2}\|v(t)\|_{A^0}+\frac{1}{2}\|v(t)\|_{A^4}\,,$$

obtaining that

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_{A^0}&\\ \leq-\left(1-2\|v(t)\|_{A^0}-\frac{c\chi p!}{2}\left(\|v(t)\|_{A^0}+2\sum_{q=1}^{p-1}\|v(t)\|_{A^0}^{q}\right)\right)\|v(t)\|_{A^4}&\\ +\frac{c\chi p!}{2}\left(\|v(t)\|_{A^0}+2\sum_{q=1}^{p-1}\|v(t)\|_{A^0}^{q}\right)\|v(t)\|_{A^0}\,.\end{aligned}$$

The smallness condition

$$1-2\|v_0\|_{A^0}-\frac{c\chi p!}{2}\left(\|v_0\|_{A^0}+2\sum_{q=1}^{p-1}\|v_0\|_{A^0}^{q}\right)>0,$$

implies that, for small times,

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_{A^0}+\left(1-2\|v_0\|_{A^0}-\frac{c\chi p!}{2}\left(\|v_0\|_{A^0}+2\sum_{q=1}^{p-1}\|v_0\|_{A^0}^{q}\right)\right)\|v(t)\|_{A^4}&\\ \leq\frac{c\chi p!}{2}\left(\|v(t)\|_{A^0}+2\sum_{q=1}^{p-1}\|v(t)\|_{A^0}^{q}\right)\|v(t)\|_{A^0}&\\ \leq\frac{c\chi p!}{2}\left(\|v_0\|_{A^0}+2\sum_{q=1}^{p-1}\|v_0\|_{A^0}^{q}\right)\|v_0\|_{A^0}\,.\end{aligned}$$

We extend this inequality to all times using a contradiction argument in time. As a consequence, we find the uniform boundedness in the following space:

$$W^{1,1}(0,T;A^0(\mathbb{T}^2))\cap L^1(0,T;A^4(\mathbb{T}^2)).$$

Invoking a Poincaré inequality in Wiener spaces, we conclude the decay estimate

$$\left\|v^{(n)}(t)\right\|_{A^0} \leq \exp\left(-\left(1-2\left\|v_0\right\|_{A^0}-c\chi p!\left(\left\|v_0\right\|_{A^0}+2\sum_{q=1}^{p-1}\left\|v_0\right\|_{A^0}^q\right)\right)t\right)\left\|v_0\right\|_{A^0},$$

for χ eventually smaller.

Compactness results: Reasoning as for the previous problem (2.1), we have that

$$\begin{aligned}
v^{(n)} &\to v && \text{a.e. } (0,T)\times\mathbb{T}^2,\\
v^{(n)} &\overset{*}{\rightharpoonup} v && \text{in } L^\infty((0,T)\times\mathbb{T}^2),\\
v^{(n)} &\overset{*}{\rightharpoonup} v && \text{in } \mathcal{M}(0,T;W^{4,\infty}(\mathbb{T}^2)),\\
v^{(n)} &\overset{*}{\rightharpoonup} v && \text{in } L^{\frac{4}{m}}(0,T;W^{m,\infty}(\mathbb{T}^2)) \quad \text{with} \quad 0<m<4,\\
v^{(n)} &\rightharpoonup v && \text{in } L^2(0,T;H^2(\mathbb{T}^2)),\\
v^{(n)} &\to v && \text{in } L^2(0,T;H^r(\mathbb{T}^2)) \quad \text{with} \quad 0\leq r<2,\\
v^{(n)} &\to v && \text{in } L^q(0,T;H^2(\mathbb{T}^2)) \quad \text{with} \quad 1\leq q<2
\end{aligned}$$

up to subsequences.

Passing to the limit: We want to take the limit in n in

$$\begin{aligned}
-\int_{\mathbb{T}^2} P_n v_0\varphi(0)\,dx &+ \iint_{\mathbb{T}^2\times(0,T)}\\
&-v^{(n)}\partial_t\varphi - v^{(n)}\nabla\Delta v^{(n)}\cdot\nabla\varphi + \chi(v^{(n)}+1)^{p-1}\Delta\varphi\,dx\,dt = 0,
\end{aligned}$$

for every $\varphi \in W^{1,1}(0,T;L^1(\mathbb{T}^2))\cap L^q(0,T;H^2(\mathbb{T}^2))$ and $1\leq q<2$. Then, using that, due to interpolation, we have strong convergence in

$$L^r(0,T;H^3),$$

and since

$$(v_n+1)^{p-1}-(v+1)^{p-1} = \sum_{q=1}^{p-1} v_n^q - v^q = (v_n - v)\sum_{q=2}^{p-1} v_n^q v^{p-1-q}$$

we deduce that

$$\begin{aligned}
&\left|\iint_{\mathbb{T}^2\times(0,T)}\left((v_n+1)^{p-1}-(v+1)^{p-1}\right)\Delta\varphi\,dx\,dt\right|\\
&\leq \sum_{q=2}^{p-1}\left\|v_n\right\|_{L^\infty(L^\infty)}^q\left\|v\right\|_{L^\infty(L^\infty)}^{p-1-q}\left\|v_n-v\right\|_{L^2(L^2)}\left\|\varphi\right\|_{L^2(H^2)}\to 0,
\end{aligned}$$

and we can pass to the limit.

4. **Conclusion.** In this paper, we have studied two fourth order nonlinear parabolic equations. These equations arise in the study of epitaxial growth and in thin films. Such quasilinear PDEs are mathematically challenging due to the high number of derivatives in the main part. For these PDEs, we have established a number of global in time existence results in the nonstandard Wiener spaces. These functional spaces allow us to take full advantage of the parabolic structure.

In particular, one of our main contributions has been the improvement of the previous global in time result contained in [10]. Indeed, the authors in [10] establish global in time results for initial data with small energy akin to the H^2 norm. However, with our techniques we can prove a global in time result imposing a size condition in the Wiener algebra A^0. The Wiener algebra shares the same scaling as L^∞, and both are critical spaces with respect to the scaling of the equation.

Regarding uniqueness, a possible approach could be the one exploited by J.-G. Liu and R. Strain in [16]: Here, with medium size assumption on $\|u_0\|_{\mathcal{A}^2}$ (see (1.6)), the authors proved the uniqueness of solutions to

$$\partial_t u = \Delta e^{-\Delta u} \qquad \text{in } (0,T)\times\mathbb{R}^N.$$

Another possible way to approach this type of problem is the one proposed by D. M. Ambrose in [1]. Here, the author proves existence and analyticity results for a fourth order problem, which describes crystal growth surfaces through fixed point techniques. The equation, in this case, is the same as [16] but in the N dimensional torus.

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