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Well-posedness and decay for a nonlinear propagation wave model in atmospheric flows

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ABSTRACT

In this note, we provide two results concerning the global well-posedness and decay of solutions to an asymptotic model describing the nonlinear wave propagation in the troposphere, namely, the morning glory phenomenon. The proof of the first result combines a pointwise estimate together with some interpolation inequalities to close the energy estimates in Sobolev spaces. The second proof relies on suitable Wiener-like functional spaces.

1. Introduction and main result

Many fascinating events that constantly test our comprehension of the dynamic and intricate atmospheric processes on Earth characterize the field of atmospheric science. Of all these mysterious events, *the morning glory* is the one weather phenomenon that has fascinated scientists for decades. Long, horizontal cloud bands that often spread across the sky to make a recognizable and arresting pattern are what define the morning glory phenomenon. Recently in [1], the authors derived from the general Navier–Stokes equation in rotating spherical coordinates a more tractable asymptotic nonlinear system that describes this wave propagation given by

$$u_t + uu_x + vu_y = \mu\Delta u + \alpha u + \beta v + F, \quad \text{in } \Omega,\ t>0, \tag{1a}$$

$$u_x + v_y = 0, \quad \text{in } \Omega,\ t>0. \tag{1b}$$

The components of the vector velocity field are denoted by u, v and $\mu \in (0,\infty)$ is the viscosity parameter. Moreover, α, β are fixed real constants that depend on the wave front distortion and a fixed reference parameter measuring the latitude. A more precise description of both constants will be given later. The force F represents a thermodynamic forcing term, which comprises the heat sources driving the motion. Eq. (1)b comprises the incompressibility condition of the flow. In (1) the spatial domain Ω is the two-dimensional channel domain

$$\Omega = \{(x,y) \text{ s.t. } x\in\mathbb{T},\ 0<y<1\},$$

and the time satisfies $t \in [0,T]$ for certain $0 < T \leq \infty$. Moreover, Eqs. (1) are subject to the following boundary conditions

$$u = 0, \quad \text{on } \partial\Omega,\ t>0, \tag{2a}$$

$$v = 0, \quad \text{on } \{y=0\},\ t>0. \tag{2b}$$

The corresponding initial-value problem consists of the system (1), (2) along with the initial condition

$$u(x,y,0) = u_0(x,y) \tag{3}$$

which is assumed to be smooth enough for the purposes of the work (cf. the statement of Theorem 1 for the precise assumptions on the initial data). As already mentioned, system (1) was originally derived in [1, equations (6.18)-(6.19)]. After a quick inspection one can readily check that the constants α and β are given in [1, equations (6.18)-(6.19)] as

$$\alpha = \sigma S, \quad \beta = \sigma\frac{C\cos(\gamma)}{d_0},$$

where $S = \sin(\theta_0 + \Phi\sin(\gamma))$ and $C = \cos\left(\theta_0 + \Phi\sin(\gamma)\right)$, see [1, equation (4.19)]. Here Φ describes the distortion along the wavefront, θ_0 is a

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fixed parameter measuring the latitude (in spherical coordinates) and γ is a fixed angle of rotation. Moreover, we have that $\sigma = \frac{2\left(\sin^2(\gamma)+C\cos^2(\gamma)\right)}{(1-C)\sin(\gamma)\cos(\gamma)}$ and d_0 denotes a positive density function.

In [1], Constantin & Johnson provided a number of exact solutions: breeze-like flows, bore-like flows as well as oscillatory-like solutions, cf. [1, §6]. The same named authors investigated the existence of travelling-wave solutions, cf. [2]. Their analysis relies on studying a nonlinear second-order ordinary differential equation by means of a global phase-space analysis using Lyapunov functions. Recently, Matioc & Roberti in [3], using an abstract quasilinear parabolic evolution framework, showed the global existence of weak solutions to (1) as well as the local existence of strong solutions. The main contribution and novelty of this article is to show the global existence and decay of classical solutions to (1) in Sobolev spaces under a smallness L^∞ assumption. Moreover, we also show a similar result in Wiener-like functional spaces.

In order to present the main result of this work, it is convenient to rewrite (1) by eliminating the variable v. Following the approach in [3], we find that integrating v in (1)b from 0 to y, we have that

$$v(x,y,t) = \int_0^y v_y(x,\xi,t)\,d\xi = -\int_0^y u_x(x,\xi,t)\,d\xi := -Tu(x,y,t),$$

where $Tf : \Omega\times\mathbb{R}^+ \to \mathbb{R}$ is given by

$$Tf(x,y,t) = \int_0^y f_x(x,\xi,t)\,d\xi.$$

This idea of using the fundamental theorem of calculus to express v in terms of u is reminiscing about the viscous primitive equations of large scale ocean and atmosphere dynamics, cf. [4]. Therefore, using this observation system (1) can be rewritten as

$$u_t + uu_x - Tuu_y = \mu\Delta u + \alpha u - \beta Tu + F, \quad \text{in } \Omega,\ t>0, \tag{4a}$$
$$Tu = \int_0^y u_x(x,\xi)\,d\xi, \tag{4b}$$
$$u(x,y,0) = u_0(x,y), \quad (x,y)\in\Omega \tag{4c}$$

supplemented with

$$u = 0, \quad \text{on } \partial\Omega,\ t>0. \tag{5}$$

We need to impose some further conditions in order to show the main result of this article. In particular, we will take $\alpha \le 0$, $\beta = 0$ and no external forcing, i.e., $F \equiv 0$. Hence, (4) becomes

$$u_t + uu_x - Tuu_y = \mu\Delta u + \alpha u, \quad \text{in } \Omega,\ t>0, \tag{6a}$$
$$Tu = \int_0^y u_x(x,\xi)\,d\xi, \tag{6b}$$
$$u(x,y,0) = u_0(x,y), \quad (x,y)\in\Omega \tag{6c}$$

supplemented with

$$u = 0, \quad \text{on } \partial\Omega,\ t>0. \tag{7}$$

As noticed in [1, §6 (b)], in the particular case of the geographical coordinates of the Gulf of Carpentaria, we have that $C\sim 0.97$, $S\sim -0.24$ and $\sigma \sim 133$ (for $\gamma = \frac{5\pi}{4}$), so that $\alpha = \sigma S \le 0$. Therefore, taking $\alpha\le 0$ is a physically justified assumption. Generically, since γ can be chosen freely, we can also take γ such that $\beta\sim 0$, for instance, $\gamma\sim\frac{\pi}{2}$. Moreover, we are assuming that the external thermodynamic force is negligible, this is, that the are not heat sources driving the motion. The three stated hypothesis on α,β and F are crucial in order to show the decay of the solution.

The main result of this work is to provide the decay of the L^∞ norm for arbitrary initial data together with a global existence of classical solutions to (6a)–(6b) under a smallness L^∞ assumption on the initial data. More precisely, the result reads as follows:

Theorem 1. *Let $u_0 \in H^1_0(\Omega)\cap H^4(\Omega)$, be a zero mean function. Let $\alpha \le 0$ and $\mu\in(0,\infty)$. Then, the Cauchy problem (6a)–(6c) possesses a unique classical solution*

$$u \in C\left([0,T); H^4(\Omega)\right)$$

for $T = T(u_0) > 0$ satisfying

$$\|u(t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}, \quad \text{for } 0\le t\le T.$$

Furthermore, if $\|u_0\|_{L^\infty}$ is sufficiently small, the unique classical solution to (6a)–(6b) satisfies

$$u\in C([0,\infty); H^1_D(\Omega))\cap C([0,\infty); H^4(\Omega))\cap L^2([0,\infty); H^5(\Omega)).$$

Remark 1. Before stating the next result, let us make the following important observations regarding Theorem 1 compared to previous well-posedness results obtained results in the literature.

- In [3], Matioc & Roberti showed two well-posedness results for system (1). First, they establish the existence and uniqueness of classical solutions to (1) for sufficiently regular initial data in $H^s_D(\Omega)$ with $1<s<2$ and external force $F\in C^{1-}(C[0,\infty); H^r(\Omega))$ for some small $r>0$. In [3], the Sobolev spaces H^s_D are given by

  $$H^s_D(\Omega) = \{u\in H^s(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

  To that purpose, they invoke an abstract quasilinear parabolic evolution framework. Moreover, they also show the existence of global weak solutions for initial data in $L^2(\Omega)$ and $F\in L^2\left((0,T);L^2(\Omega)\right)$. The proof uses a Galerkin scheme together with proper a priori estimates to pass to the limit.
- Our main contributions are three-fold: First, since the local solutions in [3] enjoy just the regularity $u\in C\left([0,T);H^s_D(\Omega)\right)$, $1<s<2$, we follow a classical approach based on estimating the time-derivative of the equation to provide the higher order regularity $u\in C\left([0,T);H^4(\Omega)\right)$. Second, we show the L^∞ decay of the classical solution (under the assumptions $\alpha\le 0,\beta=0$ and $F\equiv 0$). Such decay is not available in the literature before and to the best of the authors' knowledge it is new. To conclude, we prove that for small initial L^∞ data, the local classical solution can be extended globally in time, i.e. $T=\infty$.
- Just after the completion of this article, the preprint [5] appeared. Compared with the results in [3], the authors shows the global existence of weak solutions for external forces $F\in L^2\left((0,T);H^{-1}(\Omega)\right)$. On the other hand, the author in [5] shows that for initial data in $H^1(\Omega)$ and $F\in L^2\left((0,T);L^2(\Omega)\right)$ there exists a global strong solution $u\in L^2\left((0,T);H^2(\Omega)\right)\cap H^1\left((0,T);L^2(\Omega)\right)$. Compared to our global existence result showed in Theorem 1, the author in [5] does not need the L^∞ smallness on the initial data to absorb the non-linear contributions. Actually, such nonlinear terms are handled claiming an odd extension method works. To the best of our knowledge it is not a priori clear why the imposed odd-parity is preserved by system (1).

In order to present the second result showed in this manuscript, let us introduce the so called Wiener-like spaces. Recalling that $\Omega = \{(x,y) \text{ s.t. } x\in\mathbb{T},\ 0<y<1\}$, let us consider the Fourier series representation

$$u(x,y) = \sum_{n\in\mathbb{Z}}\sum_{m\ge 1}\hat u(n,m)e^{inx}\sin(m\pi y).$$

Using this, we can consider the following Wiener-like spaces $\tilde{\mathcal{A}}^s$ for $s\ge 0$ given by

$$\tilde{\mathcal{A}}^s(\Omega) = \left\{ u = \sum_{n\in\mathbb{Z}}\sum_{m\ge 1}\hat u(n,m)e^{inx}\sin(m\pi y) : \|u\|_{\tilde{\mathcal{A}}^s} = \sum_{n\in\mathbb{Z}}\sum_{m\ge 1}(|n|^s+|m|^s)|\hat u(n,m)| < \infty \right\}.$$

These spaces will allow us to achieve maximal parabolic regularity. The properties of Wiener-like spaces have been investigated in [6] and the references therein. In particular, we observe that

$$\tilde{A}^s(\Omega) \subset C^s(\Omega),$$

and that they form a Banach scale of Banach algebras. Let ∂ be a first order differential operator, then we have that

$$\left\|\partial^\ell u\right\|_{\tilde{A}^0} \leq C \|u\|_{\tilde{A}^\ell}, \ \ell \geq 0.$$

In particular, we have that

$$\|\partial_x u\|_{\tilde{A}^0} \leq \|u\|_{\tilde{A}^1}, \quad \left\|\partial_y u\right\|_{\tilde{A}^0} \leq \pi \|u\|_{\tilde{A}^1}, \quad \|\Delta u\|_{\tilde{A}^0} \geq \|u\|_{\tilde{A}^2}. \tag{8}$$

Moreover, for $0 \leq s < r$, $\theta = \frac{s}{r}$ the following interpolation inequalities

$$\|u\|_{\tilde{A}^s} \leq C_\theta \|u\|_{\tilde{A}^0}^{1-\theta} \|u\|_{\tilde{A}^r}^{\theta}$$

hold. In particular, we have that

$$\|u\|_{\tilde{A}^1} \leq \|u\|_{\tilde{A}^0}^{\frac{1}{2}} \|u\|_{\tilde{A}^2}^{\frac{1}{2}}. \tag{9}$$

Let us state the second result shown in this work.

Theorem 2. *Let $u_0 \in \tilde{A}^0(\Omega)$ be a zero mean function. Let $\alpha, \beta \in \mathbb{R}$ and $\mu \in (0, \infty)$ such that the relation*

$$\alpha + \frac{\beta^2}{2\mu} \leq 0,$$

holds. Then, if $u_0 \in \tilde{A}^0(\Omega)$ is such that

$$(1+\pi) \|u_0\|_{\tilde{A}^0} < \frac{\mu}{2}, \tag{10}$$

there exists a unique global in time solution to (4) satisfying $u \in C([0,\infty); \tilde{A}^0(\Omega)) \cap L^1([0,\infty); \tilde{A}^2(\Omega))$.

Remark 2. In order to show Theorem 2 we need milder restrictions on the parameter α, β and μ. In particular, we can take $\beta = 0$ and $\alpha \leq 0$ as in Theorem 1, however this is not the only choice. Moreover, this is the first result in Wiener-type spaces for system (4).

Notation

For $m \in \mathbb{N}$, the natural inhomogeneous and homogeneous Sobolev norms are defined by

$$\|f\|^2_{H^m(\Omega)} := \|f\|^2_{L^2(\Omega)} + \|\partial^m f\|^2_{L^2(\Omega)}, \quad \|f\|^2_{\dot{H}^m(\Omega)} := \|\partial^m f\|^2_{L^2(\Omega)},$$

respectively. We will use $z = (x, y)$ to denote an element in Ω. Moreover, throughout the paper $C = C(\cdot)$ will denote a positive constant that may depend on fixed parameters (but independent of time and the projection parameter N) and can change from line to line.

2. Proof of Theorem 1

We divide the proof of Theorem 1 into several steps.

Step 1: Local in time solution d'aprés matioc & roberti [3]. As stated in Remark 1, invoking the result by Matioc & Roberti [3] there exists a unique local solution u to (6a)–(6c) such that $u \in C\left([0,T); H^s_D(\Omega)\right)$, $1 < s < 2$. Moreover, from such regularity we can also extract the parabolic contribution implying $u \in L^2\left((0,T); H^{1+s}(\Omega)\right)$. However, in order to apply a pointwise L^∞ type estimate to (6a)–(6c), we need to show higher regularity for the solution u. More precisely, we will show that $u \in C([0,T); H^4(\Omega))$.

Step 2: The regularized approximate problem and higher-order a priori estimates. To show the higher order estimate, we derive appropriate energy estimates combined with a suitable approximation procedure, given by

$$u_t + P_N(P_N u P_N u_x) - P_N(P_N T u P_N u_y) = \mu \Delta P_N u + \alpha P_N u, \tag{11}$$

where

$$P_N u = \sum_{n=-N}^{N} \sum_{m\geq 1}^{N} \hat{u}(n,m) e^{inx} \sin(m\pi y), \quad N \in \mathbb{N} \cup \{0\}.$$

Observe that we can use Picard's theorem in these finite dimensional spaces to prove the local existence of an analytical approximate solution to (11), so, in particular, $u \in C^\infty([0,T) \times \Omega)$. Hence, every computation is justified and we just focus on deriving the desired a priori estimates.

Let us start, by showing the evolution of the L^2 estimate for u. First, using the fact that P_N commutes with derivatives and recalling the definition of Tu in (6b) we find that

$$\begin{aligned}\int_\Omega P_N(P_N T u P_N u_y) u \, dz &= -\frac{1}{2}\int_{\mathbb{T}}\int_0^1 P_N u_x (P_N u)^2 \, dy \, dx \\ &= -\frac{1}{6}\int_\Omega \partial_x (P_N u)^3 \, dz = 0.\end{aligned} \tag{12}$$

Hence, taking the inner product of (11) with u, integrating by parts, and making use of the cancellation estimate (12) together with the sign assumption on α we obtain that

$$\operatorname*{esssup}_{t\leq T} \|u\|^2_{L^2} \leq C \|u_0\|^2_{L^2}. \tag{13}$$

Furthermore, we also achieve the parabolic gain

$$\int_0^T \|\nabla P_N u(\tau)\|^2_{L^2} \, d\tau \leq C \|u_0\|^2_{L^2}. \tag{14}$$

To obtain higher regularity, we follow a classical approach based on taking time-derivatives of the problem. Indeed, deriving in time (11) we find that

$$\begin{aligned}u_{tt} + P_N(P_N u_t P_N u_x + P_N u P_N u_{xt}) - P_N(P_N T u_t P_N u_y + P_N T u P_N u_{yt}) \\ = \mu \Delta P_N u_t + \alpha P_N u_t,\end{aligned} \tag{15}$$

with the same boundary conditions in $\partial\Omega$. Repeating the same arguments as for the L^2 estimate (13), i.e., testing (15) against u_t and integrating by parts we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2} + \mu\|\nabla P_N u_t\|^2_{L^2} &= -\frac{1}{2}\int_\Omega P_N u_x (P_N u_t)^2 \, dz \\ + \int_\Omega \left(P_N T u_t P_N u_y P_N u_t - \frac{1}{2}\left(P_N T u\right)_y (P_N u_t)^2\right) dz &+ \alpha \int_\Omega |P_N u_t|^2 \, dz.\end{aligned} \tag{16}$$

Therefore, using the sign assumption on α, the cancellation property (12) and invoking Hölder's inequality we obtain that

$$\begin{aligned}\frac{d}{dt}\|u_t\|^2_{L^2} + \mu\|\nabla P_N u_t\|^2_{L^2} &\leq C\|P_N u_t\|^2_{L^2}\|P_N \nabla u\|_{L^\infty} \\ &+ \int_\Omega |P_N T u_t P_N u_y P_N u_t| \, dz.\end{aligned} \tag{17}$$

Moreover, using the definition of Tu given in (6b) combined with Hölder's and Young's inequality we readily check that

$$\begin{aligned}\int_\Omega |P_N T u_t P_N u_y P_N u_t| \, dz &\leq C \|P_N \nabla u_t\|_{L^2} \|P_N \nabla u\|_{L^\infty} \|P_N u_t\|_{L^2} \\ &\leq \frac{\mu}{2} \|P_N \nabla u_t\|^2_{L^2} + C\|P_N \nabla u\|^2_{L^\infty}\|P_N u_t\|^2_{L^2}.\end{aligned} \tag{18}$$

Hence, combining both (17)–(18) we conclude that

$$\begin{aligned}\frac{d}{dt}\|u_t\|^2_{L^2} &\leq C\|P_N u_t\|^2_{L^2}\left(\|P_N \nabla u\|_{L^\infty} + \|P_N \nabla u\|^2_{L^\infty}\right) \\ &\leq C\|P_N u_t\|^2_{L^2}(\|P_N u\|_{H^{1+s}} + \|P_N u\|^2_{H^{1+s}}),\end{aligned}$$

where in the last inequality we have used Sobolev embedding $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ for $1 < s < 2$. Using Grönwall's inequality combined with the fact that

$$u \in C\left([0,T); H^s_D(\Omega)\right) \cap L^2\left((0,T); H^{1+s}(\Omega)\right),$$

by Matioc & Roberti [3], see *Step 1*, we infer

$$\begin{aligned}\|u_t\|^2_{L^2} &\le C\|u_t|_{t=0}\|^2_{L^2}\exp\left(\int_0^t (\|P_N u(\tau)\|_{H^{1+s}} + \|P_N u(\tau)\|^2_{H^{1+s}})\, d\tau\right)\\ &\le C\left(\|u_t|_{t=0}\|_{L^2}, \|u_0\|_{H^s}\right)\exp\left(\sqrt{t}\right).\end{aligned} \tag{19}$$

Moreover, evaluating Eq. (11) at $t = 0$ we have that

$$\begin{aligned}\|u_t|_{t=0}\|_{L^2} &\le C\Big(\|P_N(P_N u_0 P_N u_x|_{t=0})\|_{L^2} + \left\|P_N(P_N T u_0 P_N u_y|_{t=0})\right\|_{L^2}\\ &\quad + \|\Delta P_N u_0\|_{L^2} + \|P_N u_0\|_{L^2}\Big)\\ &\le C\left(\|P_N u_0\|^2_{H^1} + \|P_N T u_0\|_{L^4}\left\|P_N u_y|_{t=0}\right\|_{L^4} + \|P_N u_0\|_{H^2}\right)\\ &\le C\left(\|u_0\|_{H^2}\right),\end{aligned} \tag{20}$$

where in the last inequality we have invoked the classical Gagliardo–Nirenberg inequality (where we have used the boundary conditions to eliminate the extra term $C\|f\|_{L^2}$)

$$\|f\|_{L^4} \le C\|f\|^{1/2}_{L^2}\|\nabla f\|^{1/2}_{L^2}, \tag{21}$$

for $f = P_N T u_0$ and $f = P_N u_y|_{t=0}$. Thus plugging (20) in (19) we can find $T > 0$ and a uniform bound in N such that

$$\operatorname*{esssup}_{t\le T}\|u_t\|_{L^2} \le C(\|u_0\|_{H^2})\exp\left(\sqrt{T}\right). \tag{22}$$

In addition, using Eq. (17) we find the parabolic regularity estimate

$$\int_0^T \|\nabla P_N u_t(\tau)\|^2_{L^2}\, d\tau \le C(\|u_0\|_{H^2})\exp\left(\sqrt{T}\right). \tag{23}$$

Furthermore, using Eq. (11) with the previous parabolic estimate (23) we obtain

$$\int_0^T \|\nabla\Delta P_N u(\tau)\|^2_{L^2}\, d\tau \le C(\|u_0\|_{H^2})\exp\left(\sqrt{T}\right). \tag{24}$$

Next, let us show how estimate (22) yields control for the H^2 norm of the solution using the structure of Eq. (11). Indeed, we have that

$$\begin{aligned}\mu\|\Delta P_N u\|_{L^2} &= \left\|u_t + P_N(P_N u P_N u_x) - P_N(P_N T u P_N u_y) - \alpha P_N u\right\|_{L^2}\\ &\le \|u_t\|_{L^2} + \underbrace{\|P_N u P_N u_x\|_{L^2}}_{\mathrm{I}_1} + \underbrace{\left\|P_N T u P_N u_y\right\|_{L^2}}_{\mathrm{I}_2} + |\alpha|\,\|P_N u\|_{L^2}.\end{aligned} \tag{25}$$

Using Hölder's and Young's inequality together with Sobolev interpolation we find that

$$\mathrm{I}_1 \le \|P_N u\|_{L^\infty}\left(\|P_N u_{xx}\|^{1/2}_{L^2}\|P_N u\|^{1/2}_{L^2}\right) \le C\|u\|_{L^\infty}\|\Delta P_N u\|_{L^2}. \tag{26}$$

Here, we use the fact that $\|P_N u\|_{L^\infty} \le C\|u\|_{L^\infty}$ for smooth functions u. On the other hand, using Jensen inequality and the Gagliardo–Nirenberg inequalities

$$\left\|f_y\right\|_{L^4} \le C\|f\|^{1/2}_{L^\infty}\left\|f_{yy}\right\|^{1/2}_{L^2}, \quad \|g_x\|_{L^4} \le C\|g\|^{1/2}_{L^\infty}\|g_{xx}\|^{1/2}_{L^2}, \tag{27}$$

for $f = P_N u$ and $g = P_N u$ we have that

$$\begin{aligned}\mathrm{I}_2 &\le \|P_N T u\|_{L^4}\left\|P_N u_y\right\|_{L^4} \le C\|P_N u_x\|_{L^4}\left\|P_N u_y\right\|_{L^4}\\ &\le C\|u\|_{L^\infty}\|\Delta P_N u\|_{L^2}.\end{aligned} \tag{28}$$

Combining (26) and (28) and recalling estimates (13) and (22), we show that

$$\mu\|\Delta P_N u\|_{L^2} \le C\left(\|u_0\|_{H^2}\right)\exp\left(\sqrt{t}\right) + C\|u\|_{L^\infty}\|\Delta P_N u\|_{L^2}. \tag{29}$$

Therefore, in order to absorb the second term with the left hand side of (29), we use the fact by continuity for $0 < \tilde{T}$ sufficiently small

$$\|u\|_{L^\infty} \le 2\|u_0\|_{L^\infty}, \quad \text{for } 0 \le t \le \tilde{T}. \tag{30}$$

Thus taking $\|u_0\|_{L^\infty}$ small enough, for instance $\|u_0\|_{L^\infty} = \frac{\mu}{4C}$ we conclude that

$$\operatorname*{esssup}_{t\le\tilde{T}}\|\Delta P_N u\|_{L^2} \le C\left(\|u_0\|_{H^2}\right)\exp\left(\sqrt{\tilde{T}}\right). \tag{31}$$

Since by (13) we also have uniform control of the L^2 norm of u, we have shown that the H^2 norm of u is bounded by

$$\operatorname*{esssup}_{t\le\tilde{T}}\|u\|_{H^2} \le C\left(\|u_0\|_{H^2}\right)\exp\left(\sqrt{\tilde{T}}\right). \tag{32}$$

Taking the inner product of (15) with $-\Delta u_t$ and integrating by parts we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|^2_{L^2} + \mu\|\Delta P_N u_t\|^2_{L^2} &= \int_\Omega \left(P_N u_t P_N u_x + P_N u P_N u_{xt}\right)P_N\Delta u_t\, dz\\ &\quad - \int_\Omega \left(P_N T u_t P_N u_y + P_N T u P_N u_{yt}\right)P_N\Delta u_t\, dz + \alpha\int_\Omega |P_N u_t|^2\, dz.\end{aligned}$$

Using the sign hypothesis on α together with Hölder's and Young's inequality (mimicking the computations in (17)–(18)) we find that

$$\begin{aligned}&\frac{d}{dt}\|\nabla u_t\|^2_{L^2} + \frac{\mu}{2}\|\Delta P_N u_t\|^2_{L^2}\\ &\quad\le C\|P_N\nabla u_t\|^2_{L^2}\left(\|P_N u\|^2_{L^\infty} + \|P_N\nabla u\|^2_{L^\infty}\right)\\ &\quad + \|P_N\nabla u\|^2_{L^\infty}\|P_N u_t\|^2_{L^2}.\end{aligned} \tag{33}$$

Invoking Grönwall's inequality and the previous parabolic gain of regularity bounds (19) and (23), we conclude

$$\operatorname*{esssup}_{t\le T}\|\nabla u_t\|_{L^2} \le C(\|\nabla u_t|_{t=0}\|_{L^2}, \|u_0\|_{H^2})\exp\left(\sqrt{T}\right).$$

However, as before taking the spatial gradient in (11) and evaluating at $t = 0$, we find that $\|\nabla u_t|_{t=0}\|_{L^2} \le C(\|u_0\|_{H^3})$ and hence

$$\operatorname*{esssup}_{t\le T}\|\nabla u_t\|_{L^2} \le C(\|u_0\|_{H^3})\exp\left(\sqrt{T}\right). \tag{34}$$

Moreover, repeating the parabolic regularity gain using Eqs. (11) and (33)

$$\int_0^T \|\Delta P_N u_t(\tau)\|^2_{L^2}d\tau + \int_0^T \|\Delta^2 P_N u(\tau)\|^2_{L^2}d\tau \le C(\|u_0\|_{H^3})\exp\left(\sqrt{T}\right).$$

In the same way that we derived the bound for the H^2 norm of u (32) using the L^2 control for u_t in (22), we can obtain H^3 norm control of u using the L^2 bound for ∇u_t (34). Indeed, using Eq. (11) we readily check that

$$\begin{aligned}\mu\|\nabla\Delta P_N u\|_{L^2} &= \left\|\nabla u_t + \nabla P_N(P_N u P_N u_x) - \nabla P_N(P_N T u P_N u_y) - \alpha\nabla P_N u\right\|_{L^2}\\ &\le C(\|\nabla u_t\|_{L^2} + \underbrace{\|\nabla(P_N u P_N u_x)\|_{L^2}}_{\mathrm{J}_1} + \underbrace{\left\|\nabla(P_N T u P_N u_y)\right\|_{L^2}}_{\mathrm{J}_2} + |\alpha|\,\|\nabla P_N u\|_{L^2}).\end{aligned} \tag{35}$$

Therefore, using Gagliardo–Nirenberg inequality (27) and the Sobolev embedding $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ for $1 < s < 2$ we find that

$$\begin{aligned}\mathrm{J}_1 &\le \|P_N\nabla u\|_{L^4}\left\|P_N u_y\right\|_{L^4} + \|P_N u\|_{L^\infty}\|\nabla P_N u_x\|_{L^2}\\ &\le C\|P_N u\|_{L^\infty}\|\Delta P_N u\|_{L^2} + \|P_N u\|^2_{L^2} \le C\|P_N u\|^2_{H^2}.\end{aligned} \tag{36}$$

For the second term we proceed in a similar way. Using once again Hölder's and Jensen's inequality we obtain

$$\mathrm{J}_2 \le C\left(\|P_N\nabla u_x\|_{L^4}\left\|P_N u_y\right\|_{L^4} + \|P_N u_x\|_{L^4}\left\|P_N\nabla u_y\right\|_{L^4}\right).$$

By means of the Gagliardo–Nirenberg inequality (21) and (27) we first notice that

$$\|P_N\nabla u_x\|_{L^4} \le C\|P_N\nabla u_x\|^{\frac{1}{2}}_{L^2}\|P_N\nabla\nabla u_x\|^{\frac{1}{2}}_{L^2} + C\|P_N\nabla u_x\|_{L^2},$$

$$\left\|P_N\nabla u_y\right\|_{L^4} \le C\left\|P_N\nabla u_y\right\|^{\frac{1}{2}}_{L^2}\left\|P_N\nabla\nabla u_y\right\|^{\frac{1}{2}}_{L^2} + C\left\|P_N\nabla u_y\right\|_{L^2},$$

$$\left\|P_N u_y\right\|_{L^4} \le C\|P_N u\|^{\frac{1}{2}}_{L^\infty}\left\|P_N u_{yy}\right\|^{\frac{1}{2}}_{L^2} + C\left\|P_N u_y\right\|_{L^2},$$

$$\|P_N u_x\|_{L^4} \le C \|P_N u\|_{L^\infty}^{\frac{1}{2}} \|P_N u_{xx}\|_{L^2}^{\frac{1}{2}} + C \|P_N u_x\|_{L^2}.$$

Therefore, following the same argument before taking $0 < t < \tilde{T}$ and using bound (32), we find that

$$\begin{aligned} J_2 &\le C \|P_N u\|_{L^\infty} \|\nabla \Delta P_N u\|_{L^2} + C \|P_N u\|_{H^2}^3 \\ &\le C \|u_0\|_{L^\infty} \|\nabla \Delta P_N u\|_{L^2} + C(\|u_0\|_{H^2}) \exp\left(\sqrt{t}\right). \end{aligned} \tag{37}$$

Collecting estimates (36)–(37), taking $\|u_0\|_{L^\infty}$ sufficiently small and invoking the previous estimate (34) we conclude that

$$\operatorname*{esssup}_{t\le \tilde{T}} \|\nabla \Delta P_N u\|_{L^2} \le C\left(\|u_0\|_{H^3}\right) \exp\left(\sqrt{\tilde{T}}\right). \tag{38}$$

Hence, estimate (38) together with the uniform bound (14) yields

$$\operatorname*{esssup}_{t\le \tilde{T}} \|u\|_{H^3} \le C\left(\|u_0\|_{H^3}\right) \exp\left(\sqrt{\tilde{T}}\right). \tag{39}$$

Taking a new time derivative of the problem we find that

$$\begin{aligned} u_{ttt} = &-P_N(P_N u_{tt} P_N u_x + P_N u P_N u_{xtt}) + P_N(P_N T u_{tt} P_N u_y \\ &+ P_N T u P_N u_{ytt}) + \mu \Delta P_N u_{tt} + \alpha P_N u_{tt} \\ &- P_N(P_N u_t P_N u_{tx} + P_N u_t P_N u_{xt}) \\ &+ P_N(P_N T u_t P_N u_{yt} + P_N T u_t P_N u_{yt}), \end{aligned}$$

and performing an L^2 energy estimate repeating the previous computations once again, we find

$$\operatorname*{esssup}_{t\le T} \|u_{tt}\|_{L^2} \le C(\|u_0\|_{H^4}) \exp\left(\sqrt{T}\right). \tag{40}$$

Bound (40) can be bootstrapped (mimicking the estimates (32) and (39)) using the structure of the Eqs. (11) and (15) to obtain the H^4 bound

$$\operatorname*{esssup}_{t\le \tilde{T}} \|u\|_{H^4} \le C(\|u_0\|_{H^4}) \exp\left(\sqrt{\tilde{T}}\right). \tag{41}$$

In particular, combining bounds (22), (34) and (40) we find uniform a priori estimates

$$\operatorname*{esssup}_{t\le \tilde{T}} \|u_t\|_{H^2} + \operatorname*{esssup}_{t\le \tilde{T}} \|u\|_{H^4} \le C(\|u_0\|_{H^4}) \exp\left(\sqrt{\tilde{T}}\right). \tag{42}$$

Recall that this estimate is only valid, as long as the bound (30) holds which for the moment can only be guaranteed for times $0 < t < \tilde{T}$.

Step 3: Passing to the limit and inherited regularity. Thanks to the previous uniform estimates, we can extract weakly converging subsequences. More precisely, owing to (42), we deduce the existence of a function $\bar{u}$ belonging to the space $L^\infty([0,\tilde{T}); H^4(\Omega))$ such that, up to the extraction of a subsequence, one has the convergence

$$u \overset{*}{\rightharpoonup} \bar{u}, \quad \text{in } L^\infty([0,\tilde{T}); H^4(\Omega)) \ \text{ for } N \to \infty.$$

Moreover, we also have that $\partial_t u$ is uniformly bounded in $L^\infty([0,\tilde{T}); H^2(\Omega))$, thus

$$\partial_t u \overset{*}{\rightharpoonup} \partial_t \bar{u}, \quad \text{in } L^\infty([0,\tilde{T}); H^2(\Omega)) \ \text{ for } N \to \infty.$$

Hence, using the compactness argument as in [7, Corollary 4], we obtain up to a subsequence that

$$u \longrightarrow \bar{u} \quad \text{in } C([0,\tilde{T}); H^s(\Omega)) \ \text{ for } N \to \infty,$$

for $2 < s < 4$. Equipped with these convergences we can pass to the limit in N via the weak formulation of the problem and find a weak solution. Furthermore, given the regularity of the limit function, such weak solution is in fact a classical solution of the original problem (6a)–(6c) supplemented with (7). By uniqueness of the limit (we avoid writing the bar notation again) the limit solution u enjoys the regularity $L^\infty([0,\tilde{T}); H^4(\Omega))$.

Step 4: The pointwise estimate and global in time solution. The time lifespan of the constructed classical solution is valid as long as the bound (30) holds true. In this section, we will show a pointwise estimate for the classical solution that demonstrates that (30) is valid for all $T > 0$. Before proceeding to the computations, notice that the mean zero condition is conserved during the existence of the solution. Indeed, integrating (6a) in Ω we find that

$$\partial_t \int_\Omega u(z,t)dz = -\int_\Omega \left(u u_x - T u u_y - \mu \Delta u - \alpha u\right)(z,t)\, dz. \tag{43}$$

Notice that the first and the third term on the right hand side in (43) are zero using the periodicity in the x variable and the boundary condition (7). Furthermore, integrating by parts in the y variable, recalling the definition (6b) and using that $u = 0$ in $\partial\Omega$ we also have that

$$\int_\Omega T u u_y \, dz = -\int_\Omega u_x u \, dz = 0.$$

Therefore,

$$\partial_t \int_\Omega u(z,t)dz = \alpha \int_\Omega u(z,t)\, dz,$$

and since by assumption $u_0(z)$ has zero mean and $\alpha \le 0$ we conclude that

$$\int_\Omega u(z,t)\, dz = 0, \quad \text{for } 0 \le t \le T. \tag{44}$$

Following, [8,9], we define

$$M(t) = \max_{z\in\Omega} u(z,t) = u(\overline{z}_t, t), \ \text{ for } t > 0, \tag{45}$$

$$m(t) = \min_{z\in\Omega} u(z,t) = u(\underline{z}_t, t), \ \text{ for } t > 0. \tag{46}$$

One can readily check that $M(t), m(t)$ are Lipschitz functions. Moreover, one can readily check that $M(t)$ satisfies

$$\begin{aligned} |M(t) - M(s)| &= \begin{cases} u(\overline{z}_t, t) - u(\overline{z}_s, s) \text{ if } M(t) > M(s) \\ u(\overline{z}_s, s) - u(\overline{z}_t, t) \text{ if } M(s) > M(t) \end{cases} \\ &\le \begin{cases} u(\overline{z}_t, t) - u(\overline{z}_t, s) \text{ if } M(t) > M(s) \\ u(\overline{z}_s, s) - u(\overline{z}_s, t) \text{ if } M(s) > M(t) \end{cases} \\ &\le \begin{cases} |\partial_t u(\overline{z}_t, \xi)||t-s| \text{ if } M(t) > M(s) \\ |\partial_t u(\overline{z}_s, \xi)||t-s| \text{ if} M(s) > M(t) \end{cases} \\ &\le \max_{\eta,\xi} |\partial_t u(\eta,\xi)||t-s|. \end{aligned}$$

Similarly

$$|m(t) - m(s)| \le \max_{\eta,\xi} |\partial_t u(\eta,\xi)||t-s|.$$

From Rademacher's theorem it holds that $M(t)$ and $m(t)$ are differentiable in t almost everywhere. Furthermore, adding and subtracting in the denominator $u(\overline{z}_{t+\delta}, t)$, we find that

$$\begin{aligned} M'(t) = \lim_{\delta\to 0} \frac{u(\overline{z}_{t+\delta}, t+\delta) - u(\overline{z}_t, t)}{\delta} &\le \lim_{\delta\to 0} \frac{u(\overline{z}_{t+\delta}, t+\delta) - u(\overline{z}_{t+\delta}, t)}{\delta} \\ &\le \partial_t u(\overline{z}_t, t). \end{aligned}$$

In a similar fashion, adding and subtracting in the denominator $u(\overline{z}_t, t+\delta)$, we obtain that

$$\begin{aligned} M'(t) = \lim_{\delta\to 0} \frac{u(\overline{z}_{t+\delta}, t+\delta) - u(\overline{z}_t, t)}{\delta} &\ge \lim_{\delta\to 0} \frac{u(\overline{z}_t, t+\delta) - u(\overline{z}_t, t)}{\delta} \\ &\ge \partial_t u(\overline{z}_t, t). \end{aligned}$$

As a consequence

$$M'(t) = \partial_t u(\overline{z}_t, t) \text{ a.e.} \tag{47}$$

Similarly

$$m'(t) = \partial_t u(\underline{z}_t, t) \text{ a.e.}$$

Therefore, using (6a) and noticing that $u_x(\overline{z}_t,t) = u_y(\overline{z}_t,t) = 0$ and $\Delta u(\overline{z}_t,t) \leq 0$ we find that

$$M'(t) \leq 0,$$

which implies that

$$M(t) \leq M(0). \tag{48}$$

Similarly, repeating the same argument and recalling that $\Delta u(\underline{z}_t,t) \geq 0$,

$$m(t) \geq m(0). \tag{49}$$

Notice that the maximum is obtained in the interior of Ω, i.e. $z \in \dot{\Omega}$ and moreover $M(t) > 0$. Otherwise, the maximum is obtained on the boundary and hence using the boundary condition (7) this implies $M(t) = 0$. Similarly, the minimum must be obtained in the interior of Ω, otherwise $m(t) = 0$ but this violates (44). As a consequence $m(t) < 0$. Hence, combining (48) and (49) we have that

$$\|u\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}, \quad \text{for } 0 \leq t \leq T. \tag{50}$$

Therefore, we have shown that the constraint (30) is satisfied for all times $T > 0$. Therefore, we can derive once again the same energy estimates as in *Step 2* for the classical solution and prove that the solution remains in the desired functional spaces for all positive time.

To obtain the uniqueness, we argue by means of a contradiction argument. If u and w are two solutions emanating from the same initial data, we consider U their difference. Then, $U = u - w$ solves

$$U_t = -uU_x + TUu_y - Uw_x + TwU_y + \mu\Delta U + \alpha U.$$

Testing against U and integrating by parts, we find

$$\frac{d}{dt}\|U\|_{L^2}^2 \leq -\mu\|\nabla U\|_{L^2}^2 + \|U\|_{L^2}\|\nabla U\|_{L^2}(\|u\|_{L^\infty} + \|w\|_{L^\infty}) + \|U_y\|_{L^2}\|TU\|_{L^2}\|u\|_{L^\infty} + 2\|U\|_{L^2}\|U_x\|_{L^2}\|w\|_{L^\infty}.$$

Using the smallness hypothesis we conclude the desired bound and the result.

3. Proof of Theorem 2

For the proof of Theorem 2 we just provide the needed *a priori* estimates. The approximation procedure to justify the regularity can be done by mimicking projecting the functions into a finite dimensional space as in the beginning of the proof of Theorem 1.

Let us start by deriving appropriate a priori energy estimates. Noticing that

$$\frac{d}{dt}\|u(t)\|_{\tilde{\mathcal{A}}^0} = \sum_{n\in\mathbb{Z}}\sum_{m\geq 1}\partial_t|\hat{u}(n,m,t)|,$$

we find that

$$\frac{d}{dt}\|u(t)\|_{\tilde{\mathcal{A}}^0} \leq \left\|uu_x\right\|_{\tilde{\mathcal{A}}^0} + \left\|Tuu_y\right\|_{\tilde{\mathcal{A}}^0} + \alpha\|u\|_{\tilde{\mathcal{A}}^0} + \beta\|Tu\|_{\tilde{\mathcal{A}}^0} - \mu\|\Delta u\|_{\tilde{\mathcal{A}}^0}.$$

Using the inequalities (8) and the Banach algebra property we obtain that

$$\frac{d}{dt}\|u(t)\|_{\tilde{\mathcal{A}}^0} \leq \|u\|_{\tilde{\mathcal{A}}^0}\|u\|_{\tilde{\mathcal{A}}^1} + \pi\|Tu\|_{\tilde{\mathcal{A}}^0}\|u\|_{\tilde{\mathcal{A}}^1} + \alpha\|u\|_{\tilde{\mathcal{A}}^0} + \beta\|Tu\|_{\tilde{\mathcal{A}}^0} - \mu\|u\|_{\tilde{\mathcal{A}}^2}.$$

Moreover, noticing that $\widehat{Tu}(n,m) = \frac{n}{m}\hat{u}(n,m)$ and the fact that $m \geq 1$ we find that

$$\|Tu\|_{\tilde{\mathcal{A}}^0} \leq \|u\|_{\tilde{\mathcal{A}}^1}.$$

Thus,

$$\frac{d}{dt}\|u(t)\|_{\tilde{\mathcal{A}}^0} \leq \|u\|_{\tilde{\mathcal{A}}^0}\|u\|_{\tilde{\mathcal{A}}^1} + \pi\|u\|_{\tilde{\mathcal{A}}^1}^2 + \alpha\|u\|_{\tilde{\mathcal{A}}^0} + \beta\|u\|_{\tilde{\mathcal{A}}^1} - \mu\|u\|_{\tilde{\mathcal{A}}^2}.$$

Invoking the interpolation inequality (9) and Young's inequality we find that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{\tilde{\mathcal{A}}^0} &\leq (1+\pi)\|u\|_{\tilde{\mathcal{A}}^0}\|u\|_{\tilde{\mathcal{A}}^2} + \alpha\|u\|_{\tilde{\mathcal{A}}^0} + \frac{\beta^2}{2\mu}\|u\|_{\tilde{\mathcal{A}}^0} - \frac{\mu}{2}\|u\|_{\tilde{\mathcal{A}}^2}\\ &\leq \left((1+\pi)\|u\|_{\tilde{\mathcal{A}}^0} - \frac{\mu}{2}\right)\|u\|_{\tilde{\mathcal{A}}^2} + \left(\alpha + \frac{\beta^2}{2\mu}\right)\|u\|_{\tilde{\mathcal{A}}^0}.\end{aligned} \tag{51}$$

Therefore, since by hypothesis $\alpha + \frac{\beta^2}{2\mu} \leq 0$, we have that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{\tilde{\mathcal{A}}^0} &\leq (1+\pi)\|u\|_{\tilde{\mathcal{A}}^0}\|u\|_{\tilde{\mathcal{A}}^2} + \alpha\|u\|_{\tilde{\mathcal{A}}^0} + \frac{\beta^2}{2\mu}\|u\|_{\tilde{\mathcal{A}}^0} - \frac{\mu}{2}\|u\|_{\tilde{\mathcal{A}}^2}\\ &\leq \left((1+\pi)\|u\|_{\tilde{\mathcal{A}}^0} - \frac{\mu}{2}\right)\|u\|_{\tilde{\mathcal{A}}^2}.\end{aligned} \tag{52}$$

Taking $\|u\|_{\tilde{\mathcal{A}}^0}$ small enough, more precisely as in (10), concludes the *a priori* estimates of the solution. The uniqueness is a consequence of the obtained regularity and a standard contradiction result, from where we conclude the desired result.

CRediT authorship contribution statement

Diego Alonso-Orán: Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization. **Rafael Granero-Belinchón:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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