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Expositiones Mathematicae 42 (2024) 125591

EXPOSITIONES MATHEMATICAE

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Dicritical foliations and semiroots of plane branches

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Received 18 April 2024; accepted 5 July 2024

Abstract

In this work we describe dicritical foliations in $(\mathbb{C}^2, 0)$ at a triple point of the resolution dual graph of an analytic plane branch $\mathcal{C}$ using its semiroots. In particular, we obtain a constructive method to present a one-parameter family $\mathcal{C}_u$ of separatrices for such foliations. As a by-product we relate the contact order between a special member of $\mathcal{C}_u$ and $\mathcal{C}$ with analytic discrete invariants of plane branches.

MSC 2010: primary 32S65; secondary 14H20; 32S10

Keywords: Dicritical foliations; Semiroots; Analytic invariants of plane branches

1. Introduction

The aim of this work is to describe a construction of foliations in $(\mathbb{C}^2, 0)$ with a dicritical component in one of the bifurcation divisor of the reduction of singularities of an irreducible plane curve (branch).

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[1] The authors are supported by the Spanish research projects PID2019-105621GB-I00 and PID2022-139631NB-I00 funded by the Agencia Estatal de Investigación - Ministerio de Ciencia e Innovación, Spain. The second author is also supported by the research project 303638/2020-6 funded by CNPq-Brazil.

https://doi.org/10.1016/j.exmath.2024.125591

0723-0869/

Previous works deal with the construction of dicritical foliations. For instance, in [7], it is proved the existence of absolutely dicritical foliations for any configuration of the exceptional divisor, that is, given a morphism $\sigma : M \to (\mathbb{C}^2, 0)$ composition of a finite number of punctual blow-ups, there exists a germ of foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ such that the transformed foliation $\sigma^*\mathcal{F}$ is completely transversal to the exceptional divisor $\sigma^{-1}(0)$. Moreover, the foliation $\mathcal{F}$ has a meromorphic first integral. In [6], the authors present a way to construct logarithmic dicritical foliations (weak logarithmic models) which share some properties with a given foliation. More precisely, a weak logarithmic model $\mathcal{L}$ for a foliation $\mathcal{F}$ is a logarithmic foliation such that the reduction of the singularities of $\mathcal{L}$ is longer than the one of $\mathcal{F}$ and coincides with it outside a "escape set" of non-singular points for $\mathcal{F}$ placed at dicritical components. Note that, this escape set depends on the analytic type of the curves defined by the equations used to write the 1-form which gives the logarithmic foliation as shown in [6, Example 19].

Our approach here is different from the works previously mentioned. The main tools of our construction are the concept of the semiroots of a branch $\mathcal{C}$, which codify part of the topological data of the curve, and a result concerning a special way to express a holomorphic 1-form in Ω^1 (Azevedo's Lemma).

Semiroots of a plane branch $\mathcal{C}$ are particular branches that allow us to determine the topological class of the curve. Zariski in [24] considered semiroots in order to relate the characteristic exponents of $\mathcal{C}$ and the minimal generators of the value semigroup associated to the branch. In [1], Abhyankar and Moh introduced particular semiroots (approximate roots) that can be used in an effective criterion of irreducibility of elements in $\mathbb{C}\{x, y\}$.

The other ingredient is the Azevedo's Lemma (see [2], Chapter 5, Proposition 2): given $n, m \in \mathbb{Z}_{>0}$, any 1-form $\omega \in \Omega^1$ can be expressed as $\omega = H_1 \cdot (nxdy - mydx) + dH_2$, with $H_1, H_2 \in \mathbb{C}\{x, y\}$. This particular way to express a 1-form has been used by other authors on topics related to plane curves, vector fields, etc. For instance, Loray in [17] presents normal forms for cuspidal singularities of analytic vector fields that correspond a particular case of Azevedo's Lemma. Bayer and Hefez (see [3]) use such expression as a tool to describe (up to analytic equivalence) plane branches such that the Milnor and Tjurina numbers differ by one or two.

In this work, we consider $\mathcal{C} := \{F = 0\}$ a plane branch, where $F \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial, with value semigroup Γ minimally generated by $\{v_0, \ldots, v_g\}$. A set $\{F_0, F_1, \ldots, F_{g+1} := F\}$ is an extended system of semiroots of F, if $F_i \in \mathbb{C}\{x\}[y]$ is monic, $deg_y F_i = \frac{v_0}{GCD(v_0, \ldots, v_i)}$ for $1 \leq i \leq g$ and $\dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\langle F, F_i\rangle} = v_i$ for $0 \leq i \leq g$ (see Section 3).

For each pair (i, j), with $0 \leq i < j \leq g$, we set $\omega_{ij} = v_i F_i dF_j - v_j F_j dF_i \in \Omega^1$ and consider the singular foliation $\mathcal{F}_\omega$ defined by

$$\omega = H_1 \omega_{ij} + dH_2, \tag{1}$$

where $H_1, H_2 \in \mathbb{C}\{x, y\} \setminus \langle F\rangle$. Note that, for $i = 0$, $j = 1$, $v_0 = n$ and $v_1 = m$ we get the expression given in Azevedo's Lemma.

In Section 3, we present the main results of this paper, Theorem 3.7 and Corollary 3.8. We give a necessary and sufficient condition to assure that the foliation $\mathcal{F}_\omega$, with ω as in (1), has a dicritical component in the last triple point of the resolution dual graph of

$\mathcal{C}$. This condition is given in terms of the intersection multiplicities of F with H_1 and H_2. Moreover, in the proof of this theorem we have a constructive method to describe a family of parameterizations for the separatrices in such dicritical component up to the desired order as illustrated in Example 3.10. The presented method in Theorem 3.7 does not make use of blowing up which is normally considered to present dicritical foliations.

Dicritical foliations and analytic invariants of irreducible plane curves are closely related. More precisely, given a parameterization $\varphi(t) = (x(t), y(t))$ of the irreducible plane curve $\mathcal{C}$ and a 1-form ω, we define the value of ω by $\nu(\omega) = ord_t(\varphi^*(\omega)) + 1$ where $\varphi^*(\omega) = \omega(\varphi(t))$. The set of differential values Λ of $\mathcal{C}$ is given by

$$\Lambda = \{\nu(\omega) \ : \ \omega \in \Omega^1\}.$$

This analytic invariant is one of the main ingredients in the analytic classification of branches (see [15,16]). When $\omega = 0$ defines a foliation $\mathcal{F} = \mathcal{F}_\omega$ in $(\mathbb{C}^2, 0)$ and $\mathcal{C}$ is not an invariant curve (separatrix) of $\mathcal{F}$, the value $\nu(\omega) - 1$ coincides with the tangency order $\tau_0(\mathcal{F}, \mathcal{C})$ of the foliation $\mathcal{F}$ with the curve $\mathcal{C}$ (see [4,8]). If we consider a hamiltonian 1-form $\omega = dg$, with $g \in \mathbb{C}\{x, y\}$ a non unit, then $\nu(dg) = (\mathcal{C}, \mathcal{D})_0$, where $(\mathcal{C}, \mathcal{D})_0$ denotes the intersection multiplicity at the origin of the curves $\mathcal{C}$ and $\mathcal{D}$, with $\mathcal{D} := \{g = 0\}$. Hence, we have that $\Gamma \setminus \{0\} \subseteq \Lambda$ where Γ is the value semigroup associated to the curve $\mathcal{C}$. Moreover, there exists a finite subset $L = \{\ell_1, \dots, \ell_k\} \subset \Lambda$ such that any $\ell \in \Lambda$ can be expressed as $\ell = \ell_i + \gamma$ for some $\gamma \in \Gamma$ and $\ell_i \in L$, that is, the set Λ is a finitely generated Γ-monomodule.

Let us consider a set of 1-forms $\{\omega_1, \dots, \omega_k\}$ such that $\nu(\omega_i) = \ell_i$. If the curve $\mathcal{C}$ has only one Puiseux pair, then the foliations defined by the 1-forms $\omega_i = 0$ are dicritical in the triple point of the resolution dual graph of the curve $\mathcal{C}$ (see [9] where properties of these 1-forms are described).

From the results in [8,10], we have that, if the foliation $\mathcal{F}$ defined by $\omega = 0$ is a non-dicritical second type foliation (see [18]), then $\tau_0(\mathcal{F}, \mathcal{C}) = \nu(\omega) - 1 = (S_\mathcal{F}, \mathcal{C})_0 - 1$, where $S_\mathcal{F}$ is the curve of separatrices of $\mathcal{F}$. Thus, if ω is a 1-form such that $\nu(\omega) \in \Lambda \setminus \Gamma$, the foliation defined by $\omega = 0$ is either dicritical or it is not a second type foliation.

In Section 4, we explore 1-forms expressed as $\omega = H_1\omega_{ij} + dH_2$ and their connection with the analytic invariant Λ. We show that the value of ω is related with the contact between the branch and a special separatrix $\mathcal{C}_\star$ of $\mathcal{F}_\omega$ (see Theorem 4.4). In particular, for curves $\mathcal{C}$ with semigroup $\langle v_0, v_1\rangle$, we show that the set Λ can be determined using dicritical foliations defined by $H_1\omega_{01} + dH_2$, or equivalently, by H_1 and the special separatrix $\mathcal{C}_\star$ (see Corollary 4.7). The separatrix $\mathcal{C}_\star$ is closely related to the concept of *analytic semiroot* introduced by Cano, Corral and Senovilla-Sanz in [9] were, as we mentioned before, geometrical properties are presented for Λ. In addition, Proposition 4.9 shows how to compute the Zariski invariant λ of $\mathcal{C}$, that is $\lambda = \min(\Lambda \setminus \Gamma) - v_0$, considering dicritical foliations in the first triple point of the resolution dual graph of $\mathcal{C}$ that extends a result by Gómez–Martínez presented for branches with value semigroup minimally generated by two elements.

2. Notations

In this section we present some classic notations. For the results about Plane Curve Theory and Foliation Theory we indicate [5,12], respectively. We denote by $\mathbb{C}\{x, y\}$ the absolutely convergent power series ring at the origin in $\mathbb{C}^2$.

A germ of an analytic plane curve $\mathcal{C}_F$ in $(\mathbb{C}^2, 0)$ is the (germ of) zero set of a reduced element $F \in \mathbb{C}\{x, y\}$ in a neighborhood at the origin. Without loss of generality (by a change of coordinates) we can consider $F \in \mathbb{C}\{x\}[y]$ a Weierstrass polynomial $F(x, y) = y^n + \sum_{i=1}^{n} A_i(x)y^{n-i}$ where n is the multiplicity of F, denoted by $mult(F)$.

If F is irreducible, we can assume that $y = 0$ is the tangent cone of the branch $\mathcal{C}_F$ and this implies that $mult(A_i(x)) > i$ for $1 \leq i \leq n$. By Newton–Puiseux theorem we can obtain $\eta\left(x^{\frac{1}{n}}\right) = \sum_{k>n} c_k x^{\frac{k}{n}} \in \mathbb{C}\left\{x^{\frac{1}{n}}\right\}$ such that $F\left(x, \eta\left(x^{\frac{1}{n}}\right)\right) = 0$ and the set of roots of F (in a neighborhood at the origin) is $\left\{\eta\left(\alpha \cdot x^{\frac{1}{n}}\right);\ \alpha \in U_n\right\}$, where $U_n = \{\alpha \in \mathbb{C};\ \alpha^n = 1\}$. In particular, we have

$$F(x, y) = \prod_{\alpha \in U_n} \left(y - \eta\left(\alpha \cdot x^{\frac{1}{n}}\right)\right). \tag{2}$$

By a Tschirnhausen transformation, i.e. by the change of coordinates $(x, y) \to \left(x, y - \frac{1}{n}A_1(x)\right)$, we can assume that $A_1(x) = 0$, or equivalently $c_k = 0$ for all $k \equiv 0 \mod n$ in $\eta\left(x^{\frac{1}{n}}\right)$.

Putting $t = x^{\frac{1}{n}}$ we obtain a Puiseux parameterization for $\mathcal{C}_F$:

$$\varphi(t) = \left(t^n, \sum_{k \geq \beta_1} c_k t^k\right), \tag{3}$$

where $\beta_1 = \min\{k;\ k \not\equiv 0 \mod n \text{ and } c_k \neq 0\}$. Moreover, we will assume that such parameterization is primitive, that is, $\varphi(t)$ cannot be reparameterized by a power of a new variable or equivalently the greatest common divisor of all exponents in $\varphi(t)$ is equal to 1.

In what follows we consider plane branches, that is, plane curves defined by an irreducible Weierstrass polynomial as (2).

There are two sequences (e_i) and (β_i) of integers associated to $\mathcal{C}_F$ and obtained by any Puiseux parameterization of $\mathcal{C}_F$:

$$\beta_0 = e_0 = n;$$
$$\beta_j = min\{i;\ i \not\equiv 0 \ mod\ e_{j-1} \text{ and } c_i \neq 0\};$$
$$e_j = GCD(e_{j-1}, \beta_j) = GCD(\beta_0, \ldots, \beta_j).$$

The elements in the increasing finite sequence $(\beta_i)_{i=0}^{g}$ are called characteristic exponents associated to the branch and such sequence completely characterizes the topological type of the curve as an immersed germ in $(\mathbb{C}^2, 0)$. The local topology of plane branches can also be determined by the value semigroup Γ_F associated to the curve $\mathcal{C}_F$. More explicitly,

$$\Gamma_F = \{I(F, G);\ G \in \mathbb{C}\{x, y\}\} \subset \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\},$$

where $I(F, G) = (\mathcal{C}_F, \mathcal{C}_G)_0$ is the intersection multiplicity of $\mathcal{C}_F$ and $\mathcal{C}_G$ at the origin that can be computed by

$$I(F, G) = dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\langle F, G\rangle} = ord_t(\varphi^*(G)),$$

and $\varphi^*(G) := G(\varphi(t))$ for a parameterization $\varphi(t)$ of $\mathcal{C}_F$ as (3).

Notice that given $F \in \mathbb{C}\{x\}[y]$ with $deg_y(F) = mult(F) = v_0 > 1$ any $G \in \mathbb{C}\{x, y\}$ can be expressed, by Weierstrass Division Theorem, as $G = QF + H$ with $H \in \mathbb{C}\{x\}[y]$ and $deg_y(H) < v_0$. As $I(F, QF + H) = I(F, H)$ we get

$$\Gamma_F = \{I(F, H);\ \ H \in \mathbb{C}\{x\}[y] \text{ with } deg_y(H) < v_0\}.$$

Zariski (in [24]) showed that the value semigroup Γ_F is minimally generated by the set of integers $\{v_0, v_1, \ldots, v_g\}$, inductively defined by

$$v_0 = \beta_0 = n, \quad v_1 = \beta_1 \quad \text{and} \quad v_i = n_{i-1}v_{i-1} + \beta_i - \beta_{i-1} \tag{4}$$

or

$$v_i = \sum_{j=0}^{i-2} \frac{e_j - e_{j+1}}{e_{i-1}} \beta_{j+1} + \beta_i \tag{5}$$

for $i = 2, \ldots, g$ where $n_0 = 1$ and $n_i = \frac{e_{i-1}}{e_i}$. It follows from the definition of n_i that $n = n_0 \cdot n_1 \cdot \ldots \cdot n_g$. We denote $\Gamma_F = \langle v_0, v_1, \ldots, v_g \rangle$ and sometimes it would be convenient to consider $\beta_{g+1} = v_{g+1} = \infty$.

The value semigroup Γ_F admits a conductor μ_F, that is, $\mu_F + \mathbb{N} \subseteq \Gamma_F$ and $\mu_F - 1 \notin \Gamma_F$. For plane branches, μ_F coincides with the Milnor number of $\mathcal{C}_F$ and

$$\mu_F = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{\langle F_x, F_y \rangle} = \sum_{l=1}^{g} (n_l - 1)v_l - v_0 + 1. \tag{6}$$

In this paper we consider germs of holomorphic singular foliations of codimension one in $(\mathbb{C}^2, 0)$ locally given by $\omega = 0$, where

$$\omega = A(x, y)dx + B(x, y)dy \in \Omega^1 := \Omega^1_{\mathbb{C}^2,0} = \mathbb{C}\{x, y\}dx + \mathbb{C}\{x, y\}dy$$

with $A, B \in \mathbb{C}\{x, y\}$, $A(0, 0) = B(0, 0) = 0$ and $GCD(A, B) = 1$. Such a foliation will be denoted by $\mathcal{F}_\omega$ and its singular locus $Sing(\mathcal{F}_\omega)$ is locally given by the common zeros of A and B.

An analytic plane branch $\mathcal{C}_F$ defined by $F = 0$ is called a separatrix (or an invariant curve) of a foliation $\mathcal{F}_\omega$ if $\omega \wedge dF = F \cdot G \cdot dx \wedge dy$, where $G \in \mathbb{C}\{x, y\}$. In particular $\mathcal{C}_F \setminus Sing(\mathcal{F}_\omega)$ is a leaf of $\mathcal{F}_\omega$.

If $\varphi(t)$ is a parameterization of $\mathcal{C}_F$ we can define the $\mathbb{C}$-linear map

$$\begin{array}{llll} \varphi^* : & \Omega^1 & \to & \mathbb{C}\{t\} \\ & \omega = Adx + Bdy & \mapsto & \varphi^*(A)x'(t) + \varphi^*(B)y'(t) \end{array} \tag{7}$$

and we have that $\varphi^*(\omega) = 0$ if and only if $\frac{\omega \wedge dF}{dx \wedge dy} \in \langle F \rangle$, that is, $\mathcal{C}_F$ is a separatrix of $\mathcal{F}_\omega$.

A singular foliation $\mathcal{F}_\omega$ is called dicritical if there is a finite sequence of blowing-ups with nonsingular invariant centers, such that this process leads to an irreducible component E_i of the exceptional divisor E that is generically transversal to the strict transform of $\mathcal{F}_\omega$. For codimension one foliations in $(\mathbb{C}^2, 0)$, the dicritical condition is equivalent to the property of having infinitely many transversal invariant curves through almost any point in E_i, or in other words there are infinitely germs of analytic curves (separatrices) containing the origin and invariant by the foliation. In this case we say that $\mathcal{F}_\omega$ is dicritical in E_i or in the point Q_i of the resolution dual graph $G(\mathcal{C}_F)$ corresponding to E_i.

In the next section, we will consider particular plane branches $\mathcal{C}_{F_i}$ such that $I(F, F_i) = v_i$ in order to define dicritical foliations in specific components of the exceptional divisor obtained by the canonical resolution of $\mathcal{C}_F$.

3. Semiroots and dicritical foliations

Azevedo, in his thesis (see [2], Chapter 5, Proposition 2), exhibits a particular way to express any 1-form in Ω^1 as we present below:

Lemma 3.1 (*Azevedo*). *Given any $n, m \in \mathbb{Z}_{>0}$ and $\omega \in \Omega^1$, there exist $H_1, H_2 \in \mathbb{C}\{x, y\}$ such that*

$$\omega = H_1 \cdot (nxdy - mydx) + dH_2. \tag{8}$$

Proof. The proof is constructive and allow us to obtain H_1 and H_2 satisfying (8). Given any $\omega = Adx + Bdy \in \Omega^1$ it is possible to prove that there exist $H_1 = \sum_{i,j\geq 0} a_{ij}x^i y^j$ and $H_2 = \sum_{i,j\geq 0} b_{ij}x^i y^j$ such that $(H_2)_x = A + myH_1$ and $(H_2)_y = B - nxH_1$. To do this, we integrate the first equation in x and substitute in the second one. Thus we obtain a recursive expression to determine the coefficients a_{ij} and b_{ij}, and consequently H_1 and H_2 (see [2], Chapter 5, Lemma 1 or [3], Proposition 2). ■

In some cases, the expression (8) can be useful to determine separatrices of $\mathcal{F}_\omega$ directly from numerical data of H_1 and H_2. The following example illustrates such a situation.

Example 3.2. In (8), let us consider $H_1 = y^a$ and $H_2 = e \cdot x^b$ with $a, b \in \mathbb{Z}_{\geq 0}$, $b \neq 0$, $e \in \mathbb{C}^*$ and $n(b-1) \neq m(a+1)$, that is, $\omega = y^a \cdot (nxdy - mydx) + d(e \cdot x^b)$. It is immediate that $x = 0$ is a separatrix of $\mathcal{F}_\omega$. Moreover, by some computations, we get that a monomial germ $\varphi(t) = (t^\alpha, ct^\beta)$ with $c \neq 0$ parameterizes a separatrix of $\mathcal{F}_\omega$ if and only if

$$\alpha = a + 1, \quad \beta = b - 1, \quad \text{and} \quad c^{a+1} = -\frac{(a+1)be}{n(b-1) - m(a+1)}.$$

Notice that $\varphi(t)$ is not necessarily a primitive parameterization.

Similarly we can obtain the description of monomial separatrices for foliations defined by $\mathcal{F}_\omega$ considering H_1 and H_2 given by other possible monomials.

In the sequel we will take 1-forms given by a similar expression in Azevedo's Lemma but considering semiroots of an irreducible Weierstrass polynomial $F \in \mathbb{C}\{x\}[y]$.

Let $\mathcal{C}_F$ be an irreducible plane curve defined by $F \in \mathbb{C}\{x\}[y]$ with semigroup $\Gamma_F = \langle v_0, \ldots, v_g \rangle$. By the minimality of the generators set $\{v_0, \ldots, v_g\}$, any element $G \in \mathbb{C}\{x, y\}$ such that $I(F, G) = v_i$ is irreducible. As $y = 0$ is the tangent cone of $\mathcal{C}_F$, it follows that $I(F, x) = v_0$.

A set $\{F_i;\ 1 \leq i \leq g+1\} \subset \mathbb{C}\{x\}[y]$ of monic polynomials satisfying

(i) $deg_y F_i = n_0 \cdot \ldots \cdot n_{i-1} = \frac{v_0}{e_{i-1}}$;

(ii) $I(F, F_i) = v_i$

is called a **system of semiroots** of F. We say that $\{F_0 := x, F_1, \ldots, F_{g+1} := F\}$ is an **extended system of semiroots** of F and F_i is an i**th semiroot**[2] of F, for $0 \leq i \leq g+1$. Moreover, for $i \neq 0$ we have that the semigroup and the characteristic exponents of $\mathcal{C}_{F_i}$ (see [21]) are

$$\Gamma_{F_i} = \left\langle \frac{v_0}{e_{i-1}}, \ldots, \frac{v_{i-1}}{e_{i-1}} \right\rangle \qquad \text{and} \qquad \left\{ \frac{\beta_0}{e_{i-1}}, \ \ldots \ , \ \frac{\beta_{i-1}}{e_{i-1}} \right\}. \tag{9}$$

If $\{F_0 = x, F_1, \ldots, F_g, F_{g+1} = F\}$ is an extended system of semiroots of F, then we have that $\{F_0 = x, F_1, \ldots, F_k, F_{k+1}\}$ is an extended system of semiroots of F_{k+1} (see [21]).

We can obtain a system of semiroots of $F \in \mathbb{C}\{x\}[y]$ by several ways, for instance considering the approximate roots introduced by Abhyankar and Moh (see [1] or [21]) or taking representatives for elements in a minimal Standard Basis of $\frac{\mathbb{C}\{x,y\}}{\langle F \rangle}$ (see [14]).

In what follows we will consider a particular system of semiroots following Zariski's approach (see [24]) obtained by a parameterization $\varphi(t) = (t^{\beta_0}, \sum_{k \geq \beta_1} c_k t^k)$ of $\mathcal{C}_F$.

Let us denote

$$\varphi_i(t) = \left(t^{\frac{\beta_0}{e_{i-1}}}, \eta_i(t) \right) := \left(t^{\frac{\beta_0}{e_{i-1}}}, \sum_{\beta_1 \leq k < \beta_i} c_k t^{\frac{k}{e_{i-1}}} \right),$$

for $i = 1, \ldots, g+1$, where $\varphi_1(t) = (t, 0)$.

Proposition 3.3 (*Zariski, [24]*). *If $F_i \in \mathbb{C}\{x\}[y]$ is the minimal polynomial of $\eta_i(x^{\frac{e_{i-1}}{\beta_0}})$ over $\mathbb{C}((x))$ where $1 \leq i \leq g+1$, then $\{F_0 := x, F_1, \ldots, F_{g+1} := F\}$ is an extended system of semiroots of F. In particular, φ_i is a Puiseux parameterization of $\mathcal{C}_{F_i}$, for $i = 1, \ldots, g+1$.*

Proof. Denoting by $m_i = \frac{\beta_0}{e_{i-1}} = \frac{v_0}{e_{i-1}}$, we have that the minimal polynomial $F_i \in \mathbb{C}\{x\}[y]$ of $\eta_i\left(x^{\frac{1}{m_i}}\right)$ over $\mathbb{C}((x))$ is given (as in (2)) by

$$F_i(x, y) = \prod_{\alpha \in U_{m_i}} (y - \eta_i(\alpha \cdot x^{\frac{1}{m_i}})) \tag{10}$$

with $U_{m_i} = \{\alpha \in \mathbb{C};\, \alpha^{m_i} = 1\}$, for $i = 1, \ldots, g+1$ (see, for instance, [24]). In particular $deg_y F_i = \frac{v_0}{e_{i-1}}$ and $I(F, F_i) = v_i$ (see the proof of Lemma 5.1). Therefore, F_i is an ith semiroot of F. ∎

In this work we take the extended system of semiroots of F obtained as in the above proposition and we call it the **canonical system of semiroots** of F.

As an immediate consequence of the classical Euclidian division algorithm, it is possible to obtain a decomposition of any element $H \in \mathbb{C}\{x\}[y]$ in terms of a system of semiroots of F.

[2] Some authors (see [21] for instance) consider the ith semiroot of F for $0 \leq i \leq g$ as a monic polynomial $F_i \in \mathbb{C}\{x\}[y]$ satisfying $deg_y F_i = \frac{v_0}{e_i}$ and $I(F, F_i) = v_{i+1}$.

Proposition 3.4 (*See [1] or [21]*). *If $\{F_0 = x, F_1, \ldots, F_{g+1} = F\}$ is an extended system of semiroots of $F \in \mathbb{C}\{x\}[y]$ then any $H \in \mathbb{C}\{x\}[y]$ has a unique expansion given by*

$$H = \sum_{\delta=(\delta_0,\ldots,\delta_{g+1})} u_\delta F_0^{\delta_0} F_1^{\delta_1} F_2^{\delta_2} \ldots F_{g+1}^{\delta_{g+1}}, \tag{11}$$

where $u_\delta \in \mathbb{C}$,

$$0 \le \delta_i < n_i = \frac{e_{i-1}}{e_i} \text{ for } i \in \{1, \ldots, g\}, 0 \le \delta_{g+1} \le \left[\frac{deg_y H}{deg_y F}\right] \tag{12}$$

and $[r]$ denotes the integral part of $r \in \mathbb{R}$. Moreover, the order in t of the terms

$$\varphi^*(F_0)^{\delta_0} \cdot \varphi^*(F_1)^{\delta_1} \cdot \varphi^*(F_2)^{\delta_2} \cdot \ldots \cdot \varphi^*(F_g)^{\delta_g}$$

are two by two distinct, where φ is a parameterization of $\mathcal{C}_F$.

By the previous result, if $H = \sum_\delta u_\delta F_0^{\delta_0} F_1^{\delta_1} \cdot \ldots \cdot F_{g+1}^{\delta_{g+1}}$ then

$$I(F, H) = \min_\delta \left\{ \sum_{i=0}^{g} \delta_i v_i;\ \ \delta = (\delta_0, \ldots, \delta_g, 0) \text{ with } u_\delta \neq 0 \right\}.$$

Remark 3.5. Notice that the expansion (11) is not necessarily a finite sum and it is a bit different of the expansion presented in [21]. In fact, according to Corollary 5.4 of [21] any $H \in \mathbb{C}\{x\}[y]$ has a unique expansion given by a *finite* sum

$$H = \sum_{(\delta_1,\ldots,\delta_{g+1})} h_{\delta_1,\ldots,\delta_{g+1}} F_1^{\delta_1} F_2^{\delta_2} \ldots F_{g+1}^{\delta_{g+1}},$$

with $h_{\delta_1,\ldots,\delta_{g+1}} \in \mathbb{C}\{x\}$ and δ_i, $1 \le i \le g+1$, satisfying the conditions (12). Writing $h_{\delta_1,\ldots,\delta_{g+1}} = \sum_{\delta=(\delta_0,\ldots,\delta_{g+1})} u_\delta F_0^{\delta_0}$ with $u_\delta \in \mathbb{C}$ we get the expansion (11). Recall that an ith semiroot for us corresponds to an $(i-1)$th semiroot in [21] for $1 \le i \le g+1$.

Considering the canonical embedded resolution $\pi : M \to (\mathbb{C}^2, 0)$ of $\mathcal{C}_F$ and $G(\mathcal{C}_F)$ the dual graph associated to it, we have that the semiroot F_i is a *curvette*[3] with respect to a component of the exceptional divisor E corresponding to the ith-endpoint of $G(\mathcal{C}_F)$ (see Fig. 1). In particular, the extended system of semiroots appears as coordinates in the embedded resolution process of $\mathcal{C}_F$ (see [21]). We denote by T_i the ith triple point in the dual graph $G(\mathcal{C}_F)$ that appears in the canonical resolution process, or equivalently, the first triple point after that F_i is desingularized, which we indicate by $\widetilde{F}_i$.

Given the canonical system of semiroots $\{F_0 = x, F_1, \ldots, F_{g+1} = F\}$ for each $0 \le i < j \le g$ we consider $\mathcal{F}_{ij}$ the singular foliation defined by

$$\omega_{ij} = v_i F_i dF_j - v_j F_j dF_i. \tag{13}$$

Notice that $\mathcal{F}_{ij}$ defines the same foliation that $d\left(\frac{F_j^{\alpha_i}}{F_i^{\alpha_j}}\right)$ where $\alpha_i = \frac{v_i}{GCD(v_i,v_j)}$, $\alpha_j = \frac{v_j}{GCD(v_i,v_j)}$ and therefore $\mathcal{F}_{ij}$ is dicritical with separatrices given by $aF_j^{\alpha_i} - bF_i^{\alpha_j} = 0$ for all $(a : b) \in \mathbb{P}^1_{\mathbb{C}}$.

[3] A *curvette* with respect to the component E_i of the exceptional divisor E is the image in $(\mathbb{C}^2, 0)$ of a smooth curve in M meeting E_i transversely in a single point which lies on no other component of E.

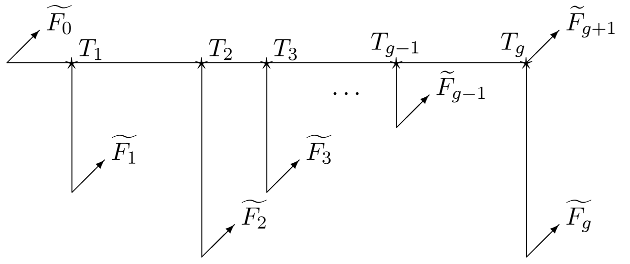

Fig. 1. Dual graph for $F_0 \cdot F_1 \cdot \ldots \cdot F_g \cdot F_{g+1}$.

Example 3.6. Let $\mathcal{C}_F$ be the plane branch defined by

$$F = (y^2 - x^3 - 2x^2y + x^4)^3 - 48x^8(y^2 - x^3 - 2x^2y + x^4) - 64x^{11} - 64x^{13}$$

with Puiseux parameterization $\varphi(t) = (t^6, t^9 + t^{12} + 2t^{13})$.

The value semigroup of $\mathcal{C}_F$ is $\Gamma = \langle 6, 9, 22\rangle$ and the canonical system of semiroots for it is

$$F_0 = x, \ \ F_1 = y, \ \ F_2 = y^2 - x^3 - 2x^2y + x^4 \ \text{ and } \ F_3 = F.$$

We have that $\omega_{01} = 6xdy - 9ydx$ admits separatrices given by $ay^2 - bx^3 = 0$, $\omega_{02} = 6xdF_2 - 22F_2dx$ with separatrices $aF_2^3 - bx^{11} = 0$ and $\omega_{12} = 9ydF_2 - 22F_2dy$ with separatrices $aF_2^9 - by^{22} = 0$, for $(a : b) \in \mathbb{P}^1_{\mathbb{C}}$.

In what follows we consider 1-forms given in a particular expression that generalizes (8). More specifically, we take $\omega = H_1 \cdot \omega_{ij} + dH_2$ where ω_{ij} is given as (13) and admitting that ω defines a foliation $\mathcal{F}_\omega$. We present a simple criterion which assures that $\mathcal{F}_\omega$ is dicritical at the ith triple point T_i of the dual graph of $\mathcal{C}_F$ and we describe a family of separatrices for it.

Our strategy is to consider initially the case $1 \le i < j = g$. The other situations are particular cases of this result by changing F by a semiroot F_{j+1} with $0 \le j < g$.

In order to obtain the results we use some technical lemmas that are presented in Section 5.

Given a plane branch $\mathcal{C}_F$ with $F \in \mathbb{C}\{x\}[y]$ as (2) and Puiseux parameterization given by $\varphi(t) = \left(t^{\beta_0}, \sum_{l \ge \beta_1} c_l t^l\right)$, we consider the family of plane branches $\mathcal{C}_{F_a}$ determined by parameterizations

$$\varphi_a(t) = \left(t^{\beta_0}, \sum_{\beta_1 \le l < \beta_g} c_l t^l + \sum_{l \ge \beta_g} a_l t^l\right) \tag{14}$$

where a_l are parameters that can be assume values in $\mathbb{C}$ and a_{β_g} nonvanishing. Notice that the coefficient c_l of t^l with $\beta_1 \le l < \beta_g$ in $\varphi_a(t)$ is precisely the coefficient of t^l in the parameterization $\varphi(t)$ of $\mathcal{C}_F$ and they are considered constant.

It is immediate that $\Gamma_{F_a} = \Gamma_F$ and $\{F_0, F_1, \ldots, F_g, F_a\}$ is an extended system of semiroots for $\mathcal{C}_{F_a}$. Moreover, if $deg_y(H) < deg_y(F) = deg_y(F_a)$ we can write $H = \sum_\delta u_\delta F_0^{\delta_0} \cdot \ldots \cdot F_g^{\delta_g} \in \mathbb{C}\{x\}[y]$ as (11) and $I(F, H) = I(F_a, H)$.

Theorem 3.7. *Let $\mathcal{F}_\omega$ be the singular holomorphic foliation defined by $\omega = H_1\omega_{ig}+dH_2$ for some $0 \le i < g$, where $H_j \in \mathbb{C}\{x\}[y]$, $\deg_y H_j < \deg_y F = v_0$ with $j = 1, 2$, $H_1 \ne 0$ and $H_2 \in \langle x, y\rangle$. Then the foliation $\mathcal{F}_\omega$ is dicritical in the last triple point T_g of the dual graph $G(\mathcal{C}_F)$ if and only if*

$$I(F, H_1) + v_i + v_g < I(F, H_2).$$

Moreover, the separatrices of $\mathcal{F}_\omega$ whose strict transform intersects transversally the component of the exceptional divisor corresponding to the triple point T_g of $G(\mathcal{C}_F)$ are plane branches parameterized by

$$\psi_u(t) = \left(t^{\beta_0}, \sum_{\beta_1 \le j < \beta_g} c_j t^j + ut^{\beta_g} + \sum_{j>\beta_g} s_j(u)t^j\right),$$

with $u \in \mathbb{C}^*$ and $s_j(u) \in \mathbb{C}(u)$.

Proof. Assume that the foliation $\mathcal{F}_\omega$ is dicritical in the last triple point T_g of the dual graph $G(\mathcal{C}_F)$. The separatrices of $\mathcal{F}_\omega$ corresponding to the triple point T_g have parameterizations as given in (14) and consequently, they satisfy the same properties as the curve $\mathcal{C}_{F_a}$ described above. Hence, if $\psi(t)$ is a parameterization of a branch of this dicritical component, then we have that

$$\psi^*\omega \equiv 0.$$

Note that, the first terms which appear in $\psi^*\omega$ are given by

$$\begin{aligned}\psi^*\omega = {} & k_1 t^{I(F,H_1)}(v_i t^{v_i} v_g t^{v_g-1} - v_g t^{v_g} v_i t^{v_i-1} + \cdots)dt \\ & + (k_2 I(F, H_2)t^{I(F,H_2)-1} + \cdots)dt\end{aligned}$$

with k_1, k_2 non-zero constants. Hence, if $I(F, H_2) \le I(F, H_1) + v_i + v_g$, then $\psi^*\omega \not\equiv 0$ against the hypothesis.

Now, assume that $I(F, H_1) + v_i + v_g < I(F, H_2)$. Let $\varphi_a(t)$ be the family of parameterizations given in (14). Thus $\varphi_a^*(\omega) = \varphi_a^*(H_1)\varphi_a^*(\omega_{ig}) + \varphi_a^*(dH_2)$. Denoting $u := a_{\beta_g}$ we will show that it is possible to take $a_i = s_i(u) \in \mathbb{C}(u)$ for every $i > \beta_g$ such that $\varphi_a^*(\omega) = 0$.

As $deg_y H_j < deg_y F$, by Proposition 3.4, if $H_j \ne 0$ then we can write $H_j = \sum_{\delta_j} b_{\delta_j} F_0^{\delta_{j0}} \cdot \ldots \cdot F_g^{\delta_{jg}}$ with $b_{\delta_j} \in \mathbb{C}$ and there exist non-negative integers $\gamma_{j0}, \ldots, \gamma_{jg}$ for $j = 1, 2$ such that

$$I(F, H_j) = I(F, F_0^{\gamma_{j0}} \cdot \ldots \cdot F_g^{\gamma_{jg}}).$$

Denoting $\mathrm{Coeff}(R(t), t^k)$ the coefficient of t^k in $R(t) \in \mathbb{C}\{t\}$, by Lemma 5.6, for any $k \ge I(F, H_1) + v_i + v_g$ we obtain that

$$\mathrm{Coeff}(\varphi_a^*(H_1\omega_{ig}), t^k) = p_k(a_{\beta_g}, \ldots, a_{k_{ig}-1}) + r_k \cdot a_{\beta_g}^{\gamma_{1g}} \cdot a_{k_{ig}} \tag{15}$$

with $r_k \in \mathbb{C}^*$ and $k_{ig} := k - I(F, H_1) - v_i - v_g + \beta_g + 1$.

For each $\prod_{l=0}^g F_l^{\delta_{2l}}$ in H_2, let us denote

$$m_{\delta_2} = \max_{0\le l\le g}\{l;\ \delta_{2l} \ne 0\} \quad \text{and} \quad I_{\delta_2} = I\left(F, \prod_{l=0}^g F_l^{\delta_{2l}}\right).$$

As $\mathrm{Coeff}(\varphi_a^*(dH_2), t^k) = \sum_{\delta_2} \mathrm{Coeff}(d(b_{\delta_2} \prod_{l=0}^{g} F_l^{\delta_{2l}}(\varphi_a)), t^k)$, by Corollary 5.4, for $k \geq I_{\delta_2}$ we get

$$\mathrm{Coeff}(d(b_{\delta_2} \prod_{l=0}^{g} F_l^{\delta_{2l}}(\varphi_a)), t^k) \in \mathbb{C}[a_{\beta_g}, \ldots, a_{\theta_{\delta_2}}] \quad \text{with} \quad \theta_{\delta_2} := k - I_{\delta_2} + \beta_{m_{\delta_2}} + 1.$$

But $\beta_{m_{\delta_2}} \leq \beta_g$ and $I_{\delta_2} \geq I(F, H_2) > I(F, H_1) + v_i + v_g$ for any δ_2, so

$$\theta_{\delta_2} = k - I_{\delta_2} + \beta_{m_{\delta_2}} + 1 < k - I(F, H_1) - v_i - v_g + \beta_g + 1 = k_{ig}$$

and, by (15), we obtain that

$$\mathrm{Coeff}(\varphi_a^*(H_1)\varphi_a^*(\omega_{ig}) + \varphi_a^*(dH_2), t^k) = P_k(a_{\beta_g}, \ldots, a_{k_{ig}-1}) + r_k \cdot a_{\beta_g}^{\gamma_{1g}} \cdot a_{k_{ig}},$$

for some polynomial $P_k(a_{\beta_g}, \ldots, a_{k_{ig}-1})$ (admitting $H_2 = 0$).

Remark that $ord_t(\varphi_a^*(\omega)) \geq \min\{ord_t(\varphi_a^*(H_1\omega_{ig})), ord_t(\varphi_a^*(dH_2))\} \geq I(F, H_1) + v_i + v_g$. In this way, $\varphi_a^*(\omega) = \varphi_a^*(H_1)\varphi_a^*(\omega_{ig}) + \varphi_a^*(dH_2) = 0$ is equivalent to solve the system

$$P_k(a_{\beta_g}, \ldots, a_{k_{ig}-1}) + r_k \cdot a_{\beta_g}^{\gamma_{1g}} \cdot a_{k_{ig}} = 0$$

for all $k \geq I(F, H_1) + v_i + v_g$. Such a solution exists and it can be obtained by the recurrence relation

$$a_{k_{ig}} = -\frac{P_k(a_{\beta_g}, \ldots, a_{k_{ig}-1})}{r_k \cdot a_{\beta_g}^{\gamma_{1g}}}, \tag{16}$$

since $r_k, a_{\beta_g} \in \mathbb{C}^*$.

In particular, taking $k = I(F, H_1) + v_i + v_g =: k_0$ in the above expression we get

$$a_{\beta_g+1} = -\left(r_{k_0} \cdot a_{\beta_g}^{\gamma_{1g}}\right)^{-1} \cdot P_{k_0}(a_{\beta_g}) \in \mathbb{C}(a_{\beta_g})$$

we vanish the coefficient of t^{k_0} in $\varphi_a^*(\omega)$.

Using the previous recurrence relation, we can vanish all terms in $\varphi_a^*(\omega)$ setting the parameters a_i in $\varphi_a(t)$ as a rational function in $\mathbb{C}(a_{\beta_g})$. Hence, considering the parameter $u := a_{\beta_g} \in \mathbb{C} \setminus \{0\}$ we get the family of parameterizations

$$\psi_u(t) := \left(t^{\beta_0}, \sum_{\beta_1 \leq j < \beta_g} c_j t^j + u t^{\beta_g} + \sum_{j > \beta_g} s_j(u) t^j\right),$$

with $s_j(u) := a_j \in \mathbb{C}(u)$ obtained in (16) and satisfying $\psi_u^*(\omega) = 0$. As $\psi_u(t)$ defines a family of plane branches with the same characteristic exponents of $\mathcal{C}_F$, every element in the family is topologically equivalent to $\mathcal{C}_F$. This allows us to conclude that the foliation defined by $\omega = 0$ is dicritical in the last triple point T_g of the dual graph $G(\mathcal{C}_F)$. ■

If we change F by a semiroot F_{j+1} for $0 \leq j < g$ in the previous theorem, then we can describe 1-forms that define dicritical foliations in any triple point T_j of the dual graph $G(\mathcal{C}_F)$.

As before, we consider $\{F_0, F_1, \ldots, F_g, F_{g+1} = F\}$ the canonical system of semiroots of F, $\Gamma_F = \langle v_0, v_1, \ldots, v_g \rangle$ and $\{\beta_0, \beta_1, \ldots, \beta_g\}$ the value semigroup and the characteristic exponents of $\mathcal{C}_F$, respectively.

Corollary 3.8. *Let $\mathcal{F}_\omega$ be the singular holomorphic foliation defined by $\omega = H_1\omega_{ij} + dH_2$ for some $0 \leq i < j \leq g$, where $H_l \in \mathbb{C}\{x\}[y]$, $\deg_y H_l < \deg_y F_{j+1} = \frac{v_0}{e_j}$ with $l = 1, 2$, $H_1 \neq 0$ and $H_2 \in \langle x, y\rangle$. The foliation $\mathcal{F}_\omega$ is dicritical in the triple point T_j of the dual graph $G(\mathcal{C}_F)$ if and only if*

$$I(F, H_1) + v_i + v_j < I(F, H_2).$$

Moreover, $\mathcal{F}_\omega$ admits a family of separatrices parameterized by

$$\psi_u(t) = \left(t^{\frac{\beta_0}{e_j}}, \sum_{\beta_1 \leq l < \beta_j} c_l t^{\frac{l}{e_j}} + u t^{\frac{\beta_j}{e_j}} + \sum_{l > \frac{\beta_j}{e_j}} s_l(u) t^l \right), \tag{17}$$

with $u \in \mathbb{C}^*$ and $s_l(u) \in \mathbb{C}(u)$.

Proof. Notice that $\{F_0, F_1, \ldots, F_j, F_{j+1}\}$ is the canonical system of semiroots for F_{j+1}. So, by (9), the value semigroup and the characteristic exponents of $\mathcal{C}_{F_{j+1}}$ are respectively, $\Gamma_{F_{j+1}} = \langle \frac{v_0}{e_j}, \ldots, \frac{v_j}{e_j}\rangle$ and $\left\{\frac{\beta_0}{e_j}, \ldots, \frac{\beta_j}{e_j}\right\}$, where $e_j = GCD(v_0, \ldots, v_j)$. In addition, $I(F_{j+1}, F_l) = \frac{v_l}{e_j} = \frac{I(F,F_l)}{e_j}$ for every $0 \leq l \leq j$.

By Proposition 3.4 any $H_l \in \mathbb{C}\{x\}[y]$ with $\deg_y H_l < \deg_y F_{j+1}$ and $l = 1, 2$ can be expressed by $H_l = \sum_\delta b_{\delta_l} F_0^{\delta_{l0}} \cdot \ldots \cdot F_j^{\delta_{lj}} \in \mathbb{C}\{x, y\}$ with $I(F_{j+1}, H_l) = I(F_{j+1}, F_0^{\gamma_{l0}} \cdot \ldots \cdot F_j^{\gamma_{lj}})$ for some non-negative integers $\gamma_{l0}, \ldots, \gamma_{lj}$. So,

$$I(F_{j+1}, H_l) = \gamma_{l0} \cdot \frac{v_0}{e_j} + \cdots + \gamma_{lj} \cdot \frac{v_j}{e_j} = \frac{\gamma_{l0} \cdot v_0 + \cdots + \gamma_{lj} \cdot v_j}{e_j} = \frac{I(F, H_l)}{e_j}.$$

Consequently, $I(F_{j+1}, H_1) + \frac{v_i}{e_j} + \frac{v_j}{e_j} < I(F_{j+1}, H_2)$ if and only if $I(F, H_1) + v_i + v_j < I(F, H_2)$.

Hence, with similar arguments as in the previous theorem, we get that if $\mathcal{F}_\omega$ is dicritical in the triple point T_j, then the condition $I(F_{j+1}, H_1) + \frac{v_i}{e_j} + \frac{v_j}{e_j} < I(F_{j+1}, H_2)$ must be fulfilled.

Considering the family given by parameterizations

$$\varphi_a(t) = \left(t^{\frac{\beta_0}{e_j}}, \sum_{\beta_1 \leq l < \beta_j} c_l t^{\frac{l}{e_j}} + \sum_{l \geq \frac{\beta_j}{e_j}} a_l t^l \right) \tag{18}$$

with $a_{\frac{\beta_j}{e_j}} \neq 0$ and proceeding with the same analysis on the coefficients of $\varphi_a^*(\omega)$ as in the previous theorem, in order to obtain $\varphi_a^*(\omega) = 0$, for all $k \geq I(F_{j+1}, H_1) + \frac{v_i}{e_j} + \frac{v_j}{e_j}$ we must have

$$P_k\left(a_{\frac{\beta_j}{e_j}}, \ldots, a_{k_{ij}-1}\right) + r_k \cdot a_{\frac{\beta_j}{e_j}}^{\gamma_{1j}} \cdot a_{k_{ij}} = 0 \tag{19}$$

where $k_{ij} := k - I(F_{j+1}, H_1) - \frac{v_i}{e_j} - \frac{v_j}{e_j} + \frac{\beta_j}{e_j} + 1$ and $r_k \in \mathbb{C}^*$. Hence, we obtain the recurrence relation

$$a_{k_{ij}} = -\left(r_k \cdot a_{\frac{\beta_j}{e_j}}^{\gamma_{1j}}\right)^{-1} \cdot P_k\left(a_{\frac{\beta_j}{e_j}}, \ldots, a_{k_{ij}-1}\right).$$

In this way, the corollary follows from the previous theorem considering the curve $\mathcal{C}_{F_{j+1}}$, $u := a_{\frac{\beta_j}{e_j}} \in \mathbb{C}^*$ and $a_l = s_l(u) \in \mathbb{C}(u)$ for $l > \frac{\beta_j}{e_j}$. ■

Notice that Theorem 3.7 and Corollary 3.8 give us a constructive and effective method to present dicritical foliations in a given triple point in the dual graph of a plane branch and to describe parameterizations for the separatrices in such dicritical component up to the desired order.

Remark 3.9. Given $\omega = H_1\omega_{ij} + dH_2$ satisfying the hypothesis of the previous corollary and $I(F, H_1) = \sum_{l=0}^{j-1} \gamma_{1l} v_l$, that is, $\gamma_{1j} = 0$ then, in (19), we obtain $P_k(a_{\frac{\beta_j}{e_j}}, \ldots, a_{k_{ij}-1}) + r_k \cdot a_{k_{ij}} = 0$ and consequently, a_i is a polynomial in $a_{\frac{\beta_j}{e_j}}$ for any $i > \frac{\beta_j}{e_j}$, since r_k is a non-zero constant. In this case, we obtain an extra separatrix for $\mathcal{F}_\omega$ taking $a_{\frac{\beta_j}{e_j}}$ with a (not necessarily primitive) parameterization

$$\psi_0(t) = \left(t^{\frac{\beta_0}{e_j}}, \sum_{\beta_1 \le l < \beta_j} c_l t^{\frac{l}{e_j}} + \sum_{l > \frac{\beta_j}{e_j}} s_l(0) t^l\right)$$

and not topologically equivalent to $\mathcal{C}_{F_{j+1}}$.

It is immediate that any irreducible factor $H \in \mathbb{C}\{x, y\}$ of H_1 and H_2 define a separatrix for $\mathcal{F}_\omega$. In addition, if F_i (respectively F_j) divides H_2, then F_i (respectively F_j) is a separatrix for $\mathcal{F}_\omega$.

The following examples illustrate the above results.

Example 3.10. Let us consider the plane branch $\mathcal{C}_F$ with semigroup $\Gamma = \langle 6, 9, 22\rangle$ as in Example 3.6. Recall that the characteristics exponents of $\mathcal{C}_F$ are $\beta_0 = 6$, $\beta_1 = 9$ and $\beta_2 = 13$.

- Notice that $\zeta_1 = (6xy)dy - (9y^2 + 5x^4)dx = y \cdot \omega_{01} - d(x^5)$ satisfies

  $$24 = 9 + 6 + 9 = I(F, H_1) + v_0 + v_1 < I(F, H_2) = 30,$$

  consequently by Corollary 3.8, $\mathcal{F}_{\zeta_1}$ is dicritical admitting a family of separatrices in the first triple point of $G(\mathcal{C}_F)$ parameterized by

  $$\psi_u(t) = \left(t^2,\ ut^3 + \frac{5}{6u}t^5 - \frac{25}{72u^3}t^7 + \frac{125}{432u^5}t^9 + \sum_{i \ge 11} q_{1,i}(u) t^i\right)$$

  with $q_{1,i}(u) \in \mathbb{C}[u^{-1}]$. Moreover, by Example 3.2, $\mathcal{F}_{\zeta_1}$ also admits the separatrices $(0, t)$, $\left(t, \frac{\sqrt{15}}{3}t^2\right)$ and $\left(t, -\frac{\sqrt{15}}{3}t^2\right)$.

- Taking $\zeta_2 = x(12y^2 - 12x^2y + x^5)dy + 2y(2x^3 + 10x^2y + x^4 - 11y^2 + 3x^5)dx = y \cdot \omega_{02} + d(x^6y)$ we have

$$37 = 9 + 6 + 22 = I(F, H_1) + v_0 + v_2 < I(F, H_2) = 45.$$

So, by Theorem 3.7, $\mathcal{F}_{\zeta_2}$ is a dicritical foliation in the last triple point of $G(\mathcal{C}_F)$. Moreover, the family

$$\psi_u(t) = \left(t^6, t^9 + t^{12} + ut^{13} - \frac{u^2}{2}t^{17} + \left(-\frac{15}{32} + \frac{u^3}{2}\right)t^{21} - \frac{1}{44}t^{24} + \sum_{i\geq 25} q_{2,i}(u)t^i\right)$$

with $q_{2,i}(u) \in \mathbb{C}[u]$ describe separatrices for $\mathcal{F}_{\zeta_2}$. Others separatrices for $\mathcal{F}_{\zeta_2}$ are $(0, t)$, $(t, 0)$ and, by Remark 3.9, $\psi_0(t) = \left(t^6, t^9 + t^{12} - \frac{15}{32}t^{21} - \frac{1}{44}t^{24} + \sum_{i\geq 9} q_{2,i}(0)t^{3i}\right)$, that is,

$$\left(t^2, t^3 + t^4 - \frac{15}{32}t^7 - \frac{1}{44}t^8 + \sum_{i\geq 9} q_{2,i}(0)t^i\right).$$

- Considering

$$\zeta_3 = \left(2x(11x^3 - 11x^4 - 2y^2) + y\left(\frac{227}{10}x^3 + \frac{33}{5}y^2 - \frac{99}{10}x^2y + \frac{33}{10}x^4\right)\right)dy + xy\left(\frac{33}{20}y + 9x\right)(-3x - 4y + 4x^2)dx$$

we can write $\zeta_3 = x \cdot \omega_{12} + d\left(\frac{33}{20}y^2F_2\right)$ with $F_2 = y^2 - x^3 - 2x^2y + x^4$. As

$$37 = 6 + 9 + 22 = I(F, H_1) + v_1 + v_2 < I(F, H_2) = 2 \cdot 9 + 22 = 40$$

the previous results ensure that $\mathcal{F}_{\zeta_3}$ is dicritical in the last triple point of $G(\mathcal{C}_F)$ and

$$\psi_u(t) = \left(t^6, t^9 + t^{12} - ut^{13} + \frac{35}{18}u^2t^{17} + \frac{473}{180}ut^{19} - \frac{748}{189}u^2t^{20} + \sum_{i\geq 21} q_{3,i}(u)t^i\right)$$

define separatrices for $\mathcal{F}_{\eta_3}$. By Remark 3.9, the curves $(t, 0)$ and $\psi_0(t) = (t^2, t^3 + t^4)$ (that is, the curve defined by $F_2 = y^2 - x^3 - 2x^2y + x^4$) are also separatrices for $\mathcal{F}_{\zeta_3}$.

Let $\mathcal{C}_F$ be the plane branch with semigroup $\Gamma = \langle 6, 9, 22\rangle$ as in Example 3.6 and

$$\zeta = (6xy)dy - (9y^2 + 4x^3)dx = y \cdot \omega_{01} - d(x^4).$$

In this case $24 = 9 + 6 + 9 = I(F, H_1) + v_0 + v_1 = I(F, H_2)$ and the foliation $\mathcal{F}_\zeta$ is not dicritical. The unique separatrix of the foliation $\mathcal{F}_\zeta$ is the curve $x = 0$. Note that $\mathcal{F}_\zeta$ is not a second type foliation: there is a saddle–node singularity in one of the corners of its reduction of singularities (see [18]).

4. Analytical invariants of $\mathcal{C}_F$ and dicritical foliations

As before, $\mathcal{C}_F$ is a plane branch defined by a Weierstrass polynomial $F \in \mathbb{C}\{x\}[y]$ with $mult(F) = v_0$ admitting a parameterization $\varphi(t) = (x(t), y(t))$. Considering

$\varphi^* : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ defined by $\varphi^*(H) = H(x(t), y(t))$ we have that $\ker\varphi^* = \langle F\rangle$ and $Im\varphi^* = \mathbb{C}\{x(t), y(t)\}$. So, we obtain the exact sequence of $\mathbb{C}$-algebras

$$\{0\} \to \langle F\rangle \hookrightarrow \mathbb{C}\{x, y\} \to \mathbb{C}\{x(t), y(t)\} \to \{0\}.$$

In this way, the local ring of $\mathcal{C}_F$ is $\mathcal{O} := \frac{\mathbb{C}\{x,y\}}{\langle F\rangle} \cong \mathbb{C}\{x(t), y(t)\} \subset \mathbb{C}\{t\} =: \overline{\mathcal{O}}$ where $\overline{\mathcal{O}}$ denotes the integral closure of $\mathcal{O}$ in its field of fractions.

Let $\nu : \overline{\mathcal{O}} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be the discrete normalized valuation given by $\nu(p(t)) = ord_t(p(t))$ for $p(t) \in \mathbb{C}\{t\}$ ($\nu(0) = \infty$) and denote $\nu(H) = \nu(\varphi^*(H))$ for $H \in \mathbb{C}\{x, y\}$. In this way, the value semigroup of $\mathcal{C}_F$ is given by $\Gamma_F = \nu(\mathcal{O})$. In addition, the conductor ideal $(\mathcal{O} : \overline{\mathcal{O}}) := \{h \in \mathcal{O};\ h\overline{\mathcal{O}} \subseteq \mathcal{O}\}$ of $\mathcal{O}$ in $\overline{\mathcal{O}}$ satisfies $(\mathcal{O} : \overline{\mathcal{O}}) = \langle t^{\mu_F}\rangle$, that is, if $p(t) \in \mathbb{C}\{t\}$ is such that $\nu(p(t)) \geq \mu_F$, then there exists $H \in \mathbb{C}\{x, y\}$ with $\varphi^*(H) = p(t)$. The integer μ_F is called the conductor of Γ_F.

If $H \in \langle x, y\rangle$ by (7) we get $ord_t(\varphi^*(dH)) = \nu(H)-1$, that is, $ord_t(\varphi^*(dH))+1 \in \Gamma_F$. In this way, given $\omega \in \Omega^1$ such that $\varphi^*(\omega) \neq 0$ we define the value of ω as $\nu(\omega) := ord_t(\varphi^*(\omega)) + 1$. Setting $\nu(\omega) = \infty$ if $\varphi^*(\omega) = 0$, we define

$$\Lambda_F := \left\{\nu(\omega);\ \omega \in \Omega^1\right\} \supseteq \Gamma_F \setminus \{0\}.$$

Remark 4.1. The set $\Lambda_F \subset \overline{\mathbb{N}}$ is an analytic invariant for $\mathcal{C}_F$ and it is the main ingredient for the analytic classification of plane branches presented in [15] (see [16] for an extended version).

In particular, the set Λ_F allows us to identify terms in a parameterization of $\mathcal{C}_F$ that can be eliminated by change of parameter and coordinates. More specifically, by Proposition 1.3.11 and Theorem 1.3.9 in [16], given a plane branch $\mathcal{C}_F$ with Puiseux parameterization $\left(t^{v_0}, \sum_{i>v_0} a_i t^i\right)$ if there exists $\omega = Adx + Bdy \in \Omega^1$ with $\nu(\omega) = k + v_0$, $A \in \langle x, y\rangle^2$ and $B \in \langle x^2, y\rangle$ then $\mathcal{C}_F$ is analytically equivalent to a plane branch with parameterization $\left(t^{v_0}, \sum_{i>v_0} b_i t^i\right)$ where $b_i = a_i$ for $i < k$ and $b_k = 0$.

We can define Λ_F by means the $\mathcal{O}$-module of Kähler differentials of $\mathcal{O}$ (or $\mathcal{C}_F$), that is,

$$\Omega_{\mathcal{O}} := \Omega_{\mathcal{O}/\mathbb{C}} = \frac{\mathcal{O}dx + \mathcal{O}dy}{\mathcal{O}(F_x dx + F_y dy)} \cong \frac{\Omega^1}{\mathcal{F}(\mathcal{C}_F)},$$

where $\mathcal{F}(\mathcal{C}_F) := F \cdot \Omega^1 + \mathbb{C}\{x, y\} \cdot dF$.

If $\eta \in \mathcal{F}(\mathcal{C}_F)$ then $\varphi^*(\eta) = 0$ and $\varphi^*(\omega + \eta) = \varphi^*(\omega)$ for any $\omega \in \Omega^1$. Thus, given $\overline{\omega} = \omega + \mathcal{F}(\mathcal{C}_F) \in \Omega_{\mathcal{O}}$ we can define $\varphi^*(\overline{\omega}) := \varphi^*(\omega)$ and $\nu(\overline{\omega}) := \nu(\omega)$. For any singular plane branch, the torsion submodule $\mathcal{T} := \{\overline{\omega} \in \Omega_{\mathcal{O}};\ h\overline{\omega} = 0 \text{ for some } h \in \mathcal{O} \setminus \{0\}\} \subset \Omega_{\mathcal{O}}$ is non trivial and we can rewrite $\mathcal{T} = \{\overline{\omega} \in \Omega_{\mathcal{O}};\ \varphi^*(\overline{\omega}) = 0\}$. In particular, we have $\frac{\Omega_{\mathcal{O}}}{\mathcal{T}} \cong \varphi^*(\Omega_{\mathcal{O}}) = \varphi^*(\Omega^1) \subset \mathbb{C}\{t\}$ and $\Lambda_F = \{\nu(\overline{\omega});\ \overline{\omega} \in \Omega_{\mathcal{O}} \setminus \mathcal{T}\}$ (see Section 7.1 in [13] and [14]).

There exists a finite subset $L = \{\ell_1, \ldots, \ell_k\} \subset \Lambda_F$ such that any $\ell \in \Lambda_F$ can be expressed as $\ell = \ell_i + \gamma$ for some $\gamma \in \Gamma_F$ and $\ell_i \in L$, that is, the set Λ_F is a finitely generated Γ_F-monomodule. A set $\mathcal{G} = \{\overline{\omega_1}, \ldots, \overline{\omega_k}\} \subset \frac{\Omega_{\mathcal{O}}}{\mathcal{T}}$ such that $\nu(\overline{\omega_i}) = \nu(\omega_i) = \ell_i \in L$ is a set of generators for $\frac{\Omega_{\mathcal{O}}}{\mathcal{T}}$ as $\mathcal{O}$-module and it is called a **Standard Basis** of $\frac{\Omega_{\mathcal{O}}}{\mathcal{T}}$.

Fixing a minimal set of generators L for Λ_F, that is, L is a set of generators for Λ_F (as Γ_F-monomodule) and $\ell_i \notin \ell_j + \Gamma_F$ for $\ell_i, \ell_j \in L$ and $i \neq j$, we call $\mathcal{G} = \{\overline{\omega_i};\ \nu(\omega_i) \in L\}$ a **Minimal Standard Basis** for $\frac{\Omega_{\mathcal{O}}}{\mathcal{T}}$. In [14] we provide an algorithm to compute a (minimal) Standard Basis $\mathcal{G}$ for $\frac{\Omega_{\mathcal{O}}}{\mathcal{T}}$ by means a parameterization $\varphi(t)$ and in Section 7.3 of [13] we describe a method to obtain $\mathcal{G}$ using F.

Remark 4.2. The set Λ_F determines and it is determined by the values of elements in the Jacobian ideal $J_F := \langle F_x, F_y \rangle$ in $\mathcal{O}$. In fact, we have the isomorphism (as $\mathcal{O}$-module)

$$\begin{array}{rccc} \Psi: & \mathcal{O}F_y + \mathcal{O}F_x & \rightarrow & \varphi^*(\Omega_{\mathcal{O}}) \cong \frac{\Omega_{\mathcal{O}}}{\mathcal{T}} \\ & AF_y + BF_x & \mapsto & \varphi^*(Adx - Bdy). \end{array}$$

Notice that $0 = dF = F_x dx + F_y dy \in \Omega_{\mathcal{O}}$ then given $\omega = Adx - Bdy \in \Omega_{\mathcal{O}}$ we have that

$$F_y \omega = (AF_y + BF_x)dx - BdF = (AF_y + BF_x)dx \quad (\text{in } \Omega_{\mathcal{O}}).$$

As $F \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial with $mult(F) = v_0$, then $\nu(F_y) = \mu_F + v_0 - 1$ (see Corollary 7.16 in [12]) and $\nu(dx) = v_0$. Thus, by the above expression, we get $\nu(AF_y + BF_x) = \nu(F_y\omega) - v_0 = \mu_F - 1 + \nu(\omega)$. In this way,

$$\nu(J_F) = \nu(\mathcal{O}F_x + \mathcal{O}F_y) = \mu_F - 1 + \Lambda_F.$$

Pol, in [20], generalizes this result for a reduced complete intersection curve.

Notice that if ℓ is an element of a minimal set of generators L for Λ_F, then any $\omega = Adx + Bdy \in \Omega^1$ such that $\nu(\omega) = \ell$ defines a foliation, that is, $GCD(A, B) = 1$. Indeed, if $A = A_1 G$ and $B = B_1 G$ with $G \in \langle x, y \rangle$ then $\ell = \nu(\omega) = \nu(G) + \nu(A_1 dx + B_1 dy)$ that is a contradiction by minimality of L.

Let $\omega = P_1 dx + P_2 dy \in \Omega^1$. By the Weierstrass Division Theorem we can express $P_2 = Q_2 F_y + A_2$ and $P_1 - Q_2 F_x = Q_1 F + A_1$, where $Q_1, Q_2 \in \mathbb{C}\{x, y\}$, $A_1, A_2 \in \mathbb{C}\{x\}[y]$, $deg_y(A_1) < deg_y(F) = v_0$ and $deg_y(A_2) < deg_y(F_y) = v_0 - 1$. Thus

$$\omega = P_1 dx + P_2 dy = A_1 dx + A_2 dy + FQ_1 dx + Q_2 dF.$$

As $FQ_1 dx + Q_2 dF \in \mathcal{F}(\mathcal{C}_F)$ we get $\nu(\omega) = \nu(A_1 dx + A_2 dy)$ and consequently

$$\Lambda_F = \{\nu(Adx + Bdy);\ A, B \in \mathbb{C}\{x\}[y] \text{ with } deg_y(A) < v_0 \text{ and } deg_y(B) < v_0 - 1\}. \tag{20}$$

Using Corollary 3.8, we can relate elements of Λ_F with the contact of $\mathcal{C}_F$ and a curve corresponding to a particular element of the family $\psi_u(t)$ as in (17).

For commodity to the reader we recall some results concerning the contact and the intersection multiplicity of plane curves (see [12] or [22]).

Consider two irreducible plane curves $\mathcal{C}_F$ and $\mathcal{C}_G$ with Puiseux parameterizations given respectively by $(t^{v_0}, \phi(t))$ and $(t^{v_0'}, \phi'(t))$. The contact $c(\mathcal{C}_F, \mathcal{C}_G)$ of $\mathcal{C}_F$ and $\mathcal{C}_G$ is defined by

$$c(\mathcal{C}_F, \mathcal{C}_G) = \max_{\gamma,\delta} \frac{ord_t(\phi(\gamma t^{v_0'}) - \phi'(\delta t^{v_0}))}{v_0 v_0'}$$

where $\gamma, \delta \in \mathbb{C}$ with $\gamma^{v_0} = 1 = \delta^{v_0'}$.

In what follows we take parameterizations of $\mathcal{C}_F$ and $\mathcal{C}_G$ such that the maximum in the previous expression is achieved.

Remark that, by definition, the series $\phi(t^{v'_0})$ and $\phi'(t^{v_0})$ coincide up to the order $c(\mathcal{C}_F,\mathcal{C}_G)v_0v'_0-1$. In addition, for any irreducible plane curve $\mathcal{C}_H$ we have that

$$c(\mathcal{C}_F,\mathcal{C}_H)\geq \min\{c(\mathcal{C}_F,\mathcal{C}_G),c(\mathcal{C}_H,\mathcal{C}_G)\}$$

and the two smallest numbers among these three coincide.

Let β_i, e_i, n_i and v_i be the integers defined at the beginning of Section 2, related to the branch $\mathcal{C}_F$. We indicate by β'_i, e'_i, n'_i and v'_i the respective integers for $\mathcal{C}_G$.

The contact of two branches $\mathcal{C}_F$ and $\mathcal{C}_G$ is related to the intersection multiplicity $I(F,G)$ in the following way (see [19]):

If $c(\mathcal{C}_F,\mathcal{C}_G)<\frac{\beta_1}{v_0}$, then $I(F,G)=c(\mathcal{C}_F,\mathcal{C}_G)v_0v'_0$. Moreover $\frac{\beta_q}{v_0}\leq c(\mathcal{C}_F,\mathcal{C}_G)<\frac{\beta_{q+1}}{v_0}$ for some $q\in\{1,\dots,g\}$ if and only if

$$\frac{I(F,G)}{v'_0}=\frac{n_qv_q+v_0\cdot c(\mathcal{C}_F,\mathcal{C}_G)-\beta_q}{n_0\cdot\ldots\cdot n_q}. \tag{21}$$

Let us consider $\omega=H_1\omega_{ij}+dH_2$ with $0\leq i<j\leq g$ satisfying the hypothesis of Corollary 3.8, that is, ω defines a dicritical foliation in the jth triple point of the dual graph of $\mathcal{C}_F$ and $\mathcal{F}_\omega$ admits a family of separatrices parameterized by $\psi_u(t)$ as (17).

Let $F_u\in\mathbb{C}\{x\}[y]$ be the irreducible Weierstrass polynomial such that $F_u(\psi_u(t))=0$. In particular, $\mathcal{C}_{F_u}$ and $\mathcal{C}_{F_{j+1}}$ are topologically equivalent and consequently they admit the same characteristic exponents $\left\{\beta'_0:=\frac{\beta_0}{e_j},\dots,\beta'_j:=\frac{\beta_j}{e_j}\right\}$ and the same value semigroup $\langle v'_0:=\frac{v_0}{e_j},\dots,v'_j:=\frac{v_j}{e_j}\rangle$.

Remark 4.3. Notice that $\frac{\beta'_j}{v'_0}=\frac{\beta_j}{v_0}\leq c(\mathcal{C}_F,\mathcal{C}_{F_u})\leq\frac{\beta_{j+1}}{v_0}$ for any $u\in\mathbb{C}^*$ and, by (17), $\frac{\beta'_j}{v'_0}=\frac{\beta_j}{v_0}=c(\mathcal{C}_F,\mathcal{C}_{F_u})$ if and only if $u\neq c_{\beta_j}$. In fact, if $\frac{\beta_{j+1}}{v_0}<c(\mathcal{C}_F,\mathcal{C}_{F_u})$ then in $\psi_u(t)$ we should have a term with exponent $\frac{\beta_{j+1}v'_0}{v_0}=\frac{\beta_{j+1}}{e_j}\notin\mathbb{Z}_{\geq 0}$, that is an absurd.

By above remark, for any $u\neq c_{\beta_j}$ we conclude, by (21), that

$$I(F,F_u)=\frac{v'_0n_jv_j}{n_0\cdot\ldots\cdot n_j}=\frac{v'_0n_jv_j}{\frac{v_0}{e_j}}=\frac{v'_0n_jv_j}{v'_0}=n_jv_j.$$

So, for $u=c_{\beta_j}$ the curve $\mathcal{C}_{F_u}$ has a special behavior. In order to simplify the notations we denote

$$\psi_\star(t):=\psi_{c_{\beta_j}}(t)\quad\text{and}\quad F_\star:=F_{c_{\beta_j}}. \tag{22}$$

If F is not a separatrix of ω, that is, $\infty\neq\nu(\omega)\in\Lambda_F$, then we can relate $c(\mathcal{C}_F,\mathcal{C}_{F_\star})$ and $\nu(\omega)$.

Theorem 4.4. *Given $\omega=H_1\omega_{ij}+dH_2$ with $0\leq i<j\leq g$ such that $I(F,H_1)+v_i+v_j<I(F,H_2)$, where $H_l\in\mathbb{C}\{x\}[y]$, $\deg_y H_l<\deg_y F_{j+1}=\frac{v_0}{e_j}$ for $l=1,2$, $H_1\neq 0$ and $H_2\in\langle x,y\rangle$ we have*

$$c(\mathcal{C}_F,\mathcal{C}_{F_\star})=\frac{\nu(\omega)-I(F,H_1)-v_i-v_j+\beta_j}{v_0}.$$

Proof. Given $\omega = A(x, y)dx + B(x, y)dy \in \Omega^1$, $\phi(t) = (x(t), y(t)) \in \mathbb{C}\{t\} \times \mathbb{C}\{t\}$ and $n \in \mathbb{Z}_{>0}$ we denote

$$(\omega\phi)(t) = A(x(t), y(t))x'(t) + B(x(t), y(t))y'(t) = \phi^*(\omega);$$

$$(\omega\phi)(t^n) = A(x(t^n), y(t^n))x'(t^n) + B(x(t^n), y(t^n))y'(t^n);$$

$$\omega(\phi(t^n)) = A(x(t^n), y(t^n))(x(t^n))' + B(x(t^n), y(t^n))(y(t^n))' = nt^{n-1}(\omega\phi)(t^n).$$

Consequently,

$$\begin{array}{r} \mathrm{Coeff}((\omega\phi)(t^n), t^{kn}) = \mathrm{Coeff}((\omega\phi)(t), t^k) \\ nt^{n-1}\mathrm{Coeff}((\omega\phi)(t^n), t^{kn}) = \mathrm{Coeff}(\omega(\phi(t^n)), t^{n(k+1)-1}). \end{array} \tag{23}$$

By the proof of Corollary 3.8, for any member of the family $\varphi_a(t)$, given in (18), we obtain that $\mathrm{Coeff}((\omega\varphi_a)(t), t^k) \in \mathbb{C}[a_{\beta'_j}, \ldots, a_\epsilon]$ where $\epsilon = k - I(F_{j+1}, H_1) + \beta'_j - v'_j - v'_i + 1$. So, by (23), the coefficients of terms with order up to ϵ in $\varphi_a(t)$, or equivalently, the coefficients of terms with order up to $v_0\epsilon$ in $\varphi_a(t^{v_0})$ determine all coefficients of terms with order up to

$$\begin{array}{ll} \epsilon + I(F_{j+1}, H_1) - \beta'_j + v'_j + v'_i - 1 & \text{in } (\omega\varphi_a)(t); \\ v_0(\epsilon + I(F_{j+1}, H_1) - \beta'_j + v'_j + v'_i - 1) & \text{in } (\omega\varphi_a)(t^{v_0}); \\ v_0(\epsilon + I(F_{j+1}, H_1) - \beta'_j + v'_j + v'_i) - 1 & \text{in } \omega(\varphi_a(t^{v_0})). \end{array}$$

As $\psi_\star(t^{v_0})$ is a member of the family $\varphi_a(t^{v_0})$ and $\mathrm{Coeff}((\omega\psi_\star)(t), t^k) = 0$ for all k, by (23), we have that $\mathrm{Coeff}((\omega\psi_\star)(t^{v_0}), t^k) = 0$ for all k. But $\psi_\star(t^{v_0})$ and $\varphi(t^{v'_0})$ coincide up to the order $v_0v'_0c(\mathcal{C}_F, \mathcal{C}_{F_\star}) - 1$ and $0 \neq \mathrm{Coeff}(\psi_\star(t^{v_0}), t^{v_0v'_0c(\mathcal{C}_F,\mathcal{C}_{F_\star})}) \neq \mathrm{Coeff}(\varphi(t^{v'_0}), t^{v_0v'_0c(\mathcal{C}_F,\mathcal{C}_{F_\star})})$, then

$$0 = \mathrm{Coeff}(\omega(\psi_\star(t^{v_0})), t^k) = \mathrm{Coeff}(\omega(\varphi(t^{v'_0})), t^k)$$

for all

$$\begin{aligned} k &< v_0(v'_0c(\mathcal{C}_F, \mathcal{C}_{F_\star}) + I(F_{j+1}, H_1) - \beta'_j + v'_j + v'_i) - 1 = v_0v'_0c(\mathcal{C}_F, \mathcal{C}_{F_\star}) \\ &\quad + v'_0(I(F, H_1) - \beta_j + v_j + v_i) - 1, \end{aligned}$$

recall that $v'_l = \frac{v_l}{e_j}$, $\beta'_l = \frac{\beta_l}{e_j}$ for $0 \leq l \leq j$ and $\frac{I(F,H_1)}{e_j} = I(F_{j+1}, H_1)$.

Moreover, we get $\mathrm{Coeff}(\omega(\varphi(t^{v'_0})), t^k) \neq 0$ for $k = v_0v'_0c(\mathcal{C}_F, \mathcal{C}_{F_\star}) + v'_0(I(F, H_1) - \beta_j + v_j + v_i) - 1$. So, by (23), $k = v'_0(ord_t((\omega\varphi)(t))) + v'_0 - 1$, that is, $k = v'_0(ord_t((\omega\varphi)(t)) + 1) - 1 = v'_0\nu(\omega) - 1$.

In this way, $v'_0\nu(\omega) - 1 = k = v_0v'_0c(\mathcal{C}_F, \mathcal{C}_{F_\star}) + v'_0(I(F, H_1) - \beta_j + v_j + v_i) - 1$, that is,

$$c(\mathcal{C}_F, \mathcal{C}_{F_\star}) = \frac{\nu(\omega) - I(F, H_1) - v_i - v_j + \beta_j}{v_0}.$$

■

Let us illustrate the previous theorem.

Example 4.5. Let $\mathcal{C}_F$ be a plane branch with semigroup $\Gamma = \langle 6, 9, 22\rangle$ and Puiseux parameterization $\varphi(t) = (t^6, t^9 + t^{12} + 2t^{13})$ as in Example 3.10.

Considering $\zeta_1 = y \cdot \omega_{01} - d(x^5)$ we have $\nu(\zeta_1) = 27$. In this case $u = c_{\beta_1} = 1$,

$$\psi_\star(t) = \left(t^2,\ t^3 + \frac{5}{6}t^5 - \frac{25}{72}t^7 + h.o.t.\right) \quad \text{and}$$
$$c(\mathcal{C}_F, \mathcal{C}_{F_\star}) = \frac{\nu(\zeta_1) - I(F, y) - v_0 - v_1 + \beta_1}{6} = 2.$$

For $\zeta_2 = y \cdot \omega_{02} + d(x^6 y)$ we have $\nu(\zeta_2) = 41$, $u = c_{\beta_2} = 2$,

$$\psi_\star(t) = \left(t^6, t^9 + t^{12} + 2t^{13} - 2t^{17} + h.o.t.\right) \quad \text{and}$$
$$c(\mathcal{C}_F, \mathcal{C}_{F_\star}) = \frac{\nu(\zeta_2) - I(F, y) - v_0 - v_2 + \beta_2}{6} = \frac{17}{6}.$$

Given $\zeta_3 = x \cdot \omega_{12} + d\left(\frac{33}{20} y^2 F_2\right)$ we get $\nu(\zeta_3) = 41$, $u = -c_{\beta_2} = -2$,

$$\psi_\star(t) = \left(t^6, t^9 + t^{12} + 2t^{13} + \frac{70}{9}t^{17} + h.o.t.\right) \quad \text{and}$$
$$c(\mathcal{C}_F, \mathcal{C}_{F_\star}) = \frac{\nu(\zeta_3) - I(F, x) - v_1 - v_2 + \beta_2}{6} = \frac{17}{6}.$$

By (21) we can determine the intersection multiplicity $I(F, F_\star)$ by means the contact $c(\mathcal{C}_F, \mathcal{C}_{F_\star})$ and consequently, by Theorem 4.4, we can relate $I(F, F_\star)$ and $\nu(\omega)$.

Corollary 4.6. *With the previous notations, we have*

$$I(F, F_\star) = \nu(\omega) - I(F, H_1) + (n_j - 1)v_j - v_i.$$

In particular, if $g = 1$ *then* $\nu(\omega) = I(F, H_1 \cdot F_\star) - (\mu_F - 1)$.

Proof. By Remark 4.3, we have $\frac{\beta_j}{v_0} < c(\mathcal{C}_F, \mathcal{C}_{F_\star}) \leq \frac{\beta_{j+1}}{v_0}$.

If $c(\mathcal{C}_F, \mathcal{C}_{F_\star}) \neq \frac{\beta_{j+1}}{v_0}$, by (21), we get

$$I(F, F_\star) = v_0' \cdot \left(\frac{n_j v_j + v_0 \cdot c(\mathcal{C}_F, \mathcal{C}_{F_\star}) - \beta_j}{n_0 \cdot \ldots \cdot n_j}\right) = \nu(\omega) - I(F, H_1) + (n_j - 1)v_j - v_i,$$

where the last equality is obtained by Theorem 4.4 remembering that $n_0 \cdot \ldots \cdot n_j = \frac{v_0}{e_j} = v_0'$.

If $c(\mathcal{C}_F, \mathcal{C}_{F_\star}) = \frac{\beta_{j+1}}{v_0}$, then by previous theorem we have $\beta_{j+1} = \nu(\omega) - I(F, H_1) - v_i - v_j + \beta_j$ that is, by (4), $\nu(\omega) - I(F, H_1) + (n_j - 1)v_j - v_i = v_{j+1}$. On the other hand, by (21), we obtain

$$I(F, F_\star) = v_0' \cdot \left(\frac{n_{j+1} v_{j+1} + \beta_{j+1} - \beta_{j+1}}{n_0 \cdot \ldots \cdot n_{j+1}}\right) = v_{j+1}$$

that gives us the result.

In particular, if $g = 1$, we have $i = 0$ and $j = g = 1$. So, by (6) we have $\nu(\omega) = I(F, H_1 \cdot F_\star) - (n_1 - 1)v_1 + v_0 = I(F, H_1 \cdot F_\star) - (\mu_F - 1)$. ■

If the value semigroup of $\mathcal{C}_F$ is $\Gamma_F = \langle v_0, v_1\rangle$, that is, $g = 1$ then Lemma 3.1 ensures that any 1-form $\eta = Adx + Bdy \in \Omega^1$ can be written as $\eta = P_1\omega_{01} + dP_2$ with

$P_1, P_2 \in \mathbb{C}\{x, y\}$. Considering $P_i = Q_i F + H_i$ such that $Q_i \in \mathbb{C}\{x, y\}$ with $H_i \in \mathbb{C}\{x\}[y]$ and $deg_y(H_i) < deg_y(F) = v_0$ we have $\eta = H_1\omega_{01} + dH_2 + F \cdot (Q_1\omega_{01} + dQ_2) + Q_2 dF$.

In this way, for any $\eta \in \Omega^1$ there exists $\omega = H_1\omega_{01} + dH_2$ with $H_i \in \mathbb{C}\{x\}[y]$ and $deg_y(H_i) < deg_y(F) = v_0$ for $i = 1, 2$ such that $\nu(\eta) = \nu(\omega)$. Moreover, if $\nu(\omega) \in \Lambda_F \setminus \Gamma_F$ then

$$I(F, H_2) = \nu(dH_2) \geq \nu(H_1 \cdot \omega_{01}) = I(F, H_1) + \nu(\omega_{01}) > I(F, H_1) + v_1 + v_0.$$

that is, if $\nu(\omega) \in \Lambda_F \setminus \Gamma_F$ and ω defines a foliation, then ω satisfies the hypothesis of Theorem 3.7.

Corollary 4.7. *Let $\mathcal{C}_F$ be a plane branch with value semigroup $\Gamma_F = \langle v_0, v_1\rangle$. If an element $\ell \in \Lambda_F \setminus \Gamma_F$ belongs to a minimal set of generators for Λ_F, then there exists a dicritical foliation $\mathcal{F}_\omega$ in the triple point of $G(\mathcal{C}_F)$ defined by $\omega = H_1 \cdot \omega_{01} + dH_2 \in \Omega^1$ with $\nu(\omega) = \ell$. In particular,*

$$\nu(\omega) + \mu_F - 1 = I(F, H_1 \cdot F_\star) = I\left(F, \frac{\omega \wedge dF}{dx \wedge dy}\right),$$

where $F_\star$ is defined in (22).

Proof. By previous comments, if $\ell \in \Lambda_F \setminus \Gamma_F$ belongs to a minimal set of generators for Λ_F, then there exists $\omega \in \Omega^1$ with $\nu(\omega) = \ell$, satisfying the hypothesis of Theorem 3.7, that is, $\mathcal{F}_\omega$ defines a dicritical foliation in the triple point of $G(\mathcal{C}_f)$. In particular, by Corollary 4.6 and Remark 4.2, we get $\nu(\omega) + \mu_F - 1 = I(F, H_1 \cdot F_\star) = I\left(F, \frac{\omega \wedge dF}{dx \wedge dy}\right)$. ■

The above result guarantees that for plane branches with value semigroup $\Gamma_F = \langle v_0, v_1\rangle$, the minimal generators for Λ_F, distinct of v_0 and v_1, can be obtained considering dicritical foliations in the triple point of $G(\mathcal{C}_F)$.

The previous corollary was obtained by Cano, Corral and Senovilla-Sanz in [9] by other methods.

4.1. The Zariski invariant of $\mathcal{C}_F$

Now we return to the general case, that is, plane branches $\mathcal{C}_F$ with value semigroup $\Gamma_F = \langle v_0, \ldots, v_g\rangle$ with $g \geq 1$. Without loss of generality, we can assume that $\mathcal{C}_F$ is defined by a Weierstrass polynomial $F \in \mathbb{C}\{x\}[y]$ satisfying $e_1 = GCD(v_0, v_1) < v_0 = I(F, x) < v_1 = I(F, y)$.

In [23], Zariski shows that $\lambda := \min(\Lambda_F \setminus \Gamma_F) - v_0$ can be computed directly by a Puiseux parameterization $(t^{\beta_0}, \sum_{i \geq \beta_1} c_i t^i)$ of $\mathcal{C}_F$. More precisely,

$$\lambda = \min\{i;\ c_i \neq 0 \text{ and } i + v_0 \notin \Gamma_F\}.$$

In addition, he proves that $\Lambda_F \setminus \Gamma_F = \emptyset$ if and only if $\mathcal{C}_F$ is analytically equivalent to $\mathcal{C}_G$ with $G = y^{v_0} - x^{v_1}$ with $GCD(v_0, v_1) = 1$. In this case we put $\lambda = \infty$.

The exponent λ is called the **Zariski invariant** of $\mathcal{C}_F$. Notice that if $\lambda \neq \infty$ then $v_1 = \beta_1 < \lambda \leq \beta_2$ and $\lambda + v_0$ is a minimal generator for Λ_F.

In what follows we present an alternative way to obtain the Zariski invariant of a plane branch using dicritical foliations in the first triple point of the dual graph $G(\mathcal{C}_F)$ as described in Corollary 3.8.

Lemma 4.8. *If λ is the Zariski invariant of a plane branch $\mathcal{C}_F$ with semigroup $\Gamma_F = \langle v_0, \ldots, v_g \rangle$, then there exist $H_1, H_2 \in \mathbb{C}\{x\}[y]$ with $deg_y H_l < n_1 = \frac{v_0}{e_1}$, H_1 a unit and $v_0 + v_1 < I(F, H_2)$ such that $\nu(H_1\omega_{01} + dH_2) = \lambda + v_0$.*

Proof. If $\lambda = \infty$ then $\Lambda_F = \Gamma_F \setminus \{0\}$ with $\Gamma = \langle v_0, v_1 \rangle$, that is, $e_1 = GCD(v_0, v_1) = 1$ and in this case $v_0 + v_1 < \nu(\omega_{01}) \in \Gamma_F \setminus \{0\}$. Then, by (20), there exists $G_1 \in \langle x, y \rangle \cap \mathbb{C}\{x\}[y]$ with $deg_y G_1 < v_0$ such that $\nu(\omega_{01}) = \nu(dG_1) < \nu(\omega_{01} - dG_1) \in \Gamma_F$.

In the same way, we obtain $G_2, \ldots, G_s \in \langle x, y \rangle \cap \mathbb{C}\{x\}[y]$ satisfying $\mu_F \leq \nu(\omega_{01} - \sum_{i=1}^{s} dG_i) \in \Gamma_F$ consequently, as $(\mathcal{O} : \overline{\mathcal{O}}) = \langle t^{\mu_F} \rangle$, there exists $G \in \langle x, y \rangle \cap \mathbb{C}\{x\}[y]$ such that $\varphi^*(dG) = \varphi^*(\omega_{01} - \sum_{i=1}^{s} dG_i)$, that is, $\omega = \omega_{01} - d(G + \sum_{i=1}^{s} G_i)$ satisfies $\varphi^*(\omega) = 0$ or equivalently $\frac{\omega \wedge dF}{dx \wedge dy} \in \langle F \rangle$. Hence, the result is obtained taking $H_1 = 1$ and $H_2 = -(G + \sum_{i=1}^{s} G_i)$

If $\infty \neq \lambda + v_0 = \nu(\eta)$, then by Lemma 3.1, we can express $\eta = A_1\omega_{01} + dA_2$ with $A_1, A_2 \in \mathbb{C}\{x, y\}$. Considering a 2-semiroot $F_2 \in \mathbb{C}\{x\}[y]$ (recall that $F_2 = F$ if $e_1 = 1$) we write $A_i = B_i F_2 + H_i$, $deg_y H_i < deg_y F_2 = \frac{v_0}{e_1}$ and

$$\eta = H_1\omega_{01} + dH_2 + B_2 dF_2 + F_2 \cdot (dB_2 + B_1\omega_{01}).$$

As $\lambda \leq \beta_2$, it follows by (4) that $\lambda + v_0 < v_2 \leq \nu(B_2 dF_2 + F_2 \cdot (dB_2 + B_1\omega_{01}))$. So, $\lambda + v_0 = \nu(\zeta)$ where $\zeta := H_1\omega_{01} + dH_2$ and, without loss of generality H_2 can be considered a non unit, that is, $H_2 \in \langle x, y \rangle$.

We must have $I(F, H_2) > I(F, H_1) + v_0 + v_1$. Indeed, if $I(F, H_2) \leq I(F, H_1) + v_0 + v_1 < \nu(H_1\omega_{01})$, then $\lambda + v_0 = \nu(\zeta) = \nu(dH_2) \in \Gamma_F$ that is a contradiction, because $\lambda + v_0 \in \Lambda_F \setminus \Gamma_F$.

Notice that $I(F, H_2) > v_0 + v_1$ implies that $H_2 \in \langle y^2 \rangle + \langle x, y \rangle^3$. In particular, $(H_2)_x \in \langle x, y \rangle^2$ and $(H_2)_y \in \langle x^2, y \rangle$.

Since $\zeta = (v_1 y H_1 + (H_2)_x)dx + (-v_0 x H_1 + (H_2)_y)dy$, if $H_1 \in \langle x, y \rangle$ then ζ is expressed as $Mdx + Ndy$ with $M \in \langle x, y \rangle^2$ and $N \in \langle x^2, y \rangle$. In this way, by Remark 4.1, the exponent $\nu(\zeta) - v_0 = \lambda$ could not be the Zariski invariant of $\mathcal{C}_F$. So, H_1 is a unit and we get the result considering $\zeta = H_1\omega_{01} + dH_2$. ■

As $\lambda + v_0$ is a minimal generator for Λ_F, any $\omega \in \Omega^1$ such that $\nu(\omega) = \lambda + v_0$ defines a foliation. In this way, by the above lemma and Corollary 3.8, we can compute the Zariski invariant for a plane branch $\mathcal{C}_F$ considering dicritical foliations in the first triple point of $G(\mathcal{C}_F)$ defined by $H_1\omega_{01} + dH_2$ with $H_1, H_2 \in \mathbb{C}\{x\}[y]$ with $deg_y H_l < n_1 = \frac{v_0}{e_1}$, H_1 a unit and $v_0 + v_1 < I(F, H_2)$, or equivalently, dicritical foliations defined by $w_{01} + H_1^{-1} dH_2 = w_{01} + P_1 dx + P_2 dy$ with $\nu(P_1 dx + P_2 dy) = \nu(H_1^{-1} dH_2) = \nu(dH_2) = I(F, H_2) > v_0 + v_1$.

Given $P_1 dx + P_2 dy \in \Omega^1$ and considering a 2-semiroot $F_2 \in \mathbb{C}\{x\}[y]$ of F we may write

$$P_2 = G_2 \cdot (F_2)_y + Q_2 \quad \text{and} \quad P_1 - G_2 \cdot (F_2)_x = G_1 F_2 + Q_1$$

with $G_1, G_2 \in \mathbb{C}\{x, y\}$, $Q_1, Q_2 \in \mathbb{C}\{x\}[y]$ with $deg_y Q_1 < deg_y F_2 = \frac{v_0}{e_1}$ and $deg_y Q_2 < deg_y (F_2)_y = \frac{v_0}{e_1} - 1$. In this way, we get

$$w_{01} + P_1 dx + P_2 dy = w_{01} + Q_1 dx + Q_2 dy + G_2 dF_2 + F_2 G_1 dx.$$

If $\nu(w_{01} + P_1 dx + P_2 dy) = \lambda + v_0$, then $\nu(Q_1 dx + Q_2 dy) > v_0 + v_1$ and $\nu(w_{01} + Q_1 dx + Q_2 dy) = \lambda + v_0$, because $\nu(G_2 dF_2 + F_2 G_1 dx) \geq v_2 > \beta_2 + v_0 \geq \lambda + v_0$.

Notice that the condition $\nu(Q_1 dx + Q_2 dy) > v_0 + v_1$ is equivalent to consider $Q_1 \in \langle x, y\rangle^2$, $mult_x Q_1(x, 0) > \frac{v_1}{v_0}$ and $Q_2 \in \langle x^2, y\rangle$. In this way, by the previous lemma and the above comments, $\lambda + v_0 = \nu(\omega)$ with ω belonging to the set

$$\mathcal{D}_1 = \left\{ \begin{matrix} \omega_{01} + Q_1 dx + Q_2 dy;\ \ Q_1 \in \langle x, y\rangle^2,\ \ Q_2 \in \langle x^2, y\rangle \text{ with} \\ deg_y Q_1 < \frac{v_0}{e_1},\ deg_y Q_2 < \frac{v_0}{e_1} - 1 \text{ and } mult_x Q_1(x, 0) > \frac{v_1}{v_0} \end{matrix} \right\}.$$

Moreover, we have the following result:

Proposition 4.9. *For any plane branch $\mathcal{C}_F$ as considered in this subsection*

$$\begin{aligned} \lambda &= \max\{\nu(\omega) - v_0;\ \ \omega \in \mathcal{D}_1\} \\ &= \max\left\{ I\left(F, \frac{\omega \wedge dF}{dx \wedge dy}\right) - (\mu_F + v_0 - 1);\ \ \omega \in \mathcal{D}_1 \right\}. \end{aligned}$$

In particular, λ is determined considering foliations $\mathcal{F}_\omega$ with $\omega \in \mathcal{D}_1$.

Proof. By the above comments, there exists $\omega = \omega_{01} + Q_1 dx + Q_2 dy \in \mathcal{D}_1$ such that $\nu(\omega) = \lambda + v_0$.

The case $\lambda = \infty$ is immediate. So, let us consider $\lambda \neq \infty$.

Suppose that there exists $\eta = Adx + Bdy \in \mathcal{D}_1$ such that $\nu(\eta) > \lambda + v_0$.

If $\nu(\eta) \in \Lambda_F \setminus \Gamma_F$, by Proposition 1.3.13 in [16], we have that $A \in \langle x, y\rangle^2$ and $B \in \langle x^2, y\rangle$, but this contradicts the fact that $\eta \in \mathcal{D}_1$.

If $\nu(\eta) \in \Gamma_F$, by Lemma 1.3.12 in [16], there exists $A_1 dx + B_1 dy \in \Omega^1$ with $A_1 \in \langle x, y\rangle^2$ and $B_1 \in \langle x^2, y\rangle$ such that $\nu(\eta) = \nu(A_1 dx + B_1 dy) < \nu(\eta - A_1 dx - B_1 dy)$. Proceeding in this way we obtain $P_1 \in \langle x, y\rangle^2$ and $P_2 \in \langle x^2, y\rangle$ such that $\eta - P_1 dx - P_2 dy \in \mathcal{D}_1$ satisfies

$$\nu(\eta - P_1 dx - P_2 dy) \in \Lambda_F \setminus \Gamma_F \quad \text{ or } \quad \nu(\eta - P_1 dx - P_2 dy) = \infty.$$

As before $\nu(\eta - P_1 dx - P_2 dy) \in \Lambda_F \setminus \Gamma_F$ contradicts the fact that $\eta - P_1 dx - P_2 dy \in \mathcal{D}_1$. On the other hand, if $\nu(\eta - P_1 dx - P_2 dy) = \infty$, as $\eta \in \mathcal{D}_1$ there exist $Q_1 \in \langle x, y\rangle^2$ and $Q_2 \in \langle x^2, y\rangle$ with $mult_x Q_1(x, 0) > \frac{v_1}{v_0}$ such that $\eta = \omega_{01} + Q_1 dx + Q_2 dy$, then $\omega_{01} = (P_1 - Q_1)dx + (P_2 - Q_2)dy$ with $P_1 - Q_1 \in \langle x, y\rangle^2$ and $P_2 - Q_2 \in \langle x^2, y\rangle$, that is an absurd.

Hence, $\lambda + v_0 = \max\{\nu(\omega);\ \omega \in \mathcal{D}_1\}$ and, by Remark 4.2, $\nu(\omega) + \mu_F - 1 = \nu\left(\frac{\omega \wedge dF}{dx \wedge dy}\right) = I\left(F, \frac{\omega \wedge dF}{dx \wedge dy}\right)$ that concludes the proof. ■

For plane branches with semigroup $\langle v_0, v_1\rangle$, the previous result was obtained by Gómez–Martínez in [11], where foliations defined by an element in $\mathcal{D}_1$ are called dicritical cuspidal foliations.

5. Technical lemmas

In what follows, given $S(t)=\sum_{i\geq i_0} a_i t^i \in \mathbb{C}\{t\}$ we denote $\mathrm{Coeff}(S(t),t^k):=a_k$, that is, the coefficient of t^k in $S(t)$. In particular, if $H\in\mathbb{C}\{x,y\}$ and $\psi(t)=(t^{i_0},\sum_{i\geq i_1} d_i t^i)$ then $\mathrm{Coeff}(\psi^*(H),t^k)$ depends polynomially on the coefficients $d_{i_1},\ldots,d_{i_k}$ for some $i_k\geq i_1$. In this case, we write $\mathrm{Coeff}(\psi^*(H),t^k)=p(d_{i_1},\ldots,d_{i_k})\in\mathbb{C}[d_{i_1},\ldots,d_{i_k}]$. If $i_k<i_1$ then we assume that $\mathbb{C}[d_{i_1},\ldots,d_{i_k}]$ is $\mathbb{C}$.

Let $\mathcal{C}_F$ be a plane branch with semigroup $\Gamma_F=\langle v_0,v_1,\ldots,v_g\rangle$ and a canonical system of semiroots $\{F_0,F_1,\ldots,F_g,F_{g+1}=F\}$ as in Proposition 3.3 with

$$\varphi_i(t)=\left(t^{\frac{\beta_0}{e_{i-1}}},\sum_{\beta_1\leq j<\beta_i} c_j t^{\frac{j}{e_{i-1}}}\right)\qquad\text{and}\qquad \varphi(t)=\left(t^{\beta_0},\sum_{j\geq\beta_1} c_j t^j\right)$$

parameterizations of $\mathcal{C}_{F_i}$ $i=1,\ldots,g$ and $\mathcal{C}_F=\mathcal{C}_{F_{g+1}}$, respectively.

Lemma 5.1. *For each $k\geq v_i$ and $i=1,\ldots,g$, we have* $\mathrm{Coeff}(\varphi^*(F_i),t^k)\in\mathbb{C}[c_{\beta_1},\ldots,c_{k-v_i+\beta_i}]$ *of degree 1 in $c_{k-v_i+\beta_i}$.*

Proof. For $1\leq i\leq g$, we take F_i as in (10), then we have

$$\begin{aligned}\varphi^*(F_i)&=\prod_{\alpha\in U_{m_i}}\left(\sum_{j\geq\beta_1} c_j t^j-\sum_{\beta_1\leq j<\beta_i} c_j\alpha^{\frac{j}{e_{i-1}}}t^j\right)\\&=\prod_{\alpha\in U_{m_i}}\left(\sum_{\beta_1\leq j<\beta_i} c_j(1-\alpha^{\frac{j}{e_{i-1}}})t^j+\sum_{j\geq\beta_i} c_j t^j\right).\end{aligned}$$

Denoting $G_s=\{\alpha\in\mathbb{C};\ \alpha^{\frac{e_s}{e_{i-1}}}=1\}$, for $0\leq s<i$, we get

$$\{1\}=G_{i-1}\subset G_{i-2}\subset\cdots\subset G_1\subset G_0=\{\alpha\in\mathbb{C};\ \alpha^{\frac{e_0}{e_{i-1}}}=1\}=U_{m_i}.$$

Notice that for each j satisfying $\beta_0\leq j<\beta_{s+1}$ and $\alpha\in G_s\setminus G_{s+1}$ we have $\alpha^{\frac{j}{e_{i-1}}}=1$, since e_s divides j. Thus

$$\begin{aligned}&\sum_{\beta_1\leq j<\beta_i} c_j(1-\alpha^{\frac{j}{e_{i-1}}})t^j+\sum_{j\geq\beta_i} c_j t^j\\&\quad=\begin{cases}c_{\beta_{s+1}}(1-\alpha^{\frac{\beta_{s+1}}{e_{i-1}}})t^{\beta_{s+1}}+h.o.t. & \text{if } \alpha\in G_s\setminus G_{s+1}\\ c_{\beta_i}t^{\beta_i}+h.o.t. & \text{if } \alpha\in G_{i-1}=\{1\},\end{cases}\end{aligned}$$

with $\alpha^{\frac{\beta_{s+1}}{e_{i-1}}}\neq 1$ if $\alpha\in G_s\setminus G_{s+1}$ (see [12], Lemma 6.8).

It follows that

$$\begin{aligned}\varphi^*(F_i) &= \left(\prod_{l=0}^{i-2}\prod_{\alpha\in G_l\setminus G_{l+1}}\left((1-\alpha^{\frac{\beta_{l+1}}{e_{i-1}}})c_{\beta_{l+1}}t^{\beta_{l+1}}+h.o.t.\right)\right)\\ &\quad\cdot\prod_{\alpha\in G_{i-1}}(c_{\beta_i}t^{\beta_i}+h.o.t.)\\ &= \prod_{l=0}^{i-2}\left(b_l t^{\frac{e_l-e_{l+1}}{e_{i-1}}\beta_{l+1}}+h.o.t.\right)\cdot\left(c_{\beta_i}t^{\beta_i}+h.o.t.\right)\\ &= \left(\prod_{l=0}^{i-2}b_l\right)c_{\beta_i}t^{v_i}+h.o.t.,\end{aligned} \tag{24}$$

where $b_l := \prod_{\alpha\in G_l\setminus G_{l+1}}(1-\alpha^{\frac{\beta_{l+1}}{e_{i-1}}})c_{\beta_{l+1}} \neq 0$ for $l=0,\ldots,i-2$ since $c_{\beta_i}\neq 0$, it is easy to verify that $\sharp(G_s\setminus G_{s+1}) = \frac{e_s-e_{s+1}}{e_{i-1}}$ for $0\le s\le i-1$, and the last equality follows by (5) since $\sum_{l=0}^{i-2}\frac{e_l-e_{l+1}}{e_{i-1}}\beta_{l+1}+\beta_i = v_i$. In fact, this is a proof that $I(F,F_i)=v_i$.

In addition, by the previous analysis, we obtain $\mathrm{Coeff}(\varphi^*(F_i),t^k)\in\mathbb{C}[c_{\beta_1},\ldots,c_\rho]$ with $k=\frac{1}{e_{i-1}}\sum_{l=0}^{i-2}(e_l-e_{l+1})\beta_{l+1}+\rho$, that is, $\rho = k-v_i+\beta_i$ and $\mathrm{Coeff}(\varphi^*(F_i),t^k)$ is a polynomial of degree one in $c_{k-v_i+\beta_i}$. More explicitly, we get

$$\mathrm{Coeff}(\varphi^*(F_i),t^k) = P_{ik}(c_{\beta_1},\ldots,c_{k-v_i+\beta_i-1})+\left(\prod_{l=0}^{i-2}b_l\right)c_{k-v_i+\beta_i} \tag{25}$$

where $P_{ik}(c_{\beta_1},\ldots,c_{k-v_i+\beta_i-1})$ is a homogeneous polynomial. ■

In what follows we consider the family $\mathcal{C}_{F_a}$ of plane branches topologically equivalent to $\mathcal{C}_F$ parameterized as (14), that is

$$\varphi_a(t) = \left(t^{\beta_0},\ \sum_{\beta_1\le l<\beta_g}c_l t^l+\sum_{l\ge\beta_g}a_l t^l\right),$$

where c_l is the coefficient of t^l in $\varphi(t)$, then it is constant for $\beta_1\le l<\beta_g$ and a_l is a complex parameter for $l\ge\beta_g$ with a_{β_g} nonvanishing.

It is immediate that $\Gamma_{F_a}=\Gamma_F=\langle v_0,v_1,\ldots,v_g\rangle$, $\{F_0,F_1,\ldots,F_g,F_a\}$ and $\{F_0,F_1,\ldots,F_g,F\}$ are the canonical system of semiroots of F_a and F, respectively. In particular, $I\left(F,\prod_{i=0}^g F_i^{\gamma_i}\right) = I\left(F_a,\prod_{i=0}^g F_i^{\gamma_i}\right)$ for any nonnegative integers γ_i.

Remark 5.2. As c_l is constant for $\beta_1\le l<\beta_g$, by Lemma 5.1 or more precisely by (25), we obtain

$$\mathrm{Coeff}(\varphi_a^*(F_i),t^k) = p_{ik}(a_{\beta_g},\ldots,a_{k-v_i+\beta_i-1})+\delta_{ik}\cdot a_{k-v_i+\beta_i}$$

with $\delta_{ik}\in\mathbb{C}^*$ for any $k\ge v_i$ and $1\le i\le g$. In particular, we get

$$\mathrm{Coeff}(\varphi_a^*(dF_i),t^k) = (k+1)\cdot(p_{i,k+1}(a_{\beta_g},\ldots,a_{k-v_i+\beta_i})+\delta_{i,k+1}\cdot a_{k-v_i+\beta_i+1}).$$

Moreover, for $k < v_i + \beta_g - \beta_i$ we have $\mathrm{Coeff}(\varphi_a^*(F_i), t^k)$ is constant, that is, it does not depend on the parameters a_l for $l \geq \beta_g$, and by (24)

$$\mathrm{Coeff}(\varphi_a^*(F_i), t^{v_i}) = \begin{cases} \delta_i & \text{if } i < g \\ \delta_g \cdot a_{\beta_g} & \text{if } i = g, \end{cases}$$

for some $\delta_i, \delta_g \in \mathbb{C}^*$.

The next lemmas are variations of Lemma 5.1.

Lemma 5.3. *Given $H = \prod_{i=0}^{j} F_i^{\gamma_i}$ with $\gamma_j \neq 0$ for some $0 \leq j \leq g$, we get $\mathrm{Coeff}(\varphi_a^*(H), t^k) \in \mathbb{C}[a_{\beta_g}, \dots, a_{k-I(F,H)+\beta_j}]$ for $k \geq I(F, H)$. In particular,*

$$\mathrm{Coeff}(\varphi_a^*(H), t^{I(F,H)}) = \begin{cases} \delta_{jH} & \text{if } j < g \\ \delta_{gH} \cdot a_{\beta_g}^{\gamma_g} & \text{if } j = g, \end{cases}$$

for some $\delta_{jH}, \delta_{gH} \in \mathbb{C}^*$.

Proof. Since $F_0 = x$, if $j = 0$ then $H = F_0^{\gamma_0}$ and $I(F, H) = I(F_a, H) = \gamma_0 \cdot v_0$ with $\varphi_a^*(H) = \varphi_a^*(F_0^{\gamma_0}) = t^{\gamma_0 v_0}$ and we get the result.

Notice that for any $\gamma_l > 0$ with $l \neq 0$, we may rewrite

$$\varphi_a^*(H) = \varphi_a^*(F_0^{\gamma_0}) \cdot \varphi_a^*(F_1^{\gamma_1}) \cdot \dots \cdot \underbrace{\varphi_a^*(F_l) \cdot \varphi_a^*(F_l^{\gamma_l - 1})} \cdot \dots \cdot \varphi_a^*(F_j^{\gamma_j}). \tag{26}$$

In order to determine $\epsilon \in \mathbb{N}$ such that $\mathrm{Coeff}(\varphi_a^*(H), t^k) \in \mathbb{C}[a_{\beta_g}, \dots, a_\epsilon]$ it is sufficient to analyze for each l, the product of a term of order γ in $\varphi_a^*(F_l)$ and all the initial terms of factors in (26) such that $k = \gamma + \sum_{i=0}^{j} \gamma_i v_i - v_l$, that is, $\gamma = k - \sum_{i=0}^{j} \gamma_i v_i + v_l = k - I(F, H) + v_l$, since $I(F, H) = I(F_a, H)$.

In this way, we can determine ϵ considering

$$\prod_{\substack{i=0 \\ i \neq l}}^{j} \left(\mathrm{Coeff}(\varphi_a^*(F_i), t^{v_i})\right)^{\gamma_i} \cdot \left(\mathrm{Coeff}(\varphi_a^*(F_l), t^{v_l})\right)^{\gamma_l - 1} \cdot \mathrm{Coeff}(\varphi_a^*(F_l), t^{k - I(F,H) + v_l}). \tag{27}$$

By Remark 5.2, the expression (27) is polynomial in $\mathbb{C}[a_{\beta_g}, \dots, a_{\gamma - v_l + \beta_l}] = \mathbb{C}[a_{\beta_g}, \dots, a_{k - I(F,H) + \beta_l}]$ of degree one in $a_{k-I(F,H)+\beta_l}$. Thus

$$\epsilon = \max_{0 \leq l \leq j} \{k - I(F, H) + \beta_l;\ \gamma_l \neq 0\} = k - I(F, H) + \beta_j.$$

By (27) and the previous remark, we have

$$\begin{aligned} \mathrm{Coeff}(\varphi_a^*(H), t^k) = {} & p_{kH}(a_{\beta_g}, \dots, a_{k-I(F,H)+\beta_j - 1}) \\ & + \begin{cases} \delta_{kH} \cdot a_{k-I(F,H)+\beta_j} & \text{if } j < g \\ \delta_{kK} \cdot a_{\beta_g}^{\gamma_g - 1} \cdot a_{k-I(F,H)+\beta_g} & \text{if } j = g \end{cases} \end{aligned}$$

for some $\delta_{kH} \in \mathbb{C}^*$. In particular,

$$\mathrm{Coeff}(\varphi_a^*(H), t^{I(F,H)}) = \begin{cases} \delta_{jH} & \text{if } j < g \\ \delta_{gH} \cdot a_{\beta_g}^{\gamma_g} & \text{if } j = g, \end{cases}$$

for some $\delta_{jH}, \delta_{gH} \in \mathbb{C}^*$. ■

As an immediate consequence of Lemma 5.3 we obtain the following result.

Corollary 5.4. *If* $H = \prod_{i=0}^{j} F_i^{\gamma_i}$ *with* $\gamma_j \neq 0$ *for some* $0 \leq j \leq g$, *then for any* $k > I(F, H)$ *we get* $\mathrm{Coeff}(\varphi_a^*(dH), t^k) \in \mathbb{C}[a_{\beta_g}, \ldots, a_{k-I(F,H)+\beta_j+1}]$.

Now we analyze $\varphi_a^*(\omega_{ig})$ for a 1-form $\omega_{ig} = v_i F_i dF_g - v_g F_g dF_i$, considered in (13) with $0 \leq i < g$. Notice that, by Remark 5.2, we get $ord_t \varphi_a^*(\omega_{ig}) \geq v_i + v_g$.

Lemma 5.5. *For* $k \geq v_i + v_g$ *we have*

$$\mathrm{Coeff}(\varphi_a^*(\omega_{ig}), t^k) = q_{ik}(a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g}) + r_{ik} \cdot a_{k-v_i-v_g+\beta_g+1},$$

with $r_{ik} \in \mathbb{C}^*$.

Proof. The highest order of a term in $\varphi_a(t)$ that contributes with the term of order k in $v_i \varphi_a^*(F_i)\varphi_a^*(dF_g)$ is determined by considering the product of the initial term $\delta_i \cdot t^{v_i}$ of $\varphi_a^*(F_i)$ with the term of order $k - v_i$ in $\varphi_a^*(dF_g)$ or the product of the initial term $v_g \cdot \delta_g \cdot a_{\beta_g} t^{v_g-1}$ of $\varphi_a^*(dF_g)$ with the term with order $k - v_g + 1$ in $\varphi_a^*(F_i)$.

By Remark 5.2, $\mathrm{Coeff}(\varphi_a^*(F_i), t^{k-v_g+1}) \in \mathbb{C}[a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_i+1}]$ and $\mathrm{Coeff}(\varphi_a^*(dF_g), t^{k-v_i}) = (k - v_i + 1) \cdot (p_{g,k-v_i+1}(a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g}) + \delta_{g,k-v_i+1} \cdot a_{k-v_i-v_g+\beta_g+1})$. In this way,

$$\mathrm{Coeff}(\varphi_a^*(F_i), t^{v_i}) \cdot \mathrm{Coeff}(\varphi_a^*(dF_g), t^{k-v_i}) \in \mathbb{C}[a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g+1}]$$

and $\mathrm{Coeff}(\varphi_a^*(dF_g), t^{v_g-1}) \cdot \mathrm{Coeff}(\varphi_a^*(F_i), t^{k-v_g+1}) \in \mathbb{C}[a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_i+1}]$.

Moreover, with the notation of Remark 5.2 and denoting $e := \delta_i \cdot \delta_{g,k-v_i+1} \neq 0$, we get

$$\mathrm{Coeff}(v_i \varphi_a^*(F_i dF_g), t^k) = Q_{1k}(a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g}) + v_i \cdot (k - v_i + 1) \cdot e \cdot a_{k-v_i-v_g+\beta_g+1}.$$

In a similar way, $\mathrm{Coeff}(v_g \varphi_a^*(F_g dF_i)) = Q_{2k}(a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g}) + v_g \cdot v_i \cdot e \cdot a_{k-v_i-v_g+\beta_g+1}$. Hence,

$$\mathrm{Coeff}(\varphi_a^*(\omega_{ig}), t^k) = q_{ik}(a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g}) + r_{ik} \cdot a_{k-v_i-v_g+\beta_g+1},$$

where $q_{ik} = Q_{1k} - Q_{2k} \in \mathbb{C}[a_{\beta_g}, \ldots, a_{k-v_i-v_g+\beta_g}]$ and $r_{ik} := e \cdot v_i \cdot (k - v_i - v_g + 1) \neq 0$. ■

In the next result, we determine the coefficient of a term in $\varphi_a^*(H \cdot \omega_{ig})$, for any $H \in \mathbb{C}\{x\}[y]$ expressed according to Proposition 3.4, that is, $H = \sum_\delta e_\delta F_0^{\delta_0} F_1^{\delta_1} \cdots F_g^{\delta_g}$ with $e_\delta \in \mathbb{C}^*$ and such that $I(F, H) = I(F, F_0^{\gamma_0} F_1^{\gamma_1} \cdots F_g^{\gamma_g})$, for some non-negative integers $\gamma_0, \gamma_1, \ldots, \gamma_g$.

Lemma 5.6. *If* $H = \sum_\delta e_\delta F_0^{\delta_0} F_1^{\delta_1} \cdots F_g^{\delta_g}$ *is as in* (11) *with* $I(F, H) = I(F, F_0^{\gamma_0} F_1^{\gamma_1} \cdots F_g^{\gamma_g})$, *then for* $k \geq I(F, H) + v_i + v_g$ *we get*

$$\mathrm{Coeff}(\varphi_a^*(H \cdot \omega_{ig}), t^k) = p_k(a_{\beta_g}, \ldots, a_{k_{ig}-1}) + r_k \cdot a_{\beta_g}^{\gamma_g} \cdot a_{k_{ig}}$$

with $k_{ig} := k - I(F, H) - v_i - v_g + \beta_g + 1$ *and some* $r_k \in \mathbb{C}^*$.

Proof. The coefficient of t^k in $\varphi_a^*(H \cdot \omega_{ig})$ is obtained by the sum of products of a term of order s_1 in $\varphi_a^*(H)$ and a term of order s_2 of $\varphi_a^*(\omega_{ig})$ such $s_1 + s_2 = k$. In this way, to prove the lemma it is sufficient to analyze such a product for the maximum possible value for s_1 or s_2.

For each element $e_\delta F_0^{\delta_0} F_1^{\delta_1} \cdots F_g^{\delta_g}$ we set

$$m_\delta := \max_{0\leq l\leq g}\{l;\ \delta_l \neq 0\} \quad \text{and} \quad I_\delta := I\left(F, \prod_{l=0}^{g} F_l^{\delta_l}\right) = I\left(F_a, \prod_{l=0}^{g} F_l^{\delta_l}\right). \tag{28}$$

Case (1) $s_1 = I(F, H) = I(F_a, H)$ and $s_2 = k - I(F, H)$, that is, the maximum possible value for s_2.

Notice that $\mathrm{Coeff}(\varphi_a^*(H), t^{I(F,H)}) = \mathrm{Coeff}(\prod_{l=0}^{g} \varphi_a^*(F_l^{\gamma_l}), t^{I(F,H)})$ and, by Lemma 5.3, it is constant if $m_\gamma < g$ and equal to $e_H \cdot a_{\beta_g}^{\gamma_g}$ with $e_H \in \mathbb{C}^*$ for $m_\gamma = g$. On the other hand, by previous lemma $\mathrm{Coeff}(\varphi_a^*(\omega_{ig}), t^{k-I(F,H)}) = p(a_{\beta_g}, \dots, a_{k_{ig}-1}) + r \cdot a_{k_{ig}}$ where $k_{ig} := k - I(F, H) - v_i - v_g + \beta_g + 1$ and $r \in \mathbb{C}^*$. Hence, $\mathrm{Coeff}(\varphi_a^*(H), t^{I(F,H)}) \cdot \mathrm{Coeff}(\varphi_a^*(\omega_{ig}), t^{k-I(F,H)})$ is expressed as

$$p_k(a_{\beta_g}, \dots, a_{k_{ig}-1}) + r_k \cdot a_{\beta_g}^{\gamma_g} \cdot a_{k_{ig}} \tag{29}$$

with $r_k := r \cdot e_H \neq 0$.

Case (2) $s_1 = k - v_i - v_g$ and $s_2 = v_i + v_g$ that is, the maximum possible value for s_1.

As $k \geq I(F, H) + v_i + v_g$, if $k = I(F, H) + v_i + v_g$ then we must have $s_1 = I(F, H)$ and $s_2 = k - I(F, H)$, that is, we are in the previous case. So, we can assume that $k > I(F, H) + v_i + v_g$.

Notice that $\mathrm{Coeff}(\varphi_a^*(H), t^{k-v_i-v_g}) = \sum_\delta e_\delta \mathrm{Coeff}(\prod_{l=0}^{g} \varphi_a^*(F_l^{\delta_l}), t^{k-v_i-v_g})$. By Lemma 5.3 and by (28), we get

$$\mathrm{Coeff}\left(\prod_{l=0}^{g} \varphi_a^*(F_l^{\delta_l}), t^{k-v_i-v_g}\right) \in \mathbb{C}[a_{\beta_g}, \dots, a_{k-v_i-v_g-I_\delta+\beta_{m_\delta}}].$$

Moreover, by Lemma 5.5, we have $\mathrm{Coeff}(\varphi_a^*(\omega_{ig}), t^{v_i+v_g}) \in \mathbb{C}[a_{\beta_g}, a_{\beta_g+1}]$. As $I_\delta \geq I(F, H)$ and $k > I(F, H) + v_i + v_g$ we have that

$$\max_\delta\{k - v_i - v_g - I_\delta + \beta_{m_\delta}, \beta_g + 1\} < k_{ig}.$$

Considering the above cases we conclude that $\mathrm{Coeff}(\varphi_a^*(H \cdot \omega_{ig}), t^k)$ is given as (29). ■

CRediT authorship contribution statement

Nuria Corral: Investigation, Writing – original draft, Writing – review & editing. **Marcelo E. Hernandes:** Investigation, Writing – original draft, Writing – review & editing. **M.E. Rodrigues Hernandes:** Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Nuria Corral reports financial support was provided by Spanish State Agency of Research. Marcelo E. Hernandes reports financial support was provided by Spanish State Agency of Research. Maria Elenice R. Hernandes reports financial support was provided by Spanish State Agency of Research. Marcelo E. Hernandes reports financial support was provided by CNPq- Brazil. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

We are grateful to Prof. Felipe Cano for all conversations and suggestions. The last two authors would like to express their gratitude to the ECSING-AFA group from Universidad de Valladolid for its hospitality during the year 2013, when the seminal ideas of this work emerged.

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