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ORIGINAL PAPER

Projections in the *J*-sums of Banach spaces

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Abstract
We study some families of projections in the J-sums of Banach spaces $J(\Phi)$ and $\hat{J}(\Phi)$ introduced by Bellenot. As an application, we show that, under some conditions, $J(\Phi)$ and $\hat{J}(\Phi)$ are subprojective, i.e., every closed infinite-dimensional subspace of either of them contains a complemented infinite-dimensional subspace.

Keywords Projections · Complemented Banach space · J-sum of Banach spaces

Mathematics Subject Classification 46B20

1 Introduction

In [15], Lindenstrauss showed that for every separable Banach space X there exists a separable Banach space Y with a monotone shrinking basis and a surjective operator Q from Y^* onto X such that $Y^{**} = i_Y(Y) \oplus Q^*(X^*)$, where $i_Y : Y \longrightarrow Y^{**}$ is the canonical embedding (hence $Y^{**} \simeq Y \oplus X^*$). This extended a previous result by James for Banach spaces with a monotone boundedly complete basis [13], and the same result was later obtained for weakly compactly generated spaces in [8].

Bellenot gave a general construction in [4] subsuming both results. Given a sequence of Banach spaces $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}_0}$ with $\dim X_0 = 0$ and a sequence of operators $\Phi = (\phi_n)_{n\in\mathbb{N}_0}$ with $\phi_n : X_n \longrightarrow X_{n+1}$ and $\|\phi_n\| \leq 1$ for every $n \in \mathbb{N}_0$, and denoting $\Pi = \prod_{n\in\mathbb{N}_0} X_n$, he defined a quantity $\|x\|_J$ for every $x = (x_n)_{n\in\mathbb{N}_0} \in \Pi$ that is similar to the norm in the classical James space J, and considered the Banach spaces $\hat{J}(\Phi) = \{x \in \Pi : \|x\|_J < \infty\}$ and $J(\Phi) = \{x \in \hat{J}(\Phi) : \lim_n \|x_n\|_n = 0\}$. With this construction, $J(\Phi)^{**}$ is isometric to $\hat{J}(\Phi)$ if all of the $(X_n)_{n\in\mathbb{N}}$ are reflexive and, given a weakly compactly generated Banach space X, the $(X_n)_{n\in\mathbb{N}}$ can be chosen so that $J(\Phi)^{**} \equiv \hat{J}(\Phi)$ and $J(\Phi)^{**}/J(\Phi)$ is isometric to X [4, Corollary 1.4].

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The main advantage of Bellenot's work over its predecessors is that it provides a construction of the space $J(\Phi)$, with an explicit definition of its norm, which allows to study the properties of $J(\Phi)$ and $\hat{J}(\Phi)$ for a given starting sequence $(X_n)_{n\in\mathbb{N}}$. This was used in [1] to find a space $J(\Phi)$ such that $J(\Phi)^{**}/J(\Phi) = \ell_1$ and a tauberian operator T in that $J(\Phi)$ whose second conjugate is not tauberian, answering a question by Kalton and Wilansky [14]. Bellenot's construction was also used in [22, Remark 4.5] to study the duality between certain operator ideals, and in [7] to show that the Kottman's constant of a Banach space and its second dual can be different.

Here we study some families of projections in the spaces $\hat{J}(\Phi)$ and $J(\Phi)$ and prove that, if each X_n is subprojective, then every closed infinite-dimensional subspace of $J(\Phi)$ contains an infinite-dimensional subspace complemented in $\hat{J}(\Phi)$, so $J(\Phi)$ itself is subprojective; as a consequence, for every separable Banach space X there exists some subprojective $J(\Phi)$ such that $J(\Phi)^{**}/J(\Phi)$ is isometric to X. If furthermore the quotient $\hat{J}(\Phi)/J(\Phi)$ is subprojective or the quotient map $\hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)/J(\Phi)$ is strictly singular, then also $\hat{J}(\Phi)$ is subprojective. We also give conditions for $J(\Phi)$ to be complemented or not in $\hat{J}(\Phi)$ and describe several examples illustrating the scope of these results.

Given an operator $T: X \longrightarrow Y$ between two Banach spaces X and Y, its kernel will be denoted by $N(T)$ and its range will be denoted by $R(T)$. T is called *strictly singular* if there is no closed infinite-dimensional subspace M of X such that the restriction $T|_M$ is an isomorphism. A Banach space X is called *subprojective* if every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X. Subprojective spaces were introduced by Whitley to find conditions for the conjugate of an operator to be strictly singular [23]. Finite-dimensional spaces are trivially subprojective, and ℓ_p $(1 \leq p < \infty)$, c_0 and L_p $(2 \leq p < \infty)$ are subprojective but L_p $(1 \leq p < 2)$ and $C([0,1])$ are not [23, Theorems 3.2 and 3.4, Corollary 3.6]; also subspaces and products of subprojective spaces are still subprojective. There has been a surge in attention to these spaces after a recent systematic study [18], such as to obtain some positive solutions to the perturbation classes problem for semi-Fredholm operators, which has a negative solution in general [9] but there are some positive answers when one of the spaces is subprojective [10] [12].

2 Definitions and basic facts

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\wp = \{\, S \subset \mathbb{N}_0 : S \text{ non-empty, finite}\,\}$. In the sequel, $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}_0}$ is a sequence of Banach spaces with $\dim X_0 = 0$ and $\Phi = (\phi_n)_{n\in\mathbb{N}_0}$ is a sequence of operators $\phi_n : X_n \longrightarrow X_{n+1}$ such that $\|\phi_n\| \leq 1$ for every $n \in \mathbb{N}_0$. If $n \leq m \in \mathbb{N}_0$, we will write $\phi_n^m = \phi_{m-1} \circ \cdots \circ \phi_{n+1} \circ \phi_n : X_n \longrightarrow X_m$, with the convention that $\phi_n^n = I_{X_n}$ is the identity on X_n; in particular, $\phi_n^{n+1} = \phi_n$ for every $n \in \mathbb{N}_0$ and $\phi_n^{m+1} = \phi_m \circ \phi_n^m$ for every $n \leq m \in \mathbb{N}_0$.

Denote $\Pi = \prod_{n\in\mathbb{N}_0} X_n$ and, given $x = (x_n)_{n\in\mathbb{N}_0} \in \Pi$ and $S = \{\, p_0 < \cdots < p_k \,\} \in \wp$, define

$$\sigma(x, S) = \left(\sum_{i=1}^{k} \left\| \phi_{p_{i-1}}^{p_i}(x_{p_{i-1}}) - x_{p_i} \right\|_{p_i}^2 \right)^{1/2}$$

$$\rho(x, S) = \left(\sum_{i=1}^{k} \left\| \phi_{p_{i-1}}^{p_i}(x_{p_{i-1}}) - x_{p_i} \right\|_{p_i}^2 + \|x_{p_k}\|_{p_k}^2 \right)^{1/2}$$

and $\|x\|_J = (1/\sqrt{2}) \sup_{S\in\wp} \rho(x, S)$. Note that $\sigma(\cdot, S)$ and $\rho(\cdot, S)$ are seminorms in Π and that $\rho(x, S)^2 = \sigma(x, S)^2 + \|x_{p_k}\|_{p_k}^2$, so $\sigma(x, S) \leq \rho(x, S)$. Also, $\sigma(x, S) = 0$ if S has a single element, $\sigma(x, S \cup T)^2 = \sigma(x, S)^2 + \sigma(x, T)^2$ if $\max S = \min T$ and $\sigma(x, S \cup T)^2 \geq \sigma(x, S)^2 + \sigma(x, T)^2$ if $\max S \leq \min T$.

Definition [4] The J-sum $J(\Phi)$ is defined as the completion of the normed space of the finitely non-zero sequences in Π with $\|\cdot\|_J$.

Proposition 2.1 [4, p. 98, Remarks 3 and 4] *Each X_m can be identified isometrically with the subspace of sequences $(x_n)_{n\in\mathbb{N}_0} \in J(\Phi)$ such that $x_n = 0$ for $n \neq m$. With this identification, $(X_n)_{n\in\mathbb{N}}$ is a bimonotone decomposition for $J(\Phi)$, i.e., $\left\|\sum_{i=p+1}^{p+q} x_i\right\|_J \leq \left\|\sum_{i=1}^{p+q+r} x_i\right\|_J$ for every $(x_n)_{n\in\mathbb{N}_0} \in J(\Phi)$ and $p, q, r \in \mathbb{N}_0$.*

In particular, $\|(x_n)_{n\in\mathbb{N}_0}\|_J = \sup_{n\in\mathbb{N}}\left\|\sum_{i=1}^{n} x_i\right\|_J = \lim_n\left\|\sum_{i=1}^{n} x_i\right\|_J$ for every $(x_n)_{n\in\mathbb{N}_0} \in J(\Phi)$.

Define now

$$\hat{J}(\Phi) = \left\{ x \in \Pi : \|x\|_J < \infty \right\},$$

which can be identified with $(X_n)^{\mathrm{LIM}}_{n\in\mathbb{N}}$, the set of all sequences $(x_n)_{n\in\mathbb{N}} \in \Pi$ such that $(\sum_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is bounded [4, p. 96], [16, Proposition 1.b.2]. This identification makes for slightly simpler notation than the direct use of $(X_n)^{\mathrm{LIM}}_{n\in\mathbb{N}}$.

The space of eventually constant sequences is

$$\Omega(\Phi) = \{ (x_n)_{n\in\mathbb{N}_0} \in \Pi : \text{there is } n \in \mathbb{N} \text{ such that } \phi_m(x_m) = x_{m+1} \text{for all } m \geq n \}.$$

Clearly $\Omega(\Phi) \subset \hat{J}(\Phi)$ and $\|(x_n)_{n\in\mathbb{N}_0}\|_\Omega = \lim_n \|x_n\|_n$ defines a seminorm on $\Omega(\Phi)$, as $\|\phi_n\| \leq 1$ for every $n \in \mathbb{N}_0$. We denote the completion of the normed space $(\Omega(\Phi)/\ker\|\cdot\|_\Omega, \|\cdot\|_\Omega)$ by $\tilde{\Omega}(\Phi)$.

Theorem 2.2 [4, Theorem 1.1]

(i) $\Omega(\Phi)$ *is dense in* $\hat{J}(\Phi)$.
(ii) *The unique extension* $\Theta\colon \hat{J}(\Phi) \longrightarrow \tilde{\Omega}(\Phi)$ *of the natural map* $\Omega(\Phi) \longrightarrow \tilde{\Omega}(\Phi)$ *is a quotient map with kernel* $J(\Phi)$.
(iii) *If each* X_n *is reflexive, then* $J(\Phi)^{**} = \hat{J}(\Phi)$*, hence* $J(\Phi)^{**}/J(\Phi)$ *is isometric to* $\tilde{\Omega}(\Phi)$.

In particular, $J(\Phi)$ can be identified with

$$\left\{ (x_n)_{n\in\mathbb{N}_0} \in \hat{J}(\Phi) : \|x_n\|_n \underset{n}{\longrightarrow} 0 \right\}.$$

Remark 2.3 It is worth noting that the choice of the ℓ_2-norm in the definition of σ and ρ above is not essential to the construction of $\hat{J}(\Phi)$; all of the results below hold equally well for any ℓ_q-norm with $1 < q < \infty$.

3 Estimates for disjoint sequences

We will adopt the following definition: If A, B are non-empty subsets of $\mathbb{N}_0$, we will write $A \ll B$ if there exists $n \in \mathbb{N}$ such that $a < n < b$ for every $a \in A$ and $b \in B$; equivalently, $\min B - \max A \geq 2$.

Definition Let $I \subseteq \mathbb{N}_0$ be an interval. We define $P_I\colon \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)$ as

$$P_I\big((x_n)_{n\in\mathbb{N}_0}\big) = (x_n \chi_I(n))_{n\in\mathbb{N}_0}.$$

Each P_I is well defined and a projection with $\|P_I\| \leq 1$ due to the following result with $m = 0$.

Proposition 3.1 *Let $I \subseteq \mathbb{N}$ be an interval, let $x = (x_n)_{n\in\mathbb{N}_0} \in \Pi$ such that $x_n = 0$ for every $n \notin I$, let $m < \min I$ and let $S \in \wp$. Then $\rho(x, S) \leq \rho(x, \{m\} \cup (S \cap I))$.*

Proof Let $\tilde{S} = S \cap [\min I, \infty) \subseteq S$; then $\rho(x, S) \leq \rho(x, \{m\} \cup \tilde{S})$. Indeed, if $\tilde{S} \subsetneq S$, then the elements in $S \setminus \tilde{S}$ do not contribute to $\rho(x, S)$ except for the possible effect of their presence on the first element of $\tilde{S}$, and then $\{m\}$ has the same effect; and, if $S = \tilde{S}$, then the addition of $\{m\}$ only prepends a (non-negative) term to $\rho(x, S)$.

Now, the elements in $\tilde{S}$ beyond the maximum of I, if any, do not contribute to $\rho(x, S)$ either, except for the fact that the last term in $\rho(x, S)$ may become smaller (as every $\|\phi_n\| \leq 1$). In any case, $\rho(x, S) \leq \rho(x, \{m\} \cup \tilde{S}) \leq \rho(x, \{m\} \cup (S \cap I))$. □

As a consequence, if $I \subset \mathbb{N}_0$ is a finite interval and $x \in R(P_I)$, then $\sup_{S\in\wp} \rho(x, S)$ is attained for some $S \subseteq [0, \max I]$.

If $n \in \mathbb{N}_0$, we define the projection $P_n = P_{\{1,\ldots,n\}}$ (as opposed to $P_{\{n\}}$, which gives the identification of X_n described in Proposition 2.1); in particular, $P_0 = P_\emptyset = 0$.

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ is said to satisfy an upper (resp., lower) p-estimate for $1 < p < \infty$ if there exists a constant $M < \infty$ such that $\left\|\sum_{j=1}^m x_j\right\|^p \leq M \sum_{j=1}^m \|x_j\|^p$ (resp., M $\left\|\sum_{j=1}^m x_j\right\|^p \geq \sum_{j=1}^m \|x_j\|^p$). The following results show that sequences in $\hat{J}(\Phi)$ with disjoint supports admit an upper 2-estimate, and also a lower 2-estimate if the supports are furthermore not adjacent.

Proposition 3.2 *[4, Theorem 1.1(III)] Let $I_1, I_2, \ldots, I_m \subseteq \mathbb{N}_0$ be disjoint intervals and let $x_j \in R(P_{I_j})$ for every $j \in \{1, 2, \ldots, m\}$. Then $\left\|\sum_{j=1}^m x_j\right\|_J^2 \leq 3 \sum_{j=1}^m \|x_j\|_J^2$.*

Proposition 3.3 *Let $I_1 \lll I_2 \lll \cdots \lll I_m \subseteq \mathbb{N}_0$ be non-empty intervals and let $x_j \in R(P_{I_j})$ for every $j \in \{1, 2, \ldots, m\}$. Then $2\left\|\sum_{j=1}^m x_j\right\|_J^2 \geq \sum_{j=1}^m \|x_j\|_J^2$.*

Proof Define $y = (y_n)_{n\in\mathbb{N}_0} = \sum_{j=1}^m x_j \in \hat{J}(\Phi)$, let $\varepsilon > 0$ and $n_1 = 0$ and pick $n_j \in (\max I_{j-1}, \min I_j)$ for every $j \in \{2, \ldots, m\}$, which is possible because $I_{j-1} \lll I_j$; note that $y_{n_j} = 0$ for all $j \in \{1, 2, \ldots, m\}$.

For every $j \in \{1, \ldots, m\}$, there exists $S_j \in \wp$ such that $\rho(x_j, S_j)^2 > 2\|x_j\|_J^2 - \varepsilon/m$ and, by Proposition 3.1, we may assume $n_j \in S_j \subseteq \{n_j\} \cup I_j$. Let then $m_j = \max S_j$, so

$$\rho(y, S_j)^2 = \sigma(y, S_j)^2 + \|y_{m_j}\|_{m_j}^2 = \sigma(y, S_j)^2 + \sigma(y, \{n_j, m_j\})^2$$

and there exists some $\tilde{S}_j$ (either S_j itself or $\{n_j, m_j\}$) such that $\sigma(y, \tilde{S}_j)^2 \geq \frac{1}{2}\rho(y, S_j)^2$, with $\max \tilde{S}_{j-1} = m_{j-1} < n_j = \min \tilde{S}_j$ if $j > 1$. Let $\tilde{S} = \bigcup_{j=1}^m \tilde{S}_j$; then

$$\begin{aligned} 2\|y\|_J^2 &\geq \rho(y, \tilde{S})^2 \geq \sigma(y, \tilde{S})^2 \geq \sum_{j=1}^m \sigma(y, \tilde{S}_j)^2 \geq \frac{1}{2}\sum_{j=1}^m \rho(y, S_j)^2 \\ &= \frac{1}{2}\sum_{j=1}^m \rho(x_j, S_j)^2 > \frac{1}{2}\sum_{j=1}^m \left(2\|x_j\|_J^2 - \varepsilon/m\right) > \sum_{j=1}^m \|x_j\|_J^2 - \varepsilon. \end{aligned}$$

As this is true for all $\varepsilon > 0$, it follows that $2\|y\|_J^2 \geq \sum_{j=1}^m \|x_j\|_J^2$ indeed. □

4 Stepping projections

We will write Λ for the set of all sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ in $\mathbb{N}_0$ such that $\alpha_n \leq \alpha_{n+1}$ and $\alpha_n \leq n$ for every $n \in \mathbb{N}_0$ and, given $\alpha = (\alpha_n)_{n\in\mathbb{N}_0} \in \Lambda$, we define a linear map $Q_\alpha \colon \Pi \longrightarrow \Pi$ as

$$Q_\alpha\big((x_n)_{n\in\mathbb{N}_0}\big) = \big(\phi^n_{\alpha_n}(x_{\alpha_n})\big)_{n\in\mathbb{N}_0}.$$

Lemma 4.1 *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}_0} \in \Lambda$, let $x \in \hat{J}(\Phi)$ and let $S \in \wp$. Then*

$$\rho(Q_\alpha(x), S) \leq \rho(x, \{\alpha_p : p \in S\}).$$

Proof Write $x = (x_n)_{n\in\mathbb{N}_0} \in \hat{J}(\Phi)$ and let $y_n = \phi^n_{\alpha_n}(x_{\alpha_n})$ for every $n \in \mathbb{N}_0$, so that $Q_\alpha(x) = (y_n)_{n\in\mathbb{N}_0}$.

Let $p < q$ be two consecutive elements of S. Then

$$\begin{aligned}\|\phi^q_p(y_p) - y_q\|_q &= \|\phi^q_p(\phi^p_{\alpha_p}(x_{\alpha_p})) - \phi^q_{\alpha_q}(x_{\alpha_q})\|_q\\ &= \|\phi^q_{\alpha_q}(\phi^{\alpha_q}_{\alpha_p}(x_{\alpha_p})) - \phi^q_{\alpha_q}(x_{\alpha_q})\|_q\\ &= \|\phi^q_{\alpha_q}(\phi^{\alpha_q}_{\alpha_p}(x_{\alpha_p}) - x_{\alpha_q})\|_q\\ &\leq \|\phi^{\alpha_q}_{\alpha_p}(x_{\alpha_p}) - x_{\alpha_q}\|_{\alpha_q};\end{aligned}$$

in particular, $\|\phi^q_p(y_p) - y_q\|_q = 0$ if $\alpha_p = \alpha_q$. Additionally, for the final term of $\rho(Q_\alpha(x), S)$, if $p = \max S$, we also have

$$\|y_p\|_p = \|\phi^p_{\alpha_p}(x_{\alpha_p})\|_p \leq \|x_{\alpha_p}\|_{\alpha_p},$$

so indeed $\rho(Q_\alpha(x), S) \leq \rho(x, \{\alpha_p : p \in S\})$. □

Proposition 4.2 *Let $\alpha \in \Lambda$. Then $Q_\alpha|_{\hat{J}(\Phi)} \colon \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)$ is a bounded linear operator and $\|Q_\alpha\| \leq 1$.*

Proof Q_α is clearly linear and $\|Q_\alpha(x)\|_J \leq \|x\|_J$ for every $x \in \hat{J}(\Phi)$ by Lemma 4.1. □

Definition Let $A \subseteq \mathbb{N}$. We define a projection $Q_A \colon \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)$ as $Q_A = Q_{\alpha_A}$ where $\alpha_A = (\alpha_n)_{n\in\mathbb{N}_0}$ is given by $\alpha_n = \max((\{0\} \cup A) \cap [0, n])$. If $n \in \mathbb{N}_0$, we define the projection $Q_n = Q_{\{1,\ldots,n\}}$ (as opposed to $Q_{\{n\}}$); in particular, $Q_0 = Q_\emptyset = 0$.

If $A = \{a_1 < a_2 < \ldots\} \subseteq \mathbb{N}$, and $A_0 = \{0\} \cup A$, then $Q_A((x_n)_{n\in\mathbb{N}_0}) = (y_m)_{m\in\mathbb{N}_0}$ is given by $y_m = \phi^m_a(x_a)$, where $a = \max A_0 \cap [0, m]$, for every $m \in \mathbb{N}$, so $Q_A(x)$ depends only on those components x_n for which $n \in A$, and $N(Q_A) = \{\, (x_n)_{n\in\mathbb{N}_0} : x_n = 0 \text{ for every } n \in A\,\}$. If $(I_j)_{j\in\mathbb{N}}$ is a sequence of non-empty intervals of $\mathbb{N}_0$ such that $I_j \ll I_{j+1}$ for every $j \in \mathbb{N}$ and $A = \mathbb{N} \setminus \bigcup_{j\in\mathbb{N}} I_j$, then $N(Q_A)$ is isomorphic to $\ell_2((R(P_{I_j}))_{j\in\mathbb{N}})$ in the natural way due to Propositions 3.2 and 3.3.

If $x = (x_m)_{m\in\mathbb{N}_0} \in \hat{J}(\Phi)$ and $n \in \mathbb{N}$, then

$$Q_n(x) = (x_1, x_2, \ldots, x_{n-1}, x_n, \phi^{n+1}_n(x_n), \phi^{n+2}_n(x_n), \ldots),$$

so Q_n is a projection (clearly $Q_n^2 = Q_n$) and the set of eventually constant sequences can be written as $\Omega(\Phi) = \bigcup_{n\in\mathbb{N}} R(Q_n)$, which is dense in $\hat{J}(\Phi)$, but this can be stated in a better form.

Proposition 4.3 *Let $x \in \hat{J}(\Phi)$. Then $Q_n(x) \underset{n}{\longrightarrow} x$.*

Proof The result is immediate for the set of eventually constant sequences $\Omega(\Phi) = \bigcup_{n\in\mathbb{N}} R(Q_n)$, as $(Q_n(x))_{n\in\mathbb{N}}$ is eventually constant for $x \in \Omega(\Phi)$, which is dense in $\hat{J}(\Phi)$ by Theorem 2.2. In fact, the proof of the denseness of $\Omega(\Phi)$ in $\hat{J}(\Phi)$ in [4, Theorem 1.1(I)] essentially proves this. □

Given $n \in \mathbb{N}$, both P_n and Q_n are projections that depend only on the first n components of x, so they have the same kernel, and their ranges are different but isometric.

Proposition 4.4 *Let $m \in \mathbb{N}$. Then $Q_m P_m = Q_m$ and $P_m Q_m = P_m$ and, for every $x \in \hat{J}(\Phi)$, $\|P_m(x)\|_J = \|Q_m(x)\|_J$.*

Proof The identities $Q_m P_m = Q_m$ and $P_m Q_m = P_m$ follow from the fact that both $P_m(x)$ and $Q_m(x)$ depend only on the first m components of $x \in \hat{J}(\Phi)$ and leave them untouched, and then

$$\|Q_m(x)\|_J = \|Q_m P_m(x)\|_J \leq \|P_m(x)\|_J = \|P_m Q_m(x)\|_J \leq \|Q_m(x)\|_J$$

from $\|P_m\| \leq 1$ and $\|Q_m\| \leq 1$. □

5 Subspaces of $J(\Phi)$ and $\hat{J}(\Phi)$

As already mentioned in Theorem 2.2, $J(\Phi)^{**}$ can be identified with $\hat{J}(\Phi)$ when every X_n is reflexive. However, even if this is not the case, $J(\Phi)^{**}$ can be identified isometrically with $\hat{J}(\Phi^{**})$, where $\Phi^{**} = (\phi_n^{**})_{n\in\mathbb{N}_0}$, [4, p. 97], and then $\hat{J}(\Phi)$ can be seen to embed isometrically into $\hat{J}(\Phi^{**}) \equiv J(\Phi)^{**}$ by way of the natural inclusion of each X_n into X_n^{**}, where each $x \in \hat{J}(\Phi)$ is identified with the weak* limit of $(P_n(x))_{n\in\mathbb{N}}$ in $J(\Phi)^{**}$ (if every X_n is reflexive, then simply $\Phi^{**} = \Phi$).

Proposition 5.1 *Let Y be a subprojective Banach space, let $T : X \longrightarrow Y$ be an operator and let M be a closed infinite-dimensional subspace of X such that $T|_M$ is not strictly singular. Then M contains an infinite-dimensional subspace complemented in X.*

Proof $T|_M$ is not strictly singular, so there exists some infinite-dimensional subspace N of M such that $T|_N$ is an isomorphism. As Y is subprojective, we can further refine N to assume that $T(N)$ is complemented in Y, and then $Y = T(N) \oplus Z$ implies $X = N \oplus T^{-1}(Z)$, so N is complemented in X. □

The following result effectively shows that $J(\Phi)$ is subprojective whenever all of the X_n are subprojective. It does a little more than this, as it proves that every closed infinite-dimensional subspace of $J(\Phi)$ contains an infinite-dimensional subspace complemented in $\hat{J}(\Phi)$, and not just in $J(\Phi)$, which has some implications that will be discussed later. Also, the result contains an analogue if all of the X_n are hereditarily ℓ_2, as then $J(\Phi)$ too is hereditarily ℓ_2; a Banach space X is called hereditarily ℓ_2 if every subspace of X contains a further subspace that is isomorphic to ℓ_2. This is already hinted at in [4] and proved in [19, 20, Lemma 3] when each X_n is finite-dimensional, and is included here because the heavy part of the proof is the same.

Theorem 5.2 *(i) If each X_n is subprojective, then every closed infinite-dimensional subspace of $J(\Phi)$ contains an infinite-dimensional subspace complemented in $\hat{J}(\Phi)$;*
(ii) if each X_n is hereditarily ℓ_2, then $J(\Phi)$ is hereditarily ℓ_2;

(iii) if each X_n is subprojective and hereditarily ℓ_2, then every closed infinite-dimensional subspace of $J(\Phi)$ contains a copy of ℓ_2 complemented in $\hat{J}(\Phi)$.

Proof Let M be a closed infinite-dimensional subspace of $J(\Phi)$. If $Q_n|_M$ is not strictly singular for some $n \in \mathbb{N}$, then $R(Q_n)$ is isometric to $R(P_n)$ by Proposition 4.4, in turn isomorphic to $\bigoplus_{i=1}^n X_i$, so, respectively for each case,

(i) M contains an infinite-dimensional subspace complemented in $\hat{J}(\Phi)$ by Proposition 5.1, since $R(Q_n)$ is isomorphic to $\bigoplus_{i=1}^n X_i$, which is subprojective [18, Proposition 2.2]; or

(ii) M contains an infinite-dimensional subspace N such that $Q_n|_N$ is an isomorphism, where $R(Q_n)$ is isomorphic to $\bigoplus_{i=1}^n X_i$, so N contains a copy of ℓ_2 as being hereditarily ℓ_2 is a three-space property [6, Theorem 3.2.d]; or

(iii) both of the above apply, so M contains a copy of ℓ_2 by (ii) and then said copy contains a further copy of ℓ_2 complemented in $\hat{J}(\Phi)$ by (i).

Otherwise, assume that $Q_n|_M$ is strictly singular for every $n \in \mathbb{N}$, in any of the cases (i), (ii) or (iii). Then, for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $x \in M$ such that $\|x\| = 1$ and $\|Q_n(x)\| < \varepsilon$, and then there is $m > n+1$ such that $\|P_{m-1}(x) - x\| < \varepsilon$. By induction, starting with an arbitrary $n_1 \in \mathbb{N}$, there exist a sequence $(n_k)_{k\in\mathbb{N}}$ of elements in $\mathbb{N}$ and a sequence $(x_k)_{k\in\mathbb{N}}$ of norm-one elements in M such that $\|Q_{n_k}(x_k)\| < 2^{-k}/6$, $n_{k+1} > n_k + 1$ and $\|P_{n_{k+1}-1}(x_k) - x_k\| < 2^{-k}/6$ for every $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, define $T_k = P_{n_{k+1}-1}(I - Q_{n_k})$ and note that, given $y = (y_n)_{n\in\mathbb{N}_0} \in \hat{J}(\Phi)$ and $m \in \mathbb{N}$, then

$$T_k(y)_m = \begin{cases} y_m - \phi_{n_k}^m(y_{n_k}), & \text{if } n_k < m < n_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

so $T_k(y)$ depends only on components $[n_k, n_{k+1})$ of y and its value lies in the range of $P_{(n_k, n_{k+1})}$. As a consequence, it is easy to check that each T_k is a projection with $\|T_k\| \le 2$ and that $T_i T_j = 0$ if $i \neq j$. If we further define $A = \{n_k : k \in \mathbb{N}\}$ and $I_k = (n_k, n_{k+1})$ for each $k \in \mathbb{N}$, it holds that

$$T_k = P_{n_{k+1}-1}(I - Q_{n_k}) = P_{I_k}(I - Q_{n_k}) = P_{I_k}(I - Q_A)$$

for every $k \in \mathbb{N}$.

Let $z_k = T_k(x_k) = P_{n_{k+1}-1}(I - Q_{n_k})(x_k) \in R(P_{I_k})$ for every $k \in \mathbb{N}$; then $\|z_k - x_k\| \le \|P_{n_{k+1}-1}(x_k) - x_k\| + \|P_{n_{k+1}-1}Q_{n_k}(x_k)\| < 2^{-k}/6 + 2^{-k}/6 = 2^{-k}/3 \le 1/6$, so $5/6 < \|z_k\| < 7/6$ for every $k \in \mathbb{N}$. If we take $x_k^* \in J(\Phi)^*$ such that $\|x_k^*\| < 6/5$ and $\langle x_k^*, z_k\rangle = 1$ for each $k \in \mathbb{N}$, then, for every $x \in \hat{J}(\Phi)$ and $k \in \mathbb{N}$, Proposition 3.3 yields

$$\begin{aligned} \sum_{i=1}^k \left|\langle x_i^*, T_i(x)\rangle\right|^2 &\le \sum_{i=1}^k 2\|T_i(x)\|_J^2 \le 4\left\|\sum_{i=1}^k T_i(x)\right\|_J^2 \\ &= 4\left\|\sum_{i=1}^k P_{I_i}(I - Q_A)(x)\right\|_J^2 \\ &= 4\left\|P_{n_{k+1}}(I - Q_A)(x)\right\|_J^2 \le 16\|x\|_J^2, \end{aligned}$$

so $S(x) = (\langle x_k^*, T_k(x)\rangle)_{k\in\mathbb{N}}$ defines a bounded operator $S\colon \hat{J}(\Phi) \longrightarrow \ell_2$ which maps, for every $j \in \mathbb{N}$,

$$S(z_j) = (\langle x_k^*, T_k(z_j)\rangle)_{k\in\mathbb{N}} = (\langle x_k^*, \delta_{jk} z_j\rangle)_{k\in\mathbb{N}} = e_j$$

As the operator $\ell_2 \longrightarrow \hat{J}(\Phi)$ that takes $(\alpha_n)_{n\in\mathbb{N}}$ to $\sum_{n=1}^{\infty} \alpha_n z_n$ (hence each e_j to z_j) is bounded by Proposition 3.2, it follows that $Z = [z_k : k \in \mathbb{N}]$ is isomorphic to ℓ_2 and complemented in $\hat{J}(\Phi)$.

Finally, let $K : \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)$ be the operator defined as $K(x) = \sum_{k=1}^{\infty} \langle x_k^*, T_k(x)\rangle (x_k - z_k)$, as in [5]; then $\|x_k^*\| \|T_k\| \|x_k - z_k\| < \frac{4}{5} 2^{-k}$ for every $k \in \mathbb{N}$, so $\sum_{k=1}^{\infty} \|x_k^*\| \|T_k\| \|x_k - z_k\| < \frac{4}{5} < 1$, which makes K well defined and $U = I + K$ an isomorphism on $\hat{J}(\Phi)$, with $K(z_k) = x_k - z_k$ and $U(z_k) = x_k$ for every $k \in \mathbb{N}$, and then $U(Z) = [x_k : k \in \mathbb{N}] \subseteq M$ is a copy of ℓ_2 complemented in $\hat{J}(\Phi)$. □

Corollary 5.3 *$J(\Phi)$ is subprojective if and only if each X_n is subprojective.*

Theorem 5.2 does not hold for subspaces of $\hat{J}(\Phi)$, as its proof relies on the fact that $P_n(x) \underset{n}{\longrightarrow} x$ for $x \in J(\Phi)$, which does not extend to $\hat{J}(\Phi)$. In fact, the comments after Proposition 6.2 show a particular case in which $J(\Phi)$ is subprojective but $\hat{J}(\Phi)$ is not, because it contains a (complemented) copy of L_1. However, the fact that any closed infinite-dimensional subspace of $J(\Phi)$ contains another infinite-dimensional subspace complemented in $\hat{J}(\Phi)$, and not merely in $J(\Phi)$, allows to show that $\hat{J}(\Phi)$ is subprojective when additional conditions on $\hat{J}(\Phi)/J(\Phi)$ are met.

Lemma 5.4 *Let X be a Banach space and let M and N be closed subspaces of X such that $M \cap N = 0$ and $M + N$ is not closed. Then there exists an automorphism $U : X \longrightarrow X$ such that $U(M) \cap N$ is infinite-dimensional.*

Proof Consider the product $M \times N$ with the product norm $\|(x, y)\| = \|x\| + \|y\|$ and the operator $T : M \times N \longrightarrow X$ defined as $T(x, y) = x - y$; then $R(T) = M + N$ is not closed, so there exist normalised sequences $(x_n)_{n\in\mathbb{N}}$ in M and $(y_n)_{n\in\mathbb{N}}$ in N such that $\|x_n - y_n\| < 2^{-n}$ for every $n \in \mathbb{N}$. Since any weak cluster point of $(x_n)_{n\in\mathbb{N}}$ must be in $M \cap N = 0$, by passing to a subsequence [2, Theorem 1.5.6] we can assume that $(x_n)_{n\in\mathbb{N}}$ is a basic sequence and that there exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* such that $\langle x_i^*, x_j\rangle = \delta_{ij}$ for every $i, j \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|x_n^*\| \|x_n - y_n\| < 1$. Then $K(x) = \sum_{n=1}^{\infty} \langle x_n^*, x\rangle (x_n - y_n)$ defines an operator $K : X \longrightarrow X$ with $\|K\| < 1$ and $U = I - K$ is an automorphism on X that maps $U(x_n) = y_n$ for every $n \in \mathbb{N}$, so $U(M) \cap N$ is infinite-dimensional. □

Proposition 5.5 *If $J(\Phi)$ and $\hat{J}(\Phi)/J(\Phi)$ are both subprojective, then $\hat{J}(\Phi)$ is subprojective.*

Note that subprojectivity is not a three-space property, in general [18, Proposition 2.8].

Proof Let M be a closed infinite-dimensional subspace of $\hat{J}(\Phi)$. If $M \cap J(\Phi)$ is infinite-dimensional, then it contains another infinite-dimensional subspace complemented in $\hat{J}(\Phi)$ by Theorem 5.2.

Otherwise, if $M \cap J(\Phi)$ is finite-dimensional, we can assume that $M \cap J(\Phi) = 0$ by passing to a further subspace if necessary. If $M + J(\Phi)$ is closed, let $Q : \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)/J(\Phi)$ be the natural quotient operator induced by $J(\Phi)$; then $Q|_M$ is an isomorphic embedding into $\hat{J}(\Phi)/J(\Phi)$, which is subprojective, so M contains an infinite-dimensional subspace complemented in $\hat{J}(\Phi)$ by Proposition 5.1.

We are left with the case where $M \cap J(\Phi) = 0$ and $M + J(\Phi)$ is not closed. By Lemma 5.4, there exists an automorphism $U : \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)$ such that $U(M) \cap J(\Phi)$ is infinite-dimensional. Let N be an infinite-dimensional subspace of $U(M) \cap J(\Phi)$ complemented in $\hat{J}(\Phi)$, which exists again by Theorem 5.2. then $U^{-1}(N) \subseteq M$ and is still complemented in $\hat{J}(\Phi)$. □

Proposition 5.6 *If $J(\Phi)$ is subprojective and the quotient map $\hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)/J(\Phi)$ is strictly singular, then $\hat{J}(\Phi)$ is subprojective.*

Proof Let M be a closed infinite-dimensional subspace of $\hat{J}(\Phi)$. If $M \cap J(\Phi)$ is infinite-dimensional, then it contains another infinite-dimensional subspace complemented in $\hat{J}(\Phi)$ by Theorem 5.2.

Otherwise, if $M \cap J(\Phi)$ is finite-dimensional, we can assume that $M \cap J(\Phi) = 0$ by passing to a further subspace if necessary, and then the strict singularity of the quotient map implies that $M + J(\Phi)$ cannot be closed. As in Proposition 5.5, by Lemma 5.4, there exists an automorphism $U\colon \hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)$ such that $U(M) \cap J(\Phi)$ is infinite-dimensional. Let N be an infinite-dimensional subspace of $U(M) \cap J(\Phi)$ complemented in $\hat{J}(\Phi)$, which exists again by Theorem 5.2; then $U^{-1}(N) \subseteq M$ and is still complemented in $\hat{J}(\Phi)$. ◻

A Banach space X is called *superprojective* if every closed infinite-codimensional subspace of X is contained in an infinite-codimensional subspace complemented in X [23] (see also [11]). There is a perfect duality between subprojectivity and superprojectivity for reflexive Banach spaces, in that a reflexive Banach space is subprojective if and only if its dual is superprojective. This does not extend to the non-reflexive case: c_0 is subprojective while ℓ_1 is not superprojective [23], and ℓ_1 is subprojective but it has a predual Y that is not superprojective [11]. It is still unclear whether the superprojectivity of a Banach space implies the subprojectivity of its dual or predual. In light of Corollary 5.3 and Proposition 5.5, it makes sense to ask whether $J(\Phi)$ would be superprojective if so were every X_n, or what additional conditions on the X_n would be required.

Question. If every X_n is superprojective, is $J(\Phi)$ superprojective?

The proof of Theorem 5.2 relies on the fact that sequences in $\hat{J}(\Phi)$ with skipped disjoint supports generate (complemented) copies of ℓ_2, due to Propositions 3.2 and 3.3. If the supports are disjoint but not skipped then only Proposition 3.2 applies, so a disjoint sequence can generate subspaces other than ℓ_2. However, the J-sum never introduces copies of ℓ_p for $p > 2$. This is already proved in [19, 20, Lemma 5] when every X_n is finite-dimensional, but this condition can be relaxed to the case where no X_n contains a copy of ℓ_p. If the construction of $J(\Phi)$ is done with an arbitrary $1 < q < \infty$ instead of 2, it is also known that $\hat{J}(\Phi)$ does not contain a copy of any ℓ_p for $p > q$ if every X_n is finite-dimensional [21, Lemma 3]; as mentioned in Remark 2.3, the results below extend to this case as well, so no space $\hat{J}(\Phi)$ constructed using any $1 < q < \infty$ contains a copy of ℓ_p for $p > q$ unless one of the X_n already does.

Lemma 5.7 *Let $(m_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers, let $m_0 = 0$ and let $x^k \in R((I - P_{m_{k-1}})Q_{m_k})$ for every $k \in \mathbb{N}$. Then $2\left\|\sum_{i=1}^k x^i\right\|_J^2 \geq \sum_{i=1}^k \|x^i\|_J^2$.*

Proof For every $k \in \mathbb{N}$, write $x^k = (x_n^k)_{n\in\mathbb{N}_0}$ and note that $P_{m_{k-1}} Q_{m_k} = Q_{m_k} P_{m_{k-1}}$, as $m_{k-1} < m_k$, so $x^k \in R((I - P_{m_{k-1}})Q_{m_k}) = R(Q_{m_k}(I - P_{m_{k-1}}))$. As a consequence, by Proposition 3.1 and Lemma 4.1,

$$\sup_{S\in\wp} \rho(x^k, S) = \sup_{S\in\wp} \rho\big(x^k, \{m_{k-1}\} \cup (S \cap (m_{k-1}, m_k])\big),$$

so there exists some $S \in \wp$ such that $m_{k-1} \in S \subseteq [m_{k-1}, m_k]$ and $\rho(x^k, S)^2 = 2\|x^k\|_J^2$; since $x_{m_{k-1}}^k = 0$, letting $p = \max S \leq m_k$, we have

$$2\|x^k\|_J^2 = \rho(x^k, S)^2 = \sigma(x^k, S)^2 + \|x_p^k\|_p^2 = \sigma(x^k, S)^2 + \sigma(x^k, \{m_{k-1}, p\})^2,$$

so there exists some $S_k \in \wp$ (either S or $\{m_{k-1}, p\}$) such that $m_{k-1} \in S_k \subseteq [m_{k-1}, m_k]$ and $\sigma(x^k, S_k)^2 \geq \|x^k\|_J^2$.

Fix now $k \in \mathbb{N}$ and consider $\sigma\big(\sum_{i=1}^k x^i, S_j\big)^2$ for a given $j \in \{1, \ldots, k\}$. For any two elements $n < p \in S_j$,

$$\phi_n^p\left(\sum_{i=1}^k x_n^i\right) - \sum_{i=1}^k x_p^i = \sum_{i=1}^k \big(\phi_n^p(x_n^i) - x_p^i\big),$$

where, for $i < j$, we have $x^i \in R(Q_{m_i})$, so $m_i \leq m_{j-1} \leq n < p$ and

$$\phi_n^p(x_n^i) - x_p^i = \phi_n^p(\phi_{m_i}^n(x_{m_i}^i)) - \phi_{m_i}^p(x_{m_i}^i) = 0$$

and, for $i > j$, we have $x^i \in R(I - P_{m_{i-1}})$, so $n < p \leq m_j \leq m_{i-1}$ and

$$\phi_n^p(x_n^i) - x_p^i = \phi_n^p(0) - 0 = 0$$

so only $i = j$ matters and

$$\phi_n^p\left(\sum_{i=1}^k x_n^i\right) - \sum_{i=1}^k x_p^i = \sum_{i=1}^k \big(\phi_n^p(x_n^i) - x_p^i\big) = \phi_n^p(x_n^j) - x_p^j,$$

hence

$$\rho\left(\sum_{i=1}^k x^i, \bigcup_{j=1}^k S_j\right)^2 \geq \sigma\left(\sum_{i=1}^k x^i, \bigcup_{j=1}^k S_j\right)^2 \geq \sum_{j=1}^k \sigma\left(\sum_{i=1}^k x^i, S_j\right)^2$$
$$= \sum_{j=1}^k \sigma(x^j, S_j)^2 \geq \sum_{j=1}^k \|x^j\|_J^2.$$

□

Proposition 5.8 *Let $p > 2$. If no X_n contains a copy of ℓ_p, then neither does $\hat{J}(\Phi)$.*

Proof Assume otherwise and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\hat{J}(\Phi)$ equivalent to the unit vector basis of ℓ_p.

Then, for every $j, m \in \mathbb{N}_0$ and $\varepsilon > 0$, $R(P_m) \simeq \oplus_{i=1}^m X_i$ does not contain copies of ℓ_p, as not containing copies of ℓ_p is a three-space property [6, Theorem 3.2.d], so $P_m|_{[x_i : j<i]}$ is strictly singular, since $[x_i : j < i]$ is still isomorphic to ℓ_p. Therefore, there exist $k \in \mathbb{N}$ with $k > j$ and $y \in [x_i : j < i \leq k]$ such that $\|y\|_J = 1$ and $\|P_m(y)\|_J < \varepsilon/2$, and then there exists $n > m$ such that $\|(I - Q_n)(I - P_m)(y)\|_J < \varepsilon/2$ by Proposition 4.3, which leads to $\|y - Q_n(I - P_m)(y)\|_J \leq \|P_m(y)\|_J + \|y - P_m(y) - Q_n(I - P_m)(y)\|_J < \varepsilon$.

By induction, given a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers and starting with $m_0 = 0$, there exist a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers and a sequence $(y_n)_{n\in\mathbb{N}}$ of norm-one elements of $\hat{J}(\Phi)$ that is a blocking of $(x_n)_{n\in\mathbb{N}}$ such that $\|y_n - Q_{m_n}(I - P_{m_{n-1}})(y_n)\|_J < \varepsilon_n$ for every $n \in \mathbb{N}$; in particular, $(y_n)_{n\in\mathbb{N}}$ is still equivalent to the unit vector basis of ℓ_p. Choosing a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ that converges to 0 fast enough, we can then assume that $(z_n)_{n\in\mathbb{N}} = \big(Q_{m_n}(I - P_{m_{n-1}})(y_n)\big)_{n\in\mathbb{N}}$ is also still equivalent to the unit vector basis of ℓ_p, where each $z_n \in R\big(Q_{m_n}(I - P_{m_{n-1}})\big) = R\big((I - P_{m_{n-1}})Q_{m_n}\big)$. But then, for any $n \in \mathbb{N}$, we would have

$$2\left\|\sum_{i=1}^n z_i\right\|_J^2 \geq \sum_{i=1}^n \|z_i\|_J^2$$

by Lemma 5.7, which is not possible for a sequence equivalent to the unit vector basis of ℓ_p with $p > 2$. □

For example, let $(p_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence in $(2, \infty)$ and let $p = \lim_n p_n$. Taking $X_n = \ell_{p_n}$ and letting $\phi_n : \ell_{p_n} \longrightarrow \ell_{p_{n+1}}$ be the natural inclusion, then $\hat{J}(\Phi)/J(\Phi)$ is isomorphic to ℓ_p (or c_0 if $p = \infty$), as $\|x\|_{p_n} \underset{n}{\longrightarrow} \|x\|_p$, so the quotient map $\hat{J}(\Phi) \longrightarrow \hat{J}(\Phi)/J(\Phi)$ is strictly singular by Proposition 5.8 and $\hat{J}(\Phi)$ is subprojective by Proposition 5.6.

6 The dense subspace chain case

Let $(Z, \|\cdot\|)$ be a Banach space. In this section, we will assume that each X_n is a closed subspace of Z such that $\bigcup_{n\in\mathbb{N}} X_n$ is dense in Z and $X_n \subseteq X_{n+1}$ for every $n \in \mathbb{N}_0$, each $\|\cdot\|_n$ is the restriction of $\|\cdot\|$ to X_n and each ϕ_n is the natural inclusion of X_n into X_{n+1}. We will write $J(X_n) = J(\Phi)$, $\hat{J}(X_n) = \hat{J}(\Phi)$ and $\Omega(X_n) = \Omega(\Phi)$ for this particular case.

Under these conditions, the mapping $S\colon \hat{J}(X_n) \longrightarrow Z$ defined as $S\big((x_n)_{n\in\mathbb{N}_0}\big) = \lim_n x_n$ is a surjective bounded linear operator with kernel $J(X_n)$ and the induced quotient map $\hat{J}(X_n)/J(X_n) \longrightarrow Z$ is an isometry [4, Corollary 1.3].

Proposition 6.1 *If Z is subprojective, then $\hat{J}(X_n)$ is subprojective.*

Proof Each $X_n \subseteq Z$ is subprojective [23, Lemma 3.1] and $\hat{J}(X_n)/J(X_n)$ is isomorphic to Z, hence subprojective, so $\hat{J}(X_n)$ is subprojective by Corollary 5.3 and Proposition 5.5. □

Given that the construction of $J(X_n)$ is made so that the quotient $\hat{J}(X_n)/J(X_n)$ is isometric to Z, it is natural to ask whether the quotient is complemented, that is, whether $\hat{J}(X_n) = J(X_n) \oplus (\hat{J}(X_n)/J(X_n)) = J(X_n) \oplus Z$. For some spaces Z, this will be impossible; for instance, $Z = c_0$ as a subspace of $\hat{J}(X_n) \equiv J(X_n)^{**}$ would make ℓ_∞ embed into $\hat{J}(X_n)$ [16, Proposition 2.e.8], which is separable ($\Omega(X_n)$ is dense in $\hat{J}(X_n)$). On the other hand, for $Z = \ell_1$ the complementation is immediate, as ℓ_1 is always complemented as a quotient [16, Proposition 2.f.7]. For other spaces, we have the following.

Proposition 6.2 *Let $(Z_n)_{n\in\mathbb{N}}$ be an unconditional Schauder decomposition of Z that satisfies a lower 2-estimate and such that $X_n = \oplus_{i=1}^n Z_i$ and let $(T_n)_{n\in\mathbb{N}}$ be the sequence of aggregate projections associated with $(Z_n)_{n\in\mathbb{N}}$ (so $R(T_n) = X_n$ and $T_n(z) \underset{n}{\longrightarrow} z$ for every $z \in Z$). Then the linear map $T\colon Z \longrightarrow \hat{J}(X_n)$ defined as $T(z) = (T_n(z))_{n\in\mathbb{N}}$ is well defined and an embedding into $\hat{J}(X_n)$, and $\hat{J}(X_n) = J(X_n) \oplus R(T) \simeq J(X_n) \oplus Z$.*

Proof Let $K < \infty$ be the suppresion constant of $(Z_n)_{n\in\mathbb{N}}$ and let $M < \infty$ be such that $\sum_{i=1}^k \|z_i\|^2 \le M\big\|\sum_{i=1}^k z_i\big\|^2$ if $(z_i)_{i=1}^k$ are disjoint with respect to $(Z_n)_{n\in\mathbb{N}}$, in that each z_i belongs to a different Z_i. Let $z \in Z$ and $S = \{\, p_0 < \cdots < p_k \,\} \in \wp$; then

$$\begin{aligned}
\rho(T(z), S)^2 &= \sum_{i=1}^k \big\|T_{p_{i-1}}(z) - T_{p_i}(z)\big\|^2 + \|T_{p_k}(z)\|^2 \\
&\le M \Big\|\sum_{i=1}^k \big(T_{p_{i-1}}(z) - T_{p_i}(z)\big)\Big\|^2 + \|T_{p_k}(z)\|^2 \\
&= M\big\|T_{p_0}(z) - T_{p_k}(z)\big\|^2 + \|T_{p_k}(z)\|^2 \\
&\le MK^2\|z\|^2 + K^2\|z\|^2 = (M+1)K^2\|z\|^2
\end{aligned}$$

so $T(z) \in \hat{J}(X_n)$ and $\|T(z)\|_J \leq \sqrt{(M+1)/2}K\|z\|$. If we define the operator $S\colon \hat{J}(X_n) \longrightarrow Z$ given by $S\big((x_n)_{n\in\mathbb{N}}\big) = \lim_n x_n$ as before, then ST is clearly the identity on Z, so TS is a projection on $\hat{J}(X_n)$ with kernel $J(X_n)$ and range $R(T)$. □

This applies, for instance, if $Z = \ell_p$, for $1 \leq p \leq 2$, and $(Z_n)_{n\in\mathbb{N}}$ is the decomposition associated to the unit vector basis, which is unconditional and satisfies a lower 2-estimate [17, Theorem 1.f.7], or to $Z = L_p$ for $1 \leq p \leq 2$ with the decomposition associated to the Haar basis (or any other unconditional basis) [3, p. 128]; in all of these cases, $\hat{J}(X_n) \simeq J(X_n) \oplus Z$.

On the other hand, if Z contains a copy of ℓ_p for $p > 2$ but no X_n contains a copy of ℓ_p, then $\hat{J}(X_n)$ does not contain copies of ℓ_p by Proposition 5.8 and $\hat{J}(X_n)$ cannot contain $\hat{J}(X_n)/J(X_n) \equiv Z$ as a subspace, so the quotient cannot be complemented. In particular, this happens for $Z = \ell_p$ itself, with $p > 2$, and its canonical finite-dimensional decomposition.

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References

1. Álvarez, T., González, M.: Some examples of Tauberian operators. Proc. Am. Math. Soc. **111**(4), 1023–1027 (1991)
2. Albiac, F., Kalton, N.J.: Topics in Banach Space Theory. Graduate Texts in Mathematics, vol. 233. Springer, New York (2006)
3. Alspach, D., Odell, E.: L_p spaces. In: Handbook of the Geometry of Banach Spaces, vol. I, pp. 123–159. North-Holland, Amsterdam (2001)
4. Bellenot, S.F.: The J-sum of Banach spaces. J. Funct. Anal. **48**(1), 95–106 (1982)
5. Bessaga, C., Pełczyński, A.: On bases and unconditional convergence of series in Banach spaces. Studia Math. **17**, 151–164 (1958)
6. Castillo, J.M.F., González, M.: Three-Space Problems in Banach Space Theory. Lecture Notes in Mathematics, vol. 1667. Springer, Berlin (1997)
7. Castillo, J.M.F., González, M., Papini, P.L.: New results on Kottman's constant. Banach J. Math. Anal. **11**(2), 348–362 (2017)
8. Davis, W.J., Figiel, T., Johnson, W.B., Pełczyński, A.: Factoring weakly compact operators. J. Funct. Anal. **17**, 311–317 (1974)
9. González, M.: The perturbation classes problem in Fredholm theory. J. Funct. Anal. **200**, 65–70 (2003)
10. González, M., Martínez-Abejón, A., Salas-Brown, M.: Perturbation classes for semi-Fredholm operators on subprojective and superprojective Banach spaces. Ann. Acad. Sci. Fenn. Math. **36**, 481–491 (2011)
11. González, M., Pello, J.: Superprojective Banach spaces. J. Math. Anal. Appl. **437**(2), 1140–1151 (2016)
12. González, M., Pello, J., Salas-Brown, M.: The perturbation classes problem for subprojective and superprojective Banach spaces. J. Math. Anal. Appl. **489**(2), 124191 (2020)
13. James, R.C.: Separable conjugate spaces. Pac. J. Math. **10**, 563–571 (1960)
14. Kalton, N., Wilansky, A.: Tauberian operators on Banach spaces. Proc. Am. Math. Soc. **57**(2), 251–255 (1976)
15. Lindenstrauss, J.: On James' paper "Separable conjugate spaces". Isr. J. Math. **9**, 279–284 (1971)
16. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I. Springer, Berlin (1977)
17. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces II. Springer, Berlin (1979)

18. Oikhberg, T., Spinu, E.: Subprojective Banach spaces. J. Math. Anal. Appl. **424**, 613–635 (2015)
19. Ostrovskiĭ, M.I.: Subspaces containing biorthogonal functionals of bases of various types. Teor. Funktsii Funktsional. Anal. i Prilozhen. **57**(1992), 115–127 (1993)
20. Ostrovskiĭ, M.I.: Subspaces containing biorthogonal functionals of bases of various types. J. Math. Sci. **77**(1), 3008–3016 (1995)
21. Ostrovskiĭ, M.I.: Quojections without Banach subspaces. Bull. Pol. Acad. Sci. Math. **44**(2), 143–146 (1996)
22. Tylli, H.-O.: Duality of the distance to closed operator ideals. Proc. R. Soc. Edinb **133A**(1), 197–212 (2003)
23. Whitley, R.J.: Strictly singular operators and their conjugates. Trans. Am. Math. Soc. **113**, 252–261 (1964)