Fragility of Kardar-Parisi-Zhang universality class in the presence of temporally correlated noise

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We study numerically a family of surface growth models that are known to be in the universality class of the Kardar-Parisi-Zhang equation when driven by uncorrelated noise. We find that, in the presence of noise with power-law temporal correlations with exponent θ, these models exhibit critical exponents that differ both quantitatively and qualitatively from model to model. The existence of a threshold value for θ below which the uncorrelated fixed point is dominant occurs for some models but not for others. In some models the dynamic exponent $z(\theta)$ is a smooth decreasing function, while it has a maximum in other cases. Despite all models sharing the same symmetries, critical exponents turn out to be strongly model dependent. Our results clearly show the fragility of the universality class concept in the presence of long-range temporally correlated noise.

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I. INTRODUCTION

Kinetic surface roughening is the term used to describe the dynamics of surfaces under the effects of random fluctuations [1–3]. The key quantity is the surface height $h(\mathbf{x}, t)$ at position $\mathbf{x} \in \mathbb{R}^d$ and time t, which is a random field with some probability distribution. If the system preserves some basic symmetries (like time translation $t \to t + t_0$, space translation $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0$, and invariance under rotations in the hyperplane of $\mathbf{x}$), which are common in nature, then translation invariance along the growth direction $h \to h + c$, for arbitrary constant c, immediately implies scale-invariant roughening [4–6]. In this case, the height-height correlations are described by power laws with some critical exponents, which values ought to depend solely on the exiting symmetries, conservation laws, and system dimension. This allows the classification of scale-invariant growth processes into universality classes [1], akin to what occurs in critical phase transitions. An important question in kinetic surface roughening theory is the robustness of universality against the presence of quenched disorder [7,8] or long-range correlations of the environmental noise [9].

One important example in the field of surface roughening is the universality class represented by the Kardar-Parisi-Zhang (KPZ) equation [10] in $d + 1$ dimensions

$$\partial_t h(\mathbf{x}, t) = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \sqrt{D}\,\eta(\mathbf{x}, t), \qquad (1)$$

where η is a Gaussian noise with zero mean $\langle \eta(\mathbf{x}, t)\rangle = 0$ and no correlations $\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t')\rangle = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$. This equation describes the roughening dynamics for a one-dimensional interface that grows by the effect of the noise term η. The first term on the right-hand side describes interface elasticity (surface tension), while the second term is a nonlinear non-Hamiltonian term associated with lateral interface growth.

The KPZ universality class plays a central role in statistical physics as a fundamental model of scaling out of equilibrium. Beyond growing interfaces, the KPZ equation appears in many very different contexts, such as directed polymers in random media [2], randomly stirred fluids [11], particle transport [12,13], driven-dissipative Bose-Einstein condensates [14], and space-time chaos [15,16], to cite a few.

KPZ equation satisfies the above mentioned $h \to h + c$ invariance, so that solutions to the KPZ equation are scale-invariant: for any scalar b, the change of variables $\mathbf{x} \to b\mathbf{x}$, $t \to b^{1/z}t$, and $h \to b^{\alpha}h$, leaves Eq. (1) unchanged for a particular choice of the d-dependent critical exponents α and z. Furthermore, the KPZ nonlinearity leads to Galilean invariance: KPZ equation remains unaffected by the transformation $\mathbf{x} \to \mathbf{x} + \lambda\epsilon t$ and $h \to h - \epsilon\mathbf{x}$ with $\epsilon \to 0$. This immediately implies the scaling relation $\alpha + z = 2$ in any dimension [1,10]. In $d = 1$ the critical exponents are known exactly: $\alpha = 1/2$ and $z = 3/2$, which can be obtained by a perturbative dynamical renormalization group (DRG) calculation [10]. Remarkably, the exact form of h in $d = 1$ has been found for several initial conditions [17,18] through several theoretical techniques that have revealed the existence of a deep connection between KPZ universality class and random matrix theory [17–20]. In higher dimensions, exact predictions do not exist and one has to rely on numerical simulations.

Soon after the introduction of the KPZ equation, the effect of correlated noise in Eq. (1) was analytically studied in a seminal paper by Medina *et al.* [9]. We focus our

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attention here in the case of solely temporal correlations where the noise in Eq. (1) has a slowly decaying correlator given by

$$\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t')\rangle = 2\delta(\mathbf{x} - \mathbf{x}')|t - t'|^{2\theta-1}, \tag{2}$$

with the index $\theta \in [0, 1/2)$ characterizing the "range" of the temporal correlation. In the $\theta \to 0$ limit the noise becomes delta correlated, while for $\theta \to 1/2$ the noise behaves effectively as a columnar disorder [21]. Using a perturbative DRG approach to analyze Eq. (1) with the noise correlation (2), Medina *et al.* made a prediction for the critical exponents in $1 + 1$ dimensions up to one-loop order in the perturbative series expansion:

$$\alpha = \begin{cases} 1/2 & \text{if } \theta < 1/6 \\ 1.69\,\theta + 0.22 & 1/6 < \theta < 1/2, \end{cases} \tag{3}$$

$$z = \begin{cases} 3/2 & \text{if } \theta < 1/6 \\ (2\alpha + 1)/(1 + 2\theta) & 1/6 < \theta < 1/2. \end{cases} \tag{4}$$

However, Medina *et al.* calculation has severe technical difficulties directly associated with the presence of temporally correlated noise. On the one hand, one immediately notes that time-correlated noise breaks Galilean invariance and the scaling relation $\alpha + z = 2$ becomes invalid, at least for $\theta > 1/6$. In DRG terms this means that the flow equation for λ has corrections in the perturbative expansion. On the other hand, and maybe more importantly, no stable fixed point is found for $\theta > 1/4$. In fact, an infinite set of singularities appear at $\theta^* = 1/4, 2/6, 3/8, \ldots$ and an increasing number $N_{\max}$ of terms contribute as θ gets larger. Medina *et al.* solved the sums numerically for various increasing values of $N_{\max}$ improving the numerical estimates of the correction. The calculation becomes increasingly difficult because $N_{\max} > 1/(1 - 2\theta) \to \infty$ as $\theta \to 1/2$. All in all, the calculation is very problematic, and it is hard to be convinced of the validity of the whole approach (see Ref. [9] for further details).

A self-consistent expansion (SCE) was employed by Katsav and Schwartz [22] to obtain the critical exponents $\alpha(\theta)$ and $z(\theta)$ as smooth functions of θ. At variance with the DRG calculation of Medina *et al.* no threshold exists, and the exponents differ from the uncorrelated case for any finite θ.

A different approach, using a perturbative functional renormalization group (FRG), was carried out by Fedorenko in the context of elastic manifolds in correlated disorder [23]. The two-loop results indicated that the dynamic exponent z is a decreasing function of θ, at variance with Eq. (4) where z is increasing with θ.

The problem was analyzed by Squizzato and Canet [24] using a nonperturbative functional renormalization group (NPFRG). They concluded that, in the presence of power-law time-correlated noise, the uncorrelated fixed point is stable for below some critical θ_{th}, while there is a new critical point above the threshold where the exponents are θ-dependent. These results are qualitatively in accordance with Medina *et al.* DRG, but in disagreement with SCE or FRG.

Early numerical explorations of the scaling behavior of the KPZ equation in the presence of long-range temporally correlated noise was carried out by Lam *et al.* [25] by using simulations of ballistic deposition (BD) in $1 + 1$ dimensions. They found a good agreement with Medina *at al.* predictions in Eqs. (3) and (4) for some noise generators, but not for others. Two noise generator algorithms could give considerably different surface exponents even if the correlator was identical. They attributed this discrepancy to extremely slow crossovers, which should vanish at large scales. The problem was revisited almost 25 years later by Song and Xia [26], who also carried out simulations of BD with correlated noise, generated by two different algorithms (one of them was identical to that of Lam *et al.* [25]) and concluded that the prediction of the DRG was in excellent agreement with simulations.

In a recent paper Alés and López [27] have shown that the scaling picture of KPZ in the presence of temporally correlated noise is much richer than previously expected. The main result was that the surface shows anomalous roughening beyond $\theta = 1/4$ and actually becomes faceted if θ is increased further. The appearance of a faceted pattern implies the existence of a new critical exponent $\alpha_s \neq \alpha$ that describes the scaling behavior of the surface power spectral density. It also implies that the standard scaling ansatz for the surface correlation function, which is the starting point of any RG analysis, has to be replaced by the *generic scaling ansatz* of Ramasco *et al.* [28]. These results were later confirmed by independent investigations and extended to $2 + 1$ dimensions [29–31]. Obviously, the original DRG approach [9] does not make any prediction about the value of the new exponent $\alpha_s(\theta)$, neither is able to explain the emergence of facets. The more recent NPFRG approach developed by Squizzato and Canet [24] was also inconclusive with respect to the faceted phase.

In this paper we revisit the problem of the universality class of the KPZ equation with temporally correlated noise. We report on extensive simulations of two different discretizations of the KPZ equation and two implementations of the BD model. Our results clearly show that the dependence of the critical exponents on the noise correlation index θ depends on microscopic details and particulars of the model. In fact, results differ, even at a qualitative level, from model to model showing that the fine details do matter and strongly affect the critical behavior. Furthermore, not only the exponents $\alpha(\theta)$ and $z(\theta)$ are qualitatively and quantitatively different for all the models, but also the surface height distribution differs. Our calculation of the kurtosis and skweness of the height distribution for the two KPZ integration schemes clearly shows that these models, although they share KPZ symmetries, do obey different height statistics.

The rest of the paper is organized as follows. In Sec. II we describe the models in detail. In Sec. III the observables and quantities of interest, critical exponents, and interface statistics are discussed. Main results are presented in Sec. IV, where a comparison of the relevant critical exponents for all the studied models is made. Finally, conclusions are summarized in Sec. V.

II. MODELS

We describe now several integration schemes we studied to investigate the universality class of KPZ with temporally correlated noise. In all cases, an adaptation of the algorithm originally proposed by Mandelbrot [32,33] was used. The details of the algorithm and parameters used in noise generation can be found in Ref. [27].

A. Numerical integration of the KPZ equation (I): Exponential correction

The first model we study consists in an explicit integration scheme for the KPZ equation with temporally correlated noise, Eqs. (1) and (2), for $\nu = D = 1$ and $\lambda = 4$,

$$h_j(t+1) = h_j(t) + \Delta t\left[\nu\,\mathcal{D}[h_j(t)] + \frac{\lambda}{2}\,\mathcal{N}[h_j(t)]\right] + \sqrt{D\Delta t}\;\eta_j(t), \tag{5}$$

where $h_j(t)$ is the interface height at position $j = 1, 2, \dots, L$ and simulations were performed for systems of size L with periodic boundary conditions. In all cases the surface is started from a initially flat profile $h_j(0) = 0$. The noise is Gaussian distributed with correlations given by

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{i,j}|t-t'|^{2\theta-1}, \tag{6}$$

with a noise correlation index $\theta \in [0, 1/2)$. The operator $\mathcal{D}$ represents the discrete Laplacian, $\mathcal{D}[h_j] = a^{-2}(h_{j+1} + h_{j-1} - 2h_j)$, where a is the lattice spacing, and $\mathcal{N}[h_j(t)]$ is the nonlinear term discretization. In this case we choose $\mathcal{N}[h_j] = f(\frac{h_{j+1}-h_{j-1}}{2a})$ with the standard smoothing function [34]

$$f(y) = \frac{1 - e^{-cy^2}}{c}, \tag{7}$$

which has been successfully used to integrate several growth equations and guarantee very stable numerical schemes for several surface growth equations [34–37], including KPZ with correlated noise [27,30]. Note that this prescription generates the usual KPZ nonlinearity $[(h_{j+1} - h_{j-1})/(2a)]^2$ at order c^0, while higher order terms, $c, c^2, c^3, \dots$, correspond to integer powers of this term—all of them consistent with the growth symmetries but with decreasing coefficients for $0 < c < 1$. This produces very stable numerical schemes in which spurious numerical instabilities associated with artificially large local gradients are effectively suppressed [27,34]. Here we use $a = 1$, $\Delta t = 10^{-3}$, and $c = 0.1$ as control parameter of the stabilization function.

B. Numerical integration of the KPZ equation (II): Numerical scheme that preserves the stationary behavior

We have also studied the numerical discretization originally devised by Lam and Shin [38] that has the property of preserving the stationary solution of the corresponding Fokker-Plank equation. To be more specific, this scheme produces a discretization of the KPZ term such that the exact steady-state probability distribution of the resulting discrete surfaces corresponds to that of the continuum. In this case, the KPZ nonlinearity is discretized as

$$\mathcal{N}[h_j] = \frac{1}{3a^2}[(h_{j+1} - h_j)^2 + (h_{j+1} - h_j)(h_j - h_{j-1}) + (h_j - h_{j-1})^2], \tag{8}$$

while the rest of the terms and conditions remain the same as in Eq. (5). We use $\nu = D = 1$ and $\lambda = 4$ in our numerical simulations, the same as before. Here we also use $a = 1$ and $\Delta t = 10^{-3}$, periodic boundary conditions, and surface evolution is always initiated from a flat state.

In addition, we have also investigated the method that was proposed by Sasamoto and Spohn [39] for the stochastic Burgers' equation

$$\partial_t u = \nu\partial_x^2 u + \lambda u\partial_x u + \sqrt{D}\partial_x \eta, \tag{9}$$

which allows us to describe the evolution of the slopes field $u = \partial_x h$ of KPZ interfaces. In this case, the KPZ discrete interface $h_j(t)$ can be recovered as

$$h_j(t) = \sum_{l=0}^{j} u_l(t) \tag{10}$$

from the numerical solutions $u_l(t)$ of the Burgers' equation. This strategy has been used with successful results to recover the KPZ behavior, for instance, in Ref. [40]. We found that the results are indistinguishable from those obtained using the method described in Eq. (8) also for correlated noise $\theta > 0$. Hence, in the following we report only on the results obtained by using (5) with the discretization (8).

C. Ballistic deposition

A wide variety of surface growth models based on particle deposition share the same symmetries as KPZ equation and are believed to belong to the same universality class. In particular, discrete growth models of the ballistic deposition (BD) type represent a typical example of KPZ scaling behavior in $1 + 1$ dimensions [1,41] (although caution must be taken in higher dimensions, at least for some BD algorithms [42]). We have also simulated several discrete algorithms for surface growth with KPZ symmetries in the presence of time correlated noise. These simulations extend previously published results by two of us [27]. We present a summary of our results for two BD models that correspond to two different implementations of the correlated noise.

The interface is an integer $h_j(t)$ that gives the height position at spatial coordinate $j = 1, 2, \dots, L$ and integer time t. Starting from a flat initial state $h_j(0) = 0$ for all i, the surface height is given by

$$h_j(t+1) = \mathrm{Max}[h_j(t) + \zeta_j(t), h_{j-1}, (t), h_{j+1}(t)],$$

where the noise $\zeta_j(t)$ can take only two values $\{0, 1\}$ and is temporally correlated with an exponent $0 < \theta < 1/2$. The algorithm is updated in parallel so that growth is attempted at all even (odd) sites at even (odd) time steps with periodic boundary conditions.

We report here on two different implementations of the correlated noise. We generate a Gaussian distributed noise $\eta_j(t)$ with the temporal correlations in Eq. (2), following Ref. [27] as before. Then we define $\zeta_j(t) = 0$ if $\eta_j(t) \leqslant 0$ and $\zeta_j(t) = 1$ if $\eta_j(t) > 0$. This corresponds to Lam *et al.* [25] implementation of BD with time-correlated noise, which we call BD-I model.

There are many possible numerical prescriptions to construct the integer noise ζ from the continuous Gaussian variable η while temporal correlations remain virtually intact. We introduce here model BD-II in which the noise is $\zeta_j(t) \leqslant 0$

if $\eta_j(t) < 0$ and $\zeta_j(t) = [[\eta_j(t)]] + 1$ if $\eta_j(t) > 0$, where $[[x]]$ means integer part of $x \in \mathbb{R}$. At variance with BD-I, in this case we have accumulation of large noise amplitudes at certain sites since a large value of ζ_j at time t is more likely to be followed by other large amplitudes at times $t+1$, $t+2$, etc., due to the time correlations. This favors large local slopes for BD-II as compared with BD-I model. In any case, note that both noise models have strictly the same correlations.

III. OBSERVABLES

In order to characterize the kinetic roughening processes from the numerical simulations of the KPZ and Burgers' equations, we measure the global roughness

$$W(t,L) = \left\langle \sqrt{\frac{1}{L}\sum_{j=1}^{L}[h_j(t) - \bar{h}(t)]^2} \right\rangle, \tag{11}$$

where L is the lateral size of the system. The scaling exponents are customarily obtained from the power-law behavior of the global roughness with time

$$W(t,L) \sim t^{\beta}, \quad t \ll t_{\rm sat} \tag{12}$$

and the saturation values $W_{\rm sat}(L) = W(t \gg t_{\rm sat}, L)$ with the system size

$$W_{\rm sat} \sim L^{\alpha}, \tag{13}$$

where $t_{\rm sat} \sim L^z$. The dynamic exponent z, which describes the spatial extent of height-height correlations can be computed just as the quotient $z = \alpha/\beta$.

Since the scaling of the interface in the presence of correlated noise is expected to be anomalous [27], we have to deal with different local and global scaling of the roughness for large values of θ. Such behavior is called *anomalous scaling* or *anomalous kinetic roughening* [43–46]. As shown by Ramasco *et al.* [28] the scaling properties of anomalously roughened surfaces can be best described in Fourier space using the power spectral density or (structure factor) $S(k,t) = \langle|\widehat{h}_k(t)|^2\rangle$ where $\widehat{h}_k(t) = L^{-1/2}\sum_{m=1}^{L} h_m(t)\exp(-ikm)$ is the discrete Fourier transform in space of the surface height in $1+1$ dimensions.

Following Ramasco *et al.* [28] in $1+1$ dimensions we expect

$$S(k,t) = k^{-(2\alpha+1)} s(kt^{1/z}), \tag{14}$$

where the most general scaling function, consistent with scale-invariant dynamics, is given by [28]

$$s(u) \sim \begin{cases} u^{2(\alpha-\alpha_s)} & \text{if } u \gg 1 \\ u^{2\alpha+1} & \text{if } u \ll 1, \end{cases} \tag{15}$$

with α being the *global* roughness exponent defined in (13) and α_s the so-called *spectral* roughness exponent [28]. Standard scaling corresponds to $\alpha_s = \alpha < 1$. However, other situations may be described within the generic scaling framework, including super-roughening and intrinsic anomalous scaling, depending on the values of α_s and α [28]. For faceted surfaces, the case of interest for us here, one has $\alpha_s > \alpha$ so that two independent roughening exponents are actually needed to completely describe the scaling properties of the surface

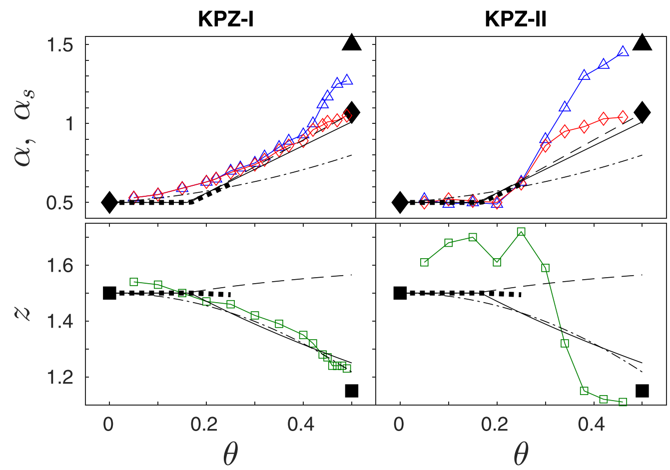

FIG. 1. Dependence with θ of the scaling exponents α (red $\diamond$), α_s (blue $\triangle$), and z (green $\square$) for KPZ with the different integration schemes described in Secs. II A and II B. Filled symbols correspond to known values for KPZ in the uncorrelated noise ($\theta = 0$) and columnar disorder ($\theta = 1/2$) limits. The latter are from numerical simulations in Ref. [48]. For comparison we also show the existing theoretical predictions: dynamic RG [9] with dashed line, SCE [22] with dotted-dashed, FRG [23] with solid line, and NPFRG [24] with dotted line.

[28]. Scaling behavior in Eqs. (14) and (15) implies that when one plots $k^{2\alpha+1}S(k,t)$ vs $kt^{1/z}$ for numerical data taken at different times t these can be collapsed into the universal scaling function (15) only for the correct choice of exponents. This is the so-called data collapse technique and provides a very systematic approach to analyze critical dynamics.

In recent times, the importance of the surface height statistics to asses the roughening universality class has been highlighted [47]. Therefore, we have also studied the field statistics by computing the time-dependent skewness

$$\mathcal{S}(t) = \frac{1}{W^3(t)}\left\langle \frac{1}{L}\sum_{j=1}^{L}[h_j(t) - \bar{h}(t)]^3 \right\rangle \tag{16}$$

and kurtosis

$$\mathcal{K}(t) = \frac{1}{W^4(t)}\left\langle \frac{1}{L}\sum_{j=1}^{L}[h_j(t) - \bar{h}(t)]^4\, dx \right\rangle, \tag{17}$$

as observables that on their own are used in order to characterize the probability distribution function (PDF) of the height fluctuations.

IV. UNIVERSALITY CLASS: NUMERICAL RESULTS

A. Scaling exponents

We now present numerically computed scaling exponents for different degree of temporal correlations θ in the four different systems assessed in this work. As mentioned above, the integration of the Burgers' equation yields to the same results as Lam and Shin discretization, model KPZ-II, in Eq. (8), so it will not be shown. In Fig. 1 we plot the critical exponents α, α_s, and z for the KPZ equation integrated with the two different discretizations that we tag as KPZ-I and KPZ-II.

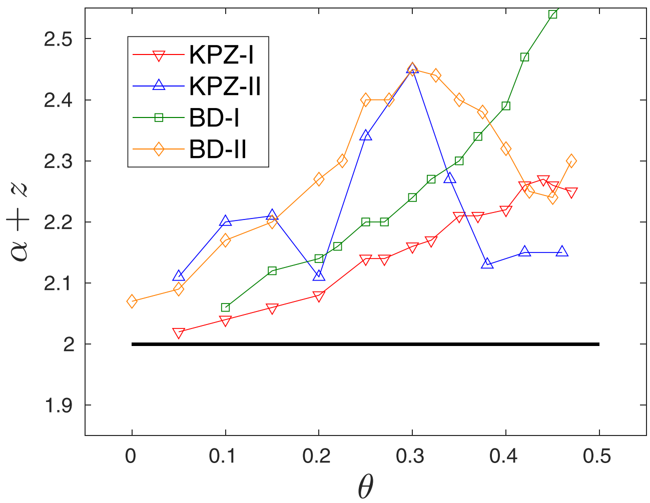

FIG. 2. Evolution of the $\alpha + z$ sum for different values of θ. The solid line corresponds to the Galilean invariance $\alpha + z = 2$.

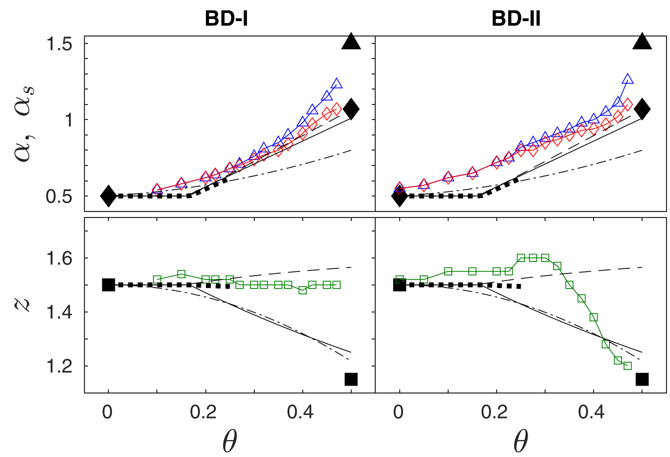

FIG. 3. Dependence with θ of the scaling exponents α (red $\diamond$), α_s (blue $\triangle$), and z (green $\square$) for BD-I and BD-II discrete models described in Sec. II C). As in Fig. 1, filled symbols correspond to known values for KPZ with white noise ($\theta = 0$) and columnar disorder ($\theta = 1/2$). Lines correspond to theoretical predictions as in Fig. 1.

The figure also shows the limiting cases $\theta = 0$ (uncorrelated noise) and $\theta \to 1/2$ (columnar noise) as a reference. In all our simulations the roughness exponent α was computed at saturation by fitting the stationary surface width to Eq. (13) for systems of size $L = 2^{10}, 2^{11}, 2^{12}$, and 2^{13}. Then, measuring the asymptotic k-dependence of the power spectral density data, $\sim k^{-(2\alpha_s+1)}$, at long times and small momenta k we can calculate the spectral roughness exponent α_s. Subsequently, numerical results for $S(k,t)$ at different times can be cast into the scaling form (15) by using standard data collapse techniques.

For both discretizations the interface develops a faceted pattern for large enough values of the noise correlation index θ, implying $\alpha_s(\theta) > \alpha(\theta)$, which was recently reported in the literature [27] as a distinctive feature associated with the interplay between the KPZ nonlinearity and the noise correlations. Let us focus our attention on the specific functional dependence of the critical exponents with the noise index θ. It is evident that the three critical exponents for KPZ-I and KPZ-II discretizations are not only quantitatively different but they also differ at qualitatively level. The two roughness exponents are monotonously increasing functions of θ for KPZ-I. In contrast, in the case of KPZ-II discretization these exponents remain nearly constant $\alpha(\theta) = \alpha_s(\theta) = 1/2$ for θ values below 0.23 approximately. Also, the spectral roughness exponent for KPZ-I discretization does not seem to converge to the columnar disorder limit value $\alpha_s = 3/2$ [21]. As for the dynamic exponent $z(\theta)$ strong differences also appear for both discretizations, which do not seem to be easily explained for two discretizations of the *same* dynamics.

In Fig. 2 we plot $\alpha + z$ in the four studied models for different values of θ. This allows us to visualize the rupture of the Galilean relation that holds for KPZ with uncorrelated noise but should be broken when the temporal correlations in the noise are present. Remarkably, the degree of violation of Galilean invariance, as measured by the deviation of $\alpha + z$ from 2, is relatively small, although clearly finite for all θ and both discretizations. This is due to the purely numerical observation that the increase of α is partially compensated by the decrease of z as θ is varied.

A similar analysis was carried out for the two BD models described in Sec. II C, and a summary of the results is shown in Fig. 3. From these plots it would be difficult to argue that both models should belong to the *same* universality class, since they share the same symmetries. Note that for BD-I model the dynamic exponent stays roughly constant around $z(\theta) \approx 3/2$ as the noise index is varied, close to its value for standard KPZ. Comparison of Figs. 1 and 3 immediately reveals the fragility of the universality concept in the presence of temporally correlated noise. It is remarkable that discrepancies in critical exponents across models appear even for relatively small values of θ, in a phase where facets are not still present (i.e., where $\alpha_s = \alpha$).

B. Fluctuation statistics

We assess the fluctuation statistics behavior by computing the skewness (16) and kurtosis (17) of the height field fluctuations for different values of θ. For standard KPZ, without temporal correlations, these quantities evolve in time during the surface dynamics from the Gaussian distribution values (linear regime) towards the Tracy-Widom (TW) distribution values (in the actual nonlinear growth phase) [47]. For finite-size systems the dynamics eventually becomes stationary for $t > t_{\text{sat}} \sim L^z$, and the statistics shows Gaussian values again due to the fluctuation-dissipation theorem [1,3]. For a flat initial condition $h_j(0) = 0,\ j = 1, 2, \ldots, L$ the skewness values evolve from $\mathcal{S} = 0$ (Gaussian) to a maximum in a plateau value at intermediate times of $\mathcal{S} \simeq 0.29$. The kurtosis exhibits an analog crossover behavior from $\mathcal{K} = 3$ (Gaussian) to $\mathcal{K} \simeq 3.16$. All this while the surface evolution is in the dynamic regime before saturation sets in, of course. These fully nonlinear KPZ regime values of $\mathcal{S}$ and $\mathcal{K}$ correspond to the so-called TW-GOE distribution, which describes the largest

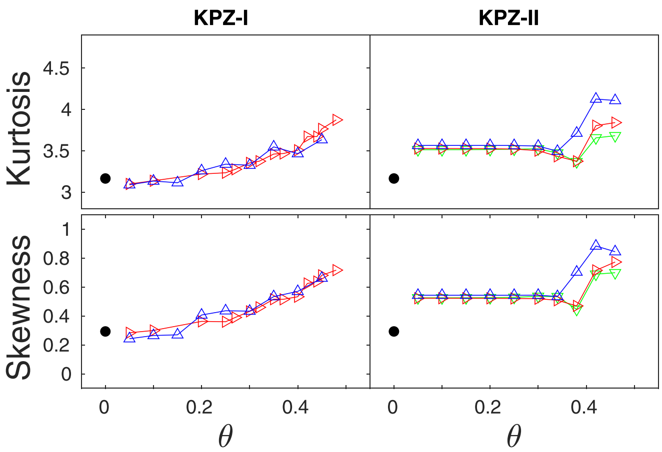

FIG. 4. Dependence with θ of the skewness and kurtosis of systems of size $L = 1024$ (green $\triangledown$), $L = 2048$ (red $\triangleright$), and $L = 4096$ (blue $\triangle$) in the three systems assessed. Filled circles correspond to the theoretical values for TW-GOE statistics: (flat initial conditon) KPZ with uncorrelated ($\theta = 0$) noise.

eigenvalue of random matrices in the Gaussian orthogonal ensemble (GOE) [47].

In our simulations of the KPZ equation (with both KPZ-I and KPZ-II discretizations) the maximum value of the skewness and kurtosis temporal series have been determined for different values of $\theta \in [0, 1/2)$. The results are shown in Fig. 4, where they are also compared with the reference values for TW-GOE, i.e., KPZ with $\theta = 0$.

Several system sizes have been considered, $L = 2^{10},\ 2^{11}$, and 2^{12}. Remarkably, for KPZ-II discretization the skewness and kurtosis seem to scale with the system size for high values of θ. This anomalous dependence of the statistics with system size appears within the range of θ values that corresponds to the faceted phase of the surface.

Comparison of the height fluctuation statistics, summarized in Fig. 4, for KPZ-I and KPZ-II discretizations again indicates strong differences in the presence of correlated noise and poses serious doubts about the independence of the critical behavior from microscopic details and the existence of universality.

V. CONCLUSIONS

We have studied a well-known family of surface growth models that, when driven by uncorrelated noise, all belong to KPZ universality class. These models included numerical integration schemes of the KPZ equation as well as ballistic particle deposition algorithms. We have analyzed the scaling behavior of these models in the presence of temporally correlated noise with a long-tail memory parametrized by the index θ. We focused on determining the critical exponents that describe the surface fluctuations (roughness) for each separate model as the noise memory range is increased.

All models studied share the emergence of a faceted regime, which was first reported in Ref. [27], for large enough values of θ. The growth of facets immediately implies, following Ramasco *et al.* [28], that an independent spectral roughness exponent $\alpha_s \neq \alpha$ enters into the scaling description. The origin of a rough faceted phase is the result of the interplay between the KPZ nonlinearity and the long memory of the driving noise.

Our main conclusion is that long-range memory in the noise driving KPZ breaks down universality: different models exhibit different critical behavior, despite they share the same fundamental symmetries. More precisely we found that the functional form of the three critical exponents $\alpha(\theta)$, $\alpha_s(\theta)$, and $z(\theta)$ is different for each model studied. Not only the functional forms are quantitatively different but also qualitative aspects change from one model to another. The dynamic exponent $z(\theta)$ may be a strictly decreasing function, go up and then down, or just stay roughly constant as θ is varied, depending on the model details. Similar differences appear in the roughness exponents. For model KPZ-II, for instance, $\alpha(\theta) = \alpha_s(\theta) = 1/2$ below $\theta_{\rm th} \approx 1/4$, while these exponents are monotonously increasing with $\theta > 0$ for other models. Our numerical results clearly show that, although critical behavior exists in all cases, the critical exponents strongly depend on the microscopic details of the particular model. This fragility of the universality class for KPZ with time-correlated noise is also reflected here in the statistics of the height fluctuations that shows very distinctive features for the two different integration schemes KPZ-I and KPZ-II. A critical inspection of existing numerical results in the literature, both from direct numerical integration by different algorithms [26,27,30,31] and from discrete deposition models [25,27,29], reveals a similar lack of consistency in the critical exponents dependence on θ, even at a purely qualitative level.

Actually, the situation with the numerical models we studied here resembles what occurs with the exiting theoretical approaches to the problem. Medina *et al.* perturbative DRG [9] and Squizzato and Canet NPFRG [24] predict the existence of a threshold $\theta_{\rm th}$ below which the exponents take the values of the uncorrelated noise case. In contrast, for the Katzav and Schwartz SCE approximation [22] and Fedorenko FRG [23] there is no trace of a threshold, with exponents smoothly varying with θ. Also, the dynamic exponent $z(\theta)$ is found to be an increasing function above $\theta_{\rm th} = 1/6$ by the perturbative DRG [9], while the other three calculation methods [22–24] lead to decreasing functions.

In conclusion, temporally correlated noise in KPZ-like growth models leads to critical behavior that is strongly dependent of model details. Our results, together with the more precise analytical approaches so far, raise serious concerns about the existence of a true universality for KPZ in the presence of time correlated noise.

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