Universidad de Cantabria

FACULTAD DE CIENCIAS

Estudio matemático de un modelo para baterías refrigeradas por fluido

(Mathematical study of a model for liquid-cooled batteries)

Trabajo de fin de Grado
para acceder al

GRADO EN MATEMÁTICAS

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Resumen

Este trabajo se centra en el estudio del sistema de ecuaciones en derivadas parciales propuesto en 2021 por D. Kato y S.J. Moura con condiciones periódicas. Se prueba la existencia de solución local por el Teorema de Picard para el problema regularizado, utilizando las estimaciones de energía y el paso al límite para extrapolar la existencia local al sistema de ecuaciones original. Por último, es comprobada la no existencia de solución global para el problema planteado.

Palabras clave: Existencia y unicidad de solución local, formación de singularidades, acoplamiento parabólico-hiperbólico, ecuaciones en derivadas parciales.

Abstract

This project focuses on the study of a partial differential equations system proposed in 2021 by D. Kato and S.J. Moura with periodic conditions. The well-posedness is proven by Picard Theorem for the regularized problem, using energy estimates and passing to the limit to generalize the local existence for the original equations system. Lastly, the non existence of a global solution for said problem is shown.

Key words: Well-posedness, singularity formation, parabolic-hyperbolic coupling, partial differential equations.

Contents

Chapter 1

Introduction

1.1 Motivation

This project studies the behaviour of a coupled battery and fluid cooling system proposed in the article [5].

We study a one dimensional parabolic-hyperbolic coupling physics based model with two states: the temperature distribution in the battery pack $u(x,t)$ and the temperature distribution in the coolant $w(x,t)$.

The system is governed by two main PDE equations:

$$u_t(x,t) = D(x,t)u_{xx}(x,t) + h(x,t,u(x,t)) + \frac{1}{R(x,t)}(w(x,t) - u(x,t)), \tag{1.1}$$

$$w_t(x,t) = -\sigma(t)w_x(x,t) + \frac{1}{R(x,t)}(u(x,t) - w(x,t)). \tag{1.2}$$

In this work, we will consider the simpler (and probably unrealistic) boundary conditions:

$$\begin{aligned} u(0,t) &= u(1,t), \\ u_x(0,t) &= u_x(1,t), \\ w(0,t) &= w(1,t), \\ w_x(0,t) &= w_x(1,t). \end{aligned}$$

Analyzing (1.1), there is a clear separation in three terms:

- $D(x,t)u_{xx}(x,t)$ denotes the Heat Equation; a second order PDE where $D(x,t)$ is the thermal diffusion coefficient within the battery pack. The characterization of this equation can be found in the following proof, obtained from [3]:

Proof. The gradient points to the maximum growth direction of a function. Knowing that heat flows from high temperature areas to the lower ones, it can be supposed that the heat flow is parallel to the gradient. Given a one-dimensional segment, we obtain

$$J = -Du_x,$$

with u the temperature of the wire.

Therefore, the temperature variation along the segment $x + \delta x$ and time δt is given by

$$\delta x u(x, t + \delta t) - \delta x u(x, t).$$

This alteration must be equal to the heat flow at the ends of the segment, and said flow adopts the following structure:

$$J(x, t)\delta t - J(x + \delta x, t)\delta t.$$

Consequently, we get to

$$\delta x u(x, t + \delta t) - \delta x u(x, t) = J(x, t)\delta t - J(x + \delta x, t)\delta t;$$

which is equivalent to

$$\frac{u(x, t + \delta t) - u(x, t)}{\delta t} = D\frac{u_x(x + \delta x, t) - u_x(x, t)}{\delta x}.$$

Now, applying Taylor's polynomial, we obtain the second order PDE

$$u_t(x, t) = Du_{xx}(x, t).$$

□

- $h(x, t, u(x, t))$ is the internal heat generation in the battery pack due to charging/ discharging. It is an exogenous term that can generally be computed from a battery model and would be a function of current imposed on the battery pack.
- $\frac{1}{R(x,t)}(w(x, t) - u(x, t))$ is the differential form of Fourier Law of Heat Transfer, where $R(x, t)$ is the thermal resistance between the battery pack and the cooling fluid.

On another side, (1.2) is formed by two terms:

- $-\sigma(t)w_x(x, t)$ is a 1D advective transport equation; a first order PDE where $\sigma(t)$ represents the transport speed of the cooling fluid. Since we restrict ourselves to the case of an incompressible fluid and uniform cross-sectional area in the cooling fluid channel, $\sigma(t)$ is constant in x. Now, the proof to the Transport Equation, extracted from [3]:

Proof. Let $\sharp w(x,t)$ be the number of particles between the points x and $x+\delta x$ at time t. Additionally, let $\sharp v(x,t)$ be the number of particles that pass through x per unit of time at a given time t.

Thus,

$$\sharp w(x,t+\delta t) = \sharp w(x,t) + \sharp v(x,t)\delta t - \sharp v(x+\delta x,t)\delta t. \tag{1.3}$$

This concept may be simplified by thinking of it as *"taking the particles originally in the tube, adding those that come in and subtracting the ones leaving"*.

Given that $w(x,t)$ is the density of particles at a point x and time t; and additionally considering $density = mass/volume$:

$$\sharp w(x,t) = w(x,t)\delta x.$$

Therefore,

$$(w(x,t+\delta t) - w(x,t))\delta x = \sharp w(x,t+\delta t) - \sharp w(x,t) \stackrel{(1.3)}{=} -(\sharp v(x+\delta x,t) - \sharp v(x,t))\delta t,$$

which is equivalent to

$$\frac{w(x,t+\delta t) - w(x,t)}{\delta t} + \frac{\sharp v(x+\delta x,t) - \sharp v(x,t)}{\delta x} = 0.$$

Using Taylor's polynomial[1]

$$w(x,t+\delta t) = w(x,t) + w_t(x,t)\delta t,$$

we obtain

$$w_t(x,t) + v_x(x,t) = 0.$$

It is indispensable to suppose that the flow of particles is related to their velocity to ensure a proper reasoning. Therefore,

$$v(x,t) = v(w(x,t)),$$

from which

$$\frac{\partial v(w(x,t))}{\partial x} = v'(w(x,t))w_x(x,t).$$

Additionally, assuming the regularity of $v'(\cdot)$ and applying Taylor, we get

$$v'(y) = q_o + q_1 y + q_2 y^2 + \cdots .$$

[1] $g(x+h) = g(x) + hg'(x) + h^2\frac{g''(x)}{2!} + \cdots$

From which we reach a first approximation of the transport equation, only valid if $|w| \ll 1$ (which can be assumed since w denotes the temperature distribution in the coolant and it should be low):

$$w_t(x,t) + q_0 w_x(x,t) = 0.$$

□

- The same linear heat transfer term as in (1.1) with the opposite sign. This change is due to the heat transfer from the battery to the coolant.

Both terms $\frac{1}{R(x,t)}(w(x,t) - u(x,t))$ and $\frac{1}{R(x,t)}(u(x,t) - w(x,t))$ correspond to the heat loss. There is need for two hypothesis in order to understand their structure:

- There is no heat loss in the exchange between the battery and the coolant. Then, the sum of both terms must be 0.
- Temperature variation is proportional to temperature difference. This means that, if the temperature is considerably high, the exchange is fast.

1.2 Sobolev spaces

Sobolev spaces [4] and their properties will constitute an important part of the development of this project. This section gathers some auxiliary characteristics of these spaces that may be required.

Definition 1.1. *The Sobolev space $H^m(\mathbb{T})$, $m \in \mathbb{Z}^+ \cup \{0\}$ is a vector space formed by functions $v \in L^2(\mathbb{T})$ such that $\partial_x^\alpha v \in L^2(\mathbb{T})$, $0 \le |\alpha| \le m$. The H^m norm is defined as*

$$||v||_{H^m} = \left(\sum_{0 \le |\alpha| \le m} ||\partial_x^\alpha v||_{L^2}^2 \right)^{\frac{1}{2}}. \tag{1.4}$$

When generalizing to the case $m = s \in \mathbb{R}$, we consider the Fourier H^s norm

$$||v||_{H^s} = \left[\sum_{0<|a|<s} \sum_{\xi=-\infty}^{\infty} (1 + |\xi|^2)^{|a|} \, |\hat{v}(\xi)|^2 \right]^{\frac{1}{2}}. \tag{1.5}$$

Now that Sobolev spaces have been given a definition, we will study some of their properties.

Lemma 1.2. *The space $H^{s+k}(\mathbb{T})$, $s > \frac{1}{2}$, $k \in \mathbb{Z}^+ \cup \{0\}$ is continuously immerse in the space $C^k(\mathbb{T})$, i.e., there exists $c > 0$ such that*

$$||v||_{C^k} \le c \, ||v||_{H^{s+k}} \quad \forall v \in H^{s+k}(\mathbb{T}). \tag{1.6}$$

Lemma 1.3. *Sobolev spaces inequalities.*

i) For every $m \in \mathbb{Z}^+ \cup \{0\}$, there exists $c > 0$ such that, for every $u, v \in L^\infty \cap H^m(\mathbb{T})$,

$$||uv||_{H^m} \leq c\{||u||_{L^\infty} ||\partial_x^m v||_{L^2} + ||\partial_x^m u||_{L^2} ||v||_{L^\infty}\}. \tag{1.7}$$

ii) For every $s > \frac{1}{2}$, $H^s(\mathbb{T})$ is a Banach algebra, i.e., there exists $c > 0$ such that, for every $u, v \in H^s(\mathbb{T})$,

$$||uv||_{H^s} \leq c\, ||u||_{H^s} ||v||_{H^s} . \tag{1.8}$$

Lemma 1.4. *Interpolation in Sobolev spaces. Given $s > 0$, there exists a constant C_s such that, for every $v \in H^s(\mathbb{T})$ and $s > s' > 0$,*

$$||v||_{H^{s'}} \leq C_s ||v||_{L^2}^{1-s'/s} ||v||_{H^s}^{s'/s} . \tag{1.9}$$

Chapter 2

Local Existence

The contents in this chapter gather the proof to the following theorem, proceeding similarly to [1].

Theorem 2.1. *Let* u_0 *and* w_0 *be initial data in* $H^3(\mathbb{T})$. *Then, there exists a time* T^* *where there is a unique solution* (u, w) *to the PDE system*

$$\begin{cases} u_t = u_{xx} + u^p + (w-u), \\ u(0) = u(1), \\ u_x(0) = u_x(1), \\ w_t = -w_x + (u-w), \\ w(0) = w(1), \\ w_x(0) = w_x(1). \end{cases} \tag{2.1}$$

The first step is to introduce an operator $\mathcal{J}$ called mollifier. This is a smooth function that, convolved with a rather irregular function, helps to smooth its sharp features. As a result, the derivatives become regular functions while it still remains close to the original nonsmooth attributes.

Given a function

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{e^{inx\pi}}{\sqrt{2}},$$

its convolution with $\mathcal{J}_\varepsilon$ holds

$$\mathcal{J}_\varepsilon * f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{e^{-\varepsilon n^2} e^{inx\pi}}{\sqrt{2}}. \tag{2.2}$$

Now, some properties of the operator $\mathcal{J}_\varepsilon$:

i) For every $f \in L^p(\mathbb{T}), g \in L^p(\mathbb{T})$

$$\int_0^1 (\mathcal{J}_\varepsilon * f) g = \int_0^1 f (\mathcal{J}_\varepsilon * g). \tag{2.3}$$

ii) For every $g \in H^s(\mathbb{T})$, $\mathcal{J}_\varepsilon * g$ converges to g in H^s and the radius of convergence at the norm H^{s-1} is linear in ε

$$\lim_{\varepsilon \searrow 0} ||\mathcal{J}_\varepsilon * g - g||_{H^s} = 0, \tag{2.4}$$

$$||\mathcal{J}_\varepsilon * g - g||_{H^{s-1}} \leq C\varepsilon \, ||g||_{H^s} . \tag{2.5}$$

iii) For every $g \in C^0(\mathbb{T})$

$$||\mathcal{J}_\varepsilon * g||_{L^\infty} \leq ||g||_{L^\infty} . \tag{2.6}$$

Furthermore,

$$||\mathcal{J}_\varepsilon * g||_{L^2} \leq ||g||_{L^2} . \tag{2.7}$$

iv) Mollifiers commute with derivatives, i.e.,

$$\partial_x^n \mathcal{J}_\varepsilon * g = \mathcal{J}_\varepsilon * \partial_x^n g, \quad \forall |n| \leq k. \tag{2.8}$$

v) For every $g \in H^m(\mathbb{T})$, $k \in \mathbb{Z}^+ \cup \{0\}$ and $\varepsilon > 0$,

$$||\mathcal{J}_\varepsilon * g||_{H^{m+k}} \leq \frac{c_{mk}}{\varepsilon^k} ||g||_{H^m} , \tag{2.9}$$

$$||\partial_x^n \mathcal{J}_\varepsilon * g||_{H^m} \leq C(n, \varepsilon) \, ||g||_{H^m} \quad \forall |n| \leq k, \tag{2.10}$$

$$||\partial_x^n \mathcal{J}_\varepsilon * g||_{L^\infty} \leq \frac{c_k}{\varepsilon^{N/2+k}} ||g||_{L^2} \quad \forall |n| \leq k. \tag{2.11}$$

The proof to these properties can be found in [6].

Hence, we denote

$$u_t^\varepsilon = \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xx}^\varepsilon + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon),$$
$$w_t^\varepsilon = \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon.$$

Composing a mollifier with an unbounded differential operator, a bounded operator is obtained. Therefore, Picard Theorem for ODEs in a Banach space can be applied to prove the existence of a solution for the regularized problem.

Theorem 2.2 (Picard Theorem on a Banach space)**.** *Let $O \subseteq \boldsymbol{B}$ be an open subset of a Banach space $\boldsymbol{B}$ and let $F : O \to \boldsymbol{B}$ be a mapping that satisfies the following parameters:*

(a) $F(X)$ maps O to $\boldsymbol{B}$.

(b) F is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subset O$ of X such that

$$||F(\tilde{X}) - F(\hat{X})||_{\boldsymbol{B}} \leq L||\tilde{X} - \hat{X}||_{\boldsymbol{B}} \qquad \text{for all} \qquad \tilde{X}, \hat{X} \in U_X.$$

Then, for any $X_0 \in O$, there exists a time T such that the ODE

$$\frac{dX}{dt} = F(X), \qquad X|_{t=0} = X_0 \in O, \tag{2.12}$$

has a unique (local) solution $X \in C^1[(-T, T); O]$.

Thus, the following proposition must be proven:

Proposition 2.3. *Given the initial conditions $(u_0, w_0) \in H^3(\mathbb{T}) \times H^3(\mathbb{T})$ it is true that, for any $\varepsilon > 0$, there exists a unique solution $(u^\varepsilon, w^\varepsilon) \in C^1([0, T_\varepsilon); H^3(\mathbb{T}) \times H^3(\mathbb{T}))$ for the ODE*

$$\begin{aligned} \frac{d(u^\varepsilon, w^\varepsilon)}{dt} &= F(u^\varepsilon, w^\varepsilon), \\ u^\varepsilon|_{t=0} &= \mathcal{J}_\varepsilon * u_0, \\ w^\varepsilon|_{t=0} &= \mathcal{J}_\varepsilon * w_0, \end{aligned} \tag{2.13}$$

where

$$F(u^\varepsilon, w^\varepsilon) = \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon) \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\epsilon_x) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon \end{pmatrix},$$

and $T_\varepsilon = T(||(u_0, w_0)||_{H^3(\mathbb{T}) \times H^3(\mathbb{T})}; \varepsilon)$.

Proof. Applying Theorem 2.2 we denote $\mathbf{B} = H^3(\mathbb{T}) \times H^3(\mathbb{T})$, $O = \{f \in H^3(\mathbb{T}) \times H^3(\mathbb{T}) : ||f||_{H^3(\mathbb{T}) \times H^3(\mathbb{T})} < \lambda\}$ where $\lambda = 2||(u_0, w_0)||_{H^3(\mathbb{T}) \times H^3(\mathbb{T})}$

$$F\colon \quad H^3(\mathbb{T}) \times H^3(\mathbb{T}) \quad \longrightarrow \quad H^3(\mathbb{T}) \times H^3(\mathbb{T}),$$

with

$$F(u^\varepsilon, w^\varepsilon) = \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon) \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_x) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon \end{pmatrix}, \tag{2.14}$$

which we may rewrite as:

$$F(u^\varepsilon, w^\varepsilon) = F_1(u^\varepsilon, w^\varepsilon) + F_2(u^\varepsilon, w^\varepsilon) + F_3(u^\varepsilon, w^\varepsilon),$$

with

$$F_1(u^\varepsilon, w^\varepsilon) = \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_x) \end{pmatrix},$$

$$F_2(u^\varepsilon, w^\varepsilon) = \begin{pmatrix} \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p \\ 0 \end{pmatrix},$$

$$F_3(u^\varepsilon, w^\varepsilon) = \begin{pmatrix} \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon) \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon \end{pmatrix}.$$

Since $(u^\varepsilon, w^\varepsilon) \in H^3(\mathbb{T}) \times H^3(\mathbb{T})$, $\mathcal{J}_\varepsilon * u^\varepsilon \in C^\infty$ and $\mathcal{J}_\varepsilon * w^\varepsilon \in C^\infty$, every term in F is well defined. Hence, it is yet to study if the second condition in Picard Theorem is satisfied.

$$\begin{aligned}
||F_1(u^\varepsilon, w^\varepsilon) - F_1(U^\varepsilon, W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})} &= \left|\left| \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_x) \end{pmatrix} - \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * U^\varepsilon_{xx} \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-W^\varepsilon_x) \end{pmatrix} \right|\right|_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \\
&= \left|\left| \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * U^\varepsilon_{xx} \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_x) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * W^\varepsilon_x \end{pmatrix} \right|\right|_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \\
&= \left|\left| \begin{pmatrix} \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (u^\varepsilon_{xx} - U^\varepsilon_{xx}) \\ \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_x + W^\varepsilon_x) \end{pmatrix} \right|\right|_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \\
&= ||\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (u^\varepsilon_{xx} - U^\varepsilon_{xx})||_{H^3(\mathbb{T})} \\
&\quad + ||\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_x + W^\varepsilon_x)||_{H^3(\mathbb{T})}.
\end{aligned}$$

Which is correct given that $||(X_1, X_2)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})} = ||X_1||_{H^3(\mathbb{T})} + ||X_2||_{H^3(\mathbb{T})}$.

According to (2.2), by differentiating the summation twice with respect to x, it is true that

$$\partial_x^2(\mathcal{J}_\varepsilon * f(x)) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{e^{-\varepsilon n^2}(-\pi^2 n^2) e^{inx\pi}}{\sqrt{2}} = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{e^{-\varepsilon n^2}\varepsilon(-\pi^2 n^2) e^{inx\pi}}{\sqrt{2}\varepsilon}. \tag{2.15}$$

Knowing the Taylor series of the exponential function is defined as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

we may write

$$\varepsilon n^2 \leq 1 + \varepsilon n^2 + \frac{\varepsilon^2 n^4}{2} + \cdots \leq e^{\varepsilon n^2}.$$

Now, using that

$$||\mathcal{J}_\varepsilon||_{H^3(\mathbb{T})} \leq C, \tag{2.16}$$

we can state

$$\begin{aligned}
||\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (u^\varepsilon_{xx} - U^\varepsilon_{xx})||_{H^3(\mathbb{T})} &\leq ||\mathcal{J}_\varepsilon * (u^\varepsilon_{xx} - U^\varepsilon_{xx})||_{H^3(\mathbb{T})} \\
&= \left|\left|\partial_x^2 \mathcal{J}_\varepsilon * (u^\varepsilon - U^\varepsilon)\right|\right|_{H^3(\mathbb{T})} \\
&\leq C(2, \varepsilon)\, ||u^\varepsilon - U^\varepsilon||_{H^3(\mathbb{T})}.
\end{aligned}$$

Following a similar procedure and differentiating (2.2) once with respect to x, we obtain

$$\partial_x(\mathcal{J}_\varepsilon * f(x)) = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{e^{-\varepsilon n^2}(i\pi n)e^{inx\pi}}{\sqrt{2}} = \sum_{n=-\infty}^{\infty} \hat{f}_n \frac{e^{-\varepsilon n^2}\sqrt{\varepsilon}(i\pi n)e^{inx\pi}}{\sqrt{2\varepsilon}}. \tag{2.17}$$

Hence,

$$\sqrt{\varepsilon}n \leq 1 + \sqrt{\varepsilon} + \varepsilon n^2 + \cdots \leq e^{\sqrt{\varepsilon}n}.$$

In consequence,

$$\begin{aligned} ||\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon + W_x^\varepsilon)||_{H^3(\mathbb{T})} &\leq ||\mathcal{J}_\varepsilon * (-w_x^\varepsilon + W_x^\varepsilon)||_{H^3(\mathbb{T})} \\ &= ||\partial_x \mathcal{J}_\varepsilon * (-w^\varepsilon + W^\varepsilon)||_{H^3(\mathbb{T})} \\ &\leq C(1,\varepsilon)\, ||(-w^\varepsilon + W^\varepsilon)||_{H^3(\mathbb{T})}. \end{aligned}$$

We have shown the second condition in Picard Theorem is held by $F_1(u^\varepsilon, w^\varepsilon)$:

$$||F_1(u^\varepsilon, w^\varepsilon) - F_1(U^\varepsilon, W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \leq C||(u^\varepsilon, w^\varepsilon) - (U^\varepsilon, W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})}.$$

The next step consists on working with $F_2(u^\varepsilon, w^\varepsilon)$. For each inequality, we will refer to every constant as C, although its value might change throughout the process.

$$\begin{aligned} ||F_2(u^\varepsilon, w^\varepsilon) - F_2(U^\varepsilon, W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})} &= \left|\left| \begin{pmatrix} \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * U^\varepsilon)^p \\ 0 \end{pmatrix} \right|\right|_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \\ &= ||\mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * U^\varepsilon)^p||_{H^3(\mathbb{T})} \\ &= ||\mathcal{J}_\varepsilon * ((\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_\varepsilon * U^\varepsilon)^p)||_{H^3(\mathbb{T})} \\ &\leq C\, ||(\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_\varepsilon * U^\varepsilon)^p||_{H^3(\mathbb{T})} \\ &\leq C \left|\left| \left(\int_0^1 p(\mu x + (1-\mu)y)^{p-1} \right) (\mathcal{J}_\varepsilon * (u^\varepsilon - U^\varepsilon)) \right|\right|_{H^3(\mathbb{T})} \\ &\leq C \left|\left| \int_0^1 p(\mu x + (1-\mu)y)^{p-1} \right|\right|_{H^3(\mathbb{T})} ||\mathcal{J}_\varepsilon * (u^\varepsilon - U^\varepsilon)||_{H^3(\mathbb{T})} \\ &\leq C(\lambda)\, ||u^\varepsilon - U^\varepsilon||_{H^3(\mathbb{T})}. \end{aligned} \tag{2.18}$$

This is correct since, according to the Fundamental Theorem of Calculus,

$$x^p - y^p = \int_0^1 \partial_\mu(\mu x + (1-\mu)y)^p d\mu = \int_0^1 p(\mu x + (1-\mu)y)^{p-1}(x-y)d\mu, \tag{2.19}$$

In addition, (2.18) is satisfied due to O being defined as a set of functions whose norm is bounded by λ, i.e.,

$$
\begin{aligned}
\left\|\int_0^1 p(\mu x+(1-\mu)y)^{p-1}\right\|_{H^3(\mathbb{T})} &\leq \|2px^{p-1}\|_{H^3(\mathbb{T})} \\
&\leq C\|x^{p-1}\|_{H^3(\mathbb{T})}, \qquad x^{p-1}\in O \\
&\leq C\lambda^{p-1}.
\end{aligned}
$$

Now, as a result for $F_3(u^\varepsilon, w^\varepsilon)$,

$$
\begin{aligned}
||F_3(u^\varepsilon,w^\varepsilon)-F_3(U^\varepsilon,W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})} &= \Bigg|\Bigg|\begin{pmatrix}\mathcal{J}_\varepsilon*(\mathcal{J}_\varepsilon*w^\varepsilon-\mathcal{J}_\varepsilon*u^\varepsilon)\\ \mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*u^\varepsilon-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*w^\varepsilon\end{pmatrix} \\
&\quad -\begin{pmatrix}\mathcal{J}_\varepsilon*(\mathcal{J}_\varepsilon*W^\varepsilon-\mathcal{J}_\varepsilon*U^\varepsilon)\\ \mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*U^\varepsilon-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*W^\varepsilon\end{pmatrix}\Bigg|\Bigg|_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \\
&= \Bigg|\Bigg|\begin{pmatrix}\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(w^\varepsilon-W^\varepsilon)-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(u^\varepsilon-U^\varepsilon)\\ \mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(u^\varepsilon-U^\varepsilon)-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(w^\varepsilon-W^\varepsilon)\end{pmatrix}\Bigg|\Bigg|_{H^3(\mathbb{T})\times H^3(\mathbb{T})} \\
&= ||\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(w^\varepsilon-W^\varepsilon)-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(u^\varepsilon-U^\varepsilon)||_{H^3(\mathbb{T})} \\
&\quad +||\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(u^\varepsilon-U^\varepsilon)-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(w^\varepsilon-W^\varepsilon)||_{H^3(\mathbb{T})} \\
&= 2\,||\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(u^\varepsilon-U^\varepsilon)-\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(w^\varepsilon-W^\varepsilon)||_{H^3(\mathbb{T})} \\
&\leq 2\Big(||\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(u^\varepsilon-U^\varepsilon)||_{H^3(\mathbb{T})} \\
&\quad +||\mathcal{J}_\varepsilon*\mathcal{J}_\varepsilon*(w^\varepsilon-W^\varepsilon)||_{H^3(\mathbb{T})}\Big) \\
&\leq 2C\Big(||u^\varepsilon-U^\varepsilon||_{H^3(\mathbb{T})}+||w^\varepsilon-W^\varepsilon||_{H^3(\mathbb{T})}\Big) \\
&= 2C\,||(u^\varepsilon,w^\varepsilon)-(U^\varepsilon,W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})}\,.
\end{aligned}
$$

Thus, taking each inequality we can state

$$
||F(u^\varepsilon,w^\varepsilon)-F(U^\varepsilon,W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})}\leq C||(u^\varepsilon,w^\varepsilon)-(U^\varepsilon,W^\varepsilon)||_{H^3(\mathbb{T})\times H^3(\mathbb{T})}.
$$

□

We have shown that there exist solutions given any $\varepsilon>0$ for a time T_ε. If the limit of $(u^\varepsilon, w^\varepsilon)$ existed along a certain period of time, said limit is the solution to (2.1). Now, we must study if every regularized solution has a common existence interval for any ε close enough to 0 (by ensuring $\liminf_{\varepsilon\to 0}T_\varepsilon\neq 0$). Otherwise, it would be possible for the time of existence of the hypothetical limit to be at $t=0$.

2.1 Energy estimates

In this subsection, we will make an estimate of the energy

$$
E(t)=||(u^\varepsilon,w^\varepsilon)||_{L^2}^2+\left|\left|\partial_x^3(u^\varepsilon,w^\varepsilon)\right|\right|_{L^2}^2, \tag{2.20}
$$

by showing it is proportionally bounded by itself, i.e.,

$$\frac{dE}{dt} \leq cE^k.$$

Taking the Fourier *Definition 1.1* of a Sobolev space:

$$\begin{aligned} ||(u^\varepsilon, w^\varepsilon)||_{H^3}^2 &= \sum_{\xi=-\infty}^{\infty} (1+|\xi|^2)^3 \left|(u^\varepsilon, \hat{w}^\varepsilon)(\xi)\right|^2 \\ &\leq C\left(||(u^\varepsilon, w^\varepsilon)||_{L^2}^2 + \left|\left|\partial_x^3 (u^\varepsilon, w^\varepsilon)\right|\right|_{L^2}^2\right). \end{aligned}$$

By definition,

$$||(u^\varepsilon, w^\varepsilon)||_{L^2}^2 = \int_{\mathbb{T}} u^\varepsilon(x,t)^2 dx + \int_{\mathbb{T}} w^\varepsilon(x,t)^2 dx.$$

Additionally, we will make use of Plancherel Theorem with real values, according to which

$$\int_{\mathbb{T}} f \mathcal{J}_\varepsilon g dx = \sum_{n=-\infty}^{\infty} \hat{f}_n \overline{e^{-\varepsilon n^2} \hat{g}_n} = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{-\varepsilon n^2} \overline{\hat{g}_n} = \int_{\mathbb{T}} \mathcal{J}_\varepsilon f g dx. \tag{2.21}$$

Therefore,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} ||(u^\varepsilon, w^\varepsilon)||_{L^2}^2 &= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} u^\varepsilon(x,t)^2 dx + \frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} w^\varepsilon(x,t)^2 dx \\ &= \int_{\mathbb{T}} u^\varepsilon(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xx}^\varepsilon + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon))dx \\ &\quad + \int_{\mathbb{T}} w^\varepsilon(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon)dx \\ &= \int_{\mathbb{T}} (\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * u_{xx}^\varepsilon) + (\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * u^\varepsilon)^p + (\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon)dx \\ &\quad + \int_{\mathbb{T}} (\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * (-w_x^\varepsilon)) + (\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * u^\varepsilon) - (\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * w^\varepsilon)dx \\ &= I_1 + I_2. \end{aligned}$$

The sum of integrals equals the integral of the sum, which implies that we can dissect each term to ease the process. Then, applying integration by parts and taking into account the periodic conditions in (2.1), we obtain

$$\begin{aligned}\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * u^\varepsilon_{xx})dx &= (\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * u^\varepsilon_x)|_0^1 - \int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)^2 dx \\ &= -\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon_x)^2 dx \\ &= -\,||(\mathcal{J}_\varepsilon * u^\varepsilon_x)||^2_{L^2} \\ &\leq 0.\end{aligned}$$

Now, working with the next term and using the Sobolev inequality (1.6)

$$\begin{aligned}\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * u^\varepsilon)^p dx &= \int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)^{p+1} dx \\ &\leq C\,||(\mathcal{J}_\varepsilon * u^\varepsilon)||^{p+1}_{C^0} \\ &\leq C\,||(\mathcal{J}_\varepsilon * u^\varepsilon)||^{p+1}_{H^3} \\ &\leq CE^{p+1}.\end{aligned}$$

Given that Hölder states

$$\int FGH \leq \|F\|_{L^p}\|G\|_{L^q}\|H\|_{L^r} \quad \text{such that} \quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1, \tag{2.22}$$

we can write

$$\begin{aligned}\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)(\mathcal{J}_\varepsilon * w^\varepsilon)dx &\leq \|\mathcal{J}_\varepsilon * u^\varepsilon\|_{L^2}\|\mathcal{J}_\varepsilon * w^\varepsilon\|_{L^2} \\ &\leq \|\mathcal{J}_\varepsilon * u^\varepsilon\|_{H^3}\|\mathcal{J}_\varepsilon * w^\varepsilon\|_{H^3} \\ &\leq C\|u^\varepsilon\|_{H^3}\|w^\varepsilon\|_{H^3} \\ &\leq CE^2.\end{aligned}$$

The last term in I_1 satisfies

$$\begin{aligned}\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)(-\mathcal{J}_\varepsilon * u^\varepsilon)dx &= -\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon)^2 dx \\ &= -\,||(\mathcal{J}_\varepsilon * u^\varepsilon)||^2_{L^2} \\ &\leq 0.\end{aligned}$$

In consequence, we have proven

$$I_1 \leq CE^{p+1} + CE^2 \leq CE^{p+1} + 1. \tag{2.23}$$

Following a similar procedure for the terms in I_2:

$$\begin{aligned}\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * (-w^\varepsilon_x))dx &= (\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * (-w^\varepsilon))|_0^1 - \int_{\mathbb{T}}(\mathcal{J}_\varepsilon * (-w^\varepsilon))(\mathcal{J}_\varepsilon * w^\varepsilon_x)dx \\ &= \frac{(\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * (-w^\varepsilon))|_0^1}{2} \\ &= 0.\end{aligned}$$

On another side,

$$\begin{aligned}\int_{\mathbb{T}} -(\mathcal{J}_\varepsilon * w^\varepsilon)(\mathcal{J}_\varepsilon * w^\varepsilon)dx &= -\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * w^\varepsilon)^2 dx \\ &= -||(\mathcal{J}_\varepsilon * w^\varepsilon)||_{L^2}^2 \\ &\leq 0.\end{aligned}$$

Thus, considering the analogous case,

$$I_2 \leq E^2. \tag{2.24}$$

Taking (2.23) and (2.24) we get

$$\frac{1}{2}\frac{d}{dt}\,||(u^\varepsilon, w^\varepsilon)||_{L^2}^2 \leq CE^{p+1} + 1 + E^2 \leq CE^{p+1} + 1. \tag{2.25}$$

Now, studying the case for $||\partial_x^3(u^\varepsilon, w^\varepsilon)||_{L^2}^2$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\left|\partial_x^3(u^\varepsilon, w^\varepsilon)\right|\right|_{L^2}^2 &= \int_{\mathbb{T}} \partial_x^3(u^\varepsilon, w^\varepsilon)\partial_x^3\partial_t(u^\varepsilon, w^\varepsilon)dx \\ &= \int_{\mathbb{T}} \partial_x^3 u^\varepsilon \partial_x^3 u_t^\varepsilon dx + \int_{\mathbb{T}} \partial_x^3 w^\varepsilon \partial_x^3 w_t^\varepsilon dx \\ &= \int_{\mathbb{T}} u_{xxx}^\varepsilon \partial_x^3(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xx}^\varepsilon + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p + \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * u^\varepsilon))dx \\ &\quad + \int_{\mathbb{T}} w_{xxx}^\varepsilon \partial_x^3(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon)dx \\ &= I_3 + I_4.\end{aligned}$$

We will start by rewriting I_3

$$\begin{aligned}I_3 &= \int_{\mathbb{T}} u_{xxx}^\varepsilon \Big(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xxxxx}^\varepsilon + \mathcal{J}_\varepsilon * \Big(p(p-1)(p-2)(\mathcal{J}_\varepsilon * u^{\varepsilon^{p-3}})(\mathcal{J}_\varepsilon * u_x^{\varepsilon^3}) \\ &\quad +3p(p-1)(\mathcal{J}_\varepsilon * u^{\varepsilon^{p-2}})(\mathcal{J}_\varepsilon * u_x^\varepsilon)(\mathcal{J}_\varepsilon * u_{xx}^\varepsilon) + p(\mathcal{J}_\varepsilon * u^{\varepsilon^{p-1}})(\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)\Big) \\ &\quad +\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w_{xxx}^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)\, dx \\ &= \int_{\mathbb{T}} (\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)(\mathcal{J}_\varepsilon * u_{xxxxx}^\varepsilon) + p(p-1)(p-2)(\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)(\mathcal{J}_\varepsilon * u^{\varepsilon^{p-3}})(\mathcal{J}_\varepsilon * u_x^{\varepsilon^3}) \\ &\quad +3p(p-1)(\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)(\mathcal{J}_\varepsilon * u^{\varepsilon^{p-2}})(\mathcal{J}_\varepsilon * u_x^\varepsilon)(\mathcal{J}_\varepsilon * u_{xx}^\varepsilon) \\ &\quad +p(\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)(\mathcal{J}_\varepsilon * u^{\varepsilon^{p-1}})(\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon) + (\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)(\mathcal{J}_\varepsilon * w_{xxx}^\varepsilon) - (\mathcal{J}_\varepsilon * u_{xxx}^\varepsilon)^2 dx \\ &= I_{3_1} + I_{3_2} + I_{3_3} + I_{3_4} + I_{3_5} + I_{3_6}.\end{aligned}$$

The first term satisfies

$$\begin{aligned}
I_{3_1} &= \mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\mathcal{J}_\varepsilon * u^\varepsilon_{xxxx}|_0^1 - \int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon_{xxxx})^2dx \\
&= -\int_{\mathbb{T}}(\mathcal{J}_\varepsilon * u^\varepsilon_{xxxx})^2dx \\
&= -||(\mathcal{J}_\varepsilon * u^\varepsilon_{xxxx})||^2_{L^2} \\
&\leq 0.
\end{aligned}$$

Applying Hölder (2.22), we may write

$$\begin{aligned}
I_{3_2} &\leq \|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2}C(p)\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-3}}\|_{L^\infty}\|\mathcal{J}_\varepsilon * u^{\varepsilon^3}_x\|_{L^\infty}\|1\|_{L^2} \\
&\leq C\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2}\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-3}}\|_{C^0}\|\mathcal{J}_\varepsilon * u^{\varepsilon^3}_x\|_{C^1} \\
&\leq C\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{H^3}\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-3}}\|_{H^3}\|\mathcal{J}_\varepsilon * u^{\varepsilon^3}\|_{H^3} \\
&\leq C\|u^\varepsilon_{xxx}\|_{H^3}\|u^{\varepsilon^{p-3}}\|_{H^3}\|u^{\varepsilon^3}\|_{H^3} \\
&\leq CEE^{p-3}E^3 \\
&= CE^{p+1}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
I_{3_3} &\leq \|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2}C(p)\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-2}}\|_{L^\infty}\|\mathcal{J}_\varepsilon * u^\varepsilon_x\|_{L^\infty}\|\mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{L^\infty}\|1\|_{L^2} \\
&\leq C\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2}\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-2}}\|_{C^0}\|\mathcal{J}_\varepsilon * u^\varepsilon_x\|_{C^1}\|\mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{C^2} \\
&\leq C\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{H^3}\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-2}}\|_{H^3}\|\mathcal{J}_\varepsilon * u^\varepsilon\|_{H^3}\|\mathcal{J}_\varepsilon * u^\varepsilon\|_{H^3} \\
&\leq C\|u^\varepsilon_{xxx}\|_{H^3}\|u^{\varepsilon^{p-2}}\|_{H^3}\|u^\varepsilon\|^2_{H^3} \\
&\leq CEE^{p-2}E^2 \\
&= CE^{p+1},
\end{aligned}$$

and

$$\begin{aligned}
I_{3_4} &\leq \|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2}C(p)\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-1}}\|_{L^\infty}\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2} \\
&\leq C\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|^2_{H^3}\|\mathcal{J}_\varepsilon * u^{\varepsilon^{p-1}}\|_{H^3} \\
&\leq CE^2E^{p-1} \\
&= CE^{p+1}.
\end{aligned}$$

On another side,

$$\begin{aligned}
I_{3_5} &\leq \|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2}\|\mathcal{J}_\varepsilon * w^\varepsilon_{xxx}\|_{L^2} \\
&\leq \|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{H^3}\|\mathcal{J}_\varepsilon * w^\varepsilon_{xxx}\|_{H^3} \\
&\leq C\|u^\varepsilon_{xxx}\|_{H^3}\|w^\varepsilon_{xxx}\|_{H^3} \\
&\leq CE^2.
\end{aligned}$$

As a consequence, using the previous reasoning to conclude $I_{3_6} \leq 0$, we have proven

$$I_3 \leq 3CE^{p+1} + CE^2 \leq CE^{p+1} + 1. \tag{2.26}$$

Now, working similarly with I_4

$$\begin{aligned} I_4 &= \int_{\mathbb{T}} w^\varepsilon_{xxx}(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w^\varepsilon_{xxxx}) + \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xxx} - \mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon_{xxx})dx \\ &= \int_{\mathbb{T}} (\mathcal{J}_\varepsilon * w^\varepsilon_{xxx})(\mathcal{J}_\varepsilon * (-w^\varepsilon_{xxxx})) + (\mathcal{J}_\varepsilon * w^\varepsilon_{xxx})(\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}) - (\mathcal{J}_\varepsilon * w^\varepsilon_{xxx})^2 dx \\ &= I_{4_1} + I_{4_2} + I_{4_3}. \end{aligned}$$

The first term satisfies

$$\begin{aligned} I_{4_1} &= \mathcal{J}_\varepsilon * w^\varepsilon_{xxx}\mathcal{J}_\varepsilon * (-w^\varepsilon_{xxx})\big|_0^1 - \int_{\mathbb{T}} \mathcal{J}_\varepsilon * (-w^\varepsilon_{xxx})\mathcal{J}_\varepsilon * w^\varepsilon_{xxxx}dx \\ &= \frac{\mathcal{J}_\varepsilon * w^\varepsilon_{xxx}\mathcal{J}_\varepsilon * (-w^\varepsilon_{xxx})\big|_0^1}{2} \\ &= 0. \end{aligned}$$

The second one holds

$$\begin{aligned} I_{4_2} &\leq \|\mathcal{J}_\varepsilon * w^\varepsilon_{xxx}\|_{L^2}\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{L^2} \\ &\leq \|\mathcal{J}_\varepsilon * w^\varepsilon_{xxx}\|_{H^3}\|\mathcal{J}_\varepsilon * u^\varepsilon_{xxx}\|_{H^3} \\ &\leq C\|w^\varepsilon_{xxx}\|_{H^3}\|u^\varepsilon_{xxx}\|_{H^3} \\ &\leq CE^2. \end{aligned}$$

With this, we can write

$$I_4 \leq CE^2. \tag{2.27}$$

So, taking (2.26) and (2.27), we get

$$\frac{1}{2}\frac{d}{dt}\left|\left|\partial_x^3(u^\varepsilon, w^\varepsilon)\right|\right|^2_{L^2} \leq CE^{p+1} + 1 + CE^2 \leq CE^{p+1} + 1. \tag{2.28}$$

Therefore, using (2.25) and (2.28), we have proven

$$\frac{dE}{dt} \leq CE^{p+1} + 1. \tag{2.29}$$

By integrating the inequality, we obtain

$$E(t) \leq 2E(0) \quad \forall t \leq T^*. \tag{2.30}$$

Observe that T^* does not depend on ε. In fact, $T^* \leq T_\varepsilon$ for every $\varepsilon > 0$.

Furthermore, there is a uniform existence time T^* for the regularization parameter. The last step consists on proving the existence of the limit of $(u^\varepsilon, w^\varepsilon)$ along the interval $[0, T^*]$.

2.2 Passing to the limit

To show there exists a limit for $(u^\varepsilon, w^\varepsilon)$ in the interval $[0, T^*]$, in this subsection we will prove it is a Cauchy sequence in a certain Banach space. This reasoning must be enough given that every Cauchy sequence is convergent in a Banach space.

Proposition 2.4. *The family $(u^\varepsilon, w^\varepsilon)$ forms a Cauchy sequence in $C([0, T^*]; L^2)$. Particularly, there exists a constant C depending on $\|(u^\varepsilon, w^\varepsilon)\|_{H^3}$ and T^*, such that, for every ε and $\tilde{\varepsilon}$*

$$\sup_{0 \le t \le T^*} \|(u^\varepsilon, w^\varepsilon) - (u^{\tilde{\varepsilon}}, w^{\tilde{\varepsilon}})\|_{L^2} \le C \max\{\varepsilon, \tilde{\varepsilon}\}. \tag{2.31}$$

Proof. It is true that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|\left|(u^\varepsilon, w^\varepsilon) - (u^{\tilde{\varepsilon}}, w^{\tilde{\varepsilon}})\right|\right|_{L^2}^2 &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |u^\varepsilon - u^{\tilde{\varepsilon}}|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |w^\varepsilon - w^{\tilde{\varepsilon}}|^2 dx \\ &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}})(u_t^\varepsilon - u_t^{\tilde{\varepsilon}}) dx + \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde{\varepsilon}})(w_t^\varepsilon - w_t^{\tilde{\varepsilon}}) dx \\ &= I_5 + I_6. \end{aligned}$$

First, working with I_5, we may rewrite it as

$$\begin{aligned} I_5 &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xx}^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u_{xx}^{\tilde{\varepsilon}}\right) dx \\ &\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^p\right) dx \\ &\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^{\tilde{\varepsilon}}\right) dx \\ &\quad - \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\right) dx \\ &= I_{5_1} + I_{5_2} + I_{5_3} + I_{5_4}. \end{aligned}$$

Selecting the first integral and introducing the crossed term $\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u_{xx}^\varepsilon$, we get

$$\begin{aligned} I_{5_1} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xx}^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u_{xx}^{\tilde{\varepsilon}}\right) dx \\ &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) (\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u_{xx}^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u_{xx}^\varepsilon) dx \\ &\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u_{xx}^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u_{xx}^{\tilde{\varepsilon}}\right) dx \\ &= I_{5_{11}} + I_{5_{12}}. \end{aligned}$$

Now, we will add another crossed term $\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}$ to each integral. Hence,

$$\begin{aligned} I_{5_{11}} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} - \mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\right) dx \\ &\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx} - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\right) dx \\ &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left((\mathcal{J}_\varepsilon - \mathcal{J}_{\tilde{\varepsilon}}) * \mathcal{J}_\varepsilon * u^\varepsilon_{xx}\right) dx \\ &\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left((\mathcal{J}_\varepsilon - \mathcal{J}_{\tilde{\varepsilon}}) * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\right) dx \\ &= I_{5_{111}} + I_{5_{112}}. \end{aligned}$$

Applying Cauchy-Schwarz inequality to $I_{5_{111}}$

$$I_{5_{111}} \leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|(\mathcal{J}_\varepsilon - \mathcal{J}_{\tilde{\varepsilon}}) * \mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{L^2}.$$

Introducing the term $\mathcal{J}_\varepsilon * u^\varepsilon_{xx}$ and applying the triangle inequality, we obtain

$$\begin{aligned} I_{5_{111}} &\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} - \mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{L^2} \\ &\quad + \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_\varepsilon * u^\varepsilon_{xx} - \mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{L^2}. \end{aligned}$$

Taking (2.5) into account, it is true that

$$\begin{aligned} I_{5_{111}} &\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left(C\varepsilon \|\mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{H^1} + C\tilde{\varepsilon} \|\mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{H^1}\right) \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_\varepsilon * u^\varepsilon_{xx}\|_{H^1} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_\varepsilon * u^\varepsilon\|_{H^3} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|u^\varepsilon\|_{H^3} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \sqrt{E(0)} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}. \end{aligned}$$

Following an analogous procedure for $I_{5_{112}}$ and using the term $\mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}$ in this case:

$$\begin{aligned} I_{5_{112}} &\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|(\mathcal{J}_\varepsilon - \mathcal{J}_{\tilde{\varepsilon}}) * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\|_{L^2} \\ &\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx} - \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\|_{L^2} \\ &\quad + \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx} - \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\|_{L^2} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx}\|_{H^1} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon\|_{H^3} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|u^\varepsilon\|_{H^3} \\ &\leq C \max(\varepsilon, \tilde{\varepsilon}) \sqrt{E(0)} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}. \end{aligned}$$

Now, for $I_{5_{12}}$, applying Plancherel Theorem (2.21) and integrating by parts:

$$\begin{aligned}
I_{5_{12}} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon_{xx} - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}_{xx} \right) dx \\
&= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon_{xx} - u^{\tilde{\varepsilon}}_{xx}) \right) dx \\
&= \int_{\mathbb{T}} \mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon - u^{\tilde{\varepsilon}}) \mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon_{xx} - u^{\tilde{\varepsilon}}_{xx}) dx \\
&= \mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon - u^{\tilde{\varepsilon}}) \mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon_x - u^{\tilde{\varepsilon}}_x)|_0^1 - \int_{\mathbb{T}} (\mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon_x - u^{\tilde{\varepsilon}}_x))^2 dx \\
&= -\int_{\mathbb{T}} (\mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon_x - u^{\tilde{\varepsilon}}_x))^2 dx \\
&\leq 0.
\end{aligned}$$

Therefore,

$$I_{5_1} \leq C \max(\varepsilon, \tilde{\varepsilon}) \sqrt{E(0)} \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2}. \tag{2.32}$$

On another side, if we add the crossed term $\mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_\varepsilon * u^\varepsilon)^p$, I_{5_2} holds,

$$\begin{aligned}
I_{5_2} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^p \right) dx \\
&= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_\varepsilon * u^\varepsilon)^p \right) dx \\
&\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^p \right) dx \\
&= I_{5_{21}} + I_{5_{22}}.
\end{aligned}$$

By adding the term $(\mathcal{J}_\varepsilon * u^\varepsilon)^p$ to $I_{5_{21}}$, we get

$$\begin{aligned}
I_{5_{21}} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_\varepsilon * u^\varepsilon)^p \right) dx \\
&\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left((\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_\varepsilon * u^\varepsilon)^p \right) dx \\
&\leq \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2} \| \mathcal{J}_\varepsilon * (\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_\varepsilon * u^\varepsilon)^p \|_{L^2} \\
&\quad + \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2} \| (\mathcal{J}_\varepsilon * u^\varepsilon)^p - \mathcal{J}_{\tilde{\varepsilon}} * (\mathcal{J}_\varepsilon * u^\varepsilon)^p \|_{L^2} \\
&\leq \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2} \left(C\varepsilon \| (\mathcal{J}_\varepsilon * u^\varepsilon)^p \|_{H^1} + C\tilde{\varepsilon} \| (\mathcal{J}_\varepsilon * u^\varepsilon)^p \|_{H^1} \right) \\
&\leq C \max(\varepsilon, \tilde{\varepsilon}) \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2} \| (\mathcal{J}_\varepsilon * u^\varepsilon)^p \|_{H^1} \\
&\leq C \max(\varepsilon, \tilde{\varepsilon}) \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2} \| (\mathcal{J}_\varepsilon * u^\varepsilon) \|_{H^1}^p \\
&\leq C \max(\varepsilon, \tilde{\varepsilon}) \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2} \| (\mathcal{J}_\varepsilon * u^\varepsilon) \|_{H^3}^p \\
&\leq C E(0)^p \max(\varepsilon, \tilde{\varepsilon}) \| u^\varepsilon - u^{\tilde{\varepsilon}} \|_{L^2}.
\end{aligned}$$

For $I_{5_{22}}$, we may apply (2.19) and obtain

$$
\begin{aligned}
I_{5_{22}} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_{\tilde{\varepsilon}} * \left((\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^p \right) \right) dx \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|\mathcal{J}_{\tilde{\varepsilon}} * \left((\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^p \right) \|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|(\mathcal{J}_\varepsilon * u^\varepsilon)^p - (\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^p\|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left\| \left(\int_0^1 p(\mu \mathcal{J}_\varepsilon * u^\varepsilon + (1-\mu)\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^{p-1} d\mu \right) \left(\mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}} \right) \right\|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left\| \int_0^1 p(\mu \mathcal{J}_\varepsilon * u^\varepsilon + (1-\mu)\mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}})^{p-1} d\mu \right\|_{L^2} \|\mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|2p{u^\varepsilon}^{p-1}\|_{H^3} \|\mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * u^{\tilde{\varepsilon}} + \mathcal{J}_\varepsilon * u^{\tilde{\varepsilon}} - \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\|_{L^2} \\
&\leq C\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \|{u^\varepsilon}^{p-1}\|_{H^3} \left(\|\mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_\varepsilon * u^{\tilde{\varepsilon}}\|_{L^2} + \|\mathcal{J}_\varepsilon * u^{\tilde{\varepsilon}} - \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\|_{L^2} \right) \\
&\leq CE(0)^{p-1} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left(\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + \|\mathcal{J}_\varepsilon * u^{\tilde{\varepsilon}} - u^{\tilde{\varepsilon}} + u^{\tilde{\varepsilon}} - \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\|_{L^2} \right) \\
&\leq CE(0)^{p-1} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left(\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + \|\mathcal{J}_\varepsilon * u^{\tilde{\varepsilon}} - u^{\tilde{\varepsilon}}\|_{L^2} + \|u^{\tilde{\varepsilon}} - \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\|_{L^2} \right) \\
&\leq CE(0)^{p-1} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left(\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + C \max(\varepsilon, \tilde{\varepsilon}) \|u^{\tilde{\varepsilon}}\|_{L^2} \right) \\
&\leq CE(0)^{p-1} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left(\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + C\sqrt{E(0)} \max(\varepsilon, \tilde{\varepsilon}) \right).
\end{aligned}
$$

As a result,

$$
I_{5_2} \leq CE(0)^p \max(\varepsilon, \tilde{\varepsilon}) \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + CE(0)^{p-1} \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} \left(\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + C\sqrt{E(0)} \max(\varepsilon, \tilde{\varepsilon}) \right).
$$

Now, for I_{5_3}, a similar procedure to the one used for I_{5_1} will be followed.

$$
\begin{aligned}
I_{5_3} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^{\tilde{\varepsilon}} \right) dx \\
&= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon \right) dx \\
&\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^{\tilde{\varepsilon}} \right) dx \\
&= I_{5_{31}} + I_{5_{32}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_{5_{31}} &= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon \right) dx \\
&= \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon \right) dx \\
&\quad + \int_{\mathbb{T}} (u^\varepsilon - u^{\tilde{\varepsilon}}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon \right) dx \\
&= I_{5_{311}} + I_{5_{312}},
\end{aligned}
$$

with

$$\begin{aligned}
I_{5_{311}} &\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon\|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_\varepsilon * w^\varepsilon\|_{L^2} \\
&\quad + \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_\varepsilon * w^\varepsilon\|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\left(C\varepsilon\|\mathcal{J}_\varepsilon * w^\varepsilon\|_{H^1} + C\tilde{\varepsilon}\|\mathcal{J}_\varepsilon * w^\varepsilon\|_{H^1}\right) \\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * w^\varepsilon\|_{H^1} \\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * w^\varepsilon\|_{H^3} \\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|w^\varepsilon\|_{H^3} \\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
I_{5_{312}} &\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon\|_{L^2} \\
&\leq \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon\|_{L^2} \\
&\quad + \|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon\|_{L^2} \\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2}.
\end{aligned}$$

On the other side,

$$\begin{aligned}
I_{5_{32}} &= \int_{\mathbb{T}}(u^\varepsilon - u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * w^{\tilde{\varepsilon}}\right)dx \\
&= \int_{\mathbb{T}}(u^\varepsilon - u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * (w^\varepsilon - w^{\tilde{\varepsilon}})\right)dx \\
&= \int_{\mathbb{T}}\mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon - u^{\tilde{\varepsilon}})\mathcal{J}_{\tilde{\varepsilon}} * (w^\varepsilon - w^{\tilde{\varepsilon}})dx \\
&\leq \|\mathcal{J}_{\tilde{\varepsilon}} * (u^\varepsilon - u^{\tilde{\varepsilon}})\|_{L^2}\|\mathcal{J}_{\tilde{\varepsilon}} * (w^\varepsilon - w^{\tilde{\varepsilon}})\|_{L^2} \\
&\leq C\|(u^\varepsilon - u^{\tilde{\varepsilon}})\|_{L^2}\|(w^\varepsilon - w^{\tilde{\varepsilon}})\|_{L^2}.
\end{aligned}$$

Therefore,

$$I_{5_3} \leq C(\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|u^\varepsilon - u^{\tilde{\varepsilon}}\|_{L^2} + \|(u^\varepsilon - u^{\tilde{\varepsilon}})\|_{L^2}\|(w^\varepsilon - w^{\tilde{\varepsilon}})\|_{L^2}). \tag{2.33}$$

Next, I_{5_4} satisfies

$$\begin{aligned}
I_{5_4} &= -\int_{\mathbb{T}}(u^\varepsilon - u^{\tilde{\varepsilon}})\left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\right)dx \\
&= -\int_{\mathbb{T}}(u^\varepsilon - u^{\tilde{\varepsilon}})\left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon\right)dx \\
&\quad - \int_{\mathbb{T}}(u^\varepsilon - u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^\varepsilon - \mathcal{J}_{\tilde{\varepsilon}} * \mathcal{J}_{\tilde{\varepsilon}} * u^{\tilde{\varepsilon}}\right)dx \\
&= I_{5_{41}} + I_{5_{42}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{5_{41}} &= -\int_{\mathbb{T}}(u^{\varepsilon}-u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\varepsilon}*\mathcal{J}_{\varepsilon}*u^{\varepsilon}-\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\right)dx\\
&= -\int_{\mathbb{T}}(u^{\varepsilon}-u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\varepsilon}*\mathcal{J}_{\varepsilon}*u^{\varepsilon}-\mathcal{J}_{\varepsilon}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\right)dx\\
&\quad -\int_{\mathbb{T}}(u^{\varepsilon}-u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\varepsilon}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}-\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\right)dx\\
&= I_{5_{411}}+I_{5_{412}},
\end{aligned}$$

holding

$$\begin{aligned}
I_{5_{411}} &\leq \|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*\mathcal{J}_{\varepsilon}*u^{\varepsilon}-\mathcal{J}_{\varepsilon}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\|_{L^2}\\
&\leq \|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*\mathcal{J}_{\varepsilon}*u^{\varepsilon}-\mathcal{J}_{\varepsilon}*u^{\varepsilon}\|_{L^2}\\
&\quad +\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}-\mathcal{J}_{\varepsilon}*u^{\varepsilon}\|_{L^2}\\
&\leq \|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\left(C\varepsilon\|\mathcal{J}_{\varepsilon}*u^{\varepsilon}\|_{H^1}+C\tilde{\varepsilon}\|\mathcal{J}_{\varepsilon}*u^{\varepsilon}\|_{H^1}\right)\\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*u^{\varepsilon}\|_{H^1}\\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*u^{\varepsilon}\|_{H^3}\\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\|u^{\varepsilon}-u^{\varepsilon}\|_{L^2}\|u^{\varepsilon}\|_{H^3}\\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
I_{5_{412}} &\leq \|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}-\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\|_{L^2}\\
&\leq \|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\varepsilon}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}-\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\|_{L^2}\\
&\quad +\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}\|\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}-\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}\|_{L^2}\\
&\leq C\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}.
\end{aligned}$$

In addition,

$$\begin{aligned}
I_{5_{42}} &= -\int_{\mathbb{T}}(u^{\varepsilon}-u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\varepsilon}-\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*u^{\tilde{\varepsilon}}\right)dx\\
&= -\int_{\mathbb{T}}(u^{\varepsilon}-u^{\tilde{\varepsilon}})\left(\mathcal{J}_{\tilde{\varepsilon}}*\mathcal{J}_{\tilde{\varepsilon}}*(u^{\varepsilon}-u^{\tilde{\varepsilon}})\right)dx\\
&= -\int_{\mathbb{T}}\mathcal{J}_{\tilde{\varepsilon}}*(u^{\varepsilon}-u^{\tilde{\varepsilon}})\mathcal{J}_{\tilde{\varepsilon}}*(u^{\varepsilon}-u^{\tilde{\varepsilon}})dx\\
&= -\|\mathcal{J}_{\tilde{\varepsilon}}*(u^{\varepsilon}-u^{\tilde{\varepsilon}})\|_{L^2}^2\\
&\leq 0.
\end{aligned}$$

Then,

$$I_{5_4}\leq C\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|u^{\varepsilon}-u^{\tilde{\varepsilon}}\|_{L^2}. \tag{2.34}$$

On another hand, moving onto I_6:

$$\begin{aligned}
I_6 &= \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon})(w_t^\varepsilon - w_t^{\tilde\varepsilon})dx \\
&= \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^{\tilde\varepsilon})\right) dx \\
&\quad + \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * u^\varepsilon - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * u^{\tilde\varepsilon}\right) dx \\
&\quad - \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * w^\varepsilon - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * w^{\tilde\varepsilon}\right) dx \\
&= I_{6_1} + I_{6_2} + I_{6_3}.
\end{aligned}$$

Following an analogous reasoning,

$$\begin{aligned}
I_{6_1} &= \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\right) dx \\
&\quad + \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^{\tilde\varepsilon})\right) dx \\
&= I_{6_{11}} + I_{6_{12}},
\end{aligned}$$

with

$$\begin{aligned}
I_{6_{11}} &= \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) - \mathcal{J}_\varepsilon * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\right) dx \\
&\quad + \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\right) dx \\
&= I_{6_{111}} + I_{6_{112}}.
\end{aligned}$$

Then,

$$\begin{aligned}
I_{6_{111}} &\leq \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) - \mathcal{J}_\varepsilon * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\|_{L^2} \\
&\leq \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_\varepsilon * (-w_x^\varepsilon) - \mathcal{J}_\varepsilon * (-w_x^\varepsilon)\|_{L^2} \\
&\quad + \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_\varepsilon * (-w_x^\varepsilon)\|_{L^2} \\
&\leq \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \left(C\varepsilon \|\mathcal{J}_\varepsilon * (-w_x^\varepsilon)\|_{H^1} + C\tilde\varepsilon \|\mathcal{J}_\varepsilon * (-w_x^\varepsilon)\|_{H^1}\right) \\
&\leq C \max(\varepsilon, \tilde\varepsilon) \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * (-w_x^\varepsilon)\|_{H^1} \\
&\leq C \max(\varepsilon, \tilde\varepsilon) \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * (-w^\varepsilon)\|_{H^3} \\
&\leq C \max(\varepsilon, \tilde\varepsilon) \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|w^\varepsilon\|_{H^3} \\
&\leq C \max(\varepsilon, \tilde\varepsilon) \sqrt{E(0)} \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
I_{6_{112}} &\leq \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\|_{L^2} \\
&\leq \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_\varepsilon * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\|_{L^2} \\
&\quad + \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} \|\mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon)\|_{L^2} \\
&\leq C \max(\varepsilon, \tilde\varepsilon) \sqrt{E(0)} \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2}.
\end{aligned}$$

To conclude,

$$\begin{aligned}
I_{6_{12}} &= \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^\varepsilon) - \mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * (-w_x^{\tilde\varepsilon}) \right) dx \\
&= \int_{\mathbb{T}} (w^\varepsilon - w^{\tilde\varepsilon}) \left(\mathcal{J}_{\tilde\varepsilon} * \mathcal{J}_{\tilde\varepsilon} * ((-w_x^\varepsilon) - (-w_x^{\tilde\varepsilon})) \right) dx \\
&= \int_{\mathbb{T}} \mathcal{J}_{\tilde\varepsilon} * (w^\varepsilon - w^{\tilde\varepsilon}) \mathcal{J}_{\tilde\varepsilon} * ((-w_x^\varepsilon) - (-w_x^{\tilde\varepsilon})) dx \\
&= (\mathcal{J}_{\tilde\varepsilon} * (w^\varepsilon - w^{\tilde\varepsilon}))(\mathcal{J}_{\tilde\varepsilon} * ((-w_x^\varepsilon) - (-w_x^{\tilde\varepsilon})))\Big|_0^1 - \int_{\mathbb{T}} \mathcal{J}_{\tilde\varepsilon} * ((-w^\varepsilon) - (-w^{\tilde\varepsilon})) \mathcal{J}_{\tilde\varepsilon} * (w_x^\varepsilon - w_x^{\tilde\varepsilon}) dx \\
&= \frac{(\mathcal{J}_{\tilde\varepsilon} * (w^\varepsilon - w^{\tilde\varepsilon}))(\mathcal{J}_{\tilde\varepsilon} * ((-w_x^\varepsilon) - (-w_x^{\tilde\varepsilon})))\Big|_0^1}{2} \\
&= 0.
\end{aligned}$$

Therefore,

$$I_{6_1} \leq C \max(\varepsilon, \tilde\varepsilon) \sqrt{E(0)} \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2}. \tag{2.35}$$

Since I_{6_2} and I_{6_3} are analogous to I_{5_3} and I_{5_4}, we may state

$$I_{6_2} \leq C(\max(\varepsilon, \tilde\varepsilon) \sqrt{E(0)} \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2} + \|(u^\varepsilon - u^{\tilde\varepsilon})\|_{L^2} \|(w^\varepsilon - w^{\tilde\varepsilon})\|_{L^2}), \tag{2.36}$$

$$I_{6_3} \leq C \max(\varepsilon, \tilde\varepsilon) \sqrt{E(0)} \|w^\varepsilon - w^{\tilde\varepsilon}\|_{L^2}. \tag{2.37}$$

Let $\mathcal{E}$ be equal to

$$\mathcal{E} = \left|\left| u^\varepsilon - u^{\tilde\varepsilon} \right|\right|_{L^2} + \left|\left| w^\varepsilon - w^{\tilde\varepsilon} \right|\right|_{L^2}. \tag{2.38}$$

Taking every inequality, it is held that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left|\left| u^\varepsilon - u^{\tilde\varepsilon} \right|\right|_{L^2}^2 &\leq CE(0)^{p-1} \left|\left| u^\varepsilon - u^{\tilde\varepsilon} \right|\right|_{L^2}^2 + C\|(u^\varepsilon - u^{\tilde\varepsilon})\|_{L^2} \|(w^\varepsilon - w^{\tilde\varepsilon})\|_{L^2} \\
&\quad + C \max(\varepsilon, \tilde\varepsilon) \left|\left| u^\varepsilon - u^{\tilde\varepsilon} \right|\right|_{L^2} \left(\sqrt{E(0)} + E(0)^p + E(0)^{p-\frac{1}{2}} \right).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{dt} \left|\left| u^\varepsilon - u^{\tilde\varepsilon} \right|\right|_{L^2} &\leq CE(0)^{p-1} \left|\left| u^\varepsilon - u^{\tilde\varepsilon} \right|\right|_{L^2} + C\|(w^\varepsilon - w^{\tilde\varepsilon})\|_{L^2} \\
&\quad + C \max(\varepsilon, \tilde\varepsilon) \left(\sqrt{E(0)} + E(0)^p + E(0)^{p-\frac{1}{2}} \right) \\
&\leq C\mathcal{E} + C \max(\varepsilon, \tilde\varepsilon).
\end{aligned} \tag{2.39}$$

On another side,

$$\frac{1}{2}\frac{d}{dt}\left|\left|w^{\varepsilon}-w^{\tilde{\varepsilon}}\right|\right|_{L^2}^2 \leq C\left(\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}\|w^{\varepsilon}-w^{\tilde{\varepsilon}}\|_{L^2}+\|(u^{\varepsilon}-u^{\tilde{\varepsilon}})\|_{L^2}\|(w^{\varepsilon}-w^{\tilde{\varepsilon}})\|_{L^2}\right).$$

Which leads to,

$$\begin{aligned}\frac{d}{dt}\left|\left|w^{\varepsilon}-w^{\tilde{\varepsilon}}\right|\right|_{L^2} &\leq C\left(\max(\varepsilon,\tilde{\varepsilon})\sqrt{E(0)}+\|(u^{\varepsilon}-u^{\tilde{\varepsilon}})\|_{L^2}\right)\\ &\leq C\mathcal{E}+C\max(\varepsilon,\tilde{\varepsilon}). \qquad (2.40)\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}\mathcal{E} &= \frac{d}{dt}\left|\left|u^{\varepsilon}-u^{\tilde{\varepsilon}}\right|\right|_{L^2}+\frac{d}{dt}\left|\left|w^{\varepsilon}-w^{\tilde{\varepsilon}}\right|\right|_{L^2}\\ &\leq C\mathcal{E}+C\max(\varepsilon,\tilde{\varepsilon}). \qquad (2.41)\end{aligned}$$

Now, if we denote

$$I = C\mathcal{E}+C\max(\varepsilon,\tilde{\varepsilon}),$$

by using (2.39) and (2.41), we may write

$$I' \leq I.$$

Solving the ODE, we obtain

$$\begin{aligned}\sup_{0\leq t\leq T^*}\mathcal{E} &\leq I(0)e^{T^*}\\ &\leq C\max(\varepsilon,\tilde{\varepsilon})e^{T^*},\end{aligned}$$

given that, at time $t=0$, $u_0^{\varepsilon}=u_0^{\tilde{\varepsilon}}$ and $w_0^{\varepsilon}=w_0^{\tilde{\varepsilon}}$.

With this, $(u^{\varepsilon},w^{\varepsilon})$ is a Cauchy sequence in the Banach space $C\{[0,T^*];L^2(\mathbb{T})\}$, so it is convergent and its limit exists in $C\{[0,T^*];L^2(\mathbb{T})\}$; i.e., we have proven the existence of a term (u,w) such that

$$\sup_{0\leq t\leq T^*}\left|\left|(u^{\varepsilon},w^{\varepsilon})-(u,w)\right|\right|_{L^2}\leq C\varepsilon.$$

The next step consists on applying the fact that the sequence $(u^{\varepsilon},w^{\varepsilon})$ is uniformly bounded in H^3 (proved in the previous subsection) to show that there exists a convergence in every intermediate norm $0<s'<3$. Then, using (1.9)

$$\left|\left|(u^{\varepsilon},w^{\varepsilon})-(u,w)\right|\right|_{H^{s'}}\leq C_s\left|\left|(u^{\varepsilon},w^{\varepsilon})-(u,w)\right|\right|_{L^2}^{1-s'/3}\left|\left|(u^{\varepsilon},w^{\varepsilon})-(u,w)\right|\right|_{H^3}^{s'/3}.$$

Obtaining

$$\sup_{0\leq t\leq T^*}\left|\left|(u^{\varepsilon},w^{\varepsilon})-(u,w)\right|\right|_{H^{s'}}\leq C_{s'}\varepsilon^{1-s'/3}E^{s'/3}(0). \qquad (2.42)$$

Thus, there is convergence for $s'<3$ in $C\{[0,T^*];H^{s'}(\mathbb{T})\}$. In sum, since $(u^{\varepsilon},w^{\varepsilon})\to(u,w)$, the limit of $(u_t^{\varepsilon},w_t^{\varepsilon})$ is (u_t,w_t). In consequence, (u,w) is the solution to the initial PDE system (2.1). □

As a result of the previous sections, we have proven the well-posedness of a solution to the regularized problem by Picard Theorem. The energy estimates obtained led us to a uniform existence time and, passing to the limit in this last subsection, the existence and uniqueness of a solution to the PDE system (2.1) until a time T^* has been shown.

Chapter 3

Nonexistence of a global solution

This chapter conforms the proof to the following theorem, applying the knowledge extracted from [2].

Theorem 3.1. *Not every initial data* $(u_0, w_0) \in H^3(\mathbb{T})$ *results in a global solution to the PDE system (2.1).*

In other words, we will show that there exists $(u_0, w_0) \in H^3(\mathbb{T})$ such that $\max u$ diverges for $p \geq 2$.

Let's define

$$J(t) = \int_0^1 u(x,t)dx \qquad \text{and} \qquad K(t) = \int_0^1 w(x,t)dx.$$

Then, the derivative of J with respect to t is

$$\begin{aligned}\frac{dJ}{dt} = \int_0^1 u_t dx &= \int_0^1 u_{xx}dx + \int_0^1 u^p dx + \int_0^1 (w-u)dx \\ &= u_x(1) - u_x(0) + \int_0^1 u^p dx + \int_0^1 w dx - \int_0^1 u dx \\ &= \int_0^1 u^p dx + \int_0^1 w dx - \int_0^1 u dx. \qquad (3.1)\end{aligned}$$

The Hölder inequality states that

$$\left| \int_\Omega \prod_{i=1}^n f_i(x)dx \right| \leq \prod_{i=1}^n ||f_i(x)||_{L^{p_i}(\Omega)}, 1 \leq p_i \leq \infty, \sum_{i=1}^n \frac{1}{p_i} = 1, \qquad (3.2)$$

with

$$L^p(\Omega) := \left\{ \text{equivalence classes } u \text{ such that } ||u||^p_{L^p(\Omega)} := \int_\Omega |u(x)|^p dx < \infty \right\}.$$

Thus,

$$J(t) = \int_0^1 u \cdot 1 dx \leq \left| \int_0^1 u \cdot 1 dx \right| \leq ||u||_{L^p} \cdot ||1||_{L^q} = \left(\int_0^1 |u(x)|^p dx \right)^{\frac{1}{p}} \cdot \left(\int_0^1 1^q dx \right)^{\frac{1}{q}}.$$

Notice that, since $p \geq 2$ and q must be greater than 1, q is a finite number so that the sum of their inverses is 1.

Therefore, by elevating everything to p,

$$J^p \leq \int_0^1 |u(x)|^p dx \cdot \left(\int_0^1 1^q dx \right)^{\frac{1}{qp}} = \int_0^1 |u(x)|^p dx \cdot \sqrt[qp]{1} = \int_0^1 |u(x)|^p dx,$$

and we obtain

$$\frac{dJ}{dt} \geq J^p - J + K. \tag{3.3}$$

Now,

$$\frac{dK}{dt} = \int_0^1 w_t dx = \int_0^1 -w_x dx + \int_0^1 u dx - \int_0^1 w dx = -w(1) + w(0) + J - K = J - K. \tag{3.4}$$

From this point on, it will be assumed that $J(0)$ and $K(0)$ are big values that verify the hypothesis

1. $\frac{J(0)}{2} > K(0) > 0$,
2. $\frac{J^p(0)}{2} > J(0)$.

In addition, we must select $K(0)$ big enough so that $K(0) >> 1$ and, in consequence, there exists a value δ such that

$$K(t) > \frac{K(0)}{2} > 0 \text{ for every } t \leq \delta.$$

Therefore, in this time interval, (3.3) satisfies

$$\frac{dJ}{dt} \geq J^p - J.$$

Given the size of $J(0)$ and the second hypothesis, it is correct to state

$$\frac{J^p(t)}{2} - J(t) > 0 \text{ for every } t \leq \delta,$$

so, consequently, the following ODE is obtained

$$\frac{dJ}{dt} \geq \frac{J^p(t)}{2} + \frac{J^p(t)}{2} - J(t) \geq \frac{J^p(t)}{2}.$$

Applying the separable variables method

$$\int \frac{dJ}{J^p} \geq \int \frac{dt}{2}$$

$$\frac{J^{1-p}}{1-p} \geq \frac{t}{2} + C. \tag{3.5}$$

From this moment on, we will replace the constant C with $\frac{J^{1-p}(0)}{1-p}$ since, for $t = 0$, the inequality (3.5) satisfies $\frac{J^{1-p}(0)}{1-p} \geq C$.

$$\frac{1}{(1-p)\frac{t}{2} + J^{1-p}(0)} \leq J^{p-1},$$

$$\frac{1}{\sqrt[p-1]{(1-p)\frac{t}{2} + J^{1-p}(0)}} \leq J. \tag{3.6}$$

Now, we may assume the following hypothesis without compromising the previous assumptions:

$$-\frac{2J^{1-p}(0)}{1-p} \leq \frac{\delta}{2}. \tag{3.7}$$

Therefore, for every $t \leq \delta$, there will always exist a time $t_1 = -\frac{2J^{1-p}(0)}{1-p}$ at which a zero appears in the denominator of (3.6), leading to an asymptote in the function, i.e., a singularity formation. With (3.7), this implies that the existence time assumed is not verified and, in fact, is smaller than δ. Thus, J diverges and we have proven there does not exist a global solution to the PDE system (2.1).

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