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On Keller-Segel systems with fractional diffusion

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Abstract

In this note we review some recent results on the parabolic-elliptic and parabolic-parabolic Keller-Segel model with fractional diffusion. Furthermore, we consider with particular emphasis the parabolic-hyperbolic case with a fractional Laplacian. In particular we prove certain new results. Namely, first, we prove the existence of solution for analytic initial data and then we establish the finite time singularity formation for initial data in this class. Finally, we obtain a new equation describing unidirectional wave propagation for the parabolic-hyperbolic Keller-Segel system.

1. The parabolic-parabolic and parabolic-elliptic Keller-Segel systems

Even if as of 2021 no "standard model" of the origin of life has yet emerged, most currently accepted models state that life arose on Earth between 3800 and 4100 million years ago. These first forms of life where single-celled organisms. For most of the History of life on Earth there were only unicellular organisms. However, now there are many different fungi, algae, plants and animals that are multicellular organisms (they are formed by aggregations of cells working together). Thus, even if we know that these single-celled organism eventually formed multicellular organisms (around 1500 millions of years ago), the origins of multicellularity are one of the most interesting topics in biology because we still do not know the details of how multicellularity arised.

A particular situation where cells form a cluster, in a process known as *cell aggregation*, arises when the motion of the cells is driven by a chemical gradient, *i.e.* the cells attempt to move towards higher (or lower) concentration of some chemical substance. This process is usually called *chemotaxis*. Then, multicellular aggregates and eventually tissue-like assemblies are formed when individual cells attach to each other as a consequence of the chemotactic movement and when this aggregation leads to subsequent cellular differentiation. That is, for instance, the case of the slime mold *Dictyostelium Discoideum* and bacterial populations, such as of *Escherichia coli* and *Salmonella typhimurium*.

A preliminary step towards a better understanding of chemotaxis and cell aggregation, was given by Keller & Segel with their 1970 classical paper [18] (see also the prior work by Patlak [21]). In this paper, Keller and Segel proposed a nonlinear system of PDEs of cross-diffusive type. After a number of simplifications, the following PDE system appears as a model of cell aggregation as a consequence of chemotactic movement:

$$\partial_t u = -(-\Delta)^{\alpha/2} u + \nabla \cdot (u \nabla v) + f(u,v) \tag{1.1}$$

$$\tau \partial_t v = \nu \Delta v + u - \lambda v. \tag{1.2}$$

Here $(-\Delta)^{\alpha/2}$ denotes the fractional laplacian, u and v denote the cells and chemical concentration, respectively, and $\tau, \nu, \lambda \geq 0$ are fixed constants. In most of the applications, the forcing is typically $f(u,v) = 0$ or $f(u) = ru(1-u)$. System (1.1)-(1.2) is known as the parabolic-parabolic Keller-Segel equation (ppKS).

Diffusions given by $-(-\Delta)^{\alpha/2} u$ for $\alpha < 2$ arise naturally when studying feeding strategies of some organisms in certain situations. For instance, these fractional diffusions have been reported for amoebas [19], microzooplancton [4], flying ants [23], fruit flies [12], jackals [2] and even humans! [22].

Following the works by T. Hillen, K. Painter & M. Winkler [17] and references therein, equations (1.1)-(1.2) also appear in the modelization of cancer invasion of healthy tissues. In particular, the model (1.1)-(1.2) with $f(u,v) = ru(1-u)$ appears as a valid approximation of the three-component urokinase plasminogen invasion.

The case when $\tau = 0$, (1.1)-(1.2) is known as the parabolic-elliptic Keller-Segel equations (peKS) and, besides its interest in regards to aggregation and chemotaxis, this equation also arises as a model of other physical phenomena. In particular, the (peKS) equation (with $\alpha = 2$, $\nu = 1$, $\lambda = 0$ and $f \equiv 0$) is, formally, similar to the two-dimensional, incompressible Navier-Stokes written in vorticity form $\omega = \text{curl} v$:

$$\partial_t \omega = \Delta \omega + \nabla \cdot (\omega \nabla^\perp \psi), \ -\Delta \psi = \omega.$$

The (peKS) equation also arises as a model of gravitational collapse and star formation [1]. Thus, every new mathematical result regarding equations (1.1)-(1.2) has potential implications in Applied Sciences.

Using the Keller-Segel model (1.1)-(1.2), aggregation is mathematically equivalent to a finite time singularity of the type

$$\lim_{t\to T}\max_x u(x,t)=\infty. \tag{1.3}$$

Then, the question we want to answer is the following: **Can the (ppKS) or the (peKS) systems develop a finite time singularity of the type of** (1.3)**?**

For the simpler case of the (peKS) equations this question is nowadays well understood. Specifically, it is known that the aggregation of cells is very sensitive to changes in the dimension of the spatial domain and the order of the diffusion, α. In particular, in the one dimensional case, (peKS) equation was first studied by Escudero [13]. He proved the global existence of solution in the case $1<\alpha\leq 2$. This result was later improved by Bournaveas & Calvez [6], where the authors proved finite time singularities for the case $0<\alpha<1$ and the existence of $K>0$ such that, for the case $\alpha=1$ and initial data satisfying the smallness restriction $\int u_0(x_1)dx_1\leq K$, there exists a global smooth solution. Furthermore, based on numerical simulations, Bournaveas & Calvez, reported the existence of finite time singularities in the case $\alpha=1$ for *large* initial data. This conjecture is in agreement with the two dimensional case and $\alpha=2$, where there are finite time singularities depending on the mass of the initial data

$$M=\int_{\mathbb{R}^2}u_0(x_1,x_2)dx_1dx_2.$$

More precisely, when $M<8\pi$, the solutions are globally defined and smooth while if $M>8\pi$ the solutions develop a finite time singularity (see for instance the paper by A. Blanchet, J. Dolbeault & B. Perthame [5] and the references therein). The singularity formation for the two dimensional case with $\alpha<2$ has been proved by D. Li, J. Rodrigo & X. Zhang [20]. Systems akin to (peKS) with a nonlinear fractional diffusion have also been studied in [16] and [7].

In a fruitful collaboration with Jan Burczak, we were able to

- **Lack of threshold behavior for the critical (**$\alpha=1$**) (peKS) equation.** One of our results [8] for the one-dimensional (peKS) equations with $\alpha=1$ is that smooth solutions can be defined for all later times, *i.e.* every initial data leads to a global solution. Furthermore, the solutions are globally bounded (in fact, they decay to the homogeneous steady state) if the initial mass is small enough (with a smallness condition of order $O(1)$). This was a very surprising result that disproved the previous conjecture by Bournaveas & Calvez [6].

- **Global existence for the supercritical (**$\alpha<d$**) (peKS) equation with logistic forcing.** In the case where a logistic forcing of the type
$$f(u)=ru(1-u),\ 0<r$$
is considered into the (peKS) equations, we proved that, if $d=1,2$ denotes the spatial dimension, we could find a range
$$d-c_1(r)<\alpha\leq d, \tag{1.4}$$
with $c_1(r)$ explicit such that there exist global in time classical solutions to the SP equations with α in the range (1.4) [9,11]. Furthermore, we could also find a second range,
$$d-c_2(r)<\alpha\leq d, \tag{1.5}$$
where $c_2(r)$ is explicit, such that there exist global in time weak solutions to the SP equations with α in the range (1.5). Note that, in d dimensions, the case $\alpha=d$ corresponds to the critical case with respect to the total mass
$$\int u(x)dx.$$
Consequently, these results are the first global well-posedness results in supercritical ranges (1.4) and (1.5) for a Keller-Segel type equation. As a consequence of these results, we obtained that aggregation is not possible for α in (1.4) if there is a logistic growth. In the one dimensional case $d=1$, these results are also interesting when compared to other classical nonlinear, one-dimensional drift-diffusion equations as the Burgers equation, where $\alpha\geq 1$ is a required for global classical solution (with large initial data) to exist.

  In the case of two spatial dimensions $d=2$, the proof of the global existence of classical solutions is based in a new delicate pointwise estimate for the two-dimensional fractional Laplacian
$$(-\Delta)^{\alpha/2}u(\bar{x}_1,\bar{x}_2)\geq C_{\alpha,\delta}\frac{u(\bar{x}_1,\bar{x}_2)^{1+\frac{\alpha}{2-\delta}}}{\|\phi\|_{C^\delta}^{1+\frac{\alpha}{2-\delta}}},$$

where $\partial_{x_1}\partial_{x_2}\phi = u$ and $(\bar{x}_1, \bar{x}_2)$ is such that $\max_{y_1,y_2} u(y_1, y_2) = u(\bar{x}_1, \bar{x}_2)$. This inequality is interesting by itself as it may be applied in many other PDE problems.

- **Dynamical properties of (ppKS) equation with logistic forcing.** In [10] we study dynamical properties of the (ppKS) system with logistic forcing. Remarkably, this model exhibits a spatio-temporal chaotic behavior, where a number of peaks emerge. We were able to prove the existence of an attractor and provide an upper bound on the number of peaks that the solution may develop. Finally, we perform a numerical analysis suggesting that there is a finite time blowup if the diffusion is weak enough, even in presence of a damping logistic term.

2. The parabolic-hyperbolic Keller-Segel system

In what follows we study the following system arising in tumor angiogenesis

$$\partial_t u = -(-\Delta)^{\alpha/2} u + \partial_x(uq), \text{ for } x \in \mathbb{T},\ t \geq 0, \tag{2.1}$$

$$\partial_t q = u^{r-1}\partial_x u, \text{ for } x \in \mathbb{T},\ t \geq 0, \tag{2.2}$$

where u is a non-negative scalar function, q is a zero-mean function, $\mathbb{T}$ denotes the domain $[-\pi, \pi]$ with periodic boundary conditions, $0 < \alpha \leq 2$, $1 \leq r \leq 2$ and $(-\Delta)^{\alpha/2} = \Lambda^\alpha$ is the fractional Laplacian.

This system was proposed by Othmers & Stevens [24] based on biological considerations as a model of tumor angiogenesis. In that context, u is the density of vascular endothelial cells and $q = \partial_x \log(v)$ where v is the concentration of the signal protein known as vascular endothelial growth factor (VEGF).

Equation (2.1) appears as a singular limit of the following Keller-Segel model of aggregation of the slime mold *Dictyostelium discoideum* [18] (see also Patlak [21])

$$\begin{cases} \partial_t u = -(-\Delta)^{\alpha/2} u - \chi\nabla\cdot(u\nabla G(v)), \\ \partial_t v = \nu\Delta v + \left(\dfrac{u^r}{r} + \lambda\right) v, \end{cases} \tag{2.3}$$

when $G(v) = \log(v)$ and the diffusion of the chemical v is negligible.

This system was studied in [14, 15]. In particular

- **Local existence and decay.** In [15] the local well-posedness for arbitrary H^3 non-negative initial data, $0 \leq \alpha \leq 2$ and $1 \leq r \leq 2$ was proved. We would like to emphasize that that the sign of the initial data plays the role of a stability condition and helps us to avoid derivative loss. Furthermore, the solution verifies the following global bound: for $r > 1$

$$\frac{\|u(t)\|_{L^r}^r}{r(r-1)} + \frac{\|q(t)\|_{L^2}^2}{2} + \frac{1}{r-1}\int_0^t \int_{\mathbb{T}} (-\Delta)^{\alpha/2} u u^{r-1} dx ds \leq \frac{\|u_0\|_{L^r}^r}{r(r-1)} + \frac{\|q_0\|_{L^2}^2}{2}, \tag{2.4}$$

while, if $r = 1$,

$$\int_{\mathbb{T}} (u\log(u) - u + 1) dx + \frac{\|q(t)\|_{L^2}^2}{2} + \int_0^t \int_{\mathbb{T}} (-\Delta)^{\alpha/2} u \log(u) dx ds \leq \int_{\mathbb{T}} (u_0 \log(u_0) - u_0 + 1) dx + \frac{\|q_0\|_{L^2}^2}{2}. \tag{2.5}$$

- **Global well-posedness for arbitrary initial data in the critical regime for $r = 2$.** Notice that the equations (2.1)-(2.2) verify the following scaling symmetry: for every $\lambda > 0$

$$u_\lambda(x,t) = \lambda^{\frac{2\alpha-2}{r}} u(\lambda x, \lambda^\alpha t), \quad q_\lambda(x,t) = \lambda^{\alpha-1} q(\lambda x, \lambda^\alpha t).$$

In [14], the global well-posedness for arbitrary H^2 non-negative initial data and the critical diffusion $\alpha = 3/2$ was proved. Similarly, the two-dimensional case is also studied for the critical value $\alpha = 2$ and global well-posedness is also presented. Due to the hyperbolic character of the equation for q, prior available global existence results of classical solution for equation (2.1) impose several assumptions [25–27] and the references therein.

Our results removed some of the previous conditions. On the one hand, we prove global existence for arbitrary data in the cases $d = 1$ and $\alpha \geq 1.5$ and $d = 2$ and $\alpha = 2$. On the other hand, in the cases where we have to impose size restrictions on the initial data, the Sobolev spaces are bigger than H^2. A question that remains open is the trend to equilibrium. From (2.4) is clear that the solution $(u(t), q(t))$ tends to the homogeneous state, namely $(\langle u_0\rangle, 0)$. However, the rate of this convergence is not clear.

2.1. Local well-posedness for analytic initial data

Equations (2.1)-(2.2) can be written as

$$\partial_t h = -\Lambda^\alpha h + \partial_x(hq) + \langle u_0\rangle \partial_x q, \text{ for } x \in \mathbb{T},\ t \geq 0, \tag{2.6}$$

$$\partial_t q = (h + \langle u_0\rangle)^{r-1}\partial_x h, \text{ for } x \in \mathbb{T},\ t \geq 0, \tag{2.7}$$

where $h = u - \langle u_0\rangle$. Without lossing generality we consider $\langle u_0\rangle = 1$. Then we have that

Theorem 1 *Define*

$$\nu(t) = 1 - \theta t,$$

for

$$\theta > 1 + \|h_0\|_{\nu(0)} + C_r(1 + \|h_0\|^{r-1}_{\nu(0)}) + \|q_0\|_{\nu(0)}.$$

Let us consider (h_0, q_0) *such that*

$$\|h(t)\|_{\nu(t)} = \sum_{n=-\infty}^{\infty} |\hat{h}(n,t)| e^{\nu(t)|n|} < \infty,$$

$$\|q(t)\|_{\nu(t)} = \sum_{n=-\infty}^{\infty} |\hat{q}(n,t)| e^{\nu(t)|n|} < \infty.$$

Then, there exist a sufficiently short time and a unique local solution which is analytic in a complex strip with sufficiently small width.

Proof We define the scale of spaces

$$\mathbb{A}_{\nu(t)} = \left\{ u \in L^2, e^{\nu(t)|n|}\hat{u}(n) \in \ell^1 \right\}$$

with norm

$$\|u\|_{\nu(t)} = \|e^{\nu(t)|n|}\hat{u}(n)\|_{\ell^1}.$$

We observe that the previous spaces are a Banach Algebra

$$\|fg\|_{\nu(t)} \leq \|f\|_{\nu(t)} \|g\|_{\nu(t)}.$$

We compute

$$\begin{aligned} \frac{d}{dt}\|F\|_{\nu(t)} &= \sum_{n=-\infty}^{\infty} \nu'(t)|n| e^{\nu(t)|n|} |\widehat{F}(n,t)| + \sum_{n=-\infty}^{\infty} e^{\nu(t)|n|} \mathrm{Re}\left(\frac{\partial}{\partial t}\widehat{F}(n,t) \frac{\overline{\widehat{F}(n,t)}}{|\widehat{F}(n,t)|} \right) \\ &\leq \sum_{n=-\infty}^{\infty} \nu'(t)|n| e^{\nu(t)|n|} |\widehat{F}(n,t)| + \left\| \frac{\partial}{\partial t} F \right\|_{\nu(t)}. \end{aligned}$$

Then, if $0 < \nu(t)$ is a decreasing function we find a regularizing contribution coming from ν'. This regularizing contribution is reflecting the fact that the strip of analyticity is shrinking. At this point it is easy to find the estimate

$$\begin{aligned} \frac{d}{dt}(\|h\|_{\nu(t)} + \|q\|_{\nu(t)}) &\leq \nu'(t)(\|\partial_x h\|_{\nu(t)} + \|\partial_x q\|_{\nu(t)}) - \|(-\Delta)^{\alpha/2} h\|_{\nu(t)} \\ &\quad + (1 + \|h\|_{\nu(t)})\|\partial_x q\|_{\nu(t)} + \|\partial_x h\|_{\nu(t)}(C_r(1 + \|h\|^{r-1}_{\nu(t)}) + \|q\|_{\nu(t)}) \\ &\leq 0, \end{aligned}$$

where in the last line we have fix

$$\theta > 1 + \|h_0\|_{\nu(0)} + C_r(1 + \|h_0\|^{r-1}_{\nu(0)}) + \|q_0\|_{\nu(0)}.$$

□

We want to remark that the previous result does not require any sign condition on h nor the parabolic term $(-\Delta)^{\alpha/2}$.

2.2. Finite time blow up for the inviscid case

We consider the inviscid system

$$\partial_t u = \partial_x(uq), \text{ for } x \in \mathbb{T},\ t \geq 0, \tag{2.8}$$

$$\partial_t q = u^{r-1}\partial_x u, \text{ for } x \in \mathbb{T},\ t \geq 0. \tag{2.9}$$

and prove the following result:

Theorem 2 *Let us consider $r = 1$. Then there exist smooth initial data such that the corresponding solution to* (2.8)-(2.9) *blows up in finite time.*

Proof Assume that $0 \leq u_0(x)$, is an even function such that $u_0(0) = \partial_x^2 u_0(0) = 0$. Assume also that q_0 is an odd function such that $\partial_x q_0(0) = 0$. We note that the symmetry is preserved, *i.e.* as long as the solution exist, $u(x,t)$ remains even and $q(x,t)$ remains odd.

The proof is similar to the one in [3]. We argue by contradiction: assume that a a global classical solution exists for this initial data. Then we want to prove that some quantity blows up. We define the following quantities

$$U_i(t) = \partial_x^i u(x,t)\Big|_{x=0},\ Q_i(t) = \partial_x^i q(x,t)\Big|_{x=0}.$$

Then we have

$$\begin{aligned}\frac{d}{dt}U_0 &= U_1 Q_0 + Q_1 U_0\\ &= Q_1 U_0,\end{aligned}$$

so

$$U_0(t) = U_0(0)e^{\int_0^t Q_1(s)ds} = 0.$$

In the same way,

$$\frac{d}{dt}Q_1 = U_2.$$

$$\begin{aligned}\frac{d}{dt}U_2 &= U_3 Q_0 + 3U_2 Q_1 + 3U_1 Q_2 + Q_3 U_0\\ &= 3U_2 Q_1\\ &= 3\frac{d}{dt}Q_1 Q_1,\end{aligned}$$

so

$$\frac{d^2}{dt^2}Q_1 = \frac{d}{dt}U_2(t) = \frac{d}{dt}\frac{3}{2}Q_1(t)^2,$$

and that implies the finite time blow up of Q_1 and U_2. □

2.3. Wave propagation

Finally we turn our attention to the wave-like form of the system (2.6)-(2.7). Indeed, we observe that the system (2.6)-(2.7) with

$$\langle u_0 \rangle = r = 1,$$

can be written as the following bidirectional non-local wave equation

$$\partial_t^2 q = -(-\Delta)^{\alpha/2}\partial_t q + \partial_x^2(\partial_x^{-1}\partial_t q q) + \partial_x^2 q. \tag{2.10}$$

Then, if ε is a small parameter and we consider the unknown

$$q = \varepsilon f,$$

we find the equation

$$\partial_t^2 f = -(-\Delta)^{\alpha/2}\partial_t f + \varepsilon\partial_x^2(\partial_x^{-1}\partial_t f f) + \partial_x^2 f. \tag{2.11}$$

To find the equation describing unidirectional waves we change to far-field variables

$$\xi = x - t, \quad \tau = \varepsilon t.$$

We can apply the chain rule to compute

$$\frac{\partial}{\partial t} f(\chi(x,t), \tau(t)), \quad \frac{\partial^2}{\partial t^2} f(\chi(x,t), \tau(t))$$

and, as a consequence, we find that

$$\partial_\xi^2 f - 2\varepsilon \partial_\tau \partial_\xi f + \varepsilon^2 \partial_\tau^2 f = (-\Delta)^{\alpha/2} \partial_\xi f - \varepsilon(-\Delta)^{\alpha/2} \partial_\tau f + \varepsilon \partial_\xi^2 (\partial_\xi^{-1}(-\partial_\xi f + \varepsilon \partial_\tau f) f) + \partial_\xi^2 f. \tag{2.12}$$

Then, if we neglect terms of order $O(\varepsilon^2)$, we obtain the asymptotic equation

$$-2\varepsilon \partial_\tau \partial_\xi f = (-\Delta)^{\alpha/2} \partial_\xi f - \varepsilon(-\Delta)^{\alpha/2} \partial_\tau f - \varepsilon \partial_\xi^2 (\partial_\xi^{-1} \partial_\xi f) f). \tag{2.13}$$

Integrating in ξ,

$$-2\varepsilon \partial_\tau f + \varepsilon(-\Delta)^{(\alpha-1)/2} H \partial_\tau f = (-\Delta)^{\alpha/2} f - 2\varepsilon f \partial_\xi f. \tag{2.14}$$

Changing back to our previous notation for the independent variables, we conclude

$$\partial_t f - \frac{1}{2}(-\Delta)^{(\alpha-1)/2} H \partial_t f = -\frac{1}{2\varepsilon}(-\Delta)^{\alpha/2} f + f \partial_x f. \tag{2.15}$$

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References

[1] Y. Ascasibar, R. Granero-Belinchón, and J. M. Moreno. An approximate treatment of gravitational collapse. *Physica D: Nonlinear Phenomena*, 262:71 – 82, 2013.

[2] R. Atkinson, C. Rhodes, D. Macdonald, and R. Anderson. Scale-free dynamics in the movement patterns of jackals. *Oikos*, 98(1):134–140, 2002.

[3] H. Bae and R. Granero-Belinchón. Singularity formation for the Serre-Green-Naghdi equations and applications to abcd-Boussinesq systems *arXiv preprint arXiv:2001.11937*, 2020.

[4] F. Bartumeus, F. Peters, S. Pueyo, C. Marrasé, and J. Catalan. Helical lévy walks: adjusting searching statistics to resource availability in microzooplankton. *Proceedings of the National Academy of Sciences*, 100(22):12771–12775, 2003.

[5] A. Blanchet, J. Dolbeault, and B. Perthame. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations*, pages No. 44, 32 pp. (electronic), 2006.

[6] N. Bournaveas and V. Calvez. The one-dimensional Keller-Segel model with fractional diffusion of cells. *Nonlinearity*, 23(4):923, 2010.

[7] J. Burczak and R. Granero-Belinchón. Boundedness of large-time solutions to a chemotaxis model with nonlocal and semilinear flux. *Topological Methods in Nonlinear Analysis.*, 47(1):369–387, 2016.

[8] J. Burczak and R. Granero-Belinchón. Critical Keller-Segel meets Burgers on $\mathbb{S}^1$: large-time smooth solutions. *Nonlinearity*, 29(12):3810, 2016.

[9] J. Burczak and R. Granero-Belinchón. Global solutions for a supercritical drift-diffusion equation. *Advances in Mathematics*, 295:334–367, 2016.

[10] J. Burczak and R. Granero-Belinchón. On a generalized doubly parabolic Keller-Segel system in one spatial dimension. *Mathematical Models and Methods in the Applied Sciences*, 26(1):111–160, 2016.

[11] J. Burczak and R. Granero-Belinchón. Suppression of blow up by a logistic source in 2 d keller-segel system with fractional dissipation. *Journal of Differential Equations*, 263(9):6115–6142, 2017.

[12] B. J. Cole. Fractal time in animal behaviour: the movement activity of drosophila. *Animal Behaviour*, 50(5):1317–1324, 1995.

[13] C. Escudero. The fractional Keller-Segel model. *Nonlinearity*, 19(12):2909, 2006.

[14] R. Granero-Belinchón. Global solutions for a hyperbolic–parabolic system of chemotaxis. *Journal of Mathematical Analysis and Applications*, 449(1):872–883, 2017.

[15] R. Granero-Belinchón. On the fractional fisher information with applications to a hyperbolic–parabolic system of chemotaxis. *Journal of Differential Equations*, 262(4):3250–3283, 2017.

[16] R. Granero-Belinchón and R. Orive-Illera. An aggregation equation with a nonlocal flux. *Nonlinear Analysis: Theory, Methods & Applications*, 108(0):260

[17] T. Hillen, K. J. Painter, and M. Winkler. Convergence of a cancer invasion model to a logistic chemotaxis model. *Math. Models Methods Appl. Sci.*, 23(1):165–198, 2013.

[18] E. Keller and L. Segel. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology*, 26(3):399–415, 1970.

[19] M. Levandowsky, B. White, and F. Schuster. Random movements of soil amebas. *Acta Protozoologica*, 36:237–248, 1997.

[20] D. Li, J. Rodrigo, and X. Zhang. Exploding solutions for a nonlocal quadratic evolution problem. *Revista Matematica Iberoamericana*, 26(1):295–332, 2010.

[21] C. Patlak. Random walk with persistence and external bias. *Bulletin of Mathematical Biology*, 15(3):311–338, 1953.

[22] D. A. Raichlen, B. M. Wood, A. D. Gordon, A. Z. Mabulla, F. W. Marlowe, and H. Pontzer. Evidence of Lévy walk foraging patterns in human hunter–gatherers. *Proceedings of the National Academy of Sciences*, 111(2):728–733, 2014.

[23] M. F. Shlesinger and J. Klafter. Lévy walks versus Lévy flights. In *On growth and form*, pages 279–283. Springer, 1986.

[24] A. Stevens, Angela and H. Othmer, SIAM Journal on Applied Mathematics 57(4):1044–1081, 1997.

[25] Z.-A. Wang, Z. Xiang, and P. Yu. Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis. *Journal of Differential Equations*, 260(3):2225–2258, 2016.

[26] Y. Zhang, Z. Tan, and M.-B. Sun. Global existence and asymptotic behavior of smooth solutions to a coupled hyperbolic-parabolic system. *Nonlinear Analysis: Real World Applications*, 14(1):465–482, 2013.

[27] H.-Y. Jin, J. Li, and Z.-A. Wang. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. *Journal of Differential Equations*, 255(2):193–219, 2013.