THE PERTURBATION CLASSES PROBLEM FOR SUBPROJECTIVE AND SUPERPROJECTIVE BANACH SPACES

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Abstract. We show that the perturbation class for the upper semi-Fredholm operators between two Banach spaces X and Y coincides with the strictly singular operators when X is subprojective and that the perturbation class for the lower semi-Fredholm operators coincides with the strictly cosingular operators when Y is superprojective. Similar results were proved in [7] under stronger conditions for X and Y.

1. Introduction

Given a class $\mathcal{A}$ of operators between Banach spaces, its perturbation class $P\mathcal{A}$ is defined as the class of all operators K such that $T+K \in \mathcal{A}$ for every $T \in \mathcal{A}$. This definition is not intrinsic, in the sense that determining whether an operator belongs to $P\mathcal{A}$ involves studying its behaviour with respect to every operator in $\mathcal{A}$. In this regard, it is useful to find an intrinsic characterisation for a perturbation class $P\mathcal{A}$, as its existence means that membership of an operator can be checked based on properties of the operator alone.

For the upper semi-Fredholm operators Φ_+, it has been long known that strictly singular operators belong to $P\Phi_+$ [9, Theorem 5.2]; an operator is strictly singular if its restriction to a closed infinite-dimensional subspace is never an isomorphism. Similarly, for the lower semi-Fredholm operators Φ_-, strictly cosingular operators belong to $P\Phi_-$ [14, Corollary 1]; an operator T is strictly cosingular if its composition QT with the quotient operator Q of a closed infinite-codimensional subspace is never a surjection. The perturbation classes problem for the semi-Fredholm operators is the question of whether these pairs of classes ($P\Phi_+$ and $\mathcal{SS}$; $P\Phi_-$ and $\mathcal{SC}$) coincide [5, page 74] [12, 26.6.12] [13, Section 3]. This question remained open for a time, but was eventually proved to have a negative answer in general: there exists a separable, reflexive Banach space Z such that $P\Phi_+(Z) \neq \mathcal{SS}(Z)$ and $P\Phi_-(Z^*) \neq \mathcal{SC}(Z^*)$ [6].

However, it is still interesting to find pairs of spaces for which the answer to the perturbation classes problem is positive, as it means that, at least for them, the relevant components of $P\Phi_+$ and $P\Phi_-$ do admit an intrinsic characterisation. There are several known such cases, including some classical results [10] [15].

In Theorems 5 and 7, we prove that the perturbation classes problem for $\Phi_+(X,Y)$ has a positive answer when X is subprojective and similarly that the perturbation classes problem for $\Phi_-(X,Y)$ has a positive answer when Y is superprojective. A Banach space X is subprojective if every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X; a Banach space X is superprojective if every closed infinite-codimensional subspace of X is contained in an infinite-codimensional subspace complemented in X. Subprojective and superprojective spaces were introduced by Whitley to study strictly singular and strictly cosingular operators [16]; see [7] for a fairly complete list of examples known at the time and [11] [8] and [4] for more recent discoveries.

Theorems 5 and 7 improve on the following, previously known results.

Theorem 1. [10] [3] *Let X and Y be Banach spaces such that $\Phi_+(X,Y)$ is not empty and Y is subprojective. Then $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$.*

Proof. Note that $\Phi_+(X,Y) \neq \emptyset$ implies that $\mathcal{SS}(X,Y) \subseteq P\Phi_+(X,Y) \subseteq \mathcal{I}n(X,Y)$ [3, Theorem 3.6] and that Y subprojective implies that $\mathcal{SS}(X,Y) = \mathcal{I}n(X,Y)$ [2, Theorems 4.3 and 4.4]. □

Theorem 2. [10] [3] *Let X and Y be Banach spaces such that $\Phi_-(X,Y)$ is not empty and X is superprojective. Then $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$.*

Proof. Note that $\Phi_-(X,Y) \neq \emptyset$ implies that $\mathcal{SC}(X,Y) \subseteq P\Phi_-(X,Y) \subseteq \mathcal{I}n(X,Y)$ [3, Theorem 3.6] and that X superprojective implies that $\mathcal{SC}(X,Y) = \mathcal{I}n(X,Y)$ [2, Theorems 4.3 and 4.4]. □

Theorem 5 is stronger than Theorem 1 because the hypotheses in Theorem 1 ($\Phi_+(X,Y) \neq \emptyset$ and Y subprojective) imply that X itself is subprojective, as they mean that a finite-codimensional subspace of X is isomorphic to a subspace of Y, and closed subspaces of subprojective spaces are subprojective too. Similarly, Theorem 7 is stronger than Theorem 2 because the hypotheses in Theorem 2 ($\Phi_-(X,Y) \neq \emptyset$ and X superprojective) imply that Y itself is superprojective, as they mean that a finite-codimensional subspace of Y is isomorphic to a quotient of X, and quotients of superprojective spaces are superprojective too.

Theorems 5 and 7 also improve on similar results obtained for spaces satisfying the formally stronger conditions of strong subprojectivity and strong superprojectivity introduced in [7]. A Banach space X is strongly subprojective if every closed infinite-dimensional subspace of X contains an infinite-dimensional subspace complemented in X with complement isomorphic to X; a Banach space X is strongly superprojective if every closed infinite-codimensional subspace of X is contained in an infinite-codimensional subspace complemented in X that is isomorphic to X. Clearly, a strongly subprojective space X is subprojective, although the question remains open as to whether there are subprojective spaces that are not strongly subprojective, and likewise for the classes of superprojective and strongly superprojective spaces.

Theorem 3. [7, Theorems 2.6 and 3.7] *Let X and Y be Banach spaces.*

(a) *If X is strongly subprojective and $\Phi_+(X,Y)$ is not empty, then $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$.*
(b) *If Y is strongly superprojective and $\Phi_-(X,Y)$ is not empty, then $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$.*

We will use standard notation. Given a (bounded, linear) operator $T\colon X \longrightarrow Y$, $N(T)$ and $R(T)$ will denote the kernel and the range of T, respectively. $\mathcal{L}(X,Y)$ will stand for the set of all operators from X to Y; if $\mathcal{A}$ is a class of operators, then $\mathcal{A}(X,Y) = \mathcal{A} \cap \mathcal{L}(X,Y)$ and $\mathcal{A}(X) = \mathcal{A}(X,X)$. If N is a closed subspace of X, we will denote the induced natural quotient operator by $Q_N\colon X \longrightarrow X/N$.

2. Results

We begin with a simple result that can be found in [1, Theorem 7.21]. We include a proof for the convenience of the reader.

Proposition 4.

(a) *If $K \in P\Phi_+(X,Y)$ and $A \in \mathcal{L}(X)$, then $KA \in P\Phi_+(X,Y)$.*
(b) *If $K \in P\Phi_-(X,Y)$ and $B \in \mathcal{L}(Y)$, then $BK \in P\Phi_-(X,Y)$.*

Proof. (a) If A is bijective, let $T \in \Phi_+(X,Y)$; then $TA^{-1} \in \Phi_+(X,Y)$, so $T + KA = (TA^{-1} + K)A \in \Phi_+(X,Y)$, hence $KA \in P\Phi_+(X,Y)$. For the general case, it is enough to note that A can be written as the sum of two bijective operators.

The proof of (b) is similar. □

The next result was already known for X strongly subprojective [7].

Theorem 5. *Let X and Y be Banach spaces such that $\Phi_+(X,Y)$ is not empty and X is subprojective. Then $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$.*

Proof. Since $\Phi_+(X,Y)$ is not empty, Y must contain some closed subspace L isomorphic to a finite-codimensional subspace of X; in particular, L must be subprojective.

Let $K \in \mathcal{L}(X,Y)\backslash\mathcal{SS}(X,Y)$; we have to show that $K \notin P\Phi_+(X,Y)$. Since K is not strictly singular, there exists a closed infinite-dimensional subspace U of X such that $K|_U$ is an isomorphism. Considering the relative positions of the subspaces $K(U)$ and L inside Y, three cases may happen:

(a) $K(U) \cap L$ is finite-dimensional and $K(U) + L$ is closed;
(b) $K(U) \cap L$ is infinite-dimensional;
(c) $K(U) \cap L$ is finite-dimensional and $K(U) + L$ is not closed.

(a) If $K(U) \cap L$ is finite-dimensional and $K(U) + L$ is closed, we can assume that $K(U) \cap L = \{0\}$ by passing to a smaller U if necessary. Since X is subprojective, we may further assume that U is complemented in X, so there exists a closed subspace M of X such that $X = U \oplus M$ and, as M is an infinite-codimensional subspace of X and L is isomorphic to a finite-codimensional subspace of X, there exists an isomorphic embedding $S\colon M \longrightarrow L$. Define an operator $T\colon X = U \oplus M \longrightarrow Y$ as $T(u + m) = -K(u) + S(m)$, where $u \in U$ and $m \in M$; then $T \in \Phi_+(X, Y)$ but $U \subseteq N(T + K)$, so $T + K \notin \Phi_+$, which proves that $K \notin P\Phi_+(X, Y)$.

(b) If $K(U) \cap L$ is infinite-dimensional, we can pass to $U \cap K^{-1}(L)$ to assume that $K(U) \subseteq L$ and, since X is subprojective, we may further assume that U is complemented in X, so there exists a projection $P\colon X \longrightarrow X$ with range U. Now, KP can be seen as an operator $KP\colon X \longrightarrow L$ that is not strictly singular, where $\Phi_+(X, L)$ is not empty and L is subprojective, so $KP \notin P\Phi_+(X, L)$ by Theorem 1. As such, $KP \notin P\Phi_+(X, Y)$ and $K \notin P\Phi_+(X, Y)$ by Proposition 4.

(c) If $K(U) \cap L$ is finite-dimensional and $K(U) + L$ is not closed, there exists a compact operator $K_1\colon X \longrightarrow Y$ such that $(K + K_1)(U) \cap L$ is infinite-dimensional [7, Theorem 2.6], and then it follows that $K + K_1 \notin P\Phi_+(X, Y)$ from case (b) and finally that $K \notin P\Phi_+(X, Y)$. □

Next we recall a technical lemma.

Lemma 6. [7, Lemma 3.5] *Let $K \in \mathcal{L}(X, Y)$ and let Y_0 be a closed subspace of Y such that $Q_{Y_0}K$ is surjective. If E is a closed subspace of X such that $K^{-1}(Y_0) \subseteq E$, then Y contains a closed subspace F such that $Y_0 \subseteq F$ and $E = K^{-1}(F)$. Moreover, if E is infinite-codimensional in X, then F is infinite-codimensional in Y.*

The next result was already known for Y strongly superprojective [7].

Theorem 7. *Let X and Y be Banach spaces such that $\Phi_-(X, Y)$ is not empty and Y is superprojective. Then $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$.*

Proof. Since $\Phi_-(X, Y)$ is not empty, X must contain some closed subspace N such that X/N is isomorphic to a finite-codimensional subspace of Y; in particular, X/N must be superprojective.

Let $K \in \mathcal{L}(X, Y) \backslash \mathcal{SC}(X, Y)$; we have to show that $K \notin P\Phi_-(X, Y)$. Since K is not strictly cosingular, there exists a closed, infinite-codimensional subspace $Z \subset Y$ such that $Q_Z K$ is surjective, where Q_Z is the natural quotient operator from Y onto Y/Z; note that this means that $R(K) + Z = Y$. Considering the relative positions of the subspaces $K^{-1}(Z)$ and N inside X, three cases may happen:

(a) $K^{-1}(Z) + N$ is finite-codimensional in X, hence closed;
(b) $\overline{K^{-1}(Z) + N}$ is infinite-codimensional in X;
(c) $\overline{K^{-1}(Z) + N}$ is finite-codimensional in X but $K^{-1}(Z) + N$ is not closed.

(a) If $K^{-1}(Z) + N$ is finite-codimensional in X, hence closed, we can assume that $K^{-1}(Z) + N = X$ by passing to a larger Z if necessary. Since Y is superprojective, we may further assume that Z is complemented in Y, so there exists a projection $P\colon Y \longrightarrow Y$ with kernel $N(P) = Z$, for which $R(P) = R(PK) = PK(N)$. Also, as Z is an infinite-codimensional complemented subspace of Y and X/N is isomorphic to a finite-codimensional subspace of Y, there exists a surjection $S\colon X/N \longrightarrow Z$, so $Z = R(SQ_N) = SQ_N\big(K^{-1}(Z)\big)$. Define an operator $T\colon X \longrightarrow Y$ as $T = SQ_N - PK$; then $N \subseteq N(SQ_N)$ and $N(PK) = K^{-1}\big(N(P)\big) = K^{-1}(Z)$ so

$$\begin{aligned} R(T) &= (SQ_N - PK)\big(K^{-1}(Z) + N\big) \\ &= SQ_N\big(K^{-1}(Z)\big) + PK(N) = Z + R(P) = Y, \end{aligned}$$

hence T is surjective and $T \in \Phi_-(X,Y)$. However, $T + K = SQ_N + (I_Y - P)K$, so $R(T+K) \subseteq Z$ and $T + K \notin \Phi_-(X,Y)$, which proves that $K \notin P\Phi_-(X,Y)$.

(b) If $\overline{K^{-1}(Z) + N}$ is infinite-codimensional in X, we can assume that $N \subseteq K^{-1}(Z)$ by passing to a larger Z if necessary using Lemma 6 and, since Y is superprojective, we may further assume that Z is complemented in Y, so there exists a projection $P\colon Y \longrightarrow Y$ with kernel $N(P) = Z$. As in the previous case, $R(PK) = R(P)$, so $PK \notin \mathcal{SC}(X,Y)$. Furthermore, $N \subseteq K^{-1}(Z) = N(PK)$, so PK factors through X/N and there exists an operator $T\colon X/N \longrightarrow Y$ such that $PK = TQ_N$, where $T \notin \mathcal{SC}(X/N,Y)$ because $PK \notin \mathcal{SC}(X,Y)$ and $\mathcal{SC}$ is a surjective operator ideal [12]. As such, since $\Phi_-(X/N,Y)$ is not empty and X/N is superprojective, it follows that $T \notin P\Phi_-(X/N,Y)$ by Theorem 2 and there exists an operator $S \in \Phi_-(X/N,Y)$ such that $S + T \notin \Phi_-(X/N,Y)$, for which

$$(S+T)Q_N = SQ_N + PK \notin \Phi_-(X,Y)$$

while $SQ_N \in \Phi_-(X,Y)$, so $PK \notin P\Phi_-(X,Y)$ and $K \notin P\Phi_-(X,Y)$ by Proposition 4.

(c) If $\overline{K^{-1}(Z) + N}$ is finite-codimensional in X but $K^{-1}(Z) + N$ is not closed, there exists a compact operator $K_1\colon X \longrightarrow Y$ such that $\overline{(K+K_1)^{-1}(Z) + N}$ is infinite-codimensional in X [7, Theorem 3.7], and then it follows that $K + K_1 \notin P\Phi_-(X,Y)$ from case (b) and finally that $K \notin P\Phi_-(X,Y)$.

□

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