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# Two classes of operators related to the perturbation classes problem

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**Abstract**
Let $\mathcal{SS}$ and $\mathcal{SC}$ be the strictly singular and the strictly cosingular operators acting between Banach spaces, and let $P\Phi_+$ and $P\Phi_+$ be the perturbation classes for the upper and the lower semi-Fredholm operators. We study two classes of operators $\Phi\mathcal{S}$ and $\Phi\mathcal{C}$ that satisfy $\mathcal{SS} \subset \Phi\mathcal{S} \subset P\Phi_+$ and $\mathcal{SC} \subset \Phi\mathcal{C} \subset P\Phi_-$. We give some conditions under which these inclusions become equalities, from which we derive some positive solutions to the perturbation classes problem for semi-Fredholm operators.

**Keywords** Perturbation classes problem · Semi-Fredholm operator · Strictly singular operator

**Mathematics Subject Classification** 47A55 · 47A53

## 1 Introduction

The perturbation classes problem asks whether the perturbation classes for the upper semi-Fredholm operators $P\Phi_+$ and the lower semi-Fredholm operators $P\Phi_-$ coincide with the classes of strictly singular operators $\mathcal{SS}$ and strictly cosingular operators $\mathcal{SC}$, respectively. This problem was raised in [9] (see also [5, 19]), and it has a positive answer in some cases [11, 13–15, 21], but the general answer is negative in both cases [10], [8, Theorem 4.5]. However, it remains interesting to find positive answers in special cases because the definitions of $\mathcal{SS}$ and $\mathcal{SC}$ are intrinsic: to check that $K$ is in

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one of them only involves the action of $K$, while to check that $K$ is in $P\Phi_+$ or $P\Phi_-$ we have to study the properties of $T + K$ for $T$ in a large set of operators.

In this paper, we consider two classes $\Phi\mathcal{S}$ and $\Phi\mathcal{C}$ introduced in [2] that satisfy

$$\mathcal{SS}(X, Y) \subset \Phi\mathcal{S}(X, Y) \subset P\Phi_+(X, Y) \quad \text{and}$$
$$\mathcal{SC}(X, Y) \subset \Phi\mathcal{C}(X, Y) \subset P\Phi_-(X, Y).$$

We study conditions on the Banach spaces $X, Y$ so that some of these four inclusions become equalities, and we derive new positive answers to the perturbation classes problem for semi-Fredholm operators. When $\Phi_+(X, Y) \neq \emptyset$, we show that $\Phi\mathcal{S}(X, Y) = P\Phi_+(X, Y)$ when $Y \times Y$ is isomorphic to $Y$, $\mathcal{SS}(X, Y) = \Phi\mathcal{S}(X, Y)$ when some quotients of $X$ embed in $Y$ (Theorem 3.1), and adding up we get conditions implying that $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$ (Theorem 3.2). When $\Phi_-(X, Y) \neq \emptyset$ we prove similar results (Theorems 4.1 and 4.2). We also state some questions concerning the classes $\Phi\mathcal{S}$ and $\Phi\mathcal{C}$.

*Notation.* Along the paper, $X$, $Y$ and $Z$ denote Banach spaces and $\mathcal{L}(X, Y)$ is the set of bounded operators from $X$ into $Y$. We write $\mathcal{L}(X)$ when $X = Y$. Given a closed subspace $M$ of $X$, we denote $J_M$ the inclusion of $M$ into $X$, and $Q_M$ the quotient map from $X$ onto $X/M$. An operator $T \in \mathcal{L}(X, Y)$ is an *isomorphism* if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in X$.

The operator $T \in \mathcal{L}(X, Y)$ is *strictly singular*, and we write $T \in \mathcal{SS}$, when there is no infinite dimensional closed subspace $M$ of $X$ such that the restriction $TJ_M$ is an isomorphism; and $T$ is *strictly cosingular*, and we write $T \in \mathcal{SC}$, when there is no infinite codimensional closed subspace $N$ of $Y$ such that $Q_N T$ is surjective. Moreover, $T$ is *upper semi-Fredholm*, $T \in \Phi_+$, when the range $R(T)$ is closed and the kernel $N(T)$ is finite dimensional; $T$ is *lower semi-Fredholm*, $T \in \Phi_-$, when $R(T)$ is finite codimensional (hence closed); $T$ is *Fredholm*, $T \in \Phi$, when it is upper and lower semi-Fredholm; and $T$ is *inessential*, $T \in \mathcal{I}n$, when $I_X - ST \in \Phi$ for each $S \in \mathcal{L}(Y, X)$.

## 2 Preliminaries

The perturbation class $P\mathcal{S}$ of a class of operators $\mathcal{S}$ is defined in terms of its components:

**Definition 2.1** Let $\mathcal{S}$ denote one of the classes $\Phi_+$, $\Phi_-$ or $\Phi$. For spaces $X, Y$ such that $\mathcal{S}(X, Y) \neq \emptyset$,

$$P\mathcal{S}(X, Y) = \{K \in \mathcal{L}(X, Y) : \text{ for each } T \in \mathcal{S}(X, Y), T + K \in \mathcal{S}\}.$$

We could define $P\mathcal{S}(X, Y) = \mathcal{L}(X, Y)$ when $\mathcal{S}(X, Y)$ is empty, but this is not useful. The components of $P\Phi$ coincide with those of the operator ideal of inessential operators $\mathcal{I}n$ when they are defined [1], but given $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X, Y)$, $S$ or $T$ in $P\Phi_+$ does not imply $ST \in P\Phi_+$, and similarly for $P\Phi_-$ [10]. However, the following result holds, and it will be useful for us.

**Proposition 2.1** *Suppose that* $K \in P\Phi_+(X, Y)$, $L \in P\Phi_-(X, Y)$, $S \in \mathcal{L}(Y)$ *and* $T \in \mathcal{L}(X)$. *Then,* $SK, KT \in P\Phi_+(X, Y)$ *and* $SL, LT \in P\Phi_-(X, Y)$.

***Proof*** Suppose that $K \in P\Phi_+(X, Y)$ and let $S \in \mathcal{L}(Y)$ and $U \in \Phi_+(X, Y)$. If $S$ is bijective, then $S^{-1}U \in \Phi_+(X, Y)$, hence $U + SK = S(S^{-1}U + K) \in \Phi_+$; thus $SK \in P\Phi_+(X, Y)$. In the general case, $S = S_1 + S_2$ with $S_1, S_2$ bijective; thus, $SK = S_1K + S_2K \in P\Phi_+(X, Y)$.

The proof of the other three results is similar. □

## 3 The perturbation class for $\Phi_+$

Given two operators $S, T \in \mathcal{L}(X, Y)$, we denote by $(S, T)$ the operator from $X$ into $Y \times Y$ defined by $(S, T)x = (Sx, Tx)$, where $Y \times Y$ is endowed with the product norm $\|(y_1, y_2)\|_1 = \|y_1\| + \|y_2\|$.

Inspired by the results of Friedman [6], the authors of [2] defined the following class of operators.

**Definition 3.1** Suppose that $\Phi_+(X, Y) \neq \emptyset$ and let $K \in \mathcal{L}(X, Y)$. We say that $K$ is $\Phi$-*singular*, and write $K \in \Phi\mathcal{S}$, when for each $S \in \mathcal{L}(X, Y)$, $(S, K) \in \Phi_+$ implies $S \in \Phi_+$.

The definition of $\Phi\mathcal{S}(X, Y)$ is similar to that of $P\Phi_+(X, Y)$, but the former one is easier to handle because the action of $S$ and $K$ is decoupled when we consider $(S, K)$ instead of $S + K$.

With our notation, [6, Theorems 3 and 4] can be stated as follows:

**Proposition 3.1** [2, Proposition 2.2] *Suppose that* $\Phi_+(X, Y) \neq \emptyset$. *Then,*

$$\mathcal{SS}(X, Y) \subset \Phi\mathcal{S}(X, Y) \subset P\Phi_+(X, Y).$$

Note that $\mathcal{SS}$ is an operator ideal but $P\Phi_+$ is not; $P\Phi_+(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$, and $P\Phi_+(X)$ is an ideal of $\mathcal{L}(X)$.

**Proposition 3.2** $\Phi\mathcal{S}(X, Y)$ *is closed in* $\mathcal{L}(X, Y)$.

***Proof*** Let $\{T_n\}$ be a sequence in $\Phi\mathcal{S}(X, Y)$ converging to $T \in \mathcal{L}(X, Y)$. Suppose that $S \in \mathcal{L}(X, Y)$ and $(S, T) \in \Phi_+(X, Y \times Y)$. Note that the sequence $(S, T_n)$ converge to $(S, T)$, because $\|(S, T_n) - (S, T)\| = \|T_n - T\|$.

Since $\Phi_+(X, Y \times Y)$ is an open set, there exists a positive integer $N$ such that $(S, T_N) \in \Phi_+(X, Y \times Y)$. Then, $T_N \in \Phi\mathcal{S}(X, Y)$ implies $S \in \Phi_+(X, Y)$. Thus $T \in \Phi\mathcal{S}(X, Y)$. □

We state some basic questions on the class $\Phi\mathcal{S}$.

**Question 3.1** *Suppose that* $\Phi_+(X, Y) \neq \emptyset$.

(a) *Is* $\Phi\mathcal{S}(X, Y)$ *a subspace of* $\mathcal{L}(X, Y)$?
(b) *Is* $\Phi\mathcal{S}$ *an operator ideal*?

(c) *Is Proposition 2.1 valid for $\Phi\mathcal{S}$?*

Answering a question in [6], an example of an operator $K \in P\Phi_+ \setminus \Phi\mathcal{S}$ was given in [2, Example 2.3], but we do not know if the other inclusion can be strict.

**Question 3.2** *Suppose that $\Phi_+(X, Y) \neq \emptyset$. Is $\mathcal{SS}(X, Y) = \Phi\mathcal{S}(X, Y)$?*

A negative answer to Question 3.2 would provide a new counterexample to the perturbation classes problem for $\Phi_+$.

Let us see that the inclusions in Proposition 3.1 become equalities in some cases.

An infinite dimensional Banach space $Y$ is isomorphic to its square, denoted $Y \times Y \simeq Y$, in many cases: $L_p(\mu)$ and $\ell_p$ $(1 \leq p \leq \infty)$, $c_0$, and $C[0, 1]$. On the other hand, James' space $J$ and some spaces of continuous functions on a compact like $C[0, \omega_1]$ are not isomorphic to their square, where $\omega_1$ is the first uncountable ordinal. See [4, 20].

**Theorem 3.1** *Suppose that the spaces $X$ and $Y$ satisfy $\Phi_+(X, Y) \neq \emptyset$.*

1. *If $Y \times Y \simeq Y$, then $\Phi\mathcal{S}(X, Y) = P\Phi_+(X, Y)$.*
2. *If every infinite dimensional subspace of $X$ has an infinite dimensional subspace $N$ such that $X/N$ embeds in $Y$, then $\mathcal{SS}(X, Y) = \Phi\mathcal{S}(X, Y)$.*

***Proof*** (1) Let $U : Y \times Y \to Y$ be a bijective isomorphism and let $V, W \in \mathcal{L}(Y)$ such that $U(y_1, y_2) = Vy_1 + Wy_2$. If $K \in P\Phi_+(X, Y)$, for each $S \in \mathcal{L}(X, Y)$ such that $(S, K) \in \Phi_+$ we have $U(S, K) = VS + WK \in \Phi_+$. By Proposition 2.1, $WK \in P\Phi_+(X, Y)$. Then, $VS \in \Phi_+$, hence $S \in \Phi_+$. Thus we conclude that $K \in \Phi\mathcal{S}(X, Y)$.

(2) Let $K \in \mathcal{L}(X, Y)$, $K \notin \mathcal{SS}$. By the hypothesis, there exists an infinite dimensional subspace $N$ of $X$ such that $KJ_N$ is an isomorphism, and there is an isomorphism $U : X/N \to Y$. Then, $S = UQ_N \in \mathcal{L}(X, Y)$ is not upper semi-Fredholm. We will prove that $K \notin \Phi\mathcal{S}$ by showing that $(S, K) \in \Phi_+$.

Recall that $\|Q_N x\| = \text{dist}(x, N)$. We can choose the isomorphism $U$ so that $\|Sx\| = \|UQ_N x\| \geq \text{dist}(x, N)$ for each $x \in X$. Moreover, there is a constant $c > 0$ such that $\|Kn\| \geq c\|n\|$ for each $n \in N$.

Let $x \in X$ with $\|x\| = 1$, and let $0 < \alpha < 1$ such that $c(1 - \alpha) = 2\|K\|\alpha$.

If $\text{dist}(x, N) \geq \alpha$, then $\|Sx\| \geq \alpha$. Otherwise, there exists $y \in N$ such that $\|x - y\| < \alpha$; hence $\|y\| > 1 - \alpha$. Therefore,

$$\|Kx\| \geq \|Ky\| - \|K(x - y)\| \geq c(1 - \alpha) - \|K\|\alpha = \|K\|\alpha.$$

Then, $\|(S, K)x\|_1 = \|Sx\| + \|Kx\| \geq \min\{\|K\|\alpha, \alpha\}$, hence $(S, K)$ is an isomorphism; in particular $(S, K) \in \Phi_+$, as we wanted to show. □

In the known examples in which $\mathcal{SS}(X, Y) \neq P\Phi_+(X, Y)$ in [8, 10], the space $Y$ has a complemented subspace which is hereditarily indecomposable in the sense of [3, 16, 17]. Therefore, the question arises.

**Question 3.3** *Suppose that $X$ and $Y$ satisfy $\Phi_+(X, Y) \neq \emptyset$ and $Y \times Y \simeq Y$.*
*Is $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$?*

A Banach space $X$ is *subprojective* if every closed infinite dimensional subspace of $X$ contains an infinite dimensional subspace complemented in $X$. The spaces $c_0$, $\ell_p$ $(1 \leq p < \infty)$ and $L_q(\mu)$ $(2 \leq q < \infty)$ are subprojective [22]. See [7, 18] for further examples.

**Corollary 3.1** *Suppose that $\Phi_+(X, Y) \neq \emptyset$ and the space $X$ is subprojective. Then, $\mathcal{SS}(X, Y) = \Phi\mathcal{S}(X, Y)$.*

***Proof*** Every closed infinite dimensional subspace of $X$ contains an infinite dimensional subspace $N$ complemented subspace in $X$; thus $X/N$ is isomorphic to the complement of $N$. Since $\Phi_+(X, Y) \neq \emptyset$, the quotient $X/N$ is isomorphic to a subspace of $Y$ and we can apply Theorem 3.1. □

The next result is a refinement of Theorem 3.1 that is proved using the previous arguments.

**Theorem 3.2** *Suppose that $\Phi_+(X, Y) \neq \emptyset$, $Y \times Y$ embeds in $Y$ and every infinite dimensional subspace of $X$ has an infinite dimensional subspace $N$ such that $X/N$ embeds in $Y$. Then, $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$.*

***Proof*** Since $Y \times Y$ embeds in $Y$, there exist isomorphisms $V, W \in \mathcal{L}(Y)$ such that $R(V) \cap R(W) = \{0\}$ and $R(V) + R(W)$ is closed. Hence, there exists $r > 0$ such that $\|y_1 + y_2\| \geq r(\|y_1\| + \|y_2\|)$ for $y_1 \in R(V)$ and $y_2 \in R(W)$, and clearly we can choose $V, W$ so that $r = 1$.

Let $K \in \mathcal{L}(X, Y)$ with $K \notin \mathcal{SS}$. Select an infinite dimensional subspace $M$ of $X$ such that $KJ_M$ is an isomorphism, and let $N$ be an infinite dimensional subspace of $M$ such that there exists an isomorphism $U : X/N \to Y$. We can assume that $\|Uz\| \geq z$ for each $z \in X/N$.

The operator $S = VUQ_N \notin \Phi_+$, and proceeding like in the proof of (2) in Theorem 3.1, we can show that $S + WK \in \Phi_+$. Then, $WK \notin P\Phi_+$, hence $K \notin P\Phi_+$, by Proposition 2.1. □

**Corollary 3.2** *If $X$ is separable, $Y \times Y$ embeds in $Y$, and $Y$ contains a copy of $C[0, 1]$, then $\mathcal{SS}(X, Y) = \Phi_+(X, Y)$.*

***Proof*** It is well known that the space $C[0, 1]$ contains a copy of each separable Banach space. □

The class $\Phi\mathcal{S}$ is injective in the following sense:

**Proposition 3.3** *Given an operator $K \in \mathcal{L}(X, Y)$ and an (into) isomorphism $L \in \mathcal{L}(Y, Y_0)$, if $LK \in \Phi\mathcal{S}(X, Y_0)$, then $K \in \Phi\mathcal{S}(X, Y)$.*

***Proof*** Let $K \in \mathcal{L}(X, Y)$ and let $L \in \mathcal{L}(X, Y_0)$ be an isomorphism into $Y_0$ such that $LK \in \Phi\mathcal{S}(X, Y_0)$. Take $S \in \mathcal{L}(X, Y)$ and suppose that $(S, K) \in \Phi_+(X, Y \times Y)$. Then, $(LS, LK) = (L \times L)(S, K) \in \Phi_+(X, Y_0 \times Y_0)$, where $(L \times L) \in \mathcal{L}(X \times X, Y_0 \times Y_0)$ is defined by $(L \times L)(x_1, x_2) = (Lx_1, Lx_2)$.

Since $LK \in \Phi\mathcal{S}(X, Y_0)$ we obtain $LS \in \Phi_+(X, Y_0)$. Therefore $S \in \Phi_+(X, Y)$, hence $K \in \Phi\mathcal{S}(X, Y)$. □

## 4 The perturbation class for $\Phi_-$

Given two operators $S, T \in \mathcal{L}(X, Y)$, we denote by $[S, T]$ the operator from $X \times X$ into $Y$ defined by $[S, T](x_1, x_2) = Sx_1 + Tx_2$, where $X \times X$ is endowed with the maximum norm $\|(x_1, x_2)\|_\infty = \max\{\|y_1\|, \|y_2\|\}$.

**Definition 4.1** Suppose that $\Phi_-(X, Y) \neq \emptyset$ and let $K \in \mathcal{L}(X, Y)$. We say that $K$ is $\Phi$-*cosingular*, and write $K \in \Phi\mathcal{C}$, when for each $S \in \mathcal{L}(X, Y)$, $[S, K] \in \Phi_-$ implies $S \in \Phi_-$.

Like in the case of $\Phi\mathcal{S}$, the definition of $\Phi\mathcal{C}(X, Y)$ is similar to that of $P\Phi_-(X, Y)$, but the former one is easier to handle because the action of $S$ and $K$ is decoupled when we consider $[S, K]$ instead of $S + K$.

**Proposition 4.1** [2, Proposition 2.5] *Suppose that $\Phi_-(X, Y) \neq \emptyset$. Then,*

$$\mathcal{SC}(X, Y) \subset \Phi\mathcal{C}(X, Y) \subset P\Phi_-(X, Y).$$

Note that $\mathcal{SC}$ is an operator ideal but $P\Phi_-$ is not; $P\Phi_-(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$, and $P\Phi_-(X)$ is an ideal of $\mathcal{L}(X)$.

**Proposition 4.2** *$\Phi\mathcal{C}(X, Y)$ is closed in $\mathcal{L}(X, Y)$.*

***Proof*** Let $\{T_n\}$ be a sequence in $\Phi\mathcal{C}(X, Y)$ converging to $T \in \mathcal{L}(X, Y)$. Suppose that $S \in \mathcal{L}(X, Y)$ and $[S, T] \in \Phi_-(X \times X, Y)$. Note that the sequence $[S, T_n]$ converge to $[S, T]$.

Since $\Phi_-(X \times X, Y)$ is an open set there exists a positive integer $N$ such that $[S, T_N] \in \Phi_-(X \times X, Y)$. Hence, $T_N \in \Phi\mathcal{C}(X, Y)$ implies $S \in \Phi_-(X, Y)$. □

**Question 4.1** *Suppose that $\Phi_-(X, Y) \neq \emptyset$.*

(a) *Is $\Phi\mathcal{C}(X, Y)$ a subspace of $\mathcal{L}(X, Y)$?*
(b) *Is $\Phi\mathcal{C}$ an operator ideal?*
(c) *Is Proposition 2.1 valid for $\Phi\mathcal{C}$?*

Answering a question in [6], an example of an operator $K \in P\Phi_- \setminus \Phi\mathcal{C}$ was given in [2], but we do not know if the other inclusion can be strict.

**Question 4.2** *Suppose that $\Phi_-(X, Y) \neq \emptyset$. Is $\mathcal{SC}(X, Y) = \Phi\mathcal{C}(X, Y)$?*

A negative answer to Question 4.2 would provide a new counterexample to the perturbation classes problem for $\Phi_-$.

Next we will show that the inclusions in Proposition 4.1 become equalities in some cases.

**Theorem 4.1** *Suppose that the spaces $X$ and $Y$ satisfy $\Phi_-(X, Y) \neq \emptyset$.*

1. *If $X \times X \simeq X$, then $\Phi\mathcal{C}(X, Y) = P\Phi_-(X, Y)$.*
2. *If every infinite codimensional closed subspace of $Y$ is contained in an infinite codimensional closed subspace $N$ which is isomorphic to a quotient of $X$, then $\Phi\mathcal{C}(X, Y) = \mathcal{SC}(X, Y)$.*

***Proof*** (1) Let $U : X \to X \times X$ be a bijective isomorphism and let $U_1, U_2 \in \mathcal{L}(X)$ such that $U(x) = (U_1x, U_2x)$. If $K \in P\Phi_-(X, Y)$, for each $S \in \mathcal{L}(X, Y)$ such that $[S, K] \in \Phi_-(X \times X, Y)$ we have $[S, K]U = SU_1 + KU_2 \in \Phi_-(X, Y \times Y)$. By Proposition 2.1, $KU_2 \in P\Phi_-(X, Y)$. Then, $SU_1 \in \Phi_-(X, Y)$, hence $S \in \Phi_-(X, Y)$. Thus we conclude that $K \in \Phi\mathcal{C}(X, Y)$.

(2) Let $K \in \mathcal{L}(X, Y)$ $K \notin \mathcal{SC}$. Then, there exists an infinite codimensional closed subspace $M$ of $Y$ such that $Q_N K$ is surjective. By the hypothesis, there exist an infinite codimensional closed subspace $N$ such that $M \subset N$ and a surjective operator $V \in \mathcal{L}(X, M)$.

Observe that $S = J_N V \in \mathcal{L}(X, Y)$ is not in $\Phi_-$. We prove that $K \notin \Phi\mathcal{C}$ by showing that $[S, K] \in \Phi_-(X \times X, Y)$.

Indeed, note that $R(S) = N$. Moreover $Q_N K$ surjective implies $R(K) + N = Y$. Since $R([S, K]) = R(S) + R(K)$, $[S, K]$ is surjective. □

In the known examples in which $\mathcal{SC}(X, Y) \neq P\Phi_-(X, Y)$ in [8, 10], the space $X$ has a complemented subspace which is hereditarily indecomposable. Therefore, the question arises.

**Question 4.3** *Suppose that* $\Phi_-(X, Y) \neq \emptyset$ *and* $X \times X \simeq X$.
*Is* $\mathcal{SC}(X, Y) = P\Phi_-(X, Y)$?

A Banach space $X$ is *superprojective* if each of its infinite codimensional closed subspaces is contained in some complemented infinite codimensional subspace. The spaces $c_0$, $\ell_p$ $(1 < p < \infty)$ and $L_q(\mu)$ $(1 < q \leq 2)$ are superprojective. See [7, 12] for further examples.

**Corollary 4.1** *Suppose that* $\Phi_-(X, Y) \neq \emptyset$ *and the space* $Y$ *is superprojective. Then,* $\Phi\mathcal{C}(X, Y) = \mathcal{SC}(X, Y)$.

***Proof*** Every closed infinite codimensional subspace of $Y$ is contained in an infinite codimensional complemented subspace $N$. Since $\Phi_-(X, Y) \neq \emptyset$, there exists $T \in \mathcal{L}(X, Y)$ with $R(T) \supset N$, and composing with the projection $P$ on $Y$ onto $N$ we get $R(PT) = N$, and we can apply Theorem 4.1. □

The following result is a refinement of the previous results in this section.

**Theorem 4.2** *Suppose that* $\Phi_-(X, Y) \neq \emptyset$, *there exists a surjection from* $X \times X$ *onto* $X$, *and every closed infinite codimensional subspace of* $Y$ *is contained in a closed infinite codimensional subspace* $N$ *which is isomorphic to a quotient of* $X$. *Then,* $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$.

***Proof*** $K \in \mathcal{L}(X, Y)$, $K \notin \mathcal{SC}$. Then, there exists an infinite codimensional subspace $N$ of $Y$ such that $Q_N K$ is surjective. By hypothesis, we can assume that there exists a surjective operator $U : X \to N$. Then, $S = J_N U \in \mathcal{L}(X, Y)$ is not in $\Phi_-$. Moreover, $[S, K]$ is surjective: $R([S, K]) = R(S) + R(K) = N + R(K) = Y$, hence $[S, K] \in \Phi_-(X \times X, Y)$.

Let $V : X \to X \times X$ be a surjection and let $V_1, V_2 \in \mathcal{L}(X)$ such that $V(x) = (V_1x, V_2x)$ for each $x \in X$. Then, $[S, K]V = SV_1 + KV_2 \in \Phi_-(X, Y)$. Since $SV_1 \notin \Phi_-$, we get $KV_2 \notin P\Phi_-$; hence $K \notin P\Phi_-$ by Proposition 2.1. □

**Corollary 4.2** *If $Y$ is separable, there exists a surjection from $X \times X$ onto $X$, and $X$ has a quotient isomorphic to $\ell_1$, then $\mathcal{SC}(X, Y) = \Phi_-(X, Y)$.*

***Proof*** It is well known that every separable Banach space is isomorphic to a quotient of $\ell_1$. □

The class $\Phi\mathcal{C}(X, Y)$ is surjective in the following sense:

**Proposition 4.3** *Given $K \in \mathcal{L}(X, Y)$ and a surjective operator $Q \in \mathcal{L}(Z, X)$, if $KQ \in \Phi\mathcal{C}(Z, Y)$, then $K \in \Phi\mathcal{C}(X, Y)$.*

***Proof*** Let $S \in \mathcal{L}(X, Y)$ such that $[S, K] \in \Phi_-(X \times X, Y)$, and let $Q : Z \to X$ be a surjective operator. Then, the operator $Q \times Q \in \mathcal{L}(Z \times Z, X \times X)$ defined by $(Q \times Q)(a, b) = (Qa, Qb)$ is surjective. Thus, $[S, K](Q \times Q) = [SQ, KQ]$ is in $\Phi_-(Z \times Z, Y)$. Since $KQ \in \Phi\mathcal{C}(Z, Y)$, we obtain $SQ \in \Phi_-(Z, Y)$, hence $S \in \Phi_-(X, Y)$, and we conclude $K \in \Phi\mathcal{C}(X, Y)$. □

The dual space $(X \times X, \|\cdot\|_\infty)^*$ can be identified with $(X^* \times X^*, \|\cdot\|_1)$ in the obvious way. Hence, the conjugate operator $[S, T]^*$ can be identified with $(S^*, T^*)$. Indeed, for $x^* \in X^*$ and $x \in X$, we have

$$\begin{aligned}\langle [S, T]^* x^*, x\rangle &= \langle x^*, [S, T]x\rangle = \langle x^*, Sx + Tx\rangle \\ &= \langle S^* x^* + T^* x^*, x\rangle = \langle (S^*, T^*)x^*, x\rangle.\end{aligned}$$

As a consequence, $[S, T] \in \Phi_-$ if and only if $(S^*, T^*) \in \Phi_+$. Similarly, $(S, T)^*$ can be identified with $[S^*, T^*]$.

The following result describes the behavior of the classes of $\Phi$-singular and $\Phi$-cosingular operators under duality.

**Proposition 4.4** *Let $K \in \mathcal{L}(X, Y)$.*

1. *If $K^* \in \Phi\mathcal{S}(Y^*, X^*)$, then $K \in \Phi\mathcal{C}(X, Y)$.*
2. *If $K^* \in \Phi\mathcal{C}(Y^*, X^*)$, then $K \in \Phi\mathcal{S}(X, Y)$.*

***Proof*** (1) Let $S \in \mathcal{L}(X, Y)$ such that $[S, K] \in \Phi_-(X \times X, Y)$. Then, $[S, K]^* \in \Phi_+$. Since $[S, K]^* \equiv (S^*, K^*)$, we have $(S^*, K^*) \in \Phi_+(Y^*, X^* \times X^*)$, and from $K^* \in \Phi\mathcal{S}(Y^*, X^*)$ we obtain $S^* \in \Phi_+(Y^*, X^*)$; therefore, $S \in \Phi_-(X, Y)$, and hence $K \in \Phi\mathcal{C}(X, Y)$.

The proof of (2) is similar. □

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## Declarations

**Conflict of interest** The authors have no competing interests to declare that are relevant to the content of this article.

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