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ORIGINAL PAPER

On symplectic Banach spaces

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Abstract
We extend and generalize the result of Kalton and Swanson (Z_2 is a symplectic Banach space with no Lagrangian subspace) by showing that all higher order Rochgberg spaces $\mathfrak{R}^{(n)}$ are symplectic Banach spaces with no Lagrangian subspaces. The nontrivial symplectic structure on Rochberg spaces of even order is the one induced by the natural duality; while the nontrivial symplectic structure on Rochberg spaces of odd order requires perturbation with a complex structure. We will also study symplectic structures on general Banach spaces and, motivated by the unexpected appearance of complex structures, we introduce and study almost symplectic structures.

Keywords Symplectic Banach space · Symplectic operator · Rochberg spaces · Kalton–Peck space · Hilbert space

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1 Introduction

A real Banach space X is said to be *symplectic* if there is a continuous alternating bilinear map $\omega : X \times X \to \mathbb{R}$ such that the induced map $L_\omega : X \to X^*$ given by $L_\omega(x)(y) = \omega(x, y)$ is an isomorphism onto. A symplectic Banach space is necessarily isomorphic to its dual and reflexive (see Lemma 2.2). During the decade of 1970 several authors drew the attention to the importance of the study of symplectic forms on Banach spaces and, more broadly, on Banach manifolds. For instance, in the proof of Weinstein [28] of an infinite dimensional

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version of the classical Darboux theorem for symplectic geometry, or in the Hamiltonian formulation of infinite dimensional mechanics due to Chernoff and Marsden [14]. See also Swanson [25, 26] for various results about symplectic structures on Banach spaces.

A motivation for this work has been the negative solution given by Kalton and Swanson [20] to the question raised by Weinstein [28] of whether every infinite dimensional symplectic Banach space is trivial. A symplectic Banach space (X, ω) is said *trivial* if there exists a reflexive Banach space Y and an isomorphism $T : X \to Y \oplus Y^*$ such that $\omega(x, y) = \Omega_Y(Tx, Ty)$ for every $x, y \in X$, where

$$\Omega_Y[(z, z^*), (w, w^*)] = w^*(z) - z^*(w).$$

In this case, $T^{-1}(Y \times \{0\})$ is, according to Definition 2.6, a *Lagrangian* subspace of (X, ω). Kalton and Swanson show that the celebrated Kalton–Peck space Z_2 (see [19]) is a symplectic space with no Lagrangian subspaces. In this paper we will consider the sequence of higher order Rochberg spaces $\mathfrak{R}^{(n)}$ [23] obtained from the scale of ℓ_p spaces, which can be considered as generalizations of both ℓ_2 and Z_2 since $\mathfrak{R}^{(1)} = \ell_2$ and $\mathfrak{R}^{(2)} = Z_2$. We will show that all these spaces $\mathfrak{R}^{(n)}$ are symplectic and contain no (infinite dimensional) Lagrangian subspaces; in other words, they admit a nontrivial symplectic structure. A remarkable point is that while the nontrivial symplectic structure on even spaces $\mathfrak{R}^{(2n)}$ is the one induced by the natural duality; the nontrivial symplectic structure on the odd spaces $\mathfrak{R}^{(2n+1)}$ requires to modify the natural duality structure with a complex structure.

A second motivation for this work is to clarify the connection between symplectic and complex structures on Banach spaces. Recall that given a real Banach space X, a linear and bounded operator $J : X \to X$ is called a *complex structure* on X if $J^2 = -Id$. In this case, the operator J induces a $\mathbb{C}$-linear structure on X in the form $ix = J(x)$. In the Hilbert space setting there is a correspondence between symplectic structures and complex structures since Weinstein [28, Prop. 5.1] proved that every symplectic structure on a Hilbert space $\mathcal{H}$ has the form $\omega(x, y) = \langle J(x), y\rangle$, for some complex structure J and an equivalent inner product on $\mathcal{H}$. The argument of Weinstein actually shows that every symplectic structure on a real Hilbert space is trivial. This correspondence is no longer valid in general (say, non-reflexive spaces may admit complex structures), but still some properties of complex structures can be studied in the context of symplectic structures. Section 8 is devoted to the study of perturbations of symplectic structures by strictly singular operators and extensions of symplectic structures on hyperplanes following the techniques of Ferenczi [15] and Ferenczi and Galego [16] about complex structures. We also prove an analogous result for symplectic structures to those of [9] for complex structures: no symplectic structure on ℓ_2 can be extended to a bilinear form on a hyperplane H of Z_2 containing it.

2 Background

Definition 2.1 Given a real Banach space X, a *symplectic form* on X is a bilinear map $\omega : X \times X \to \mathbb{R}$ satisfying the following conditions:

(1) ω is continuous: there exists $K > 0$ such that $|\omega(x, y)| \le K\|x\|\|y\|$ for every $x, y \in X$.
(2) ω is alternating: $\omega(x, y) = -\omega(y, x)$ for all $x, y \in X$.
(3) The induced map $L_\omega : X \to X^*$ given by $L_\omega(x)(y) = \omega(x, y)$ is an isomorphism of X onto X^*.

In this case, the pair (X, ω), where X is a Banach space and ω is a symplectic form on X, is called a symplectic Banach space. We say that X is *symplectic* if there exists a symplectic form ω on X.

The following result is from the pioneering work of Kalton and Swanson [20]

Lemma 2.2 *A continuous alternating bilinear map ω on a real Banach space X is symplectic if and only if X is reflexive and $L_\omega : X \to X^*$ is an isomorphism into.*

We include the proof for the sake of completeness:

Proof If (X, ω) is symplectic, then $L_\omega^* : X^{**} \to X^*$ is also an isomorphism onto and if $x \in X \subseteq X^{**}$ then $L_\omega^*(x) = -L_\omega(x)$. So $X = X^{**}$ in the canonical embedding. Assume that X is reflexive and that $L_\omega : X \to X^*$ is an isomorphism into. Suppose that there exists $f \in X^* \setminus \mathrm{Im}(L_\omega)$. By the Hahn-Banach theorem in combination with reflexivity there exists $y \in X$ such that $L_\omega(x)(y) = 0$ for every $x \in X$ and $f(y) \neq 0$. Then $\omega(x, y) = 0$ for every $x \in X$ and so $L_\omega(y) = 0$ which contradicts that L_ω is injective. □

Thus, a real Banach space X is symplectic if and only if X is reflexive and there exists $\alpha : X \to X^*$ isomorphism into such that $\alpha^* = -\alpha$ (where $\alpha^* : X^{**} \to X^*$ is the adjoint of α with the canonical identification). In this case $\omega(x, y) = \alpha(x)(y)$ is a symplectic form on X. These results justify the following definition.

Definition 2.3 Let X be a real reflexive Banach space. An isomorphism $\alpha : X \to X^*$ is said to be a *symplectic isomorphism* if $\alpha^* = -\alpha$.

The basic examples of symplectic Banach spaces known so far are:

- Finite dimensional symplectic spaces are even dimensional.
- *Standard structure:* If X is a real reflexive Banach space and
$$\omega((e, e^*), (f, f^*)) = f^*(e) - e^*(f)$$
then $(X \oplus X^*, \omega)$ is a symplectic space.
- Infinite dimensional Hilbert spaces are symplectic.
- The Kalton–Peck twisted Hilbert spaces $\ell_2(\phi)$ introduced in [19] are symplectic (it follows from the proof of [19, Theorem 5.1] that $\ell_2(\phi)^*$ is isomorphic to $\ell_2(-\phi)$ and the definition of the isomorphism T on [19, Page 18]).

Only symplectic structures on Hilbert spaces admit a simple description.

Lemma 2.4 *Let $(\mathcal{H}, \langle\, ,\, \rangle)$ be a real Hilbert space and ω be a continuous bilinear form on $\mathcal{H}$. Then ω is a symplectic form on $\mathcal{H}$ if and only if there exists an isomorphism $\alpha : \mathcal{H} \to \mathcal{H}$ with Hilbert-adjoint $\alpha^* = -\alpha$ such that $\omega(x, y) = \langle \alpha(x), y \rangle$ for all $x, y \in \mathcal{H}$.*

We pass to define the equality notion for symplectic forms.

Definition 2.5 Two symplectic Banach spaces (X_1, ω_1) and (X_2, ω_2) are equivalent if there is an onto isomorphism $T : X_1 \to X_2$ such that $\omega_2(Tx, Ty) = \omega_1(x, y)$. If ω_1, ω_2 are two symplectic structures on a real Banach space X we will write $\omega_1 \sim \omega_2$ to denote they are equivalent.

Hence the symplectic spaces (X_1, ω_1) and (X_2, ω_2) are equivalent if and only if there exists an isomorphism $T : X_1 \to X_2$ such that $L_{\omega_1} = T^* L_{\omega_2} T$, where L_{ω_i} is the respective isomorphism of X_i onto X_i^*, $i = 0, 1$.

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad T \quad} & X_2 \\ {\scriptstyle L_{\omega_1}}\big\downarrow & & \big\downarrow{\scriptstyle L_{\omega_2}} \\ X_1^* & \xleftarrow[\quad T^* \quad]{} & X_2^* \end{array}$$

The following notions will also be useful:

Definition 2.6 Let (X, ω) be a symplectic Banach space and let F be a closed subspace of X.

- The symplectic orthogonal (or symplectic annihilator) of F is the linear subspace $F^\omega = \{x \in X : \omega(x, y) = 0 \text{ for all } y \in F\}$.
- F is symplectic if $F \neq \{0\}$ and $(F, \omega|_{F\times F})$ is a symplectic Banach space.
- F is isotropic if $\omega(x, y) = 0$ for every $x, y \in F$, i.e., $F \subseteq F^\omega$.
- F is Lagrangian if it is isotropic and if X can be written as a topological direct sum $X = F \oplus G$ for an isotropic subspace G of X.
- The symplectic structure (X, ω) is *trivial* if X contains a Lagrangian subspace.

Observe that for a closed subspace $F = (F^\omega)^\omega$. By Zorn's lemma every symplectic structure admits a maximal isotropic subspace. Also, if (X, ω) is trivial and $X = F \oplus G$, for closed isotropic subspaces F and G of X, then G is isomorphic to F^* and the symplectic space (X, ω) is equivalent to the standard structure on $F \oplus F^*$ (see [17, page 99]).

Lemma 2.7 *Let (X, ω) be a real symplectic space. A closed subspace F of X is symplectic if and only if $X = F \oplus F^\omega$.*

Proof Denote $L_F = L_{\omega|_{F\times F}}$, then L_F is injective if and only if $F \cap F^\omega = \{0\}$. Suppose that F is symplectic. For every $x \in X$ there exists $f \in F$ such that $L_F(f) = L_\omega(x)_{|F}$. It follows that $x - f \in F^\omega$ and hence $X = F \oplus F^\omega$. Conversely, assume that $X = F \oplus F^\omega$. Given $\phi \in F^*$, let $x \in X$ such that $\phi = L_\omega(x)_{|F}$. If we write $x = f + g$ with $f \in F$ and $g \in F^\omega$, then $L_F(f) = \phi$. We conclude that L_F is an isomorphism and hence F is symplectic. □

From here it immediately follows:

Corollary 2.8 *Let (X, ω) be a real symplectic space. A closed subspace F of X is symplectic if and only if F^ω is symplectic. If a closed finite codimensional subspace F of X is symplectic then the dimension of X/F is even.*

As an immediate consequence, no real symplectic structure extends from an hyperplane to the whole space. Moreover

Lemma 2.9 *Let (X, ω) be a real symplectic space and let H be a closed hyperplane of X. Then $H^\omega \subseteq H$ and* $\dim(H^\omega) = 1$. *Moreover, there exists a closed symplectic subspace $H' \subseteq H$ with* $\dim(H/H') = 1$.

Proof Let $g \in X^*$ with $H = \ker g$. Since X is symplectic, there exists $x_0 \neq 0$ in X such that $L_\omega(x_0) = g$. For every $h \in H$, we have $\omega(x_0, h) = L_\omega(x_0)(h) = g(h) = 0$. Hence $x_0 \in H^\omega$.

On the other hand, $g(x_0) = \omega(x_0, x_0) = 0$, and therefore $x_0 \in H$. Now for every $x \in H^\omega$, we have $\ker g \subseteq \ker L_\omega(x)$. It follows that $L_\omega(x) = \lambda L_\omega(x_0)$ for some constant $\lambda \in \mathbb{R}$. We conclude that $x = \lambda x_0$ and then $\dim(H^\omega) = 1$. For the moreover part, let $X = H \oplus \mathrm{span}\{x_1\}$. The argument above implies that there exists a closed hyperplane H_1 of X containing x_1 such that $H = \{x_1\}^\omega$. Consider $H' = H \cap H_1$. It follows that $X = H' \oplus \mathrm{span}\{x_0, x_1\}$ and since $\omega(x_0, x_1) \neq 0$ we have that $\{x_0, x_1\}^\omega = H'$. We conclude that H' is symplectic by using Lemma 2.7. □

3 About the symplectic structure of Z_2

The only known "nontrivial" symplectic space is the Kalton–Peck space Z_2 [19]. The space Z_2 can be presented either in its natural quasinorm as a twisted Hilbert space, in its natural quasinorm as a derived space or in its natural norm as a Rochberg space. And there are other possibilites. All those norms and quasinorms are of course equivalent. We define Z_2 as the space $\ell_2 \oplus_{KP} \ell_2 = \{(\omega, x) \in \ell_\infty \times \ell_2 : \omega - KPx \in \ell_2\}$ where KP is the Kalton–Peck map $KPx(n) = x(n) \log \frac{|x(n)|}{\|x\|_2}$, for every $n \in \mathbb{N}$ (with the convention $0 = 0 \log 0$). The space Z_2 endowed with the quasinorm $\|(\omega, x)\| = \|\omega - KPx\|_2 + \|x\|_2$ is a quasi-Banach space, and this quasinorm is equivalent to a norm [17]. The space Z_2 is a twisted Hilbert space in the sense that there is an exact sequence

$$0 \longrightarrow \ell_2 \xrightarrow{\jmath} Z_2 \xrightarrow{Q} \ell_2 \longrightarrow 0$$

with inclusion $\jmath(y) = (y, 0)$ and quotient map $Q(\omega, x) = x$. The space Z_2 is isomorphic to its dual (see also [4] for a detailed study of this question) but is not isomorphic to a Hilbert space. Z_2 is a superreflexive Banach space (as any twisted Hilbert space) with a basis $(u_n)_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ by $u_{2n-1} = (e_n, 0)$ and $u_{2n} = (0, e_n)$, where $(e_n)_n$ is the canonical basis of ℓ_2 [19, Th. 4.10]. Z_2 does not admit unconditional basis [21]. It however has an Unconditional Finite Dimensional Decomposition into the 2-dimensional subspaces $X_n = \mathrm{span}\{(e_n, 0), (0, e_n)\}$. An operator $T \in \mathfrak{L}(Z_2)$ is either strictly singular (that is, not an isomorphism on any infinite dimensional subspace of Z_2) or an isomorphism on a complemented copy of Z_2. An operator $T \in \mathfrak{L}(Z_2)$ is strictly singular if and only if $T\jmath$ is strictly singular [19].

The most remarkable fact for our purposes is that Z_2 is isomorphic to its dual. It follows from [5, Proposition 5.5] that Z_2 in its Rochberg norm is isometric to its dual. Also the spaces $\ell_2 \oplus_{KP} \ell_2$ and $\ell_2 \oplus_{-KP} \ell_2$ are isometric via any of the maps $(x, y) \to (-x, y)$ or $(x, y) \to (x, -y)$, and in duality via the bilinear form [17]

$$\langle (x, y), (x', y') \rangle = \langle x, y' \rangle_{\ell_2} + \langle y, x' \rangle_{\ell_2}.$$

More precisely, the bilinear antisymmetric form $\lhd \cdot, \cdot \rhd : Z_2 \times Z_2 \to \mathbb{R}$ given by

$$\lhd (x, y), (x', y') \rhd = \langle x, y' \rangle_{\ell_2} - \langle y, x' \rangle_{\ell_2}.$$

is such that $D : Z_2 \to Z_2^*$ given by $D(a)[b] = \lhd a, b \rhd$ is an isomorphism. Accordingly, Z_2 is isomorphic to Z_2^* and

Proposition 3.1 *Z_2 is a symplectic Banach space.*

Any reflexive Banach space X that is isomorphic to its dual and to its square admits a renorming $(X, |\cdot|)$ such that $(X, |\cdot|)$ and $(X^*, |\cdot|^*)$ are isometric: if $\tau : X \to X \oplus_2 X^*$ is an isomorphism, set $|x| = \|\tau x\|$. The same occurs to Z_2.

Definition 3.2 The symplectic adjoint $T^+ : Z_2 \to Z_2$ of an operator $T \in \mathfrak{L}(Z_2)$ is defined assigning to each $y \in Z_2$ the only vector $T^+ y$ such that, for all $x \in Z_2$,

$$\lhd\ T^+ y, x \rhd = \lhd\ y, Tx \rhd . \tag{1}$$

Indeed, T^+ exists since the map $x \to \lhd\ y, Tx \rhd$ defines a continuous functional on Z_2. By Proposition 3.1, there exists an unique $y' \in Z_2$ so that $\lhd\ y', x \rhd = \lhd\ y, Tx \rhd$ for all $x \in Z_2$. This defines a linear involution $+ : \mathcal{B}(Z_2) \to \mathcal{B}(Z_2)$ such that T^+ identifies with the usual dual map T^*. The map T^+ is bounded whenever T is bounded since $\frac{1}{3}\|T\| \leq \|T^+\| \leq 3\|T\|$, which can be proved using that $\|D\| \leq 3$ and $\|D^{-1}\| \leq 1$ (see [4]). Moreover, there is a commutative diagram

$$\begin{array}{ccc} Z_2 & \xrightarrow{\ D\ } & Z_2^* \\ {\scriptstyle T^+}\downarrow & & \downarrow{\scriptstyle T^*} \\ Z_2 & \xrightarrow{\ D\ } & Z_2^* \end{array} \tag{2}$$

where $D : Z_2 \to Z_2^*$ is the isomorphism given in Proposition 3.1. Indeed, given $x \in Z_2$, one has $DT^+x = \lhd\ T^+x, \cdot \rhd \in Z_2^*$ and the other way around gives $T^*Dx = T^*\big(\lhd\ x, \cdot\ \rhd\big) = \lhd\ x, T(\cdot) \rhd \in Z_2^*$, and both functionals coincide by (3.2). It follows that $T^+ = D^{-1}T^*D$. This duality was fully exploited by Kalton [17] and Kalton and Swanson [20].

3.1 Matrix representation for T^+

Bounded operators $T : X \to X$ defined on reflexive Banach spaces with basis $(e_i)_{i\in\mathbb{N}}$ admit a matrix representation (a_{ij}) in the sense that $T(e_i) = \sum_{j=1}^{\infty} a_{ij}e_j$ in terms of such basis. Indeed, the canonical duality between X and X^* given by $\langle e_i, e_j^* \rangle = \delta_{ij}$ yields $a_{ij} = \langle T(e_i), e_j^* \rangle$. Taking into account the identities $\langle T^*(e_i^*), e_j^{**} \rangle = \langle T^*(e_i^*), e_j \rangle = T^*(e_i^*)(e_j) = e_i^*(Te_j) = \langle e_i^*, T(e_j) \rangle = \langle T(e_j), e_i^* \rangle$ it is then clear that the matrix representation of T^* is just the transpose of that of T.

The symplectic form $\lhd \cdot, \cdot \rhd$ of Z_2 described above defines the matrix $(\lhd\ u_i, u_j \rhd)_{ij}$, namely

$$D = \begin{pmatrix} \lhd\ u_1, u_1 \rhd & \lhd\ u_1, u_2 \rhd & \lhd\ u_1, u_3 \rhd & \dots \\ \lhd\ u_2, u_1 \rhd & \lhd\ u_2, u_2 \rhd & \lhd\ u_2, u_3 \rhd & \dots \\ \lhd\ u_3, u_1 \rhd & \lhd\ u_3, u_2 \rhd & \lhd\ u_3, u_3 \rhd & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

Note that the matrix $-D$ corresponds precisely to the isomorphism $D : Z_2 \to Z_2^*$ when represented by the bases $(u_n)_n$ and $(u_n^*)_n$; so D^{-1} corresponds to the matrix D. Now assume that T admits a matrix representation given by coefficients $(a_{i,j})_{i,j}$. Since the representation of T^* is just the transpose of T and $T^+ = D^{-1}T^*D$, it follows that the matrix representation of T^+ is given by the product

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{41} & \cdots \\ a_{12} & a_{22} & a_{32} & b_{42} & \cdots \\ a_{13} & a_{23} & a_{33} & a_{43} & \cdots \\ a_{14} & a_{24} & a_{34} & a_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which gives that

$$T^+ = \begin{pmatrix} a_{22} & -a_{12} & a_{42} & -a_{32} & \cdots \\ -a_{21} & a_{11} & -a_{41} & a_{31} & \cdots \\ a_{24} & -a_{14} & a_{44} & -a_{34} & \cdots \\ -a_{23} & a_{13} & -a_{43} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A different matrix representation for an operator $T \in \mathfrak{L}(Z_2)$ is considered in [13] $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta$ are linear maps $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$. The relationship between this 2×2 representation and the infinite matrix representation described above can be seen the following way: the matrix $\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ corresponds to the representation of Z_2 by means of the 2-dimensional decomposition given by $X_n = \operatorname{span}\{(e_n, 0), (0, e_n)\}$. Since the infinite matrix representation is given by the basis $(u_n)_n$ defined by $u_{2n-1} = (e_n, 0)$ and $u_{2n} = (0, e_n)$, it follows that

$$\alpha = \mathfrak{a}_{2i-1,2j-1}, \quad \delta = \mathfrak{a}_{2i,2j-1}, \quad \beta = \mathfrak{a}_{2i-1,2j} \quad \text{and } \gamma = \mathfrak{a}_{2i,2j},$$

where, for instance, $\mathfrak{a}_{2i-1,2j-1}$ represents the infinite matrix formed by the odd columns and odd rows of the matrix $(a_{ij})_{i,j}$ representing T.

It is easy to check now that if $T = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$ is a bounded operator on Z_2 then $T^+ = \begin{pmatrix} \gamma^* & -\beta^* \\ -\delta^* & \alpha^* \end{pmatrix}$. The matrix $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ represents the bounded operator $Z_2 \stackrel{Q}{\to} \ell_2 \stackrel{\jmath}{\to} Z_2$ and $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}$. Consequently, $(\jmath Q)^+(\jmath Q) = (\jmath Q)^2 = 0$. Thus, $\jmath Q$ provides an example of bounded operator $T \in \mathfrak{L}(Z_2)$ such that the identity $\|T^+T\| = \|T\|\,\|T^+\|$ fails.

4 Symplectic transformations on Z_2

Definition 4.1 An operator $T : Z_2 \to Z_2$ will be called a *symplectic transformation* if it preserves the symplectic form, in the sense that

$$\lhd\ T(x), T(y) \rhd = \lhd x, y \rhd, \quad \text{for all } x, y \in Z_2. \tag{4}$$

An operator T is a symplectic transformation if and only if $T^+T = I$ (here I is the identity): indeed, for all $x, y \in Z_2$ we have

$$\lhd T^+Tx, y \ \rhd - \lhd\ x, y \rhd = \lhd (T^+T - I)x, y \rhd = 0,$$

and thus we deduce from Proposition 3.1 that $(T^+T - I)x = 0$ for all $x \in Z_2$. The other implication is clear. From this we obtain:

Proposition 4.2 *Bounded symplectic transformations on Z_2 have complemented range.*

Unbounded symplectic transformations on Z_2 are possible: just set the linear map $L(e_n, 0) = (e_n, 0)$ and $L(0, e_n) = (ne_n, e_n)$. Indeed, L preserves $\lhd \cdot, \cdot \rhd$ by checking on the basis elements, and it is unbounded since $\|L(e_n, e_n)\| = \|(n+1)e_n - KP(e_n)\|_2 + \|e_n\|_2 = n+2$, for every $n \in \mathbb{N}$. From now on we will only consider bounded symplectic transformations and refer to them simply as *symplectic transformations*. Let us show some natural examples.

Definition 4.3 An operator $\eta : \ell_2 \to \ell_2$ is said to be an *operator on the scale* if there is $p > 2$ such that η also acts linear and boundedly $\eta : \ell_p \to \ell_p$ as well as $\eta : \ell_{p^*} \to \ell_{p*}$. It will be called an isometric operator on the scale if both $\eta : \ell_p \to \ell_p$ and $\eta : \ell_{p^*} \to \ell_{p*}$ are into isometries.

A result of Banach [1] establishes that $U\big((x_n)_n\big) = (\varepsilon_n x_{\pi(n)})$, where $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation and $|\varepsilon_n| = 1$ for all $n \in \mathbb{N}$, are the only examples of surjective isometric operators on the scale. One of the forms of the Commutator Theorem, see [7, 11] for details, is that if η is an operator on the scale then $\tau_\eta = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$ defines a bounded operator on Z_2; see also [13].

Proposition 4.4 *Suppose that $\alpha : \ell_2 \to \ell_2$ is an operator acting on the scale and $\beta \in \mathfrak{L}(\ell_2)$. Then the upper triangular operator $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ is a symplectic transformation if and only if α is an isometry on ℓ_2 and $\alpha^*\beta$ is selfadjoint.*

Proof First note that for any bounded operator $\beta \in \mathfrak{L}(\ell_2)$, the operator $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ is bounded on Z_2 because coincides with the composition $Z_2 \stackrel{Q}{\to} \ell_2 \stackrel{\beta}{\to} \ell_2 \stackrel{j}{\to} Z_2$. Since $\alpha : \ell_2 \to \ell_2$ acts on the scale, we have that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

and both operators on the right side are bounded, hence the upper triangular operator is bounded. Once that boundness is settled, note that

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}^+ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^* & -\beta^* \\ 0 & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\beta - \beta^*\alpha \\ 0 & \alpha^*\alpha \end{pmatrix}.$$

We conclude that $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ is a symplectic transformation if and only if $\alpha^*\alpha = I$ (i.e. an isometry on ℓ_2) and $\alpha^*\beta = \beta^*\alpha = (\alpha^*\beta)^*$. □

Corollary 4.5 *Let $U \in \mathfrak{L}(\ell_2)$ be any isometry that acts on the scale and $T \in \mathfrak{L}(\ell_2)$ a selfadjoint operator. Then $\begin{pmatrix} U & UT \\ 0 & U \end{pmatrix}$ is a symplectic transformation.*

4.1 Polar decompositions

A specially remarkable instance occurs when one sets the polar decomposition $T = UP = U(T^*T)^{1/2}$ of an operator $T \in \mathfrak{L}(\ell_2)$.

Proposition 4.6 *Let $T \in \mathfrak{L}(\ell_2)$ be an operator and $T = UP = U(T^*T)^{1/2}$ its polar decomposition. If U is an operator on the scale then $\begin{pmatrix} U & T \\ O & U \end{pmatrix}$ is a symplectic transformation on Z_2.*

Proof By Proposition 4.4 we just have to recall that $U^*U = I$ and $U^*T = P$ is selfadjoint. □

4.2 Diagonal operators

Let $\sigma \in \ell_\infty$. The diagonal operator $\sigma\big((x_n)_n\big) = (\sigma_n x_n)_n$ is an operator on the scale and it therefore induces the operator $\tau_\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ on Z_2. The operator τ_σ is a symplectic transformation if and only if

$$\tau_\sigma^+ \tau_\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}^+ \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} \sigma^* & 0 \\ 0 & \sigma^* \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

If σ is a sequence of reals, the operator σ is selfadjoint this means that $\sigma_n \sigma_n = 1$ for all $n \geq 1$. Thus, τ_σ is a symplectic transformation if and only if $\sigma \in \{-1, 1\}^{\mathbb{N}}$.

4.3 Shift operators

The right-shift operator $r((x_n)_n) = \big((x_{n-1})_n\big)$ is an isometric operator on the scale and therefore $R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in \mathfrak{L}(Z_2)$ is an isometry on Z_2 with 2-codimensional range. The adjoint $\ell = r^*$ is the left-shift operator $\ell((x_n)_n) = \big((x_{n+1})_n\big)$, which is also an operator on the scale and therefore $L = \begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix} \in \mathfrak{L}(Z_2)$. It follows from Proposition 4.4 that R is a symplectic transformation (see also below), while L is not symplectic because it is not injective. The comments at the end of the previous section imply that $R^+ = L$ and thus $LR = R^+R = I$.

4.4 Block operators

Let $\mathfrak{u}$ be a sequence $(u_n)_n$ of disjointly supported normalized blocks in ℓ_2, that we can understand as the operator $\mathfrak{u} : \ell_2 \to \ell_2$ given by $\mathfrak{u}(x) = \sum x_n u_n$. In general $\mathfrak{u}$ is not an operator on the scale and $\begin{pmatrix} \mathfrak{u} & 0 \\ 0 & \mathfrak{u} \end{pmatrix}$ is not an operator in Z_2. The *block operator* $T_\mathfrak{u} : Z_2 \to Z_2$ is defined as $T_\mathfrak{u}(e_n, 0) = (u_n, 0)$ and $T_\mathfrak{u}(0, e_n) = (KPu_n, u_n)$, namely

$$T_\mathfrak{u} = \begin{pmatrix} \mathfrak{u} & KP\mathfrak{u} \\ 0 & \mathfrak{u} \end{pmatrix}, \tag{5}$$

where $KP\mathfrak{u} : \ell_2 \to \mathbb{R}^{\mathbb{N}}$ denotes the linear map defined by $KP\mathfrak{u}(e_n) = KP(u_n)$.

The operators $T_\mathfrak{u}$ are symplectic transformations, and the proof for this can be followed in detail in [4, Section 10.9: The Properties of Z_2 explained by itself]. We will prove the general case in Sect. 6. The idea is that equation (4) is equivalent to $T_\mathfrak{u}^+ T_\mathfrak{u} = I$ or else to $D = T_\mathfrak{u}^* D T_\mathfrak{u}$, since $T^+ = D^{-1} T^* D$ (see the comment after diagram (2)) and this follows, after a few cumbersome computations, from the equality

$$\left\langle u_j, \sum_i x_i KP(u_i) \right\rangle = \left\langle KP(u_j), \sum_i x_i u_i \right\rangle.$$

The operator $T_{\mathrm{u}} T_{\mathrm{u}}^{+}$ therefore defines a projection onto $T_{\mathrm{u}}[Z_2]$ and this shows that T_{u} is an into isometry (with respect to the usual quasi-norm) with complemented range. All these results are from [17].

Observe that R can be regarded as a block operator with the choice of $(u_n)_{n\in\mathbb{N}} = (e_{n+1})_{n\in\mathbb{N}}$ since $KPe_n = 0$.

4.5 Transvections

Throughout this section (X, ω) will denote a symplectic Banach space, in particular Z_2. The *symplectic group* of (X, ω) is the subgroup $\mathrm{Sp}(X, \omega)$ of $\mathrm{GL}(X)$ of all symplectic automorphisms:

$$\mathrm{Sp}(X, \omega) = \{T \in \mathrm{GL}(X) \colon \omega(Tx, Ty) = \omega(x, y) \text{ for all } x, y \in X\}.$$

We will denote by $\mathrm{Sp}(Z_2)$ the symplectic group of Z_2 endowed with the symplectic form of Prop. 3.1. Observe that $\mathrm{Sp}(Z_2)$ is not a bounded subgroup. Indeed, if $D_a : \ell_2 \to \ell_2$ is a diagonal operator on ℓ_2 given by some real $a \in \ell_\infty$ then D_a is selfadjoint and $\begin{pmatrix} I & D_a \\ 0 & I \end{pmatrix}$ is an invertible symplectic transformation with norm $\|a\|_\infty + 1$.

Definition 4.7 Let $\lambda \in \mathbb{K}$ and $u \in X$. The *transvection* associated to λ and u is the linear map $\mathcal{T}_{u,\lambda}$ given by $\mathcal{T}_{u,\lambda}(x) = x + \lambda\omega(x, u)u, \quad$ for each $x \in X$.

Recall that for a subset $U \subset X$, we denote by $U^\omega = \{x \in X \colon \omega(x, u) = 0,$ for all $u \in U\}$ the symplectic annihilator of U. For any $u \in X$, let us denote by u^ω the anihilator of $\{u\}$. By duality of Lemma 2.9, the annihilator u^ω of a line defined by any $u \in X$ is an hyperplane of X. It follows that a transvection $\mathcal{T}_{u,\lambda}$ is the identity on the hyperplane u^ω and induces the identity on the corresponding quotient X/u^ω (as $\mathcal{T}_{u,\lambda}(x) - x \in u^\omega$ for each $x \in X$).

Lemma 4.8 *Transvections are symplectic transformations.*

Proof. In the first place, transvections are linear due as the symplectic form $\omega(\cdot,\cdot)$ is bilinear. Given $x \in X$, boundedness follows by

$$\begin{aligned}\|\mathcal{T}_{u,\lambda}(x)\|_X &\leq \|x\|_X + |\lambda|\,|\omega(x,u)|\|u\|_X \\ &\leq \left(1 + |\lambda|\,\|\omega\|\,\|u\|_X^2\right)\|x\|_X.\end{aligned}$$

Now taking $x, y \in X$ we deduce that

$$\begin{aligned}\omega(\mathcal{T}_{u,\lambda}(x), \mathcal{T}_{u,\lambda}(y)) &= \omega(x + \lambda\omega(x,u)u,\, y + \lambda\omega(y,u)u) \\ &= \omega(x,y) + \lambda\omega(y,u)\omega(x,u) - \lambda\omega(x,u)\omega(y,u) \\ &= \omega(x,y).\end{aligned}$$

□

Moreover, observe that $\mathcal{T}_{u,\lambda}\mathcal{T}_{u,\mu} = \mathcal{T}_{u,\lambda+\mu}$ and $\mathcal{T}_{au,\lambda} = \mathcal{T}_{u,a^2\lambda}$ as immediately follows from the definition of transvection, which shows that, given $u \in X$, $\mathscr{T}_u = \{\mathcal{T}_{u,\lambda} \colon \lambda \in \mathbb{K}\}$ is a subgroup of $\mathrm{Sp}(X)$ and the map $\lambda \in (\mathbb{K}, +) \mapsto \mathcal{T}_{u,\lambda}$ defines an isomorphism of groups.

5 Rochberg spaces are symplectic

Consider the complex interpolation method applied to the scale (ℓ_∞, ℓ_1) (see the classical [2]; or else [4]). It is well known that it provides the space $(\ell_\infty, \ell_1)_\theta = \ell_{\theta^{-1}}$ for $0 < \theta < 1$. In particular, $(\ell_\infty, \ell_1)_{1/2} = \ell_2$. Let U be the strip $\{z \in \mathbb{C} : 0 < \Re z < 1\}$ on the complex plane. Recall that the associated Calderón space $\mathcal{C}$ to the scale (ℓ_∞, ℓ_1) is the space of continuous functions $f : \overline{U} \to \ell_\infty$ which are bounded on $\overline{U}$, analytic on U and satisfy the boundary conditions that, $f(ti) \in \ell_\infty$ and $f(1+it) \in \ell_1$ for every $t \in \mathbb{R}$ and $\|f\|_{\mathcal{C}} = \sup\{\|f(k+it)\| : t \in \mathbb{R},\ k = 0, 1\} < \infty$. The Rochberg spaces [23] obtained at 1/2 are defined as

$$\mathfrak{R}^{(n)} = \{(x_{n-1}, \ldots, x_1, x_0) \in \ell_\infty^n : x_i = f^{(i)}(1/2)/i!,\ \text{for some } f \in \mathcal{C},\ 0 \le i \le n-1\}.$$

These spaces can be considered as generalizations of ℓ_2. Indeed, $\mathfrak{R}^{(1)} = \ell_2$ and it was Kalton who noticed that $\mathfrak{R}^{(2)} = Z_2$ (see [6, 11] for additional information). To show that Rochberg spaces are symplectic we need first to know that they are isomorphic to their duals in the following form taken from [5].

Proposition 5.1 *Consider for each $n \ge 1$ the continuous bilinear map $\omega_n : \mathfrak{R}^{(n)} \times \mathfrak{R}^{(n)} \to \mathbb{R}$ given by*

$$\omega_n\big((x_{n-1}, \ldots, x_0), (y_{n-1}, \ldots, y_0)\big) = \sum_{i+j=n-1} (-1)^i \langle x_i, y_j \rangle.$$

The induced operator $D_n : \mathfrak{R}^{(n)} \to \mathfrak{R}^{(n)^*}$ given by $D_n(x)(y) = \omega_n(x, y)$ is an isomorphism onto.

That this duality makes Rochberg spaces symplectic for even n, as it occurs with Z_2, is somehow unexpected. Surprisingly enough, odd Rochberg spaces are also symplectic, but not in the same way as even Rochberg spaces. To see why, observe that in the Hilbert space case we know that there is a correspondence between complex and symplectic structures: if ω is a symplectic form, then there is a complex structure J such that $\omega(x, y) = \langle x, J(y)\rangle$. In this way, a symplectic structure is obtained "twisting" the natural duality with a complex structure and this approach generalizes to higher order odd Rochberg spaces; i.e., a complex structure on $\mathfrak{R}^{(n)}$ may be used to induce a perturbation on ω_n and define a symplectic structure.

Theorem 5.2 *All Rochberg spaces are symplectic.*

Proof Observe that ω_n is alternating if and only if n is even; so, the result holds for even n.

For n odd, consider a complex structure σ on ℓ_2 that is an operator on the scale; say, $\sigma(x) = (-x_2, x_1, -x_4, x_3, \ldots)$, so that the induced diagonal operator τ_σ is bounded on Z_2. The generalized form of the commutator theorem, see [5, 7], shows that the $n \times n$ matrix diagonal operator still acts boundedly on the corresponding $\mathfrak{R}^{(n)}$. We will continue calling τ_σ this diagonal operator. We define the bilinear map

$$\begin{aligned}\overline{\omega_n}\big((x_{n-1}, \ldots, x_0), (y_{n-1}, \ldots, y_0)\big) &= \omega_n\big((x_{n-1}, \ldots, x_0), \tau_\sigma(y_{n-1}, \ldots, y_0)\big)\\ &= \sum_{i+j=n-1} (-1)^i \langle x_i, \sigma y_j \rangle.\end{aligned}$$

This map is now alternating due to the fact that $\sigma^* = -\sigma$. Indeed,

$$\begin{aligned}
\overline{\omega_n}\big((x_{n-1},\dots,x_0),(y_{n-1},\dots,y_0)\big) &= \sum_{i+j=n-1} (-1)^i \langle x_i, \sigma y_j\rangle \\
&= \sum_{i+j=n-1} (-1)^i \langle \sigma^* x_i, y_j\rangle \\
&= \sum_{i+j=n-1} (-1)^i\,(-1)\langle \sigma x_i, y_j\rangle \\
&= (-1)\sum_{i+j=n-1} (-1)^i \langle y_j, \sigma x_i\rangle \\
&= (-1)\sum_{j+i=n-1} (-1)^i\,(-1)^{i+j} \langle y_j, \sigma x_i\rangle \\
&= (-1)\sum_{i+j=n-1} (-1)^j \langle y_j, \sigma x_i\rangle \\
&= -\overline{\omega_n}\big((y_{n-1},\dots,y_0),(x_{n-1},\dots,x_0)\big).
\end{aligned}$$

Boundedness follows from the boundness of ω_n and τ_σ:

$$\big|\overline{\omega_n}(x,y)\big| = \big|\omega_n(x,\tau_\sigma y)\big| \le K\,\|x\|\,\|\tau_\sigma y\| \le C\|x\|\,\|y\|.$$

To obtain that $(\mathfrak{R}^{(n)},\overline{\omega_n})$ is symplectic it suffices to show that the induced linear map $L_{\overline{\omega_n}} : \mathfrak{R}^{(n)} \to \mathfrak{R}^{(n)*}$ is an isomorphism onto. Assume that there exists $x \in \mathfrak{R}^{(n)}$ such that $L_{\overline{\omega_n}}(x)(y) = 0$ for all $y \in \mathfrak{R}^{(n)}$. Thus $L_{\omega_n}(x)(\tau_\sigma y) = 0$ for all $y \in \mathfrak{R}^{(n)}$. Taking into account that τ_σ is invertible in $\mathfrak{R}^{(n)}$, it follows that $L_{\omega_n}(x)(y) = 0$ for all $y \in \mathfrak{R}^{(n)}$, so that $x = 0$. Moreover, as τ_σ is an isomorphism, it is clear that $\overline{\omega_n}$ has closed range because ω_n has closed range. □

6 Block operators on Rochberg spaces are symplectic

A sequence $\mathfrak{u} = (u_n)_{n\in\mathbb{N}}$ of normalized blocks in ℓ_2 induces a multiplication operator $u : \ell_2 \to \ell_2$ given by $u(e_n) = u_n$ and, as we showed in Sect. 4.4, a block operator $\begin{pmatrix}\mathfrak{u} & 2\mathfrak{u}\log\mathfrak{u}\\ 0 & \mathfrak{u}\end{pmatrix}$ in Z_2. The higher order generalizations of block operators were obtained in [10, Theorem 7.2] as the operators

$$T_{\mathfrak{u},n} = \begin{pmatrix}
\mathfrak{u} & 2\mathfrak{u}\log\mathfrak{u} & 2\mathfrak{u}\log^2\mathfrak{u} & \cdots & \frac{2^{n-1}}{(n-1)!}\mathfrak{u}\log^{n-1}\mathfrak{u}\\
0 & \mathfrak{u} & 2\mathfrak{u}\log\mathfrak{u} & 2\mathfrak{u}\log^2\mathfrak{u} & \cdots\\
0 & 0 & \mathfrak{u} & 2\mathfrak{u}\log\mathfrak{u} & 2\mathfrak{u}\log^2\mathfrak{u}\\
0 & 0 & 0 & \mathfrak{u} & 2\mathfrak{u}\log\mathfrak{u}\\
0 & 0 & 0 & 0 & \mathfrak{u}
\end{pmatrix}$$

that act boundedly $T_{\mathfrak{u},n} : \mathfrak{R}^{(n)} \to \mathfrak{R}^{(n)}$. We will shorten the name to $T_{\mathfrak{u}}$ when no confusion is possible about which is n. Given $T \in \mathfrak{L}(\mathfrak{R}^{(n)})$, recall that T^+ always denotes the symplectic adjoint of an operator T, namely $\omega_n(T^+x, y) = \omega_n(x, Ty)$. If $T \in \mathfrak{L}(\mathfrak{R}^{(2n-1)})$ then we will denote $T^\sharp$ the symplectic adjoint with respect to $\overline{\omega}_{2n-1}$, namely $\overline{\omega}_{2n-1}(T^\sharp x, y) = \overline{\omega}_{2n-1}(x, Ty)$. We have:

Lemma 6.1 *Let* $T = \begin{pmatrix} A_0^0 & A_0^1 & A_0^2 & \cdots & A_0^{n-1} \\ 0 & A_1^0 & A_1^1 & A_1^2 & \cdots \\ 0 & 0 & A_2^0 & A_2^1 & A_{n-3}^0 \\ 0 & 0 & 0 & A_3^0 & A_{n-2}^1 \\ 0 & 0 & 0 & 0 & A_{n-1}^0 \end{pmatrix}$ *be an upper triangular operator on* $\mathfrak{R}^{(n)}$.

- $T^+ = \begin{pmatrix} A_{n-1}^{0*} & -A_{n-2}^{1*} & A_{n-3}^{2*} & \cdots & (-1)^{n-1} A_0^{n-1*} \\ 0 & A_{n-2}^{0*} & -A_{n-3}^{1*} & A_{n-4}^{2*} & \cdots \\ 0 & 0 & A_{n-3}^{0*} & -A_{n-4}^{1*} & A_0^{2*} \\ 0 & 0 & 0 & A_{n-4}^{0*} & -A_0^{1*} \\ 0 & 0 & 0 & 0 & A_0^{0*} \end{pmatrix}.$
- *If n is odd,* $T^\sharp = -T^+\tau_\sigma$.

Proof. The first part can be obtained by plain induction. The second part is simple: since $\overline{\omega}_n(x, Ty) = \omega_n(x, \tau_\sigma Ty)$ then

$$T^\sharp = (\tau_\sigma T)^+ = T^+\tau_\sigma^+ = T^+(\tau_\sigma)^* = -T^+\tau_\sigma. \qquad \square$$

We prove now that block operators are symplectic.

Proposition 6.2 *Let* $D_n : \mathfrak{R}^{(n)} \to \mathfrak{R}^{(n)^*}$ *be the duality isomorphism* $D_n(x)(y) = \omega_n(x, y)$ *from Proposition 5.1. One has* $T_{\mathrm{u}}^* D_n T_{\mathrm{u}} = D_n$ *or, equivalently,* $\omega_n(T_{\mathrm{u}}x, T_{\mathrm{u}}y) = \omega_n(x, y)$.

Proof For integers $i \in \mathbb{N}$ and $1 \le k \le n$, let us denote by $x_{i,k}$ the vector of $\mathfrak{R}^{(n)}$ having e_i at the k^{th} position and zeroes in the other coordinates (that is, $x_{i,k} = (x_{n-1}, \ldots, x_0)$ where the only non-zero coordinate is $x_{n-k} = e_i$). It is enough to prove that

$$\omega_n(T_{\mathrm{u}}(x_{i,k}), T_{\mathrm{u}}(x_{j,l})) = \omega_n(x_{i,k}, x_{j,l}). \tag{6}$$

First, suppose that $k + l = n + 1$. Then by definition

$$\begin{aligned} &\omega_n(T_{\mathrm{u}}(x_{i,k}), T_{\mathrm{u}}(x_{j,l})) \\ &\quad = \omega_n\left(\left(\frac{2^{k-1}}{(k-1)!}u_i \log^{k-1}|u_i|, \ldots, u_i, \ldots, 0\right), \left(\frac{2^{l-1}}{(l-1)!}u_j \log^{l-1}|u_j|, \ldots, u_j, \ldots, 0\right)\right) \\ &\quad = (-1)^{n-k}\langle u_i, u_j\rangle = \omega_n(x_{i,k}, x_{j,l}). \end{aligned}$$

If $k + l < n + 1$ then (6) cancels out as we are multiplying by zeroes. If $k + l > n + 1$ then $\omega_n(x_{i,k}, x_{j,l}) = 0$ and (6) becomes, after setting $m = k + l - (n + 1)$,

$$\begin{aligned} &\omega_n(T_{\mathrm{u}}(x_{i,k}), T_{\mathrm{u}}(x_{j,l})) \\ &\quad = (-1)^{n-k}\left\langle u_i, \frac{2^m}{m!}u_j \log^m|u_j|\right\rangle + (-1)^{n-k+1}\left\langle \frac{2^1}{1!}u_i \log|u_i|, \frac{2^{m-1}}{(m-1)!}u_j \log^{m-1}|u_j|\right\rangle \\ &\qquad + \cdots + (-1)^{n-k+m}\left\langle \frac{2^m}{m!}u_i \log^m|u_i|, u_j\right\rangle \\ &\quad = (-1)^{n-k}\sum_{p=0}^{m}(-1)^p\left\langle \frac{2^p}{p!}u_i \log^p|u_i|, \frac{2^{m-p}}{(m-p)!}u_j \log^{m-p}|u_j|\right\rangle. \end{aligned} \tag{7}$$

If $i \neq j$ then all summands in (7) are null because $\langle u_i, u_j \rangle = 0$ and the result follows. If $i = j$ then (7) becomes

$$(-1)^{n-k} \log^m |u_i| \left[\sum_{p=0}^{m} (-1)^p \frac{2^m}{p!(m-p)!} \right] = (-1)^{n-k} \log^m |u_i| \left[\sum_{p=0}^{m} (-1)^p \frac{2^m}{m!} \binom{m}{p} \right]$$
$$= (-1)^{n-k} \log^m |u_i| \frac{2^m}{m!} \left[\sum_{p=0}^{m} (-1)^p \binom{m}{p} \right].$$

Now, the Binomial Theorem $0 = (1-1)^m = \sum_{k=0}^{m} \binom{m}{k} 1^{m-k} (-1)^k$ cancels out all terms. □

We conclude this section with several technical lemmata of independent interest about generalized block operators.

Lemma 6.3 *The range $T_{\mathfrak{u},n}[\mathfrak{R}^{(n)}]$ is isomorphic to $\mathfrak{R}^{(n)}$ and complemented in $\mathfrak{R}^{(n)}$.*

Proof The proof follows the arguments of [8, 20]. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{R}_{n-1} & \longrightarrow & \mathfrak{R}_n & \longrightarrow & \ell_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_{\mathfrak{u},n-1}} & & \downarrow{\scriptstyle T_{\mathfrak{u},n}} & & \downarrow{\scriptstyle T_{\mathfrak{u},1}} & & \\ 0 & \longrightarrow & \mathfrak{R}_{n-1} & \longrightarrow & \mathfrak{R}_n & \longrightarrow & \ell_2 & \longrightarrow & 0 \end{array}$$

By a general 3-space property (see [4]), since $T_{\mathfrak{u},1} = \tau_{\mathfrak{u}}$ is an isometry, $T_{\mathfrak{u},n}$ must be an into isometry. Thus, $T_{\mathfrak{u},n}[\mathfrak{R}^{(n)}]$ is an isometric copy of $\mathfrak{R}^{(n)}$. This isometric copy is complemented because of the identity $(T_{\mathfrak{u},n})^* D_n T_{\mathfrak{u},n} = D_n$ from Proposition 6.2. Thus, $T_{\mathfrak{u},n} D_n^{-1} (T_{\mathfrak{u},n})^* D_n$ is a projection of $\mathfrak{R}^{(n)}$ onto the range of $T_{\mathfrak{u},n}$. □

We now extend the classical result about the behaviour of operators on Z_2 due to Kalton [17, Lemma 6] to higher order Rochberg spaces.

Lemma 6.4 *If $T : \mathfrak{R}^{(n)} \to \mathfrak{R}^{(n)}$ $(n > 1)$ is not strictly singular then there exists $\alpha \neq 0$ and block operators $T_{\mathfrak{u}}$ and $T_{\mathfrak{v}}$ such that $TT_{\mathfrak{u}} - \alpha T_{\mathfrak{v}}$ is strictly singular.*

Proof Let us recall from [6] that the canonical exact sequence

$$0 \longrightarrow \ell_2 \longrightarrow \mathfrak{R}^{(n)} \longrightarrow \mathfrak{R}^{(n-1)} \longrightarrow 0 \qquad (8)$$

has strictly singular quotient map. Therefore, there exists $\alpha \neq 0$ and normalized block basic sequences $\mathfrak{u} = (u_n)_n$ and $\mathfrak{v} = (v_n)_n$ in ℓ_2 such that

$$T(u_n, 0, 0, \ldots, 0) = \alpha(v_n, 0, 0, \ldots, 0) + w_n, \quad \text{for each } n \in \mathbb{N},$$

where $\sum_n \|w_n\| < \infty$. Now just take the block operators $T_{\mathfrak{u}}$, $T_{\mathfrak{v}}$ induced by those sequences and consider the following operator $\widehat{K} : \ell_2 \to \mathfrak{R}^{(n)}$ given by

$$\widehat{K}(x, 0, \ldots, 0) = \sum_{n=1}^{\infty} (e_n, 0, \ldots, 0)^* \big((x, 0, \ldots, 0)\big) w_n, \quad \text{for each } (x, 0, \ldots, 0) \in \ell_2 \subset \mathfrak{R}^{(n)}.$$

Since $\sum_{n=1}^{\infty} \|(e_n, 0, \ldots, 0)^*\| \|w_n\| = \sum_{n=1}^{\infty} \|w_n\| < \infty$, we have that $\widehat{K}$ is nuclear. If we denote by $K : \mathfrak{R}^{(n)} \to \mathfrak{R}^{(n)}$ the natural nuclear extension to $\mathfrak{R}^{(n)}$, then K is compact and $TT_{\mathfrak{u}} - \alpha T_{\mathfrak{v}} - K = 0$ on the canonical copy of ℓ_2 in $\mathfrak{R}^{(n)}$ given by (8) and thus $TT_{\mathfrak{u}} - \alpha T_{\mathfrak{v}}$ must be strictly singular. □

An important consequence of this perturbation result is:

Lemma 6.5 *Let $n > 1$ and $T \in \mathfrak{L}\left(\mathfrak{R}^{(n)}\right)$.*

- *If T^+T is strictly singular then T is strictly singular.*
- *If n is odd and $T^\sharp T$ is strictly singular then T is strictly singular.*

Proof Let $T \in \mathfrak{L}\left(\mathfrak{R}^{(n)}\right)$ be a non-strictly singular operator. By the previous lemma, there exists $\alpha \neq 0$ and block operators $T_{\mathfrak{u}}, T_{\mathfrak{v}}$ such that $TT_{\mathfrak{u}} = \alpha T_{\mathfrak{v}} - S$ with S strictly singular. Therefore

$$T_{\mathfrak{u}}^+ T^+ T T_{\mathfrak{u}} = (TT_{\mathfrak{u}})^+ TT_{\mathfrak{u}} = (\alpha T_{\mathfrak{v}}^+ - S^+)(\alpha T_{\mathfrak{v}} - S) = \alpha' T_{\mathfrak{v}}^+ T_{\mathfrak{v}} + S' = \alpha' I + S',$$

and therefore T^+T is not strictly singular. If n is odd, we have

$$T_{\mathfrak{u}}^\sharp T^\sharp T T_{\mathfrak{u}} = (TT_{\mathfrak{u}})^\sharp TT_{\mathfrak{u}} = (\alpha T_{\mathfrak{v}}^\sharp - S^\sharp)(\alpha T_{\mathfrak{v}} - S) = \alpha' T_{\mathfrak{v}}^\sharp T_{\mathfrak{v}} + S' = -\alpha' T_{\mathfrak{v}}^+ \tau_\sigma T_{\mathfrak{v}} + S'$$

by Proposition 6.1. This means that if $T^\sharp T$ is strictly singular, then $T_{\mathfrak{v}}^+ \tau_\sigma T_{\mathfrak{v}}$ must be strictly singular, but since $\tau_\sigma T_{\mathfrak{v}}$ is invertible, $T_{\mathfrak{v}}^+$ must be strictly singular, as well as $T_{\mathfrak{v}}$, which is a contradiction since block operators are never strictly singular. □

7 Rochberg spaces do not contain Lagrangian subspaces

We now extend the Kalton–Swanson theorem [20] showing that the symplectic structures of Rochberg spaces we have defined are not trivial.

Theorem 7.1 *$\mathfrak{R}^{(n)}$ $(n > 1)$ has no Lagrangian subspace.*

Proof Let T be a projection onto an infinite dimensional isotropic subspace. If n is even, $T^+T = 0$, so T must be strictly singular and thus every complemented isotropic subspace must be finite dimensional. If n is odd, $T^\sharp T = 0$ and then T must be also strictly singular. □

Some of the authors of this paper conjecture that Rochberg spaces obtained from a reflexive Banach space X such that $X \cap \overline{X}^*$ is dense in both X and $\overline{X}^*$ and $(X, \overline{X}^*)_{1/2}$ is isometric to a Hilbert space are symplectic. Conditions to obtain $(X, \overline{X}^*)_{1/2} = \ell_2$ are in [27] (see also [11]). By [5], see also [24, Proposition 2.11], the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_2^* & \longrightarrow & (\mathfrak{R}^{(2)})^* & \longrightarrow & \ell_2^* & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & \ell_2^* & \longrightarrow & \mathfrak{R}^{*(2)} & \longrightarrow & \ell_2^* & \longrightarrow & 0 \end{array} \tag{9}$$

with $T(x^*, y^*)(x, y) = \langle x^*, y\rangle + \langle y^*, x\rangle$ for all $(x^*, y^*) \in d(X_\theta^*)$ and all $(x, y) \in dX_\theta$ is commutative, which shows that the corresponding $\mathfrak{R}^{(2)}$ is symplectic. The assertion generalizes to "even Rochberg spaces $\mathfrak{R}^{(2n)}$ are symplectic". Moreover, assume that both X and X^* have a common complex structure, i.e., an operator σ bounded in both X and X^*satisfying that $\sigma^2 = -I$. Then, reasoning as in Theorem 5.2, we deduce that also odd Rochberg spaces $\mathfrak{R}^{(2n-1)}$ are symplectic. However, it seems difficult to prove that such symplectic structures are non-trivial. If one wants to adapt the proof for the couple (ℓ_1, ℓ_∞) to the general case one should prove that the corresponding block operators

$$\begin{pmatrix} u & \Omega^{(1)}(u) & \Omega^{(2)}(u) & \dots & \Omega^{(n-1)}(u) \\ 0 & u & \Omega^{(1)}(u) & \Omega^{(2)}(u) & \dots \\ 0 & 0 & u & \Omega^{(1)}(u) & \Omega^{(2)}(u) \\ \dots & 0 & 0 & u & \Omega^{(1)}(u) \\ 0 & \dots & 0 & 0 & u \end{pmatrix}$$

where $\Omega^{(n)}$ are the corresponding n-differentials (see [23]) are symplectic operators. See also [10] to determine how they can be calculated. In our Kalton–Peck case, $\Omega^{(n)}x = \frac{2^n}{n!}x \log^n x$ for normalized x. In general, it is perfectly possible (think about the case of weighted Hilbert spaces [8]) that all iterated Rochberg spaces are actually Hilbert spaces and thus their symplectic structures are trivial.

8 Almost symplectic structures

This section is motivated by the study [9] on complex structures on Z_2. Let us recall that a linear bounded operator $J : X \to X$ defined on a real Banach space is a *complex structure* if $J^2 = -I$; and that a complex structure J on X yields a $\mathbb{C}$-linear structure on X by declaring $(\alpha + i\beta)x = \alpha x + \beta J(x)$. The resulting complex space will be denoted X^J and it is a Banach space equipped with the norm $|||x||| = \sup_{\theta \in [0,2\pi]} \|\cos\theta x + \sin\theta J(x)\|$.

The following result of Weinstein [28] shows that symplectic structures on real Hilbert spaces are obtained from complex structures.

Lemma 8.1 *Let $\mathcal{H}$ be a real Hilbert space. For every symplectic form ω in $\mathcal{H}$ there exist a complex structure J on $\mathcal{H}$ and an equivalent inner product $\langle\, ,\, \rangle_R$ on $\mathcal{H}$ such that $\omega(x, y) = \langle x, Jy\rangle_R$ for every $x, y \in \mathcal{H}$.*

Consequently, all symplectic structures on a real Hilbert space are equivalent to the standard one. Let us prove that a complex structure on a real Hilbert space induces a symplectic structure.

Lemma 8.2 *Let $\mathcal{H}$ be a real Hilbert space and J be a complex structure on $\mathcal{H}$. Then there exist an equivalent inner product $\langle\, ,\, \rangle_R$ on $\mathcal{H}$ such that $\omega(x, y) = \langle x, Jy\rangle_R$ defines a symplectic form ω on $\mathcal{H}$.*

Proof Let J be a complex structure on $\mathcal{H}$. Let us take $R = I + J^*J$. Then $\langle x, y\rangle_R = \langle x, Ry\rangle$ defines an equivalent inner product on $\mathcal{H}$ for which J is an isometry and therefore a unitary operator. The Hilbert-adjoint of J with respect to this inner product is $J^{-1} = -J$. Then $\omega(x, y) := \langle x, Jy\rangle_R$ is symplectic on $\mathcal{H}$. □

It is straightforward that a complex structure on a hyperplane of any Banach space cannot be extended to a complex structure on the whole space. The same situation occurs for symplectic structures (Corollary 2.8). It was shown in [9] that no complex structure on ℓ_2 can be extended to a complex structure on a hyperplane of Z_2 containing it. We now observe that an analogous result holds for symplectic structures (Corollary 8.5).

Definition 8.3 Let X and Y be Banach spaces and $j : Y \to X$ be an isomorphism into. A bilinear map Ω on X extends a bilinear map ω on Y *through* j when $\Omega(jx, jy) = \omega(x, y)$ for every $x, y \in Y$. Equivalently Ω extend ω through j if the diagram is commutative

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \\ L_\omega \big\downarrow & & \big\downarrow L_\Omega \\ Y^* & \xleftarrow{j^*} & X^* \end{array} \tag{10}$$

Essentially following [12, Prop. 3.1] we obtain

Proposition 8.4 *Let X and Y be Banach spaces and $j : Y \to X$ be an isomorphism into. If a symplectic structure ω on Y can be extended to a bilinear form Ω on X, then $j(Y)$ is complemented on X.*

Proof Indeed, $jL_\omega^{-1}j^*L_\Omega$ would be a projection onto $j[Y]$ for symplectic extensions. □

Since Z_2 does not contain complemented copies of ℓ_2 [19, Corollary 6.7] we have

Corollary 8.5 *No symplectic structure on ℓ_2 can be extended to a bilinear form on a hyperplane H of Z_2 through any embedding $j : \ell_2 \to H$.*

We now observe that a symplectic structure on an hyperplane induces an almost symplectic structure on the space in the following sense:

Definition 8.6 Let X be a real (complex, resp.) Banach space and let $\alpha : X \to X^*$ ($\alpha : X \to \overline{X}^*$, resp.) be an isomorphism. We say that α is almost symplectic if $\alpha + \alpha^*$ ($\alpha + \overline{\alpha}^*$, resp.) is strictly singular. We will say that X admits an almost symplectic structure if there exists an almost symplectic isomorphism $X \to X^*$ ($X \to \overline{X}^*$, resp.).

Recall that for a given Banach space X and F a subspace of X the annihilator of F is the closed subspace of X^* defined by $F^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in F\}$.

Now, if β is a symplectic isomorphism on a hyperplane H of X and we identify $H^* = X^*/H^\perp$, then we can consider the isomorphism extension $\alpha : H \oplus [e] \to H^\perp \oplus [e]^\perp$ given by $\alpha(h + \lambda e) = \beta(h) + \lambda e^*$, for some $e^* \in H^\perp$ with $e^*(e) = 1$. Quite clearly $\alpha + \alpha^*$ is a rank one operator. We prove now the converse:

Proposition 8.7 *Let X be a real or complex Banach space admitting an almost symplectic structure. Then either X or its hyperplanes admit a symplectic structure. In particular, X is reflexive.*

Proof In the real case, consider $\alpha : X \to X^*$ an almost symplectic isomorphism and let $s : X \to X^*$ be a strictly singular operator such that $\alpha + \alpha^* = s$. Then, denoting by $\beta = \alpha - s/2$, we have $\beta^* = -\beta$ and that β is a Fredholm operator with index 0. By Fredholm theory there exist closed subspaces $X_0 \subseteq X$ and $Y_0 \subseteq X^*$ such that $E = X_0 \oplus \ker\beta$ and $X^* = Y_0 \oplus F$, where $\ker\beta$ and F are finite dimensional subspaces with the same dimension and such that the restriction $\gamma := \beta_{|X_0} : X_0 \to Y_0$ is an isomorphism onto.

Observe that that $Y_0 = (\ker\beta)^\perp$. Indeed, let $\phi \in Y_0$ and $x \in \ker\beta$. Let $x_0 \in X_0$ be such that $\phi = \beta(x_0)$, then $\phi(x) = \beta(x_0)(x) = -\beta(x)(x_0) = 0$. Hence $Y_0 \subseteq (\ker\beta)^\perp$. The equality holds since $\dim(X^*/(\ker\beta)^\perp) = \dim(X^*/Y_0)$. Then we can assume that $F = X_0^\perp$. Let us take $\{x_1, \ldots, x_n\}$ and $\{\phi_1, \ldots, \phi_n\}$ basis of $\ker\beta$ and $X_0^\perp$, respectively, such that $\phi_i(x_j) = \delta_{ij}$ for all $1 \le i, j \le n$. Suppose now that n is even and consider the map $\psi : \ker\beta \to X_0^\perp$ defined by $\psi(x_{2k-1}) = \phi_{2k}$ and $\psi(x_{2k}) = -\phi_{2k-1}$ for $k = 1, \ldots, n/2$. It follows that the map $\Gamma : X_0 \oplus \ker\beta \to (\ker\beta)^\perp \oplus X_0^\perp$ defined by the matrix $\begin{pmatrix} \gamma & 0 \\ 0 & \psi \end{pmatrix}$ is a symplectic isomorphism on X. When n is odd, the previous construction gives us a symplectic structure for an hyperplane of X. The proof for the complex case is completely analogous. □

A similar proof implies that if X is a Banach space (over $\mathbb{R}$ or $\mathbb{C}$) and $\alpha : X \to X^*$ ($\alpha : X \to \overline{X}^*$, resp.) is an isomorphism such that $\alpha - \alpha^*$ ($\alpha - \overline{\alpha}^*$, resp.) is strictly singular then there exists an isomorphism $\beta : X \to X^*$ ($\beta : X \to \overline{X}^*$, resp.) such that $\beta^* = \beta$ ($\overline{\beta}^* = \beta$, resp.) and such that $\beta - \alpha$ is strictly singular. In both the real and complex case, we call such isomorphism β a *Hermitian structure* on X.

Proposition 8.8 *Let X be a reflexive Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) isomorphic to its dual (dual conjugate when $\mathbb{K} = \mathbb{C}$) such that every operator $T : X \to X$ is of the form $\lambda I + S$ for some $\lambda \in \mathbb{K}$ and S strictly singular. Then either X or its hyperplanes admit a symplectic structure or X admits a Hermitian structure.*

Proof For the real case, let α be an isomorphism onto the dual. Using the canonical identification of $(\alpha)^{-1}\alpha^* : X^{**} \to X$ as an operator on X, we have by reflexivity that $\alpha^* = \lambda\alpha + s$ where s is strictly singular. It follows that $\alpha = \lambda\alpha^* + s^*$, and then $\lambda^2 = 1$. If $\lambda = -1$ we have an almost symplectic structure. Then by the previous proposition X or its hyperplanes admit symplectic structure, and if $\lambda = 1$ we have a Hermitian structure.

In the complex case we may assume an isomorphism α of X onto $\overline{X}^*$, then $\overline{\alpha}^* = \lambda\alpha + s$ and we get that $|\lambda| = 1$. If $\lambda = e^{i\theta}$ then by taking $\mu = e^{i(\theta-\pi)/2}$ we have

$$\overline{(\mu\alpha)}^* = \overline{\mu}\overline{\alpha}^* = \overline{\mu}\lambda\alpha + s_1 = -\mu\alpha + s_1,$$

for a strictly singular operator s_1. Hence we obtain an almost symplectic structure on X. By taking instead $\mu = e^{i\theta/2}$ we have an Hermitian structure on X. □

When such spaces or their hyperplanes admit a symplectic structure, then it cannot be trivial, since it would rely on writing $X = Y \oplus Y^*$ and a space with the $(\lambda I + S)$-property cannot have nontrivial complemented subspaces. It is an open question the existence of spaces satisfying the hypothesis of Proposition 8.8.

Proposition 8.9 *Let X be a reflexive real Banach space. Let $\alpha : X \to X^*$ be a symplectic isomorphism and $s : X \to X^*$ be a strictly singular operator. If $\alpha + s$ is also a symplectic isomorphism then α and $\alpha + s$ are equivalent.*

Proof Recall that $\widehat{X}$ denotes the usual complexification of a real Banach space X. If $T : X \to Y$ is an operator then $\widehat{T}$ denotes the respective induced operator from $\widehat{X}$ to $\widehat{Y}$. Let us consider the spectrum of a real operator τ as the spectrum of its complexification, and denote it by $\sigma(\widehat{\tau})$. Now set $S = \alpha^{-1}s$, then $\widehat{S} = \widehat{\alpha}^{-1}\widehat{s}$ is also strictly singular. Consider Γ a rectangular with vertical and horizontal edges, rectifiable, conjugation-invariant, simple closed curve, contained in the open unit disk, and such that $\Gamma \cap \sigma(\widehat{S}) = \emptyset$. Denote by U the bounded open connected component of $\mathbb{C} \setminus \Gamma$, and by V the unbounded open domain of $\mathbb{C} \setminus \Gamma$. Let $\widehat{P}$ be the spectral projection of $\widehat{S}$ associated to $\sigma(\widehat{S}) \cap U$. By a general argument (see [16, Prop. 10]), $\widehat{P}$ is induced by a real operator P on X. Let also $\widehat{Q}$ be the spectral projection associated to $\sigma(\widehat{S}) \cap V$. Then $\widehat{S} = \widehat{S}\widehat{P} + \widehat{S}\widehat{Q}$.

The spectral radius of $\widehat{S}\widehat{P}$ is strictly smaller than one. Then the series $\sum_{n\geq 1} a_n(\widehat{S}\widehat{P})^n$ converges to an operator $\widehat{R}$, where $\sum_{n\geq 1} a_n z^n$ converges to $2(-1 + (1+z)^{1/2})$ for every $|z| < 1$. Since the coefficients of the series are reals, then $\widehat{R}$ is induced by a real operator $R = \sum_{n\geq 1} a_n(SP)^n$ on X which is strictly singular. It follows that

$$\widehat{R} + \frac{1}{4}\widehat{R}^2 = \widehat{S}\widehat{P}. \tag{11}$$

We also have that $(\widehat{\alpha}\widehat{R})^* = -\widehat{\alpha}\widehat{R}$. Indeed, $(\widehat{\alpha}(\widehat{S}\widehat{P})^n)^* = [(\widehat{s}\widehat{P})^*\widehat{\alpha}^{-1*}]^n\widehat{\alpha}^* = -(\widehat{s}\widehat{P}\widehat{\alpha}^{-1})^n\widehat{\alpha} = -\widehat{\alpha}(\widehat{\alpha}^{-1}(\widehat{s}\widehat{P}))^n = -\widehat{\alpha}(\widehat{S}\widehat{P})^n$ for every $n \in \mathbb{N}$, where we used that $\widehat{s}^* = -\widehat{s}$ and hence $(\widehat{s}\widehat{P})^* = -\widehat{s}\widehat{P}$ with the canonical identification. Therefore

$$\begin{aligned}\left(\widehat{P}+\frac{1}{2}\widehat{R}\right)^*\widehat{\alpha}\left(\widehat{P}+\frac{1}{2}\widehat{R}\right) &= \left(\left(I+\frac{1}{2}\widehat{R}\right)\widehat{P}\right)^*\widehat{\alpha}\left(I+\frac{1}{2}\widehat{R}\right)\widehat{P}\\ &= \widehat{P}^*\left(I+\frac{1}{2}\widehat{R}^*\right)\left(\widehat{\alpha}+\frac{1}{2}\widehat{\alpha}\widehat{R}\right)\widehat{P}\\ &= \widehat{P}^*\left(\widehat{\alpha}+\frac{1}{2}\widehat{\alpha}\widehat{R}+\frac{1}{2}\widehat{R}^*\widehat{\alpha}+\frac{1}{4}\widehat{R}^*\widehat{\alpha}\widehat{R}\right)\widehat{P}\\ &= \widehat{P}^*\left(\widehat{\alpha}+\widehat{\alpha}\widehat{R}+\frac{1}{4}\widehat{\alpha}\widehat{R}^2\right)\widehat{P}\\ &= \widehat{P}^*\left(\widehat{\alpha}+\widehat{s}\right)\widehat{P}. \end{aligned} \tag{12}$$

Let $X_0 = PX$ and consider $T_1 := I + \frac{1}{2}R$. Since $\beta := T_1^*\alpha T_1 = \alpha + \mathfrak{s}$, where $\mathfrak{s}$ is strictly singular and $\beta^* = -\beta$ it follows from the proof of Proposition 8.7 that for any closed subspace $Z \subseteq X$ such that $X = Z \oplus \ker\beta$ the restriction $\beta_{|Z} : Z \to (\ker\beta)^\perp$ is symplectic. Now observe that T_1 is Fredholm of index 0 and $\ker T_1 \subseteq \ker\beta$. Then we can write $X = X_1 \oplus \ker\beta$ where the restriction of T_1 to X_1 is an isomorphism onto its image. We may assume by Lemma 2.9 that $X_1 \subseteq X_0$ and that $\gamma = \beta_{|X_1} : X_1 \to (\ker\beta)^\perp$ is a symplectic isomorphism.

Let us denote by Ω and ω the symplectic forms associated to $\alpha + s$ and α, respectively. Equation (12) implies that $P^*(T_1^*\alpha T_1)P = P^*(\alpha + s)P$. Then for every $x, y \in X_1$ we have $\Omega(x, y) = P^*(\alpha+s)P(x)(y) = \beta(x)(y)$ and therefore X_1 is a symplectic subspace of (X, Ω). Analogously, T_1X_1 is a symplectic subspace of (X, ω). Hence by Corollary 2.8 we can write $X = X_1 \oplus X_1^\Omega$ and $X = T_1X_1 \oplus (T_1X_1)^\omega$ where X_1^Ω and $(T_1X_1)^\omega$ are finite dimensional symplectic subspaces with the same dimension. Then there exists an isomorphism $T_2 : X_1^\Omega \to (T_1X_1)^\omega$ such that $\omega(T_2x, T_2y) = \Omega(x, y)$ for all $x, y \in X_1^\Omega$. Hence the isomorphism $T : X_1 \oplus X_1^\Omega \to T_1X_1 \oplus (T_1X_1)^\omega$ represented by the matrix $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ satisfies $T^*\alpha T = \alpha + s$. □

We obtain now a result analogous to [16, Prop. 8] for symplectic structures on Hilbert spaces. Recall that a bilinear map $T : X \times X \to \mathbb{R}$ on a Banach space is said to be compact if its associated operator $L_T : X \to X^*$ is compact. First we need the following lemma.

Lemma 8.10 *The spectrum of a symplectic isomorphism $\alpha : \mathcal{H} \to \mathcal{H}$ on a real Hilbert space $\mathcal{H}$ has only imaginary values.*

Proof Let $(\widehat{\mathcal{H}}, \langle,\rangle_{\mathbb{C}})$ be the complexification of $(\mathcal{H}, \langle,\rangle)$. It follows that $\widehat{\alpha}^* = -\widehat{\alpha}$. Let $\lambda \in \partial\sigma(\widehat{\alpha})$. By [22, Corollary 3.1] there is a normalized sequence $(x_n)_n$ such that $\lim_{n\to\infty}(\widehat{\alpha}(x_n) - \lambda x_n) = 0$. Since $(\widehat{\alpha} - \lambda I)^* = -\widehat{\alpha} - \overline{\lambda}I$ commutes with $\widehat{\alpha} - \lambda I$ then $\|(\widehat{\alpha} - \lambda I)^*(x)\| = \|(\widehat{\alpha} - \lambda I)x\|$ for all $x \in \widehat{\mathcal{H}}$. Therefore $\lim_{n\to\infty} -(\lambda + \overline{\lambda})x_n = \lim_{n\to\infty}(-\widehat{\alpha}(x_n) - \overline{\lambda}x_n) = 0$ and thus $\Re(\lambda) = 0$. □

Proposition 8.11 *Let $\mathcal{H}$ be a real infinite-dimensional Hilbert space and ω be a symplectic structure on $\mathcal{H}$. Then there do not exist bounded bilinear maps Ω and κ on $X = \mathbb{R} \oplus \mathcal{H}$ with κ compact, Ω an extension of ω and such that $\Omega + \kappa$ is symplectic on X.*

Proof Let $J \in \mathfrak{L}(\mathcal{H})$ be a complex structure given by Lemma 8.1 such that $\omega(x, y) = \langle Jx, y\rangle$ for an equivalent inner product $\langle\, ,\, \rangle$ on $\mathcal{H}$. Let X be equipped with the inner product determined by $\langle(a, x), (b, y)\rangle_X = ab + \langle x, y\rangle$, and let A be the operator on X defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$. Suppose that there exists a compact operator K on X such that $A + K$ is a symplectic isomorphism on X and consider the function from $[0, 1]$ into $\mathfrak{L}(\widehat{X})$ given by $T_\mu = \widehat{A} + \mu\widehat{K}$. As in the proof of [16, Prop. 8], denote by $E(\lambda, T_\mu)$ the spectral projection associated to every $\lambda \in \sigma(T_\mu)$. Let $n(\mu) = \sum_{\lambda\in\mathbb{R}\cap\sigma(T_\mu)} \mathrm{rk}(E(\lambda, T_\mu))$ and let $I_0 = \{\mu \in [0, 1] : n(\mu) \text{ is even}\}$, $I_1 = \{\mu \in [0, 1] : n(\mu) \text{ is odd}\}$.

The operator $\widehat{A}$ is defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \widehat{J} \end{pmatrix}$, therefore has only one real eigenvalue, with associated spectral projection of dimension 1, thus $0 \in I_1$. On the other hand, by Lemma 8.10 we have that T_1 does not have real eigenvalues and therefore $1 \in I_0$. Following essentially the same arguments of [16, Prop. 8], we obtain that I_0 and I_1 are open, and partition $[0, 1]$. Since they are non-empty, we get a contradiction. □

The spectral arguments of the proof of [16, Prop. 8] do not apply when we considering symplectic isomorphisms on Banach spaces. We do not know whether Proposition 8.11 holds for general Banach spaces.

9 Open problems

Question 9.1 *Does every twisted Hilbert space admit a symplectic structure?*

It is not even known if every twisted Hilbert space is isomorphic to its dual.

The spectral group of a symplectic Banach space (X, ω) is the subgroup of operators T in $\mathrm{GL}(X)$ such that $\omega(Tx, Ty) = \omega(x, y)$ for every $x, y \in E$.

Question 9.2 *Is the spectral group of the Kalton–Peck space Z_2 connected?*

Given a complex structure J on a real Banach space X, the complex space X^{-J} is called the *conjugate* of X^J. Bourgain [3] and Kalton [18] (among others) constructed examples of complex structures for which X^J and X^{-J} are not $\mathbb{C}$-isomorphic. These results motivate the following question.

Question 9.3 (Conjugate problem) *Let ω be a symplectic structure on a real Banach space X. Is $\omega \sim -\omega$?*

If the symplectic structure ω is trivial then it is easy to see that $\omega \sim -\omega$. The nontrivial symplectic structure ω on the Kalton–Peck space induced by Proposition 3.1 is equivalent to $-\omega$, since $(x, y) \overset{T}{\mapsto} (-x, y)$ is an isometry such that $T^*\omega T = -\omega$. A similar argument shows that the symplectic structure on Rochberg spaces $\mathfrak{R}^{(n)}$ defined by Theorem 5.2 is equivalent to its 'conjugate'.

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