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Disjointly non-singular operators: Extensions and local variations ☆

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ABSTRACT

The disjointly non-singular (DN-S) operators $T \in \mathrm{L}(E, Y)$ from a Banach lattice E to a Banach space Y are those operators which are strictly singular in no closed subspace generated by a disjoint sequence of non-zero vectors. When E is order continuous with a weak unit, E can be represented as a dense ideal in some $L_1(\mu)$ space, and we show that each $T \in$ DN-S(E, Y) admits an extension $\overline{T} \in$ DN-S$(L_1(\mu), \mathrm{PO})$, where PO is certain Banach space, from which we derive that both T and T^{**} are tauberian operators and that the operator $T^{co} : E^{**}/E \to Y^{**}/Y$ induced by T^{**} is an (into) isomorphism. Also, using a local variation of the notion of DN-S operator, we show that the ultrapowers of $T \in$ DN-S(E, Y) are also DN-S operators. Moreover, when E contains no copies of c_0 and admits a weak unit, we show that $T \in$ DN-S(E, Y) implies $T^{**} \in$ DN-S(E^{**}, Y^{**}).

1. Introduction

In a Banach lattice E we can consider two kinds of closed subspaces: those generated by a disjoint sequence of non-zero vectors, and those that are at a positive distance of every normalized disjoint sequence. The later ones are called *dispersed subspaces* in [12]. In the study of operators acting on E it is useful to consider their action on these kinds of subspaces (see [7]). The *disjointly strictly singular* operators (DSS operators, for short) were introduced in [18] as those operators $T : E \to Y$ from a Banach lattice E into a Banach space Y such that T is an isomorphism on no closed subspace of E generated by a disjoint sequence of non-zero vectors. These operators have been applied to the study of the structure of Banach lattices (see [6] and references therein). More recently, the disjointly non-singular operators (DN-S operators, for short)

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where introduced in [12] as those operators $T : E \to Y$ from a Banach lattice E to a Banach space Y that are strictly singular in no closed subspace of E generated by a disjoint sequence of non-zero vectors. The DN-S operators have also been studied in [1] and [13]. Note that the kernel of a DN-S operator is a dispersed subspace.

By [10, Theorem 2], an operator $T : L_1 \to Y$ is DN-S if and only it is tauberian in the sense of Kalton and Wilansky [20]. In this case the second conjugate $T^{**} : L_1^{**} \to Y^{**}$ and the ultrapowers $T_{\mathcal{U}} : (L_1)_{\mathcal{U}} \to Y_{\mathcal{U}}$ are also DN-S, and the operator $T^{co} : L_1^{**}/L_1 \to Y^{**}/Y$ induced by T^{**} is an (into) isomorphism; see [10,11].

In this paper we extend these results for $E = L_1$ to the operators in DN-S(E, Y) when E is order continuous with a weak unit. Our main tool is the fact that in this case E admits a representation as a dense sublattice of some $L_1(\mu)$ space with μ a probability measure. We characterize the operators in DN-S(E, Y) in terms of their action over the normalized sequences (x_n) in E satisfying $\lim_{n\to\infty} \mu(\operatorname{supp} x_n) = 0$. As a consequence, T is an isomorphism on the closed band E_A of E generated by a measurable set A when $\mu(A)$ is small enough. Moreover, using the push-out construction, we show that every operator $T \in$ DN-S(E, Y) admits an extension $\overline{T} \in$ DN-S$(L_1(\mu), \mathrm{PO})$, where PO is the push-out Banach space. From this result, we derive that each $T \in$ DN-S(E, Y) is a tauberian operator such that T^{**} is tauberian and T^{co} is an (into) isomorphism. Also, using a local variation of the notion of DN-S operator, we prove that the class of DN-S operators is preserved by ultrapowers, we give an example showing that it is not preserved by ultraproducts, and we introduce and study the (n, r)-dispersed subpaces, a local variation of the notion of dispersed subspace. Moreover, when E contains no copies of c_0 and admits a weak unit, we show that $T \in$ DN-S(E, Y) implies $T^{**} \in$ DN-S.

Notations. Throughout the paper X and Y are Banach spaces, E is a Banach lattice and $E_+ = \{x \in E : x \geq 0\}$. The unit sphere of X is $S_X = \{x \in X : \|x\| = 1\}$, and for a sequence (x_n) in X, $[x_n]$ denotes the closed subspace generated by (x_n). We also denote $\mathrm{d}(E) = \{(x_n) \subset E \setminus \{0\} : (x_n) \text{ disjoint}\}$, and $\mathrm{dn}(E) = \{(x_n) \subset S_E : (x_n) \text{ disjoint}\}$.

Operators always are linear and continuous, and $\mathrm{L}(X, Y)$ denotes the set of all operators from X into Y. Given $T \in \mathrm{L}(X, Y)$, $N(T)$ is the kernel of T, $R(T)$ is the range of T, and we denote by T_M the restriction of $T \in \mathrm{L}(X, Y)$ to a closed subspace M of X.

An operator $T \in \mathrm{L}(X, Y)$ is *strictly singular* if there is no closed infinite dimensional subspace M of X such that T_M is an isomorphism; the operator T is *upper semi-Fredholm* if $N(T)$ is finite dimensional and $R(T)$ is closed; and T is *tauberian* if its second conjugate $T^{**} : X^{**} \to Y^{**}$ satisfies $T^{**-1}(Y) = X$ [20]; equivalently, if the operator $T^{co} : X^{**}/X \to Y^{**}/Y$ induced by T^{**} is injective. We refer to [15] for the properties of T^{co}.

2. Preliminaries

An operator $T \in \mathrm{L}(E, Y)$ is *disjointly strictly singular*, and we write $T \in$ DSS(E, Y), if there is no $(x_n) \in \mathrm{d}(E)$ such that $T_{[x_n]}$ is an isomorphism. The class DSS was introduced by Hernández and Rodríguez-Salinas in [18] and [17]. The operator T is *disjointly non-singular*, and we write $T \in$ DN-S(E, Y), if there is no $(x_n) \in \mathrm{d}(E)$ such that $T_{[x_n]}$ is strictly singular. The class DN-S was introduced in [12], and studied in [1] and [13]. Note that DN-S(L_1, Y) is the set of tauberian operators from L_1 into Y (see [10,12]). We refer to [14] and [11] for information on tauberian operators. A closed subspace M of E is *dispersed* if there is no $(x_n) \in \mathrm{dn}(E)$ such that $\lim_{n\to\infty} \operatorname{dist}(x_n, M) = 0$.

A sequence (x_n) in E is *unbounded norm convergent (or un-convergent)* to $x \in E$ if $(|x_n - x| \wedge u)$ converges in norm to 0 for each $u \in E_+$ [23]. In this case we write $x_n \xrightarrow{un} x$.

The disjointly non-singular operators can be characterized as follows.

Theorem 2.1. *[12, Theorems 2.8 and 2.10] For an operator $T \in L(E,Y)$, the following assertions are equivalent:*

(1) *T is disjointly non-singular.*
(2) *For every $(x_n) \in d(E)$, the restriction $T_{[x_n]}$ is an upper semi-Fredholm operator.*
(3) *For every $(x_n) \in dn(E)$, $\liminf_{n\to\infty} \|Tx_n\| > 0$.*
(4) *For every compact operator $S \in L(E,Y)$, $N(T+S)$ is dispersed.*

Theorem 2.2. *[1, Theorem 5.3] Suppose that E is order continuous. For $T \in L(E,Y)$, the following assertions are equivalent:*

(1) *T is disjointly non-singular.*
(2) *For no normalized un-null sequence (x_n) we have $\lim_{n\to\infty} \|Tx_n\| = 0$.*
(3) *There exists $r > 0$ such that for every $(x_n) \in dn(E)$, $\liminf_{n\to\infty} \|Tx_n\| > r$.*

A Banach lattice E is *order continuous* if every net in E decreasing in order to 0 converges in norm to 0; and a *weak unit* in E is a vector $e \in E_+$ such that $\||x| \wedge e\| = 0$ implies $x = 0$. We refer to [21,24] for information on order continuous Banach lattices.

2.1. Representation of order continuous Banach lattices

In [21, Theorem 1.b.14] it is shown that every order continuous Banach lattice E with a weak unit admits a representation as a Köthe function space, in the sense that there exists a probability space (Ω, Σ, μ) so that

- $L_\infty(\mu) \subset E \subset L_1(\mu)$ with E dense in $L_1(\mu)$ and $L_\infty(\mu)$ dense in E,
- $\|f\|_1 \le \|f\|_E \le 2\|f\|_\infty$ when $f \in L_\infty(\mu)$,
- the order in E coincides with the one induced by $L_1(\mu)$.

In the paper, one such representation is fixed for each E order continuous with a weak unit. For vectors in $L_1(\mu)$, we denote by $x_n \xrightarrow{\mu} x$ the *convergence in measure.*

Among the order continuous Banach lattices with a weak unit we have some rearrangement invariant (r.i., for short) function spaces on $(0,1)$. Besides $L_p(0,1)$ $(1 \le p < \infty)$, the most commonly used r.i. function spaces on $(0,1)$ are the Orlicz spaces and the Lorentz spaces (see [21, Section 2a]). Below we give a brief description of the second ones.

Example 2.3. Let $1 \le p < \infty$ and let W be a positive non-increasing continuous function on $(0,1]$ so that $\lim_{t\to 0} W(t) = \infty$ and $\int_0^1 W(t) = 1$. The *Lorentz function space* $L_{W,p}(0,1)$ is the space of all measurable functions f on $(0,1)$ such that

$$\|f\|_{W,p} = \left(\int_0^1 f^*(t)^p W(t) dt \right)^{1/p} < \infty,$$

where f^* is the decreasing rearrangement of $|f|$.

The space $L_{W,p}(0,1)$ is a r.i. function space on $(0,1)$ different from $L_1(0,1)$.

The following result will be useful.

Lemma 2.4. [3, Corollary 2.12, Theorem 4.6] *Let E be an order continuous Banach lattice with a weak unit e, and let $(x_n) \subset E$. Then the following statements are equivalent:*

(1) $x_n \xrightarrow{un} 0$.
(2) $(|x_n| \wedge e)$ *converges in norm to* 0.
(3) $x_n \xrightarrow{\mu} 0$.

For an order continuous Banach lattice E with a weak unit e, we define the *support* of $x \in E$ as $\operatorname{supp} x = \{t \in \Omega : x(t) \neq 0\}$.

Corollary 2.5. *Suppose that E is order continuous with a weak unit. Then each sequence (x_k) in E with $\lim_{n\to\infty} \mu(\operatorname{supp} x_n) = 0$ is un-convergent to 0.*

Proof. Note that $\lim_{n\to\infty} \mu(\operatorname{supp} x_n) = 0$ implies $x_n \xrightarrow{\mu} 0$. □

2.2. Ultraproducts of spaces and operators

Let I be a set admitting a non-trivial ultrafilter $\mathcal{U}$ and let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be families of Banach spaces. The ultraproduct $(X_i)_{\mathcal{U}}$ of $(X_i)_{i\in I}$ is defined as the quotient of $\ell_\infty(I, X_i)$ by the closed subspace

$$N_{\mathcal{U}}(X_i) = \{(x_i) \in \ell_\infty(I, X_i) : \lim_{i\to\mathcal{U}} \|x_i\| = 0\}.$$

The element of $(X_i)_{\mathcal{U}}$ which has $(x_i) \in \ell_\infty(I, X_i)$ as a representative is denoted $[(x_i)]$.

When $X_i = X$ for each $i \in I$, we denote the ultraproduct by $X_{\mathcal{U}}$, and we call it an ultrapower of X.

If each X_i is a Banach lattice then $(X_i)_{\mathcal{U}}$ has a natural structure of Banach lattice: $[(x_i)] \leq [(y_i)]$ if there exists $(z_i) \in N_{\mathcal{U}}(X_i)$ such that $x_i + z_i \leq y_i$ for each $i \in I$.

If $(T_i)_{i\in I}$ is a bounded family of operators with $T_i \in \mathrm{L}(X_i, Y_i)$ for each $i \in I$, the ultraproduct $(T_i)_{\mathcal{U}} \in \mathrm{L}((X_i)_{\mathcal{U}}, (Y_i)_{\mathcal{U}})$ is defined by $(T_i)_{\mathcal{U}}[(x_i)] = [(T_i x_i)]$. When $T_i = T$ for each $i \in I$, we write $T_{\mathcal{U}}$ which is called an ultrapower of T. We refer to [4, Chapter 8] or [16] for additional information on ultraproducts of spaces and operators.

3. Disjointly non-singular operators

We begin with a complement to Theorem 2.2.

Theorem 3.1. *Let E be an order continuous Banach lattice with a weak unit. For $T \in L(E, Y)$, the following assertions are equivalent:*

(1) *T is disjointly non-singular.*
(2) *There exists $r > 0$ such that for every (x_n) in S_E with $\lim_{n\to\infty} \mu(\operatorname{supp} x_n) = 0$, $\liminf_{n\to\infty} \|Tx_n\| > r$.*
(3) *For every (x_n) in S_E with $\lim_{n\to\infty} \mu(\operatorname{supp} x_n) = 0$, $\liminf_{n\to\infty} \|Tx_n\| > 0$.*
(4) *There is $r > 0$ such that for every $x \in S_E$ with $\mu(\operatorname{supp} x) < r$ we have $\|Tx\| > r$.*

Proof. (1)⇒(2) Suppose that T is disjointly non-singular. Without loss of generality, we can assume that $\|T\| = 1$. By Theorem 2.2, there is $r > 0$ such that for every disjoint sequence (z_n) in S_E, $\liminf_{n\to\infty} \|Tz_n\| > r$.

If (2) fails, then we can find a sequence (x_n) in S_E with $\lim_{n\to\infty} \mu(\operatorname{supp} x_n) = 0$ and $\liminf_{n\to\infty} \|Tx_n\| < r/2$. Passing to a subsequence if necessary, we can assume that $\limsup_{n\to\infty} \|Tx_n\| < r/2$ and

$\sum_{n=1}^{\infty} \mu(\operatorname{supp} x_n) < \infty$. We denote $A_n = \cup_{k=n}^{\infty} \operatorname{supp} x_n$ and $B_n = \Omega \setminus A_n$. Then $\mu(A_n) \to 0$, (B_n) increases to Ω and, since E is order continuous, $(x\chi_{B_n})$ converges in norm to x for every $x \in E$ [24, Theorem 1.1].

First, we choose $n_1 > 1$ such that $\|x_1 - x_1\chi_{B_{n_1}}\| < 1/2$, and denote $y_1 = x_1\chi_{B_{n_1}}$. Note that $\|Tx_1 - Ty_1\| < 1/2$ and $|y_1| \wedge |x_j| = 0$ for $j \geq n_1$.

Next, we choose $n_2 > n_1$ such that $\|x_{n_1} - x_{n_1}\chi_{B_{n_2}}\| < 1/3$, and denote $y_2 = x_{n_1}\chi_{B_{n_2}}$. Note that $\|Tx_{n_1} - Ty_2\| < 1/3$ and $|y_i| \wedge |x_j| = 0$ for $i = 1, 2$ and $j \geq n_2$.

Continuing in this way we obtain a disjoint sequence (y_n) such that $\|y_n\| \to 1$ as $n \to \infty$ and $\limsup_{n\to\infty} \|Ty_n\| < r/2$. Thus taking $z_n = y_n/\|y_n\|$, we obtain a normalized disjoint sequence (z_n) with $\limsup_{n\to\infty} \|Tz_n\| < r/2$, and we get a contradiction.

(2)⇒(3) is trivial.

(3)⇒(4) If (4) fails, we can find a sequence (x_n) in S_E with $\mu(\operatorname{supp} x_n) < 1/n$ and $\|Tx_n\| < 1/n$. So (3) also fails.

(4)⇒(1) For every disjoint sequence (x_n) in S_E, $\liminf_{n\to\infty} \|Tx_n\| \geq r$. Thus Theorem 2.2 implies that T is disjointly non-singular. □

If E is order continuous with a weak unit, for each measurable set A with $\mu(A) > 0$, the set $E_A = \{x \in E : \operatorname{supp} x \subset A\}$ is a closed band in E.

From part (4) of Theorem 3.1 we derive:

Corollary 3.2. *If E is order continuous with a weak unit and $T \in DN\text{-}S(E, Y)$, then there exists $r > 0$ such that, when A is a measurable set with $0 < \mu(A) < r$, the restriction of T to E_A is an isomorphism.*

Corollary 3.3. *Let E be an order continuous r.i. function space on $[0, 1]$ and suppose that $DN\text{-}S(E, Y) \neq \emptyset$. Then Y contains a subspace isomorphic to E.*

Proof. Note that the characteristic function $\chi_{[0,1]}$ is a weak unit in E. Moreover, as in [21, Section 2.b], for $0 < s < \infty$ we consider the linear map D_s defined on the space of measurable functions on $[0, 1]$ by

$$(D_s f)(t) = \begin{cases} f(t/s), & t \leq \min\{1, s\} \\ 0, & s < t \leq 1 \quad (\text{in case } s < 1). \end{cases}$$

Clearly D_s has norm one on $L_\infty[0, 1]$ and norm s on $L_1[0, 1]$. Thus, a result of Calderón (see [21, Theorem 2.a.10]) implies that D_s is bounded on E with norm $\leq \max\{1, s\}$.

If $1 < s < \infty$ and $r = s^{-1}$, then D_r is injective and $D_r D_s f = \chi_{[0,r]} f$ (see [21, Section 2.b]); hence D_r is an isomorphism of E onto $E_{[0,r]}$. Moreover, if $T \in \text{DN-S}(E, Y)$ then Corollary 3.2 implies that T is an isomorphism on $E_{[0,r]}$ for r small enough. □

4. Push-outs and DN-S operators

Suppose that E is order continuous with a weak unit. We denote by $\jmath : E \longrightarrow L_1(\mu)$ the inclusion of E into $L_1(\mu)$, which is a (continuous) operator.

Given an operator $T : E \longrightarrow Y$, the *push-out diagram* for the pair $\jmath, T$ is

$$\begin{array}{ccc} E & \xrightarrow{T} & Y \\ {\scriptstyle \jmath}\big\downarrow & & \big\downarrow{\scriptstyle \bar{\jmath}} \\ L_1(\mu) & \xrightarrow{\overline{T}} & \text{PO} \end{array}$$

where $\Delta = \{(Tx, -\jmath x) : x \in E\}$ is a subspace of $Y \oplus_1 L_1(\mu)$ with closure $\overline{\Delta}$, PO is the quotient $(Y \oplus_1 L_1(\mu))/\overline{\Delta}$, and the operators $\bar{\jmath}$ and $\overline{T}$ are defined by $\bar{\jmath}y = (y, 0) + \overline{\Delta}$ and $\overline{T}f = (0, f) + \overline{\Delta}$. See [2, Section 1.3].

Note that $\bar{\jmath}$ and $\overline{T}$ are continuous because they are restrictions of the quotient map onto PO, and the push-out diagram is commutative: $\bar{\jmath}T = \overline{T}\jmath$.

Proposition 4.1. *Let E be an order continuous Banach lattice with a weak unit.*

(1) *The inclusion $\jmath : E \to L_1(\mu)$ is disjointly strictly singular ($\jmath \in$ DSS) if and only if for every $(x_n) \in dn(E)$, $\|\jmath x_n\|_1 \to 0$ as $n \to \infty$.*
(2) *If E is an r.i. function space on $(0, 1)$ different from $L_1(0, 1)$ then $\jmath : E \to L_1(0, 1)$ is always DSS.*

Proof. (1) For the direct implication, suppose that $(x_n) \in \mathrm{dn}(E)$ and $C = \inf_n \|\jmath x_n\|_1 > 0$. Note that (x_n) is an unconditional basic sequence and

$$\left\|\jmath\left(\sum_{i=1}^\infty a_i x_i\right)\right\|_1 = \left\|\sum_{i=1}^\infty a_i \jmath x_i\right\|_1 = \sum_{i=1}^\infty |a_i| \cdot \|\jmath x_i\|_1$$

because $(\jmath x_i)$ is a disjoint sequence in $L_1(\mu)$. Therefore,

$$\left\|\jmath\left(\sum_{i=1}^\infty a_i x_i\right)\right\|_1 \geq C \sum_{i=1}^\infty |a_i| \geq C \left\|\sum_{i=1}^\infty a_i x_i\right\|,$$

and $\jmath$ is an isomorphism on $[x_n]$.

The converse implication is immediate.

(2) is proved in [8, Corollary 4.4]. □

The following result is mentioned without proof in [5], Proposition 1.1 and post comment. For the convenience of the reader we will sketch a proof.

Lemma 4.2. *Let E be an order continuous Banach lattice with a weak unit.*

(1) *For every closed subspace N of E, the restriction of $\jmath$ to N is an isomorphism, or N is not dispersed.*
(2) *For every sequence (x_n) in S_E, $\inf \|x_n\|_1 > 0$ or there exists a subsequence (x_{n_k}) and a disjoint sequence (z_k) in E such that $\|x_{n_k} - z_k\| \to 0$ as $k \to \infty$.*

Proof. (1) is [21, Proposition 1.c.8] in disguise. Next we include part of the proof in [21] which will be helpful to prove (2).

For $x \in E$ and $\varepsilon > 0$, put $\sigma(x, \varepsilon) = \{\omega \in \Omega : |x(\omega)| \geq \varepsilon\|x\|\}$ and consider the set $M(\varepsilon) = \{x \in E : \mu(\sigma(x, \varepsilon)) \geq \varepsilon\}$. If $N \subset M(\varepsilon)$ for some $\varepsilon > 0$ then $\|y\|_1 \geq \varepsilon^2\|y\|$ for every $y \in N$, hence $\jmath|_N$ is an isomorphism. Otherwise we can find $(z_n) \subset S_N$ such that $z_n \notin M(2^{-n})$ for all n, and there is a subsequence (z_{n_i}) and a disjoint sequence (y_i) in E such that $\|z_{n_i} - y_i\| \to 0$ as $i \to \infty$. Hence N is not dispersed.

(2) If (x_n) in S_E and $\inf \|x_n\|_{L_1} = 0$, then we can find a subsequence (x_{n_k}) with $\|x_{n_k}\|_1 < 2^{-2k}$ for each k. Thus $x_{n_k} \notin M(2^{-k})$ and, as in the proof of (1), passing to a further subsequence if necessary, we can find a disjoint sequence (z_k) in E such that $\|x_{n_k} - z_k\| \to 0$ as $k \to \infty$. □

Remark 4.3. By Proposition 4.1, both alternatives in Lemma 4.2 become dichotomies if and only if $\jmath : E \to L_1(\mu)$ is DSS.

Indeed, if $\jmath \notin$ DSS then $\jmath$ is an isomorphism on the closed subspace generated by a normalized disjoint sequence, which is not dispersed. Conversely, if there exists a closed subspace N which is not dispersed and $\jmath|_N$ is an isomorphism, then each normalized disjoint sequence (x_n) with $\lim_{n\to\infty} \mathrm{dist}(x_n, N) = 0$ has a subsequence (x_{n_k}) such that $\jmath$ is an isomorphism on $[x_{n_k}]$; hence $\jmath \notin \mathrm{DSS}$.

We consider the injective operator $D : E \to Y \oplus_1 L_1(\mu)$ defined by $Dx = (Tx, -\jmath x)$.

Proposition 4.4. *Suppose that E is an order continuous Banach lattice with a weak unit, and let $T \in L(E, Y)$.*

(1) *If $T \in$ DN-S, then Δ is a closed subspace of $Y \oplus_1 L_1(\mu)$ and $\bar{\jmath}$ is injective.*
(2) *If $\jmath : E \to L_1(\mu)$ is DSS and Δ is closed in $Y \oplus_1 L_1(\mu)$ then $T \in$ DN-S.*

Proof. (1) Suppose that $\Delta = R(D)$ is not closed. Then there is a sequence (x_n) in S_E such that $\|Dx_n\| = \|Tx_n\| + \|x_n\|_1 \to 0$ as $n \to \infty$. By part (2) in Lemma 4.2, there is a subsequence (x_{n_k}) and a disjoint sequence (z_k) in E such that $\|x_{n_k} - z_k\| \to 0$. Hence $\|Tz_k\| \to 0$, and Theorem 2.1 implies that $T \notin$ DN-S.

Also, $\bar{\jmath}y = 0$ implies $(y, 0) \in \overline{\Delta} = \Delta$; thus $(y, 0) = (Tx, -\jmath x)$ for some $x \in E$. Since $\jmath$ is injective, $x = 0$ and $y = Tx = 0$. Hence $\bar{\jmath}$ is injective.

(2) Suppose that $T \notin$ DN-S. Then there exists $(x_n) \in \mathrm{dn}(E)$ such that $\|Tx_n\| \to 0$. Since $\|\jmath x_n\|_1 \to 0$, $R(D) = \Delta$ is non-closed. □

When both $\jmath$ and $\bar{\jmath}$ are injective, we can see $\overline{T}$ as an extension of T.

Theorem 4.5. *Suppose that E is an order continuous Banach lattice with a weak unit, and let $T \in$ DN-S(E, Y).*

(1) *$\overline{T} \in$ DN-S$(L_1(\mu), \mathrm{PO})$; equivalently, $\overline{T}$ is tauberian.*
(2) *T is a tauberian operator.*
(3) *T^{**} is tauberian and T^{co} is an (into) isomorphism.*

Proof. Since $T \in$ DN-S(E, Y), the subspace Δ is closed by Proposition 4.4.

(1) Let (f_n) be a disjoint sequence in $S_{L_1(\mu)}$. Then $\overline{T}f_n = (0, f_n) + \Delta$ and

$$\|\overline{T}f_n\| = \inf_{x\in E} \|Tx\| + \|f_n - \jmath x\|_{L_1}.$$

Since $\liminf_{n\to\infty} \|f_n - \jmath x\|_{L_1} \geq 1$ for each $x \in E$, we get $\liminf_{n\to\infty} \|\overline{T}f_n\|_{L_1} \geq 1$, hence $\overline{T} \in$ DN-S by Theorem 2.1.

(2) Note that $\overline{T}f = 0 \Leftrightarrow (0, f) \in \Delta \Leftrightarrow f = \jmath x$ for some $x \in E$ and $Tx = 0$. Then $N(\overline{T}) = \jmath(N(T))$, which is closed. Hence $\jmath|_{N(T)}$ is an isomorphism onto $N(\overline{T})$. Since $\overline{T}$ is tauberian, $N(\overline{T})$ is reflexive, hence so is $N(T)$.

Now, if $T \in$ DN-S and $S \in \mathrm{L}(E, F)$ is compact, then $T + S \in$ DN-S [12, Corollary]. Therefore $N(T + S)$ is reflexive for each compact S, hence T is tauberian by the main result of [14].

(3) The argument we gave in the proof of (2) shows that each $T \in$ DN-S(E, Y) is supertauberian in the sense of [9], because each reflexive subspace of $L_1(\mu)$ is superreflexive and supertauberian operators admit a perturbative characterization: $T \in \mathrm{L}(X, Y)$ is supertauberian if and only if $N(T + K)$ is superreflexive for each compact operator $K \in \mathrm{L}(X, Y)$ [9, Theorem 15]. Moreover, if T is supertauberian then T^{co} is an (into) isomorphism [11, Proposition 6.5.3], and the last fact implies T^{**} tauberian because $(T^{co})^{**} \equiv (T^{**})^{co}$ is injective in this case. □

Question 1. *Suppose that E is an order continuous Banach lattice with a weak unit.*

Is it true that $\overline{T} \in$ DN-S$(L_1(\mu), \mathrm{PO})$ implies $T \in$ DN-S(E, Y)?

We conjecture that, under the hypothesis of Theorem 4.5, $T^{**} \in$ DN-S. Next we prove a special case of this conjecture. Observe that, for E, containing no copies of c_0 is slightly stronger than being order continuous [24, Chapter 7].

Proposition 4.6. *Suppose that E is a Banach lattice with a weak unit that contains no copies of c_0, and let $T \in$ DN-S(E, Y). Then $T^{**} \in$ DN-S(E^{**}, Y^{**}).*

Proof. Since E contains no copies of c_0, the canonical copy of E in E^{**} is a projection band [21, Theorem 1.c.4]. Thus, denoting $E^{\perp} = \{z \in E^{**} : |x| \wedge |z| = 0 \text{ for each } x \in E\}$, we have that $E^{**} = E \oplus E^{\perp}$. Let P denote the projection on E^{**} onto E with kernel $E^{\perp}$, and let $q : Y^{**} \to Y^{**}/Y$ denote the quotient map.

By part (3) in Theorem 4.5, T^{co} is an isomorphism (into); hence so is qT^{**} on $E^{\perp}$. Therefore, given a normalized disjoint sequence (z_n) in E^{**} and denoting $x_n = Pz_n$ and $y_n = (I - P)z_n$, the sequence (x_n) is disjoint in E and there exists $C > 0$ such that $\|T^{**}z_n\| \geq C \max\{\|Tx_n\|, \|qT^{**}y_n\|\}$ for each n. Hence $\liminf \|T^{**}z_n\| > 0$, and we conclude $T^{**} \in$ DN-S. □

5. Ultraproducts of operators

Here we prove the stability of the class of DN-S operators under ultrapowers when E is order continuous with a weak unit. The following local variation of the notion of DN-S operator will be useful.

Definition 5.1. Let $n \in \mathbb{N}$ and $r > 0$. An operator $T \in \mathrm{L}(E, Y)$ is in the class DN-S$_{n,r}$ if for each normalized disjoint $(x_i)_{i=1}^n$ in E we have $\max_{1 \leq i \leq n} \|Tx_i\| \geq r$.

Next we state a characterization of the class DN-S$_{n,r}$ that was given in the proof of [19, Lemma 2.2] for $E = Y$ a L_1 space. For $x \in E$ we write $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$.

Proposition 5.2. *Suppose that E is a Banach lattice and $T \in L(E, Y)$. Then $T \in$ DN-S$_{n,r}$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $x_1, \ldots, x_n \in S_E$ and $\||x_i| \wedge |x_j|\| < \delta$ for $1 \leq i < j \leq n$ then $\max_{1 \leq i \leq n} \|Tx_i\| > r - \varepsilon$.*

Proof. For the direct implication, if $x_1, \ldots, x_n \in S_E$ satisfy $\||x_i| \wedge |x_j|\| < \delta$ for $i \neq j$, we define $z_i = z_i^+ - z_i^-$ by

$$z_i^+ = x_i^+ - \big(x_i^+ \wedge (\vee_{j \neq i} x_j^+)\big) \quad \text{and} \quad z_i^- = x_i^- - \big(x_i^- \wedge (\vee_{j \neq i} x_j^-)\big).$$

Then the vectors z_i are pairwise disjoint and $1 - 2n\delta \leq \|z_i\| \leq 1$. Applying the DN-S$_{n,r}$ condition to $(z_i/\|z_i\|)_{i=1}^n$ we get $\max_{1 \leq i \leq n} \|Tx_i\| > r - \varepsilon$ if $\delta = \delta(\varepsilon, n, \|T\|)$ is small enough.

The converse implication is immediate. □

The next result was proved in [19] for operators acting on a L_1 space using Kakutani's representation theorem.

Proposition 5.3. *Suppose that E is order continuous with a weak unit. An operator $T \in L(E, Y)$ is in DN-S if and only if $T \in$ DN-S$_{n,r}$ for some $n \in \mathbb{N}$ and $r > 0$.*

Proof. If $T \in$ DN-S$_{n,r}$, then for every $(x_n) \in \mathrm{dn}(E)$, $\liminf_{n \to \infty} \|Tx_n\| \geq r$. Thus, by Theorem 2.1, $T \in$ DN-S.

Conversely, suppose that $T \in$ DN-S$_{n,r}$ for no pair $n \in \mathbb{N}$ and $r > 0$. Then for each $n \in \mathbb{N}$ we can find a normalized disjoint $(x_i^n)_{i=1}^n$ with $\max_{1 \leq i \leq n} \|Tx_i^n\| \leq 1/n$. This fact contradicts (4) in Theorem 3.1, hence $T \notin$ DN-S. □

As a consequence, DN-S$_{n,r}$ is stable under ultraproducts:

Proposition 5.4. *Suppose that E_i is order continuous with a weak unit for each $i \in I$. Let $\mathcal{U}$ be a non-trivial ultrafilter on I. If $(T_i)_{i\in I}$ is a bounded family with $T_i \in$ DN-S$_{n,r}(E_i, Y_i)$ for each $i \in I$ then $(T_i)_\mathcal{U} \in$ DN-S$_{n,r}$.*

Proof. Two vectors $[(x_i)], [(y_i)]$ in $(E_i)_\mathcal{U}$ are disjoint if and only if $\lim_{i\to\mathcal{U}} \| |x_i| \wedge |y_i| \| = 0$. In this case, for each $\delta > 0$, $\{i \in I : \| |x_i| \wedge |y_i| \| < \delta\} \in \mathcal{U}$; hence we can choose the representatives $(x_i), (y_i)$ so that $\| |x_i| \wedge |y_i| \| < \delta$ for every $i \in I$. Since $(T_i)_{i\in I}$ is bounded, for each $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon, n)$ in Proposition 5.2 which is valid for all T_i, and conclude that $(T_i)_\mathcal{U} \in$ DN-S$_{n,r}$. □

Corollary 5.5. *Suppose that E is order continuous with a weak unit, and let $\mathcal{U}$ be a non-trivial ultrafilter. If $T \in$ DN-S(E, Y) then the ultrapower $T_\mathcal{U} \in$ DN-S.*

As a consequence of the following observation, we shall show that the class of DN-S operators is not stable under ultraproducts, by constructing a sequence $(q_k) \subset$ DN-S such that $(q_k) \subset$ DN-S$_{n,r}$ for no pair (n, r).

Remark 5.6. It follows from [12, Proposition 2.12] that a closed subspace M of E is dispersed if and only if the quotient map $q : E \to E/M$ is a DN-S operator.

Example 5.7. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. By [21, Corollary 2.f.5], for each $k \in \mathbb{N}$ with $k \geq 2$, there exists a subspace M_k of $L_1 \equiv L_1(0, 1)$ isometric to $L_{1+1/k}$. Since reflexive subspaces of $L_1(0, 1)$ are dispersed, if $q_k : L_1 \to L_1/M_k$ is the quotient map, then $q_k \in$ DN-S for each $k \in \mathbb{N}$ by Remark 5.6. Let us see that $(q_k)_\mathcal{U} \notin$ DN-S.

Since $(q_k)_\mathcal{U}$ acts on $(L_1)_\mathcal{U}$, which is a $L_1(\mu)$ space [16, Theorem 3.3], it is enough to show that $\ker(q_k)_\mathcal{U}$ is not reflexive, and this is true because it is not isomorphic to a subspace of $L_q(\mu)$ for some $q > 1$, by the main result of [22].

The previous example also shows that the class of dispersed subspaces is not stable under ultraproducts: each $\ker q_k$ is dispersed, but $\ker(q_k)_\mathcal{U}$ is not. However, we can prove the stability for a local variation of the notion of dispersed subspace.

Definition 5.8. Suppose that E is order continuous with a weak unit, and let $n \in \mathbb{N}$ and $r > 0$. A closed subspace M of E is (n, r)-*dispersed* if for each disjoint set $\{x_1, \dots, x_n\}$ in S_E there exists $i \in \{1, \dots, n\}$ so that $\text{dist}(x_i, M) \geq r$.

In the conditions of Definition 5.8, a closed subspace M of E is (n, r)-dispersed if and only if the quotient map onto E/M is a DN-S$_{n,r}$ operator. Therefore, by Proposition 5.3, M is dispersed if and only if it is (n, r)-dispersed for some n and r.

Proposition 5.9. *Suppose that E_i is order continuous with a weak unit for each $i \in I$. Let $\mathcal{U}$ be a non-trivial ultrafilter on I. If for each $i \in I$, M_i is a (n, r)-dispersed subspace then $(M_i)_\mathcal{U}$ is a (n, r)-dispersed subspace of $(E_i)_\mathcal{U}$.*

Proof. It is a direct consequence of Proposition 5.4 and Remark 5.6. □

Observe that $T \in$ DN-S$_{n,r}$ and $c > 0$ implies $cT \in$ DN-S$_{n,cr}$; hence $N(T)$ is not necessarily (n, r)-dispersed, and there is no perturbative characterization for $T \in$ DN-S$_{n,r}$.

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