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On Gegenbauer Point Processes on the Unit Interval

Carlos Beltrán[1] · **Antonia Delgado**[2] · **Lidia Fernández**[2] ·
Joaquín Sánchez-Lara[2]

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Abstract

In this paper we compute the logarithmic energy of points in the unit interval [-1,1] chosen from different Gegenbauer Determinantal Point Processes. We check that all the different families of Gegenbauer polynomials yield the same asymptotic result to third order, we compute exactly the value for Chebyshev polynomials and we give a closed expression for the minimal possible logarithmic energy. The comparison suggests that DPPs cannot match the value of the minimum beyond the third asymptotic term.

Keywords Point distributions · Gegenbauer polynomials · Determinantal point processes · Fekete points

Mathematics Subject Classification (2010) 33C45 · 60G55

1 Introduction

Let $\mathrm{K} \subseteq \mathbb{R}^d$ be an infinite compact set. The logarithmic energy of n points $x_1, \ldots, x_n \in \mathrm{K}$, $x_i \neq x_j$ for $i \neq j$ is

$$\mathrm{E}_{\log}(x_1, \ldots, x_n) = -\sum_{i \neq j} \log \|x_i - x_j\|. \tag{1}$$

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✉ Carlos Beltrán
beltranc@unican.es

[1] Universidad de Cantabria, Santander, Spain

[2] Universidad de Granada, Granada, Spain

The minimum value of this energy is then

$$\mathcal{E}_{log}(\mathrm{K}, n) = \min_{x_1,\ldots,x_n \in \mathrm{K}} \mathrm{E}_{\log}(x_1, \ldots, x_n).$$

Points which minimize the logarithmic energy (i.e. points with energy equal to $\mathcal{E}_{log}(\mathrm{K}, n)$) are called Fekete points of K. Note that an alternative formulation is: a collection of n points in K is a set of Fekete points if the product of their mutual distances is as large as it can be.

The study of Fekete points is an area of very active research, see the classical surveys [19, 24] or the recent and very complete monography [9]. The case that $\mathrm{K} = \mathbb{S}^2 \subseteq \mathbb{R}^3$ is the unit sphere is the topic of problem number 7 in Smale's list [21] (the problem was actually posed by Shub and Smale in [20]). In particular, it is of greatest interest to describe the value of $\mathcal{E}_{log}(\mathrm{K}, n)$.

This problem of finding Fekete points is fully solved if $\mathrm{K} = [-1, 1]$ is the unit interval since Fejer [12]: the unique minimizer of Eq. 1 is the set consisting on the 2 extremes of the interval, plus the $n-2$ zeros of a Gegenbauer polynomial, namely $C_{n-2}^{3/2}$. Recall that for any fixed $\lambda > 0$, the sequence of Gegenbauer polynomials C_n^λ is a sequence of polynomials of degree $n = 0, 1, 2, \ldots$ such that

$$C_n^\lambda(1) = \binom{2\lambda + n - 1}{n}, \quad \int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x) w^\lambda(x)\, dx = 0 \quad \text{for } m \neq n,$$

where

$$w^\lambda(x) = \frac{\Gamma(\lambda + 1)}{\sqrt{\pi}\,\Gamma(\lambda + 1/2)}(1 - x^2)^{\lambda - 1/2}. \tag{2}$$

That is to say, they are a sequence of orthogonal polynomials w.r.t. the weight $w^\lambda(x)$. We have not found in the literature an explicit value of the minimal logarithmic energy in the unit interval[1] (which is a nontrivial calculation from the description of Fekete points). This is our first main result:

Theorem 1 *Let $\epsilon_n = \mathcal{E}_{log}([-1, 1], n)$ be the energy of n Fekete points in the interval* $[-1, 1]$. *Then,*

$$\epsilon_n = -n(n-1)\log(2) - 3(n-1)\log(n-1) - n\log(n) - 4\sum_{j=1}^{n-2} j\log(j) + \sum_{j=1}^{2n-2} j\log(j)$$

$$= \log(2)n^2 - n\log(n) - 2\log(2)n - \frac{1}{4}\log(n) + O(1).$$

Zeros of sequences of orthogonal polynomials are known to be well–distributed according to different criteria, and more generally in other contexts zeros of functions of increasing degree exhibit good separation properties and attain somehow low values of the logarithmic energy. For example, the eigenvalues of $n \times n$ matrices of the form $A^{-1}B$ where A, B are random with complex Gaussian entries (called the *spherical ensemble* [2, 17]) taken onto the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ through the inverse stereographic projection have, on the average, a logarithmic energy that matches the minimum value $\mathcal{E}_{log}(\mathbb{S}^2, n)$ up to second order asymptotics. The zeros of random polynomials whose coefficients are complex Gaussians with some carefully chosen variances, sent again to the sphere with the same projection, have even better properties regarding logarithmic energy, see [4].

[1] While this paper was being refereed, we learnt that J. S. Brauchart had came to an alternative expression for the same quantity, see [10].

However in general for other sets it is not possible to find families of functions whose zeros have these good properties, and it is thus quite common to take alternative approaches. A very popular method is that of Determinantal Point Processes (DPPs) that we briefly describe now (see [16] for a detailed description of the theory).

Let $n \geq 1$. We endow the infinite compact set K with some measure σ, and we let $\mathcal{H} \subseteq L^2(\mathrm{K}, \sigma)$ be some $(n+1)$–dimensional linear subspace of the space of squared integrable functions defined on K. Following [16], the Macci-Soshnikov theorem implies that there exists a random point process associated with $\mathcal{H}$ that has the following properties:

1. With probability 1, the random process outputs $n+1$ different points in K.
2. For any given σ–integrable function $f : \mathrm{K} \times \mathrm{K} \to \mathbb{R}$ (possibly undefined in the diagonal), the expected value of the sum $\sum_{i \neq j} f(x_i, x_j)$ when $x_0, \ldots, x_n$ are the output of the random process, is given by the formula

$$\mathrm{E}\left(\sum_{i \neq j} f(x_i, x_j)\right) = \iint_{x,y \in \mathrm{K}} (K_{\mathcal{H}}(x,x) K_{\mathcal{H}}(y,y) - |K_{\mathcal{H}}(x,y)|^2) f(x,y)\, d\sigma(x)\, d\sigma(y), \tag{3}$$

where $K_{\mathcal{H}} : \mathrm{K} \times \mathrm{K} \to \mathbb{R}$ is the projection kernel onto $\mathcal{H}$. In other words, if $f_0, \ldots, f_n$ is an orthogonal basis of $\mathcal{H}$ then

$$K_{\mathcal{H}}(x, y) = \sum_{j=0}^{n} f_j(x) f_j(y), \quad x, y \in \mathrm{K}.$$

The second property implies that the random points exhibit some repulsion, and has been used to give upper bounds on the minimum value of the energy $\mathcal{E}_{\log}(\mathrm{K}, n)$ (and other energies) for different sets: in [2] for the 2–sphere $\mathbb{S}^2$, in [7] for the d–sphere $\mathbb{S}^d$, in [5] (see also [8]) for the complex projective space, in [3] for 2–point homogeneous spaces, in [18] for the flat torus and in [6] for the rotation group $SO(3)$.

Despite all this success, an important gap in the theory remains open: how do DPPs compare to optimal point distribution in the case of the unit interval? Namely, although in that case we know exactly the position of the Fekete points and the value of the minimal energy (Theorem 1), it seems that no deep study has been made of the value of the energy of points coming from the DPPs in that interval (see [15] for some computations that would lead to first order asymptotics), and a comparison of the power of this technique with the exact solution is in order.

In this paper we study the following questions:

I) Given a measure σ in $[-1, 1]$, the most straightforward choice for the $(n+1)$–dimensional subspace $\mathcal{H} \subseteq L^2([-1, 1], \sigma)$ is the subspace generated by the first $n+1$ orthogonal polynomials associated to σ. Which is the optimal choice of σ? Namely, which choice of σ gives the smallest value for Eq. 3 with $f(x_i, x_j) = -\log \|x_i - x_j\|$?

II) Fekete points are actually the 2 extreme points, plus the zeros of a Gegenbauer polynomial. Is it better also to take the extreme points and then the points coming from a DPP, or is it better to just take the points coming from the DPP?

III) A collection of natural choices for σ is given by $d\sigma = w^\lambda dx$ where $\lambda \in (-1/2, \infty)$, w^λ is given by Eq. 2, and dx is the Lebesgue measure in $[-1, 1]$. The associated subspace for $n+1$ points is then generated by the first $n+1$ Gegenbauer polynomials C_k^λ, $0 \leq k \leq n$. How does the expected energy of points coming from these processes, computed via Eq. 3, compare to the minimal energy ϵ_{n+1} given in Theorem 1?

We now state our second main result which solves item III) and its corollary that gives a partial answer to item II).

Theorem 2 *Let* $d\sigma = w^\lambda dx$, $\lambda \in (-1/2, \infty)$ *fixed, and let* $\mathcal{H}_{n+1}$ *be the subspace generated by the first* $n+1$ *orthogonal polynomials associated to* σ. *In other words,*

$$\mathcal{H}_{n+1} = Span(C_0^\lambda, \ldots, C_n^\lambda).$$

Then, the expected value of the logarithmic energy of the corresponding DPP is

$$\mathrm{E}(\lambda, n+1) = (n+1)^2 \log 2 - (n+1)\log(n+1) + (1-\gamma - 2\log 2)n + o(n),$$

where γ *is the Euler–Mascheroni constant. In particular, the dependence on* λ *falls into the remainder* $o(n)$. *Moreover, in the case* $\lambda = 0$ *(which corresponds to Chebyshev polynomials) we get the exact expression*

$$\begin{aligned}\mathrm{E}(0, n+1) &= (n+1)^2 \log 2 - \left((n+1)\log 2 + \frac{3}{4}H_n + nH_{2n-1} + \frac{H_{2n}}{2} - n + \frac{1}{2}\right) \\ &= (n+1)^2 \log 2 - (n+1)\log(n+1) + (1-\gamma - 2\log 2)n - \frac{1}{4}\log n + O(1),\end{aligned}$$

where $H_n = 1 + 1/2 + \cdots + 1/n$ *is the n–th harmonic number.*

Comparing the asymptotics of Theorem 2 with ϵ_{n+1} from Theorem 1 we conclude that the points generated by the DPP have a greater constant in the $O(n)$ term: $1-\gamma-2\log 2$ versus $-2\log 2$. Thus, there is an excess of $(1-\gamma)n \approx 0.423n$. It thus seems that the power of the technique of DPPs is insufficient to fit the correct $O(n)$ term in the minimal energy expression.

Our partial answer to item II) above is:

Corollary 3 *Let* $\lambda \in (-1/2, \infty)$ *and consider the two following point processes in the unit interval:*

1. *Generate* $n+3$ *points using the point process of Theorem 2.*
2. *Generate* $n+1$ *points using the point process of Theorem 2, and add the two extremes* ± 1 *of the unit interval.*

Then, the expected energy of the two point processes is equal up to $o(n)$. *Moreover, if* $\lambda = 0$ *then the first process has smaller energy than the second one, the difference being in the* $O(\log n)$ *term.*

We finish the introduction suggesting a solution to the optimal measure problem:

Conjecture 4 *The answer to Question I) above is: for any fixed* $n \geq 1$, *the optimal measure is* $d\sigma = w^\lambda dx$ *for some* $\lambda \in (-1/2, \infty)$. *The value of* λ *may depend on* n.

The rest of the paper is structured as follows. In Section 2 we prove the first main result (Theorem 1). Later, in Section 3 we present the integrals that will be involved in our second main result (Theorem 2), which will be proved for $\lambda = 0$ and $\lambda \neq 0$ in Sections 4 and 5, respectively. In Section 6 the proof of Corollary 3 is given. Finally, we collect in Appendices A and B a number of auxiliary results which are used in the rest of the sections.

2 Proof of Theorem 1

Recall Jacobi's polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}x^n + O(x^{n-1})$$

that are a generalization of Gegenbauer's polynomials, in particular satisfying $C_{n-2}^{3/2} = \frac{n}{2}P_{n-2}^{(1,1)}$, see for example [1] where we also find the following classical facts:

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!} \qquad \Rightarrow \qquad P_{n-2}^{(1,1)}(1) = \frac{(2)_{n-2}}{(n-2)!} = n-1.$$

Let κ be the leading coefficient of $P_{n-2}^{(1,1)}$,

$$\kappa = \frac{1}{2^{n-2}}\binom{2n-2}{n-2} = \frac{1}{2^{n-2}}\frac{(n+1)_{n-2}}{(n-2)!}.$$

We need to analyze the energy of the n point set consisting of $x_1 = -1$, $x_n = 1$, and $x_2, \dots, x_{n-1}$ the zeros of $P_{n-2}^{(1,1)}$, that is

$$\begin{aligned}
\epsilon_n &= \sum_{i\neq j}\log\frac{1}{|x_i-x_j|} = \\
&= -2\log 2 - 2\log\prod_{i=2}^{n-1}(1-x_i) - 2\log\prod_{i=2}^{n-1}(1+x_i) - 2\log\prod_{1<i<j<n}|x_i-x_j| \\
&= -2\log 2 - 2\log(P_{n-2}^{(1,1)}(1)) - 2\log((-1)^{n-2}P_{n-2}^{(1,1)}(-1)) + 4\log\kappa - 2\log\prod_{1<i<j<n}|x_i-x_j| \\
&= -2\log 2 - 4\log(P_{n-2}^{(1,1)}(1)) + 4\log\kappa - 2\log\prod_{1<i<j<n}|x_i-x_j| \\
&= -2\log 2 - 4\log(n-1) + 4\log\kappa - 2\log\prod_{1<i<j<n}|x_i-x_j| \\
&= -2\log 2 - 4\log(n-1) + (4+2(n-2)-2)\log\kappa - \log\left(\kappa^{2(n-2)-2}\prod_{1<i<j<n}(x_i-x_j)^2\right) \\
&= -2\log 2 - 4\log(n-1) + 2(n-1)\log\kappa - \log D_{n-2}^{(1,1)},
\end{aligned}$$

where

$$D_{n-2}^{(1,1)} = 2^{-(n-2)(n-3)}\prod_{j=1}^{n-2} j^{j-2n+6}(j+1)^{2j-2}(n+j)^{n-2-j}$$

is the discriminant of $P_{n-2}^{(1,1)}$, whose value is known (see [22, Th. 6.71, p. 143]). Plugging this and the value of κ in the formula for ϵ_n we thus conclude:

$$\begin{aligned}
\epsilon_n &= -n(n-1)\log 2 - 4\log(n-1) + 2(n-1)\left(\sum_{j=1}^{n-2}\log(n+j) - \sum_{j=1}^{n-2}\log(j)\right) \\
&\quad - \sum_{j=1}^{n-2}\Big((j-2n+6)\log(j) + 2(j-1)\log(j+1) + (n-2-j)\log(n+j)\Big).
\end{aligned}$$

Expanding and simplifying this expression we get the simpler formula claimed in the theorem. Using

$$\log(n-a) = \log(n) + \log\left(\frac{n-a}{n}\right) = \log(n) + \log\left(1 - \frac{a}{n}\right) = \log(n) - \frac{a}{n} + O(n^{-2})$$

and Lemma 24 we also get the value for the asymptotic expansion of ϵ_n, finishing the proof of the theorem.

3 First Integral Formulas

In order to compute the expected value of the logarithmic energy for points coming from the DPPs, we need to write down the integral in Eq. 3 for our choice of σ. To do so, we first need an orthonormal sequence of polynomials. Fix $\lambda \in (-1/2, \infty)$, $\lambda \neq 0$, and consider the normalized Gegenbauer polynomials

$$\widehat{C}_n^\lambda = \gamma_n^\lambda C_n^\lambda, \qquad \text{with} \quad \gamma_n^\lambda = \sqrt{\frac{n!(n+\lambda)}{\lambda(2\lambda)_n}},$$

which describe an orthonormal basis of the subspace they span.

Recall the following identities

$$\begin{aligned} \frac{d}{dx} C_n^\lambda(x) &= 2\lambda C_{n-1}^{\lambda+1}(x), \\ \frac{d}{dx}\left[C_{n-1}^{\lambda+1}(x) w^{\lambda+1}(x)\right] &= -\frac{n(n+2\lambda)(\lambda+1)}{\lambda(2\lambda+1)} C_n^\lambda(x) w^\lambda(x). \end{aligned}$$

We can now compute the kernel K_n^λ (for the point process of $n+1$ points) using the Christoffel–Darboux summation formula (see [22]):

$$\begin{aligned} K_n^\lambda(x, y) &= \sum_{j=0}^{n} \widehat{C}_j^\lambda(x)\widehat{C}_j^\lambda(y) = \sum_{j=0}^{n} \frac{j!(j+\lambda)}{\lambda(2\lambda)_j} C_j^\lambda(x) C_j^\lambda(y) \\ &= \frac{(n+1)!}{2\lambda(2\lambda)_n} \frac{C_{n+1}^\lambda(x)C_n^\lambda(y) - C_n^\lambda(x)C_{n+1}^\lambda(y)}{x-y}. \end{aligned}$$

The formula above is valid for $x \neq y$. The case $x = y$ is obtained by taking limits:

$$\begin{aligned} K_n^\lambda(x, x) &= \frac{(n+1)!}{2\lambda(2\lambda)_n}\left(C_n^\lambda(x)(C_{n+1}^\lambda)'(x) - C_{n+1}^\lambda(x)(C_n^\lambda)'(x)\right) \\ &= \frac{(n+1)!}{(2\lambda)_n}\left(C_n^\lambda(x)C_n^{\lambda+1}(x) - C_{n+1}^\lambda(x)C_{n-1}^{\lambda+1}(x)\right). \end{aligned}$$

The case $\lambda = 0$, corresponding to $w(x) = w^0(x) = \pi^{-1}(1-x^2)^{-1/2}$, must be done independently since some of the expressions above make no sense if $\lambda = 0$. The orthogonal polynomials are just the classical Chebyshev polynomials after some normalization:

$$\widehat{C}_n(x) = \widehat{C}_n^0(x) = \widehat{C}_n^\lambda(x)\big|_{\lambda=0} = \begin{cases} 1, & \text{if } n = 0, \\ \sqrt{2}T_n(x) = \sqrt{2}\cos(n \arccos x), & \text{if } n \geq 1. \end{cases}$$

The kernel also admits a simpler expression

$$K_n^0(\cos\theta,\cos\varphi) = \sum_{j=0}^{n} \widehat{C}_j^0(\cos\theta)\widehat{C}_j^0(\cos\varphi) = 1+2\sum_{j=1}^{n}\cos(j\theta)\cos(j\varphi),$$

$$K_n^0(\cos\theta,\cos\theta) = 1+2\sum_{j=1}^{n}\cos^2(j\theta) = 1+n+\frac{\cos((n+1)\theta)\sin(n\theta)}{\sin\theta},$$

where for the last sum we have used [14, Sec. 1.35]. According to Eq. 3 we aim to compute, for any given $n \geq 2$ and $\lambda \in (-1/2,\infty)$, the integrals

$$L_1 = L_1(\lambda,n) = \iint_{[-1,1]^2} K_n^\lambda(x,x)K_n^\lambda(y,y)w^\lambda(x)w^\lambda(y)\log\frac{1}{|x-y|}\,d(x,y),$$

$$L_2 = L_2(\lambda,n) = \iint_{[-1,1]^2} K_n^\lambda(x,y)^2 w^\lambda(x)w^\lambda(y)\log\frac{1}{|x-y|}\,d(x,y).$$

The expected value computed in Theorem 2 is equal to $L_1 - L_2$. In the case $\lambda = 0$ we succeed in computing these integrals exactly. In the rest of the cases, we get to an asymptotic value up to $o(n)$.

4 Proof of Theorem 2 for $\lambda = 0$

We first compute L_1 and L_2 for the case $\lambda = 0$. For simplicity, we omit in this section the dependence on λ. Note that

$$\begin{aligned} L_1 &= \iint_{[-1,1]^2} K_n(x,x)K_n(y,y)\log\frac{1}{|x-y|}w(x)w(y)\,d(x,y) \\ &= \log 2\iint_{[-1,1]^2} K_n(x,x)K_n(y,y)w(x)w(y)\,d(x,y) \\ &\quad + \iint_{[-1,1]^2} K_n(x,x)K_n(y,y)\log\frac{1}{2|x-y|}w(x)w(y)\,d(x,y) \\ &= (n+1)^2\log 2 + \sum_{k,\ell=0}^{n}\mathcal{J}_{k,\ell}, \end{aligned}$$

where

$$\mathcal{J}_{k,\ell} = \iint_{[-1,1]^2} \widehat{C}_k(x)^2\widehat{C}_\ell(y)^2 w(x)w(y)\log\frac{1}{2|x-y|}\,d(x,y).$$

Proposition 5 *Let $k,\ell \geq 0$. Then.*

$$\begin{cases} \mathcal{J}_{k,k} = \frac{1}{4k}, & k \geq 1, \\ \mathcal{J}_{k,\ell} = 0, & \text{for any other choice of } k \text{ and } \ell. \end{cases}$$

Proof We first prove the case $k \neq \ell,\ k,\ell \geq 1$. Note that

$$\mathcal{J}_{k,\ell} = \frac{4}{\pi^2}\iint_{[-1,1]^2}\frac{\cos^2(k\arccos(x))\cos^2(\ell\arccos(y))}{\sqrt{1-x^2}\sqrt{1-y^2}}\log\frac{1}{2|x-y|}\,d(x,y).$$

Applying Lemma 26 with $f(x, y) = \cos^2(k \arccos(x)) \cos^2(\ell \arccos(y))$, the inner integral in the right hand side of the claim in that lemma becomes

$$\begin{aligned} U &= \int_{-\pi}^{\pi} \cos^2(k\theta) \cos^2(\ell\theta + \ell\alpha)\, d\theta \\ &= \int_{-\pi}^{\pi} \cos^2(k\theta) \left(\cos(\ell\theta)\cos(\ell\alpha) - \sin(\ell\theta)\sin(\ell\alpha)\right)^2 \, d\theta, \end{aligned}$$

then, it suffices to check that U is a constant independent of α. Indeed, expanding the term in the parenthesis we easily get

$$U = \frac{\pi \cos^2(\ell\alpha)}{2} + \frac{\pi \sin^2(\ell\alpha)}{2} = \frac{\pi}{2}.$$

The case $k = 0, \ell \geq 1$ (equiv. $k \geq 1, \ell = 0$) is very similar. The integral this time is

$$U = \int_{-\pi}^{\pi} \cos(\ell\theta + \ell\alpha)^2 \, d\theta = \pi,$$

proving the proposition in that case. The case $k, \ell = 0$ is even easier since we are just integrating a constant. The remaining case is $k = \ell \geq 1$, for which we have

$$U = \int_{-\pi}^{\pi} \cos^2(k\theta) \cos^2(k\theta + k\alpha)\, d\theta = \frac{\pi}{2} + \frac{\pi}{4}\cos(2k\alpha).$$

From Lemma 26 we get to

$$\mathcal{J}_{k,k} = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2k\alpha) \log \tfrac{1}{\sqrt{2-2\cos\alpha}}\, d\alpha \overset{\text{Lemma 27}}{=} \tfrac{1}{4k}.$$

□

As a direct consequence of the previous results we get

Corollary 6 *If* $\lambda = 0$, *then*

$$L_1 = (n+1)^2 \log 2 + \frac{H_n}{4}.$$

Lemma 7

$$L_2 = (n+1)\log 2 + H_n + nH_{2n-1} + \tfrac{H_{2n}}{2} - n + \tfrac{1}{2}.$$

Proof Note that

$$K_n(x, y)^2 = \left(\sum_{k=0}^{n} \widehat{C}_k(x)\widehat{C}_k(y)\right)^2 = \sum_{k,\ell=0}^{n} \widehat{C}_k(x)\widehat{C}_k(y)\widehat{C}_\ell(x)\widehat{C}_\ell(y),$$

which leads us to $L_2 = P_0 + P + 2Q + 2R + S$ where:

$$\begin{aligned} P_0 &= \log 2 \iint_{[-1,1]^2} K_n(x, y)^2 w(x)w(y)\, d(x, y) = (n+1)\log 2, \\ P &= \iint_{[-1,1]^2} \log \frac{1}{2|x-y|} w(x)w(y)\, d(x, y) = \mathcal{J}_{0,0} = 0, \end{aligned}$$

$$Q = \frac{2}{\pi^2}\sum_{k=1}^{n}\iint_{[-1,1]^2}\frac{\cos(k\arccos(x))\cos(k\arccos(y))}{\sqrt{1-x^2}\sqrt{1-y^2}}\log\frac{1}{2|x-y|}\,d(x,y),$$
$$R = \frac{4}{\pi^2}\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\iint_{[-1,1]^2}\frac{f_{k,\ell}(x,y)}{\sqrt{1-x^2}\sqrt{1-y^2}}\log\frac{1}{2|x-y|}\,d(x,y),$$
$$S = \frac{4}{\pi^2}\sum_{k=1}^{n}\iint_{[-1,1]^2}\frac{\cos^2(k\arccos(x))\cos^2(k\arccos(y))}{\sqrt{1-x^2}\sqrt{1-y^2}}\log\frac{1}{2|x-y|}\,d(x,y)$$
$$= \sum_{k=1}^{n}\mathcal{J}_{k,k} \overset{\text{Prop. 5}}{=} \frac{H_n}{4},$$

and

$$f_{k,\ell}(x,y) = \cos(k\arccos(x))\cos(k\arccos(y))\cos(\ell\arccos(x))\cos(\ell\arccos(y)).$$

From Lemma 26 we have for $n \geq 1$:

$$Q = \sum_{k=1}^{n}\frac{1}{\pi^2}\int_{-\pi}^{\pi}d\alpha\log\frac{1}{\sqrt{2-2\cos\alpha}}\int_{-\pi}^{\pi}\cos(k\theta)\cos(k\theta+k\alpha)\,d\theta$$
$$= \sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(k\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha \overset{\text{Lemma 27}}{=} \sum_{k=1}^{n}\frac{1}{k} = H_n.$$

On the other hand, also from Lemma 26,

$$R=\sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\frac{2}{\pi^2}\int_{-\pi}^{\pi}d\alpha\log\frac{1}{\sqrt{2-2\cos\alpha}}\int_{-\pi}^{\pi}\cos(k\theta)\cos(k\theta+k\alpha)\cos(\ell\theta)\cos(\ell\theta+\ell\alpha)\,d\theta$$
$$= \sum_{k=2}^{n}\sum_{\ell=1}^{k-1}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(k\alpha)\cos(\ell\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha.$$

We have computed these integrals in Lemma 27 getting:

$$\begin{aligned} 2R &= \sum_{k=2}^{n}\sum_{l=1}^{k-1}\left(\frac{1}{k-\ell}+\frac{1}{k+\ell}\right) \\ &= \sum_{k=2}^{n}\left(H_{2k-1}-\frac{1}{k}\right) = 1-H_n+\sum_{k=2}^{n}H_{2k-1} \\ &\overset{\text{Lemma 28}}{=} nH_{2n-1}+\frac{H_{2n}}{2}-\frac{5H_n}{4}-n+\frac{1}{2}. \end{aligned}$$

All in one, we have proved that

$$L_2 = (n+1)\log 2 + H_n + nH_{2n-1} + \tfrac{H_{2n}}{2} - n + \tfrac{1}{2},$$

as wanted. □

The proof of Theorem 2 is now complete for $\lambda = 0$ since we just need to write down $L_1 - L_2$. The asymptotic expression for $\mathrm{E}(0, n+1)$ is obtained from the one for the harmonic number.

5 Proof of Theorem 2 for General $\lambda \neq 0$

We will compute $L_1(\lambda, n)$ and $L_2(\lambda, n)$ in two propositions:

Proposition 8 *For any* $\lambda > -1/2$ *and* $n \geq 2$,

$$(n+1)^2 \log 2 \leq L_1(\lambda, n) \leq (n+1)^2 \log 2 + o(n).$$

Proof See Section 5.1. □

Proposition 9 *For any* $\lambda > -1/2$, *and as* $n \to \infty$

$$L_2(\lambda, n) = n \log(n) + (\gamma + 2\log(2) - 1)n + o(n)\,.$$

Proof See Section 5.2. □

In the rest of the paper, $Q(\lambda)$ is some constant depending only on λ, its value may change from one appearance to another and we do not care about it.

5.1 The Value of L_1

The integral L_1 is indeed the energy of a measure, so we can make use of the terminology related with this topic. The mutual energy of a pair of (possibly signed) measures μ and ν is given by

$$I(\mu, \nu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y)\,,$$

where the integral is assumed to exist, and the energy of a measure μ is $I(\mu, \mu)$ which is usually denoted simply by $I(\mu)$. The potential of a measure μ is defined by

$$V^{\mu}(x) = \int \log \frac{1}{|x-y|} d\mu(y)\,.$$

Thus we have the following obvious relations

$$I(\mu, \nu) = \int V^{\mu}(x) d\nu(x) = \int V^{\nu}(x) d\mu(x)\,, \qquad I(\mu) = \int V^{\mu}(x) d\mu(x)\,.$$

The following well known lemma gives a formula for the energy of a linear combination of two (possibly signed) measures.

Lemma 10 *The following identities hold*

$$I(\mu + \nu) = I(\mu) + 2I(\mu, \nu) + I(\nu)\,, \qquad I(t\mu) = t^2 I(\mu)\,,$$

where $t \in \mathbb{R}$.

The equilibrium measure in $[-1, 1]$, which from now on we denote by μ, is the unique minimizer of the energy among all the measures of total mass 1 and also is the unique (unitary) measure such that its potential is constant, $V^{\mu}(x) = \log(2)$ for $x \in [-1, 1]$. Its density is

$$\frac{d\mu(x)}{dx} = \frac{1}{\pi\sqrt{1-x^2}}\,,$$

and the fact that the potential is constant (the constant is called Robin's constant) reads

$$\int_{-1}^{1} \frac{\log|x-y|^{-1}}{\pi\sqrt{1-x^2}}\,dx = \log 2, \forall\, y \in [-1,1]. \tag{4}$$

In other words, for any other probability measure $d\nu$, the logarithmic energy given by

$$\iint_{x,y\in[-1,1]} \log\frac{1}{|x-y|}\,d\nu(x)\,d\nu(y)$$

is greater than log 2. Now, since $(n+1)^{-1}K_n^\lambda(x,x)w^\lambda(x)$ is the density of a probability measure, the lower bound of Proposition 8 follows immediately since $I((n+1)\mu) = (n+1)^2\log(2)$. For the upper bound we are going to consider the equilibrium measure μ, the measures ν_n such that

$$\frac{d\nu_n}{dx} = K_n^\lambda(x)w^\lambda(x)\,,$$

and $\varepsilon_n = \nu_n - (n+1)\mu$. Both ν_n and $(n+1)\mu$ are positive measures of total mass $(n+1)$, hence ε_n is a signed measure of total mass 0.

The first step is to use the decomposition $\nu_n = (n+1)\mu + \varepsilon_n$. Thus

$$L_1(n,\lambda) = I(\nu_n) = I\left((n+1)\mu + \varepsilon_n\right) = I\left((n+1)\mu\right) + 2I\left((n+1)\mu, \varepsilon_n\right) + I(\varepsilon_n)\,.$$

The mutual energy of $(n+1)\mu$ and ε_n vanishes since

$$I((n+1)\mu,\varepsilon_n) = (n+1)I(\mu,\varepsilon_n) = (n+1)\int V^\mu(x)d\varepsilon_n(x) = (n+1)\log(2)\int d\varepsilon_n(x) = 0.$$

The consequence is that we can write

$$L_1(n,\lambda) = (n+1)^2\log(2) + I(\varepsilon_n)\,,$$

and then we have to prove that $I(\varepsilon_n) = o(n)$. Now, from lemmas 18 and 21 we have that

$$|d\varepsilon_n(x)| \le \begin{cases} \frac{Q(\lambda)\log n}{1-x^2}, & x\in[-1+n^{-2}, 1-n^{-2}],\\ \frac{Q(\lambda)n^{2\lambda+1}(1-x^2)^\lambda}{\sqrt{1-x^2}} + \frac{Q(\lambda)n}{\sqrt{1-x^2}} \le \frac{Q(\lambda)n}{\sqrt{1-x^2}}, & x\in[1-n^{-2},1],\ \lambda\ge 0\\ \frac{Q(\lambda)n^{2\lambda+1}(1-x^2)^\lambda}{\sqrt{1-x^2}} + \frac{Q(\lambda)n}{\sqrt{1-x^2}} \le \frac{Q(\lambda)n^{2\lambda+1}}{\sqrt{1-x^2}^{1-2\lambda}}, & x\in[1-n^{-2},1],\ \lambda<0 \end{cases}$$

and using a simple symmetry argument we have

$$I(\varepsilon_n) \le 4A_n + 4B_n + C_n$$

where the definition of A_n, B_n depends on the cases $\lambda \ge 0$ and $\lambda < 0$. For the first one:

$$\begin{aligned} A_n &= \iint_{x,y\in[1-n^{-2},1]} \left|\log\frac{1}{|x-y|}\right| \frac{Q(\lambda)n^2}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy,\\ B_n &= \iint_{x\in[-1+n^{-2},1-n^{-2}],y\in[1-n^{-2},1]} \left|\log\frac{1}{|x-y|}\right| \frac{Q(\lambda)n\log n}{(1-x^2)\sqrt{1-y^2}}\,dx\,dy,\\ C_n &= \iint_{x,y\in[-1+n^{-2},1-n^{-2}]} \left|\log\frac{1}{|x-y|}\right| \frac{Q(\lambda)(\log n)^2}{(1-x^2)(1-y^2)}\,dx\,dy\\ &\le O(\sqrt{n}\log(n)^3), \end{aligned}$$

the last from Lemma 19 and some little arithmetic. Also from Lemma 19 we have

$$\begin{aligned} B_n &\le Q(\lambda)n^{3/2}\log n\int_{1-n^{-2}}^{1} \frac{1}{\sqrt{1-y^2}}\,dy\\ &\le Q(\lambda)n^{3/2}\log n\sqrt{1-(1-n^{-2})^2} \le Q(\lambda)n^{1/2}\log n. \end{aligned}$$

Finally, the change of variables $x = 1 - t$, $y = 1 - s$ gives

$$\begin{aligned} A_n &\le Q(\lambda)n^2 \int_0^{n^{-2}} dt \int_0^t \frac{1}{\sqrt{2t - t^2}\sqrt{2s - s^2}} \log \frac{1}{t - s}\, ds \\ &\le Q(\lambda)n^2 \int_0^{n^{-2}} dt \int_0^t \frac{1}{\sqrt{ts}} \log \frac{1}{t - s}\, ds \\ &= -Q(\lambda)^2 n^2 \int_0^{n^{-2}} 2(\log t + 2\log 2 - 2) dt = O(\log n). \end{aligned}$$

In the case $\lambda \in (-1/2, 0)$ the definitions of A_n and B_n are slightly different but they satisfy similar bounds: A_n equals

$$\iint_{x,y\in[1-n^{-2},1]} \left|\log \frac{1}{|x-y|}\right| \frac{Q(\lambda)n^{4\lambda+2}}{\sqrt{1-x^2}^{1-2\lambda}\sqrt{1-y^2}^{1-2\lambda}}\, dx\, dy \le Q(\lambda)n^{(2+4\lambda)/(3-2\lambda)} = o(n),$$

where we have used Lemma 19. Finally, the definition of B_n if $\lambda \in (-1/2, 0)$ is

$$\iint_{x\in[-1+n^{-2},1-n^{-2}],\, y\in[1-n^{-2},1]} \left|\log \frac{1}{|x - y|}\right| \frac{Q(\lambda)n^{2\lambda+1}\log n}{(1-x^2)\sqrt{1-y^2}^{1-2\lambda}}\, dx\, dy = O(\sqrt{n}\log n),$$

again from Lemma 19. This finishes the proof of Proposition 8.

5.2 The Value of L_2

We devote this section to the proof of Proposition 9. To this end, we consider the following overlapping regions (see Fig. 1):

$$D_{n,\alpha} = \left\{ (x, y) \in [-1, 1]^2 \;:\; \begin{array}{l} |\arccos x - \arccos y| \le 2n^{-\alpha} \\ 4n^{-\alpha} \le \arccos x + \arccos y \le 2\pi - 4n^{-\alpha} \end{array} \right\},$$

$$T_{n,\alpha} = \{(x, y) \in [-1, 1]^2 \;:\; \arccos y - \arccos x \ge 2n^{-\alpha}\}.$$

The border line of $T_{n,\alpha}$ with the rest of $[-1, 1]^2$ is an arc joining $(x, y) = (-1+c_n n^{-2\alpha}, -1)$ with $(x, y) = (1, 1 - c_n n^{-2\alpha})$ along the graphic of a convex increasing function, where $c_n > 0$ and $c_n \to 2$.

We show that the main term in the asymptotics comes from the region $D_{n,\alpha}$.

Let us commence the proof with some auxiliary results about the kernel $K_n^\lambda(x, y)$, some of them interesting on their own. In these auxiliary results we consider a general parameter α, which belongs to some interval that changes from a result to another, so it is sometimes in $[0, 1]$ or $(1/2, 1/(1 - 2\lambda))$ or $(0, 3/4)$. In the proof of Proposition 9 we will choose $\alpha = 1/2 + \epsilon$ for some small ϵ depending on λ, so that all these auxiliary results will apply.

Proposition 11 *Let us take $x = \cos\theta$ and $y = \cos\sigma$. For any $\lambda > -1/2$, $\alpha \in [0, 1]$ and $c > 0$:*

(i) *For $(x, y) \in [-1 + cn^{-2\alpha}, 1 - cn^{-2\alpha}]$,*

$$\left| K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)} - \frac{1}{2\pi\sqrt{\sin\theta}\sqrt{\sin\theta}} \frac{\sin((n + \lambda + 1/2)(\theta - \sigma))}{\sin((\theta - \sigma)/2)} \right| \le \frac{Q(\lambda, c)n^\alpha \log n}{\sqrt{\sin\theta}\sqrt{\sin\sigma}}.$$

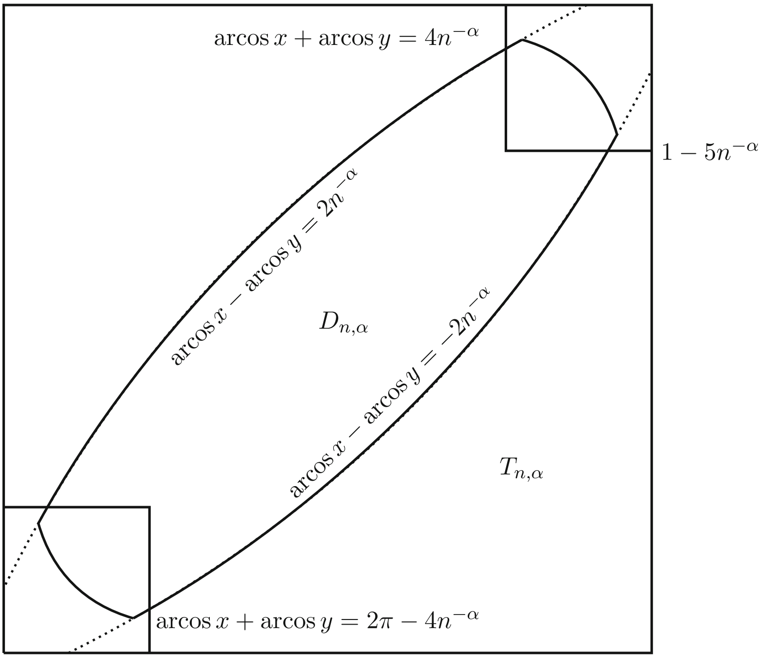

Fig. 1 Regions on the square $[-1, 1]^2$

(ii) *For* $\lambda \geq 0$ *and* $x \neq y$ *both in* $[-1, 1]$*:*

$$\left|K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \leq \frac{Q(\lambda)}{|x-y|\sqrt{\sin\theta}\sqrt{\sin\sigma}} .$$

In the case $\lambda \in (-1/2, 0)$ *this last inequality is also valid but only for* $x \neq y$ *both in* $[-1 + cn^{-2}, 1 - cn^{-2}]$ *and additionally:*

$$\left|K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \leq \frac{Q(\lambda)(n\sin\theta)^\lambda}{|x-y|\sqrt{\sin\theta}\sqrt{\sin\sigma}} ,$$

for $(x, y) \in [1 - cn^{-2}, 1] \times [-1 + cn^{-2}, 1 - cn^{-2}]$.

(iii) *For* $(x, y) \in [-1, 1]^2$ *and* $\lambda \geq 0$

$$\left|K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \leq \frac{Q(\lambda)n}{\sqrt{\sin\theta}\sqrt{\sin\sigma}} .$$

If $-1/2 < \lambda < 0$ *and* $(x, y) \in [1 - cn^{-2\alpha}, 1] \times [-1, -1 + cn^{-2\alpha}]$ *or* $(x, y) \in [1 - cn^{-2\alpha}, 1]^2$*, then*

$$\left|K_n^\lambda(x, y)\right| \leq Q(\lambda)n^{2\alpha\lambda+1} ,$$

while if $(x, y) \in [1 - cn^{-2}, 1] \times [-1 + cn^{-2}, 1 - cn^{-2}]$*:*

$$\left|K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \leq \frac{Q(\lambda)n}{\sqrt{\sin\theta}\sqrt{\sin\sigma}}(n \sin\theta)^\lambda .$$

Proof Here, we will use some auxiliary results which are collected in the appendices at the end of the paper. Let us prove *(i)*. Starting from Lemma 20, we sum from $k = 0$ to n

$$\begin{aligned}
&\left| K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)} - \sum_{k=0}^{n} \frac{2\cos\left((k+\lambda)\theta - \lambda\pi/2\right)\cos\left((k+\lambda)\sigma - \lambda\pi/2\right)}{\pi\sqrt{\sin\theta}\sqrt{\sin\sigma}} \right| \\
&\le \sum_{k=0}^{n} \frac{Q(\lambda)}{k\sin^{3/2}\theta}\frac{T(k,\theta)}{1+T(k,\theta)}\frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\sigma}}\left|\cos((k+\lambda)\sigma - \lambda\pi/2)\right| \\
&\quad + \sum_{k=0}^{n} \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\theta}}\left|\cos((k+\lambda)\theta - \lambda\pi/2)\right| \frac{Q(\lambda)}{k\sin^{3/2}\sigma}\frac{T(k,\sigma)}{1+T(k,\sigma)} \\
&\quad + \sum_{k=0}^{n} \frac{Q(\lambda)}{k\sin^{3/2}\theta}\frac{T(k,\theta)}{1+T(k,\theta)}\frac{Q(\lambda)}{k\sin^{3/2}\sigma}\frac{T(k,\sigma)}{1+T(k,\sigma)} \\
&\le \frac{Q(\lambda)}{\sqrt{\sin\theta}\sqrt{\sin\sigma}}\left(\sum_{k=1}^{n}\frac{1}{k\sin\theta} + \sum_{k=1}^{n}\frac{1}{k\sin\sigma} + \sum_{k=1}^{n}\frac{1}{k\sin\theta}\frac{1}{k\sin\sigma}\frac{T(k,\sigma)}{1+T(k,\sigma)}\right).
\end{aligned}$$

For $\lambda \ge 0$ we have $T(k,\sigma) = k\sin\sigma$ and

$$\frac{1}{k\sin\sigma}\frac{T(k,\sigma)}{1+T(k,\sigma)} = \frac{1}{1+k\sin\sigma} < 1.$$

For $\lambda \in (-1/2, 0)$ we have $T(k,\sigma) = (k\sin\sigma)^{\lambda+1}$ and

$$\frac{1}{k\sin\sigma}\frac{T(k,\sigma)}{1+T(k,\sigma)} = \frac{(k\sin\sigma)^\lambda}{1+(k\sin\sigma)^{\lambda+1}} < (k\sin\sigma)^\lambda.$$

In both cases, when taking $\theta, \sigma \in [cn^{-\alpha}, \pi - cn^{-\alpha}]$, the term in the parenthesis is at most like $n^\alpha \log n$. Note that in the $\lambda \in (-1/2, 0)$ case we use that $\alpha(1-\lambda)+\lambda \le \alpha$. Finally, with the help of Lemma 30,

$$\begin{aligned}
&\left| K_n^\lambda(x, y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)} - \frac{1}{2\pi\sqrt{\sin\theta}\sqrt{\sin\sigma}}\frac{\sin((n+\lambda+1/2)(\theta-\sigma))}{\sin((\theta-\sigma)/2)} \right| \\
&\le \frac{1}{2\pi\sqrt{\sin\theta}\sqrt{\sin\sigma}}\left(\left|\frac{\sin((n+\lambda+1/2)(\theta+\sigma)-\lambda\pi)}{\sin((\theta+\sigma)/2)}\right| + \left|\frac{\sin((\lambda-1/2)(\theta+\sigma)-\lambda\pi)}{\sin((\theta+\sigma)/2)}\right|\right. \\
&\quad \left. + \left|\frac{\sin((\lambda-1/2)(\theta-\sigma))}{\sin((\theta-\sigma)/2)}\right| + Q(\lambda)n^\alpha \log n\right) \\
&\le \frac{1}{2\pi\sqrt{\sin\theta}\sqrt{\sin\sigma}}\left(Q(\lambda)n^\alpha + Q(\lambda)n^\alpha + Q(\lambda) + Q(\lambda)n^\alpha\log n\right) \le Q(\lambda)n^\alpha \log n\,,
\end{aligned}$$

and *(i)* is proved (the constant $Q(\lambda)$ in the argument depends on c so we denote it by $Q(\lambda, c)$ in the proposition).

Inequalities in *(ii)* are consequence of the Christoffel-Darboux summation formula (see [22]) and some bounds for the Gegenbauer polynomials. First inequality in Eq. 18 holds for all $\theta \in [0, \pi]$ if $\lambda \geq 0$ and also for $\theta \in [n^{-1}, \pi - n^{-1}]$ if $-1/2 < \lambda < 0$, so in these cases

$$\begin{aligned}
&\left|K_n^\lambda(x,y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \\
&= \left|\frac{(n+1)!}{2\lambda(2\lambda)_n}\frac{C_{n+1}^\lambda(x)C_n^\lambda(y)-C_n^\lambda(x)C_{n+1}^\lambda(y)}{x-y}\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \\
&\leq Q(\lambda)n^{2-2\lambda}\frac{n^{2\lambda-2}\sin^{-\lambda}\theta\sin^{-\lambda}\sigma}{|x-y|}\sin^{\lambda-1/2}\theta\sin^{\lambda-1/2}\sigma \leq \frac{Q(\lambda)}{|x-y|\sqrt{\sin\theta}\sqrt{\sin\sigma}}.
\end{aligned}$$

If $-1/2 < \lambda < 0$ and $(x, y) \in [1 - cn^{-2}, 1] \times [-1 + cn^{-2}, 1 - cn^{-2}]$, then by the second inequality in Eq. 18

$$\begin{aligned}
&\left|K_n^\lambda(x,y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \\
&= \left|\frac{(n+1)!}{2\lambda(2\lambda)_n}\frac{C_{n+1}^\lambda(x)C_n^\lambda(y)-C_n^\lambda(x)C_{n+1}^\lambda(y)}{x-y}\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \\
&\leq Q(\lambda)n^{2-2\lambda}\frac{n^{2\lambda-1}n^{\lambda-1}\sin^{-\lambda}\sigma}{|x-y|}\sin^{\lambda-1/2}\theta\sin^{\lambda-1/2}\sigma \leq \frac{Q(\lambda)(n\sin\theta)^\lambda}{|x-y|\sqrt{\sin\theta}\sqrt{\sin\sigma}}.
\end{aligned}$$

Then *(ii)* is proved.

Finally *(iii)* is quite direct: if $\lambda \geq 0$, from first inequality in Eq. 18 (which is also valid for $\theta \in [0, \pi]$)

$$\begin{aligned}
&\left|K_n^\lambda(x,y)\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)}\right| \\
&\leq Q(\lambda)\sum_{k=0}^n\left(\gamma_k^\lambda\right)^2\left|C_k^\lambda(\cos\theta)C_k^\lambda(\cos\sigma)\right|\sin^{\lambda-1/2}\theta\sin^{\lambda-1/2}\sigma \\
&\leq Q(\lambda)\sum_{k=0}^n k^{2-2\lambda}\sin^{-\lambda}\theta k^{\lambda-1}\sin^{-\lambda}\sigma k^{\lambda-1}\sin^{\lambda-1/2}\theta\sin^{\lambda-1/2}\sigma \leq \frac{Q(\lambda)n}{\sqrt{\sin\theta}\sqrt{\sin\sigma}}.
\end{aligned}$$

If $-1/2 < \lambda < 0$, combining both inequalities in Eq. 18 one gets

$$\left|C_k^\lambda(\cos\theta)\right| \leq Q(\lambda)n^{(\alpha+1)\lambda-1}, \qquad \theta \in [0, dn^{-\alpha}]$$

for some d. Hence if $x \in [1 - cn^{-2\alpha}, 1]$ and $y \in [1 - cn^{-2\alpha}, 1]$ or $y \in [-1, -1 + cn^{-2\alpha}]$:

$$\left|K_n^\lambda(x,y)\right| \leq \sum_{k=0}^n\left(\gamma_k^\lambda\right)^2\left|C_k^\lambda(\cos\theta)C_k^\lambda(\cos\sigma)\right| \leq \sum_{k=0}^n Q(\lambda)k^{2-2\lambda}k^{2(\alpha+1)\lambda-2} \leq Q(\lambda)n^{2\alpha\lambda+1},$$

and if $(x, y) \in [1 - cn^{-2}, 1] \times [-1+cn^{-2}, 1 - cn^{-2}]$, again from Eq. 18:

$$\begin{aligned}
&\left|K_n^\lambda(x,y)\right|\sqrt{w^\lambda(x)}\sqrt{w^\lambda(y)} \leq Q(\lambda)\sum_{k=0}^n\left(\gamma_k^\lambda\right)^2\left|C_k^\lambda(\cos\theta)C_k^\lambda(\cos\sigma)\right|\sin^{\lambda-1/2}\theta\sin^{\lambda-1/2}\sigma \\
&\leq \sum_{k=0}^n Q(\lambda)k^{2-2\lambda}k^{2\lambda-1}\sin^{-\lambda}\sigma k^{\lambda-1}\sin^{\lambda-1/2}\theta\sin^{\lambda-1/2}\sigma = \frac{Q(\lambda)\sin^\lambda\theta}{\sqrt{\sin\theta}\sqrt{\sin\sigma}}\sum_{k=0}^n k^\lambda \\
&\leq \frac{Q(\lambda)n(n\sin\theta)^\lambda}{\sqrt{\sin\theta}\sqrt{\sin\sigma}}.
\end{aligned}$$

□

We now can establish a series of lemmas which reduce the region from where the main terms in the asymptotics of Proposition 9 come.

Lemma 12 *For* $\alpha \in (1/2, 1]$, $\alpha_0 \in (0, 1/2)$ *such that* $\alpha + \alpha_0 > 1$, $c > 1$ *and* $d > 0$

$$\iint_{[1-n^{-2\alpha},1]\times[1-dn^{-2\alpha_0},1-cn^{-2\alpha}]} \left|\log \frac{1}{|x-y|}\right| K_n^\lambda(x,y)^2 w^\lambda(x) w^\lambda(y)\, d(x,y) = o(n).$$

Proof Let I be our integral. In this region it is clear that $\left|\log \frac{1}{|x-y|}\right| \le Q \log(n)$ where Q is a constant depending on α and α_0. In the case $\lambda \ge 0$, to estimate I, we use *(iii)* in Proposition 11 and take into account that

$$\int_{1-n^{-2\alpha}}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \le Qn^{-\alpha} \qquad \int_{1-dn^{-2\alpha_0}}^{1-cn^{-2\alpha}} \frac{1}{\sqrt{1-y^2}}\, dy \le Qn^{-\alpha_0},$$

for some constant Q, in order to get

$$\begin{aligned} I &\le Q(\lambda)\log(n) n^2 \int_{1-n^{-2\alpha}}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \int_{1-dn^{-2\alpha_0}}^{1-cn^{-2\alpha}} \frac{1}{\sqrt{1-y^2}}\, dy \\ &\le Q(\lambda)\log(n) n^2 n^{-\alpha} n^{-\alpha_0} = o(n). \end{aligned}$$

In the case $\lambda \in (-1/2, 0)$ we use the inequality

$$\int_{1-n^{-2\alpha}}^{1} \frac{(n\sqrt{1-x^2})^{2\lambda}}{\sqrt{1-x^2}}\, dx \le Q(\lambda) n^{2\lambda} \int_0^{kn^{-\alpha}} \theta^{2\lambda}\, d\theta \le Q(\lambda) n^{2\lambda(1-\alpha)-\alpha}$$

and the last one in *(iii)* of Proposition 11 to obtain

$$\begin{aligned} I &\le Q(\lambda)\log(n) n^2 \int_{1-n^{-2\alpha}}^{1} \frac{(n\sqrt{1-x^2})^{2\lambda}}{\sqrt{1-x^2}}\, dx \int_{1-dn^{-2\alpha_0}}^{1-cn^{-2\alpha}} \frac{1}{\sqrt{1-y^2}}\, dy \\ &\le Q(\lambda)\log(n) n^{2+2\lambda(1-\alpha)-\alpha-\alpha_0} = o(n). \end{aligned}$$

□

Lemma 13 *Let* $n > 1$, $\lambda \in (-1/2, 0)$, $\alpha \in (1/2, 1)$ *and* $c > 0$. *For any* $y \in [-1, 1]$*:*

$$\int_{1-cn^{-2\alpha}}^{1} \left|\log|x-y| w^\lambda(x)\right| dx \le Q(\lambda) n^{\alpha(1-2\lambda-2/q)}, \quad 1 < q < \frac{2}{1-2\lambda},$$

and

$$\int_{1-cn^{-2\alpha}}^{1} w^\lambda(x)\, dx \le Q(\lambda) n^{\alpha(-1-2\lambda)}.$$

Proof Using Holder's inequality the first integral is bounded by:

$$\begin{aligned} I &\le Q(q,\lambda) \left(\int_{1-cn^{-2\alpha}}^{1} \frac{1}{(1-x^2)^{(1/2-\lambda)q}} dx \right)^{1/q} \\ &\le Q(q,\lambda) \left(\int_{1-cn^{-2\alpha}}^{1} \frac{1}{(1-x)^{(1/2-\lambda)q}} dx \right)^{1/q} = Q(q,\lambda) n^{\alpha(1-2\lambda-2/q)} \end{aligned}$$

where $(1/2-\lambda)q < 1$. For the last integral:

$$\int_{1-cn^{-2\alpha}}^{1} (1-x^2)^{\lambda-1/2}\, dx \le Q(\lambda) \int_{1-cn^{-2\alpha}}^{1} (1-x)^{\lambda-1/2}\, dx \le Q(\lambda) n^{\alpha(-1-2\lambda)}.$$

□

Lemma 14 *For $\alpha \in (1/2, 1]$ and $c > 0$,*

$$\iint_{[1-cn^{-2\alpha},1]^2} \left|\log \frac{1}{|x-y|}\right| K_n^\lambda(x,y)^2 w^\lambda(x) w^\lambda(y)\, d(x,y) = o(n).$$

Proof For $\lambda > 0$, using Proposition 11 *(iii)* we get:

$$K_n^\lambda(x,y)^2 w^\lambda(x) w^\lambda(y) \le \frac{Q(\lambda) n^2}{(1-x^2)^{1/2}(1-y^2)^{1/2}}\,.$$

Therefore,

$$\iint_{[1-cn^{-2\alpha},1]^2} \left|\log \frac{1}{|x-y|}\right| K_n^\lambda(x,y)^2 w^\lambda(x) w^\lambda(y)\, d(x,y)$$
$$\le Q(\lambda) n^2 \iint_{[1-cn^{-2\alpha},1]^2} \left|\log \frac{1}{|x-y|}\right| \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\, d(x,y) \le Q(\lambda) n^{2(1-\alpha)} \log(n)\,.$$

The last inequality holds in the same logic that the one used to bound A_n in the proof of Proposition 8.

For $\lambda < 0$, using again *(iii)* of Proposition 11, $K_n^\lambda(x,y)^2 \le Q(\lambda) n^{4\alpha\lambda+2}$, then from Lemma 13:

$$\iint_{[1-cn^{-2\alpha},1]^2} \left|\log \frac{1}{|x-y|}\right| K_n^\lambda(x,y)^2 w^\lambda(x) w^\lambda(y)\, d(x,y)$$
$$\le Q(\lambda) n^{4\alpha\lambda+2} \iint_{[1-cn^{-2\alpha},1]^2} \left|\log \frac{1}{|x-y|}\right| \frac{1}{\sqrt{1-x^2}^{\,1-2\lambda}\sqrt{1-y^2}^{\,1-2\lambda}}\, d(x,y)$$
$$\le Q(\lambda) n^{4\alpha\lambda+2} n^{\alpha(1-2\lambda-2/q)} \int_{[1-cn^{-2\alpha},1]} \frac{1}{\sqrt{1-y^2}^{\,1-2\lambda}} dy$$
$$\le Q(\lambda) n^{2\alpha\lambda+2+\alpha-2\alpha/q} \int_{[1-cn^{-2\alpha},1]} \frac{1}{\sqrt{1-y}^{\,1-2\lambda}} dy$$
$$\le Q(\lambda) n^{2\alpha\lambda+2+\alpha-2\alpha/q} n^{\alpha(-1-2\lambda)} = Q(\lambda) n^{2(1-\alpha/q)}.$$

Choosing small enough $1 < q < 2\alpha$, we have $2(1-\alpha/q) < 1$ and the result follows. □

The last regions whose integrals are $o(n)$ at most are going to be of the type $T_{n,\alpha}$ (see Fig. 1 in page 14).

Lemma 15 *Let $\lambda > -1/2$ and $\alpha \in (0, 3/4)$, then as $n \to \infty$*

$$\left|\iint_{T_{n,\alpha}} \log \frac{1}{|x-y|} K_n^\lambda(x,y)^2 w^\lambda(x) w^\lambda(y) dx dy\right| = o(n)\,.$$

Proof We prove it first for $\alpha < 1/2$ and $\lambda \geq 0$. The change of variables

$$(u, v) = \left(\frac{\theta - \sigma}{2}, \frac{\theta + \sigma}{2}\right) = \left(\frac{\arccos(x) - \arccos(y)}{2}, \frac{\arccos(x) + \arccos(y)}{2}\right), \quad (5)$$

transforms $T_{n,\alpha}$ into a triangle $T^*_{n,\alpha} = \{(u, v) : u \leq -n^{-\alpha}, u + v \geq 0, -u + v \leq \pi\}$, whose vertices are $(-\pi/2, \pi/2)$, $(-n^{-\alpha}, n^{-\alpha})$ and $(-n^{-\alpha}, \pi - n^{-\alpha})$. Also

$$|x - y| = |2\sin((\theta + \sigma)/2)\sin((\theta - \sigma)/2)| = |2\sin(v)\sin(u)|.$$

Then, first inequality in *(ii)* of Proposition 11 implies that the absolute value of our integral is less than

$$Q(\lambda)\iint_{T^*_{n,\alpha}} \log(n)\frac{1}{\sin^2(v)\sin^2(u)}d(u, v) \leq Q(\lambda)\log(n)\iint_{T^*_{n,\alpha}} \frac{1}{v^2(\pi - v)^2u^2}d(u, v)$$
$$\leq Q(\lambda)\log(n)n^{2\alpha} \leq o(n),$$

so we have proved the lemma in this case. If $\lambda \in (-1/2, 0)$ and still $\alpha < 1/2$, we consider the subsets of $T_{n,\alpha}$

$$\begin{aligned} A &= \{(x, y) \in T_{n,\alpha} : |x| \leq 1 - n^{-2}, |y| \leq 1 - n^{-2}\}, \\ B &= \{(x, y) \in T_{n,\alpha} : x \geq 1 - n^{-2}, |y| \leq 1 - n^{-2}\}, \\ \bar{B} &= \{(x, y) \in T_{n,\alpha} : |x| \leq 1 - n^{-2}, y < -1 + n^{-2}\}, \\ C &= \{(x, y) \in T_{n,\alpha} : x \geq 1 - n^{-2}, y < -1 + n^{-2}\}, \end{aligned}$$

so that by symmetry between B and $\bar{B}$,

$$\begin{aligned} &\left|\iint_{T_{n,\alpha}} \log\frac{1}{|x - y|}K_n^\lambda(x, y)^2 w^\lambda(x)w^\lambda(y)dxdy\right| \\ &\leq \left|\iint_A \log\frac{1}{|x - y|}K_n^\lambda(x, y)^2 w^\lambda(x)w^\lambda(y)dxdy\right| \\ &\quad +2\left|\iint_B \log\frac{1}{|x - y|}K_n^\lambda(x, y)^2 w^\lambda(x)w^\lambda(y)dxdy\right| \\ &\quad +\left|\iint_C \log\frac{1}{|x - y|}K_n^\lambda(x, y)^2 w^\lambda(x)w^\lambda(y)dxdy\right|, \end{aligned}$$

and we have to prove that these three integrals are $o(n)$. The integral in A can be bounded as in the case $\lambda \geq 0$. The integral in C can be proved to be also $o(n)$ at most by using the same arguments than in Lemma 14 (it is even easier since the logarithmic term does not appear). By *(ii)* of Proposition 11, the absolute value of our integral in B is less than

$$\begin{aligned} &Q(\lambda)\log(n)\iint_B \frac{(n\sin\theta)^{2\lambda}}{|x - y|^2\sqrt{1 - x^2}\sqrt{1 - y^2}}dxdy \\ &\leq Q(\lambda)\log(n)n^{2\lambda}n^{4\alpha}\iint_B \frac{w^\lambda(x)}{\sqrt{1 - y^2}}dxdy \\ &\leq Q(\lambda)\log(n)n^{2\lambda+4\alpha}n^{-2\lambda-1} = Q(\lambda)\log(n)n^{4\alpha-1} = o(n), \end{aligned}$$

since $\alpha < 1/2$. Hence we have proved the lemma for $\alpha \in (0, 1/2)$ and any $\lambda > -1/2$.

Suppose now $\alpha \in (1/2, 3/4)$ (the case $\alpha = 1/2$ can be deduced from this case) and take $\alpha_0 \in (0, 1/2)$ such that $\alpha + \alpha_0 > 1$, $2\alpha - \alpha_0 < 1$ (observe that these conditions are not void since $\alpha < 3/4$). Consider the following subsets of $T_{n,\alpha}$: T_{n,α_0} and

$$E = \{(x, y) \in T_{n,\alpha} : 2n^{-\alpha_0} \geq \arccos y - \arccos x, x \geq 1 - n^{-2\alpha}\},$$
$$F = \{(x, y) \in T_{n,\alpha} : 2n^{-\alpha_0} \geq \arccos y - \arccos x, x \leq 1 - n^{-2\alpha}, y > -1 + n^{-2\alpha}\}.$$

We have already proved that the integral in T_{n,α_0} is at most $o(n)$ and Lemma 12 shows that the integral in E is also $o(n)$ at most (because $\alpha + \alpha_0 > 1$ and E is included in the rectangle $[1 - n^{-2\alpha}, 1] \times [1 - 3n^{-2\alpha_0}, 1 - n^{-2\alpha}]$). The integral in F can be bounded using *(i)* of Proposition 11:

$$\left| \iint_F \log\left(\frac{1}{|x - y|}\right) K_n^\lambda(x, y)^2 w^\lambda(x) w^\lambda(y) dx dy \right|$$
$$\leq Q(\lambda) \log(n) \iint_F \left(\left| \frac{1}{\sin((\theta - \sigma)/2)} \right| + n^\alpha \log(n) \right)^2 \frac{1}{\sqrt{1 - x^2}\sqrt{1 - y^2}} dx dy$$
$$\leq Q(\lambda) n^{2\alpha} \log^3(n) \iint_F \frac{1}{\sqrt{1 - x^2}\sqrt{1 - y^2}} dx dy \leq Q(\lambda) n^{2\alpha} \log^3(n) n^{-\alpha_0} = o(n),$$

because we have chosen $\alpha_0 > 2\alpha - 1$. Now the lemma follows by gathering these inequalities and taking into account the symmetry. □

To finish with the estimation of L_2, we need to study this integral over the last remaining region, the diagonal $D_{n,\alpha}$ (see Fig. 1) which is where the dominant terms will lie. We will devote the rest of this section to prove the following

Proposition 16 *For* $\alpha \in (1/2, 1)$ *it holds*

$$I_{D_{n,\alpha}} = \iint_{D_{n,\alpha}} \log \frac{1}{|x - y|} K_n^\lambda(x, y)^2 \omega^\lambda(x) \omega^\lambda(y) dx dy = n \log n + (-1 + \gamma + 2 \log 2) n + o(n).$$

Proof Using Proposition 11 *(i)* together with the change of variables Eq. 5 performed in the proof of Lemma 15, the integral in the proposition can be written as

$$I_{D_{n,\alpha}} = \frac{1}{\pi^2} \iint_R \log \frac{1}{2 \sin v \sin u} \left[\frac{\sin\big((2n + 2\lambda + 1)u\big)}{\sin u} + O\big(n^\alpha \log n\big) \right]^2 du dv,$$

where the integration region is the rectangle $R = [0, n^{-\alpha}] \times [2n^{-\alpha}, \pi - 2n^{-\alpha}]$. Expanding the squared term, this integral splits into three terms involving three integrals, namely

$$I_{D_{n,\alpha}} = I_1 O(n^{2\alpha} \log^2 n) + I_2 O(n^\alpha \log n) + I_3. \tag{6}$$

We are going to treat each one of these integrals separately. We shall see that the first two terms are $o(n)$ and also that I_3 contains all the highest order terms. Throughout the proof, we will use a constant Q which is independent on n and might not be the same from one appearing to another. We start with the first integral

$$I_1 = \iint_R \log \frac{1}{2 \sin v \sin u} du dv,$$

which can be bounded as

$$|I_1| \leq \iint_R \log 2\, dudv + \iint_R \log \frac{1}{\sin v} dudv + \iint_R \log \frac{1}{\sin u} dudv$$
$$\leq Q(n^{-\alpha} + n^{-\alpha} + n^{-\alpha}\log n) \leq Q(n^{-\alpha}\log n),$$

where we have used that $\log(\sin v)$ is an integrable function on $[0, \pi]$ and also that $\log(1/\sin u) \leq Q \log(1/u)$ for $u \in [0, n^{-\alpha}]$. Then, we directly have that the first term in Eq. 6 is $o(n)$ since $\alpha < 1$.

Now we deal with the second integral

$$I_2 = \iint_R \log \frac{1}{2\sin v \sin u} \frac{\sin((2n+2\lambda+1)u)}{\sin u} dudv.$$

Observe that the logarithmic term is always positive since $2\sin(v)\sin(u) < 1$ on R for n sufficiently large. Then, we can bound

$$|I_2| \leq \iint_R \log \frac{1}{2\sin v} \frac{|\sin((2n+2\lambda+1)u)|}{\sin u} dudv + \iint_R \log \frac{1}{\sin u} \frac{|\sin((2n+2\lambda+1)u)|}{\sin u} dudv$$
$$\leq o(1) \int_0^{n^{-\alpha}} \frac{|\sin((2n+2\lambda+1)u)|}{\sin u} du + Q \int_0^{n^{-\alpha}} \log \frac{1}{\sin u} \frac{|\sin((2n+2\lambda+1)u)|}{\sin u} du,$$

where, we have used Lemma 31 for the estimation of the first term. Taking into account that we can bound

$$\frac{|\sin((2n+2\lambda+1)u)|}{\sin u} \leq Qn, \quad \log \frac{1}{\sin u} \leq Q \log \frac{1}{u}, \quad u \in [0, n^{-1}], \tag{7}$$

and also

$$\frac{|\sin((2n+2\lambda+1)u)|}{\sin u} \leq Q\frac{1}{u}, \quad \log \frac{1}{\sin u} \leq Q \log \frac{1}{u}, \quad u \in [n^{-1}, n^{-\alpha}], \tag{8}$$

we split each of the above integrals into these two intervals obtaining

$$|I_2| \leq Q\left(n \int_0^{n^{-1}} du + \int_{n^{-1}}^{n^{-\alpha}} \frac{1}{u} du + n \int_0^{n^{-1}} \log \frac{1}{u} du + \int_{n^{-1}}^{n^{-\alpha}} \log \frac{1}{u}\frac{1}{u} du\right)$$
$$\leq Q\big(1 + \log n + \log n + \log^2 n\big) \leq Q \log^2 n.$$

Thus, the second term in Eq. 6 is $o(n)$ since $\alpha < 1$.

For the third integral, we split the logarithm factor into two terms and so we can write

$$I_3 = \tfrac{1}{\pi^2} \iint_R \log \frac{1}{2\sin v \sin u} \frac{\sin^2((2n+2\lambda+1)u)}{\sin^2 u} dudv = I_4 + I_5.$$

Let us continue with

$$I_4 = \tfrac{1}{\pi^2} \iint_R \log \frac{1}{2\sin v} \frac{\sin^2((2n+2\lambda+1)u)}{\sin^2 u} dudv \leq o(1) \int_0^{n^{-\alpha}} \frac{\sin^2((2n+2\lambda+1)u)}{\sin^2 u} du,$$

where we have used again Lemma 31. Using the same ideas as in I_2, we can split the integration interval, and using the bounds Eqs. 7 and 8 we get

$$I_4 \leq o(1) \left(\int_0^{n^{-1}} n^2 du + \int_{n^{-1}}^{n^{-\alpha}} \frac{1}{u^2} du\right) \leq o(1)n = o(n).$$

Finally, we will deal with the remaining integral,

$$
\begin{aligned}
I_5 &= \frac{1}{\pi^2}\iint_R \log\frac{1}{\sin u}\,\frac{\sin^2\big((2n+2\lambda+1)u\big)}{\sin^2 u}\,du\,dv \\
&= \frac{\pi-4n^{-\alpha}}{\pi^2}\int_0^{n^{-\alpha}} \log\frac{1}{\sin u}\,\frac{\sin^2\big((2n+2\lambda+1)u\big)}{\sin^2 u}\,du. \qquad (9)
\end{aligned}
$$

We are not going to find a bound, since in this case we are interested in getting exactly the highest order term. First, using Taylor expansion of $u/\sin u$ around 0, we have

$$\frac{1}{\sin u} = \frac{1}{u}\big(1+O(n^{-2\alpha})\big), \quad u\in[0,n^{-\alpha}],$$

from where

$$\frac{1}{\sin^2 u} = \frac{1}{u^2}\big(1+O(n^{-2\alpha})\big), \quad \log\frac{1}{\sin u} = \log\frac{1}{u} + O(n^{-2\alpha}), \quad u\in[0,n^{-\alpha}].$$

We plug these last two identities in Eq. 9, getting

$$
\begin{aligned}
I_5 = {} & \tfrac{1}{\pi}\big(1+O(n^{-\alpha})\big)\int_0^{n^{-\alpha}} \log\frac{1}{u}\,\frac{\sin^2((2n+2\lambda+1)u)}{u^2}\,du \\
& + O(n^{-2\alpha})\int_0^{n^{-\alpha}} \frac{\sin^2\big((2n+2\lambda+1)u\big)}{u^2}\,du.
\end{aligned} \qquad (10)
$$

These two integrals can be easily computed using the Sine Integral function, defined as

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(t)}{t}\,dt,$$

which has the property

$$\frac{d}{du}\left(t\,\mathrm{Si}(2tu) - \frac{\sin^2(tu)}{u}\right) = \frac{\sin^2(tu)}{u^2} \qquad (11)$$

and it satisfies the asymptotics (see [1, (5.2.8), (5.2.34), (5.2.35)])

$$\mathrm{Si}(x) = \frac{\pi}{2} + O\left(\frac{1}{x}\right) \ \text{ as } \ x\to+\infty.$$

With these two properties, the integral on the second term of Eq. 10 becomes,

$$
\begin{aligned}
&\int_0^{n^{-\alpha}} \frac{\sin^2\big((2n+2\lambda+1)u\big)}{u^2}\,du \\
&= (2n+2\lambda+1)\,\mathrm{Si}\big(2(2n+2\lambda+1)n^{-\alpha}\big) - \frac{\sin^2\big((2n+2\lambda+1)n^{-\alpha}\big)}{n^{-\alpha}} \\
&= (2n+2\lambda+1)\big(\pi/2 + O(n^{\alpha-1})\big) + O(n^{\alpha}) = n\pi + O(n^{\alpha}). \qquad (12)
\end{aligned}
$$

Now, for the integral on the first term of Eq. 10, we perform integration by parts using Eq. 11, followed by simplification with Eq. 12 and the asymptotic of the Sine Function, obtaining

$$
\begin{aligned}
\int_0^{n^{-\alpha}} \log\frac{1}{u}\,\frac{\sin^2\big((2n+2\lambda+1)u\big)}{u^2}\,du = {} & \alpha\pi n\log n - \pi n \\
& + (2n+2\lambda+1)\int_0^{n^{-\alpha}} \frac{\mathrm{Si}\big(2(2n+2\lambda+1)u\big)}{u}\,du + O(n^{\alpha}\log n).
\end{aligned} \qquad (13)
$$

This integral above can be computed using Lemma 32, which gives

$$\int_0^{n^{-\alpha}} \frac{\mathrm{Si}\big(2(2n+2\lambda+1)u\big)}{u}du = \frac{\pi}{2}\log\big(2(2n+2\lambda+1)n^{-\alpha}\big) + \frac{\gamma\pi}{2} + O(n^{\alpha-1}). \quad (14)$$

Then, with Eqs. 12, 13 and 14 plugged into Eq. 10, and after some straightforward computations we get

$$I_5 = n\log n + (-1+\gamma+2\log 2)n + O(n^{\max(\alpha,1-\alpha)}\log n).$$

Now Proposition 16 is proved since $\alpha \in (1/2, 1)$. This finishes the proof of Proposition 9. □

6 Proof of Corollary 3

Let us consider the process described in the second point of Corollary 3. The value of the expected energy for this case is clearly $-2\log 2 + E(\lambda, n+1) + 2L_3$ where L_3 accounts for the energy corresponding to the crossed terms of the DPP with $n+1$ points and the extremes. In other words, L_3 is as given in the following result.

Theorem 17 *Let $L_3 = L_3(\lambda, n)$ be defined by*

$$\begin{aligned} L_3 &= \int_{-1}^1 K_n^\lambda(x,x)\log\left(\frac{1}{1+x}\right)w^\lambda(x)dx + \int_{-1}^1 K_n^\lambda(x,x)\log\left(\frac{1}{1-x}\right)w^\lambda(x)dx \\ &= \int_{-1}^1 K_n^\lambda(x,x)\log\left(\frac{1}{1-x^2}\right)w^\lambda(x)dx. \end{aligned}$$

Then,

$$\begin{aligned} L_3 &= (n+1)\left(\psi(n+\lambda+1) - \psi(\lambda+1/2)\right) \\ &\quad -(n+2\lambda)\big(\psi(n+\lambda+1/2) - \psi(\lambda+1/2) - 2\psi(2n+2\lambda+1) + 2\psi(n+2\lambda+1)\big), \end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. In particular, for any fixed $\lambda \in (-1/2, \infty)$, we have

$$L_3 = 2n\log(2) - (2\lambda - 1)\log n + O(1).$$

We prove this theorem later. First, let us finish the proof of Corollary 3. From Theorem 2, the energy of the first process in the corollary (generate $n+3$ points with the DPP) is

$$E(\lambda, n+3) = (n+3)^2\log 2 - (n+3)\log(n+3) + (1-\gamma-2\log 2)n + o(n). \quad (15)$$

From the same theorem and from Theorem 17, the energy of the second process described in Corollary 3 is

$$\begin{aligned} &-2\log 2 + E(\lambda, n+1) + 2L_3(\lambda, n) = \\ &\quad (n+1)^2\log 2 - (n+1)\log(n+1) + (1-\gamma-2\log 2)n + 4n\log(2) + o(n). \quad (16) \end{aligned}$$

It is an easy exercise to check that Eqs. 15 and 16 describe the same asymptotics up to $o(n)$, as claimed by the corollary. In the case $\lambda = 0$ we have exact values for L_1, L_2 and L_3 so we can compare directly the expressions $E(0, n+3)$ and $-2\log 2 + E(0, n+1) + 2L_3(0, n)$. It is straightforward to see that the first process has smaller energy than the second one, and that the difference is in the $O(\log n)$ term. This finishes the proof of Corollary 3 and it only remains to prove Theorem 17.

6.1 Proof of Theorem 17

Note that

$$L_3 = -2 \sum_{k=0}^{n} \int_{-1}^{1} \widehat{C}_k^\lambda(x)^2 w^\lambda(x) \log(1-x)\, dx. \tag{17}$$

The case $\lambda = 0$ can be done directly. In that case, we use

$$-2\log(1-x) = 2\log 2 + 4\log\frac{1}{\sqrt{2-2x}},$$

and combine Eq. 17 with Lemma 22 to get

$$L_3 = 2(n+1)\log 2 + H_n.$$

It is easy to check that this equals the expression in Theorem 17 (use [1, 6.3.8]).

We now center in the case that $\lambda \neq 0$. From Eq. 17 and Lemma 23 we have

$$L_3(\lambda, n) = -2\sum_{k=0}^{n}\left(-2\psi(2\lambda+2k)+\psi(2\lambda+k)+\log 2+\psi(\lambda+k+1/2)-\frac{1}{2k+2\lambda}\right).$$

We have to see that this is equal to the expression in Theorem 17, which we do by induction on n. The case $n=0$ reduces to [1, 6.3.8]. Moreover, by induction hypotheses we need to check that

$$4\psi(2\lambda+2n)-2\psi(2\lambda+n)-2\log 2-2\psi(\lambda+n+1/2)+\frac{1}{n+\lambda} = A+B,$$

where

$$\begin{aligned} A &= (n+1)\left(\psi(n+\lambda+1)-\psi(\lambda+1/2)\right) - n\left(\psi(n+\lambda)-\psi(\lambda+1/2)\right), \\ B &= (n+2\lambda-1)\big(\psi(n+\lambda-1/2)-\psi(\lambda+1/2)-2\psi(2n+2\lambda-1)+2\psi(n+2\lambda)\big) \\ &\quad -(n+2\lambda)\big(\psi(n+\lambda+1/2)-\psi(\lambda+1/2)-2\psi(2n+2\lambda+1)+2\psi(n+2\lambda+1)\big). \end{aligned}$$

This is a simple yet tedious exercise using that $\psi(z+1) = \psi(z) + 1/z$ and [1, 6.3.8].

Appendix A: Bounds and Integrals Involving Gegenabuer Polynomials

In this appendix we state some technical lemmas that have been used in the proof of the main results. Recall from [22, Eq. (7.33.6)] the following: for $n \geq 1$ and $\lambda \in (-1/2, \infty)$,

$$\left|C_n^\lambda(\cos\theta)\right| \leq \begin{cases} Q(\lambda)\sin^{-\lambda}\theta\, n^{\lambda-1}, & cn^{-1} \leq \theta \leq \pi - cn^{-1} \\ Q(\lambda)n^{2\lambda-1}, & 0 \leq \theta \leq cn^{-1} \end{cases}, \tag{18}$$

for $c > 0$ fixed and some constant $Q(\lambda)$. Observe that when $\lambda \geq 0$, both inequalities, though less sharp, hold in $[0, \pi]$. The following lemma follows easily:

Lemma 18 *Let $\lambda > -1/2$ and $n \geq 1$. Then, for $x \in [-1, 1]$ such that $0 \leq \arccos x \leq 2n^{-1}$ (this holds in particular if $1 - n^{-2} \leq x \leq 1$), we have*

$$K_n^\lambda(x, x) w^\lambda(x) \leq Q(\lambda) n^{2\lambda+1}(1-x^2)^{\lambda-1/2}.$$

Proof Recall that in the range of $x = \cos\theta$, for $k \geq 1$ we have

$$\hat{C}_k^\lambda(\cos\theta)^2 = \frac{k!(k+\lambda)\Gamma(2\lambda)}{\lambda\Gamma(2\lambda+k)} C_k^\lambda(\cos\theta)^2 \leq \frac{Q(\lambda)}{k^{2\lambda-2}} C_k^\lambda(\cos\theta)^2 \overset{(18)}{\leq} Q(\lambda)k^{2\lambda},$$

where we have used some standard estimates on the Gamma function. Then,

$$K_n^\lambda(x,x) = \sum_{k=0}^{n} \hat{C}_k^\lambda(\cos\theta)^2 \leq Q(\lambda)\sum_{k=1}^{n} k^{2\lambda}.$$

For all $\lambda \in (-1/2, \infty)$, the sum is bounded above by $Q(\lambda)n^{2\lambda+1}$, and we are done. □

The following is almost immediate:

Lemma 19 *Let $n > 1$. There exists a constant $c > 0$ such that for any $y \in [-1, 1]$:*

$$\int_{-1}^{1} \left| \frac{\log|x-y|}{\sqrt{1-x^2}} \right| dx \leq 3\pi \log 2, \quad \int_{-1+n^{-2}}^{1-n^{-2}} \left| \frac{\log|x-y|}{1-x^2} \right| dx \leq c\sqrt{n}.$$

If additionally we have $\lambda \in (-1/2, 0)$, then

$$\int_{1-n^{-2}}^{1} \left| \frac{\log|x-y|}{\sqrt{1-x^2}^{1-2\lambda}} \right| dx \leq Q(\lambda)n^{-1-2\lambda+(2+4\lambda)/(3-2\lambda)},$$

and

$$\int_{1-n^{-2}}^{1} \frac{1}{\sqrt{1-x^2}^{1-2\lambda}} dx \leq Q(\lambda)n^{-1-2\lambda}.$$

Proof Recall that

$$\int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}} dx = 1, \quad \int_{-1}^{1} \frac{\log|x-y|}{\pi\sqrt{1-x^2}} dx = -\log 2, \quad \forall y \in [-1, 1],$$

the last from Eq. 4. Since for any $x, y \in [-1, 1]$ we have $\log \frac{|x-y|}{2} \leq 0$, we conclude

$$\begin{aligned}
\int_{-1}^{1} \left| \frac{\log|x-y|}{\pi\sqrt{1-x^2}} \right| dx &= \int_{-1}^{1} \left| \frac{\log 2 + \log\frac{|x-y|}{2}}{\pi\sqrt{1-x^2}} \right| dx \\
&\leq \int_{-1}^{1} \frac{\log 2}{\pi\sqrt{1-x^2}} dx - \int_{-1}^{1} \frac{\log\frac{|x-y|}{2}}{\pi\sqrt{1-x^2}} dx \\
&= 2\int_{-1}^{1} \frac{\log 2}{\pi\sqrt{1-x^2}} dx - \int_{-1}^{1} \frac{\log|x-y|}{\pi\sqrt{1-x^2}} dx = 3\log 2.
\end{aligned}$$

The first claim of the lemma follows. For the second one, let J be the integral we want to estimate. From Holder's inequality,

$$J \leq \left(\int_{-1+n^{-2}}^{1-n^{-2}} |\log|x-y||^4 \, dx \right)^{1/4} \left(\int_{-1+n^{-2}}^{1-n^{-2}} \frac{1}{(1-x^2)^{4/3}} dx \right)^{3/4}.$$

The first of these two integrals is bounded above by some universal constant. The second one is at most

$$\left(2\int_{-1+n^{-2}}^{0} \frac{1}{(1+x)^{4/3}} dx \right)^{3/4} \leq c\sqrt{n}.$$

For the third integral in the lemma (that we denote $I(\lambda, n)$), again Holder's inequality yields the upper bound (independent of y):

$$I(\lambda, n) \leq Q(q,\lambda)\left(\int_{1-n^{-2}}^{1} \frac{1}{\sqrt{1-x^2}^{(1-2\lambda)q}}\,dx\right)^{1/q}$$
$$\leq Q(q,\lambda)\left(\int_{1-n^{-2}}^{1} \frac{1}{(1-x)^{(1/2-\lambda)q}}\,dx\right)^{1/q} \leq Q(q,\lambda)n^{1-2\lambda-2/q},$$

where $q > 1$ is any positive number such that $(1-2\lambda)q < 2$ and $Q(q,\lambda)$ plays the same role as $Q(\lambda)$ but in this case it may depend on q. We choose $q = (3-2\lambda)/(2-4\lambda)$ and the third inequality follows. The last one is even more elementary:

$$\int_{1-n^{-2}}^{1} \frac{1}{\sqrt{1-x^2}^{1-2\lambda}}\,dx \leq Q(\lambda)\int_{1-n^{-2}}^{1} \frac{1}{(1-x)^{1/2-\lambda}}\,dx \leq Q(\lambda)n^{-1-2\lambda}.$$

□

Lemma 20 *For $\lambda > -1/2$, $n \geq 1$ and $\theta \in [0,\pi]$*

$$\left|\widehat{C}_n^\lambda(\cos\theta)\sqrt{w^\lambda(\cos\theta)} - \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\theta}}\cos\left((n+\lambda)\theta - \lambda\pi/2\right)\right| \leq \frac{Q(\lambda)}{n\sin^{3/2}\theta}\min(1,T), \tag{19}$$

where $Q(\lambda)$ is some constant depending only on λ and

$$T = T(n,\theta) = \begin{cases} n\sin\theta, & \text{if } \lambda \geq 0 \\ (n\sin\theta)^{\lambda+1}, & \text{otherwise.} \end{cases}$$

One can change $\min(1,T)$ *to* $T/(T+1)$ *if desired.*

Proof The classical asymptotic results for Gegenbauer polynomials [22, Eq. (8.21.18)] yield:

$$\left|C_n^\lambda(\cos\theta) - \frac{2^\lambda S_n^\lambda}{\sqrt{\pi n}\sin^\lambda\theta}\cos\left((n+\lambda)\theta - \lambda\pi/2\right)\right| \leq \frac{|S_n^\lambda|Q(\lambda)}{n^{3/2}\sin^{\lambda+1}\theta} \leq \frac{Q(\lambda)n^{\lambda-2}}{\sin^{\lambda+1}\theta},$$

valid for $\theta \in [n^{-1}, \pi - n^{-1}]$, where $S_n^\lambda = \frac{\Gamma(n+2\lambda)\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\Gamma(\lambda+n+1/2)}$ satisfies $|S_n^\lambda| \leq Q(\lambda)n^{\lambda-1/2}$. Then, for the orthonormal polynomials we have

$$\left|\widehat{C}_n^\lambda(\cos\theta) - \frac{\gamma_n^\lambda 2^\lambda S_n^\lambda}{\sqrt{\pi n}\sin^\lambda\theta}\cos\left((n+\lambda)\theta - \lambda\pi/2\right)\right| \leq \frac{\gamma_n^\lambda Q(\lambda)n^{\lambda-2}}{\sin^{\lambda+1}\theta} \leq \frac{Q(\lambda)}{n\sin^{\lambda+1}\theta}.$$

Using

$$\left|\frac{\gamma_n^\lambda 2^\lambda S_n^\lambda}{\sqrt{\pi n}} - \sqrt{\frac{2\Gamma(\lambda+1/2)}{\sqrt{\pi}\Gamma(\lambda+1)}}\right| \leq \frac{Q(\lambda)}{n},$$

and multiplying by $\sqrt{w^\lambda}$ we get

$$\left|\widehat{C}_n^\lambda(\cos\theta)\sqrt{w^\lambda(\cos\theta)} - \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\theta}}\cos\left((n+\lambda)\theta - \lambda\pi/2\right)\right| \leq \frac{Q(\lambda)}{n\sin^{3/2}\theta} + \frac{Q(\lambda)}{n\sqrt{\sin\theta}},$$

so Eq. 19 is proved for $\theta \in [n^{-1}, \pi - n^{-1}]$.

Now if $\theta \in [0, n^{-1}] \cup [\pi - n^{-1}, \pi]$, using Eq. 18 we obtain

$$
\begin{aligned}
& n \sin^{3/2}\theta \left| \widehat{C}_n^{\lambda}(\cos\theta)\sqrt{w^{\lambda}(\cos\theta)} - \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\theta}} \cos\left((n+\lambda)\theta - \lambda\pi/2\right) \right| \\
& \leq n \sin^{3/2}\theta\, Q(\lambda) \left(\gamma_n^{\lambda} n^{2\lambda-1} \sin^{\lambda-1/2}\theta + \frac{1}{\sqrt{\sin\theta}} \right) \\
& \leq n \sin^{3/2}\theta\, Q(\lambda) \left(n^{\lambda} \sin^{\lambda-1/2}\theta + \frac{1}{\sqrt{\sin\theta}} \right) \\
& \leq Q(\lambda) \left((n\sin\theta)^{\lambda+1} + n\sin\theta \right),
\end{aligned}
$$

which is bounded above by $n\sin\theta$ if $\lambda \geq 0$ and by $(n\sin\theta)^{\lambda+1}$ otherwise, so Eq. 19 is valid for $\theta \in [0, \pi]$. □

Lemma 21 *Let $\lambda > -1/2$ and let $n \geq 2$, then*

$$
\left| K_n^{\lambda}(x, x) w^{\lambda}(x) - \frac{n+1}{\pi\sqrt{1-x^2}} \right| \leq \frac{Q(\lambda)\log n}{1-x^2}, \quad x \in [-1, 1]
$$

where $Q(\lambda)$ is some constant depending only on λ.

Proof For $\theta = \arccos(x) \in [0, \pi]$ and $k \geq 1$, by Lemma 20

$$
\begin{aligned}
& \left| \widehat{C}_k^{\lambda}(\cos\theta)^2 w^{\lambda}(\cos\theta) - \frac{2}{\pi \sin\theta} \cos^2((k+\lambda)\theta - \lambda\pi/2) \right| \\
& \leq \frac{Q(\lambda)}{k \sin^{3/2}\theta} \frac{T(k,\theta)}{T(k,\theta)+1} \left(\left|\widehat{C}_k^{\lambda}(\cos\theta)\right| \sqrt{w^{\lambda}(\cos\theta)} + \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{\sin\theta}} \right) \\
& \leq \frac{Q(\lambda)}{k \sin^{3/2}\theta} \frac{T(k,\theta)}{T(k,\theta)+1} \left(\frac{Q(\lambda)}{k \sin^{3/2}\theta} \frac{T(k,\theta)}{T(k,\theta)+1} + \frac{1}{\sqrt{\sin\theta}} \right) \\
& \leq \frac{Q(\lambda)}{(k+1)^2 \sin^3\theta} \left(\frac{T(k,\theta)}{T(k,\theta)+1} \right)^2 + \frac{Q(\lambda)}{(k+1)\sin^2\theta} \left(\frac{T(k,\theta)}{T(k,\theta)+1} \right),
\end{aligned}
$$

where we have used Lemma 20 and that the normalization coefficient γ_k^{λ} behaves like $k^{1-\lambda}$. Observe that this inequality is also valid for $k = 0$. Then, from Lemma 29 we conclude

$$
\begin{aligned}
K_n^{\lambda}(x, x) w^{\lambda}(x) &= \sum_{k=0}^{n} \widehat{C}_k^{\lambda}(x)^2 w^{\lambda}(x) \\
&\leq \frac{n+1}{\pi \sin\theta} + \sum_{k=0}^{n} \frac{Q(\lambda)}{(k+1)^2 \sin^3\theta} \left(\frac{T(k,\theta)}{T(k,\theta)+1} \right)^2 \\
&\quad + \sum_{k=0}^{n} \frac{Q(\lambda)}{(k+1)\sin^2\theta} \left(\frac{T(k,\theta)}{T(k,\theta)+1} \right).
\end{aligned}
$$

If $\lambda \geq 0$ we can bound the last sums by

$$\frac{Q(\lambda)}{\sin\theta}\sum_{k=0}^{n}\frac{1}{(1+k\sin\theta)^2} \leq \frac{Q(\lambda)}{\sin\theta}\int_0^n \frac{1}{(1+x\sin\theta)^2}\,dx \leq \frac{Q(\lambda)}{\sin\theta}\int_0^n \frac{1}{1+x\sin\theta}\,dx,$$

$$\frac{Q(\lambda)}{\sin\theta}\sum_{k=0}^{n}\frac{1}{1+k\sin\theta} \leq \frac{Q(\lambda)}{\sin\theta}\int_0^n \frac{1}{1+x\sin\theta}\,dx \leq \frac{Q(\lambda)\log(1+n\sin\theta)}{\sin^2\theta} \leq \frac{Q(\lambda)\log n}{\sin^2\theta}.$$

If $\lambda < 0$ the corresponding bounds are

$$\frac{Q(\lambda)}{\sin^{1-2\lambda}\theta}\sum_{k=0}^{n}\frac{k^{2\lambda+2}}{(k+1)^2(1+k^{\lambda+1}\sin^{\lambda+1}\theta)^2} \leq \frac{Q(\lambda)}{\sin^{1-2\lambda}\theta}\int_0^n \frac{x^{2\lambda}}{(1+x^{\lambda+1}\sin^{\lambda+1}\theta)^2}\,dx \leq \frac{Q(\lambda)}{\sin^2\theta},$$

(the last by dividing the integration interval with midpoint $1/\sin\theta$ if n is greater than this quantity), and

$$\frac{Q(\lambda)}{\sin^{1-\lambda}\theta}\sum_{k=0}^{n}\frac{k^{\lambda+1}}{(k+1)(1+k^{\lambda+1}\sin^{\lambda+1}\theta)} \leq \frac{Q(\lambda)}{\sin^{1-\lambda}\theta}\int_0^n \frac{x^{\lambda}}{(1+x^{\lambda+1}\sin^{\lambda+1}\theta)}\,dx = \frac{Q(\lambda)\log(1+(n\sin\theta)^{\lambda+1})}{\sin^2\theta} \leq \frac{Q(\lambda)\log n}{\sin^2\theta},$$

as we wanted. The reciprocal inequality

$$K_n^{\lambda}(x,x)w^{\lambda}(x) \geq \frac{n+1}{\pi\sin\theta} - \frac{Q(\lambda)\log n}{\sin^2\theta}$$

is proved the same way (now, the error bounds have a minus sign). □

Lemma 22 *Let* $\lambda = 0$. *Then,*

$$\sum_{k=0}^{n}\int_{-1}^{1}\widehat{C}_k^0(x)^2 w^0(x)\log\frac{1}{\sqrt{2-2x}}\,dx = \frac{H_n}{4}.$$

Proof With the change of variables $x = \cos\theta$, the integral of the lemma becomes

$$\frac{1}{2\pi}\sum_{k=0}^{n}\int_{-\pi}^{\pi}\widehat{C}_k^0(\cos\theta)^2\log\frac{1}{\sqrt{2-2\cos\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\frac{1}{\sqrt{2-2\cos\theta}}\,d\theta + \sum_{k=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(k\theta)\log\frac{1}{\sqrt{2-2\cos\theta}}\,d\theta = \frac{H_n}{4},$$

the last from lemmas 25 and 27. □

Lemma 23 *The following equality holds for all* $\lambda \in (-1/2, 0) \cup (0, \infty)$ *and integer* $k \geq 0$*:*

$$\int_{-1}^{1} C_k^\lambda(x)^2(1-x^2)^{\lambda-1/2}\log(1-x)\,dx = \frac{\pi 2^{1-2\lambda}\Gamma(2\lambda+k)}{k!(k+\lambda)\Gamma(\lambda)^2}$$
$$\times\left(-2\psi(2\lambda+2k)+\psi(2\lambda+k)+\log 2+\psi(\lambda+k+1/2)-\frac{1}{2k+2\lambda}\right).$$

In other words (from the definition of $\widehat{C}_k^\lambda$ *and* w^λ*),*

$$\int_{-1}^{1} \widehat{C}_k^\lambda(x)^2 w^\lambda(x)\log(1-x)\,dx$$
$$=\left(-2\psi(2\lambda+2k)+\psi(2\lambda+k)+\log 2+\psi(\lambda+k+1/2)-\frac{1}{2k+2\lambda}\right).$$

Proof A change of variables $x \to 1-x$ shows that the integral in the lemma equals

$$\frac{1}{2}\int_{-1}^{1} C_k^\lambda(x)^2(1-x^2)^{\lambda-1/2}\log(1-x^2)\,dx,$$

that has been computed in [23, Theorem 3]. The expression in [23, Theorem 3] and ours are equivalent (use [1, 6.3.8]). □

Appendix B: Some Integrals and Sums

We have used some technical results that we include here for the reader's convenience.

Lemma 24

$$\sum_{j=1}^{n} j\log(j) = \frac{1}{2}n^2\log n - \frac{1}{4}n^2 + \frac{1}{2}n\log n + \frac{1}{12}\log n + O(1).$$

Proof Let

$$\begin{aligned} S_n := \sum_{j=1}^{n} j\log(j) &= \log\left(\prod_{j=1}^{n} j^j\right) = \log\left(\prod_{j=1}^{n}\prod_{k=1}^{j} j\right) \\ &= \log\left(\prod_{k=1}^{n}\prod_{j=k}^{n} j\right) = \log\left(\prod_{k=1}^{n}\frac{n!}{(k-1)!}\right) \\ &= n\log(n!) - \log\left(\prod_{k=1}^{n}(k-1)!\right) = n\log\Gamma(n+1) - \log G(n+1), \end{aligned}$$

where $G(n) = (n-2)!(n-3)!\dots 1!$ is Barnes G-function, also called the double gamma function. The asymptotics of $G(z)$ for $z \to +\infty$ is known (see [13, Theorem 1] or [11, 5.17.5]):

$$\log(G(z+1)) = \frac{1}{4}z^2 + z\log\Gamma(z+1) - \frac{1}{2}z^2\log z - \frac{1}{2}z\log z - \frac{1}{12}\log z + O(1).$$

We thus have proved that

$$\begin{aligned} S_n &= n\log\Gamma(n+1) - \left(\frac{1}{4}n^2 + n\log\Gamma(n+1) - \frac{1}{2}n^2\log n - \frac{1}{2}n\log n - \frac{1}{12}\log n + O(1)\right) \\ &= \frac{1}{2}n^2\log(n) - \frac{1}{4}n^2 + \frac{1}{2}n\log(n) + \frac{1}{12}\log(n) + O(1). \end{aligned}$$

□

Lemma 25 *We have*

$$\int_{-\pi}^{\pi} \log\frac{1}{\sqrt{2-2\cos\alpha}}\, d\alpha = 0.$$

Proof First, translate the integration interval to $[0, 2\pi]$ and after that combine the change of variables $\alpha = 2\pi x$ with [14, 4.384 (3)]. □

Lemma 26 *Let* $f : [-1, 1]^2 \to \mathbb{R}$ *be a continuous function. Then,*

$$\begin{aligned} &\int_{[-1,1]^2} \frac{f(x,y)}{\sqrt{1-x^2}\sqrt{1-y^2}} \log\frac{1}{2|x-y|}\, d(x,y) \\ &= \frac{1}{2}\int_{-\pi}^{\pi} d\alpha \log\frac{1}{\sqrt{2-2\cos\alpha}} \int_{-\pi}^{\pi} f\left(\cos\theta, \cos(\theta+\alpha)\right)\, d\theta. \end{aligned}$$

In particular, from Lemma 25, if $\int_{-\pi}^{\pi} f\left(\cos\theta, \cos(\theta+\alpha)\right) d\theta$ *is constant (i.e. if its value does not depend on α) then the integral of the lemma is 0.*

Proof Denote by I the integral in the lemma. The change of variables $x = \cos\theta$, $y = \cos\phi$ yields

$$\begin{aligned} 4I &= 4\int_{[0,\pi]^2} f(\cos\theta,\cos\phi)\log\tfrac{1}{2|\cos\theta-\cos\phi|}\, d(\theta,\phi) \\ &= \int_{[-\pi,\pi]^2} f(\cos\theta,\cos\phi)\log\tfrac{1}{2|\cos\theta-\cos\phi|}\, d(\theta,\phi) \\ &= \int_{[-\pi,\pi]^2} f(\cos\theta,\cos\phi)\log\tfrac{1}{|e^{i\theta}-e^{i\phi}|}\, d(\theta,\phi) \\ &\quad + \int_{[-\pi,\pi]^2} f(\cos\theta,\cos\phi)\log\tfrac{1}{|e^{i\theta}-e^{-i\phi}|}\, d(\theta,\phi), \end{aligned}$$

where we have used the following classical fact:

$$2|\cos\theta - \cos\phi| = \left|e^{i\theta} - e^{i\phi}\right| \left|e^{i\theta} - e^{-i\phi}\right|.$$

The change of variables $\phi \to -\phi$ shows that the last two integrals above are equal and hence we have

$$2I = \int_{[-\pi,\pi]^2} f(\cos\theta,\cos\phi)\log\frac{1}{\left|e^{i\theta}-e^{i\phi}\right|}\, d(\theta,\phi).$$

We write this last expression as an integral in the product $[-\pi, \pi] \times S^1$, where S^1 is the unit circle, getting

$$\begin{aligned} 2I &= \int_{-\pi}^{\pi} d\theta \int_{z\in S^1} f(\cos\theta, Re(z))\log\frac{1}{\left|e^{i\theta}-z\right|}\frac{dz}{iz} \\ &= \int_{-\pi}^{\pi} d\theta \int_{w\in S^1} f\left(\cos\theta, Re(e^{i\theta}w)\right)\log\frac{1}{\left|e^{i\theta}-e^{i\theta}w\right|}\frac{dw}{iw} \\ &= \int_{w\in S^1} \frac{dw}{iw}\log\frac{1}{|1-w|}\int_{-\pi}^{\pi} f\left(\cos\theta, Re(e^{i\theta}w)\right)\, d\theta. \end{aligned}$$

We have applied the isometry $S^1 \to S^1$ given by $z \to w = e^{-i\theta} z$. Parametrizing again the unit circle by $w = e^{i\alpha}$ we get to

$$2I = \int_{\alpha\in[-\pi,\pi]} d\alpha \log \frac{1}{\sqrt{2-2\cos\alpha}} \int_{-\pi}^{\pi} f(\cos\theta, \cos(\theta+\alpha))\, d\theta,$$

proving the lemma. □

Lemma 27 *For any integer $k \geq 1$ we have*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha = \frac{1}{k}, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(k\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha = \frac{1}{4k}.$$

Moreover, for integers $k > \ell \geq 1$ we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k\alpha)\cos(\ell\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha = \frac{1}{2}\left(\frac{1}{k-\ell}+\frac{1}{k+\ell}\right).$$

Proof For the first integral, we proceed as in Lemma 25

$$2\int_0^1 \cos(2k\pi x)\log\frac{1}{2\sin(\pi x)} \overset{[14,\ 4.384\ (3)]}{=} \frac{1}{k}.$$

For the second integral we consider

$$x = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(k\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha,$$
$$y = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(k\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha,$$

and we note that

$$x + y = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha \overset{\text{Lemma 25}}{=} 0,$$
$$x - y = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2k\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha = \frac{1}{2k},$$

and adding these two equalities gives the desired result. For the last claim we similarly consider

$$x = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(k\alpha)\cos(\ell\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha,$$
$$y = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(k\alpha)\sin(\ell\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha,$$

which readily gives

$$x + y = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos((k-\ell)\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha = \frac{1}{k-\ell},$$
$$x - y = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos((k+\ell)\alpha)\log\frac{1}{\sqrt{2-2\cos\alpha}}\,d\alpha = \frac{1}{k+\ell},$$

and again adding these equalities gives the last integral of the lemma. □

Lemma 28 *For $n \geq 2$,*

$$\sum_{k=2}^{n} H_{2k-1} = nH_{2n-1} + \frac{H_{2n}}{2} - \frac{H_n}{4} - n - \frac{1}{2}.$$

Proof This is an easy exercise of induction. □

Lemma 29 *The following equality holds:*

$$\sum_{k=0}^{n} \cos^2(k\theta+\alpha) = \frac{n}{2} + \frac{O(1)}{\sin\theta} = \frac{n+1}{2} + \frac{O(1)}{\sin\theta}.$$

Proof From the double angle formulas, the sum in the lemma is equal to $A - B + C$ where

$$\begin{aligned} A &= \cos^2(\alpha)\sum_{k=0}^{n}\cos^2(k\theta), \\ B &= 2\cos(\alpha)\sin(\alpha)\sum_{k=0}^{n}\cos(k\theta)\sin(k\theta) = \frac{\sin(2\alpha)}{2}\sum_{k=0}^{n}\sin(2k\theta), \\ C &= \sin^2(\alpha)\sum_{k=0}^{n}\sin^2(k\theta). \end{aligned}$$

These three sums are known, see [14, Sec. 1.34, 1.35], yielding the following value for the sum in the lemma:

$$\cos^2(\alpha) + \frac{n}{2} + \cos(2\alpha)\frac{\cos((n+1)\theta)\sin(n\theta)}{2\sin\theta} - \frac{\sin(2\alpha)}{2}\frac{\sin((n+1)\theta)\sin(n\theta)}{\sin\theta}.$$

We are done. □

Lemma 30 *The following identities hold:*

$$\sum_{k=0}^{n}\cos(ak+b) = \frac{1}{2\sin(a/2)}\left(\sin\left(an+\frac{a}{2}+b\right)-\sin\left(b-\frac{a}{2}\right)\right),$$

$$\begin{aligned} \sum_{k=0}^{n}\cos(ak+b)\cos(ck+d) =& \frac{1}{4\sin\left(\frac{a+c}{2}\right)}\left(\sin\left((a+c)n+\frac{a+c}{2}+b+d\right)+\sin\left(\frac{a+c}{2}-b-d\right)\right) \\ &+\frac{1}{4\sin\left(\frac{a-c}{2}\right)}\left(\sin\left((a-c)n+\frac{a-c}{2}+b-d\right)+\sin\left(\frac{a-c}{2}-b+d\right)\right). \end{aligned}$$

Proof First identity is direct consequence of [14, 1.341.3] and the identities for the product of a pair of trigonometric functions. Second identity is consequence of the identities for the product of two trigonometric functions, and first identity. □

Lemma 31 *The following identity holds:*

$$\int_0^{\pi} \log\frac{1}{2\sin v}dv = 0.$$

Proof Using the change of variable $x = \cos(v)$ and taking into account that we can write $\sqrt{1-x^2} = |x-1|^{1/2}\cdot|x+1|^{1/2}$ we get

$$\frac{1}{\pi}\int_0^{\pi}\log\frac{1}{2\sin v}dv = -\log 2 + \int_{-1}^{1}\log\frac{1}{\sqrt{1-x^2}}\frac{dx}{\pi\sqrt{1-x^2}} = -\log 2 + \frac{V(1)+V(-1)}{2},$$

where $V(x)$ is the equilibrium measure potential, which constantly equals $\log 2$. So the integral vanishes. □

Lemma 32 *Let* $\mathrm{Si}(t)$ *the Integral Sine function. The following asymptotic expansion holds:*

$$\int_0^x \frac{Si(t)}{t} dt = \frac{\pi}{2}\log x + \frac{\gamma\pi}{2} + O(x^{-1}), \qquad \text{as } x \to \infty,$$

being γ *the Euler constant.*

Proof We will obtain this identity by means of the imaginary part of a complex line integral. Let us take $x > 0$ and consider $C_x = \{z \in \mathbb{C} : |z| = x; \operatorname{Im} z > 0\}$ parametrized counterclockwise. Then,

$$\int_{-x}^{x} \frac{1}{t}\int_0^t \frac{e^{iz}-1}{z} dz\, dt + \int_{C_x} \frac{1}{t}\int_0^t \frac{e^{iz}-1}{z} dz\, dt = 0 \tag{20}$$

since we are integrating an entire function along a closed curve. We now split the integral over the semicircle within three other ones,

$$\begin{aligned}\int_{C_x} \frac{1}{t}\int_0^t \frac{e^{iz}-1}{z} dz\, dt &= \int_{C_x} \frac{1}{t}\int_0^x \frac{e^{iz}-1}{z} dz\, dt + \int_{C_x} \frac{1}{t}\int_x^t \frac{e^{iz}}{z} dz\, dt - \int_{C_x} \frac{1}{t}\int_x^t \frac{1}{z} dz\, dt\\ &= I_1 + I_2 - I_3.\end{aligned}$$

Let us work with each one of these integrals.

$$I_1 = (\log(-x) - \log(x))\int_0^x \frac{e^{iz}-1}{z} dz = i\pi\int_0^x \frac{e^{iz}-1}{z} dz.$$

For the second integral, we parametrize C_x as $t = xe^{i\theta}$ and after the change of variable $z = xe^{i\sigma}$, standard computations lead to

$$I_2 = -\int_0^{\pi}\int_0^{\theta} \exp(ixe^{i\sigma}) d\sigma\, d\theta,$$

which can be bounded as

$$|I_2| \le \int_0^{\pi}\int_0^{\theta} e^{-x\sin\sigma} d\sigma\, d\theta = \int_0^{\pi}\int_{\sigma}^{\pi} e^{-x\sin\sigma} d\theta\, d\sigma \le 2\pi\int_0^{\pi/2} e^{-x\sin\sigma} d\sigma.$$

Now we use the Jordan's inequality: $\sin\sigma \ge 2\sigma/\pi$ for $\sigma \in (0, \pi/2)$, getting

$$|I_2| \le 2\pi\int_0^{\pi/2} e^{-2x\sigma/\pi} d\sigma = \frac{\pi^2}{x}(1 - e^{-x}) = O(x^{-1}) \quad \text{as} \quad x \to +\infty.$$

The last integral can be computed directly using the parametrization $t = xe^{i\theta}$,

$$I_3 = \int_{C_x} \frac{1}{t}(\log t - \log x) dt = -\frac{\pi^2}{2}.$$

Then, Eq. 20 reads as

$$\int_{-x}^{x} \frac{1}{t}\int_0^t \frac{e^{iz}-1}{z} dz\, dt + i\pi\int_0^x \frac{e^{iz}-1}{z} dz + O(x^{-1}) + \frac{\pi^2}{2} = 0,$$

from where, taking imaginary part we get

$$2\int_0^x \frac{\mathrm{Si}(t)}{t} dt + \pi\int_0^x \frac{\cos(z)-1}{z} dz + O(x^{-1}) = 0.$$

Finally, in the second term we use the asymptotics of the integral cosine function ([1, 5.2.2, 5.2.9, 5.2.34 and 5.2.35]),

$$\mathrm{Ci}(x) = \gamma + \log x + \int_0^x \frac{\cos(z) - 1}{z} dz = O(x^{-1}) \quad \text{as} \quad x \to +\infty,$$

and then the announced result is proved. □

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