Received: 3 November 2021 | Revised: 22 September 2022 | Accepted: 4 January 2023

DOI: 10.1002/mana.202100589

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Asymptotic stability of the spectrum of a parametric family of homogenization problems associated with a perforated waveguide

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Funding information
MICINN, Grant/Award Numbers: PGC2018-098178-B-I00, PID2020-114703GB-I00; Severo Ochoa Programme for Centres of Excellence in R&D, Grant/Award Number: CEX2019-000904-S

Abstract

In this paper, we provide uniform bounds for convergence rates of the low frequencies of a parametric family of problems for the Laplace operator posed on a rectangular perforated domain of the plane of height H. The perforations are periodically placed along the ordinate axis at a distance $O(\varepsilon)$ between them, where ε is a parameter that converges toward zero. Another parameter η, the Floquet-parameter, ranges in the interval $[-\pi,\pi]$. The boundary conditions are quasi-periodicity conditions on the lateral sides of the rectangle and Neumann over the rest. We obtain precise bounds for convergence rates which are uniform on both parameters ε and η and strongly depend on H. As a model problem associated with a waveguide, one of the main difficulties in our analysis comes near the nodes of the limit dispersion curves.

KEYWORDS

band-gap structure, double periodicity, homogenization, Neumann–Laplace operator, perforated media, spectral gaps, spectral perturbations, waveguide

MSC (2020)

35B27, 35P05, 47A55, 35J25, 47A10

1 | INTRODUCTION

In this section, we formulate the spectral perturbation problem which constitutes a homogenization problem for the Laplacian in a periodically perforated rectangular domain, εH being the period, with $\varepsilon \ll 1$ and $H > 0$; H is the height of the rectangle. The so-called *Floquet-parameter* $\eta \in [-\pi,\pi]$ arises on the lateral boundary conditions. On the rest of

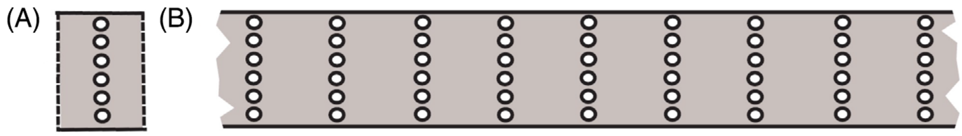

FIGURE 1 (A) The perforated domain Ω^ε. (B) The perforated strip Π^ε.

the boundary, Neumann conditions are imposed. The perforations are periodically placed along the ordinate axis. In Section 1.2, we introduce the homogenized problem which is still a η-dependent problem. Some background on the problem and the structure of the paper are discussed in Section 1.3. Our aim is to study the asymptotic behavior of the spectrum, as $\varepsilon \to 0$, in its dependence on the other parameters η and H.

1.1 | The parametric family of homogenization spectral problems

Let H be a positive parameter and Ω^0 be the rectangle

$$\Omega^0 = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, |x_1| < 1/2, \quad x_2 \in (0, H)\}. \tag{1.1}$$

Let ω be a domain in the plane $\mathbb{R}^2$ which is bounded by a smooth simple closed curve $\partial\omega$ and has the compact closure $\overline{\omega} = \omega \cup \partial\omega$ inside Ω^0. We introduce the perforated domain Ω^ε, see Figure 1A), obtained from Ω^0 by removing the family of holes

$$\omega^\varepsilon(k) = \{x \,:\, \varepsilon^{-1}(x_1, x_2 - \varepsilon k H) \in \omega\}, \quad k = 0, \dots, N-1, \tag{1.2}$$

which are distributed periodically along the line $x_1 = 0$. Each hole is homothetic to ω of ratio ε and translation of $\varepsilon\omega = \omega^\varepsilon(0)$. Namely,

$$\Omega^\varepsilon = \Omega^0 \setminus \overline{\omega^\varepsilon} \quad \text{where} \quad \omega^\varepsilon = \bigcup_{k=0}^{N-1} \omega^\varepsilon(k). \tag{1.3}$$

Here, ε is a small positive parameter and N is a big natural number, both related by $N = \varepsilon^{-1}$. The period is εH with $\varepsilon \ll 1$.

In the domain Ω^ε, we consider the spectral problem consisting in the partial differential equation

$$-\Delta_x U^\varepsilon(x;\eta) = \Lambda^\varepsilon(\eta) U^\varepsilon(x;\eta), \;\; x \in \Omega^\varepsilon, \tag{1.4}$$

the quasi-periodicity conditions on the vertical sides of Ω^ε

$$\begin{aligned} U^\varepsilon\left(\frac{1}{2}, x_2; \eta\right) &= \mathrm{e}^{\mathrm{i}\eta} U^\varepsilon\left(-\frac{1}{2}, x_2; \eta\right), \;\; x_2 \in (0, H), \\ \frac{\partial U^\varepsilon}{\partial x_1}\left(\frac{1}{2}, x_2; \eta\right) &= \mathrm{e}^{\mathrm{i}\eta} \frac{\partial U^\varepsilon}{\partial x_1}\left(-\frac{1}{2}, x_2; \eta\right), \;\; x_2 \in (0, H), \end{aligned} \tag{1.5}$$

and the Neumann condition on the remaining part of the boundary of Ω^ε

$$\partial_\nu U^\varepsilon(x;\eta) = 0, \;\; x \in \{x \in \partial\Omega^\varepsilon \,:\, |x_1| < 1/2\}. \tag{1.6}$$

Here, $\eta \in [-\pi, \pi]$ is the Floquet-parameter, ∂_ν denotes the directional derivative along the outward normal ν. $\Lambda^\varepsilon(\eta)$ and $U^\varepsilon(\cdot;\eta)$, respectively, denote the eigenvalues and eigenfunctions which depend on both the perturbation parameter and the Floquet-parameter. We address the asymptotic behavior of $(\Lambda^\varepsilon(\eta), U^\varepsilon(\cdot;\eta))$ as $\varepsilon \to 0$.

The variational formulation of the problem (1.4)–(1.6) reads (see, e.g., [12]): find $\Lambda^\varepsilon(\eta)$ and $U^\varepsilon(\cdot;\eta) \in H^{1,\eta}_{per}(\Omega^\varepsilon)$, $U^\varepsilon(\cdot;\eta) \neq 0$, satisfying

$$(\nabla_x U^\varepsilon(\cdot;\eta), \nabla_x V^\varepsilon)_{\Omega^\varepsilon} = \Lambda^\varepsilon(\eta)(U^\varepsilon(\cdot;\eta), V^\varepsilon)_{\Omega^\varepsilon} \quad \forall V^\varepsilon \in H^{1,\eta}_{per}(\Omega^\varepsilon), \tag{1.7}$$

where $H^{1,\eta}_{per}(\Omega^\varepsilon)$ is the Sobolev space of functions in $H^1(\Omega^\varepsilon)$ satisfying the quasi-periodicity conditions (1.5), while $(\cdot,\cdot)_{\Omega^\varepsilon}$ stands for the scalar product in $L^2(\Omega^\varepsilon)$.

In view of the compact embedding $H^1(\Omega^\varepsilon) \subset L^2(\Omega^\varepsilon)$ problem (1.7) has a discrete spectrum constituting the unbounded monotone sequence of eigenvalues (cf. [4, Section 10.2] and [29, Section 4.5]),

$$0 \le \Lambda_1^\varepsilon(\eta) \le \Lambda_2^\varepsilon(\eta) \le \cdots \le \Lambda_p^\varepsilon(\eta) \le \cdots \to +\infty, \quad \text{as } p \to +\infty, \tag{1.8}$$

which are repeated according to their multiplicities. Also, the corresponding eigenfunctions $\{U_p^\varepsilon(\cdot;\eta)\}_{p=1}^\infty$ are chosen to form an orthonormal basis in $L^2(\Omega^\varepsilon)$.

Furthermore, the functions

$$[-\pi,\pi] \ni \eta \;\mapsto\; \Lambda_p^\varepsilon(\eta), \quad p \in \mathbb{N}, \tag{1.9}$$

are continuous and 2π-periodic. This last assertion is due to the fact that problem (1.4)–(1.6) is the model problem associated with a waveguide: a periodically perforate Neumann strip recently considered in the literature (cf. (1.17), Figure 1B) and [9]). For the sake of completeness, in order to outline the interest of the problem under consideration (1.4)–(1.6), as well as its properties we introduce briefly this waveguide in Section 1.3.

1.2 | The parametric family of homogenized problems

By analogy with the homogenization of perforated domains along manifolds with Neumann boundary conditions (see, e.g., [15] for spectral problems, and [18, 31] for other close problems), we easily see that the homogenized problem is a spectral problem for the Laplacian, posed in the rectangle Ω^0, cf. (1.1), with the Neumann (1.11) (quasi-periodicity (1.12), respectively) conditions on the horizontal (vertical, respectively) sides of the rectangle. For the readers convenience, we introduce here this problem and provide the explicit formulas for the eigenvalues and eigenfunctions in terms of the parameters η and H. The convergence of the spectrum of (1.4)–(1.6), as $\varepsilon \to 0$, will be outlined in Section 2 (cf. Corollary 2.3) as a consequence of a more general result (cf. Theorem 2.2).

In Ω^0, we set the *homogenized spectral problem*

$$-\Delta_x U^0(x;\eta) = \Lambda^0(\eta) U^0(x;\eta), \;\; x \in \Omega^0, \tag{1.10}$$

$$\frac{\partial U^0}{\partial x_2}(x_1,0;\eta) = \frac{\partial U^0}{\partial x_2}(x_1,H;\eta) = 0, \;\; x_1 \in \left(-\frac{1}{2},\frac{1}{2}\right), \tag{1.11}$$

$$\begin{aligned} U^0\left(\frac{1}{2},x_2;\eta\right) &= e^{i\eta} U^0\left(-\frac{1}{2},x_2;\eta\right), \\ \frac{\partial U^0}{\partial x_1}\left(\frac{1}{2},x_2;\eta\right) &= e^{i\eta}\frac{\partial U^0}{\partial x_1}\left(-\frac{1}{2},x_2;\eta\right), \;\; x_2 \in (0,H), \end{aligned} \tag{1.12}$$

where $\Lambda^0(\eta)$ and $U^0(\cdot;\eta)$ denote the eigenvalues and corresponding eigenfunctions.

Problem (1.10)–(1.12) admits the variational formulation: find $\Lambda^0(\eta)$ and $U^0(\cdot;\eta) \neq 0$, $U^0(\cdot;\eta) \in H^{1,\eta}_{per}(\Omega^0)$, satisfying

$$(\nabla_x U^0(\cdot;\eta), \nabla_x V)_{\Omega^0} = \Lambda^0(\eta)(U^0(\cdot;\eta), V)_{\Omega^0} \quad \forall V \in H^{1,\eta}_{\rm per}(\Omega^0). \tag{1.13}$$

As is well known, it has a discrete spectrum, which we can write in an increasing order

$$0 \le \Lambda_1^0(\eta) \le \Lambda_2^0(\eta) \le \cdots \le \Lambda_p^0(\eta) \le \cdots \to +\infty, \quad \text{as } p \to +\infty, \tag{1.14}$$

where the convention of repeated eigenvalues has been adopted. Also, the corresponding eigenfunctions $\{U_p^0(\cdot;\eta)\}_{p=1}^\infty$ are chosen to form an orthonormal basis in $L^2(\Omega^0)$. Furthermore, the functions

$$[-\pi,\pi] \ni \eta \;\mapsto\; \Lambda_p^0(\eta), \quad p \in \mathbb{N}, \tag{1.15}$$

are continuous and 2π-periodic.

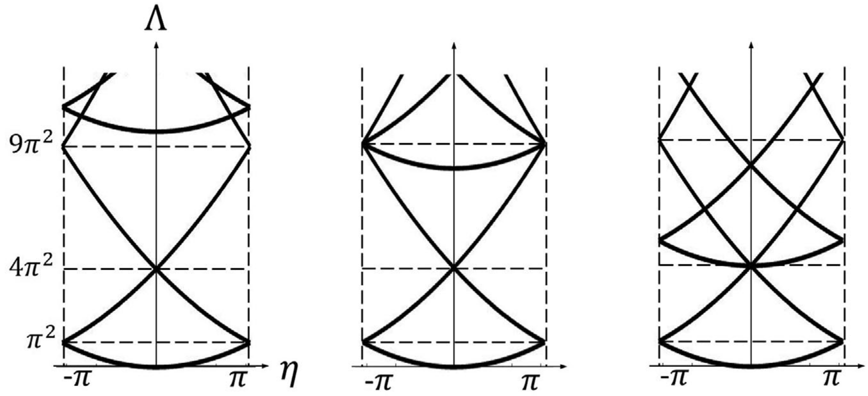

FIGURE 2 The first limit dispersion curves in the cases $H < 1/3$, $H = 1/\sqrt{8}$ and $H = 1/2$.

Explicit formulas for eigenvalues and eigenfunctions of (1.10)–(1.12) are obtained by separation of variables:

$$\Lambda^0_{jk}(\eta) = (\eta + 2\pi j)^2 + \pi^2\frac{k^2}{H^2}, \qquad U^0_{jk}(x;\eta) = \mathrm{e}^{\mathrm{i}(\eta+2\pi j)x_1}\cos\left(\pi k\frac{x_2}{H}\right), \quad j \in \mathbb{Z},\ k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{1.16}$$

It should be mentioned that re-numeration of the eigenvalues in (1.16) is needed to compose the monotone sequence (1.14).

Note that formulas (1.16) are of great interest to draw the *limit dispersion curves* for different values of H. Recall that these curves are the graphs of $\Lambda^0_p(\eta)$, for $\eta \in [-\pi, \pi]$. Figures 2 and 3 show the different possibilities of behaviors of such curves depending on the value of H. Also, they give an idea of what we can expect for the behavior of the perturbed dispersion curves, see, for instance, the second row in Figure 3.

1.3 | The state-of-the-art and the structure of the paper

First, let us introduce a problem closely related to (1.4)–(1.6): a Neumann problem for the Laplace operator in a periodically perforate strip with a double periodicity.

Extending Ω^ε (cf. (1.3) and Figure 1A)) by periodicity along the x_1-axis, we create the unbounded perforated strip Π^ε (see Figure 1B)):

$$\Pi^\varepsilon = \mathbb{R}\times(0,H)\setminus\bigcup_{j\in\mathbb{Z}}\bigcup_{k=0}^{N-1}\overline{\omega^\varepsilon(j,k)}$$

where $\omega^\varepsilon(j,k) = \{x : \varepsilon^{-1}(x_1 - j, x_2 - \varepsilon kH) \in \omega\}$ with $j \in \mathbb{Z},\ k = 0, 1, \dots, N-1$. In the waveguide Π^ε, we consider the Neumann spectral problem

$$\begin{aligned} -\Delta u^\varepsilon(x) &= \lambda^\varepsilon u^\varepsilon(x),\ x \in \Pi^\varepsilon, \\ \partial_\nu u^\varepsilon(x) &= 0,\ x \in \partial\Pi^\varepsilon. \end{aligned} \tag{1.17}$$

Then, applying the Floquet–Bloch–Gelfand transform (see, for instance, [7, 11, 25, 28, 32])

$$u^\varepsilon(x) \ \mapsto\ U^\varepsilon(x;\eta) = \frac{1}{\sqrt{2\pi}}\sum_{p\in\mathbb{Z}}\mathrm{e}^{-\mathrm{i}p\eta}u^\varepsilon(x_1 + p, x_2),$$

problem (1.17) converts into the boundary value problem (1.4)–(1.6) in Ω^ε.

The spectrum of the operator on the Hilbert space $L^2(\Pi^\varepsilon)$ associated with problem (1.17) is given by

$$\sigma^\varepsilon = \bigcup_{p\in\mathbb{N}}\beta^\varepsilon_p \tag{1.18}$$

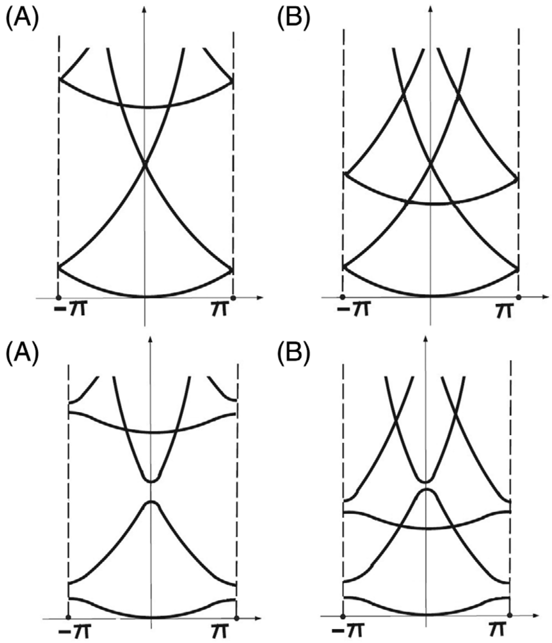

FIGURE 3 Above, the first six dispersion curves of the homogenized problem for some values of H, (A) $H \in (1/\sqrt{8}, 1/2)$, (B) $H \in (1/2, 1)$. Below, some possible dispersion curves of the perturbed problem for the same values of H.

where

$$\beta_p^\varepsilon = \{\Lambda_p^\varepsilon(\eta) \,:\, \eta \in [-\pi, \pi]\} \subset \overline{\mathbb{R}_+}. \tag{1.19}$$

As a consequence of the previously mentioned continuity of $\Lambda^\varepsilon(\eta)$ in (1.9), the sets β_p^ε are closed, connected, and finite segments. Formulas (1.18) and (1.19) for the spectrum of problem (1.17) are well known in the framework of the Floquet–Bloch–Gelfand theory (cf. [1, 3, 6, 9, 11, 25, 28–30, 32]).

Therefore, to study the asymptotic behavior of the spectrum of (1.4)–(1.6) becomes essential to detect the band gap structure of the spectrum (1.18). Opening gaps at the low-frequency level have been broached in [9] under certain symmetry restrictions for the holes (further specifying, the so-called *mirror symmetry*, see Equation (2.14) and Figure 4): roughly speaking, this involves controlling the total number of perturbed eigenvalues inside certain boxes of the band $[-\pi, \pi] \times [0, \infty)$ in the coordinate axis (η, Λ). Nevertheless, the technique in [9] does not allow to obtain uniform bounds for discrepancies between the dispersion curves $\{\Lambda_p^\varepsilon(\eta) \,:\, \eta \in [-\pi, \pi]\}$ and $\{\Lambda_p^0(\eta) \,:\, \eta \in [-\pi, \pi]\}$ for different values of p. Instead, the technique in this paper can contribute to discover how to open these gaps. However, we do not deal with this task but with obtaining the above-mentioned bounds.

Comparing with former papers in the literature, we mention [8–10, 23] as the closest papers. References [23] and [9] address opening gaps in a perforated waveguide for the Laplace operator with Dirichlet and Neumann boundary conditions, respectively. Asymptotic stability for the spectrum of problem (1.4)–(1.6) but changing the Neumann conditions for the Dirichlet ones has been addressed in [10], while (1.4)–(1.6) appears as a limit case of the stiff problem considered in [8]. It happens that for the Dirichlet condition in [23] the limit dispersion curves do not depend on η and the analysis in [10] becomes much more simplified than for Neumann.

It is worth mentioning that the explicit formulas (1.16) allow us to obtain the precise multiplicity of each eigenvalue $\Lambda_p^0(\eta)$ for each $p = 1, 2, \ldots$ and for each $\eta \in [-\pi, \pi]$. This proves to be essential for our analysis in Sections 3–5 which provides the precise number of eigenvalues of the perturbed problem at a distance $O(\varepsilon)$ of the eigenvalues of the homogenized problem (cf. Figure 7). It becomes particularly complicated near the points $\eta_{0,p}$ in which the limit eigenvalues $\Lambda_p^0(\eta)$ change the multiplicity from 1 to 2 or 3 (cf. Figures 2, 5 and 7, and Remark 4.4). Note the different behavior of the limit dispersion curves depending on H; let us outline the following cases:

$$a) \quad H \in \left(0, \frac{1}{\sqrt{8}}\right), \qquad b) \quad H \in \left[\frac{1}{\sqrt{8}}, \frac{1}{2}\right), \qquad c) \quad H \in \left[\frac{1}{2}, 1\right), \qquad d) \quad H \in [1, +\infty),$$

and refer to Figures 2 and 3 to realize the difference for certain of these cases. As a matter of fact, limit eigenvalues of multiplicity 3 appear first for larger values of H, the first one being $\Lambda = 4\pi^2$ when $H = 1/2$. For values of H greater than $1/2$, the parabolic dispersion curves that are translations of $\Lambda = \eta^2$ appear earlier, with a minimum in the interval $[\pi^2, 4\pi^2)$ when $H \in (1/2, 1]$ and much less when $H > 1$ (the minimum closer to 0) which complicates the trusses-nodes structure; cf. Figure 2 and the first row of Figure 3. This is the main reason why our final result (cf. Theorem 5.1) provides uniform bounds for discrepancies, in different intervals, depending on H, and we focus our attention on the first eigenvalues.

Let us describe the structure of the paper.

In Section 2, we provide some preliminary results valid for any geometry of the holes of smooth boundaries. First, we obtain upper bounds for the eigenvalues of the perturbed problem (1.4)–(1.6), namely,

$$\Lambda_m^\varepsilon(\eta) \leq \Lambda_m^0(\eta) + c_m\varepsilon \quad \text{for} \quad \varepsilon \leq \varepsilon_m, \quad \eta \in [-\pi, \pi],$$

that show how they are controlled by those of the homogenized problem (1.10)–(1.12). Then, in Section 2.1, we introduce a result of convergence of the eigenvalues and corresponding eigenfunctions toward those of the homogenized problem with conservation of the multiplicity, which also allows a perturbation of the Floquet-parameter (see Theorem 2.2 and Corollary 2.3),

$$\Lambda_m^{\varepsilon_l}(\eta_l) \to \Lambda_m^0(\hat{\eta}), \quad \text{as} \quad (\varepsilon_l, \eta_l) \xrightarrow{l\to\infty} (0, \hat{\eta}). \tag{1.20}$$

This shows a strong stability of both problems on the parameter η which arises in the quasi-periodicity conditions, cf. (1.5) and (1.12). The result has recently been introduced in the literature of model problems for waveguides, cf. [9, 10], and proves to be essential for obtaining uniform bounds (in the perturbation and Floquet parameters) for convergence rates between the eigenvalues of the perturbation and limit problems. In Section 2.2, we introduce a boundary layer problem posed in an unbounded strip (cf. (2.5)–(2.9)); it has been obtained by means of asymptotic expansions in [9].

In Sections 3 and 4, we obtain complementary results on the asymptotics of the eigenvalues which provide bounds for the distance between the dispersion curves of the limit problem and those of the homogenization problem. To do it, we use a well-known result on *almost eigenvalues and eigenfunctions* from the spectral perturbation theory, cf. Lemma 3.1, which implies the construction of families of quasimodes from the solutions of the homogenized problem and the boundary layer problem, cf. (3.19)–(3.21). Some restrictions are performed both on the geometry of the hole and on the choice of the limiting eigenvalues (cf. (2.14) and (3.7)). Avoiding these restrictions implies that further terms of asymptotic expansions as well as further boundary layer functions are necessary, and the computations become excessively high. Summarizing, we can set:

$$|\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta) - \Lambda_m^0(\eta)| < C_0\varepsilon, \qquad \text{for } \varepsilon \leq \varepsilon_0,\ \eta \in [-\pi, \pi],\ \text{and some } p(\varepsilon, \eta, m) \geq m,$$

when $m = 1$ and $H > 0$, when $m = 2$ and $H \in (0, 1/2)$ and when $m = 3$ and $H \in (0, 1/\sqrt{8})$. The same results, for η ranging in smaller intervals (dependent on m and H) are obtained in Section 4 for higher values of m (see Theorems 4.1 and 4.2).

At this stage, we cannot assert that $p(\varepsilon, \eta, m) = m$, and this is the aim of Section 5, where we combine the results in Sections 2 and 4 with different arguments of contradiction involving convergence (1.20). We obtain

$$|\Lambda_m^\varepsilon(\eta) - \Lambda_m^0(\eta)| < C_0\varepsilon, \qquad \text{with } \varepsilon \leq \varepsilon_0 \text{ and } \eta \in [-\pi, \pi],$$

for the same values of m and H. Above and throughout the paper, ε_m, c_m and C_m, with $m \in \mathbb{N}_0$, denote certain positive constants independent of both variables ε and η.

Due to the complexity of the trusses–nodes structure of the dispersion curves (cf. Figures 2 and 3), in the theorems, we have imposed restrictions on the index m depending on H, which can affect the intervals where the Floquet-parameter ranges, but the technique can be extended to further values of m.

Finally, it should be emphasized that the method here developed can be applied to many problems arising in waveguide theory, cf. [2, 5, 19–21] among others.

2 | PRELIMINARY RESULTS: CONVERGENCE AND BOUNDARY LAYER

In Section 2.1, we state a result of convergence for the spectrum of the ε-dependent problem toward that of the homogenized one which allows the perturbation of the Floquet-parameter, cf. Theorem 2.2. This result extends the usual convergence of the spectrum for each fixed $\eta \in [-\pi, \pi]$, cf. Corollary 2.3, and becomes necessary to derive our result in Section 5. In Section 2.2, we introduce a boundary layer problem and its solution which proves useful, in Section 3, for the construction of new approximations to groups of eigenfunctions of (1.4)–(1.6) corresponding to eigenvalues in small intervals.

First, we obtain some estimates for the eigenvalues of the perturbation problem which relate homogenization and homogenized problems, as a consequence of the comparison of both spectra.

Proposition 2.1. *For fixed $m = 1, 2, \ldots$, let $\Lambda_m^\varepsilon(\eta)$ and $\Lambda_m^0(\eta)$ be the eigenvalues in the sequence (1.8) and (1.14), respectively. Then, there exist positive constants c_m and ε_m, independent of ε and η, such that*

$$\Lambda_m^\varepsilon(\eta) \le \Lambda_m^0(\eta) + c_m\varepsilon, \quad \text{for } \varepsilon \in (0, \varepsilon_m], \quad \eta \in [-\pi, \pi]. \tag{2.1}$$

Proof. Let us apply the minimax principle. For each $m = 1, 2, \ldots$ and $\eta \in [-\pi, \pi]$, we write

$$\Lambda_m^\varepsilon(\eta) = \min_{E_m^\varepsilon \subset H^{1,\eta}_{\text{per}}(\Omega^\varepsilon)} \max_{V \in E_m^\varepsilon, V \neq 0} \frac{(\nabla_x V, \nabla_x V)_{\Omega^\varepsilon}}{(V, V)_{\Omega^\varepsilon}}, \tag{2.2}$$

where the minimum is computed over the set of subspaces E_m^ε of $H^{1,\eta}_{\text{per}}(\Omega^\varepsilon)$ with dimension m.

To prove (2.1), we take a particular subspace $E_m^{\varepsilon,*}$ that we construct as the linear space

$$E_m^{\varepsilon,*} = \langle U_1^0(\cdot;\eta)\big|_{\Omega^\varepsilon}, \ldots, U_m^0(\cdot;\eta)\big|_{\Omega^\varepsilon}\rangle,$$

where $\{U_k^0(\cdot;\eta)\}_{k=1}^m$ are the eigenfunctions of Equation (1.13), corresponding to the eigenvalues $\{\Lambda_k^0\}_{k=1}^m$ which are orthonormal in $L^2(\Omega^0)$.

For $V \in E_m^{\varepsilon,*}$ without any restriction we can assume that $\|V;\, L^2(\Omega^0)\|^2 = 1$ (cf. [2, 8] for the idea in different problems). Thus, we have $V = \alpha_1 U_1^0(\cdot;\eta) + \cdots + \alpha_m U_m^0(\cdot;\eta)$ for some constants α_i such that $\|V;\, L^2(\Omega^0)\|^2 = \alpha_1^2 + \cdots + \alpha_m^2 = 1$. Therefore, we write

$$(\nabla_x V, \nabla_x V)_{\Omega^\varepsilon} \le (\nabla_x V, \nabla_x V)_{\Omega^0} = \alpha_1^2 \Lambda_1^0(\eta) + \cdots + \alpha_m^2 \Lambda_m^0(\eta) \le \Lambda_m^0(\eta).$$

Also, we can write

$$(V, V)_{\Omega^\varepsilon} = (V, V)_{\Omega^0} - (V, V)_{\Omega^0 \setminus \Omega^\varepsilon} = 1 - (V, V)_{\Omega^0 \setminus \Omega^\varepsilon} \ge 1 - \widetilde{c}_m \varepsilon,$$

for a certain constant $\widetilde{c}_m$ independent of η and sufficiently small ε. Indeed, to show the last inequality, we use the estimate $\|V;\, L^2(\Omega^0 \cap \{|x_1| < \varepsilon\})\|^2 \le C\varepsilon \|V;\, H^1(\Omega^0)\|^2, \forall V \in H^1(\Omega^0)$, see, for example, Lemma 2.4 in [18], and consequently,

$$(V, V)_{\Omega^0 \setminus \Omega^\varepsilon} \le c\varepsilon((\nabla V, \nabla V)_{\Omega^0} + (V, V)_{\Omega^0}) \le c\varepsilon(\Lambda_m^0(\eta) + 1) \le \widetilde{c}_m \varepsilon.$$

Hence, (2.2) leads us to

$$\Lambda_m^\varepsilon(\eta) \le \max_{V \in E_m^{\varepsilon,*}, V \neq 0} \frac{(\nabla_x V, \nabla_x V)_{\Omega^\varepsilon}}{(V, V)_{\Omega^\varepsilon}} \le \frac{\Lambda_m^0(\eta)}{1 - \widetilde{c}_m \varepsilon} \le \Lambda_m^0(\eta) + c_m \varepsilon,$$

for some constant c_m and ε small enough. This shows assertion (2.1) and the proposition is proved. □

2.1 | Convergence for eigenvalues

Based on extensions of eigenfunctions in perforated domains (see, e.g., Section I.4.2 in [26]), the continuity on the Floquet-parameter of the mappings (1.9) and (1.15), and some contradiction arguments, the following result has been proved in [9].

Theorem 2.2. *For each subsequence $\{(\varepsilon_r,\eta_r)\}_{r=1}^{\infty}$ such that $\varepsilon_r \to 0$ and $\eta_r \to \hat{\eta} \in [-\pi,\pi]$, as $r \to \infty$, the eigenvalues $\Lambda_m^{\varepsilon_r}(\eta_r)$ of problem (1.4)–(1.6) in the sequence (1.8), when $(\varepsilon,\eta) \equiv (\varepsilon_r,\eta_r)$, converge toward the eigenvalues of problem (1.10)–(1.12) for $\eta \equiv \hat{\eta}$, as $r \to \infty$, and there is conservation of the multiplicity. Namely, for each $m = 1,2,\ldots$, the convergence*

$$\Lambda_m^{\varepsilon_r}(\eta_r) \to \Lambda_m^0(\hat{\eta}), \qquad \text{as } r \to \infty, \tag{2.3}$$

holds, where $\Lambda_m^0(\hat{\eta})$ is the m-th eigenvalue in the sequence (1.14) of eigenvalues of Equations (1.10)–(1.12) for $\eta \equiv \hat{\eta}$.

In addition, we can extract a subsequence, still denoted by ε_r, such that some suitable extension of the eigenfunctions $\{U_m^{\varepsilon_r}(\cdot;\eta_r)\}_{m=1}^{\infty}$ normalized in $L^2(\Omega^{\varepsilon_r})$, $\{\hat{U}_m^{\varepsilon_r}(\cdot;\hat{\eta})\}_{m=1}^{\infty}$, converge in $L^2(\Omega^0)$, as $r \to \infty$, toward the eigenfunctions $\{U_m^0(\cdot;\hat{\eta})\}_{m=1}^{\infty}$ of (1.10)–(1.12) which form an orthonormal basis of $L^2(\Omega^0)$.

Theorem 2.2 shows a certain stability of the limit of the spectrum of (1.4)–(1.6) under any perturbation of the Floquet-parameter η.

Corollary 2.3. *For any $\eta \in [-\pi,\pi]$, the eigenvalues $\Lambda_m^{\varepsilon}(\eta)$ of problem (1.7) in the sequence (1.8) converge toward the eigenvalues of problem (1.10)–(1.12) in the sequence (1.14), namely,*

$$\Lambda_m^{\varepsilon}(\eta) \;\to \Lambda_m^0(\eta) \quad \text{ as } \;\varepsilon \to 0, \tag{2.4}$$

and there is conservation of the multiplicity. In addition, for each sequence, we can extract a subsequence, still denoted by ε, such that some suitable extension of the eigenfunctions $\{U_m^{\varepsilon}(\cdot;\eta)\}_{m=1}^{\infty}$ normalized in $L^2(\Omega^{\varepsilon})$, $\{\hat{U}_m^{\varepsilon}\}_{m=1}^{\infty}$, converge in $L^2(\Omega^0)$, as $\varepsilon \to 0$, toward the eigenfunctions $\{U_m^0(\cdot;\eta)\}_{m=1}^{\infty}$ of Equations (1.10)–(1.12) which form an orthonormal basis of $L^2(\Omega^0)$.

This corollary states the classical spectral convergence for problem (1.7); its proof is an immediate consequence of Theorem 2.2.

2.2 | The boundary layer problem

As usual in two-scale boundary homogenization, near the perforation string $\omega^{\varepsilon}(0),\ldots,\omega^{\varepsilon}(N-1) \subset \Omega^0$, there appears a boundary layer which is described in the stretched coordinates

$$\xi = (\xi_1,\xi_2) = \varepsilon^{-1}(x_1, x_2 - \varepsilon k H). \tag{2.5}$$

Using these auxiliary coordinates, we introduce a boundary layer problem and its solution in the unbounded perforated strip

$$\Xi := \{x \in \mathbb{R} \times (0,H)\} \setminus \overline{\omega}\,. \tag{2.6}$$

Obviously, for each $k = 0,\ldots,N-1$, the change of variable (2.5) transforms $\omega^{\varepsilon}(k)$ into ω, cf. (1.2). The proof of the results of this section can be found in [9]; cf. also [23, 24] for the technique and further references.

Let us consider the function W_0^1 to be harmonic in Ξ,

$$-\Delta_{\xi} W_0^1(\xi) = 0,\;\; \xi \in \Xi, \tag{2.7}$$

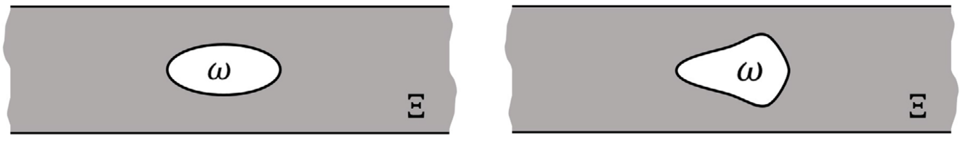

FIGURE 4 The strip Ξ with two different possible geometries for the hole ω.

with the periodicity conditions

$$W_0^1(\xi_1, H) = W_0^1(\xi_1, 0), \ \frac{\partial W_0^1}{\partial \xi_2}(\xi_1, H) = \frac{\partial W_0^1}{\partial \xi_2}(\xi_1, 0), \ \xi_1 \in \mathbb{R}, \tag{2.8}$$

and the nonhomogeneous Neumann condition on the boundary of the hole $\overline{\omega}$

$$\partial_{\nu(\xi)} W_0^1(\xi) = -\partial_{\nu(\xi)}\xi_1 = -\nu_1(\xi), \ \xi \in \partial\omega. \tag{2.9}$$

Here, $\nu(\xi) = (\nu_1(\xi), \nu_2(\xi))$ denote the outward (with respect to Ξ) normal vector on $\partial\omega$.

Also, for convenience, we introduce here the cut-off functions, $\chi_\pm \in C^\infty(\mathbb{R})$,

$$\chi_\pm(t) = 1 \text{ for } \pm t > 2R \text{ and } \chi_\pm(t) = 0 \text{ for } \pm t < R, \tag{2.10}$$

with a fixed $R > 0$ satisfying

$$\overline{\omega} \subset \Xi(R) := \{\xi \in \Xi : |\xi_1| < R\}. \tag{2.11}$$

The variational formulation of (2.7)–(2.9) reads: find $W_0^1 \in \mathcal{H}^1_{\text{per}}(\Xi)$ satisfying

$$(\nabla_\xi W_0^1, \nabla_\xi V)_\Xi = (-\nu_1(\xi), V)_{\partial\omega} \quad \forall V \in \mathcal{H}^1_{\text{per}}(\Xi), \tag{2.12}$$

where the space $\mathcal{H}^1_{\text{per}}(\Xi)$ denotes the completion of the linear space $C^\infty_{\text{per}}(\overline{\Xi})$ (of the infinitely differentiable H-periodic in ξ_2 functions with compact supports) in the norm

$$\|W; \mathcal{H}^1_{\text{per}}(\Xi)\| = \left(\|\nabla_\xi W; L^2(\Xi)\|^2 + \|W; L^2(\Xi(2R))\|^2\right)^{1/2}.$$

Since the compatibility condition, $(-\nu_1(\xi), 1)_{\partial\omega} = 0$, is satisfied, problem (2.12) has a unique solution $W_0^1 \in \mathcal{H}^1_{\text{per}}(\Xi)$ which is uniquely defined up to an additive constant. Moreover, since the boundary $\partial\omega$ is smooth, this solution is infinitely differentiable in $\overline{\Xi}$ and the Fourier method (cf. [9, 14, 17]), in particular, gives the decomposition

$$W_0^1(\xi) = \chi_-(\xi_1)C_- + \chi_+(\xi_1)C_+ + \widetilde{W}_0^1(\xi) \tag{2.13}$$

with the exponentially decaying remainder $\widetilde{W}_0^1$, and some constants C_+ and C_- which can also depend on R, cf. (2.10) and (2.11).

Note that the above results hold for any smooth hole. In addition, under the assumption of *mirror symmetry*:

$$\omega = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : (\xi_1, H - \xi_2) \in \omega\}, \tag{2.14}$$

see, for instance, Figure 4, the function W_0^1 is even in the $\xi_2 - H/2$ variable and satisfies

$$\frac{\partial W_0^1}{\partial \xi_2}(\xi_1, 0) = \frac{\partial W_0^1}{\partial \xi_2}(\xi_1, H) = 0, \ \xi_1 \in \mathbb{R}, \tag{2.15}$$

see [9] for a proof.

3 | ASYMPTOTIC FORMULAS FOR CONVERGENCE RATES

In this and next sections, we obtain some important complementary results of (2.4) on the asymptotics of the perturbed dispersion curves, at a low-frequency range. This implies constructing the so-called *almost eigenvalues and eigenfunctions* or *quasimodes* of a compact operator associated with (1.4)–(1.6). In order to do this, we set an abstract framework for the perturbation problem and construct the approximations from the eigenfunction of the homogenized problem and a boundary layer function (cf. Section 2.2). We obtain asymptotic formulas for eigenvalues and estimates when the limit eigenvalues have corresponding eigenfunctions which do not depend on x_2, cf. (1.16), (3.7), and (3.8). These formulas do not allow us to predict the precise index of the eigenvalue in the sequence (1.8), which will be determined in Section 5 using the results of Section 2. Another restriction that simplifies computations in this section comes from the geometry of the perforation: we assume the mirror symmetry (cf. Figure 4, (2.14) and (2.15)).

First, we reformulate the spectral problem (1.4)–(1.6) in terms of operators on Hilbert spaces. Let $\mathcal{H}^\varepsilon(\eta)$ denote the space $H^{1,\eta}_{\text{per}}(\Omega^\varepsilon)$ equipped with the usual scalar product in $H^1(\Omega^\varepsilon)$. We denote by $\langle \cdot,\cdot\rangle_{\eta\varepsilon}$ this scalar product and by $\|U^\varepsilon; \mathcal{H}^\varepsilon(\eta)\|$ the associated norm.

In the space $\mathcal{H}^\varepsilon(\eta)$, we define the compact, positive, and symmetric operator $\mathcal{B}^\varepsilon(\eta)$, as

$$\langle \mathcal{B}^\varepsilon(\eta)U^\varepsilon, V^\varepsilon\rangle_{\eta\varepsilon} = (U^\varepsilon, V^\varepsilon)_{\Omega^\varepsilon} \quad \forall U^\varepsilon, V^\varepsilon \in H^{1,\eta}_{\text{per}}(\Omega^\varepsilon). \tag{3.1}$$

It is self-evident that the variational formulation (1.7) of (1.4)–(1.6) can be re-written as follows:

$$\mathcal{B}^\varepsilon(\eta)U^\varepsilon(\eta) = M^\varepsilon(\eta)U^\varepsilon(\eta) \quad \text{in} \quad \mathcal{H}^\varepsilon(\eta), \tag{3.2}$$

with the new spectral parameter

$$M^\varepsilon(\eta) = (1+\Lambda^\varepsilon(\eta))^{-1}. \tag{3.3}$$

The following result, which we state for the specific operator $\mathcal{B}^\varepsilon(\eta)$ in (3.1), is based on the lemma on almost eigenvalues and eigenfuntions from the spectral perturbation theory, cf. [33], [26, Section 3.1], [22, Section 7.1], and [13, Section 5.32].

Lemma 3.1. *Let $M^\varepsilon_{as}(\eta) \in \mathbb{R}$ and $U^\varepsilon_{as}(\eta) \in \mathcal{H}^\varepsilon(\eta) \setminus \{0\}$ verify the equality*

$$\|\mathcal{B}^\varepsilon(\eta)U^\varepsilon_{as}(\eta) - M^\varepsilon_{as}(\eta)U^\varepsilon_{as}(\eta); \mathcal{H}^\varepsilon(\eta)\| = \delta^\varepsilon \|U^\varepsilon_{as}(\eta); \mathcal{H}^\varepsilon(\eta)\|. \tag{3.4}$$

Then, there exists an eigenvalue $M^\varepsilon(\eta)$ of the operator $\mathcal{B}^\varepsilon(\eta)$ such that

$$|M^\varepsilon(\eta) - M^\varepsilon_{as}(\eta)| \le \delta^\varepsilon. \tag{3.5}$$

Moreover, for any $\delta^\varepsilon_1 > \delta^\varepsilon$, there exist coefficients $\alpha^\varepsilon_J, \dots, \alpha^\varepsilon_{J+K-1}$, satisfying

$$\left\| \|U^\varepsilon_{as}(\eta); \mathcal{H}^\varepsilon(\eta)\|^{-1} U^\varepsilon_{as}(\cdot;\eta) - \sum_{j=J}^{J+K-1} \alpha^\varepsilon_j U^\varepsilon_j(\eta); \mathcal{H}^\varepsilon(\eta) \right\| \le 2\frac{\delta^\varepsilon}{\delta^\varepsilon_1}, \qquad \sum_{j=J}^{J+K-1} (\alpha^\varepsilon_j)^2 = 1, \tag{3.6}$$

where $M^\varepsilon_J(\eta), \dots, M^\varepsilon_{J+K-1}(\eta)$ are all the eigenvalues of the operator $\mathcal{B}^\varepsilon(\eta)$ in the interval $[M^\varepsilon_{as}(\eta) - \delta^\varepsilon_1, M^\varepsilon_{as}(\eta) + \delta^\varepsilon_1]$, and $U^\varepsilon_J(\cdot;\eta), \dots, U^\varepsilon_{J+K-1}(\cdot;\eta)$ are the corresponding eigenvectors normalized by

$$\langle U^\varepsilon_i(\cdot;\eta), U^\varepsilon_j(\cdot;\eta)\rangle_{\eta\varepsilon} = \delta_{ij}, \quad i,j = J, \dots, J+K-1,$$

being δ_{ij} the Kronecker delta.

The pair $(M^\varepsilon_{as}(\eta),\ \|U^\varepsilon_{as}(\eta); \mathcal{H}^\varepsilon(\eta)\|^{-1} U^\varepsilon_{as}(\eta))$ in Lemma 3.1 is the so-called quasimode of operator $\mathcal{B}^\varepsilon(\eta)$ with remainder δ^ε. It approaches eigenvalues and eigenfunctions of the operator $\mathcal{B}^\varepsilon(\eta)$ as stated in (3.5) and (3.6). Also, if there is no confusion, the function $\|U^\varepsilon_{as}(\eta); \mathcal{H}^\varepsilon(\eta)\|^{-1} U^\varepsilon_{as}(\eta)$ is referred to as the quasimode.

In what follows, throughout the section, we construct $M^\varepsilon_{as}(\eta)$ and $U^\varepsilon_{as}(\eta)$ and obtain a bound for δ^ε in (3.4), cf. (3.13). The relation of $M^\varepsilon_{as}(\eta)$ with the true eigenvalues in the sequence (1.8) will be given in Section 4.

For $j = 0, 1, 2, \dots$, let

$$\Lambda^0_{\pm,j}(\eta) = (\eta \pm 2\pi j)^2 \tag{3.7}$$

be eigenvalues in (1.16) corresponding to a fixed Floquet parameter $\eta \in [-\pi, \pi]$. From the explicit computations, the corresponding eigenfunctions

$$U^0_{\pm,j}(x_1;\eta) = \mathrm{e}^{\mathrm{i}(\eta \pm 2\pi j)x_1}, \tag{3.8}$$

do not depend on the x_2-variable. Obviously, for the different signs, values in (3.7) and functions in (3.8), coincide only for $j = 0$. Namely, for $j = 0$, we have the first eigenvalue in the increasing sequence (1.14) and the corresponding eigenfunction:

$$\Lambda^0_1(\eta) = \Lambda^0_{\pm,0}(\eta) = \eta^2, \quad U^0_1(\cdot;\eta) = U^0_{\pm,0}(x_1;\eta) = \mathrm{e}^{\mathrm{i}\eta x_1}. \tag{3.9}$$

According to the notation in (3.3) we take

$$M^0_{\pm,j}(\eta) = (1 + \Lambda^0_{\pm,j}(\eta))^{-1}$$

as an approximate eigenvalue ($M^\varepsilon_{as}(\eta)$ for $\pm$, respectively), and

$$U^\varepsilon_{\pm,j}(x;\eta) = X^\varepsilon(x_1)U^0_{\pm,j}(x_1;\eta) + (1 - X^\varepsilon(x_1))\left(U^0_{\pm,j}(0;\eta) + x_1\frac{\partial U^0_{\pm,j}}{\partial x_1}(0;\eta)\right) + \varepsilon\chi_0(x_1)\frac{\partial U^0_{\pm,j}}{\partial x_1}(0;\eta)W^1_0\left(\frac{x}{\varepsilon}\right), \tag{3.10}$$

as an approximate eigenfunction $U^\varepsilon_{as}(\eta)$ which we construct from $U^0_{\pm,j}$ in (3.8) and $W^1_0(\xi)$, the bounded harmonics in Ξ, see (2.6), (2.7)–(2.9) and (2.13). Here,

$$X^\varepsilon(x_1) = 1 - \chi_+(x_1/\varepsilon) - \chi_-(x_1/\varepsilon), \quad \text{and,} \quad \chi_0 \in C^\infty(\mathbb{R}),\ \chi_0(x_1) = 1 \text{ for } |x_1| \le 1/6,\ \chi_0(x_1) = 0 \text{ for } |x_1| \ge 1/3, \tag{3.11}$$

where the even smooth cut-off functions $\chi_\pm$ are defined by (2.10). It can be easily verified that $U^\varepsilon_{\pm,j}(x;\eta) \in \mathcal{H}^\varepsilon(\eta)$.

Note that, depending on H, $j \in \mathbb{N}_0$ and $\eta \in [-\pi,\pi]$, the eigenvalues $M^0_{\pm,j}(\eta)$ can be simple, or have a multiplicity greater than or equal to 2. Obviously, once H and j are fixed, also the eigenvalue number in the sequence (1.14) may change depending on η. Below, we fix j and the sign plus or minus, and for brevity, we omit to write the η dependence of function $U^\varepsilon_{\pm,j}$.

In order to apply Lemma 3.1, we multiply (3.4) by $\|U^\varepsilon_{as}(\eta);\mathcal{H}^\varepsilon(\eta)\|^{-1}$, write $\delta^\varepsilon \equiv \delta^\varepsilon_{\pm,j}(\eta)$ and obtain the remainder

$$\delta^\varepsilon_{\pm,j}(\eta) := \|U^\varepsilon_{\pm,j};\mathcal{H}^\varepsilon(\eta)\|^{-1}\|\mathcal{B}^\varepsilon(\eta)U^\varepsilon_{\pm,j} - M^0_{\pm,j}(\eta)U^\varepsilon_{\pm,j};\mathcal{H}^\varepsilon(\eta)\|, \tag{3.12}$$

for which we obtain the uniform estimate:

$$\delta^\varepsilon_{\pm,j}(\eta) \le c_j\varepsilon, \quad \text{for } \varepsilon \le \varepsilon_j, \tag{3.13}$$

where c_j and ε_j are two positive constants independent of η and ε.

To prove (3.13), first, let us show the almost orthogonality property for the family of functions constructed in (3.10):

$$|\langle U^\varepsilon_{+,j}, U^\varepsilon_{-,j}\rangle_{\eta\varepsilon}| \le C_j\varepsilon^{1/2} \qquad \text{for } \varepsilon < \varepsilon_j,\ \eta \in [-\pi,\pi], \quad j \in \mathbb{N}, \tag{3.14}$$

$$|\langle U^\varepsilon_{\pm,j}, U^\varepsilon_{\mp,k}\rangle_{\eta\varepsilon}| \le C_{j,k}\varepsilon^{1/2} \qquad \text{for } \varepsilon < \varepsilon_{j,k},\ \eta \in [-\pi,\pi], \quad j,k \in \mathbb{N}_0,\ j \ne k, \tag{3.15}$$

where $C_j, C_{j,k}, \varepsilon_j$, and $\varepsilon_{j,k}$ are some positive constants independent of ε and η; recall that $\langle\cdot,\cdot\rangle_{\eta\varepsilon}$ denotes the scalar product in $\mathcal{H}^\varepsilon(\eta)$.

Let us prove (3.14). Owing to the orthogonality of the functions $U^0_{\pm,j}$ in $L^2(\Omega^0)$ and $H^1(\Omega^0)$, we write

$$\begin{aligned}|\langle U^\varepsilon_{+,j}, U^\varepsilon_{-,j}\rangle_{\eta\varepsilon}| =&(\nabla_x(U^\varepsilon_{+,j} - U^0_{+,j}), \nabla_x U^\varepsilon_{-,j})_{\Omega^\varepsilon} + (\nabla_x U^0_{+,j}, \nabla_x(U^\varepsilon_{-,j} - U^0_{-,j}))_{\Omega^\varepsilon}\\ &- (\nabla_x U^0_{+,j}, \nabla_x U^0_{-,j})_{\Omega^0\setminus\Omega^\varepsilon} + (U^\varepsilon_{+,j} - U^0_{+,j}, U^\varepsilon_{-,j})_{\Omega^\varepsilon}\\ &+ (U^0_{+,j}, U^\varepsilon_{-,j} - U^0_{-,j})_{\Omega^\varepsilon} - (U^0_{+,j}, U^0_{-,j})_{\Omega^0\setminus\Omega^\varepsilon}.\end{aligned} \tag{3.16}$$

In addition, by the definition (3.10) of $U^\varepsilon_{\pm,j}$, we have

$$\begin{aligned}\|U^\varepsilon_{\pm,j} - U^0_{\pm,j};\mathcal{H}^\varepsilon(\eta)\| \le &\left\|(1 - X^\varepsilon(x_1))\left(U^0_{\pm,j}(x_1;\eta) - U^0_{\pm,j}(0;\eta) - x_1\frac{\partial U^0_{\pm,j}}{\partial x_1}(0;\eta)\right);\mathcal{H}^\varepsilon(\eta)\right\|\\ &+\left\|\varepsilon\frac{\partial U^0_{\pm,j}}{\partial x_1}(0;\eta)W^1_0\left(\frac{x}{\varepsilon}\right);\mathcal{H}^\varepsilon(\eta)\right\|.\end{aligned}$$

Thus, since $U^0_{\pm,j}$ are smooth functions, the support of $1 - X^\varepsilon$ is contained in $\{|x_1| < 2R\varepsilon\}$ and $W^1_0 \in \mathcal{H}^1_{per}(\Xi)$, we have

$$\|U^\varepsilon_{\pm,j} - U^0_{\pm,j}; \mathcal{H}^\varepsilon(\eta)\| \leq \widehat{C}_j \varepsilon^{1/2}, \tag{3.17}$$

where, on account of (3.11), we have applied the Taylor formula for $U^0_{\pm,j}(x_1, \eta)$, while $|x_1| \leq O(\varepsilon)$, the change of variable (2.5), and the periodicity of $W^1_0(\xi)$ in the ξ_2-direction. Also $\widehat{C}_j$ is a constant independent of ε and η. Now, using (3.17), the smoothness of $U^0_{\pm,j}(x_1, \eta)$ and the fact that $|\Omega^0 \setminus \Omega^\varepsilon| = O(\varepsilon)$, we obtain

$$\|U^\varepsilon_{\pm,j}; \mathcal{H}^\varepsilon(\eta)\|^2 \xrightarrow{\varepsilon \to 0} \|U^0_{\pm,j}; L^2(\Omega^0)\|^2 + \|\nabla_x U^0_{\pm,j}; L^2(\Omega^0)\|^2 = (1 + \Lambda^0_{\pm,j}(\eta))H. \tag{3.18}$$

Hence, gathering (3.16), (3.17), (3.18) and using that $|\Omega^0 \setminus \Omega^\varepsilon| = O(\varepsilon)$, we get (3.14).

Rewriting the proof above, with minor modifications, for each k and j, $\varkappa \neq j$, we obtain the four estimates in (3.15).

Then, for each sign plus or minus, index $j \in \mathbb{N}_0$ and $\eta \in [-\pi, \pi]$, let us introduce

$$U^\varepsilon_{as,\pm,j}(\eta) := \|U^\varepsilon_{\pm,j}(\eta); \mathcal{H}^\varepsilon(\eta)\|^{-1} U^\varepsilon_{\pm,j}(\eta) \tag{3.19}$$

to be the quasimode constructed from the eigenfunction corresponding with $\Lambda^0_{\pm,j}(\eta)$, cf. (3.10) and (3.8). From (3.14), (3.15), and (3.18), we get the almost orthonormality conditions:

$$|\langle U^\varepsilon_{as,+,j}, U^\varepsilon_{as,-,j}\rangle_{\eta\varepsilon}| \leq \widetilde{C}_j \varepsilon^{1/2} \qquad \text{for } \varepsilon < \widetilde{\varepsilon}_j,\ \eta \in [-\pi, \pi], \quad j \in \mathbb{N}, \tag{3.20}$$

$$|\langle U^\varepsilon_{as,\pm,j}, U^\varepsilon_{as,\mp,k}\rangle_{\eta\varepsilon} - \delta_{kj}| \leq \widetilde{C}_{j,k} \varepsilon^{1/2} \qquad \text{for } \varepsilon < \widetilde{\varepsilon}_{j,k},\ \eta \in [-\pi, \pi], \quad j, k \in \mathbb{N}_0, \tag{3.21}$$

where $\widetilde{C}_j$, $\widetilde{C}_{j,k}$, $\widetilde{\varepsilon}_j$ and $\widetilde{\varepsilon}_{j,k}$ are some positive constants independent of ε and η.

Finally, we write the following estimate:

$$\|\mathcal{B}^\varepsilon(\eta) U^\varepsilon_{\pm,j} - M^0_{\pm,j}(\eta) U^\varepsilon_{\pm,j}; \mathcal{H}^\varepsilon(\eta)\| \leq \widehat{c}_j \varepsilon, \qquad \text{for } \varepsilon \leq \widehat{\varepsilon}_j$$

(for a certain positive constant $\widehat{c}_j$ independent of ε and η), whose proof involves cumbersome computations that we avoid introducing them here: it follows rewriting the proof in [9] with minor modifications. This estimate together with the convergence (3.18) allows us to write the uniform bound for the remainder (3.13).

Now, applying Lemma 3.1 gives us an eigenvalue $M^\varepsilon_{\pm,j}(\eta)$ of the operator $\mathcal{B}^\varepsilon(\eta)$ such that

$$|M^\varepsilon_{\pm,j}(\eta) - M^0_{\pm,j}(\eta)| \leq c_j \varepsilon, \qquad \text{for } \varepsilon \leq \varepsilon_j,\ \eta \in [-\pi, \pi], \tag{3.22}$$

where the factor $c_j > 0$ and the constant $\varepsilon_j > 0$ are independent of η and ε. Recalling (3.3), we derive from (3.22) that

$$|\Lambda^\varepsilon_{\pm,j}(\eta) - \Lambda^0_{\pm,j}(\eta)| \leq c_j \varepsilon (1 + \Lambda^0_{\pm,j}(\eta))(1 + \Lambda^\varepsilon_{\pm,j}(\eta)), \tag{3.23}$$

and, hence

$$(1 + \Lambda^\varepsilon_{\pm,j}(\eta))(1 - c_j \varepsilon (1 + \Lambda^0_{\pm,j}(\eta)) \leq 1 + \Lambda^0_{\pm,j}(\eta).$$

Let us set

$$\widetilde{\varepsilon}_j := \frac{1}{2c_j(1 + 4\pi^2 j^2)} \text{ when } j = 1, 2, \ldots \quad \text{while } \widetilde{\varepsilon}_0 := \frac{1}{2c_0(1 + \pi^2)} \text{ when } j = 0.$$

Then, for $\varepsilon \leq \min(\varepsilon_j, \widetilde{\varepsilon}_j)$, we have $(1 - c_j \varepsilon (1 + \Lambda^0_{\pm,j}(\eta)) > 1/2$ and therefore

$$|\Lambda^\varepsilon_{\pm,j}(\eta) - \Lambda^0_{\pm,j}(\eta)| \leq 2c_j \varepsilon (1 + \Lambda^0_{\pm,j}(\eta))^2 \leq C_j \varepsilon \qquad \forall \eta \in [-\pi, \pi]. \tag{3.24}$$

Estimates (3.22) and (3.24) become essential in what follows to derive bounds for convergence rates between the eigenvalues in sequences (1.8) and (1.14).

4 | THE ALMOST EIGENVALUES AND THE TRUE EIGENVALUES

In this section, we provide a certain relation between the eigenvalue $\Lambda^{\varepsilon}_{\pm,j}(\eta)$ given by Lemma 3.1, which satisfies (3.24), and an eigenvalue in the increasing sequence (1.8). To state Theorems 4.1 and 4.2 and Corollary 4.3, we use the results in Section 3 and arguments on families of almost orthogonal functions approaching the eigenfunctions, cf. [16, 23, 27] for the technique in very different problems. See also [13, Section 5.32] in this connection.

Based on definitions (3.7)–(3.9) and estimates (3.22) and (3.24), we show the following result.

Theorem 4.1. *For each eigenvalue $\Lambda^0_m(\eta)$ of problem (1.10)–(1.12) in the sequence (1.14), with $m = 1, 2, 3, \ldots$, and such that a corresponding eigenfunction does not depend of the x_2-variable for η in an interval $[\mathfrak{a}, \mathfrak{b}] \subset [-\pi, \pi]$, there is at least one eigenvalue $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,m)}(\eta)$ of problem (1.4)–(1.6) satisfying*

$$|\Lambda^{\varepsilon}_{p(\varepsilon,\eta,m)}(\eta) - \Lambda^0_m(\eta)| < C_m\varepsilon \qquad \forall\ \eta \in [\mathfrak{a}, \mathfrak{b}],\ \varepsilon \le \varepsilon_m, \tag{4.1}$$

where C_m and ε_m are certain positive constants that are independent of ε and η. Moreover, for intervals $[\mathfrak{a}, \mathfrak{b}]$ which does not contain η_0 such that the multiplicity of $\Lambda^0_m(\eta_0)$ is greater than 1, $p(\varepsilon, \eta, m) \ge m$.

Proof. Since by hypothesis, $\Lambda^0_m(\eta)$ coincides with some $\Lambda^0_{\pm,j}(\eta)$ in $[\mathfrak{a}, \mathfrak{b}]$ for some sign plus or minus and some $j \equiv j(m)$, we set $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,m)}(\eta) = \Lambda^{\varepsilon}_{\pm,j}(\eta)$, for the same sign and j, where it should be noted that the sign could change in subintervals of $[\mathfrak{a}, \mathfrak{b}]$. Therefore, (3.24) provides (4.1). The fact that $p(\varepsilon, \eta, m)$ be greater than or equal to m is due to the estimate (2.1), and the theorem is proved. □

It should be emphasized that while Corollary 2.3 provides approaches between all the eigenvalues of (1.4)–(1.6) and (1.10)–(1.12), Theorem 4.1 only provides estimates for discrepancies between certain eigenvalues of the homogenization problem (1.4)–(1.6) and certain eigenvalues of the homogenized problem (1.10)–(1.12) for η in certain intervals which can coincide with $[-\pi, \pi]$ depending on m and H (see Figures 2 and 3). Nevertheless, we cannot assure that all the eigenvalues (1.14) enjoy of such an approach. In addition, we cannot assure yet that the number $p(\varepsilon, \eta, m)$ coincides with m. To show this rigorously can only be done for a few values of m, always depending on the value of H. This is due to the difficulty in ordering the eigenvalues (3.7) in the monotone sequence (1.14). In particular, to outline the difficulty, we show that $p(\varepsilon, \eta, m) = m$ for the values of $m = 1, 2, 3$ (cf. Theorem 5.1). Next theorem provides the preliminary result $p(\varepsilon, \eta, m) \ge m$ for η in a neighborhood of η_0 such that $\Lambda^0_m(\eta_0)$ is a multiple eigenvalue of (1.10)–(1.12). However, on account of (3.20)–(3.24), the process can be continued for the values of m arising in the statement of Theorem 4.1, provided that the $\{\Lambda_p(\eta)\}_{p=1}^m$ has a corresponding eigenfunction depending only on x_1, see, for example, Figure 2 when $H < 1/\sqrt{8}$, and Remark 4.4.

For the sake of simplicity in the proof of the next theorem, while $m = 1, 2, 3$, it proves useful to write (3.22) and (3.24) as

$$|M^{\varepsilon}_{\pm,j}(\eta) - M^0_{\pm,j}(\eta)| \le c\varepsilon \qquad \forall \eta \in [-\pi, \pi],\ \varepsilon \le \varepsilon_0 \tag{4.2}$$

and

$$|\Lambda^{\varepsilon}_{\pm,j}(\eta) - \Lambda^0_{\pm,j}(\eta)| \le C\varepsilon \qquad \forall \eta \in [-\pi, \pi],\ \varepsilon \le \varepsilon_0, \tag{4.3}$$

for certain positive constants ε_0, c and C. Consequently, for any $c_r \ge c$ the interval $[M^0_{\pm,j}(\eta) - c_r\varepsilon, M^0_{\pm,j}(\eta) + c_r\varepsilon]$ contains at least one eigenvalue $M^{\varepsilon}_{\pm,j}(\eta)$ and the value of ε_0 can be replaced by $\varepsilon_{0,r}$ in order that (4.3) be satisfied for a certain constant C_r. Namely,

$$|M^{\varepsilon}_{\pm,j}(\eta) - M^0_{\pm,j}(\eta)| \le c_r\varepsilon \qquad \forall \eta \in [-\pi, \pi],\ \varepsilon \le \varepsilon_{0,r} \tag{4.4}$$

and

$$|\Lambda^{\varepsilon}_{\pm,j}(\eta) - \Lambda^0_{\pm,j}(\eta)| \le C_r\varepsilon \qquad \forall \eta \in [-\pi, \pi],\ \varepsilon \le \varepsilon_{0,r}. \tag{4.5}$$

Note that the above formulas contain the case where $j = 0$, cf. (3.9):

$$|(1+\Lambda^\varepsilon_{\pm,0}(\eta))^{-1} - (1+\eta^2)^{-1}| \le c_r\varepsilon \qquad \forall \eta \in [-\pi,\pi],\ \varepsilon \le \varepsilon_{0,r}, \tag{4.6}$$

and

$$|\Lambda^\varepsilon_{\pm,0}(\eta) - \eta^2| \le C_r\varepsilon \qquad \forall \eta \in [-\pi,\pi],\ \ \varepsilon \le \varepsilon_{0,r}. \tag{4.7}$$

Similarly, for simplicity, for a fixed $m = 1,2,3$ and $\eta \in [\mathfrak{a},\mathfrak{b}] \subset [-\pi,\pi]$, we avoid writing corresponding signs and index $j \equiv j(m)$ and we denote by $U^\varepsilon_{as,m}(\eta)$ and $U^\varepsilon_{as,m+1}(\eta)$ the two quasimodes in (3.19) constructed from the two eigenfunction corresponding to $\Lambda^0_m(\eta)$ and $\Lambda^0_{m+1}(\eta)$, cf. (3.10) and (3.8), for the associated j and sign plus or minus. On account of (3.20) and (3.21), they satisfy

$$\langle U^\varepsilon_{as,m}(\eta)\,,\, U^\varepsilon_{as,m+1}(\eta)\rangle_{\eta\varepsilon} = \widetilde{C}\sqrt{\varepsilon} \to 0 \quad \text{ and } \quad \|U^\varepsilon_{as,m}(\eta);\mathcal{H}^\varepsilon(\eta)\|^2 = 1, \quad m = 1,2,3. \tag{4.8}$$

Theorem 4.2. *There exist constants $\varepsilon_0 > 0$ and C_0 independent of ε and $\eta \in [\mathfrak{a},\mathfrak{b}] \subseteq [-\pi,\pi]$ such that for each eigenvalue $\Lambda^0_m(\eta)$ of problem (1.10)–(1.12) in the sequence (1.14), with $m = 1,2,3$, and a corresponding eigenfunction depending only on the x_1-variable, there is an eigenvalue $\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta)$ of problem (1.4)–(1.6) satisfying*

$$|\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta) - \Lambda^0_m(\eta)| < C_0\varepsilon \qquad \textit{for } m = 1,2,3,\ \ 0 < \varepsilon < \varepsilon_0 \ \textit{ and } \ \eta \in [\mathfrak{a},\mathfrak{b}] \subset [-\pi,\pi], \tag{4.9}$$

where $p(\varepsilon,\eta,m) \ge m$. Depending on H and m, the interval $[\mathfrak{a},\mathfrak{b}]$ can coincide with the whole $[-\pi,\pi]$ or with any interval which does not contain abscises of intersecting points of the dispersion curves with one of the corresponding eigenfunctions depending on x_2.

In particular, for $m = 1$ and $H > 0$, for $m = 2$ and $H \in (0,1/2)$ and for $m = 3$ and $H \in (0,1/\sqrt{8})$ we get

$$|\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta) - \Lambda^0_m(\eta)| < C_0\varepsilon, \qquad \textit{for } 0 < \varepsilon < \varepsilon_0,\ \eta \in [-\pi,\pi],\ \textit{ and } \ p(\varepsilon,\eta,m) \ge m. \tag{4.10}$$

Proof. The existence of such an index $p(\varepsilon,\eta,m)$ has been proved in Theorem 4.1. For each m, let us show that $p(\varepsilon,\eta,m) \ge m$, even if $[\mathfrak{a},\mathfrak{b}]$ contains points η_0 such that $\Lambda_m(\eta^0) = \Lambda_{m+1}(\eta^0)$. We divide the proof into several steps depending on m.

Step (1): *The case when $m = 1$.* The result $p(\varepsilon,\eta,m) \ge m$ is self-evident when $m = 1$, and $[\mathfrak{a},\mathfrak{b}] = [-\pi,\pi]$. In addition, when $\eta = \pm\pi$ the multiplicity of $\Lambda^0_1(\pm\pi)$ is 2, and $(\pm\pi,\pi^2)$ are the only points of the limit dispersion curve $\Lambda^0_1(\eta)$, where the multiplicity is greater than 1. That is to say, for $\eta = \pm\pi$, the first eigenvalue of the limit problem is double: $\Lambda^0_1(\pm\pi) = \Lambda^0_2(\pm\pi) = \pi^2$. The corresponding eigenfunctions are $U^0_1(-\pi) = \mathrm{e}^{-i\pi x_1}$ and $U^0_2(-\pi) = \mathrm{e}^{i\pi x_1}$ when $\eta = -\pi$, while they are $U^0_1(\pi) = \mathrm{e}^{i\pi x_1}$ and $U^0_2(\pi) = \mathrm{e}^{-i\pi x_1}$ when $\eta = \pi$, and we can prove that $p(\varepsilon,\pm\pi,2) \ge 2$. In fact, let us show that there are at least two eigenvalues of Equations (1.4)–(1.6) satisfying (4.6). Since both points can be treated in the same way, let us proceed with $\eta = -\pi$. The constructed quasimodes $U^\varepsilon_{as,1}(-\pi)$ and $U^\varepsilon_{as,2}(-\pi)$ satisfy the condition (4.8). It is self-evident that actually, $U^\varepsilon_{as,1}(-\pi) := U^\varepsilon_{as,\pm,0}(-\pi)$ and $U^\varepsilon_{as,2}(-\pi) = U^\varepsilon_{as,+,1}(-\pi)$, cf. (3.9), (3.7), (3.8), and (3.10), and, further specifying,

$$U^\varepsilon_{as,1}(-\pi) = \|U^\varepsilon_{\pm,0}(-\pi);\mathcal{H}^\varepsilon(-\pi)\|^{-1}U^\varepsilon_{\pm,0}(-\pi), \quad U^\varepsilon_{as,2}(-\pi) = \|U^\varepsilon_{+,1}(-\pi);\mathcal{H}^\varepsilon(-\pi)\|^{-1}U^\varepsilon_{+,1}(-\pi).$$

For any $c_r \ge c$ (cf. (4.4) and (4.5)) let us consider all the eigenvalues

$$\{M^\varepsilon_J(-\pi),\dots,M^\varepsilon_{J+K-1}(-\pi)\}$$

in $[(1+\Lambda^0_1(-\pi))^{-1} - c_r\varepsilon,\ (1+\Lambda^0_1(-\pi))^{-1} + c_r\varepsilon]$ and the corresponding eigenfunctions $\{U^\varepsilon_J(\cdot;-\pi),\dots,U^\varepsilon_{J+K-1}(\cdot;-\pi)\}$. Using the bound (3.6) in Lemma 3.1, we get

$$\left\| U^\varepsilon_{as,l}(-\pi) - \sum_{i=J}^{J+K-1}\alpha^\varepsilon_{i,l}U^\varepsilon_i(\cdot;-\pi);\mathcal{H}^\varepsilon(-\pi)\right\| \le 2\frac{c}{c_r}, \qquad \sum_{i=J}^{J+K-1}|\alpha^\varepsilon_{i,l}|^2 = 1, \quad l = 1,2.$$

Let us show that $\widetilde{U}^\varepsilon_1 := \sum_{i=J}^{J+K-1}\alpha^\varepsilon_{i,1}U^\varepsilon_i(\cdot;-\pi)$ and $\widetilde{U}^\varepsilon_2 := \sum_{i=J}^{J+K-1}\alpha^\varepsilon_{i,2}U^\varepsilon_i(\cdot;-\pi)$ are linearly independent functions, and consequently, among the sequence $\{M^\varepsilon_J(-\pi),\dots,M^\varepsilon_{J+K-1}(-\pi)\}$ there are at least two eigenvalues of (1.4)–(1.6) with total multiplicity greater than or equal to 2.

Indeed, using the above estimate and (4.8), we write

$$\left|\langle \widetilde{U}_1^\varepsilon, \widetilde{U}_2^\varepsilon\rangle_{-\pi\varepsilon}\right| \leq \left|\langle \widetilde{U}_1^\varepsilon, \widetilde{U}_2^\varepsilon - U^\varepsilon_{as,2}(-\pi)\rangle_{-\pi\varepsilon}\right| + \left|\langle \widetilde{U}_1^\varepsilon - U^\varepsilon_{as,1}(-\pi), U^\varepsilon_{as,2}(-\pi)\rangle_{-\pi\varepsilon}\right|$$
$$+\left|\langle U^\varepsilon_{as,1}(-\pi), U^\varepsilon_{as,2}(-\pi)\rangle_{-\pi\varepsilon}\right| \leq 2\frac{c}{c_r} + 2\frac{c}{c_r} + \widetilde{C}\sqrt{\varepsilon}. \tag{4.11}$$

Now, assuming that $\widetilde{U}_1^\varepsilon$ and $\widetilde{U}_2^\varepsilon$ are two linearly dependent functions, without any restriction, we can write $\alpha\widetilde{U}_1^\varepsilon = \widetilde{U}_2^\varepsilon$ for some $\alpha \neq 0$, $|\alpha| = 1$, and taking the scalar product with $\widetilde{U}_1^\varepsilon$ we have

$$|\alpha|\left|\langle \widetilde{U}_1^\varepsilon, \widetilde{U}_1^\varepsilon\rangle_{-\pi\varepsilon}\right| = 1 = \left|\langle \widetilde{U}_2^\varepsilon, \widetilde{U}_1^\varepsilon\rangle_{-\pi\varepsilon}\right| \leq 2\frac{c}{c_r} + 2\frac{c}{c_r} + \widetilde{C}\sqrt{\varepsilon}.$$

Consequently, it suffices to take $c_r := c_{r_0}$ and $\varepsilon_0 := \varepsilon_{0,r_0}$ such that

$$4\frac{c}{c_{r_0}} + \widetilde{C}\sqrt{\varepsilon_{0,r_0}} < 1$$

to get a contradiction. Thus, we have also shown that $p(\varepsilon, -\pi, 2) \geq 2$ and $p(\varepsilon, \pi, 2) \geq 2$, and we can fix constants $C_0 := C_{r_0}$ and $\varepsilon_0 := \varepsilon_{0,r_0}$ in (4.9) to obtain $p(\varepsilon, \eta, 1) \geq 1\ \forall \eta \in [\mathfrak{a}, \mathfrak{b}] \equiv [-\pi, \pi]$ while at the points $\eta = \pm\pi$ where the multiplicity of $\Lambda_1^0(\eta)$ is two, also the index $p(\varepsilon, \pm\pi, 2)$ is greater than or equal to 2.

Step (2): The case when $m = 2$. Let us proceed with $m = 2$, in such a way that also $[\mathfrak{a}, \mathfrak{b}] = [-\pi, \pi]$, namely, $H \in (0, 1/2)$, cf. Figures 2 and 3A. Now, since $\Lambda_2^0(-\pi) = \Lambda_1^0(-\pi) = \pi^2$ and $\Lambda_2^0(+\pi) = \Lambda_1^0(+\pi) = \pi^2$, we can avoid the two points since the result is the same as for $\Lambda_1^0(\pm\pi)$ and $p(\varepsilon, \pm\pi, 2) \geq 2$.

For each $\varepsilon \leq \varepsilon_0$, let us consider the abscises $\tilde{a}_{\pm\pi,\varepsilon}$ of the two points in which the ε-neighborhood of $(1 + \Lambda_2^0(\eta))^{-1}$ and the ε-neighborhood of $(1 + \Lambda_1^0(\eta))^{-1}$ cut across each other. Further specifying, we consider the cut points of the lines

$$(1 + \Lambda_1^0(\eta))^{-1} - c_{r_0}\varepsilon \quad \text{and} \quad (1 + \Lambda_2^0(\eta))^{-1} + c_{r_0}\varepsilon, \tag{4.12}$$

which we denote by $\tilde{a}_{-\pi,\varepsilon}$ and $\tilde{a}_{\pi,\varepsilon}$, respectively (see Figures 5 and 6).

Consider η in the interval $(\tilde{a}_{-\pi,\varepsilon}, \tilde{a}_{\pi,\varepsilon})$ since the ε-neighborhoods do not intersect, formulas (4.4) and (4.5) when $j = 1$ and (4.6) and (4.7) provide two eigenvalues of (1.4)–(1.6) which cannot coincide and the result holds true, namely $p(\varepsilon, \eta, 2) \geq 2\ \forall \eta \in (\tilde{a}_{-\pi,\varepsilon}, \tilde{a}_{\pi,\varepsilon})$.

Consider now η in the interval $(-\pi, \tilde{a}_{-\pi,\varepsilon}]$ and $\varepsilon \leq \varepsilon_0$. In this case, both ε-neighborhoods intersect and we take a larger interval $[(1 + \Lambda_2^0(\eta))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda_2^0(\eta))^{-1} + 3c_{r_0}\varepsilon]$ which contains $[(1 + \Lambda_1^0(\eta))^{-1} - c_{r_0}\varepsilon, (1 + \Lambda_1^0(\eta))^{-1} + c_{r_0}\varepsilon]$ and therefore, it contains at least one eigenvalue of the operator $\mathcal{B}^\varepsilon(\eta)$ in (3.2). Let us show that it contains another different eigenvalue of this operator and the result of the theorem is also true for $[\mathfrak{a}, \mathfrak{b}] = [-\pi, \pi]$ when $H \in (0, 1/2)$. Here, and below throughout the proof, by different we understand that the total multiplicity of these two eigenvalues is greater than or equal to 2, or equivalently, that they have two corresponding entries in the sequence (1.8) of eigenvalues of (1.4)–(1.6).

Indeed, formulas (4.4) and (4.6) provide two eigenvalues of (1.4)–(1.6) which, by now, could coincide. But, we have constructed two quasimodes $U^\varepsilon_{as,1}(\eta)$ and $U^\varepsilon_{as,2}(\eta)$ satisfying (4.8). Let us show that they approach two linear combination of eigenfunctions of (1.4)–(1.6) which are linearly independent functions. To do this, we consider all the possible eigenvalues of the operator $\mathcal{B}^\varepsilon(\eta)$ in the interval $[(1 + \Lambda_2^0(\eta))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda_2^0(\eta))^{-1} + 3c_{r_0}\varepsilon]$, $\{M_i^\varepsilon(\eta)\}_{i=J_\eta}^{J_\eta+K_\eta-1}$ and the corresponding eigenfunctions, and use (3.6) in Lemma 3.1; we get

$$\left\|U^\varepsilon_{as,l}(\eta) - \sum_{i=J_\eta}^{J_\eta+K_\eta-1} \alpha^\varepsilon_{i,l,\eta} U_i^\varepsilon(\cdot;\eta); \mathcal{H}^\varepsilon(\eta)\right\| \leq 2\frac{c}{3c_{r_0}}, \qquad \sum_{i=J_\eta}^{J_\eta+K_\eta-1} |\alpha^\varepsilon_{i,l,\eta}|^2 = 1, \quad l = 1, 2,$$

and proceed as above for $\eta = -\pi$, cf. (4.11), by contradiction, to obtain the result: namely, we claim that there are at least two different eigenvalues of the operator $\mathcal{B}^\varepsilon(\eta)$ in $[(1 + \Lambda_2^0(\eta))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda_2^0(\eta))^{-1} + 3c_{r_0}\varepsilon]$. Then, we fix constants C_0 and ε_0 from $3c_{r_0}$ and $\varepsilon_0 := \varepsilon_{r_0}$ in (4.9), cf. (3.23) and (3.24), to obtain $p(\varepsilon, \eta, 2)$ greater than or equal to 2, $\forall \eta \in [-\pi, \pi]$.

In addition, when $0 < H < 1/2$ (cf. Figures 2 and 3), we can show that at the point $\eta \equiv 0$, where $\Lambda_2^0(0) = \Lambda_3^0(0) = 4\pi^2$, the index $p(\varepsilon, 0, 3)$ satisfies $p(\varepsilon, 0, 3) \geq 3$. Indeed, consider all the eigenvalues of the operator $\mathcal{B}^\varepsilon(\eta)$ when $\eta \equiv 0$ in $[(1 + \Lambda_2^0(0))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda_2^0(0))^{-1} + 3c_{r_0}\varepsilon]$. Formula (3.22) provide two eigenvalues of (1.4)–(1.6) which, by now, could coincide. But, we have constructed two quasimodes $U^\varepsilon_{as,2}(0)$ and $U^\varepsilon_{as,3}(0)$ satisfying (4.8). Let us show that the two mentioned eigenvalues are different.

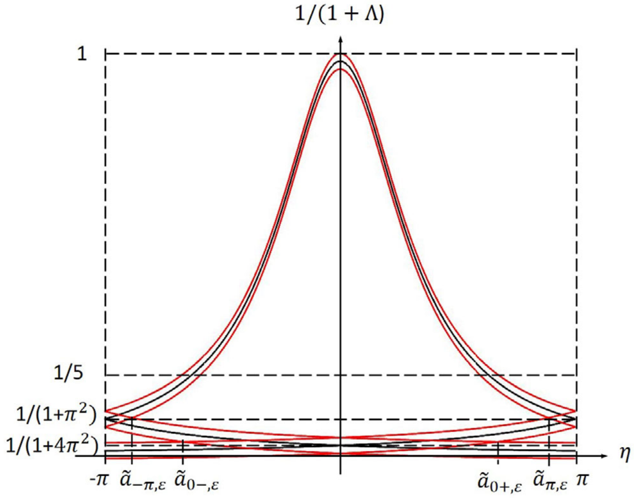

FIGURE 5 Graphics of the curves $(1+\Lambda_1^0(\eta))^{-1}$ and $(1+\Lambda_2^0(\eta))^{-1}$ when $H \in (0,1/2)$. Associated ε-neighborhoods are the regions between the surrounding red lines.

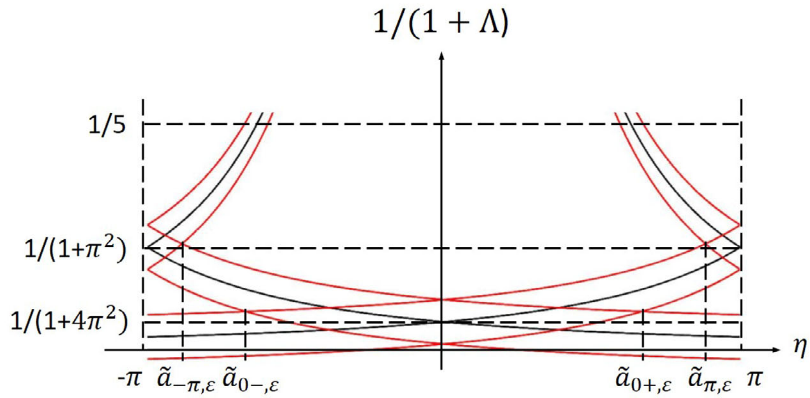

FIGURE 6 Figure 5 after stretching the ordinate axis: the curves $(1+\Lambda_1^0(\eta))^{-1}$ and $(1+\Lambda_2^0(\eta))^{-1}$ and the associated ε-neighborhood between the red lines.

To do this, we consider all the possible eigenvalues

$$\{M^\varepsilon_{J_0}(0), \dots, M^\varepsilon_{J_0+K_0-1}(0)\}$$

in $[(1+\Lambda_2^0(0))^{-1} - 3c_{r_0}\varepsilon, (1+\Lambda_2^0(0))^{-1} + 3c_{r_0}\varepsilon]$ with the corresponding eigenfunctions $\{U^\varepsilon_{J_0}(\cdot;0), \dots, U^\varepsilon_{J_0+K_0-1}(\cdot;0)\}$, and use the bounds for discrepancies in Lemma 3.1, cf. (3.6); we get

$$\left\| U^\varepsilon_{as,l}(0) - \sum_{i=J_0}^{J_0+K_0-1} \alpha^\varepsilon_{i,l,0} U^\varepsilon_i(\cdot;0); \mathcal{H}^\varepsilon(0) \right\| \le 2\frac{c}{3c_{r_0}}, \qquad \sum_{i=J_0}^{J_0+K_0-1} |\alpha^\varepsilon_{i,l,0}|^2 = 1, \quad l = 2,3,$$

and proceed as for $\eta = -\pi$, by contradiction, cf. (4.11), finding two linearly independent functions

$$\left\{ \sum_{j=J_0}^{J_0+K_0-1} \alpha^\varepsilon_{j,2,0} U^\varepsilon_j(\cdot;0), \sum_{j=J_0}^{J_0+K_0-1} \alpha^\varepsilon_{j,3,0} U^\varepsilon_j(\cdot;0) \right\}.$$

Thus, for $\eta \equiv 0$ there are at least two different eigenvalues of the operator $\mathcal{B}^\varepsilon(\eta)$, in the interval $[(1+\Lambda_2^0(0))^{-1} - 3c_{r_0}\varepsilon, (1+\Lambda_2^0(0))^{-1} + 3c_{r_0}\varepsilon]$.

Consequently, we obtain $p(\varepsilon, 0, 2) \geq 2$ and $p(\varepsilon, 0, 3) \geq 3$. This comes from the fact that the value $\Lambda^{\varepsilon}_{p(\varepsilon,0,1)}(0)$ cannot coincide with $\Lambda^{\varepsilon}_{p(\varepsilon,0,2)}(0)$ or $\Lambda^{\varepsilon}_{p(\varepsilon,0,3)}(0)$.

Now, in the case where $H \geq 1/2$, cf. Figures 3B and 2, considering any interval $[\mathfrak{a}, \mathfrak{b}] = [-\pi, \mathfrak{b}_0]$ or $[\mathfrak{a}, \mathfrak{b}] = [\mathfrak{a}_0, \pi]$ which does not contain the abscises η_0 of the intersecting points of the dispersion curves $(\eta \pm 2\pi)^2$ and $\eta^2 + \pi^2/H^2$, the result of the theorem holds true. Namely $p(\varepsilon, \eta, 2) \geq 2$ for any $\eta \in [-\pi, \mathfrak{b}_0] \cup [\mathfrak{a}_0, \pi]$, is a consequence of the results above for $m = 1$, $m = 2$ and Theorem 4.1.

Step (3): The case when $m = 3$. When $H < 1/2$, the process can be continued for $m = 3$, cf. Figures 3A and 2. Let us show that $p(\varepsilon, \eta, 3) \geq 3$ when $\eta \in [\mathfrak{a}, \mathfrak{b}]$ with $0 \in [\mathfrak{a}, \mathfrak{b}] \subset [a_0, b_0] \subset [-\pi, \pi]$, (a_0, b_0) being the largest interval which does not contain abscises of intersecting points of the curves $((\eta + 2\pi)^2 + 1)^{-1}$ and $((\eta^2 + \pi^2/H^2) + 1)^{-1}$ nor of $((\eta - 2\pi)^2 + 1)^{-1}$ and $((\eta^2 + \pi^2/H^2) + 1)^{-1}$.

Indeed, let us consider the abscises $\tilde{a}_{0\pm,\varepsilon}$ of the cut points of the ε-neighborhood of the curves $(\Lambda^0_2(\eta) + 1)^{-1}$ and $(\Lambda^0_3(\eta) + 1)^{-1}$ near $\eta = 0$, namely, near the double eigenvalue $(1 + \Lambda^0_2(0))^{-1} = (1 + \Lambda^0_3(0))^{-1} = 1/(1 + 4\pi^2)$ of $\mathcal{B}^{\varepsilon}(0)$. Further specifying, $\tilde{a}_{0-,\varepsilon}$ and $\tilde{a}_{0+,\varepsilon}$ denote abscissa of the intersecting points of the curves

$$(1 + \Lambda^0_2(\eta))^{-1} - c_{r_0}\varepsilon \quad \text{and} \quad (1 + \Lambda^0_3(\eta))^{-1} + c_{r_0}\varepsilon,$$

with $\tilde{a}_{0-,\varepsilon} < \tilde{a}_{0+,\varepsilon}$; see Figures 5 and 6.

In the case where $\mathfrak{a} > -\pi$ (similarly, $\mathfrak{b} < \pi$), it is clear that for $\eta \in [\mathfrak{a}, \tilde{a}_{0-,\varepsilon})$ (similarly, $\eta \in (\tilde{a}_{0+,\varepsilon}, \mathfrak{b}]$) the eigenvalues $\Lambda^0_i(\eta)$ with $i = 1, 2, 3$ are simple, (4.2) and (4.6) guarantee at least one eigenvalue $(\Lambda^{\varepsilon}_p(\eta) + 1)^{-1}$ in the ε-neighborhood of each $(\Lambda^0_i(\eta) + 1)^{-1}$, and since these neighborhoods do not intersect, there are at least three different eigenvalues $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,i)}(\eta)$, $i = 1, 2, 3$ (their total multiplicity is greater than or equal to 3). Therefore, $p(\varepsilon, \eta, 3) \geq 3$ for $\eta \in [\mathfrak{a}, \tilde{a}_{0-,\varepsilon}) \cup (\tilde{a}_{0+,\varepsilon}, \mathfrak{b}]$. Also, if $\mathfrak{a} = -\pi$ (similarly, $\mathfrak{b} = \pi$), it has been proved above that in an ε-neighborhood of $(\Lambda^0_1(\eta) + 1)^{-1}$ there are at least two eigenvalues of the operator $\mathcal{B}^{\varepsilon}(\eta)$ with total multiplicity greater than or equal to 2, and again $p(\varepsilon, \eta, 3) \geq 3$.

Next, let us prove that the result holds for $\eta \in [\tilde{a}_{0-,\varepsilon}, \tilde{a}_{0+,\varepsilon}]$.

As for $\eta = 0$, the result $p(\varepsilon, 0, 3) \geq 3$ has been proved above, in the previous step. Because of the symmetry, it suffices to analyze in further detail the value of $p(\varepsilon, \eta, 3)$ for the case where $\eta \in [\tilde{a}_{0-,\varepsilon}, 0)$. We use the idea in the previous step for $\Lambda^0_2(\eta)$ with η near $-\pi$. That is, now, both ε-neighborhoods intersect and we take a larger interval $[(1 + \Lambda^0_3(\eta))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda^0_3(\eta))^{-1} + 3c_{r_0}\varepsilon]$ which contains $[(1 + \Lambda^0_2(\eta))^{-1} - c_{r_0}\varepsilon, (1 + \Lambda^0_2(\eta))^{-1} + c_{r_0}\varepsilon]$ and therefore, it contains at least one eigenvalue of the operator $\mathcal{B}^{\varepsilon}(\eta)$. Let us show that it contains another different eigenvalue of this operator.

Indeed, formula (4.2), with $j = 1$, provide two eigenvalues of the problem (1.4)–(1.6) which, by now, could coincide. But, we have constructed two quasimodes $U^{\varepsilon}_{as,2}(\eta)$ and $U^{\varepsilon}_{as,3}(\eta)$ satisfying (4.8). Let us derive that they approach to two linear combination of eigenfunctions of (1.4)–(1.6) which are linearly independent functions. To do this, we consider all the possible eigenvalues of $\mathcal{B}^{\varepsilon}(\eta)$ in $[(1 + \Lambda^0_3(\eta))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda^0_3(\eta))^{-1} + 3c_{r_0}\varepsilon]$, $\{M^{\varepsilon}_i(\eta)\}_{i=J'_\eta}^{J'_\eta + K'_\eta - 1}$ and the corresponding eigenfunctions, and use Lemma 3.1, cf. (3.6); we get

$$\left\| U^{\varepsilon}_{as,l}(\eta) - \sum_{i=J'_\eta}^{J'_\eta + K'_\eta - 1} \dot{\alpha}^{\varepsilon}_{i,l,\eta} U^{\varepsilon}_i(\cdot; \eta); \mathcal{H}^{\varepsilon}(\eta) \right\| \leq 2\frac{c}{3c_{r_0}}, \qquad \sum_{i=J'_\eta}^{J'_\eta + K'_\eta - 1} |\dot{\alpha}^{\varepsilon}_{i,l,\eta}|^2 = 1, \quad l = 2, 3.$$

and proceed as above for $\eta = -\pi$ in (4.11), by contradiction, to obtain the result. Thus, there are at least two different eigenvalues of the operator $\mathcal{B}^{\varepsilon}(\eta)$ in $[(1 + \Lambda^0_3(\eta))^{-1} - 3c_{r_0}\varepsilon, (1 + \Lambda^0_3(\eta))^{-1} + 3c_{r_0}\varepsilon]$. Since, for $\eta \in [\tilde{a}_{0-,\varepsilon}, \tilde{a}_{0+,\varepsilon}]$, there is another eigenvalue in the ε-neighborhood for $(\Lambda^0_1(\eta) + 1)^{-1}$ which does not cut those ε-neighborhoods under consideration for $M^0_{\pm,1}(\eta)$, it follows that we have obtained at least three eigenvalues of (1.4)–(1.6) with total multiplicity greater than or equal to 3. Again, we fix the constants C_0 from $3c_{r_0}$ and $\varepsilon_0 := \varepsilon_{r_0}$ in (4.9) to obtain the value $p(\varepsilon, \eta, 3) \geq 3$ in the statement of the theorem.

As a matter of fact, $[\mathfrak{a}, \mathfrak{b}] \equiv [-\pi, \pi]$ for $H < 1/\sqrt{8}$, see Figure 2. Therefore, (4.10) and all the statements of the theorem hold true. □

As a consequence of the proof of Theorem 4.2 when dealing with small neighborhoods of eigenvalues of the homogenized problem of multiplicity 2, we can state the following result.

Corollary 4.3. *Under the hypothesis of Theorem 4.2, assume that for a certain $m \in \{1, 2, 3\}$ and a certain $\eta^0 \in [\mathfrak{a}, \mathfrak{b}]$, $\Lambda^0_m(\eta^0) = \Lambda^0_{m+1}(\eta^0)$. Then, for $\varepsilon \leq \varepsilon_0$, there are $a_{\eta^0_-,\varepsilon}$ and $a_{\eta^0_+,\varepsilon}$ such that for $\eta \in [a_{\eta^0_-,\varepsilon}, a_{\eta^0_+,\varepsilon}] \subset [\mathfrak{a}, \mathfrak{b}]$, at least two eigenvalues*

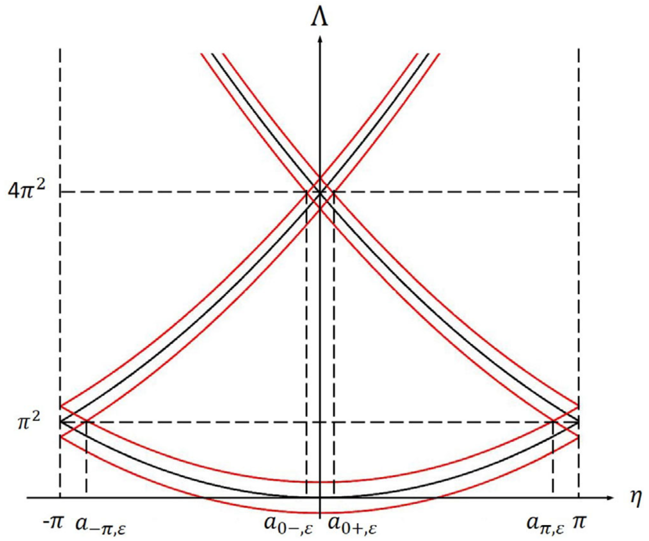

FIGURE 7 Graphics of the limit dispersion curves $\Lambda_1^0(\eta)$ and $\Lambda_2^0(\eta)$ when $H \in (0, 1/2)$. Associated ε-neighborhoods are the regions between the surrounding red lines.

$\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta)$ and $\Lambda^\varepsilon_{p(\varepsilon,\eta,m)+1}(\eta)$ satisfy Equation (4.9), and consequently,

$$\left|\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta) - \Lambda^0_j(\eta)\right| \leq C_0\varepsilon, \quad \left|\Lambda^\varepsilon_{p(\varepsilon,\eta,m)+1}(\eta) - \Lambda^0_j(\eta)\right| \leq C_0\varepsilon, \quad \forall \eta \in [a_{\eta^0_-,\varepsilon}, a_{\eta^0_+,\varepsilon}],\ \varepsilon \leq \varepsilon_0, \tag{4.13}$$

while $j = m, m+1$. The edges $a_{\eta^0_-,\varepsilon}$ and $a_{\eta^0_+,\varepsilon}$ of the interval can be determined from an ε-neighborhood of $(1 + \Lambda^0_m(\eta))^{-1}$ and $(1 + \Lambda^0_{m+1}(\eta))^{-1}$ and both converge toward η^0, as $\varepsilon \to 0$.

Remark 4.4. Comparing results in Theorems 4.1 and 4.2, it proves useful to observe that at the points η_0 such that $\Lambda^0_m(\eta_0) = \Lambda^0_{m+1}(\eta_0)$, the result in Theorem 4.1 provides at least one eigenvalue $\Lambda^\varepsilon_{p(\varepsilon,\eta_0,m)}(\eta)$ with $p(\varepsilon, \eta_0, m) \geq m$, but it does not take into account the multiplicity. That is to say, it does not ensure that there is another different index $p(\varepsilon, \eta_0, m+1)$ such that $p(\varepsilon, \eta_0, m+1) \geq m+1$, and the same can happen in a small neighborhood of η_0. This is provided by Theorem 4.2. The main difficulty in the proof of Theorem 4.2 arises when we are in an ε-neighborhood of the collision points of the two dispersion curves $(\eta_0, \Lambda^0_m(\eta_0)) \equiv (\eta_0, \Lambda^0_{m+1}(\eta^0))$, points in which the multiplicity of each branch $\Lambda^0_m(\eta)$ and $\Lambda^0_{m+1}(\eta)$ changes from one to two (see Figure 7). In these neighborhoods also the proof in Theorem 4.2 ensures that there are two different indexes, namely, at least the eigenvalues $\Lambda^\varepsilon_{p(\varepsilon,\eta,m)}(\eta)$ and $\Lambda^\varepsilon_{p(\varepsilon,\eta,m)+1}(\eta)$ satisfy (4.9) for η ranging in a small neighborhood of η_0, cf. Corollary 4.3.

Also, in connection with the last step of the proof of Theorem 4.2, it is worth noting that, again, depending on the value of $H < 1/\sqrt{8}$, the reasoning can be continued for other values of m even greater than 3 (cf. Figure 2), and the statement of the theorem holds true for further values of m. However, the process relies on the complex trusses–nodes structure for the curves $(1 + \Lambda^0_m(\eta))^{-1}$ and computations become cumbersome, this being the reason for avoiding them in this paper. □

5 | CONVERGENCE RATES FOR LOW-FREQUENCY DISPERSION CURVES

In this section, we prove the main result of the paper. We show that the index $p(\varepsilon, \eta, m) \geq m$ found in the previous section coincides with m, and as a consequence, we provide bounds for discrepancies between the eigenvalues of the sequences (1.8) and (1.14) which are uniform in both parameters ε and η.

Theorem 5.1. *Let m be fixed, m ranging in $\{1, 2, 3\}$ and such that $\Lambda^0_m(\eta)$ is an eigenvalue of problem (1.10)–(1.12) in the sequence (1.14) having a corresponding eigenfunction independent of x_2 for η in an interval $[\mathfrak{a}, \mathfrak{b}] \subseteq [-\pi, \pi]$.*

i) *Assume that the interval* $[\mathfrak{a}, \mathfrak{b}]$ *does not contain* η^0 *such that* $\Lambda^0_m(\eta^0) = \Lambda^0_{m+1}(\eta^0)$*, then, we show that there exist positive constants* ε_0 *and* C_0 *independent of* η *and* ε *such that the eigenvalue* $\Lambda^\varepsilon_m(\eta)$ *of problem (1.4)–(1.6) in the sequence (1.8) is the only one meeting the estimate*

$$|\Lambda^\varepsilon_m(\eta) - \Lambda^0_m(\eta)| \le C_0\varepsilon, \quad \forall \varepsilon \le \varepsilon_0,\ \eta \in [\mathfrak{a}, \mathfrak{b}]. \tag{5.1}$$

ii) *If for some* $\eta_0 \in [\mathfrak{a}, \mathfrak{b}]$*,* m *satisfies* $\Lambda^0_m(\eta^0) = \Lambda^0_{m+1}(\eta^0) < \Lambda^0_{m+2}(\eta^0)$*, then, there are exactly two eigenvalues* $\Lambda^\varepsilon_m(\eta)$ *and* $\Lambda^\varepsilon_{m+1}(\eta)$ *meeting the discrepancy (5.1) for* $\eta \in [a_{\eta^0_-,\varepsilon}, a_{\eta^0_+,\varepsilon}] \cap [\mathfrak{a}, \mathfrak{b}]$*, where* $[a_{\eta^0_-,\varepsilon}, a_{\eta^0_+,\varepsilon}]$ *is a small neighborhood of* η_0*. The edges* $a_{\eta^0_-,\varepsilon}$ *and* $a_{\eta^0_+,\varepsilon}$ *can be determined from an* ε*-neighborhood of the dispersion curves* $\Lambda^0_m(\eta)$ *and* $\Lambda^0_{m+1}(\eta)$ *and both converge toward* η^0*, as* $\varepsilon \to 0$*.*

iii) *In particular, for* $m = 1$ *and* $H > 0$*, for* $m = 2$ *and* $H \in (0, 1/2)$ *and for* $m = 3$ *and* $H \in (0, 1/\sqrt{8})$ *we have*

$$|\Lambda^\varepsilon_m(\eta) - \Lambda^0_m(\eta)| < C_0\varepsilon, \qquad \text{for } \varepsilon \le \varepsilon_0,\ \eta \in [-\pi, \pi]. \tag{5.2}$$

Proof. Let us proceed with the proof in three steps, where we use the bounds in Theorems 4.1 and 4.2.

Step (1): The case where $\forall \eta \in [\mathfrak{a}, \mathfrak{b}]$, $\Lambda_m(\eta) < \Lambda_{m+1}(\eta)$. Let us show that $p(\varepsilon, \eta, m)$ arising in (4.1) verifies $p(\varepsilon, \eta, m) = m$ when $\eta \in [\mathfrak{a}, \mathfrak{b}]$, and consequently (5.1) also holds.

We proceed by contradiction, denying (5.1), while, on account of Theorem 4.1, $p(\varepsilon, \eta, m) \ge m$. This implies that there exists $\eta^* \in [\mathfrak{a}, \mathfrak{b}]$ such that there is $\varepsilon_{\eta^*} \le \varepsilon_m$ for which $p(\varepsilon_{\eta^*}, \eta^*, m) \ge m + 1$ and (5.1) does not hold. By the hypothesis,

$$\Lambda^0_{m+1}(\eta^*) > \Lambda^0_m(\eta^*).$$

First of all, we observe that, for such a η^*, the number of ε_{η^*} that we can select above must constitute a finite number, because otherwise, we can take a subsequence of $\varepsilon_{\eta^*,l} \to 0$ as $l \to \infty$ and then, from (4.1), or equivalently from (4.9) we write

$$\Lambda^{\varepsilon_{\eta^*,l}}_{m+1}(\eta^*) \le \Lambda^{\varepsilon_{\eta^*,l}}_{p(\varepsilon_{\eta^*,l},\eta^*,m)}(\eta^*) \le \Lambda^0_m(\eta^*) + C_0\varepsilon_{\eta^*,l}. \tag{5.3}$$

Now, on account of (2.4), taking limits, as $l \to \infty$, we get a contradiction:

$$\Lambda^0_{m+1}(\eta^*) \le \Lambda^0_m(\eta^*).$$

Consequently, for each η^* for which (5.1) does not hold, there is at the most a finite number $\varepsilon_{\eta^*,1}, \varepsilon_{\eta^*,2}, \dots \varepsilon_{\eta^*,k_{\eta^*}}$ for which (5.1) does not hold. In addition, if there is only one such η^* taking $\varepsilon^*_m = min(\varepsilon_m, \varepsilon_{\eta^*,1}, \varepsilon_{\eta^*,2}, \dots \varepsilon_{\eta^*,k_{\eta^*}})$, then inequality (5.1) holds for $\varepsilon \le \varepsilon^*_m$. The same happens if there is only a finite number of η^* for which (5.1) does not hold.

Thus, denying (5.1) must imply that there is at least one sequence $\{\eta^*_r\}_{r=1}^\infty$ that converge toward some $\hat{\eta} \in [\mathfrak{a}, \mathfrak{b}]$, as $r \to \infty$, such that (5.1) is not satisfied for $\varepsilon_{\eta^*_r,1}, \varepsilon_{\eta^*_r,2}, \dots \varepsilon_{\eta^*_r,k_{\eta^*_r}}$, $r = 1, 2, \dots$ while (4.1) holds. Without any restriction we can assume that there is also a sequence of $\varepsilon_{\eta^*_r}$ converging toward zero as $r \to \infty$, Indeed, let us explain this latter assertion in further detail. For the set $\mathcal{J} := \{\eta^* \in [\mathfrak{a}, \mathfrak{b}] : (5.1) \text{ is not satisfied } \} \subset [-\pi, \pi]$, we consider the associated set of parameters constructed above:

$$\mathcal{E} := \{\varepsilon_{\eta^*,1}, \varepsilon_{\eta^*,2}, \dots \varepsilon_{\eta^*,k_{\eta^*}}\}_{\eta^* \in \mathcal{J}}.$$

Either $\mathcal{E}$ has a lower bound $\varepsilon^{**}_m > 0$ or we can extract a sequence $\{\varepsilon_{\eta^*_r}\}_{r=1}^\infty$ converging toward zero as $r \to \infty$, each term $\varepsilon_{\eta^*_r}$ being associated with a certain value $\eta^*_r \in \mathcal{J}$. In the first case, (5.1) holds for $\varepsilon \le \varepsilon^*_m := \min(\varepsilon^{**}_m, \varepsilon_m)$ and the proof is completed. In the second case, since the sequence $\{\eta^*_r\}_{r=1}^\infty$ is bounded from above and below, by subsequences, we can construct the above-mentioned sequence, still denoted by r,

$$(\eta^*_r, \varepsilon_{\eta^*_r}) \to (\hat{\eta}, 0) \text{ as } r \to \infty.$$

Let us show that this last assertion leads us to a contradiction.

To this end, we notice that, as in (5.3), from (4.9) we can write

$$\Lambda^{\varepsilon_{\eta^*_r},}_{m+1}(\eta^*_r) \le \Lambda^{\varepsilon_{\eta^*_r}}_{p(\varepsilon_{\eta^*_r},\eta^*_r,m)}(\eta^*_r) \le \Lambda^0_m(\eta^*_r) + C_0\varepsilon_{\eta^*_r}, \tag{5.4}$$

for the corresponding sequence of eigenvalues. Taking into account Theorem 2.2, cf. (2.3), and the continuity of the application (1.15), we take limits in (5.4), as $r \to \infty$, and get $\Lambda^0_{m+1}(\hat{\eta}) \le \Lambda^0_m(\hat{\eta})$. Therefore, when the limiting eigenvalue $\Lambda^0_m(\hat{\eta})$

is simple, as happens by the hypothesis on the dispersion curves and the interval $[\mathfrak{a}, \mathfrak{b}]$, we get a contradiction. The contradiction process also shows that there are no more eigenvalues $\Lambda_p^\varepsilon(\eta)$ satisfying (5.1), since by Theorem 4.1 (equivalently, by the hypothesis) p should be greater than or equal to m.

Step (2): the case where $\eta^0 \in [\mathfrak{a}, \mathfrak{b}]$ *with* $\Lambda_m^0(\eta^0) = \Lambda_{m+1}^0(\eta^0)$. Let us start with $m = 1$.

By the hypothesis on the limiting dispersion curves, there are only two possibilities for the curves $\Lambda_1^0(\eta)$ and $\Lambda_2^0(\eta)$, which is to have a common point for the values of $\eta_0 = \pm\pi$, $[\mathfrak{a}, \mathfrak{b}]$ being either $[-\pi, \mathfrak{b}]$ when $\eta_0 = -\pi$ or $[\mathfrak{a}, \pi]$ when $\eta_0 = \pi$. The two nodes are respectively $(-\pi, \pi^2)$ and (π, π^2) and this holds for any $H > 0$. Let us start by $\eta = -\pi$ and consider the points

$$a_{-\pi,\varepsilon} = -\pi + \alpha_0\varepsilon \quad \text{and} \quad a_{\pi,\varepsilon} = \pi - \alpha_0\varepsilon, \quad \text{with } \alpha_0 = \frac{C_0}{2\pi}, \quad \varepsilon \le \varepsilon_0,$$

which are the abscises of intersecting points of the ε-neighborhood lines of the dispersion curves: $\Lambda_1^0(\eta) + C_0\varepsilon$ and $\Lambda_2^0(\eta) - C_0\varepsilon$ near $\pm\pi$ (cf. (4.12) to compare). Since for $\eta \in [-\pi, a_{-\pi,\varepsilon}]$ the eigenvalues $\Lambda_1^0(\eta)$ and $\Lambda_2^0(\eta)$ are at a distance $O(\varepsilon)$ between them, it suffices to show

$$|\Lambda_1^\varepsilon(\eta) - \Lambda_1^0(\eta)| \le C_0\varepsilon, \quad |\Lambda_2^\varepsilon(\eta) - \Lambda_1^0(\eta)| \le C_0\varepsilon, \quad \forall \varepsilon \le \varepsilon_0,\ \eta \in [-\pi, a_{-\pi,\varepsilon}], \tag{5.5}$$

with C_0 being a positive constant independent of ε and η.

Considering Theorem 4.2 (see Step (2)), it has been shown that for each $\eta \in [-\pi, a_{-\pi,\varepsilon}]$ there are at least two eigenvalues $\Lambda^\varepsilon_{p(\varepsilon,\eta,1)}$ and $\Lambda^\varepsilon_{p(\varepsilon,\eta,1)+1}$ satisfying $p(\varepsilon, \eta, 1) \ge 1$ and

$$|\Lambda^\varepsilon_{p(\varepsilon,\eta,1)}(\eta) - \Lambda_1^0(\eta)| \le C_0\varepsilon, \quad |\Lambda^\varepsilon_{p(\varepsilon,\eta,1)+1} - \Lambda_1^0(\eta)| \le C_0\varepsilon, \quad \forall \varepsilon \le \varepsilon_0,\ \eta \in [-\pi, a_{-\pi,\varepsilon}].$$

Let us assume that (5.5) does not hold, which means that there is $\varepsilon^* < \varepsilon_0$ for which we cannot ensure that $p(\varepsilon^*, \eta_{\varepsilon^*}, 1)$ is equal to 1 for some $\eta_{\varepsilon^*} \in [-\pi, a_{-\pi,\varepsilon^*}]$, namely, $p(\varepsilon^*, \eta_{\varepsilon^*}, 1) \ge 2$ and $p(\varepsilon^*, \eta_{\varepsilon^*}, 1) + 1 \ge 3$. If the set of ε^* for which (5.5) does not hold is finite, taking the minimum, we can take the ε^{**} in such a way that (5.5) holds for $\varepsilon \le \varepsilon^{**}$. Therefore, the worst situation happens when we can extract an infinitesimal sequence of ε_r^* and $\eta_r^* := \eta_{\varepsilon_r^*} \in [-\pi, a_{-\pi,\varepsilon_r^*}]$ for which $p(\varepsilon_r^*, \eta_r^*, 1) \ge 2$ and therefore $p(\varepsilon_r^*, \eta_r^*, 1) + 1 \ge 3$. If so, $(\varepsilon_{\eta_r^*}, \eta_r^*) \to (0, -\pi)$ and, as in (5.4), write the corresponding chain of inequalities

$$\Lambda_3^{\varepsilon_{\eta_r^*}}(\eta_r^*) \le \Lambda^{\varepsilon_{\eta_r^*}}_{p(\varepsilon_{\eta_r^*}, \eta_r^*, 1)+1}(\eta_r^*) \le \Lambda_1^0(\eta_r^*) + C_0\varepsilon_{\eta_r^*},$$

cf. (4.13). Taking limits and applying Theorem 2.2, we obtain the following contradiction:

$$\Lambda_3^0(-\pi) \le \Lambda_1^0(-\pi) = \Lambda_2^0(-\pi) < \Lambda_3^0(-\pi). \tag{5.6}$$

Consequently, $p(\varepsilon, \eta, 1) = 1$ for $\varepsilon \le \varepsilon_0$, $\eta \in [-\pi, a_{-\pi,\varepsilon}]$ and Equation (5.5) holds. Since in this interval $\Lambda_1^0(\eta)$ and $\Lambda_2^0(\eta)$ are at distance $O(\varepsilon)$, also (5.1) holds for $m = 1$ and $m = 2$. In addition, there cannot be more eigenvalues $\Lambda_p^\varepsilon(\eta)$ satisfying (5.1) because of the same argument of contradiction above: indeed, another eigenvalue should compulsorily be $\Lambda_3^\varepsilon(\eta)$ for certain ε and η ranging in converging subsequences, which leads to the contradiction (5.6).

In a similar way, we proceed with the other possible node of the limiting dispersion curves, that is, with the point $(0, 4\pi^2)$ when $m = 2$ and $H < 1/2$. By the hypothesis on the dispersion curves, there is only one possibility for the curves $\Lambda_2^0(\eta)$ and $\Lambda_3^0(\eta)$ to cut across each other. This possibility means to have a common point at $\eta_0 = 0$ and $0 \in [\mathfrak{a}, \mathfrak{b}] \subset [-\pi, \pi]$.

We consider the points

$$a_{0-,\varepsilon} = -\frac{C_0\varepsilon}{4\pi} \quad \text{and} \quad a_{0+,\varepsilon} = \frac{C_0\varepsilon}{4\pi},$$

which are intersecting points of the ε-neighborhood lines of the dispersion curves $\Lambda_2^0(\eta) + C_0\varepsilon$ and $\Lambda_3^0(\eta) - C_0\varepsilon$ near $\eta = 0$, cf. Figure 7.

Since for $\eta \in [a_{0-,\varepsilon}, a_{0+,\varepsilon}]$ the eigenvalues $\Lambda_2^0(\eta)$ and $\Lambda_3^0(\eta)$ are at a distance $O(\varepsilon)$ between them, it suffices to show

$$|\Lambda_2^\varepsilon(\eta) - \Lambda_2^0(\eta)| \le C_0\varepsilon, \quad |\Lambda_3^\varepsilon(\eta) - \Lambda_2^0(\eta)| \le C_0\varepsilon, \quad \forall \varepsilon \le \varepsilon_0,\ \eta \in [a_{0-,\varepsilon}, a_{0+,\varepsilon}]. \tag{5.7}$$

Also, in Theorem 4.2 (see Step (3), and (4.13)), it has been shown that for each $\eta \in [a_{0-,\varepsilon}, a_{0+,\varepsilon}]$ there are at least two eigenvalues $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,2)}(\eta)$ and $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,2)+1}(\eta)$ satisfying $p(\varepsilon,\eta,2) \geq 2$ and

$$|\Lambda^{\varepsilon}_{p(\varepsilon,\eta,2)}(\eta) - \Lambda^0_2(\eta)| \leq C_0\varepsilon, \quad |\Lambda^{\varepsilon}_{p(\varepsilon,\eta,2)+1}(\eta) - \Lambda^0_2(\eta)| \leq C_0\varepsilon, \quad \forall \varepsilon \leq \varepsilon_0,\ \eta \in [a_{0-,\varepsilon}, a_{0+,\varepsilon}].$$

Let us assume that (5.7) does not hold, which means that there is ε^* and $\eta_{\varepsilon^*} \in [a_{0-,\varepsilon^*}, a_{0+,\varepsilon^*}]$ for which we cannot ensure that $p(\varepsilon^*, \eta_{\varepsilon^*}, 2)$ is equal to 2 , namely, $p(\varepsilon^*, \eta_{\varepsilon^*}, 2) \geq 3$ and $p(\varepsilon^*, \eta_{\varepsilon^*}, 2) + 1 \geq 4$. As in the reasoning above (in the first step), the worst situation happens when we can extract a sequence $(\varepsilon_{\eta_r^*}, \eta_r^*) \to (0,0)$ and, as in (5.4), we write

$$\Lambda_4^{\varepsilon_{\eta_r^*}}(\eta_r^*) \leq \Lambda^{\varepsilon_{\eta_r^*}}_{p(\varepsilon_{\eta_r^*},\eta_r^*,2)+1}(\eta_r^*) \leq \Lambda^0_2(\eta_r^*) + C_0\varepsilon_{\eta_r^*}.$$

Taking limits and applying Theorem 2.2, we obtain the following contradiction:

$$\Lambda^0_4(0) \leq \Lambda^0_2(0) = \Lambda^0_3(0) < \Lambda^0_4(0). \tag{5.8}$$

Consequently, $p(\varepsilon,\eta,2) = 2$ for $\varepsilon \leq \varepsilon_0$, $\eta \in [a_{0-,\varepsilon}, a_{0+,\varepsilon}]$ and (5.7) holds as well as (5.1) for $m = 2$ and $m = 3$. In addition, there cannot be more eigenvalues $\Lambda^{\varepsilon}_p(\eta)$ satisfying (5.1) because of the same argument of contradiction above: indeed, another eigenvalue should compulsorily be $\Lambda^{\varepsilon}_4(\eta)$ for certain ε and η ranging in subsequences, which leads to the contradiction (5.8).

It should be noted that when numbering the eigenvalues (1.16) in the sequence (1.14), the eigenvalue number m can change depending on the values of $\eta \in [-\pi, \pi]$ and $H > 0$. The rest of the node points when $m \geq 2$ can be treated in a similar way, and we avoid introducing here the cumbersome computations. Also, as a matter of fact, if $H < 1/\sqrt{8}$ the process can be continued for values of m greater than 3.

Step (3): Let us show (5.2) for the different values of m. Now, from the previous steps of this proof, we note that while $\eta \in [-\pi, 0]$, the points where estimate (5.2) has not been proved yet are those for $\eta \in [a_{-\pi,\varepsilon}, \mathfrak{a}]$ when $m = 1, 2$, or $\eta \in [\mathfrak{a}, a_{0-,\varepsilon}]$ when $m = 2, 3$, with any $\mathfrak{a}$ such that $-\pi < \mathfrak{a} < 0$. The same can be said while $\eta \in [0, \pi]$, with $\eta \in [\mathfrak{b}, a_{\pi,\varepsilon}]$ and $m = 1, 2$, or $\eta \in [a_{0+,\varepsilon}, \mathfrak{b}]$ when $m = 2, 3$, with any $\mathfrak{b}$ such that $0 < \mathfrak{b} < \pi$.

Let us start with $m = 1, 2$, and $\eta \in [-\pi, \mathfrak{a}] \supsetneqq [a_{-\pi,\varepsilon}, \mathfrak{a}]$. Theorem 4.2 ensures that (4.9) holds for $m = 2$ with $p(\varepsilon,\eta,2) \geq 2$. We can apply the same contradiction argument used above which leads to a certain sequence

$$(\eta_r^*, \varepsilon_{\eta_r^*}) \to (\hat{\eta}, 0) \text{ as } r \to \infty, \qquad \text{for some } \hat{\eta} \in [-\pi, \mathfrak{a}],$$

for which $p(\varepsilon_{\eta_r^*}, \eta_r^*, 2) \geq 3$. Actually, taking limits in

$$\Lambda_3^{\varepsilon_{\eta_r^*}}(\eta_r^*) \leq \Lambda^{\varepsilon_{\eta_r^*}}_{p(\varepsilon_{\eta_r^*},\eta_r^*,1)}(\eta_r^*) \leq \Lambda^0_2(\eta_r^*) + C_0\varepsilon_{\eta_r^*}$$

cf. (2.3), we obtain $\Lambda^0_3(\hat{\eta}) \leq \Lambda^0_2(\hat{\eta})$ which is a contradiction for $\hat{\eta} \in [-\pi, \mathfrak{a}] \subset [-\pi, 0)$. Thus, (5.2) holds for $m = 2$ and $\eta \in [-\pi, \mathfrak{a}] \subset [-\pi, 0)$.

For $m = 1$, we observe that the construction of the endpoints in the second step applies if we replace the abscise $a_{-\pi,\varepsilon}$ by $\tilde{a}_{-\pi,\varepsilon}$ arising in the second step of the proof of Theorem 4.2 (see Figures 5–7). The construction in Theorem 4.2 ensures that, for $\eta \in [\tilde{a}_{-\pi,\varepsilon}, \mathfrak{a}]$, there are at least two different eigenvalues: $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,1)}(\eta)$ satisfying (4.10) with $m = 1$, and $\Lambda^{\varepsilon}_{p(\varepsilon,\eta,2)}(\eta)$ satisfying (4.10) with $m = 2$. Since we have proved above that $p(\varepsilon,\eta,2) = 2$, also $p(\varepsilon,\eta,1) = 1$. Thus, (5.2) also holds for $m = 1$ and $\eta \in [-\pi, \mathfrak{a}]$, and therefore we have obtained that for $H < 1/2$ bounds (5.2) hold for $m = 1, 2$.

We proceed in a similar way in the interval $\eta \in [\mathfrak{a}, a_{0-,\varepsilon}]$ when $m = 2, 3$, arguing by contradiction for a $p(\varepsilon,\eta,3) \geq 4$ and then, combining the second step of the proof of this theorem with the third step in the proof of Theorem 4.2. Thus, we obtain that for $H < 1/\sqrt{8}$ bounds (5.2) hold for $m = 1, 2, 3$.

Therefore, the result of the last statement holds and the theorem is proved. □

ACKNOWLEDGMENTS

The work has been partially supported by MICINN through PGC2018-098178-B-I00, PID2020-114703GB-I00 and Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000904-S).

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How to cite this article: D. Gómez, S. A. Nazarov, R. Orive-Illera, and M.-E. Pérez-Martínez, *Asymptotic stability of the spectrum of a parametric family of homogenization problems associated with a perforated waveguide*, Math. Nachr. **296** (2023), 4888–4910. https://doi.org/10.1002/mana.202100589