Relativistic thermodynamics on conveyor belt

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Abstract

Two thermodynamic processes are analysed by using a relativistic four-vector fundamental equation formalism: the launching of a projectile by forces produced by chemical reactions inside a cannon (equivalent to a person launching a ball) placed on a moving platform, a mechanical energy production process, and the Joule-Kelvin process implemented on a conveyor belt. Each process is first studied in frame S, in which the device – the cannon or the porous plug –, is at rest, obtaining its four-vector fundamental equation $\mathrm{d}E^\mu = \delta W^\mu + \delta Q^\mu$. Using the Lorentz transformation, the corresponding four-vector equation in frame $\bar{\mathrm{S}}$ – in which the processes are carried out on a moving platform or a conveyor belt – $\mathrm{d}\bar{E}^\mu = \delta\bar{W}^\mu + \delta\bar{Q}^\mu$, is obtained, obeying Einstein's principle of relativity. For the relativistic description to be coherent, one has to assign linear momentum to the non-mechanical energies and consider their variations in the corresponding Newton's second law equation and the relativistic non-simultaneity and conveyor belt effects. In relativistic thermodynamics, Newton's second law and the first law of thermodynamics are not independent equations. It is shown that entropy variations and fuel consumption are frame independent magnitudes.

1 Introduction

Relativistic thermodynamics. During the 20th century, the so-called 'relativistic thermodynamics' [1, 2] was conceived to obtain relativistic temperature transformations between inertial reference frames

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(e.g., T to $\bar{T}$ between frames S and $\bar{\text{S}}$, with relative speed V) [3]. On temperature transformations proposals – $\bar{T} = \gamma_V^{-1} T$ (Planck) [4], $\bar{T} = \gamma_V T$ (Ott) [5], or $\bar{T} = T$ (Landsberg) [6] [$\gamma_V = (1 - V^2/c^2)^{-1/2}$] –, the scientific community did not reach consensus [7, 8]. Working on two different formulations of relativity – the synchronous formulation [9] and the asynchronous formulation [10] – also did not help to correctly pose the relativistic temperature transformation problem. In this summary on the relativistic thermodynamics story, only a small number of the reach bibliography [11, 12] has been cited. Recently, in the 21st century, there has been some renewed interest in the subject [13, 14].

Throughout this work, relativistic thermodynamics will be understood as the description of a thermodynamic process carried out on a conveyor belt, or wagon, in motion relative to an observer $\bar{\text{S}}$, at rest on the ground, by means of Einstein's special theory of relativity equations and methods: in the asynchronous formulation [7], by using a four-vector formalism [2]. The description of the process in frame $\bar{\text{S}}$ will be obtained from that in S, at rest with respect to the conveyor belt and transforming the corresponding four-vector equation by application of the Lorentz transformation [15, 16].

To apply Einstein's inertia of energy principle [17], some thermodynamic quantities (e.g., heat) will require a quantum mechanical-statistical description [18, 19].

Principle of relativity and thermodynamics. The principle of relativity (Huygens-Galileo principle [20, Ch. 16]) does not play a significant role in Newtonian physics [21, p. 451] and even less in thermodynamics. Although the change of reference frame is considered in describing processes involving a conveyor belt [22], the thermodynamic aspects of such processes where mechanical energy dissipation is present, are hardly considered [23].

The scientific community has not been traditionally interested in describing thermodynamic processes taking place on moving frames, as is the case of a conveyor belt. The lack of mechanical considerations, something usual when analyzing thermodynamic processes, entails that they cannot be properly described in different frames, making difficult to relate the descriptions of these processes with the principle of relativity. The Joule-Kelvin (JK) – or Joule-Thomson – process is an exception to that comment [24]. However, it is not clear how equations describing the JK process are transformed from frame S, where the porous plug is at rest with respect to observers [25, p. 3]), to frame $\bar{\text{S}}$ where the whole device is in motion [26] (Fig. 1). In classical physics, there is no direct (i.e., algorithmic) procedure allowing the transformation from the

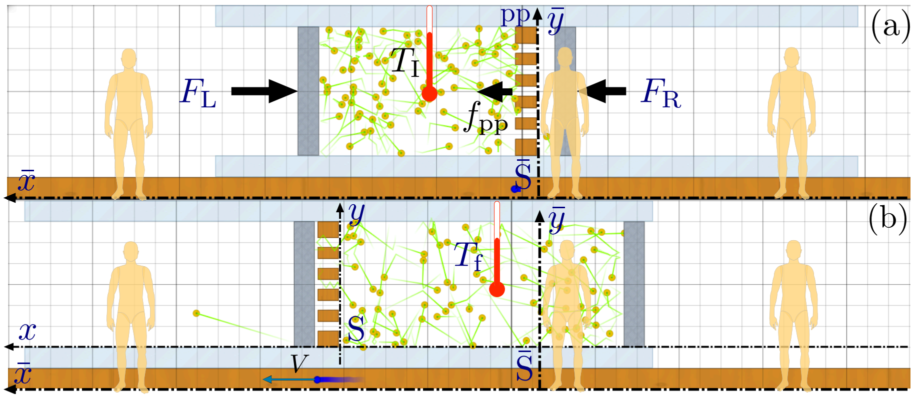

Figure 1: Joule-Kelvin process on a conveyor belt. In frame S (xy), at rest relative to the conveyor belt, the porous plug (pp) is fixed. The whole device (frame S) moves with velocity $\bar{\mathbf{v}}_{\rm cb} = (-V, 0, 0)$ with respect to frame $\bar{\rm S}$, in the standard configuration. (a) One mole of gas, at temperature $T_{\rm I}$, is enclosed into a cylinder of section A; with pressure $P_{\rm L} = F_{\rm L}/A$ exerted on it. The porous plug applies force $f_{\rm pp}$ on the gas. (b) The fluid is forced to flow through the porous plug, expanding against pressure $P_{\rm R} = F_{\rm R}/A$, being $P_{\rm R} < P_{\rm L}$, with final temperature $T_{\rm f}$.

equations in frame S to those in frame $\bar{\rm S}$ and vice versa.

Analyzing the process of a person, located on a moving wagon (frame S), with velocity $\bar{\mathbf{v}}_{\rm pl} = (V, 0, 0)$ relative to frame $\bar{\rm S}$, throwing an object (Fig. 2), only mechanical aspects are considered (basically the description of kinetic energy production and work done in both frames, S and $\bar{\rm S}$, as well as their internal coherence [27]). Thermodynamic aspects such as launcher's free energy consumption [e.g., ATP (adenosine triphosphate) wasted in its muscles, previously obtained from food] are not considered. If any, transformations of quantities between frames are performed intuitively.

The absence of thermodynamical considerations in mechanical processes and vice versa, may be in part due to the fact that progress in classical physics [28] has led both disciplines as separate subjects [29]. Moreover, both of them, in general, ignore the atomic structure of matter and its constituent elementary particles such as electric charges (protons and electrons), as well as their interactions, interchanging electromagnetic radiation (photons [30], even for low-speed processes).

The purpose of this article is to show that an Einsteinian relativistic analysis allows to pose the equations of a process directly, by means of a Minowski's four-vector fundamental equation, implicitly containing Newton's second law (NSL) and the first law of thermodynamics (FLT),

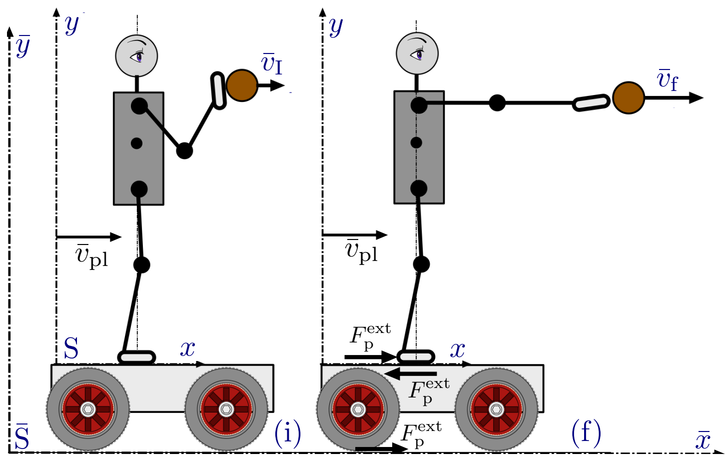

Figure 2: Person standing on a moving wagon, throwing a ball. Frame $\bar{\rm S}$ ($\bar{x}\bar{y}$): person on wagon moving with velocity $\bar{\mathbf{v}}_{\rm pl} = (V, 0, 0)$. Frame S ($xy$): person at rest relative to the wagon. The ball, with initial velocity $\bar{\mathbf{v}}_{\rm I} = (\bar{v}_{\rm I}, 0, 0)$, reaches final velocity $\bar{\mathbf{v}}_{\rm f} = (\bar{v}_{\rm f}, 0, 0)$ in $\bar{\rm S}$, and $\mathbf{v}_{\rm f} = (v_{\rm f}, 0, 0)$ in S. A force $F_{\rm p}^{\rm ext}$ is exerted on the person preventing it from moving relative to the wagon when it throws the ball. An external agent (motor) must exert the same force on the wagon to keep it moving at constant speed.

allowing the transformation between frames by using the Lorentz transformation [19], thus obtaining the process description in different inertial frames, according to Einstein's special theory of relativity postulates.

Einsteinian principle of relativity. The principle of relativity (Poincaré-Einstein principle [31, p. 86]) constitutes the first postulate of Einstein's special theory of relativity, playing a fundamental role in its development [23]. Under the Einsteinian relativistic formalism, the equations must be expressed by four-vectors, which in turn are composed of a vector, $\mathbf{a} = (a_1, a_2, a_3)$, and a scalar, $a_{\rm t}$, into a matrix structure, called Minkowski four-vector (a contra-variant 4-vector is a column matrix, denoted by a Latin letter with a Greek index; a co-variant 4-vector is a row matrix, denoted by a Greek sub-index), as $a^\mu \equiv (\mathbf{a}, a_{\rm t}) = (a_1, a_2, a_3, a_{\rm t})$ [32] (for typographical reasons a contra-variant four-vector column can be written as a matrix row, still with Greek superscript; see below). A usual process in mechanics or thermodynamics can be described as a four-vector fundamental equation [17], allowing the possibility of switching between inertial frames by direct application of the Lorentz transformation [33], explicitly guaranteeing the principle of relativity fulfilment. In this work, thermodynamical processes on conveyor belt will be analyzed using a relativistic Einstein-Minkowski-Lorentz formalism [34, 1, 2].

The four-tensor Lorentz transformation between inertial frames S and $\bar{\mathrm{S}}$, assuming $\bar{\mathrm{S}}$ moves with constant velocity $\mathbf{v}_{\bar{\mathrm{S}}|\mathrm{S}} = (V, 0, 0)$ with respect to S, is given by

$$\mathcal{L}^{\mu}_{\nu}(V) = \begin{pmatrix} \gamma_V & 0 & 0 & -\beta_V\gamma_V \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_V\gamma_V & 0 & 0 & \gamma_V \end{pmatrix} , \tag{1}$$

with $\mathcal{L}^{\mu}_{\sigma}(V)\mathcal{L}^{\sigma}_{\nu}(-V) = 1^{\mu}_{\nu}$ (Einstein's summation convention), the Lorentz factor $\gamma_V = (1-\beta_V^2)^{-1/2}$, $\beta_V = V/c$ and c the light-speed in vacuum.

In the asynchronous formulation of relativity [35], a contra-variant four-vector in frame S, with $A^{\mu} \equiv (\mathbf{A}, A_4) = (A_1, A_2, A_3, A_4)$, is transformed into the corresponding four-vector in frame $\bar{\mathrm{S}}$, $\bar{A}^{\mu} \equiv (\bar{\mathbf{A}}, \bar{A}_4) = (\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, by applying the Lorentz transformation, $\bar{A}^{\mu} = \mathcal{L}^{\mu}_{\nu}(V)A^{\nu}$ [36]:

$$\bar{A}^{\mu} = \mathcal{L}^{\mu}_{\nu}(V) \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_t \end{pmatrix} = \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_t \end{pmatrix} = \begin{pmatrix} \gamma_V(A_1 - \beta_V A_t) \\ A_2 \\ A_3 \\ \gamma_V(A_t - \beta_V A_1) \end{pmatrix} . \tag{2}$$

Since the transformation is required to be symmetric, $A^{\mu} = \mathcal{L}^{\mu}_{\nu}(-V)\bar{A}^{\nu}$ must be satisfied, with $\bar{\mathbf{v}}_{\mathrm{S}|\bar{\mathrm{S}}} = (-V, 0, 0)$ [37]. Defining the norm of four-vector $\bar{A}^{\mu}$ [38, p. 639] as,

$$\|\bar{A}^{\mu}\| \equiv \left[\bar{A}_t^2 - (\bar{A}_1^2 + \bar{A}_2^2 + \bar{A}_3^2)\right]^{1/2} , \tag{3}$$

it can be concluded that $\|\bar{A}^{\mu}\| = \|A^{\mu}\|$: the norm of a four-vector is Lorentz invariant. In other words, the norm of a physical quantity expressed as a four-vector, is an intrinsic property, regardless of the frame. For example, the norm of the momentum–energy four-vector $E^{\mu} = (c\mathbf{p}, E)$, with $\mathbf{p} = c\gamma_v\mathcal{M}\mathbf{v}$, and $E = \gamma_v\mathcal{M}c^2$ (see below), is $\|E^{\mu}\| = \mathcal{M}c^2$, its energy at rest (the norm of other four-vectors such as heat, Q^{μ}, or fuel consumption, ΔG^{μ}_{ξ}, will be discussed later).

Four-vector fundamental equation in S. As a convenient frame, we choose frame S where forces are simultaneously applied [37] and the thermal reservoir surrounding the system is at rest. For a process involving a finite system and a body with near-infinite inertia at rest such as the floor or a wall, acting as a surrounding thermal reservoir, the four-vector fundamental equation in S is given by [17]:

$$E^{\mu}_{\mathrm{f}} - E^{\mu}_{\mathrm{I}} = W^{\mu}_{\mathrm{ext}} + Q^{\mu} . \tag{4}$$

This equation must be obtainable for well-posed processes. In Eq. (4), $E_{\rm f}^{\mu}$ and $E_{\rm I}^{\mu}$ are the final and initial linear-momentum–energy four-vectors respectively. The term $W_{\rm ext}^{\mu} = \Sigma_{\rm k=1}^{\rm r} W_{\rm k}^{\mu}$ is the linear-impulse–work four-vector, expressed as the sum of linear-impulse–work four-vectors associated with (r) external forces simultaneously applied on the system in S.

Energy exchanged as heat is characterized in S as a set of thermal photons, with frequency ν (monochromatic approximation [39]). Adding over all photons emitted in S during the process as opposed pairs of thermal photons (see below), the total linear momentum, $\mathbf{p}_{\rm ph} = \Sigma_{\rm i} {\rm p}_{\rm ph|i} = 0$, and total energy, $E_{\rm ph} = \Sigma_{\rm i} {\rm e}_{\rm ph|i} = N_{\rm ph} h\nu$, are obtained, being $N_{\rm ph}$ the number of thermal photons emitted [23]. The energy exchanged as heat has in S linear-momentum–energy four-vector $Q^{\mu} = (0, 0, 0, -E_{\rm ph})$, where the convention for the sign in the energy term will be negative for emission and positive for absorbtion.

The thermodynamical meaning for heat, from a classical point of view, is given by the norm of heat four-vector, $Q \equiv \|Q^{\mu}\| = N_{\rm ph} h\nu$ [19]. The entropy change for the thermal reservoir surrounding the system, at temperature $T_{\rm F}$, $\Delta S_{\rm F} = -\|Q^{\mu}\|/T_{\rm F}$, can be found by imposing the maximum entropy principle [23]. The entropy change of the system, its surroundings and the universe, are obtained by using classical thermodynamics methods.

Four-vector fundamental equation in $\bar{\rm S}$. Assuming frame $\bar{\rm S}$ moves with velocity $\mathbf{v}_{\bar{\rm S}|{\rm S}} = (V, 0, 0)$ in the standard configuration relative to S, we want to obtain the equations describing the process in $\bar{\rm S}$. Due to the near-infinite inertia of bodies at rest in S (floor, platform or wall), they are described in $\bar{\rm S}$ as moving, with velocity $\bar{\mathbf{v}}_{{\rm S}|\bar{\rm S}} = (-V, 0, 0)$, behaving as a conveyor belt. By applying the Lorentz transformation $\mathcal{L}_{\nu}^{\mu}(V)$ onto the four-vector fundamental equation in S [see Eq. (4)],

$$\mathcal{L}_{\nu}^{\mu}(V)[E_{\rm f}^{\nu} - E_{\rm I}^{\nu} = W_{\rm ext}^{\nu} + Q^{\nu}] \to \bar{E}_{\rm f}^{\mu} - \bar{E}_{\rm I}^{\mu} = \bar{W}_{\rm ext}^{\mu} + \bar{Q}^{\mu}\,, \qquad (5)$$

the corresponding four-vector fundamental equation describing the process in $\bar{\rm S}$ is obtained. The Lorentz transformation procedure followed to switch equations between different frames (asynchronous formulation of relativity [10]), guarantees the fulfilment of special theory of relativity postulates: the principle of relativity, with equations having the same functional form in frames S and $\bar{\rm S}$, and the same light-speed c in any frame. The four-momentum of an extended body is an operative four-vector only if the body is not subjected to tidal forces (i.e., the body is isolated). No aspects related to general relativity will be considered at this point [40, pp. 142-6]. Hereinafter, it will be assumed that each body can be considered as isolated, so that the Lorentz transformations, as expressed in Eq. (5) hold.

The remainder of the article is organized as follows. In Sec. 2, a process with mechanical energy production – launching of a projectile by a chemical load inside a cannon placed on a moving wagon –, is described (by using the same formalism, mechanical energy dissipation processes have been described in Ref. [16]). In Sec. 3, the Joule Kelvin process for a Van der Waals gas is described when the whole experimental device is located on a moving conveyor belt. Finally, in Sec. 4, some conclusions about relativistic thermodynamics are discussed.

2 Launching of a projectile on a conveyor belt

A process equivalent to that of a person on a moving wagon throwing a ball, which must spent chemical energy as ATP – previously produced by food consumption into their cells (Krebs cycle) – into their muscles, shown in Fig. 2, but somewhat easier to visualise and analyse, is that of a projectile (ball) launched by a cannon fixed to a wagon.

Figure 3 sketches a process in which a chemical load, for example, a stoichiometric mixture of n_ξ moles of H_2 and $n_\xi/2$ moles of O_2, located inside a cannon fixed to a wagon, launches a projectile. In ground's frame $\bar{\text{S}}$, where the wagon and the cannon move with velocity $\bar{\mathbf{v}}_{\text{S}|\bar{\text{S}}} = (V, 0, 0)$, the projectile, with initial speed $\bar{v}_{\text{I}} = V$, reaches final speed $\bar{v}_{\text{f}}$.

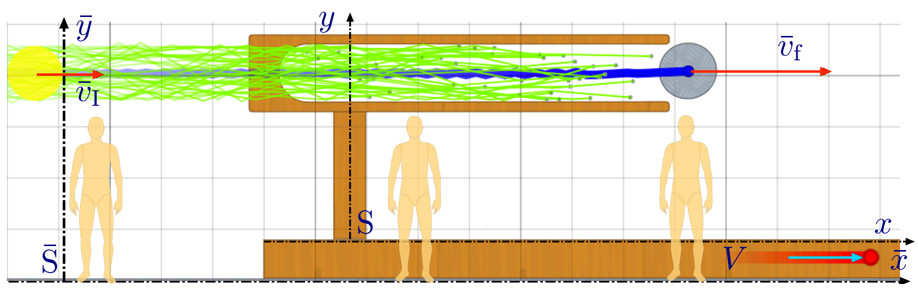

Figure 3: Observer frame $\bar{\text{S}}$ $(\bar{x}\bar{y})$. A moving wagon – frame S (xy) – carries a cannon with a chemical load and projectile inside, moving with velocity $\mathbf{v}_{\text{wg}} = (V, 0, 0)$ with respect to $\bar{\text{S}}$. The projectile, with initial speed $\bar{v}_{\text{I}} = V$ in $\bar{\text{S}}$, is launched from the wagon. When leaving the muzzle, the final speeds are v_{f} (frame S) and $\bar{v}_{\text{f}}$ (frame $\bar{\text{S}}$).

Internal energy and inertia. A solid system (i.e., almost zero coefficient of compressibility, $\kappa_T \approx 0$, see Sec. 3 for a gas), with chemical composition ξ (e.g., n moles of $\xi \equiv \text{Fe}$) and temperature T, has, according to Einstein's inertia of energy principle [19], internal energy E_0 and inertia $\mathcal{M}$ given by:

$$E_0(\xi, T) = \Sigma_{\rm j} m_{\rm j} c^2 - \left|\tilde{U}_{\rm N} + \tilde{U}_{\rm A} + \tilde{U}_{\rm M}\right| + \int_0^T n c_{\mathcal{V}}(T') {\rm d}T' \,, \tag{6a}$$

$$\mathcal{M}(\xi, T) = c^{-2} E_0(\xi, T) \,, \tag{6b}$$

where $m_{\rm j}$ is the mass of the (j)-th elementary particle (proton, neutron, electron) components, $\tilde{U}_\xi = \tilde{U}_{\rm N} + \tilde{U}_{\rm A} + \tilde{U}_{\rm M}$ is the sum of binding energies (nuclear, atomic, molecular or crystal) contributing to chemical composition and structure, and $c_{\mathcal{V}}(T)$ is the constant-volume molar specific heat. In frame S_0 where the system is at rest, with linear momentum $\mathbf{p}_0 = (0, 0, 0)$ and energy E_0, its state four-vector E_0^μ – linear-momentum–energy four-vector – is $E_0^\mu \equiv (\mathbf{p}_0, E_0) = (0, 0, 0, E_0)$. In frame $\hat{\mathrm{S}}$, in the standard configuration with respect to S_0, with speed $-\hat{v}$, the linear-momentum–energy four-vector becomes $\hat{E}^\mu = \mathcal{L}^\mu_\nu(-\hat{v}) E_0^\nu$, with $\hat{E}^\mu \equiv (\hat{\mathbf{p}}, \hat{E}) = (c\gamma_{\hat{v}}\mathcal{M}\hat{v}, 0, 0, \gamma_{\hat{v}} E_0)$, imposing the inertia of energy principle, with linear momentum $\hat{\mathbf{p}} = \gamma_{\hat{v}}\mathcal{M}\hat{\mathbf{v}}$ and energy $\hat{E} = \gamma_{\hat{v}} E_0$.

2.1 Frame S

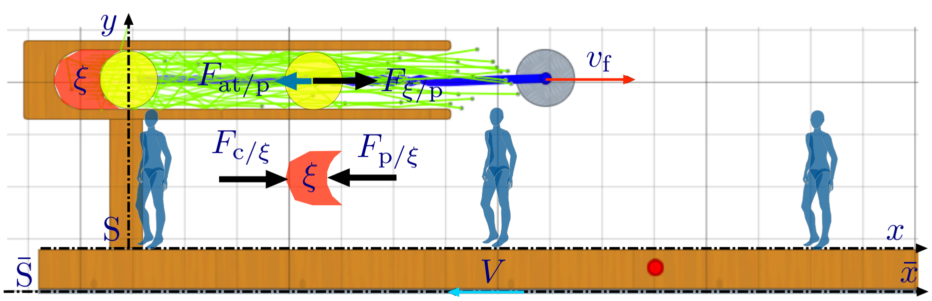

Figure 4: Observer frame S(xy). Cannon at rest and ground moving with speed $-V$ with respect to the wagon. Gases produced by a chemical reaction hit and launch a ball. When leaving the muzzle, the final speeds are $v_{\rm f}$ (frame S) and $\bar{v}_{\rm f}$ (frame $\bar{\rm S}$). On chemicals (ξ), forces $F_{{\rm c}/\xi}$ and $F_{{\rm p}/\xi}$ are applied.

Let us consider how an observer at rest over the wagon (frame S) describes this process. For the observer in S, a projectile with initial speed $v_{\rm I} = 0$, is moved by the products of a chemical reaction [$n_\xi H_2 + (n_\xi/2) O_2 \rightarrow n_\xi H_2O$, for example], which, by striking on it, cause it to reach the muzzle with speed $v_{\rm f}$ after travelling distance $x_{\rm p} \approx L_{\rm cn}$ under a constant (average) force $F_{\xi/{\rm p}}$. The internal energy variation for this chemical reaction is given by:

$$\Delta U_\xi = n_\xi \Delta u_\xi \,, \tag{7a}$$

$$\Delta u_\xi = N_\mathrm{A} \big\{ 2e(\mathrm{O-H}) - [e(\mathrm{H-H}) + \frac{1}{2} e(\mathrm{O-O})] \big\} \,, \tag{7b}$$

where N_A is Avogadro's number and $e(\mathrm{H-H})$, $e(\mathrm{O-O})$, $e(\mathrm{O-H})$ are bonding energies for H−H, O−O and O−H chemical bonds, respectively.

The key point in the thermodynamic description of the process is that not all chemical reaction internal energy variation, $\Delta U_\xi < 0$, can be transformed to mechanical energy (i.e., kinetic energy in this case; if the barrel presents pitch angle $\alpha > 0$ with the ground, the work performed on the projectile to overcome Earth's gravitational attraction must be considered). In the presence of atmosphere, at pressure P, part of the internal energy variation during the chemical reaction must be used to expand the products against that pressure, with enthalpy variation, $\Delta H_\xi = \Delta U_\xi + P\Delta V_\xi$, where $\Delta V_\xi = A x_\mathrm{p} > 0$ is the volume variation due to chemical reaction and A is the barrel's section.

During the chemical reaction, a decrease in entropy $\Delta S_\xi < 0$ takes place. The second law of thermodynamics requires that at least, a minimum amount of heat, $Q^\mathrm{min} = -T\Delta S_\xi$, must be transferred to the thermal reservoir surrounding the system at temperature T, thus increasing its entropy by $\Delta S_\mathrm{F} = -\big|Q^\mathrm{min}\big|/T > 0$, ensuring the entropy-of-the-universe for the process does not decrease, $\Delta S_\mathrm{U} = \Delta S_\xi + \Delta S_\mathrm{F} \geq 0$.

Inertia of the projectile. The internal energy and inertia of the projectile (assumed it is composed of n moles) are given by:

$$E_0(\mathrm{Fe}, T) = \Sigma_\mathrm{j} m_\mathrm{j} c^2 - \left|\tilde{U}_\mathrm{N} + \tilde{U}_\mathrm{A} + \tilde{U}_\mathrm{C}\right| + n\hat{c}_\mathcal{V} T \,, \tag{8a}$$

$$\mathcal{M}_\mathrm{p}(\mathrm{Fe}, T) = c^{-2} E_0 \,, \tag{8b}$$

being $\hat{c}_\mathcal{V}$ its mean $[0, T]$ K molar heat capacity.

Projectile system. The following forces are applied on the projectile:

1. Force $F_{\xi/\mathrm{p}}$, exerted by chemicals; this force exerts momentum and performs work $W_{\xi/\mathrm{p}} = F_{\xi/\mathrm{p}} x_\mathrm{p}$. This force travels length x_p and, for a given chemical reaction, depends on the rate of fuel consumption.

2. Force $F_{\mathrm{at/p}} = PA$, due to atmospheric pressure, exerted by the atmosphere, opposed to the displacement of the projectile. This force exerts momentum and performs work $W_{\mathrm{at/p}} = -PAx_\mathrm{p}$. No friction will be assumed between projectile and the cannon.

As a four-vector fundamental equation in frame S, one has:

$$\mathrm{d}E^{\mu} = \Sigma_{\mathrm{k}} \delta W^{\mu}_{\mathrm{k}} + \delta Q^{\mu} \,, \tag{9}$$

where subindex (k) stands for the set of external forces exerted on the projectile. Focusing on the process sketched in Fig. 4, assuming constant forces, the four-vector fundamental equation for the projectile becomes:

$$E^{\mu}_{\mathrm{p|f}} - E^{\mu}_{\mathrm{p|I}} = W^{\mu}_{\xi/\mathrm{p}} + W^{\mu}_{\mathrm{at/p}} \,. \tag{10}$$

In matrix form:

$$\begin{pmatrix} c\gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}}v_{\mathrm{f}} \\ 0 \\ 0 \\ \gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}}c^2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathcal{M}_{\mathrm{p}}c^2 \end{pmatrix} = \begin{pmatrix} cF_{\xi/\mathrm{p}}t_0 \\ 0 \\ 0 \\ F_{\xi/\mathrm{p}}x_{\mathrm{p}} \end{pmatrix} + \begin{pmatrix} -cPAt_0 \\ 0 \\ 0 \\ -PAx_{\mathrm{p}} \end{pmatrix} . \tag{11}$$

Matching by components in Eq. (11), the following equations are obtained:

Newton's second law. Matching elements of the first row in Eq. (11), with $\mathcal{M}_{\mathrm{p}}\mathrm{d}(\gamma_{v_{\mathrm{f}}}v_{\mathrm{f}}) = (F_{\xi/\mathrm{p}} - PA)\mathrm{d}t$, one has ($v_{\mathrm{I}} = 0$):

$$c\gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}}v_{\mathrm{f}} = c(F_{\xi/\mathrm{p}} - PA)t_0. \tag{12}$$

Newton's second law complementary dynamical relationship (NSL-CDR). By using $v\mathrm{d}(\gamma_v v) = c^2\mathrm{d}(\gamma_v)$, from Newton's second law:

$$(\gamma_{v_{\mathrm{f}}} - 1)\mathcal{M}_{\mathrm{p}}c^2 = (F_{\xi/\mathrm{p}} - PA)x_{\mathrm{p}} \,. \tag{13}$$

First law of thermodynamics. Matching elements of the fourth row in Eq. (11) (temporal components), from $\mathcal{M}_{\mathrm{p}}\mathrm{d}(\gamma_v c^2) = (F_{\xi/\mathrm{p}} - PA)\mathrm{d}x$, again, Eq. (13) is obtained, where the net force, $F_{\mathrm{p}} = (F_{\xi/\mathrm{p}} - PA)$, applied on the projectile is related to its kinetic energy variation. Since the projectile can be considered as a particle (no deformation under pressure has been assumed), NSL-CDR and FLT equations become the same.

Fuel system. The mass of the fuel is taken as negligible (relativistic fuel). In frame S its velocity is considered to be unchanged. The following forces are applied on the fuel:

1. Force $F_{\mathrm{c}/\xi}$, exerted by the base of the barrel (cylinder) on the fuel. This force exerts momentum, and does not perform work $W_{\mathrm{c}/\xi} = F_{\mathrm{c}/\xi}x_{\mathrm{c}} = 0$, since its point of application does not move.

2. Force $F_{\mathrm{p}/\xi}$, exerted by the projectile on the reaction products. This force exerts impulse and performs work, $W_{\mathrm{p}/\xi} = -F_{\mathrm{p}/\xi} x_{\mathrm{p}}$ (see Fig. 4).

Regarding the heat exchanged during the launching, it is considered that only the (minimum) heat $Q^{\min} = T\Delta S_\xi$ must be emitted as thermal photons to the surroundings thermal reservoir, in order to compensate the entropy decrease due to chemical reaction, thus ensuring no decrease of the entropy of the universe. Under this assumption, force $F_{\xi/\mathrm{p}}$ will be maximum, and the projectile will reach the maximum possible speed for the same amount of fuel.

Regarding – by quantum statistical-mechanics considerations – the energy exchanged as heat as an ensemble of pairs of thermal photons, $\mathrm{c} - \tilde{\mathrm{c}}$, emitted in opposite directions,

$$e^\mu_{\mathrm{c}} = \begin{pmatrix} h\nu\cos\theta_{\mathrm{r}} \\ h\nu\sin\theta_{\mathrm{r}} \\ 0 \\ -h\nu \end{pmatrix},\ e^\mu_{\tilde{\mathrm{c}}} = \begin{pmatrix} -h\nu\cos\theta_{\mathrm{r}} \\ -h\nu\sin\theta_{\mathrm{r}} \\ 0 \\ -h\nu \end{pmatrix} \tag{14}$$

one has

$$Q^\mu_\xi = \Sigma_{\mathrm{c}-\tilde{\mathrm{c}}}(e^\mu_{\mathrm{c}} + e^\mu_{\tilde{\mathrm{c}}}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ T\Delta S_\xi \end{pmatrix}, \tag{15}$$

where subindex $\mathrm{c} - \tilde{\mathrm{c}}$ stands for the set of N_{ph} thermal photons emitted as opposed pairs to the surroundings, with $N_{\mathrm{ph}} h\nu \geq T\Delta S_\xi$. Note that although $\|e^\mu_{\mathrm{c}}\| = 0$, $\|Q^\mu_\xi\| = T\Delta S_\xi$ (correlated photons).

The four-vector fundamental equation for the chemical reaction is thus:

$$E^\mu_{\xi|\mathrm{f}} - E^\mu_{\xi|\mathrm{I}} = W^\mu_{\mathrm{c}/\xi} + W^\mu_{\mathrm{p}/\xi} + Q^\mu_\xi\,. \tag{16}$$

In matrix form:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ U_{\xi|\mathrm{f}} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ U_{\xi|\mathrm{I}} \end{pmatrix} = \begin{pmatrix} cF_{\mathrm{c}/\xi}t_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -cF_{\mathrm{p}/\xi}t_0 \\ 0 \\ 0 \\ -F_{\mathrm{p}/\xi}x_{\mathrm{p}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ T\Delta S_\xi \end{pmatrix}. \tag{17}$$

If frictional forces want to be incorporated (e.g., between the projectile and the barrel), a linear-impulse–work 4-vector for the frictional force needs to be incorporated, $W^\mu_{\mathrm{R}} = (c\mathbf{f}_{\mathrm{R}}t_0, 0)$ (note that now frictional force, $\mathbf{f}_{\mathrm{R}}$, does not perform work) and a heat four-vector, $Q^\mu_{\mathrm{R}} = (0, 0, 0, Q_{\mathrm{R}})$, for thermal photons with zero net linear-momentum. In this case, Eq. (16) becomes:

$$E^{\mu}_{\xi|\mathrm{f}} - E^{\mu}_{\xi|\mathrm{I}} = W^{\mu}_{\mathrm{c}/\xi} + W^{\mu}_{\mathrm{p}/\xi} + W^{\mu}_{\mathrm{R}} + Q^{\mu}_{\xi} + Q^{\mu}_{\mathrm{R}}\,,$$

with $Q_{\mathrm{R}} = -f_{\mathrm{R}} x_{\mathrm{p}}$.

Matching by components in Eq. (17), the following equations are obtained:

Newton's second law. Matching elements of the first row in Eq. (17), one has

$$0 = cF_{\mathrm{c}/\xi} t_0 - cF_{\mathrm{p}/\xi} t_0 \longrightarrow F_{\mathrm{c}/\xi} = F_{\mathrm{p}/\xi}\,. \tag{18}$$

The force $F_{\mathrm{c}/\xi}$ exerted by the cannon on the chemicals is equal to the force $F_{\mathrm{p}/\xi}$ exerted by the projectile on the chemicals.

First law of thermodynamics. Matching elements of the fourth row in Eq. (17) (temporal components), one has:

$$U_{\xi|\mathrm{f}} - U_{\xi|\mathrm{I}} = -F_{\mathrm{p}/\xi} x_{\mathrm{p}} + T\Delta S_{\xi}\,. \tag{19}$$

Projectile + fuel system. Adding over the projectile–fuel system [i.e., adding Eqs. (10) and (16)], taking into account that $F_{\xi/\mathrm{p}}$ and $F_{\mathrm{p}/\xi}$ are internal forces of this system (thus disappearing from the description since their contributions to the net impulse and work cancel each other out), one has the four-vector fundamental equation,

$$E^{\mu}_{(\mathrm{p}+\xi)|\mathrm{f}} - E^{\mu}_{(\mathrm{p}+\xi)|\mathrm{I}} = W^{\mu}_{\mathrm{at}/\mathrm{p}} + W^{\mu}_{\mathrm{c}/\xi} + Q^{\mu}_{\xi}\,. \tag{20}$$

In matrix form:

$$\begin{aligned} &\begin{pmatrix} c\gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}} v_{\mathrm{f}} \\ 0 \\ 0 \\ \gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}} c^2 + U_{\xi|\mathrm{f}} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathcal{M}_{\mathrm{p}} c^2 + U_{\xi|\mathrm{I}} \end{pmatrix} = \\ &= \begin{pmatrix} -cPAt_0 \\ 0 \\ 0 \\ -PAx_{\mathrm{p}} \end{pmatrix} + \begin{pmatrix} cF_{\mathrm{c}/\xi} t_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ T\Delta S_{\xi} \end{pmatrix}\,, \end{aligned} \tag{21}$$

where $Ax_{\mathrm{p}} = \Delta V_{\xi}$ and $T\Delta S_{\xi} = Q_{\xi}$.

Matching by components in Eq. (21), the following equations are obtained:

Linear-impulse–linear-momentum variation equation. Matching components of the first row in Eq. (21), one has NSL,

$$\gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}} v_{\mathrm{f}} = (F_{\mathrm{c}/\xi} - PA) t_0 \longrightarrow \gamma_{v_{\mathrm{f}}}\mathcal{M}_{\mathrm{p}} v_{\mathrm{f}} = (F_{\xi/\mathrm{p}} - PA) t_0\,. \tag{22}$$

The corresponding NSL-CDR is thus,

$$(\gamma_{v_\mathrm{f}} - 1)\mathcal{M}_\mathrm{p} c^2 = (F_{\xi/\mathrm{p}} - PA)x_\mathrm{p} = F_{\xi/\mathrm{p}} x_\mathrm{p} - P\Delta V_\xi \,. \tag{23}$$

First law of thermodynamics. Matching components of the fourth row in Eq. (21), one has FLT,

$$\begin{aligned} \gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p} c^2 - \mathcal{M}_\mathrm{p} c^2 + U_{\xi|\mathrm{f}} - U_{\xi|\mathrm{I}} = -P\Delta V_\xi + T\Delta S_\xi \;\; \longrightarrow \\ \longrightarrow \;\; \gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p} c^2 - \mathcal{M}_\mathrm{p} c^2 = -\Delta G_\xi \,, \end{aligned} \tag{24}$$

with $\Delta G_\xi = \Delta U_\xi + P\Delta V_\xi - T\Delta S_\xi$, the Gibbs free energy change of the chemical reaction. From Eq. (24), the following relations are obtained:

$$(\gamma_{v_\mathrm{max}} - 1)\mathcal{M}_\mathrm{p} c^2 = -n_\xi \Delta g_\xi \,, \tag{25a}$$

$$(\gamma_{v_\mathrm{f}} - 1)\mathcal{M}_\mathrm{p} c^2 \leq -n_\xi \Delta g_\xi \,, \tag{25b}$$

$$-\Delta G_\xi = F_{\xi/\mathrm{p}} x_\mathrm{p} - P\Delta V_\xi \,, \tag{25c}$$

$$F_{\xi/\mathrm{p}} x_\mathrm{p} = -(\Delta U_\xi - T\Delta V_\xi) = -\Delta F_\xi \,, \tag{25d}$$

$$\Delta K_\mathrm{p} + P\Delta V_\xi \leq -\Delta F_\xi \,. \tag{25e}$$

Equations (25a) and (25b) arise from the proper definition of the Gibbs free enthalpy potential function variation, which can be used to calculate the maximum amount of mechanical energy obtainable by a thermodynamically closed system at constant temperature and pressure. Equation (25c) can be deduced by combining the definition of the Gibbs potential and the FLT. Equation (25d) can be obtained from Eq. (25c), by introducing the Helmholtz free energy function, $F \equiv U - TS$, which is the thermodynamical potential measuring the useful work obtainable from a closed system at constant temperature (including the atmospheric work), being its change given by $\Delta F_\xi = \Delta U_\xi - T\Delta S_\xi$. It is important to realize that $F_{\xi/\mathrm{p}}$ in Eq. (25d) refers to a force (i.e., the force exerted by the fuel on the projectile) whilst F_ξ refers to chemical reaction Helmholtz free energy function. Equation (25e) reflects the fact that work performed by chemical origin forces (including work performed against the atmospheric pressure), equals the decrease of the chemical reaction Helmholtz free energy function. Finally, Eq. (25a) states that mechanical energy production (i.e., the final kinetic energy of the projectile, plus gravitational potential energy, if any), equals the decrease of the chemical reaction Gibbs free enthalpy function.

2.2 Frame $\bar{\mathrm{S}}$

The equation of the process for an observer in $\bar{\mathrm{S}}$ can be directly obtained by applying the Lorentz transformation,

$$\mathcal{L}^{\mu}_{\nu}(-V) \equiv \begin{pmatrix} \gamma_V & 0 & 0 & \beta_V\gamma_V \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_V\gamma_V & 0 & 0 & \gamma_V \end{pmatrix} , \tag{26}$$

since for the observer in S, frame $\bar{\mathrm{S}}$ moves with velocity $\mathbf{v}_{\bar{\mathrm{S}}|\mathrm{S}} = (-V, 0, 0)$ in the standard configuration.

Projectile system. In frame $\bar{\mathrm{S}}$, the four-vector fundamental equation is obtained by applying the Lorentz transformation, $\mathcal{L}^{\mu}_{\nu}(-V)$, to the corresponding equation in S [Eq. (10)]:

$$\begin{aligned} \mathcal{L}^{\mu}_{\nu}(-V)[E^{\nu}_{\mathrm{p|f}} - E^{\nu}_{\mathrm{p|I}} = W^{\nu}_{\xi/\mathrm{p}} + W^{\nu}_{\mathrm{at/p}}] \rightarrow \\ \rightarrow \bar{E}^{\mu}_{\mathrm{p|f}} - \bar{E}^{\mu}_{\mathrm{p|I}} = \bar{W}^{\mu}_{\xi/\mathrm{p}} + \bar{W}^{\mu}_{\mathrm{at/p}} \,. \end{aligned} \tag{27}$$

In matrix form:

$$\begin{pmatrix} c\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}\bar{v}_\mathrm{f} \\ 0 \\ 0 \\ \gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}c^2 \end{pmatrix} - \begin{pmatrix} c\gamma_V\mathcal{M}_\mathrm{p}V \\ 0 \\ 0 \\ \gamma_V\mathcal{M}_\mathrm{p}c^2 \end{pmatrix} = \begin{pmatrix} cF_{\xi/\mathrm{p}}\bar{t}_0 \\ 0 \\ 0 \\ F_{\xi/\mathrm{p}}\bar{x}_\mathrm{p} \end{pmatrix} + \begin{pmatrix} -cPA\bar{t}_0 \\ 0 \\ 0 \\ -PA\bar{x}_\mathrm{p} \end{pmatrix} , \tag{28}$$

with space-time transformations:

$$\bar{x}_\mathrm{p} = \gamma_V(x_\mathrm{p} + Vt_0) \,, \tag{29a}$$

$$\bar{t}_0 = \gamma_V[t_0 + (V/c^2)x_\mathrm{p}] \,. \tag{29b}$$

From the following relationships:

$$\gamma_{\bar{v}_\mathrm{f}}\bar{v}_\mathrm{f} = \gamma_{v_\mathrm{f}}\gamma_V(v_\mathrm{f} + V) \,, \tag{30a}$$

$$\gamma_{\bar{v}_\mathrm{f}} = \gamma_{v_\mathrm{f}}\gamma_V(1 + v_\mathrm{f}V/c^2) \,, \tag{30b}$$

the velocity transformation

$$\bar{v}_\mathrm{f} = \frac{v_\mathrm{f} + V}{1 + v_\mathrm{f}V/c^2} \,, \tag{31}$$

is obtained. The observer in $\bar{\mathrm{S}}$ describes forces as in S, applied during the time interval $\bar{t}_0$, with displacement $\bar{x}_\mathrm{p}$ of its point of application. The projectile, with initial velocity $\bar{v}_\mathrm{I} = V$, reaches final velocity $\bar{v}_\mathrm{f}$.

This description is equivalent to the description in S since any equation in $\bar{\mathrm{S}}$ can be expressed as a linear combination of equations in S [41, p. 1], which is a consequence of the Lorentz transformation. As an example, for NSL in $\bar{\mathrm{S}}$ one has:

$$\begin{aligned} c\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}\bar{v}_\mathrm{f} - c\gamma_V\mathcal{M}_\mathrm{p}V = c\left(F_{\xi/\mathrm{p}} - PA\right)\bar{t}_0 \rightarrow \\ \rightarrow \gamma_V[c\gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p}v_\mathrm{f} = c\left(F_{\xi/\mathrm{p}} - PA\right)t_0] + \\ + \beta_V\gamma_V[\gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p}c^2 = \left(F_{\xi/\mathrm{p}} - PA\right)x_\mathrm{p}]\,, \end{aligned} \tag{32}$$

as a linear combination of two equations in S. The same is true for the equations in S, each of which can be written as a linear combination of equations in $\bar{\mathrm{S}}$:

$$\mathcal{L}^\mu_\nu(V)[\bar{E}^\nu_{\mathrm{p|f}} - \bar{E}^\nu_{\mathrm{p|I}} = \bar{W}^\nu_{\xi/\mathrm{p}} + \bar{W}^\nu_{\mathrm{at/p}}] \rightarrow E^\mu_{\mathrm{p|f}} - E^\mu_{\mathrm{p|I}} = W^\mu_{\xi/\mathrm{p}} + W^\mu_{\mathrm{at/p}}\,, \tag{33}$$

since observer in $\bar{\mathrm{S}}$ sees frame S moving with velocity $\bar{\mathbf{v}}_{\mathrm{S|\bar{S}}} = (V, 0, 0)$, in the standard configuration:

$$\begin{aligned} c\gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p}v_\mathrm{f} = c\left(F_{\xi/\mathrm{p}} - PA\right)t_0 \rightarrow \\ \rightarrow \gamma_V[c\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}\bar{v}_\mathrm{f} - c\gamma_V\mathcal{M}_\mathrm{p}V = c\left(F_{\xi/\mathrm{p}} - PA\right)\bar{t}_0] - \\ - \beta_V\gamma_V[\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}c^2 - \gamma_V\mathcal{M}_\mathrm{p}c^2 = \left(F_{\xi/\mathrm{p}} - PA\right)\bar{x}_\mathrm{p}]\,. \end{aligned} \tag{34}$$

This transformational symmetry between frames is required by the principle of relativity and is not fulfilled in the Newtonian description, since the equation of NSL is the same in any frame (time is absolute in this description) and cannot be written as a linear combination of equations including the first law of thermodynamics.

Fuel system. The four-vector fundamental equation for the fuel in frame $\bar{\mathrm{S}}$ is obtained by applying the Lorentz transformation to the corresponding equation in S [Eq. (16)]:

$$\begin{aligned} \mathcal{L}^\mu_\nu(-V)[E^\nu_{\xi|\mathrm{f}} - E^\nu_{\xi|\mathrm{I}} = W^\nu_{\mathrm{c}/\xi} + W^\nu_{\mathrm{p}/\xi} + Q^\nu_\xi] \rightarrow \\ \rightarrow \bar{E}^\mu_{\xi|\mathrm{f}} - \bar{E}^\mu_{\xi|\mathrm{I}} = \bar{W}^\mu_{\mathrm{c}/\xi} + \bar{W}^\mu_{\mathrm{p}/\xi} + \bar{Q}^\mu_\xi. \end{aligned} \tag{35}$$

In matrix form:

$$\begin{aligned} &\begin{pmatrix} c\gamma_V(c^{-2}U_{\xi|\mathrm{f}})V \\ 0 \\ 0 \\ \gamma_V U_{\xi|\mathrm{f}} \end{pmatrix} - \begin{pmatrix} c\gamma_V(c^{-2}U_{\xi|\mathrm{I}})V \\ 0 \\ 0 \\ \gamma_V U_{\xi|\mathrm{I}} \end{pmatrix} = \\ &= \begin{pmatrix} cF_{\mathrm{c}/\xi}\gamma_V t_0 \\ 0 \\ 0 \\ F_{\mathrm{c}/\xi}\gamma_V V t_0 \end{pmatrix} + \begin{pmatrix} -cF_{\mathrm{p}/\xi}\bar{t}_0 \\ 0 \\ 0 \\ -F_{\mathrm{p}/\xi}\bar{x}_\mathrm{p} \end{pmatrix} + \begin{pmatrix} c\gamma_V(c^{-2}N_\mathrm{ph}h\nu)V \\ 0 \\ 0 \\ \gamma_V T\Delta S_\xi \end{pmatrix}, \end{aligned} \tag{36}$$

with space-time transformations given by Eqs. (29a) and (29b).

The description of the process in $\bar{\mathrm{S}}$ has the following characteristics:

1. *Inertia of energy.* Although the fuel is considered massless in S – even though it contains energy in their chemical bonds –, in $\bar{\mathrm{S}}$, due to the Lorentz transformation with the resulting inertia of energy principle, the fuel also has inertia, $\mathcal{M}_\xi = c^{-2} U_\xi$, contributing to the linear momentum of the system. Fuel's inertia changes during the process.

2. *Non-simultaneity effect.* Forces applied simultaneously in S, are neither applied simultaneously nor during the same time interval in $\bar{\mathrm{S}}$. As an example, forces $F_{\mathrm{c}/\xi}$ and $\bar{F}_{\mathrm{p}/\xi}$ are applied during time intervals $\bar{t}_{\mathrm{c}/\xi} = \gamma_V t_0$ and $\bar{t}_0$ respectively.

3. *Conveyor belt effect.* Forces not performing work in S have associated work in $\bar{\mathrm{S}}$, performed by the external agent keeping wagon speed constant. As an example, force $F_{\mathrm{c}/\xi}$ does not perform work in S, having associated work $W_{\mathrm{c}/\xi} = F_{\mathrm{c}/\xi} V \bar{t}_{\mathrm{c}/\xi}$ in $\bar{\mathrm{S}}$.

4. *Doppler and aberration effects.* From

$$\bar{Q}^\mu \equiv \begin{pmatrix} c\bar{p}_{\mathrm{ph}} \\ 0 \\ 0 \\ \bar{E}_{\mathrm{ph}} \end{pmatrix} = \begin{pmatrix} c\gamma_V (c^{-2} N_{\mathrm{ph}} h\nu) V \\ 0 \\ 0 \\ \gamma_V T \Delta S_\xi \end{pmatrix} , \qquad (37)$$

photons having zero net linear momentum in S have non-zero linear momentum in $\bar{\mathrm{S}}$, fulfilling the inertia of energy principle. This is explained – by transforming all four-vectors e^μ_{c} and $e^\mu_{\bar{\mathrm{c}}}$ from S to $\bar{\mathrm{S}}$ – by relativistic Doppler and aberration effects. In $\bar{\mathrm{S}}$, photons vary their frequencies and directions with respect to the those in S. Therefore, photons in $\bar{\mathrm{S}}$ transport energy $\bar{E}_{\mathrm{ph}} = \gamma_V^2 Q$ with net linear momentum, $\bar{p}_{\mathrm{ph}} = \gamma_V (c^{-2} N_{\mathrm{ph}} h\nu) V$, corresponding to an amount of inertia, $\mathcal{M}_{\mathrm{ph}} = c^{-2} N_{\mathrm{ph}} h\nu$. This inertia becomes internal energy in S, where its linear momentum is zero. The norm

$$\|\bar{Q}^\mu\| \equiv (\bar{E}^2_{\mathrm{ph}} - c^2 \bar{p}^2_{\mathrm{ph}})^{1/2} = (\gamma_V^2 Q^2 - \beta_V^2 \gamma_V^2 Q^2)^{1/2} = Q \, , \qquad (38)$$

takes into consideration that not all energy, $\bar{E}_{\mathrm{ph}} = \gamma_V Q$, interchanged with the surroundings in $\bar{\mathrm{S}}$, contributes to the entropy change of the universe during the process. Energy $\bar{E}_{\mathrm{ph}}$ has an associated linear momentum, $\bar{p}_{\mathrm{ph}}$, and therefore does not fully contribute to the entropy change of the universe since it is a kind of ordered energy [42]. It is obtained that an entropy variation is a relativistic invariant.

Following a Newtonian description, these effects disappear except the conveyor belt effect.

Projectile + fuel system. The four-vector fundamental equation for the projectile–fuel system in frame $\bar{\mathrm{S}}$ is obtained by applying the Lorentz transformation to the corresponding equation in S [Eq. (20)]:

$$\begin{aligned} \mathcal{L}^{\mu}_{\nu}(-V)[E^{\nu}_{(\mathrm{p}+\xi)|\mathrm{f}} - E^{\nu}_{(\mathrm{p}+\xi)|\mathrm{I}} = W^{\nu}_{\mathrm{at/p}} + W^{\nu}_{\mathrm{c}/\xi} + Q^{\nu}_{\xi}] \rightarrow \\ \rightarrow \bar{E}^{\mu}_{(\mathrm{p}+\xi)|\mathrm{f}} - \bar{E}^{\mu}_{(\mathrm{p}+\xi)|\mathrm{I}} = \bar{W}^{\mu}_{\mathrm{at/p}} + \bar{W}^{\mu}_{\mathrm{c}/\xi} + \bar{Q}^{\mu}_{\xi}\,. \end{aligned} \tag{39}$$

In matrix form:

$$\begin{aligned} &\begin{pmatrix} c\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}\bar{v}_\mathrm{f} + c\gamma_V[c^{-2}U_{\xi|\mathrm{f}}]V \\ 0 \\ 0 \\ \gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}c^2 + \gamma_V U_{\xi|\mathrm{f}} \end{pmatrix} - \begin{pmatrix} c\gamma_V\mathcal{M}_\mathrm{p}c^2 + c\gamma_V[c^{-2}U_{\xi|\mathrm{I}}]V \\ 0 \\ 0 \\ \gamma_V\mathcal{M}_\mathrm{p}c^2 + \gamma_V U_{\xi|\mathrm{I}} \end{pmatrix} = \\ &= \begin{pmatrix} -cPA\bar{t}_\mathrm{p} \\ 0 \\ 0 \\ -PA\bar{x}_\mathrm{p} \end{pmatrix} + \begin{pmatrix} cF_{\mathrm{c}/\xi}\bar{t}_{\mathrm{c}/\xi} \\ 0 \\ 0 \\ F_{\mathrm{c}/\xi}\bar{x}_{\mathrm{c}/\xi} \end{pmatrix} + \begin{pmatrix} c\gamma_V[c^{-2}T\Delta S_\xi]V \\ 0 \\ 0 \\ \gamma_V T\Delta S_\xi\,, \end{pmatrix}\,. \end{aligned} \tag{40}$$

Matching by components in Eq. (40), again, the equations for NSL and FLT are obtained.

For this projectile–fuel system, descriptions in S and $\bar{\mathrm{S}}$ are equivalent since an equation in $\bar{\mathrm{S}}$ can be expressed as a linear combination of equations in S, via the Lorentz transformation. As an example, for the FLT in $\bar{\mathrm{S}}$ [i.e., matching by components elements of the fourth row in Eq. (40)], we have:

$$\begin{aligned} \gamma_{\bar{v}_\mathrm{f}}\mathcal{M}_\mathrm{p}c^2 + \gamma_V U_{\xi|\mathrm{f}} - (\gamma_V\mathcal{M}_\mathrm{p}c^2 + \gamma_V U_{\xi|\mathrm{I}}) = -PA\bar{x}_\mathrm{p} + F_{\mathrm{c}/\xi}\hat{x}_0 + \gamma_V T\Delta S_\xi \rightarrow \\ \rightarrow \beta_V\gamma_V\left[\gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p}v_\mathrm{f} = F_{\xi/\mathrm{p}}t_0 - PAt_0\right] + \\ +\gamma_V\left[\gamma_{v_\mathrm{f}}\mathcal{M}_\mathrm{p}c^2 - \mathcal{M}_\mathrm{p}c^2 + (U_{\xi|\mathrm{f}} - U_{\xi|\mathrm{I}}) = -PAx_\mathrm{p} + T\Delta S_\xi\right]\,, \end{aligned} \tag{41}$$

linear combination of equations in S. Symmetrically, an equation in S is a linear combination of equations in $\bar{\mathrm{S}}$. The equations of the FLT in S and $\bar{\mathrm{S}}$ are given by:

$$(\gamma_{v_\mathrm{f}} - 1)\mathcal{M}c^2 = -n_\xi\Delta g_\xi\,, \tag{42a}$$

$$(\gamma_{\bar{v}_\mathrm{f}} - 1)\mathcal{M}c^2 - (\gamma_{\bar{v}_\mathrm{I}} - 1)\mathcal{M}c^2 = F_{\xi/\mathrm{p}}V\gamma_V t_0 - n_\xi\gamma_V\Delta g_\xi\,. \tag{42b}$$

Fuel-consumption four-vectors in S and $\bar{\mathrm{S}}$ are given by:

$$\Delta G_\xi^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\Delta G_\xi \end{pmatrix} \, , \, \Delta \bar{G}_\xi^\mu = \begin{pmatrix} c\gamma_V \left[c^{-2}\Delta G_\xi\right] V \\ 0 \\ 0 \\ -\gamma_V \Delta G_\xi \end{pmatrix} \, , \qquad (43)$$

having identical norm, $\|\Delta G_\xi^\mu\| = \|\Delta \bar{G}_\xi^\mu\| = n_\xi \Delta g_\xi$. Since fuel-consumption (either in S or in $\bar{\mathrm{S}}$) is defined as the norm of the corresponding four-vector, $\|\Delta G_\xi^\mu\|$ or $\|\Delta \bar{G}_\xi^\mu\|$ – by subtracting from energy $\gamma_V \Delta G_\xi$ the amount not devoted to produce mechanical energy –, it can be concluded that it is a frame-independent magnitude, with $\bar{n}_\xi = n_\xi$ [43].

Let be $u_\mu = c^{-1} V^\mu = (c^{-1} V \gamma_V, 0, 0, \gamma_V)$ the velocity covariant unit four-vector operator. When this operator is applied on the fundamental four-vector equation in $\bar{\mathrm{S}}$, the FLT is obtained in frame S:

$$u_\mu [\bar{E}_\mathrm{f}^\mu - \bar{E}_\mathrm{I}^\mu = \Sigma_\mathrm{k} \bar{W}_\mathrm{k}^\mu + \bar{Q}^\mu] \rightarrow E_\mathrm{f} - E_\mathrm{I} = \Sigma_\mathrm{k} W_\mathrm{k} + Q \, .$$

On the other hand, when the covariant unit four-vector, $\hat{u}_\mu = (\gamma_V, 0, 0, (V/c)\gamma_V)$, is applied, the NSL is then obtained in S [44]:

$$\hat{u}_\mu [\bar{E}_\mathrm{f}^\mu - \bar{E}_\mathrm{I}^\mu = \Sigma_\mathrm{k} \bar{W}_\mathrm{k}^\mu + \bar{Q}^\mu] \rightarrow p_\mathrm{f} - p_\mathrm{I} = \Sigma_\mathrm{k} I_\mathrm{k} \mathrm{d}t \, .$$

These properties are equivalent to:

$$\mathcal{L}_\nu^\mu (-V) [\bar{E}_\mathrm{f}^\mu - \bar{E}_\mathrm{I}^\mu = \Sigma_\mathrm{k} \bar{W}_\mathrm{k}^\mu + \bar{Q}^\mu] \rightarrow E_\mathrm{f}^\mu - E_\mathrm{I}^\mu = \Sigma_\mathrm{k} W_\mathrm{k}^\mu + Q^\mu \, ,$$

since the four-vector equation in S contains both the NSL and the FLT for the process.

The equation for the FLT in S in the classical low velocity limit is given by:

$$\frac{1}{2} \mathcal{M} v_\mathrm{f}^2 = -n_\xi \Delta g_\xi \, . \qquad (44)$$

The kinetic energy of the projectile produced during the process is associated to the decrease in chemical reaction Gibb's free enthalpy function. In $\bar{\mathrm{S}}$, with the wagon moving at constant velocity relative to S, one might think in equation $\mathcal{M}\bar{v}_\mathrm{f}^2/2 - \mathcal{M}\bar{v}_\mathrm{I}^2/2 \equiv -n_\xi \Delta g_\xi$. In fact, the correct equation in $\bar{\mathrm{S}}$ is given by:

$$\frac{1}{2} \mathcal{M} \bar{v}_\mathrm{f}^2 - \frac{1}{2} \mathcal{M} \bar{v}_\mathrm{I}^2 = F_{\xi/\mathrm{p}} V t_0 - n_\xi \Delta g_\xi \, , \qquad (45)$$

thus incorporating the work term of the conveyor belt effect, $W_\mathrm{cb} = F_{\xi/\mathrm{p}} V t_0$. This work has to be done on the conveyor belt, and has to be incorporated during the launching in order to prevent any change in its velocity. Even in Newtonian physics, observers in different inertial frames measure the same fuel-consumption ($\bar{n}_\xi = n_\xi$).

3 Joule-Kelvin process on wagon

Let us consider the relativistic description of the Joule-Kelvin (JK) process [24] depicted in Fig. (5).

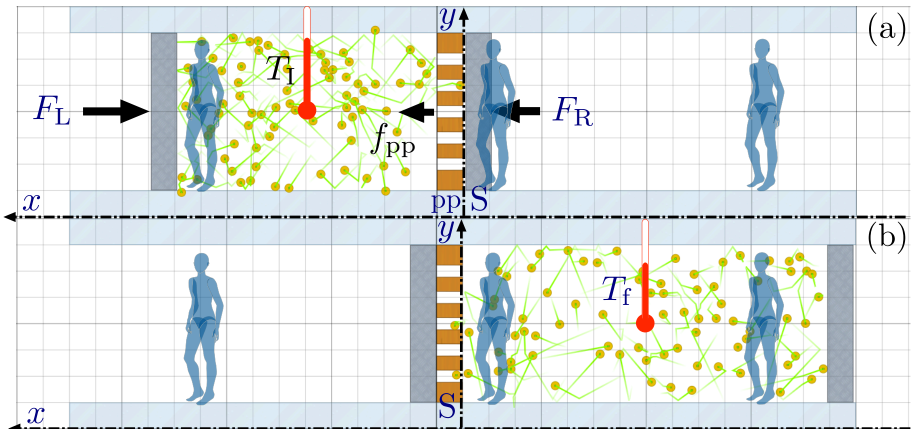

Figure 5: Joule-Kelvin process in Frame S (xy) with porous plug at rest. (a) One mole of gas, enclosed into a cylinder with section A, at temperature T_I, has volume $\mathcal{V}_\mathrm{I} \equiv \mathcal{V}_\mathrm{L}$; pressure P_L is exerted on it. (b) The gas flows through the porous plug, expanding against pressure P_R. In its final state, the gas, at temperature T_f, has volume $\mathcal{V}_\mathrm{f} \equiv \mathcal{V}_\mathrm{R}$.

In thermodynamics, the JK process – or porous plug (pp) process [45] – has been analysed in different inertial frames [26], discussing its relation to the classical principle of relativity. In Ref. [24], equations for a JK process taking place on a moving wagon, frame $\bar{\mathrm{S}}$ (Fig. 1) in standard configuration with respect to frame S (fixed to the wagon), in which the pp is at rest (see Fig. 5), are presented.

Figure 1 sketches a process in which a fluid is enclosed in a cylinder, with a fixed porous plug inside and closed by two pistons, L and R. By applying force F_L on the left piston to the right, exerting pressure $P_\mathrm{L} = F_\mathrm{L}/A$, one mole of fluid is forced to flow through the porous plug, expanding against pressure $P_\mathrm{R} = F_\mathrm{R}/A$. Force F_R is exerted on the right piston to the left, with $P_\mathrm{R} < P_\mathrm{L}$. The process is assumed to evolve adiabatically (i.e., cylinder walls are thermally isolated). When the initial temperature of the fluid, T_I, is appropriate (i.e., below the maximum inversion temperature of the gas, T_in [46, pp. 277-286]), the final temperature will be $T_\mathrm{f} < T_\mathrm{I}$, resulting in its cooling.

For the JK effect to take place, a real (non ideal) gas such as a Van der Waals (VdW) gas must be used since an ideal gas does not present this effect. Therefore, a gas with short range interactions between atoms or molecules and with internal energy, $U(\xi, T, \mathcal{V})$, depending on volume $\mathcal{V}$ will be necessary. For n moles of a given VdW gas (e.g., $\xi \equiv N_2$), the thermal equation of state is given by [47]:

$$P(\xi, T, \mathcal{V}) = \frac{nRT}{\mathcal{V} - nb} - \frac{n^2 a}{\mathcal{V}^2}\,, \tag{46}$$

where the ξ dependence is implicitly given in the VdW gas characteristic constants a and b [48, pp. 55-56]. For simplicity, one mole ($n = 1$) of VdW gas will be assumed. The internal energy of the gas, $E_0(T, \mathcal{V})$, is given by [32]:

$$\begin{aligned} E_0(\xi, T, \mathcal{V}) = \Sigma_{\rm i} m_{\rm i} c^2 - \left|\tilde{U}_{\rm N} + \tilde{U}_{\rm A} + \tilde{U}_{\xi}\right| + \\ + \int_0^T \left(\frac{\partial U}{\partial T}\right)_{\mathcal{V}} \mathrm{d}T + \int_{\mathcal{V}_{\rm r}}^{\mathcal{V}} \left(\frac{\partial U}{\partial \mathcal{V}}\right)_T \mathrm{d}\mathcal{V}\,, \end{aligned} \tag{47}$$

where $\mathcal{M}(\xi, T, \mathcal{V}) = c^{-2} E_0(\xi, T, \mathcal{V})$ is the inertia of the gas (according to Einstein's inertia of energy principle); $m_{\rm i}$ is the mass of the i-th elementary constituent particle (proton, neutron, electron) and $\tilde{U}_{\rm s}$ are binding nuclear (s = N), atomic (s = A) and molecular (s = ξ) energies respectively. For the VdW gas internal energy coefficients, one has:

$$\left(\frac{\partial U}{\partial T}\right)_{\mathcal{V}} = c_{\mathcal{V}}\,, \tag{48a}$$

$$\left(\frac{\partial U}{\partial \mathcal{V}}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_{\mathcal{V}} - P = \frac{a}{\mathcal{V}^2}\,. \tag{48b}$$

Taking as reference volume $\mathcal{V}_{\rm r} \approx \infty$, the internal energy becomes:

$$E_0(\xi, T, \mathcal{V}) = \mathcal{M}(\xi, T, \mathcal{V}) c^2 = \Sigma_{\rm i} m_{\rm i} c^2 - \left|\Sigma_{\rm s} \tilde{U}_{\rm s}\right| + c_{\mathcal{V}} T - \frac{a}{\mathcal{V}}\,. \tag{49}$$

The JK thermodynamic coefficient $\mu_{\rm JK}$, giving the rate of change of temperature with respect to pressure in a constant enthalpy H process [45] is given by:

$$\mu_{\rm JK} \equiv \left(\frac{\partial T}{\partial P}\right)_H = -\frac{(\partial H/\partial P)_T}{(\partial H/\partial T)_P}\,. \tag{50}$$

By approaching $RT/P\mathcal{V} \approx 1$ when convenient, $P(T, \mathcal{V}) \approx (RT/\mathcal{V})[1 + (b/V)] - aP/(\mathcal{V}RT)$. Rearranging, the VdW gas thermal equation of state $\mathcal{V}(T, P)$ can be expressed as:

$$\mathcal{V}(T, P) \approx \frac{RT}{P} + b - \frac{a}{RT}\,. \tag{51}$$

By application of the thermodynamics formalism [49, pp. 165-172] to thermal equation, $\mathcal{V}(T,P)$, with $\mathrm{d}T = \mu_{\mathrm{JK}}\mathrm{d}P$, one has

$$\left(\frac{\partial H}{\partial T}\right)_P = c_P \approx c_{\mathcal{V}} + R\,, \tag{52a}$$

$$\left(\frac{\partial H}{\partial P}\right)_T = -T\left(\frac{\partial \mathcal{V}}{\partial T}\right)_P + \mathcal{V} = b - \frac{2a}{RT}\,, \tag{52b}$$

$$\mathrm{d}T = \frac{b-(2a/RT)}{c_P}\mathrm{d}P \;\longrightarrow\; \mu_{\mathrm{JK}} = \frac{b-(2a/RT)}{c_P}\,, \tag{52c}$$

for the VdW gas approximation.

Frame S. In this frame, forces F_{L}, F_{R} and f_{pp} are simultaneously applied on the fluid, during time interval $[0,t_0]$. Initial and final volumes are $\mathcal{V}_{\mathrm{L}} = AL_{\mathrm{L}}$ and $\mathcal{V}_{\mathrm{R}} = AL_{\mathrm{R}}$ respectively. Forces F_{L} and F_{R} exert linear impulse and perform work on the fluid. The corresponding impulse–work four-vectors are given by $W^{\mu}_{\mathrm{L}} = (cF_{\mathrm{L}}t_0, 0, 0, F_{\mathrm{L}}L_{\mathrm{L}})$ and $W^{\mu}_{\mathrm{R}} = (-cF_{\mathrm{R}}t_0, 0, 0, -F_{\mathrm{R}}L_{\mathrm{R}})$ respectively. Force f_{pp} exerts linear impulse and performs no work on the fluid [26], with impulse–work four-vector $W^{\mu}_{\mathrm{pp}} = (cf_{\mathrm{pp}}t_0, 0, 0, 0)$. The four-vector fundamental equation for the JK process in frame S ($Q = 0$, no heat involved) is thus:

$$E^{\mu}_{\mathrm{f}} - E^{\mu}_{\mathrm{I}} = W^{\mu}_{\mathrm{L}} + W^{\mu}_{\mathrm{R}} + W^{\mu}_{\mathrm{pp}}\,. \tag{53}$$

In matrix form:

$$\begin{aligned}\begin{pmatrix} c\gamma_{v_{\mathrm{f}}}\mathcal{M}(\xi,T_{\mathrm{f}},\mathcal{V}_{\mathrm{R}})v_{\mathrm{f}} \\ 0 \\ 0 \\ \gamma_{v_{\mathrm{f}}}\mathcal{M}(\xi,T_{\mathrm{f}},\mathcal{V}_{\mathrm{R}})c^2 \end{pmatrix} - \begin{pmatrix} c\gamma_{v_{\mathrm{I}}}\mathcal{M}(\xi,T_{\mathrm{I}},\mathcal{V}_{\mathrm{L}})v_{\mathrm{I}} \\ 0 \\ 0 \\ \gamma_{v_{\mathrm{I}}}\mathcal{M}(\xi,T_{\mathrm{I}},\mathcal{V}_{\mathrm{L}})c^2 \end{pmatrix} = \\ = \begin{pmatrix} cP_{\mathrm{L}}At_0 \\ 0 \\ 0 \\ P_{\mathrm{L}}\mathcal{V}_{\mathrm{L}} \end{pmatrix} + \begin{pmatrix} -cP_{\mathrm{R}}At_0 \\ 0 \\ 0 \\ -P_{\mathrm{R}}\mathcal{V}_{\mathrm{R}} \end{pmatrix} + \begin{pmatrix} cf_{\mathrm{pp}}t_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,.\end{aligned} \tag{54}$$

The sign of force f_{pp} will be determined by the process. Matching components of the first row in Eq. (54) the NSL equation for the process is obtained:

$$\gamma_{v_{\mathrm{f}}}\mathcal{M}(\xi,T_{\mathrm{f}},\mathcal{V}_{\mathrm{R}})v_{\mathrm{f}} - \gamma_{v_{\mathrm{I}}}\mathcal{M}(\xi,T_{\mathrm{I}},\mathcal{V}_{\mathrm{L}})v_{\mathrm{I}} = (P_{\mathrm{L}}A - P_{\mathrm{R}}A + f_{\mathrm{pp}})t_0\,. \tag{55}$$

Assuming $v_{\mathrm{f}} \approx v_{\mathrm{I}}$, the expression for f_{pp} is obtained:

$$f_{\mathrm{pp}} \approx -(P_{\mathrm{L}} - P_{\mathrm{R}})A < 0\,. \tag{56}$$

From the enthalpy $H = U + P\mathcal{V}$, with $\mathrm{d}H = T\mathrm{d}S + \mathcal{V}\mathrm{d}P$, the expression for the entropy variation during the JK process is:

$$\mathrm{d}S = \left(\frac{\partial S}{\partial P}\right)_H \mathrm{d}P \longrightarrow \Delta S_\mathrm{U} = \Delta S \approx -\frac{R}{P_\mathrm{L}}(P_\mathrm{R} - P_\mathrm{L}) > 0\,. \qquad (57)$$

The force f_pp exerted on the gas by the porous plug, performing no work, is usually not considered into the descriptions of JK processes.

Matching components of the fourth row in Eq. (54) the FLT equation for the process is obtained, in the low kinetic energy limit ($\gamma_{v_\mathrm{f}} = \gamma_{v_\mathrm{I}} \approx 1$):

$$\mathcal{M}(\xi, T_\mathrm{f}, \mathcal{V}_\mathrm{R})c^2 - \mathcal{M}(\xi, T_\mathrm{I}, \mathcal{V}_\mathrm{L})c^2 \approx P_\mathrm{L}\mathcal{V}_\mathrm{L} - P_\mathrm{R}\mathcal{V}_\mathrm{R}\,, \qquad (58a)$$

$$c_\mathcal{V}(T_\mathrm{f} - T_\mathrm{I}) - a\left(\mathcal{V}_\mathrm{R}^{-1} - \mathcal{V}_\mathrm{L}^{-1}\right) = P_\mathrm{L}\mathcal{V}_\mathrm{L} - P_\mathrm{R}\mathcal{V}_\mathrm{R}\,, \qquad (58b)$$

$$c_\mathcal{V}T_\mathrm{f} - a\mathcal{V}_\mathrm{R}^{-1} + P_\mathrm{R}\mathcal{V}_\mathrm{R} = c_\mathcal{V}T_\mathrm{I} - a\mathcal{V}_\mathrm{L}^{-1} + P_\mathrm{L}\mathcal{V}_\mathrm{L}\,, \qquad (58c)$$

with $T_\mathrm{f} = T_\mathrm{R}$ and $T_\mathrm{I} = T_\mathrm{L}$. From the enthalpy definition:

$$H_\mathrm{L} \equiv U_\mathrm{L} + P_\mathrm{L}\mathcal{V}_\mathrm{L} = U_\mathrm{R} + P_\mathrm{R}\mathcal{V}_\mathrm{R} \equiv H_\mathrm{R}, \qquad (59)$$

Eq. (58c) is obtained again. For this quasi-static approach, enthalpy remains constant during the JK process. From Eq. (46), assuming $\mathcal{V} = RT/P$, $b \ll V$ and $n = 1$, the following relations are found:

$$P\mathcal{V} \approx RT\left(1 + \frac{b}{\mathcal{V}}\right) - \frac{aP}{RT}\,, \qquad (60a)$$

$$c_\mathcal{V}T - \frac{a}{\mathcal{V}} + P\mathcal{V} \approx c_\mathcal{V}T - \frac{aP}{RT} + RT\left(1 + \frac{b}{\mathcal{V}}\right) - \frac{aP}{RT} =$$
$$= (c_\mathcal{V} + R)T + \left(b - \frac{2a}{RT}\right)P\,, \qquad (60b)$$

where a pressure-temperature relationship along the JK process is obtained. For a finite process, with close initial and final temperatures $T_\mathrm{R} \approx T_\mathrm{L}$, from Eq. (59), with $c_P \approx c_V + R$ and Eq. (60b), one has:

$$T_\mathrm{f} - T_\mathrm{I} \approx -\frac{b - (2a/RT_\mathrm{I})}{c_P}(P_\mathrm{R} - P_\mathrm{L})\,. \qquad (61)$$

If the initial temperature T_I is below the maximum inversion temperature of the JK process, T_in (with $T_\mathrm{in} = 2a/Rb$ for a VdW gas [50]), the final temperature of the fluid will be lower than the initial one, $T_\mathrm{f} < T_\mathrm{I}$ (cooling JK process).

Frame $\bar{\mathrm{S}}$. The four-vector fundamental equation in frame $\bar{\mathrm{S}}$ is straightforwardly obtained from the corresponding one in S [Eq. (53)]:

$$\mathcal{L}^\mu_\nu(V)\left[E^\nu_\mathrm{f} - E^\nu_\mathrm{I} = W^\nu_\mathrm{L} + W^\nu_\mathrm{R} + W^\nu_\mathrm{pp}\right] \rightarrow$$
$$\rightarrow \bar{E}^\mu_\mathrm{f} - \bar{E}^\mu_\mathrm{I} = \bar{W}^\mu_\mathrm{L} + \bar{W}^\mu_\mathrm{R} + \bar{W}^\mu_\mathrm{pp}\,. \qquad (62)$$

In matrix form,

$$\begin{pmatrix} c\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}(\xi,T_\mathrm{f},\mathcal{V}_\mathrm{R})\bar{v}_\mathrm{f} \\ 0 \\ 0 \\ \gamma_{\bar{v}_\mathrm{f}}\mathcal{M}(\xi,T_\mathrm{f},\mathcal{V}_\mathrm{R})c^2 \end{pmatrix} - \begin{pmatrix} c\gamma_{\bar{v}_\mathrm{I}}\mathcal{M}(\xi,T_\mathrm{I},\mathcal{V}_\mathrm{L})\bar{v}_\mathrm{I} \\ 0 \\ 0 \\ \gamma_{\bar{v}_\mathrm{I}}\mathcal{M}(\xi,T_\mathrm{I},\mathcal{V}_\mathrm{L})c^2 \end{pmatrix} =$$

$$= \begin{pmatrix} cP_\mathrm{L}A\bar{t}_\mathrm{L} \\ 0 \\ 0 \\ P_\mathrm{L}A\bar{L}_\mathrm{L} \end{pmatrix} + \begin{pmatrix} -cP_\mathrm{R}A\bar{t}_\mathrm{R} \\ 0 \\ 0 \\ -P_\mathrm{R}A\bar{L}_\mathrm{R} \end{pmatrix} + \begin{pmatrix} -cf_\mathrm{pp}\bar{t}_\mathrm{pp} \\ 0 \\ 0 \\ f_\mathrm{pp}V\bar{t}_\mathrm{pp} \end{pmatrix}, \tag{63}$$

with the following force application time lapses:

$$\bar{t}_\mathrm{L} = \gamma_V[t_0 - (V/c^2)L_\mathrm{L}]\,, \tag{64a}$$
$$\bar{t}_\mathrm{R} = \gamma_V[t_0 - (V/c^2)L_\mathrm{R}]\,, \tag{64b}$$
$$\bar{t}_\mathrm{pp} = \gamma_V t_0\,. \tag{64c}$$

Forces are applied in frame $\bar{\mathrm{S}}$ during different time intervals (asynchronous formulation). Displacements for pistons and porous plug in $\bar{\mathrm{S}}$ are given by:

$$\bar{L}_\mathrm{L} = \gamma_V(L_\mathrm{L} - Vt_0)\,, \tag{65a}$$
$$\bar{L}_\mathrm{R} = \gamma_V(L_\mathrm{R} - Vt_0)\,, \tag{65b}$$
$$\bar{L}_\mathrm{pp} = -\gamma_V V t_0\,. \tag{65c}$$

The work performed by the external agent keeping the porous plug in motion, $\bar{W}_\mathrm{pp} = f_\mathrm{pp}V\gamma_V t_0$, is directly obtained through the application of the formalism (conveyor belt effect). Regarding length transformations for L_L and L_R, it is worth mentioning that the length contraction effect is a feature of the synchronous formulation of relativity and therefore it has not been considered [51].

Matching components of the first row in Eq. (63), the NSL equation for the process in frame $\bar{\mathrm{S}}$ is obtained:

$$\begin{aligned} &c\gamma_{\bar{v}_\mathrm{f}}\mathcal{M}(\xi,T_\mathrm{f},\mathcal{V}_\mathrm{R})\bar{v}_\mathrm{f} - c\gamma_{\bar{v}_\mathrm{I}}\mathcal{M}(\xi,T_\mathrm{I},\mathcal{V}_\mathrm{L})\bar{v}_\mathrm{I} = cP_\mathrm{L}A\bar{t}_\mathrm{L} - cP_\mathrm{R}A\bar{t}_\mathrm{R} - cf_\mathrm{pp}\bar{t}_\mathrm{pp}\,, \\ \rightarrow\, &\gamma_V\left[c\gamma_{v_\mathrm{f}}\mathcal{M}(\xi,T_\mathrm{f},\mathcal{V}_\mathrm{R})v_\mathrm{f} - c\gamma_{v_\mathrm{I}}\mathcal{M}(\xi,T_\mathrm{I},\mathcal{V}_\mathrm{L})v_\mathrm{I} = cP_\mathrm{L}At_0 - cP_\mathrm{R}At_0 + cf_\mathrm{pp}t_0\right] - \\ &\beta_V\gamma_V\left[\gamma_{v_\mathrm{f}}\mathcal{M}(\xi,T_\mathrm{f},\mathcal{V}_\mathrm{R})c^2 - \gamma_{v_\mathrm{I}}\mathcal{M}(\xi,T_\mathrm{I},\mathcal{V}_\mathrm{L})c^2 = P_\mathrm{L}AL_\mathrm{L} - P_\mathrm{R}AL_\mathrm{R}\right]. \end{aligned} \tag{66}$$

Einstein's inertia of energy principle allows to write NSL in frame $\bar{\mathrm{S}}$, [Eq. (66)], as a linear combination of the corresponding NSL (times γ_V) and the FLT (times $-\beta_V\gamma_V$) equations describing the JK process in frame S. Under the Newtonian approximation, NSL equations are the same in both S and $\bar{\text{S}}$.

Matching components of the fourth row in Eq. (63), the FLT equation for the JK process in frame $\bar{\text{S}}$ is obtained under the assumption $\bar{v}_{\text{f}} \approx \bar{v}_{\text{I}} \approx 0$:

$$\mathcal{M}(\xi, T_{\text{f}}, \mathcal{V}_{\text{R}})c^2 - \mathcal{M}(\xi, T_{\text{I}}, \mathcal{V}_{\text{L}})c^2 = P_{\text{L}} A \bar{L}_{\text{L}} - P_{\text{R}} A \bar{L}_{\text{R}} + \gamma_V f_{\text{pp}} V t_0 \,, \tag{67a}$$

$$c_V (T_{\text{f}} - T_{\text{I}}) + a \left(\mathcal{V}_{\text{I}}^{-1} - \mathcal{V}_{\text{f}}^{-1} \right) \approx P_{\text{L}} \mathcal{V}_{\text{L}} - P_{\text{R}} \mathcal{V}_{\text{R}} \,. \tag{67b}$$

The same temperature variation is obtained as in S, and the same final temperature T_{f} [23]. This result is obtained by the conveyor belt effect term (work $\bar{W}_{\text{m}} = \gamma_V f_{\text{pp}} V t_0$ performed on the porous plug in $\bar{\text{S}}$). It is not difficult to check that the FLT Eq. (67a) in frame $\bar{\text{S}}$ can be expressed as linear combination of NSL (times $-\beta_V \gamma_V$) and the FLT (times γ_V) equations describing the JK process in frame S.

4 Conclusions

A relativistic formalism, able to correctly describe processes in mechanics or in thermodynamics, taking place on a conveyor belt or moving platform, has been developed. The process is first described in frame S where the conveyor belt is at rest and the corresponding four-vector fundamental equation is obtained, relating initial and final system states in terms of linear-momentum–energy, linear-impulse–work and heat four-vectors. Therefore, conservative, restriction and dissipative forces can be directly distinguished. By applying this formalism, Newton's second law and the first law of thermodynamics arise naturally, and the classical equations are recovered in the low-speed limit. Newton's second law complementary dynamical relationship and thermal effects equation are also obtained. Then, by applying the Lorentz transformation operator onto the previously obtained equation in S, the corresponding one describing the process in frame $\bar{\text{S}}$, with the conveyor belt in motion, is obtained, ensuring its correctness and preserving the principle of relativity.

This relativistic thermodynamics formalism has been applied to two processes: the launching of a projectile subjected to forces produced by chemical reactions inside a cannon (equivalent to a person launching a ball), and the well-known thermodynamical Joule-Kelvin process taking place on a moving platform or conveyor belt.

Problem-solving by using the special theory of relativity, in particular, the Einstein-Minkowski-Lorentz formalism considered in this work, overcomes the classical mechanics-thermodynamics separation and can deal with problems requiring the laws of both disciplines. In relativity, there is no difference between mechanical and thermodynamical processes since there is a mechanical-thermodynamical description of the inertia and internal energy of the system, through which both are related. Moreover, heat is described under a mechanical-statistical way, providing, if any, an associated linear momentum. Newton's second law incorporates the change in linear momentum of non-mechanical energies (previously, only those corresponding to mechanical energies were considered) while the first law of thermodynamics incorporates relativistic effects, including the conveyor belt effect.

For a given process, the relativistic Newton's second law and the first law of thermodynamics are not independent equations. They can be transformed between frames as the components of a space-time four-vector and can be obtained directly through the spatial and temporal components respectively. In frame S, forces are simultaneously applied – during the same time interval – and the thermal reservoir surrounding the system (conveyor belt included) is at rest. Under this asynchronous formulation, any variation in thermal energy of the system contributes to its inertia and linear momentum variation in frame $\bar{\text{S}}$. Heat also contributes to the linear momentum variation of the system (a result explained from Doppler and aberration relativistic effects). Work performed by the external agent keeping the belt moving at a constant speed (conveyor belt effect) is incorporated into the first law of thermodynamics for the process in frame $\bar{\text{S}}$. The principle of relativity is correctly and symmetrically fulfilled.

Under this relativistic context (asynchronous formulation with four-vectors) it does not make sense to ask about relativistic temperature transformation, since it lacks an associated four-vector. There is no T^μ four-vector, the closest one being the linear momentum–energy four-vector E^μ for a set of thermal photons enclosed in a cavity [23]. However, in transient relativistic thermodynamics, an inverse temperature four-vector in a covariant form, $\beta_\mu = T^{-1}u_\mu$, has been postulated, where u_μ is the velocity 4-vector of the system [52]. This four-vector has not been considered in the present work. Since temperature is an intensive quantity, which can be measured with a temperature sensor at rest in S, and transmitted – via wifi, for example – to other inertial observers, it has to be considered, at any given time, as a constant of each thermodynamic system in equilibrium [53, 6] (the same comment is valid por pressure P, a variable on which there was early consensus that it was frame-independent). The heat exchanged is defined as the norm of heat four-vector Q^μ, which is a relativistic invariant, thus entropy variation (of the system, of the surroundings and of the universe) is a relativistic invariant (Planck's original ansatz about relativistic thermodynamics [4]). Furthermore, it is shown that fuel-consumption is also frame-independent.

In the low-speed limit ($v/c \to 0$), classical results are recovered, including the conveyor-belt effect (no Doppler and aberration effects for heat), and Newton's second law is the same in all inertial frames (absolute time).

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