A four-tensor momenta equation for rolling physics

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Abstract

Relativistic four-tensor equation $\mathrm{d}J^{\mu\nu} = M^{\mu\nu}\mathrm{d}t$ is developed to analyse linear translation with rotation processes. The postulated cause-effect four-tensor equation, a relativistic generalisation for classical angular-impulse–angular-momentum variation equation $\mathrm{d}J = M\mathrm{d}t$, includes the Poinsot-Euler rotation (angular-impulse–angular-momentum variation) equation, Newton's second law (linear-impulse–linear-momentum variation equation), and thermodynamics first law (work–energy equation). This four-tensor formalism is applied to describe three linear translation with rotation processes: a ring rolling on the floor by a horizontal force linear impulse and torque, fulfilling the rolling condition (mechanical energy conservation), a spinning ring placed on the ground until achieved the rolling condition (mechanical energy dissipation by friction), and a fireworks wheel ascending an incline (mechanical energy production by decreasing a thermodynamic potential).

1 Introduction

Scientific theories, e.g., Newton's second law, are always subject to revision. They are useful in guiding scientific research until an experiment proves them wrong and new knowledge is gained. Although provisional and premature when first proposed, scientific theories are necessary and must be stated at some point. The scientific approach benefits from the results obtained through these theories [1], and equations developed by earlier physicists should be updated as new knowledge is acquired [2]. In this sense, and according to J A Wheeler, the ultimate goal of physicists would be to express the laws of physics in the language of space and time [3].

In this work, we apply the advances of Einstein's special theory of relativity [4] compared to classical physics to develop a four-tensor momenta equation to analyze rolling – linear translation with rotation (T&R) – processes [5]. With this four-tensor momenta equation formalism we will discuss rolling processes in mechanical energy conservation, mechanical energy dissipation by friction and mechanical energy production by a chemical reaction [6].

Linear translation. Four-vector fundamental equation. A relativistic description of a linear translation process for a rigid enough body, e.g., a block descending an incline,

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can be carried out by four-vector fundamental equation [7]

$$\mathrm{d}E^{\mu} = \delta W^{\mu} + \delta Q^{\mu}\,, \tag{1}$$

a relativistic generalisation of classical first law of thermodynamics $\mathrm{d}U = \delta W + \delta Q$ [8, pp. 1four-5] (U for internal energy, $\delta W = -P\mathrm{d}V$ for work, being work done on the system when $\mathrm{d}V < 0$, and Q for heat). Matrix Eq. (1) includes relativistic Newton's second law and energy equations [9], generalising classical Newton second law (vectors are noted in bold) $\mathrm{d}\mathbf{p} = \mathbf{F}\mathrm{d}t$ – and then its complementary dynamical relationship $\mathrm{d}K_{\mathrm{cm}} = \mathbf{F}\cdot\mathrm{d}\mathbf{x}_{\mathrm{cm}}$ (cm stands for centre-of-mass) [10] – and the energy equation, i.e., generalising thermodynamics first law equation $\mathrm{d}E = \delta W + \delta Q$, with total energy E differential $\mathrm{d}E = \mathrm{d}K_{\mathrm{cm}} + \mathrm{d}U$ [11]; K_{cm} refers to the centre of mass kinetic energy defined as $K_{\mathrm{cm}} = \frac{1}{2}Mv_{\mathrm{cm}}^2$ and $\mathrm{d}U$ refers to thermal internal energy variation, usually $\mathrm{d}U = Mc_V\mathrm{d}T$ with c_V the specific heat at constan volume and T absolute temperature.

Rotation. Poinsot-Euler rotation equation. A pure planar rotation process – i.e., no cm linear translation – relativistic description for a system with axial symmetry, e.g., a ring with moment of inertia I, can be carried out by using a relativistic angular-impulse–angular-momentum variation equation [vectors noted with Greek letters, (angle) θ, (angular velocity) ω, (torque) Γ, for example, shall not be written in bold]

$$I\mathrm{d}[\chi(\omega)\omega] = \Gamma\mathrm{d}t\,, \tag{2}$$

with relativistic rotational kinetic energy variation equation [12]

$$\mathcal{M}\mathrm{d}[\zeta(\omega)c^2] = \Gamma\mathrm{d}\theta\,,$$

being $\chi(\omega)$ – moment of inertia – and $\zeta(\omega)$ – inertia – rotating body characteristic functions (see below). Equation (2) generalises the classical angular-impulse–angular-momentum variation equation, Poinsot-Euler rotation equation $I\mathrm{d}\omega = \Gamma\mathrm{d}t$ – and then its complementary dynamical relationship $\mathrm{d}K_{\mathrm{rt}} = \Gamma\mathrm{d}\theta$, where K_{rt} is classical rotational kinetic energy [13] –.

Rolling processes. Processes of interest and with no straightforward description [14] are those in translation with rotation (T&R in what follows), i.e., rolling processes [15] – (Fig. 1). When a T&R process evolves by mechanical energy dissipation, e.g., by friction or with mechanical energy production, e.g., by a chemical reaction, decreasing its Gibbs free enthalpy function, these thermal effects require the laws of thermodynamics to be involved in their complete description.

For such a rolling process, relativistic requirements demand a covariant description in terms of a four-tensor equation [16],

$$\mathrm{d}J^{\mu\nu} = M^{\mu\nu}\mathrm{d}t\,,$$

simultaneously incorporating Newton's second law for linear translation, the Poinsot-Euler's rotation equation and the first law of thermodynamics. Einstein's inertia of energy principle [9], will be the foundations for achieving such a four-tensor equation together with the two four-vector cross-product generalisation as a four-tensor [12].

A rolling ring process is sketched in Fig. 1. In this paper, we will use rings (2D) as rolling bodies; the underlying physics is essentially the same as for a disk, cylinder or sphere; the ring characteristic functions (see below) are mathematically simpler than those for disks, cylinders or spheres [17].

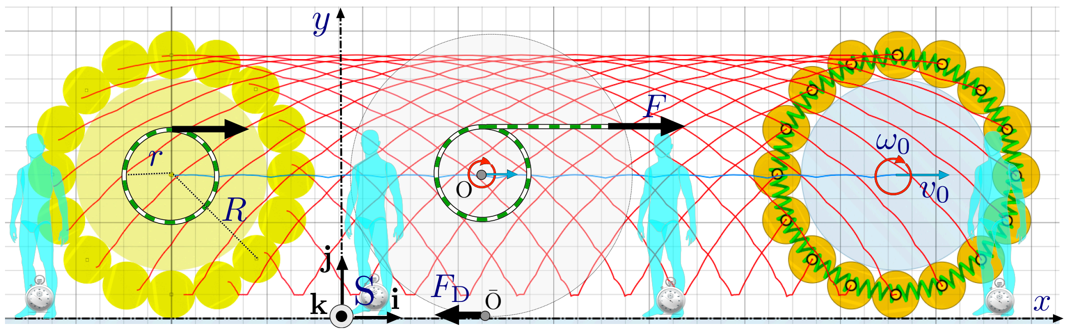

Figure 1: Frame S (x, y, z, t) (at rest relative to the ground). The ring consists of n_r elements (small solid disks). Force F is applied to the end of a rope unwinding from a radius r circle. Force F_D, exerted by the ground on the ring edge, is required by the rolling condition $v = R\omega$. Point O is located at ring's centre. Point Ō is located at ring-ground contact point. The trajectory of a point located at ring's edge is a cycloid.

For mathematical simplicity, in this work only processes in planar rotation will be considered.

Planar rotation. Reference frame (frame in what follows) axes are right-hand oriented, with versors satisfying $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (Fig. 1). The forces are applied in the (xy) plane, the ring elements rolling with linear velocities in plane (xy); ring angular momentum and angular velocity point to z axis; angular momentum z axis direction does not vary during a process.

The ring, composed by solid elements (small disks) joined by elastic springs (see below), rolls due to application of force $\mathbf{F} = (F, 0, 0)$ to the end of a rope previously wrapped around a radius r circumference, centred around the axis perpendicular to the ring passing through point O. Force $\mathbf{F}_\mathrm{D} = (-F_\mathrm{D}, 0, 0)$, exerted by the ground on ring edge, is required to fulfil the rolling condition.

In classical physics, three basic equations are used to solve a T&R exercise: Newton's second law (NSL), the Poinsot-Euler rotation (PER) equation and the first law of thermodynamics (FLT). In general, for a given process, these equations are not mutually independent. Thus, when point O, about which momenta are usually taken, is changed to point Ō (see Fig. 1), the PER equation for point Ō ($I_{\bar{\mathrm{O}}} = I_\mathrm{R} + MR^2$ parallel-axis theorem), given by:

$$I_{\bar{\mathrm{O}}} \mathrm{d}\omega = F(r + R)\mathrm{d}t \,,$$

is linear combination of NSL and the PER equations for point O, i.e.,

$$\begin{aligned} (I_\mathrm{R} + MR^2)\mathrm{d}\omega &= F(r+R)\mathrm{d}t \;\rightarrow && (3)\\ I_\mathrm{R}\mathrm{d}\omega &= (Fr + F_\mathrm{D}R)\mathrm{d}t \;+ && (4)\\ R\big[M\mathrm{d}v_\mathrm{cm} &= (F - F_\mathrm{D})\mathrm{d}t\big]\,, && (5) \end{aligned}$$

[where $\rightarrow$ means that Eq. (3) is sum (+), i.e., linear combination of Eqs. (4) and (5), and square brackets in Eq. (5) means that R affects the whole equation, i.e., it multiplies both sides of the equation; this criteria for rightarrow ($\rightarrow$), plus sign (+) and angular brackets ([]) will be used in what follows], with $\omega R = v_\mathrm{cm}$, where v_cm is ring centre-of-mass linear speed. Also, when frame S, with the ground at rest, is changed by frame $\bar{\mathrm{S}}$, in the standard configuration with velocity $\mathbf{V} = (V, 0, 0)$ to S, the FLT equation in frame $\bar{\mathrm{S}}$, given by

$$\frac{1}{2}M\bar{v}_0^2 - \frac{1}{2}MV^2 + \frac{1}{2}I_\mathrm{R}\omega_0^2 = F(\bar{x}_0 + r\theta_0) + F_\mathrm{D}Vt_0 \,,$$

(this is a mechanical energy conservation equation due thermal effects absence, for constant F and F_D forces) –with conveyor belt effect work term $\bar{W}_\mathrm{D} = F_\mathrm{D} V t_0$ [18]– is linear combination of NSL and the FLT equations in frame S [9], i.e.,

$$\begin{aligned}
\frac{1}{2}M(v_0 - V)^2 - \frac{1}{2}MV^2 + \frac{1}{2}I_\mathrm{R}\omega_0^2 &= F[(x_0 - Vt_0) + r\theta_0] + F_\mathrm{D} V t_0 \;\rightarrow \\
-V\big[Mv_0 &= (F - F_\mathrm{D})t_0\big] \;+ \\
\frac{1}{2}Mv_0^2 + \frac{1}{2}I_\mathrm{R}\omega_0^2 &= F(x_0 + r\theta_0)\,,
\end{aligned}$$

with $\omega_0 R = v_0$, by using Galilean transformations for space ($\bar{x}_0$) and speed ($\bar{v}_0$).

But solving problems in physics is not just a matter of applying formulas. Generally, it is necessary to ensure the problem is well-posed beforehand. Because nature applies forces locally on each body component element, the macroscopic description of a process carried out by an extended, composite body can only be solved when the problem is well posed, i.e., when the many-body problem is reduced to a one-body problem [19].

In addition, the use of a covariant formalism when writing equations is unavoidable for any process, ensuring the equivalence in any inertial frame, regardless of how the process is described. Equivalent here means that every equation in frame S is a linear combination of equations in frame $\bar{\mathrm{S}}$ and vice-versa [20, p. 1], and that these transformations are carried out directly (i.e., by a mathematical algorithm, not by hand). In classical physics, equations are not guaranteed to be covariant, i.e., there is no direct procedure for transforming the equations describing a process from inertial frame S to frame $\bar{\mathrm{S}}$ (such transformations must be done by hand). The four-vector fundamental equation (1) contains NSL and FLT equations and transforms from frame to frame by the Lorentz transformation (see below).

Physics equations relativistic requirements. Classical physics assumes the existence of non-deformable, rigid bodies, with compressibility coefficient $\kappa_S = 0$. The rigid-body concept is helpful in theoretical physics: the body does not deform, there are no internal energy variations, and forces exerted on it are distributed among every body element in such a way they all acquire the same linear speed $v \equiv v_\mathrm{cm}$ (and $\mathrm{d}v_\mathrm{cm}$), coincident with body centre-of-mass speed, and the same angular velocity ω (and $\mathrm{d}\omega$).

In contrast with these classical assumptions, some of then usually not explicit, the relativistic solution for T&R exercises – i.e., rolling process relativistic description – for an extended, composite body, a ring in our case, requires some essential issues to consider.

(i) *No rigid-body assumption.* There is no rigid-body [21]. In a solid with near-zero compressibility coefficient $\kappa_S \approx 0$, sound speed v_s could be larger than light speed c, $v_\mathrm{s} > c$ [22]. Then, in relativity, the ring is described as a chain of solid elements, in our case, disks (2D), joined by elastic, able to deform, massless springs [23]. With the ring in its minimum elastic potential energy configuration, a spring (which can be a quantum oscillator in its fundamental state) does not change its elastic energy during a rigid rolling process (see below).

(ii) *Inertia of energy principle.* Every kind of internal energy contributes to ring internal energy E_0, and to its inertia, given by $\mathcal{M} = c^{-2}E_0$ [9], a quantity that will be involved in both ring linear and angular momenta. This postulate, necessary to build coherent relativistic theories, constitutes the relativistic bridge between classical mechanics and thermodynamics. For a rolling ring, rotational kinetic energy K_rt is internal energy and angular velocity variation $\mathrm{d}\omega$ will contribute to its linear momentum variation $\mathrm{d}p$. This circumstance must be considered when obtaining a rolling ring linear-momentum–energy four-vector $E^\mu(v,\omega)$, with linear speed v while spinning with angular speed ω. This crossover effect between translation and rotation is a non classical physics effect, i. e., characteristic of relativity (see below).

(iii) *Locality.* Relativity is a local theory: a force cannot be described as performing mechanical effects on elements other than those to which it is applied; a force must be located on the particular element where equations want to be obtained in which that force comes into play. When this description is not available – e.g., a half-filled glycerine cylindrical body can roll down an incline [19] or a satellite, subjected to intense tidal forces, orbiting a planet [24, pp. 121-7]–, the problem is ill-posed, and functional equations cannot be obtained.

(iv) *Rigid rolling.* For a process to be considered well-posed, it must be admitted that external, classically identified forces (i.e., macroscopic forces) must be distributed among the ring elements so that their relative distances are not modified, preventing elastic energy from being stored into ring springs; i.e., the ring moves as a whole in rigid translation [25] and rotates around its centre in rigid rotation [12]. This behaviour will be achieved, without considering completely stiff springs, when identical equations can be written for each ring element, all of them varying its linear and angular velocities ($\mathrm{d}v$ and $\mathrm{d}\omega$) by the same amount. Any process with this behaviour is well-posed.

(v) *Functional equations.* A fundamental difficulty in this relativistic context is reducing a many-body problem to a one-body problem. Thus, it must be possible to obtain a functional equation – whose predictions can be tested in an experiment (real or simulated [26]) –, with a single linear translation speed v and a single angular speed ω, as the sum over ring elements equations. Therefore, for well-posed processes, ring elements (and springs) must be identical.

Finally, to comply with the relativistic requirements mentioned above and to preserve the relations between the NSL, PER and FLT equations, it seems necessary to formulate a single equation, which should be formulated in terms of four-tensor momenta, simultaneously incorporating these three equations.

This article is organised as follows. In Sec. 2 the linear-momentum–energy four-vector for a rolling ring is obtained by addition of linear-momentum–energy four-vectors associated to all component elements of the ring. In Sec. 3 relativistic locality is implemented for the classically identified forces distributed on ring elements, obtaining linear-impulse–work four-vectors associated to forces. In Sec. 4 the four-tensor momenta is defined for two four-vectors, and angular-impulse and torque four-tensors are obtained for a ring element and for the whole ring. In Sec. 5 the four-tensor momenta equation for T&R processes with mechanical energy conservation is presented. In Sec. 6 the formalism is applied to a process evolving with mechanical energy conservation. In Sec. 7, the four-tensor momenta for thermal photons is obtained. In Sec. 8, a general four-tensor momenta equation, including thermal effects, is presented. In Sec. 9, a process with mechanical energy dissipation by friction is analysed. In Sec. 10 a process with mechanical energy production by chemical reactions is discussed. In Sec. 11 some conclusions on the theoretical and practical interest of this four-tensor formalism are presented.

2 Rolling ring linear-momentum–energy four-vector

The inertia of energy principle, with ring rotational kinetic energy contributing to its inertia, mixes linear translation and rotation descriptions. Therefore, several steps must be taken to obtain linear-momentum–energy four-vector $E^{\mu}(v,\omega)$ for a rolling ring as the sum over element (c) linear-momentum–energy four-vectors $E^\mu_{\rm c}(v,\omega)$.

Non-rotating ring internal energy and inertia. The ring is considered to consist of $n_{\rm r}$ solid elements joined by elastic (rigid enough) springs. For ring element (s), chemical composition ξ (e.g., n moles of solid Fe), its rest energy $E_{0|{\rm s}}$ and inertia $\mathcal{M}_{\rm s}$ are [27]:

$$\begin{aligned} E_{0|{\rm s}}(\xi) &= \Sigma_{\rm j} m_{\rm j} c^2 - |\tilde{U}_{\rm s}(\xi)| + nc_\xi T \approx \Sigma_{\rm j} m_{\rm j} c^2 - |\tilde{U}_{\rm s}(\xi)| \\ \mathcal{M}_{\rm s} &= c^{-2} E_{0|{\rm s}}\,, \end{aligned}$$

with $m_{\rm j}$ the mass of the component elementary particles (protons, neutrons, electrons); binding energy $\tilde{U}_{\rm s}(\xi) = U_{\rm N} + U_{\rm A} + U_{\rm M} < 0$, which includes nuclear (N), atomic (A) and molecular (M) binding energies [28], with temperature $T \approx 0$ K. A ring element (c) does not vary its temperature for the processes to be considered. Ring elements have the same $\mathcal{M}_{\rm s}$ inertia – homogeneous ring –. Plain inertia $\mathcal{M}$ is defined as $\mathcal{M} = \Sigma_{\rm s=1}^{\rm s=n_r} \mathcal{M}_{\rm s} = n_{\rm r}\mathcal{M}_{\rm s}$ [29].

Element and ring. Linear momentum and energy. When ring element (c) moves, in a frame S, for example, with linear velocity $\mathbf{v}_{\rm c}$, its linear momentum is $\mathbf{p}_{\rm c} = \gamma_{v_{\rm c}} \mathcal{M}_{\rm s} \mathbf{v}_{\rm c}$, where $\gamma_v = (1 - v^2/c^2)^{-1/2}$ (Lorentz factor), and its (total) energy is $E_{\rm c} = \gamma_{v_{\rm c}} \mathcal{M}_{\rm s} c^2$, with kinetic energy given by $K_{\rm c} \equiv E_{\rm c} - E_{0|{\rm c}}(\xi) = (\gamma_{v_{\rm c}} - 1)\mathcal{M}_{\rm s} c^2$.

Ring centre frame $\hat{\rm S}$. Let frame $\hat{\rm S}$ move instantaneously with ring centre [Fig. 2(a)]. All distances between elements remain unchanged. Springs do not store elastic energy. At instant t_0 every element rotates with angular velocity $\omega(t_0)$ and linear speed $v = \omega R$. Rotating element (c) inertia is then $\mathcal{M}_{\rm c}(\omega) = \gamma(\omega R)\mathcal{M}_{\rm s}$, with $\gamma(\omega R) = [1 - (\omega R)^2/c^2]^{-1/2}$ (Lorentz factor) [to avoid confusion with γ_v].

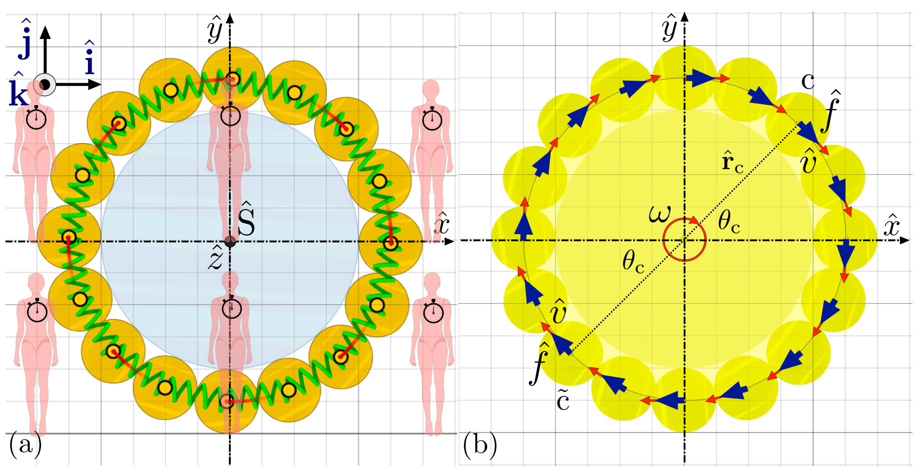

Figure 2: Ring centre (instantaneous) frame $\hat{\rm S}(\hat{x},\hat{y},\hat{z},\hat{t})$. (a) Each element of the ring spins with angular speed ω around its centre. The linear velocity of element (c) is $\hat{\mathbf{v}}_{\rm c} = \hat{\mathbf{r}}_{\rm c} \times \omega$. Ring net linear momentum is $\hat{\mathbf{p}} = \Sigma_{\rm c=1}^{\rm c=n_r} \hat{\mathbf{p}}_{\rm c} = 0$. (b) Distribution of velocities and forces.

An observer (a set of devices located at the nodes of the three-dimensional lattice forming the frame, each located by its coordinates, and equipped, in particular, with synchronised clocks) in $\hat{\rm S}$ describes a rotating ring with angular speed $\omega \equiv \omega(t_0)$. In $\hat{\rm S}$, ring element (c), vector-localised $\hat{\mathbf{r}}_{\rm c} = (R\cos\theta_{\rm c}, R\sin\theta_{\rm c}, 0)$, has linear velocity $\hat{\mathbf{v}}_{\rm c} = (\omega R \sin\theta_{\rm c}, -\omega R\cos\theta_{\rm c}, 0)$, and angular momentum $\hat{\mathbf{J}}_{\rm c} = \mathcal{M}_{\rm s}(\hat{\mathbf{r}}_{\rm c} \times \hat{\mathbf{v}}_{\rm c}) = -\gamma(\omega R)\mathcal{M}_{\rm s} R^2 \omega \hat{\mathbf{k}}$. By adding over all elements, ring angular momentum is $\hat{\mathbf{J}} = \Sigma_{\rm c} \hat{\mathbf{J}}_{\rm c} = -\gamma(\omega R)\mathcal{M} R^2 \omega \hat{\mathbf{k}}$.

Due to relativistic locality requirement, observer $\hat{\mathrm{S}}$ admits that the same force modulus $\hat{f} = |\hat{\mathbf{f}}|$, e.g., $\mathbf{f} = (\hat{f}\sin\theta_\mathrm{c}, -\hat{f}\cos\theta_\mathrm{c}, 0)$ for element (c), is applied on each ring element to justify the measured angular acceleration [Fig. 2(b)].

Ring linear-momentum–energy four-vector in frame $\hat{\mathrm{S}}$. In frame $\hat{\mathrm{S}}$, element (c), located by space-time four-vector [for typographical reasons, a contra-variant four-vector can be written as a row four-vector, keeping a Greek letter as index] $\hat{r}^\mu_\mathrm{c} \equiv (\hat{\mathbf{r}}_\mathrm{c}, c\hat{t})$, with $\mathrm{d}\theta = -\omega\mathrm{d}\hat{t}$, the velocity four-vector $\hat{v}^\mu_\mathrm{c}$ is given by:

$$\hat{r}^\mu_\mathrm{c} \equiv \begin{pmatrix} R\cos\theta_\mathrm{c} \\ R\sin\theta_\mathrm{c} \\ 0 \\ c\hat{t} \end{pmatrix}, \; \hat{v}^\mu_\mathrm{c} \equiv \frac{\mathrm{d}\hat{r}^\mu_\mathrm{c}}{\mathrm{d}\tau} = \begin{pmatrix} \gamma(\omega R)\omega R\sin\theta_\mathrm{c} \\ -\gamma(\omega R)\omega R\cos\theta_\mathrm{c} \\ 0 \\ \gamma(\omega R)c \end{pmatrix},$$

with element (c) proper time $\mathrm{d}\tau = \gamma^{-1}(\omega R)\mathrm{d}\hat{t}$. For symmetry, and to simplify calculations, it is assumed that there is another element ($\tilde{\mathrm{c}}$), opposite to (c) by ring centre (Fig. 2), localised by vector $\hat{\mathbf{r}}_{\tilde{\mathrm{c}}} = (-R\cos\theta_\mathrm{c}, -R\sin\theta_\mathrm{c}, 0)$, with linear velocity $\hat{\mathbf{v}}_{\tilde{\mathrm{c}}} = (-\omega R\sin\theta_\mathrm{c}, \omega R\cos\theta_\mathrm{c}, 0)$. In frame $\hat{\mathrm{S}}$, the linear-momentum–energy four-vectors for elements ($\mathrm{c}\tilde{\mathrm{c}}$) are given by:

$$\hat{E}^\mu_\mathrm{c} = \begin{pmatrix} c\gamma(\omega R)\mathcal{M}_\mathrm{s}\omega R\sin\theta_\mathrm{c} \\ -c\gamma(\omega R)\mathcal{M}_\mathrm{s}\omega R\cos\theta_\mathrm{c} \\ 0 \\ \gamma(\omega R)\mathcal{M}_\mathrm{s}c^2 \end{pmatrix}, \; \hat{E}^\mu_{\tilde{\mathrm{c}}} = \begin{pmatrix} -c\gamma(\omega R)\mathcal{M}_\mathrm{s}\omega R\sin\theta_\mathrm{c} \\ c\gamma(\omega R)\mathcal{M}_\mathrm{s}\omega R\cos\theta_\mathrm{c} \\ 0 \\ \gamma(\omega R)\mathcal{M}_\mathrm{s}c^2 \end{pmatrix}. \tag{6}$$

For element (c), localised by $\hat{\mathbf{r}}_\mathrm{c} = (R\cos\theta_\mathrm{c}, R\sin\theta_\mathrm{c}, 0)$,

$$\begin{aligned} \hat{\mathbf{p}}_\mathrm{c} &= (\gamma(\omega R)\mathcal{M}_\mathrm{s}\sin\theta_\mathrm{c}R\omega, -\gamma(\omega R)\mathcal{M}_\mathrm{s}\cos\theta_\mathrm{c}R\omega, 0)\,, \\ \hat{\mathbf{J}}_\mathrm{c} &= \hat{\mathbf{r}}_\mathrm{c} \times \hat{\mathbf{p}}_\mathrm{c} = -\gamma(\omega R)\mathcal{M}_\mathrm{s}R^2\omega\hat{\mathbf{k}}\,. \\ \hat{E}_\mathrm{c} &= \gamma(\omega R)\mathcal{M}_\mathrm{s}c^2\,. \end{aligned}$$

For element ($\tilde{\mathrm{c}}$), localised by $\hat{\mathbf{r}}_{\tilde{\mathrm{c}}} = -\hat{\mathbf{r}}_\mathrm{c}$,

$$\begin{aligned} \hat{\mathbf{p}}_{\tilde{\mathrm{c}}} &= (-\gamma(\omega R)\mathcal{M}_\mathrm{s}\sin\theta_\mathrm{c}R\omega, \gamma(\omega R)\mathcal{M}_\mathrm{s}\cos\theta_\mathrm{c}R\omega, 0)\,, \\ \hat{\mathbf{J}}_{\tilde{\mathrm{c}}} &= \hat{\mathbf{r}}_{\tilde{\mathrm{c}}} \times \hat{\mathbf{p}}_{\tilde{\mathrm{c}}} = \hat{\mathbf{J}}_\mathrm{c} \\ \hat{E}_{\tilde{\mathrm{c}}} &= \gamma(\omega R)\mathcal{M}_\mathrm{s}c^2 = \hat{E}_\mathrm{c}\,. \end{aligned}$$

Rotating ring characteristics functions. The inertia $\mathcal{M}(\omega)$ and moment of inertia $I(\omega)$ of rotating bodies are greater than those of non-rotating bodies (inertia of energy principle).

A body at rest as a whole (zero linear and angular velocity) has inertia $\mathcal{M} = c^{-2}E(0)$, where $E(0)$ is its internal energy. Inertia $\mathcal{M}(\omega)$ and moment of inertia $I_\mathrm{B}(\omega)$ functions for axial symmetry bodies composed by n_r elements, rotating with angular velocity ω, are respectively given by [30]:

$$\mathcal{M}(\omega) = \Sigma_{\mathrm{c}=1}^{\mathrm{c}=\mathrm{n_r}}\mathcal{M}_\mathrm{s}\gamma_{v_\mathrm{c}} = \mathcal{M}\zeta_\mathrm{B}(\omega) = c^{-2}E(\omega)\,, \tag{7}$$

$$I_\mathrm{B}(\omega) = \Sigma_{\mathrm{c}=1}^{\mathrm{c}=\mathrm{n_r}}\mathcal{M}_\mathrm{s}r_\mathrm{c}^2\gamma_{v_\mathrm{c}} = I_\mathrm{B}\chi_\mathrm{B}(\omega)\,, \tag{8}$$

with rest moment of inertia I_B, and plain inertia $\mathcal{M} = n_\mathrm{r}\mathcal{M}_\mathrm{s}$, being $\chi_\mathrm{B}(\omega)$ and $\zeta_\mathrm{B}(\omega)$ body characteristic functions [17].

For a ring, with non-rotating moment of inertia $I_{\rm R} = \mathcal{M}R^2$, its characteristic functions are

$$\chi_{\rm R}(\omega) = \gamma(\omega R)\,,\, \zeta_{\rm R}(\omega) = \gamma(\omega R)\,.$$

Ring linear-momentum–energy four-vector $\hat{E}^\mu$ in frame $\hat{\rm S}$ is obtained as a sum over element pairs (c$\tilde{\rm c}$):

$$\hat{E}^\mu(\omega) = \Sigma_{\rm c\tilde{c}}(\hat{E}^\mu_{\rm c} + \hat{E}^\mu_{\tilde{\rm c}}) = \left(0, 0, 0, \gamma(\omega R)\mathcal{M}c^2\right)\,. \quad (9)$$

Four-vector $\hat{E}^\mu(\omega)$ describes a homogeneous ring, its centre at rest, $\hat{\mathbf{p}} = 0$, with internal energy $\hat{E} = \mathcal{M}(\omega)c^2$, with $\mathcal{M}(\omega) \equiv \mathcal{M}\zeta_{\rm R}(\omega)$: rotational kinetic energy contributes to ring internal energy and inertia. For the ring in $\hat{\rm S}$ [31] :

$$\begin{aligned}
\hat{\mathbf{p}} &= \Sigma_{\rm c\tilde{c}}(\hat{\mathbf{p}}_{\rm c} + \hat{\mathbf{p}}_{\tilde{\rm c}}) = 0\,,\\
\hat{\mathbf{J}} &= \Sigma_{\rm c\tilde{c}}(\hat{\mathbf{J}}_{\rm c} + \hat{\mathbf{J}}_{\tilde{\rm c}}) = -\gamma(\omega R)\mathcal{M}R^2\omega\mathbf{k}\,.\\
\hat{E}(\omega) &= \gamma(\omega R)\mathcal{M}c^2\,,\\
\hat{K}_{\rm rt} &= [\gamma(\omega R) - 1]\mathcal{M}c^2\,.
\end{aligned}$$

Rotational kinetic energy $\hat{K}_{\rm rt}$ is ring internal energy. By the inertia of energy principle, this energy contributes to its inertia. Therefore, for a rolling ring, $\hat{K}_{\rm rt}$ contributes to its linear momentum.

The Lorentz transformation. The four-tensor $\mathcal{L}^\mu_\nu(V)$ for the Lorentz transformation between frames in the standard configuration with velocity $\mathbf{V} = (V, 0, 0)$ is:

$$\mathcal{L}^\mu_\nu(V) \equiv \begin{pmatrix} \gamma_V & 0 & 0 & -\beta_V\gamma_V \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_V\gamma_V & 0 & 0 & \gamma_V \end{pmatrix} \quad (10)$$

being $\beta_V = V/c$, $\gamma_V = [1 - \beta_V^2]^{-1/2}$.

The transformation of a four-vector $\hat{A}^\mu$ from frame $\hat{\rm S}$ to frame S – floor at rest –, moving with velocity $\hat{\mathbf{V}} = (-v, 0, 0)$ relative to $\hat{\rm S}$, is given by [32]: $A^\mu = \mathcal{L}^\mu_\nu(-v)\hat{A}^\mu$.

Ring linear-momentum–energy four-vector in frame S**.** In frame S, rest floor frame, moving with (instantaneous) velocity $\hat{\mathbf{V}} = (-v, 0, 0)$ relative to $\hat{\rm S}$ (Fig. 3), the linear-momentum–energy four-vectors $E^\mu_{\rm c}$ and $E^\mu_{\tilde{\rm c}}$ for elements (c$\tilde{\rm c}$) are given by:

$$E^\mu_{\rm c}(\omega, v) = \mathcal{L}^\mu_\nu(-v)\hat{E}^\nu_{\rm c} \approx \begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}[v + \omega R\sin\theta_{\rm c}] \\ -c\gamma(\omega R)\mathcal{M}_{\rm s}[\omega R\cos\theta_{\rm c}] \\ 0 \\ \gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}c^2 \end{pmatrix}\,, \quad (11)$$

$$E^\mu_{\tilde{\rm c}}(\omega, v) = \mathcal{L}^\mu_\nu(-v)\hat{E}^\nu_{\tilde{\rm c}} \approx \begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}(v - \omega R\sin\theta_{\rm c}) \\ c\gamma(\omega R)\mathcal{M}_{\rm s}[\omega R\cos\theta_{\rm c}] \\ 0 \\ \gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}c^2 \end{pmatrix}\,. \quad (12)$$

Order c^{-2} terms (cross relativistic translation-rotation effect by the inertia of inertia principle) have been neglected in Eqs. (11)-(12) (i.e., by assuming $(1-v^2/c^2)^{-1}[1-(\omega R)^2/c^2]^{-1} \approx [1 - (v + \omega R)^2/c^2]^{-1}$).

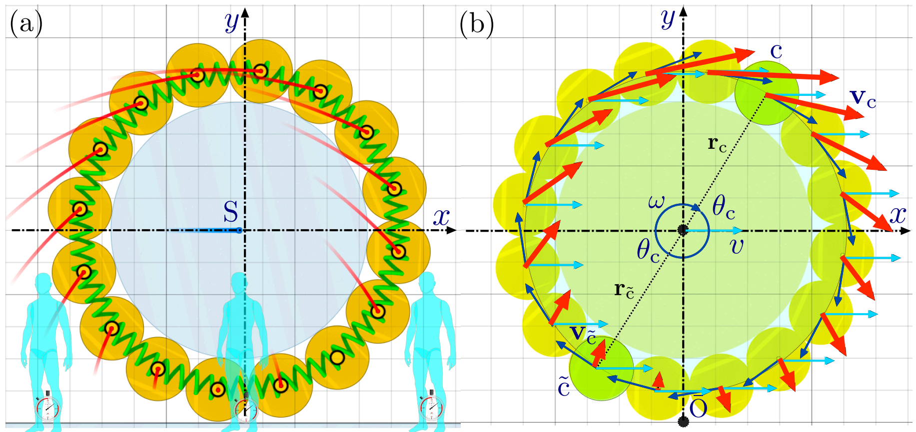

Figure 3: (a) Rolling ring in frame S(x, y, z, t) with floor at rest. (b) Velocity distribution on elements by velocities composition. The contact ring-floor contact point $\bar{\text{O}}$ has zero speed. At time $t = 0$ origin of frames S and $\hat{\text{S}}$ are coincident.

For element (c), with $\mathbf{r}_\text{c} = \hat{\mathbf{r}}_\text{c}$,

$$\begin{aligned}
\mathbf{p}_\text{c} &= (\gamma(\omega R)\mathcal{M}_\text{s}(v + \sin\theta_\text{c} R\omega), -\gamma(\omega R)\mathcal{M}_\text{s}\cos\theta_\text{c} R\omega, 0)\,, \\
\mathbf{J}_\text{c} &= \mathbf{r}_\text{c} \times \mathbf{p}_\text{c}\,, \\
E_\text{c}(\omega, v) &= \gamma_v \gamma(\omega R)\mathcal{M}_\text{s} c^2\,.
\end{aligned}$$

For element ($\tilde{\text{c}}$), with $\mathbf{r}_{\tilde{\text{c}}} = -\mathbf{r}_\text{c}$,

$$\begin{aligned}
\mathbf{p}_{\tilde{\text{c}}} &= (\gamma(\omega R)\mathcal{M}_\text{s}[v - \sin\theta_\text{c} R\omega], \gamma(\omega R)\mathcal{M}_\text{s}\cos\theta_\text{c} R\omega, 0)\,, \\
\mathbf{J}_{\tilde{\text{c}}} &= \mathbf{r}_{\tilde{\text{c}}} \times \mathbf{p}_{\tilde{\text{c}}}\,, \\
E_{\tilde{\text{c}}}(\omega, v) &= \gamma_v \gamma(\omega R)\mathcal{M}_\text{s} c^2\,.
\end{aligned}$$

The ring linear-momentum–energy four-vector E^μ in frame S, in which the ring is in T&R, is obtained by adding over pairs (c$\tilde{\text{c}}$):

$$\begin{aligned}
E^\mu(\omega, v) &= \Sigma_{\text{c}\tilde{\text{c}}}(E^\mu_\text{c} + E^\mu_{\tilde{\text{c}}}) = \begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}v \\ 0 \\ 0 \\ \gamma_v\gamma(\omega R)\mathcal{M}c^2 \end{pmatrix}\,, \\
E^\mu(\omega, v) &= \mathcal{L}^\mu_\nu(-v)\hat{E}^\nu(\omega)\,,
\end{aligned} \tag{13}$$

or by apply on $\hat{E}^\nu(\omega)$ the Lorentz transformation: the formalism has coherence in the rigid rolling (T&R) consideration. By the inertia of energy principle, there are translation-rotation cross-effects that do not appear in classical physics.

Four-vector $E^\mu(\omega, v)$ describes [33]:

(i) a rotating ring with inertia (in rotation) $\mathcal{M}(\omega) = \gamma(\omega R)\mathcal{M}$ moving its centre with linear velocity $\mathbf{v} = (v, 0, 0)$, linear momentum $\mathbf{p}$, total energy E, and kinetic energy K, given by:

$$\begin{aligned}
\mathbf{p} &= \Sigma_{\mathrm{c\tilde{c}}}(\mathbf{p}_{\mathrm{c}} + \mathbf{p}_{\tilde{\mathrm{c}}}) = (\gamma_v \gamma(\omega R)\mathcal{M}v, 0, 0)\,, \\
E &= \gamma_v \gamma(\omega R)\mathcal{M}c^2 = \gamma_v \mathcal{M}(\omega)c^2\,, \\
\mathcal{M}(v,\omega) &= \gamma_v \gamma(\omega R)\mathcal{M}\,, \\
K &= [\gamma_v \gamma(\omega R) - 1]\mathcal{M}c^2\,,
\end{aligned}$$

(ii) or a moving ring with inertia $\mathcal{M}(v) = \gamma_v \mathcal{M}$ spinning with angular velocity ω around its centre, with angular momentum $\mathbf{J}$:

$$\mathbf{J} = \Sigma_{\mathrm{c\tilde{c}}}(\mathbf{J}_{\mathrm{c}} + \mathbf{J}_{\tilde{\mathrm{c}}}) = -\gamma_v \gamma(\omega R)\mathcal{M}R^2\omega\mathbf{k}\,. \tag{14}$$

Low speed limit. In the low-speed limit $v/c \to 0$, or $\omega R/c \to 0$

$$\lim_{v/c\to 0} \gamma_v = 1\,,\ \lim_{v/c\to 0} (\gamma_v - 1)c^2 = \frac{1}{2}v^2\,, \tag{15}$$

$$\lim_{\omega R/c\to 0} \gamma(\omega R) = 1\,,\ \lim_{\omega R/c\to 0} [\gamma(\omega R) - 1]c^2 = \frac{1}{2}R^2\omega^2\,, \tag{16}$$

one has $[\mathcal{M}(\omega) = \mathcal{M}(v) \approx \mathcal{M}]$, $I_{\mathrm{R}} = \mathcal{M}R^2$,

$$\lim_{v\to 0,\omega\to 0} (\gamma_v - 1)\mathcal{M}(\omega)c^2 + [\gamma(\omega R) - 1]\mathcal{M}(v)c^2 = \frac{1}{2}\mathcal{M}v^2 + \frac{1}{2}I_{\mathrm{R}}\omega^2\,,$$

for a rolling ring total kinetic energy.

Ring centre-of-inertia velocity. For a homogeneous ring with n_{r} elements (c), with linear velocity $\mathbf{v}_{\mathrm{c}}$, its centre-of-inertia velocity $\mathbf{v}_{\mathrm{ci}}$ is defined as [34]:

$$\mathbf{v}_{\mathrm{ci}} \equiv \frac{\Sigma_{\mathrm{c=1}}^{\mathrm{c=n_r}} \gamma_{v_{\mathrm{c}}} \mathcal{M}_{\mathrm{s}} \mathbf{v}_{\mathrm{c}}}{\Sigma_{\mathrm{c=1}}^{\mathrm{c=n_r}} \gamma_{v_{\mathrm{c}}} \mathcal{M}_{\mathrm{s}}} = \frac{\mathbf{p}}{c^{-2}E}\,. \tag{17}$$

Then, $\mathbf{v}_{\mathrm{c}} = \mathbf{v}_{\mathrm{ci}} + \mathbf{r}_{\mathrm{c}} \times \omega$, and with $\gamma_{v_{\mathrm{c}}} \approx \gamma_{v_{\mathrm{ci}}}\gamma(\omega R)$, ring linear momentum and energy are given by [35]:

$$\begin{aligned}
\mathbf{p} &= \gamma_{v_{\mathrm{ci}}} \mathcal{M}(\omega) v_{\mathrm{ci}}\,, \\
E &= \gamma_{v_{\mathrm{ci}}} \mathcal{M}(\omega) c^2\,.
\end{aligned}$$

Then, $\mathcal{M}^2(\omega)c^4 = E^2 - c^2(\mathbf{p}\cdot\mathbf{p})$ [36].

Speed v entering into the ring-as-a-whole linear translation equations (see below) is the ring centre-of-inertia speed, $v \equiv v_{\mathrm{ci}}$; then $\mathrm{d}x_{\mathrm{ci}} = v_{\mathrm{ci}}\mathrm{d}t$. In operational equations, system centre-of-inertia x_{ci} is its geometric centre.

3 Force distribution on ring elements

Each force must be identified as being applied on a ring element due to relativistic locality requirement; i.e., any force must be located on the ring element on which it is applied, avoiding instantaneous action at distance descriptions; the effects – on linear momentum, angular momentum, and energy variations – of any force applied at a given point, are not allowed to be noticed, or exerted, at different points [37].

For force $\mathbf{f}_{\mathrm{k|c}} = (f_{x|\mathrm{k;c}}, f_{y|\mathrm{k;c}}, 0)$, applied on element (c), the following elements must be determined:

(i) time interval $[t_0, t_0 + \mathrm{d}t]$, during which the force is applied,

(ii) element (c) on which the force is exerted, located by vector $\mathbf{r}_{\mathrm{c}} = (x_{\mathrm{c}}, y_{\mathrm{c}}, 0)$, and four-vector r^{μ}_{c},

(iii) force application point velocity $\mathbf{u}_{\mathrm{k|c}} = (u_{x|\mathrm{k;c}}, u_{y|\mathrm{k;c}}, 0)$ (this velocity may be different from that of the element onto which it is applied).

Then, the following quantities can be determined:

(i) Linear impulse, $\delta\mathbf{I}_{\mathrm{k|c}} = \mathbf{f}_{\mathrm{k|c}}\mathrm{d}t$ – every force exerts linear impulse (time always runs) –.

(ii) Angular impulse, $\mathrm{d}\mathbf{M}_{\mathrm{k|c}} = (\mathbf{r}_{\mathrm{c}} \times \mathbf{f}_{\mathrm{k|c}})\mathrm{d}t$ – some forces do not exert angular impulse–.

(iii) Work performed, $\delta W_{\mathrm{k|c}} = (\mathbf{f}_{\mathrm{k|c}} \cdot \mathbf{u}_{\mathrm{k|c}})\mathrm{d}t$, $\delta W_{\mathrm{k|c}} = \mathbf{f}_{\mathrm{k|c}} \cdot \mathrm{d}\mathbf{x}|\mathbf{c}_{\mathrm{k}}$, with $\mathrm{d}\mathbf{x}_{\mathrm{k|c}} = \mathbf{u}_{\mathrm{k|c}}\mathrm{d}t$ – some forces do not perform work–.

The macroscopic forces that can be identified as being applied on the ring (e.g., classically, e.g., $\mathbf{F}$ and $\mathbf{F}_{\mathrm{D}}$ in Fig. 1), must be distributed among the homogeneous ring elements (Fig. 4), and must have the following characteristics: (i) on each element, the same set of distributed forces must be applied, (ii) the resultant of the distributed forces must be the resultant of macroscopic forces; every force exerts linear impulse; (iii) the resultant of the distributed torques must be the resultant torque of macroscopic forces; some forces do not exert torque; (iv) the resultant work performed by distributed forces must be the work performed by the classical macroscopic forces; some forces do not perform work. Forces are simultaneously applied in frame S during time interval $[t_0, t_0 + \mathrm{d}t]$.

Force linear-impulse-work four-vector. For force $\mathbf{f}_{\mathrm{k|c}} = (f_{x\mathrm{k;c}}, f_{y\mathrm{k;c}}, 0)$, whose application point displaces with velocity $\mathbf{u}_{\mathrm{k|c}} = (u_{x\mathrm{k|c}}, u_{y\mathrm{k|c}}, 0)$, one has, respectively, the Minkowski's force-power and linear-impulse–work four-vectors:

$$f^{\mu}_{\mathrm{k|c}} = \begin{pmatrix} \gamma_v f_{x|\mathrm{k;c}} \\ \gamma_v f_{y|\mathrm{k;c}} \\ 0 \\ c^{-1}\gamma_v(\mathbf{f}_{\mathrm{k|c}} \cdot \mathbf{u}_{\mathrm{k|c}}) \end{pmatrix}, \; \delta W^{\mu}_{\mathrm{k|c}} \equiv c f^{\mu}_{\mathrm{k|c}} \mathrm{d}\tau_{\mathrm{k|c}} = \begin{pmatrix} c f_{x|\mathrm{k;c}}\mathrm{d}t \\ c f_{y|\mathrm{k;c}}\mathrm{d}t \\ 0 \\ (\mathbf{f}_{\mathrm{k|c}} \cdot \mathbf{u}_{\mathrm{k|c}})\mathrm{d}t \end{pmatrix},$$

with force application point differential proper time, $\mathrm{d}\tau_{\mathrm{k|c}} = \gamma^{-1}_{u_{\mathrm{k|c}}}\mathrm{d}t$, being $\mathrm{d}t$ the time interval measured by the ensemble of clocks synchronized in frame S.

Forces distribution on ring elements. It must be possible to hypothesise how (classical, macroscopic) forces are distributed and exerted on the ring elements to achieve rigid rolling conditions and to pose equations for each ring element. This demand means one must propose a model about forces applied to each element according to previous requirements. In addition to physical intuition, the observed distribution of velocities by elements should greatly help in this matter.

We are going to use forces $\mathbf{F} = (F, 0, 0)$ and $\mathbf{F}_{\mathrm{D}} = (-F_{\mathrm{D}}, 0, 0)$ in Fig. 1 to explain in detail how the distribution of forces by elements is carried out. Forces $\mathbf{G} + \mathbf{N} = 0$ will not be considered. Macroscopic and distributed forces are given in Fig. 4(a) and (b) respectively.

1. Force $\mathbf{F}$ is assumed exerts linear impulse $\delta\mathbf{I} = \mathbf{F}\mathrm{d}t$, produces angular impulse $\delta\mathbf{M} = (\mathbf{r} \times \mathbf{F})\mathrm{d}t$, with lever arm $\mathbf{r} = (r, 0, 0)$, and performs work $\delta W_F = (\mathbf{F} \cdot \mathbf{u}_F)\mathrm{d}t$, where $\mathbf{u}_F$ is the force application point velocity, $\mathbf{u}_F = \mathbf{v}_{\mathrm{ci}} + \mathbf{r} \times \omega$, with $\mathbf{v}_{\mathrm{ci}}$ the velocity of ring geometric centre and ω the angular velocity [38]. Force $\mathbf{F}$ will enter into Newton's second law equation for linear translation ($\mathbf{v}_{\mathrm{ci}}$), the Poinsot-Euler equation for rotation (ω), and the first law of thermodynamics equation.

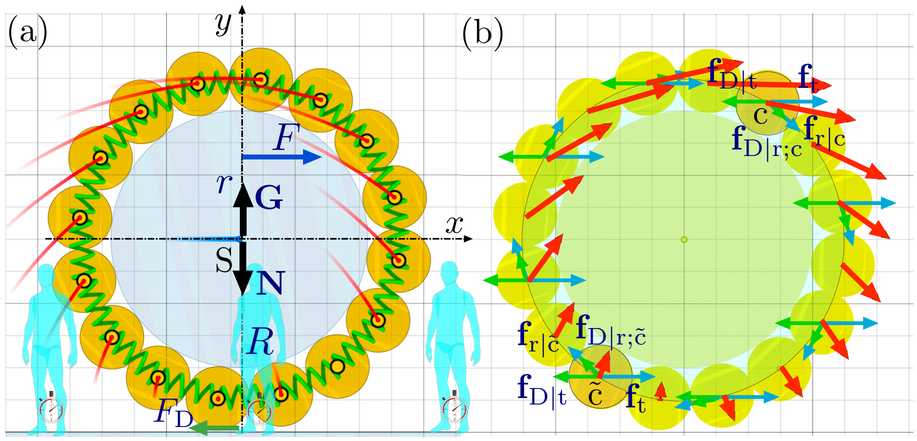

Figure 4: (a) Macroscopic forces. (b) Each force is distributed among ring elements so that net resultant force, net torque and work, are identical to those corresponding to macroscopic forces F and F_{D}. As $\mathbf{G}+\mathbf{N}=0$, gravitational and normal forces are not considered.

Force $\mathbf{F}$ will give rise to two forces, $\mathbf{f}$ on translation, the same for any element, and $\mathbf{f}_{\mathrm{r|c}}$, on rotation, different for different elements (c); force $\mathbf{f}$, contributes to linear translation, and force $\mathbf{f}_{\mathrm{r|c}}$, contributes to rotation; both forces perform work, contributing to generic element (c) total kinetic energy variation.

1A. Linear-impulse–work four-vector $\delta W_{\mathrm{t}}^{\mu}$ for force $\mathbf{f}=(n_{\mathrm{r}}^{-1}F,0,0)$, modulus $\left|\mathbf{f}\right|\equiv f=n_{\mathrm{r}}^{-1}F$, is:

$$\delta W_{\mathrm{t}}^{\mu}=\begin{pmatrix} c\mathbf{f}\mathrm{d}t \\ \delta W_{\mathrm{t}}\end{pmatrix};\ \delta W_{\mathrm{t}}^{\mu}\equiv \dot{W}_{\mathrm{t}}^{\mu}\mathrm{d}t=\begin{pmatrix} cf \\ 0 \\ 0 \\ fv\end{pmatrix}\mathrm{d}t\,, \tag{18}$$

with $\delta W_{\mathrm{t}}^{\mu}=\dot{W}_{\mathrm{t}}^{\mu}\mathrm{d}t$, since its application point moves with ring centre. Each ring element has the same four-vector $\delta W_{\mathrm{t}}^{\mu}$.

1B. Forces $\mathbf{f}_{\mathrm{r|c}}=(|\mathbf{f}_{\mathrm{r|c}}|\sin\theta_{\mathrm{c}},-|\mathbf{f}_{\mathrm{r|c}}|\cos\theta_{\mathrm{c}},0)$ and $\mathbf{f}_{\mathrm{r|\tilde{c}}}=(-|\mathbf{f}_{\mathrm{r|c}}|\sin\theta_{\mathrm{c}},|\mathbf{f}_{\mathrm{r|c}}|\cos\theta_{\mathrm{c}},0)$, with modulus $|\mathbf{f}_{\mathrm{r|c}}|\equiv f_{\mathrm{r|c}}=n_{\mathrm{r}}^{-1}F(r/R)$, applied on elements (c) and ($\tilde{\mathrm{c}}$), have an associated linear-impulse–work four-vectors $\delta W_{\mathrm{r|c}}^{\mu}\equiv(c\mathbf{f}_{\mathrm{r|c}}\mathrm{d}t,\delta W_{\mathrm{r|c}})$ and $\delta W_{\mathrm{r|\tilde{c}}}^{\mu}\equiv(c\mathbf{f}_{\mathrm{r|\tilde{c}}}\mathrm{d}t,\delta W_{\mathrm{r|\tilde{c}}})$,

$$\delta W_{\mathrm{r|c}}^{\mu}=\begin{pmatrix} f_{\mathrm{r|c}}\sin\theta_{\mathrm{c}}\mathrm{d}t \\ -f_{\mathrm{r|c}}\cos\theta_{\mathrm{c}}\mathrm{d}t \\ 0 \\ f_{\mathrm{r|c}}R\omega\mathrm{d}t\end{pmatrix},\ \delta W_{\mathrm{r|\tilde{c}}}^{\mu}\equiv\dot{W}_{\mathrm{r|\tilde{c}}}^{\mu}\mathrm{d}t=\begin{pmatrix} -f_{\mathrm{r|c}}\sin\theta_{\mathrm{c}} \\ f_{\mathrm{r|c}}\cos\theta_{\mathrm{c}} \\ 0 \\ f_{\mathrm{r|c}}R\omega\end{pmatrix}\mathrm{d}t\,.$$

With force lever arm $\mathbf{r}_{\mathrm{c}}=(R\cos\theta_{\mathrm{c}},R\sin\theta_{\mathrm{c}},0)$ for element (c), one has:

$$\begin{aligned}\Sigma_{\mathrm{c=1}}^{\mathrm{c=n_r}}\mathbf{f} &= \mathbf{F}\,,\\ \Sigma_{\mathrm{c=1}}^{\mathrm{c=n_r}}(\mathbf{r}_{\mathrm{c}}\times\mathbf{f}_{\mathrm{r|c}}) &= Fr\mathbf{k}\,.\\ \Sigma_{\mathrm{c=1}}^{\mathrm{c=n_r}}(fv+f_{\mathrm{r|c}}R\omega)\mathrm{d}t &= F(v+r\omega)\mathrm{d}t\,.\end{aligned}$$

Net linear impulse, net angular impulse, and work performed by these distributed forces are the same as those of the original force $\mathbf{F}$, as demanded. By distributing the global, macroscopic forces among ring elements in this way allows to write equations for any element, fulfilling relativistic locality.

2. Force $\mathbf{F}_{\rm D}$ exerts linear impulse $\delta\mathbf{I}_{\rm D}=\mathbf{F}_{\rm D}{\rm d}t$, exerts angular impulse $\delta\mathbf{M}_{\rm D}=(\mathbf{R}_{\rm D}\times\mathbf{F}_{\rm D}){\rm d}t$, with lever arm vector $\mathbf{R}_{\rm D}=(-R,0,0)$ to point O, and does not perform work, i.e., $\mathbf{F}_{\rm D}\cdot\mathbf{v}_{\rm D}=0$, since its application point is always instantly at rest; ring rim speed at point $\bar{\rm O}$ is $v_{\rm D}=v_{\rm cm}-\omega R=0$. This force enters in NSL and PER equations, but not in the FLT equation.

Force $\mathbf{F}_{\rm D}$ is distributed into two forces applied to each of ring elements (Fig. 4):

2A. Force $\mathbf{f}_{\rm D|t}=(-f_{\rm D|t},0,0)$, modulus $|\mathbf{f}_{\rm D|t}|\equiv f_{\rm D}=n_{\rm r}^{-1}F_{\rm D}$, contributes to ring element linear translation. This force does not perform work. Then, $\mathbf{f}_{\rm D|t}=(-n_{\rm r}^{-1}F_{\rm D},0,0)$, with four-vector $\delta W^{\mu}_{\rm D|t}$

$$\delta W^{\mu}_{\rm D|t}=\begin{pmatrix} c\mathbf{f}_{\rm D}{\rm d}t \\ 0\end{pmatrix}\ ;\ \delta W^{\mu}_{\rm D|t}\equiv\dot{W}^{\mu}_{\rm D|t}{\rm d}t=\begin{pmatrix} cf_{\rm D}\\ 0\\ 0\\ 0\end{pmatrix}{\rm d}t\,, \tag{19}$$

The same linear-impulse-work four-vector $\delta W^{\mu}_{\rm D|t}$ is found for each ring element.

2B. Force $\mathbf{f}_{\rm D|r;c}=(f_{\rm D}\sin\theta_{\rm c},-f_{\rm D}\cos\theta_{\rm c},0)$, with modulus $|\mathbf{f}_{\rm D|r;c}|\equiv f_{\rm D}=n_{\rm r}^{-1}F_{\rm D}$, does not perform work, contributing to the rotation of element (c) around ring centre:

$$\delta W^{\mu}_{\rm D|r;c}=\begin{pmatrix} f_{\rm D}\sin\theta_{\rm c}{\rm d}t\\ -f_{\rm D}\cos\theta_{\rm c}{\rm d}t\\ 0\\ 0\end{pmatrix}\,,\ \dot{W}^{\mu}_{\rm D|r;\tilde{c}}{\rm d}t=\begin{pmatrix} -f_{\rm D}\sin\theta_{\rm c}\\ f_{\rm D}\cos\theta_{\rm c}\\ 0\\ 0\end{pmatrix}{\rm d}t\,.$$

For this distributed forces one has:

$$\begin{aligned}\Sigma_{\rm c=1}^{\rm c=n_r}\mathbf{f}_{\rm D} &= \mathbf{F}_{\rm D}\,,\\ \Sigma_{\rm c=1}^{\rm c=n_r}(\mathbf{r}_{\rm c}\times\mathbf{f}_{\rm D|r;c}) &= F_{\rm D}R\mathbf{k}\,.\end{aligned}$$

By the linear-impulse-work four-vectors for forces distributed by ring elements one obtains ring elements (c) and ($\tilde{\rm c}$) linear-impulse–work four-vectors:

$$\begin{aligned}\delta W^{\mu}_{\rm c} &\equiv \dot{W}^{\mu}_{\rm c}{\rm d}t=(\dot{W}^{\mu}_{\rm t}+\dot{W}^{\mu}_{\rm r|c}+\dot{W}^{\mu}_{\rm D|t}+\dot{W}^{\mu}_{\rm D|r;c}){\rm d}t=\\ &= \begin{pmatrix} c(f+f_{\rm r|c}\sin\theta_{\rm c}-f_{\rm D}+f_{\rm D|r}\sin\theta_{\rm c})\\ -c(f_{\rm r|c}\cos\theta_{\rm c}+f_{\rm D|r}\cos\theta_{\rm c})\\ 0\\ (fv+f_{\rm r|c}R\omega)\end{pmatrix}{\rm d}t\,,\end{aligned} \tag{20}$$

$$\begin{aligned}\delta W^{\mu}_{\tilde{\rm c}} &\equiv \dot{W}^{\mu}_{\tilde{\rm c}}{\rm d}t=(\dot{W}^{\mu}_{\rm t}+\dot{W}^{\mu}_{\rm r|\tilde{c}}+\dot{W}^{\mu}_{\rm D|t}+\dot{W}^{\mu}_{\rm D|r;\tilde{c}}){\rm d}t=\\ &= \begin{pmatrix} c(f-f_{\rm r|c}\sin\theta_{\rm c}-f_{\rm D}-f_{\rm D|r}\sin\theta_{\rm c})\\ c(f_{\rm r|c}\cos\theta_{\rm c}+f_{\rm D|r}\cos\theta_{\rm c})\\ 0\\ (fv+f_{\rm r|c}R\omega)\end{pmatrix}{\rm d}t\,.\end{aligned} \tag{21}$$

Observing these relationships, by $(\delta W^{\mu}_{\rm c}+\delta W^{\mu}_{\tilde{\rm c}})$ it is easy to check which forces will contribute to ring linear translation. Then, for the whole ring, its linear-impulse–work four-vector is

$$\delta W^{\mu}=\Sigma_{\rm c\tilde{c}}(\delta W^{\mu}_{\rm c}+\delta W^{\mu}_{\tilde{\rm c}})=\begin{pmatrix} c(\mathbf{F}-\mathbf{F}_{\rm D})\\ F(v+r\omega)\end{pmatrix}{\rm d}t\,.$$

4 Two four-vectors four-tensor momenta definition

Four-tensor formulations have played an essential role in theoretical physics. Shortly after Einstein published their papers on the special theory of relativity, Minkowski developed a four-tensor formalism for electro-magnetism [39, Ch. 13]; Einstein developed the general theory of relativity [40, Ch. 6] formalism in terms of four-tensors.

For linear translation processes, which in classical physics are described by a vector equation (NSL) and a scalar equation – (FLT), including vectors through the scalar-product work term–, a four-vector (rank 1 tensor) equation meets mathematical and physical (principle of relativity) requirements previously stated. For T&R processes, to which an angular momentum equation, including vectors cross–product term, (PER) is added, mathematical and physical requirements imply the description of a rolling process in terms of a rank 2 tensor (a four-tensor in what follows)[41, pp. 133-134].

In classical physics, NSL and FLT are independent equations. In relativity, the four-vector fundamental equation relates NSL and FLT for a process, such that, for example, the NSL in a moving frame is linear combination of NSL and FLT in the proper frame through the Lorentz transformation. Given such a relation between four-vectors, five equations are obtained: NSL, NSL-CDR, FLT, the heat equation and the entropy variation equation. On the one hand, NSL is "thermodynamized", i.e., quantities with no associated linear momentum in classical physics such as heat, in relativity, in a moving frame ($\bar{\mathrm{S}}$) have associated linear momentum, which is incorporated into NSL in $\bar{\mathrm{S}}$. On the other hand, thermodynamics is "mechanised" in relativity the same way as in classical physics.

In which follows, given two four-vectors cross-product, $A^\mu \otimes B^\mu$, proper definition – two four-vectors cross-product definition can be given by an anti-symmetric four-tensor –, for a given process, a four-tensor momenta equation can be proposed based on its four-vector fundamental equation, such that seven equations are obtained for the process: NSL, NSL-CDR, PER, PER-CDR, FLT, heat and entropy variation equations.

Cross-product classical definition: a vector. Let two three-dimensional vectors be $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Cross-product $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, with $\mathbf{c} = (c_1, c_2, c_3)$, is defined as:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3)\mathbf{i} + (a_3 b_1 - b_3 a_1)\mathbf{j} + (a_1 b_2 - b_1 a_2)\mathbf{k}\,,$$

with

$$c_1 = a_2 b_3 - b_2 a_3\,,\; c_2 = a_3 b_1 - b_3 a_1,\; c_3 = a_1 b_2 - b_1 a_2\,.$$

This definition for the cross-product ($\times$) comes from the following: force and its displacement scalar product ($\cdot$), allows for the calculation of the portion of work done by the force that is related to the variation in the kinetic energy of the body's center of mass and the change in direction of the system's movement. Additionally, the cross-product between torque and rotated angle can determine the work done by the force in a direction perpendicular to the body's displacement. If an object changes its velocity, it can do so by either altering its modulus or direction. In the former case, the object will not rotate, while in the latter, it will rotate. The cross-product is used to calculate changes in rotation.

Cross-product classical definition: (3×3) anti-symmetric tensor. The cross-product can be defined in terms of the following anti-symmetric matrix:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{pmatrix} ,$$

or 3×3 tensor momenta, with $\otimes$ operator, with:

$$\begin{aligned} C^{\mathrm{rs}} &= a_{\mathrm{r}} b_{\mathrm{s}} - b_{\mathrm{r}} a_{\mathrm{s}} = -C^{\mathrm{sr}} , \\ C^{\mathrm{rr}} &= 0 . \end{aligned}$$

The following terms are identified:

$$\begin{aligned} C^{12} &= a_1 b_2 - b_1 a_2 = \quad c_3 , \\ C^{13} &= a_1 b_3 - b_3 a_1 = -c_2 , \\ C^{23} &= a_2 b_3 - b_2 a_3 = \quad c_1 . \end{aligned}$$

Two four-vectors cross-product: (4×4) four-tensor momenta. The above definition for the cross-product between vectors ($\otimes$) can be generalised to four-vectors. A contra-variant four-vector A^μ, noted by a Greek letter superscript, is formed combining a vector $\mathbf{a}$ and a related scalar a_t, in column matrix such as $A^\mu \equiv (\mathbf{a}, a_t)$. Given four-vectors A^μ and B^μ,

$$A^\mu \equiv \begin{pmatrix} \mathbf{a} \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} , B^\mu \equiv \begin{pmatrix} \mathbf{b} \\ b_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} , \tag{22}$$

the anti-symmetric 4×4-tensor (four-tensor in which follows) $C^{\nu\mu}$ is defined as:

$$C^{\nu\mu} \equiv A^\nu \otimes B^\mu = \begin{pmatrix} C^{11} & C^{12} & C^{13} & C^{14} \\ C^{21} & C^{22} & C^{23} & C^{24} \\ C^{31} & C^{32} & C^{33} & C^{34} \\ C^{41} & C^{42} & C^{43} & C^{44} \end{pmatrix} , \tag{23}$$

with components:

$$C^{\mathrm{ij}} = a_{\mathrm{i}} b_{\mathrm{j}} - b_{\mathrm{i}} a_{\mathrm{j}} , C^{\mathrm{ij}} = -C^{\mathrm{ji}} .$$

The operator ($\otimes$) is the four-vector generalisation of the cross-product between vectors ($\times$).

The components in the four-tensor momenta that correspond to those of the 3-tensor momenta are easy to interpret (that is the reason it is done this way). The purely thermodynamic components should be interpreted (see below): terms involving ct_0 will be related to NSL, satisfying the requirement that forces must be simultaneously applied , and terms of work and energy will be related to FLT, fulfilling the locality requirement: force must be applied on the element whose mechanical state it modify.

This definition will also apply to photons, such that thermal photons are emitted with zero linear and angular momentum (Clausius requirement).

Four-tensor components C^{ij} include three kinds of momentum: (i) cross-product $\mathbf{a}\times\mathbf{b}$ components, (ii) vector-scalar product components, e.g., $b_4\mathbf{a}$, and (iii) product of a vector component times a scalar, e. g., b_1a_4. The non-zero $C^{\nu\mu}$ components are:

$$\begin{aligned}
C^{12} &= a_1b_2-b_1a_2=-C^{21}\,,\; C^{13}=a_1b_3-b_1a_3=-C^{31}\,,\; C^{14}=a_1b_4-b_1a_4=-C^{41}\,.\\
C^{23} &= a_2b_3-b_2a_3=-C^{23}\,,\; C^{24}=a_2b_4-b_2a_4=-C^{42}\,.\\
C^{34} &= a_3b_4-b_3a_4=-C^{43}\,.
\end{aligned}$$

The non-zero components of the 3×3 tensor momenta for the two vectors cross product are:

$$C^{12} = a_1b_2-b_1a_2\equiv \quad c_z\,,\; C^{13}=a_1b_3-b_1a_3\equiv -c_y\,,\; C^{23}=a_2b_3-b_2a_3\equiv \quad c_x\,.$$

Given this definition of four-tensor momenta for four-vectors, the two vectors cross product of the two vectors entering the four-vectors is recovered at the specified positions. The components of $C^{\nu\mu}$ with index 4 are relativistic components:

$$C^{14}\mathbf{i}+C^{24}\mathbf{j}+C^{34}\mathbf{k}=\mathbf{a}b_4-a_4\mathbf{b}\,, \tag{24}$$

not entering in classical physics. This components should be interpreted (see below).

A. Angular impulse four-tensor. When $A^\mu\equiv r^\mu$ is an space-time four-vector, and $B^\mu\equiv p^\mu$ (or $B^\mu\equiv E^\mu$) is a linear-momentum–energy four-vector, their cross-product ($\otimes$) will be a kind of angular-momentum four-tensor. Given a ring element (c) linear-momentum–energy four-vector E^μ_{c} in T&R, the corresponding angular momentum four-tensor $J^{\mu\nu}_{\mathrm{c}}$ can be obtained. For instance, with element (c) four-vector E^μ_{c}, given by Eq. (11), and provided the space-time lever arm four-vector $r^\mu_{\mathrm{c}}=(\mathbf{r}_{\mathrm{c}},ct_0)$, the element (c) angular momentum four-tensor $J^{\mu\nu}_{\mathrm{c}}$ is given by:

$$J^{\mu\nu}_{\mathrm{c}}\equiv r^\mu_{\mathrm{c}}\otimes E^\nu_{\mathrm{c}}\;=$$
$$=\begin{pmatrix} R\cos\theta_{\mathrm{c}}\\ R\sin\theta_{\mathrm{c}}\\ 0\\ ct_0\end{pmatrix}\otimes\begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}(v+\omega R\sin\theta_{\mathrm{c}})\\ -c\gamma(\omega R)\mathcal{M}_{\mathrm{s}}(\omega R\cos\theta_{\mathrm{c}})\\ 0\\ \gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}c^2\end{pmatrix}\,,\; J^{\mu\nu}_{\mathrm{c}}\equiv\begin{pmatrix} 0 & J_{z|\mathrm{c}} & 0 & N_{x|\mathrm{c}}\\ -J_{z|\mathrm{c}} & 0 & 0 & N_{y|\mathrm{c}}\\ 0&0&0&0\\ -N_{x|\mathrm{c}} & -N_{y|\mathrm{c}} & 0 & 0\end{pmatrix}.$$

with four-tensor components $J_{z|\mathrm{c}}$, $N_{x|\mathrm{c}}$ and $N_{y|\mathrm{c}}$ easily obtained:

$$\begin{aligned}
J_{z|\mathrm{c}} &= (R\cos\theta_{\mathrm{c}})[-c\gamma(\omega R)\mathcal{M}_{\mathrm{s}}(\omega R\cos\theta_{\mathrm{c}})]-(R\sin\theta_{\mathrm{c}})[c\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}(v+\omega R\sin\theta_{\mathrm{c}})]\,,\\
N_{x|\mathrm{c}} &= (R\cos\theta_{\mathrm{c}})[\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}c^2]-(ct_0)[c\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}(v+\omega R\sin\theta_{\mathrm{c}})]\,,\\
N_{y|\mathrm{c}} &= (R\sin\theta_{\mathrm{c}})[\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}c^2]-(ct_0)[-c\gamma(\omega R)\mathcal{M}_{\mathrm{s}}(\omega R\cos\theta_{\mathrm{c}})]\,.
\end{aligned}$$

For element (c̃), with $E^\mu_{\tilde{\mathrm{c}}}$ given by Eq. (12), located by space-time four-vector $r^\mu_{\tilde{\mathrm{c}}}=(-\mathbf{r}_{\mathrm{c}},ct_0)$, one has

$$J^{\mu\nu}_{\tilde{\mathrm{c}}}\equiv r^\mu_{\tilde{\mathrm{c}}}\otimes E^\nu_{\tilde{\mathrm{c}}}\;=$$
$$=\begin{pmatrix} -R\cos\theta_{\mathrm{c}}\\ -R\sin\theta_{\mathrm{c}}\\ 0\\ ct_0\end{pmatrix}\otimes\begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}[v-\omega R\sin\theta_{\mathrm{c}}]\\ c\gamma(\omega R)\mathcal{M}_{\mathrm{s}}[\omega R\cos\theta_{\mathrm{c}}]\\ 0\\ \gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}c^2\end{pmatrix}\,,\; J^{\mu\nu}_{\tilde{\mathrm{c}}}\equiv\begin{pmatrix} 0 & J_{z|\tilde{\mathrm{c}}} & 0 & N_{x|\tilde{\mathrm{c}}}\\ -J_{z|\tilde{\mathrm{c}}} & 0 & 0 & N_{y|\tilde{\mathrm{c}}}\\ 0&0&0&0\\ -N_{x|\tilde{\mathrm{c}}} & -N_{y|\tilde{\mathrm{c}}} & 0 & 0\end{pmatrix}.$$

with the corresponding $J_{z|\tilde{c}}$, $N_{x|\tilde{c}}$ and $N_{y|\tilde{c}}$. Observing these relationships, it is easy to check by $J_{z|c} + J_{z|\tilde{c}}$ which four-tensor components will contribute to ring angular momentum.

Adding over pairs ($c\tilde{c}$), the angular-momentum four-tensor for the ring $J^{\mu\nu}(v,\omega)$ is obtained as:

$$J^{\mu\nu}(\omega,v) = \Sigma_{c\tilde{c}}(J^{\mu\nu}_{c} + J^{\mu\nu}_{\tilde{c}}) = \begin{pmatrix} 0 & J_z & 0 & N_x \\ -J_z & 0 & 0 & N_y \\ 0 & 0 & 0 & 0 \\ -N_x & -N_y & 0 & 0 \end{pmatrix},$$

with four-tensor $J^{\mu\nu}(\omega,v)$ components:

$$\begin{aligned} J_z &= \gamma_v\gamma(\omega R)\mathcal{M}R^2\omega \approx \mathcal{M}R^2\omega\,, \\ N_x &= \Sigma_c(R\cos\theta_c)[\gamma_v\gamma(\omega R)\mathcal{M}_s c^2] - (ct_0)[c\gamma_v\gamma(\omega R)\mathcal{M}v]\,, \\ N_y &= \Sigma_c(R\sin\theta_c)[\gamma_v\gamma(\omega R)\mathcal{M}_s c^2] - (ct_0)[0]\,. \\ N_x\mathbf{i} + N_y\mathbf{j} &\approx \Sigma_c\mathbf{r}_c[\gamma_v\gamma(\omega R)\mathcal{M}_s c^2] - (ct_0)(c\mathcal{M}\mathbf{v})\,. \end{aligned}$$

Angular momentum, linear momentum and energy. The following quantities can be found inside ring angular momentum four-tensor $J^{\mu\nu}(\omega,v)$:

(i) *Angular momentum.* The $J^{\mu\nu}(\omega,v)$ C^{12}-component, $J_z \approx \mathcal{M}R^2\omega$, corresponds (low speed limit) to ring angular momentum.

(ii) *Linear momentum.* Linear momentum $\mathbf{p} = (\mathbf{p}_x, \mathbf{p}_y, 0) = \gamma(\omega R)\mathcal{M}(\gamma_v v_{x|ci}, \gamma_v v_{y|ci}, 0)$, is involved in four-tensor components N_x, N_y, multiplied by temporal lever arm ct_0. For process sketched in Fig. 1, one has $\mathbf{p} \approx \mathcal{M}\mathbf{v}$.

(iii) *Energy.* Element (c) energy $E_c(v,\omega) = \gamma_v\gamma(\omega R)\mathcal{M}_s c^2$, is involved in four-tensor components N_x, N_y and multiplied by vector $\mathbf{r}_c$.

B. Torque four-tensor. Let $A^\mu \equiv r^\mu$ be a space-time four-vector, and $B^\mu \equiv F^\mu$ (or $B^\mu \equiv \dot{W}^\mu$) be a linear-impulse–work four-vector. Their cross-product ($\otimes$) will be a torque four-tensor. Given the macroscopic forces distributed over ring elements, the associated torque four-tensor can be obtained.

Torque four-tensor $M^{\mu\nu}_c$ exerted on a generic element (c) is obtained by adding over torque four-tensors $M^{\mu\nu}_{k|c}$ for the (k) forces applied on it,

$$M^{\mu\nu}_c = \Sigma_k M^{\mu\nu}_{k|c} = r^\mu_c \otimes \Sigma_k \dot{W}^\nu_{k|c}\,. \tag{25}$$

With element (c) four-vector linear-impulse-work δW^μ_c, given by Eq. (20), and provided the space-time lever arm four-vector $r^\mu_c = (\mathbf{r}_c, ct_0)$, the element (c) torque four-tensor $M^{\mu\nu}_c$ is given by:

$$M^{\mu\nu}_c \equiv r^\mu_c \otimes \dot{W}^\nu_c =$$

$$= \begin{pmatrix} R\cos\theta_c \\ R\sin\theta_c \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c(f + f_{r|c}\sin\theta_c - f_D + f_{D|r}\sin\theta_c) \\ -c(f_{r|c}\cos\theta_c + f_{D|r}\cos\theta_c) \\ 0 \\ fv + f_{r|c}R\omega \end{pmatrix},\ M^{\mu\nu}_c = \begin{pmatrix} 0 & M_{z|c} & 0 & G_{x|c} \\ -M_{z|c} & 0 & 0 & G_{y|c} \\ 0 & 0 & 0 & 0 \\ -G_{x|c} & -G_{y|c} & 0 & 0 \end{pmatrix},$$

with $M^{\mu\nu}_c$ components $M_{z|c}$, $G_{x|c}$ and $G_{y|c}$:

$$\begin{aligned} M_{z|c} &= (R\cos\theta_c)[-c(f_{r|c}\cos\theta_c + f_{D|r}\cos\theta_c)] - (R\sin\theta_c)[c(f + f_{r|c}\sin\theta_c - f_D + f_{D|r}\sin\theta_c)]\,, \\ G_{x|c} &= \Sigma_c(R\cos\theta_c)[fv + f_{r|c}R\omega] - (ct_0)[c(f + f_{r|c}\sin\theta_c - f_D + f_{D|r}\sin\theta_c)]\,, \\ G_{y|c} &= \Sigma_c(R\sin\theta_c)[fv + f_{r|c}R\omega] - (ct_0)[-c(f_{r|c}\cos\theta_c + f_{D|r}\cos\theta_c)]\,. \end{aligned}$$

For element ($\tilde{c}$) with four-vector linear-impulse-work $\delta W^{\mu}_{\tilde{c}}$, given by Eq. (21), one has:

$$M^{\mu\nu}_{\tilde{c}} \equiv r^{\mu}_{\tilde{c}} \otimes \dot{W}^{\nu}_{\tilde{c}} =$$

$$= \begin{pmatrix} -R\cos\theta_{\rm c} \\ -R\sin\theta_{\rm c} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c(f - f_{\rm r|c}\sin\theta_{\rm c} - f_{\rm D} - f_{\rm D|r}\sin\theta_{\rm c}) \\ c(f_{\rm r|c}\cos\theta_{\rm c} + f_{\rm D|r}\cos\theta_{\rm c}) \\ 0 \\ fv + f_{\rm r|c}R\omega \end{pmatrix} , \; M^{\mu\nu}_{\tilde{c}} = \begin{pmatrix} 0 & M_{z|\tilde{c}} & 0 & G_{x|\tilde{c}} \\ -M_{z|\tilde{c}} & 0 & 0 & G_{y|\tilde{c}} \\ 0 & 0 & 0 & 0 \\ -G_{x|\tilde{c}} & -G_{y|\tilde{c}} & 0 & 0 \end{pmatrix}$$

with $M^{\mu\nu}_{\tilde{c}}$ components $M_{z|\tilde{c}}$, $G_{x|\tilde{c}}$ and $G_{y|\tilde{c}}$. Observing these relationships, it is easy to check which forces will contribute to ring rotation.

For the whole ring, the angular impulse four-tensor $M^{\mu\nu}$ is obtained as:

$$M^{\mu\nu} = \Sigma_{\rm c\tilde{c}}(M^{\mu\nu}_{\rm c} + M^{\mu\nu}_{\tilde{c}}) = \begin{pmatrix} 0 & M_z & 0 & G_x \\ -M_z & 0 & 0 & G_y \\ 0 & 0 & 0 & 0 \\ -G_x & -G_y & 0 & 0 \end{pmatrix} ,$$

with momenta four-tensor components:

$$\begin{aligned} M_z &= (Fr + F_{\rm D}R)\,, \\ G_x &= \Sigma_{\rm c}(R\cos\theta_{\rm c})[fv + f_{\rm r|c}R\omega] - (ct_0)[c(F - F_{\rm D})]\,, \\ G_y &= \Sigma_{\rm c}(R\sin\theta_{\rm c})[fv + f_{\rm r|c}R\omega] - (ct_0)[0]\,. \\ G_x\mathbf{i} + G_y\mathbf{j} &= \Sigma_{\rm c}\mathbf{r}_{\rm c}[fv + f_{\rm r|c}R\omega] - (ct_0)[c(\mathbf{F} - \mathbf{F}_{\rm D})\,. \end{aligned}$$

Angular impulse, linear impulse and work. The following quantities can be found inside ring torque four-tensor momenta:

(i) *Angular impulse (torque).* The four-tensor $M^{\mu\nu}$ C^{12}-component M_z corresponds to torque $\Gamma = (Fr + F_D R)$ exerted on ring.

(ii) *Linear impulse (resultant force).* Forces $\mathbf{F}$ and $\mathbf{F}_D$, linear impulse on ring is $\delta\mathbf{I} = (F - F_{\rm D}, 0, 0)\mathrm{d}t$, involved in components G_x and G_y, multiplied by temporal lever arm ct_0.

(iii) *Work (power).* On element (c) located by vector $\mathbf{r}_{\rm c}$ power $\dot{W}_{\rm c} = fv + f_{\rm r|c}R\omega$ is performed.

5 Four-tensor momenta equation for rolling processes

For a well-posed process, a set of four-vector fundamental equations, one for each ring generic element (c), on which (k) forces are exerted, are

$$\mathrm{d}E^{\mu}_{\rm c} = \delta W^{\mu}_{\rm c}\,,$$

being $\delta W^{\mu}_{\rm c} = \Sigma_{\rm k}\delta W^{\mu}_{\rm k|c}$. Forces are simultaneously applied during time interval $[t_0, t_0 + \mathrm{d}t]$. Similarly, a set of four-tensor momenta equations, one for each ring generic element (c), on which (k) forces are exerted, are

$$\mathrm{d}(r^{\mu}_{\rm c} \otimes E^{\mu}_{\rm c}) = (r^{\mu}_{\rm c} \otimes \dot{W}^{\mu}_{\rm c})\mathrm{d}t\,,$$

being $\delta W^{\mu}_{\rm c} = \dot{W}_{\rm c}^{\mu}\mathrm{d}t$, where $\mathrm{d}t$ is common to ring elements. Torques are simultaneously applied during time interval $[t_0, t_0 + \mathrm{d}t]$. Summations reflect forces superposition principle:

force effects – linear impulse, angular impulse, work – does not depend on another exerted force.

One body functional equations. In order to obtain operational equations, the posed many-body problem for ring elements must be reduced to a one-body problem (i.e., the ring). Thus, equations for elements should be possible to be added, thus obtaining an equation for the ring as a whole, depending on its centre of inertia and angular velocity.

Adding over ring elements,

$$\mathrm{d}(\Sigma_{\mathrm{c}} J_{\mathrm{c}}^{\mu\nu}) = (\Sigma_{\mathrm{c}} M_{\mathrm{c}}^{\mu\nu})\mathrm{d}t \,\rightarrow\, \mathrm{d}J^{\mu\nu} = M^{\mu\nu}\mathrm{d}t\,, \tag{26}$$

and the four-tensor momenta equation [16] is then:

$$\mathrm{d}\begin{pmatrix} 0 & J_z & 0 & N_x \\ -J_z & 0 & 0 & N_y \\ 0 & 0 & 0 & 0 \\ -N_x & -N_y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & M_z & 0 & G_x \\ -M_z & 0 & 0 & G_y \\ 0 & 0 & 0 & 0 \\ -G_x & -G_y & 0 & 0 \end{pmatrix}\mathrm{d}t\,.$$

The cause (torque, resultant force and power) acting during time interval $[t_0, t_0+\mathrm{d}t]$ (angular momentum, linear momentum, work), has the effect of varying the system mechanical state (combined angular momentum, linear momentum and energy).

For a given system and process, e.g., the process sketched in Fig. 1, the four-tensor momenta equation $\mathrm{d}J^{\mu\nu} = M^{\mu\nu}\mathrm{d}t$ establishes a cause-effect relationship between the linear-momentum–angular-momentum–energy four-tensor variation $\mathrm{d}J^{\mu\nu}$ (effect) and the linear-impulse–angular-impulse–work four-tensor $M^{\mu\nu}\mathrm{d}t$ exerted during time interval $[t_0, t_0+\mathrm{d}t]$, where t is frame (S) proper time, measured by its set of synchronised clocks.

5.1 Relationships from four-tensor momenta equation

The equality between four-tensors implies equality component by component or by a linear combination of components.

Four-tensor equation, spatial components. From four-tensor spatial components, the PER equation for the process, and then, its PER-CDR, are directly obtained.

PER. From four-tensor C^{12} spatial (angular) components one has:

$$\mathrm{d}(\Sigma_{\mathrm{c}=1}^{\mathrm{c}=\mathrm{n_r}} J_{z|\mathrm{c}}) = (\Sigma_{\mathrm{c}=1}^{\mathrm{c}=\mathrm{n_r}} M_{z|\mathrm{c}})\mathrm{d}t\,, \tag{27}$$

$$\mathrm{d}J_z = M_z\mathrm{d}t\,. \tag{28}$$

Assuming $\gamma_v \approx 1$ (rotation is decoupled from translation), the angular-impulse–angular-momentum variation equation is obtained:

$$I_{\mathrm{R}}\mathrm{d}[\chi_{\mathrm{R}}(\omega)\omega] \approx \Gamma^{\mathrm{ext}}\mathrm{d}t\,, \tag{29}$$

with external torque: $\Gamma^{\mathrm{ext}} = \Sigma_{\mathrm{k}}(\mathbf{r}_{\mathrm{k}} \times \mathbf{F}_{\mathrm{k}})$, where $\mathbf{F}_{\mathrm{k}}$ is k-th external force and $\mathbf{r}_{\mathrm{k}}$ its lever arm relative to ring centre O. Equation (29) is the Poinsot-Euler rotation equation for the process. This equation, through the torques, considers the positions where macroscopic forces are applied. The same force with another lever arm produces different angular impulse.

For a finite process, with constant torque applied during time interval $[t_0, t_0+\Delta t]$,

$$I_{\mathrm{R}}[\chi_{\mathrm{R}}(\omega_{\mathrm{f}})\omega_{\mathrm{f}}] - I_{\mathrm{R}}[\chi_{\mathrm{R}}(\omega_{\mathrm{I}})\omega_{\mathrm{I}}] = \Gamma^{\mathrm{ext}}\Delta t\,, \tag{30}$$

with resultant external torque $\Gamma^{\rm ext} = \Sigma_{\rm k}\Gamma_{\rm k}$: the same torque combination produces the same angular momentum variation.

Equation (30) takes into account the fact that when the (k)-th constant force is applied outside the centre-of-inertia displacement path, its application point has an additional displacement $\mathrm{d}x_{\rm r} = r_{\rm k}\omega\mathrm{d}t$ relative (in addition) to the centre-of-inertia displacement $\mathrm{d}x = v_{\rm ci}\mathrm{d}t$, during time interval $\mathrm{d}t$; Therefore, an additional work is performed, provided by the external agent that ensures a constant force applied (Fig. 5).

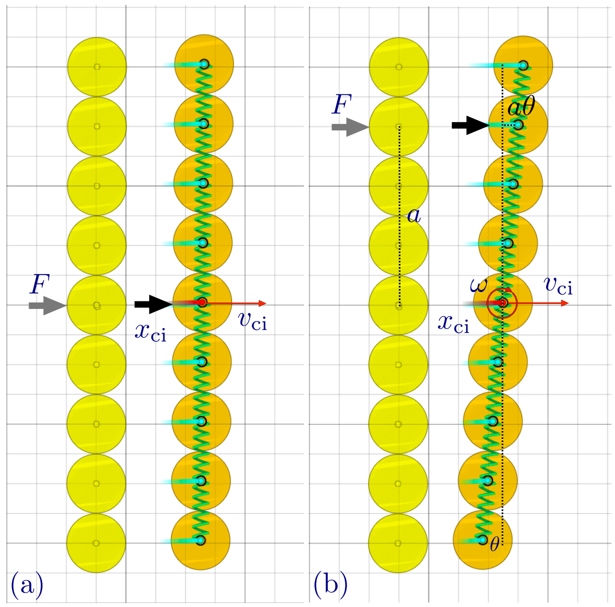

Figure 5: (a) Force F acts along a line through the centre of the bar, during time interval Δt, moving distance $x_{\rm ci}$ and reaching speed $v_{\rm ci}$. (b) Force F acts along a line at distance a from the centre of the bar, during time interval Δt, moving distance $\tilde{x} = x_{\rm ci} + a\theta$, with an extra displacement $\hat{x} = a\theta$. The bar rotates angle θ, reaching angular velocity ω (and centre speed $v_{\rm ci}$), with force application point speed $u = v_{\rm ci} + a\omega$.

PER-CDR. The PER-CDR equation relates rotational kinetic energy variation with external torques pseudo-work $\mathrm{d}K_{\rm rt} = \mathrm{p}W_{\rm rt}$. From PER Eq. (29), by using relationship [43]:

$$I_{\rm R}\omega\mathrm{d}[\chi_{\rm R}(\omega)\omega] = \mathcal{M}\mathrm{d}[\zeta_{\rm R}(\omega)c^2]\,, \tag{31}$$

[relationship Eq. (31) cannot be used with Eq. (29) if $\gamma_v \neq 1$) its PER-CDR is obtained as:

$$\begin{aligned} \mathcal{M}\mathrm{d}[\zeta_{\rm R}(\omega)c^2] &= \Gamma^{\rm ext}\mathrm{d}\theta\,, & (32)\\ \mathcal{M}\{[\zeta_{\rm R}(\omega_{\rm f})c^2] - [\zeta_{\rm R}(\omega_{\rm I})c^2]\} &= \Gamma^{\rm ext}\Delta\theta\,, & (33) \end{aligned}$$

with $\mathrm{d}\theta = \omega\mathrm{d}t$. This PER-CDR Eq. (33), which is not directly obtained from the four-tensor momenta equation, will assign kinetic energy to that additional work: the rotational kinetic energy. Product $\mathrm{p}W_{\rm r} \equiv \Gamma\mathrm{d}\theta$ will, in general, not be work but pseudo-work; e.g., a dissipative force torque does not perform work (see below). Equation (32) is the pseudo-work equation for ring rotation.

Four-tensor equation, mixed components. From relationship (proposed by the intuition provided by classical physics)

$$\mathrm{d}N_x\mathbf{i} + \mathrm{d}N_y\mathbf{j} = G_x\mathrm{d}t\mathbf{i} + G_y\mathrm{d}t\mathbf{j}\,, \tag{34}$$

several equations will be obtained, leading to NSL and to the FLT for the process.

NSL. Matching terms with common factor ct_0 and unit vector $\mathbf{i}$ or $\mathbf{j}$ in Eq. (34):

$$\begin{aligned} \mathbf{i}ct_0\big\{\mathcal{M}_\mathrm{s}\mathrm{d}[\gamma_v\gamma(\omega R)v_x] &= \Sigma_\mathrm{k}F_{x|\mathrm{k;c}}\mathrm{d}t\big\}\,, & (35)\\ \mathbf{j}ct_0\big\{\mathcal{M}_\mathrm{s}\mathrm{d}[\gamma_v\gamma(\omega R)v_y] &= \Sigma_\mathrm{k}F_{y|\mathrm{k;c}}\mathrm{d}t\big\}\,. & (36) \end{aligned}$$

By adding over forces exerted on element (c):

$$\mathcal{M}_\mathrm{s}\mathrm{d}[\gamma_v\gamma(\omega R)\mathbf{v}] = \Sigma_\mathrm{k}\mathbf{F}_{\mathrm{k|c}}\mathrm{d}t\,. \tag{37}$$

By adding over ring element pairs (c$\tilde{\mathrm{c}}$):

$$\mathcal{M}\mathrm{d}[\gamma_v\gamma(\omega R)\mathbf{v}] = \mathbf{F}^\mathrm{ext}\mathrm{d}t\,, \tag{38}$$

being external resultant force $\mathbf{F}^\mathrm{ext} = \Sigma_\mathrm{c}\Sigma_\mathrm{k}\mathbf{F}_{\mathrm{k|c}}$, and $v \equiv v_\mathrm{ci}$ the ring centre speed.

Relations (35)-(36) check whether forces are simultaneously applied during time interval $[t_0, t_0 + \mathrm{d}t]$. Otherwise, this equation will detect the inconsistency. It is assumed that forces can be added to obtain a resultant characterizing ring centre motion. Equation (38) represents Newton's second law for the process.

Assuming $\gamma(\omega R) \approx 1$ (translation is decoupled from rotation), for a finite process, with constant forces applied during time interval $[t_0, t_0 + \Delta t]$,

$$\mathcal{M}(\gamma_{v_\mathrm{f}}\mathbf{v}_\mathrm{f}) - \mathcal{M}(\gamma_{v_\mathrm{I}}\mathbf{v}_\mathrm{I}) = \mathbf{F}^\mathrm{ext}\Delta t\,. \tag{39}$$

Equation (39) gathers all forces applied on the ring, considering their resultant applied on its centre during time interval $[t_0, t_0 + \Delta t]$, whether or not the force is conservative.

NSL-CDR. From NSL equation, by using relationship:

$$\mathbf{v}\cdot\mathrm{d}(\gamma_v\mathbf{v}) = \mathrm{d}(\gamma_v c^2)\,, \tag{40}$$

[relationship Eq. (40) cannot be applied to Eq. (37) if $\gamma(\omega R) \neq 1$] its NSL-CDR is obtained as:

$$\begin{aligned} \mathcal{M}\mathrm{d}(\gamma_v c^2) &= \mathbf{F}^\mathrm{ext}\cdot\mathrm{d}\mathbf{x}\,, & (41)\\ \mathcal{M}(\gamma_{v_\mathrm{f}} - \gamma_{v_\mathrm{I}})c^2 &= \mathbf{F}^\mathrm{ext}\cdot\Delta\mathbf{x}\,, & (42) \end{aligned}$$

where $\mathrm{d}\mathbf{x} = \mathbf{v}\mathrm{d}t$ ($\mathrm{d}\mathbf{x}_\mathrm{ci} = \mathbf{v}_\mathrm{ci}\mathrm{d}t$), with external resultant force $\mathbf{F}^\mathrm{ext} = \Sigma_\mathrm{c}\Sigma_\mathrm{k}\mathbf{F}_{\mathrm{k|c}}$. The NSL-CDR equation Eq. (41), which is not directly obtained from the four-tensor equation, assigns ring centre-of-inertia kinetic energy variation to force $\mathbf{F}^\mathrm{ext}$ times ring centre-of-inertia displacement (some forces displacement may not be ring centre displacement, e.g., force $\mathbf{F}_\mathrm{D}$); force-displacement product $\mathrm{p}W \equiv \mathbf{F}^\mathrm{ext}\cdot\Delta\mathbf{x}$ will not be, in general, performed work but includes pseudo-work terms. Equation (42) is, in general, the pseudo-work equation for body linear translation.

FLT. Matching terms with common location vector $\mathbf{r}_\mathrm{c}$ in Eq. (34), one has the relationship:

$$\mathbf{r}_\mathrm{c}\big\{\mathcal{M}_\mathrm{s}\mathrm{d}[\gamma_v\gamma(\omega R)c^2]\big\} = \mathbf{r}_\mathrm{c}\big[\Sigma_\mathrm{k}(\mathbf{F}_{\mathrm{k|c}}\cdot\mathbf{v}_{\mathrm{k|c}})\mathrm{d}t\big]\,, \tag{43}$$

one for system element. Each equation corresponds to a ring element and checks whether the force acting on it is really localised on it. Specifically, any force must be exerted directly on one element and any change in its energy must be due to work performed by distributed forces applied on it. Assuming that force $\mathbf{F}_\mathrm{k}$ is exerted on a point other than the ring element, then $\mathbf{r}_\mathrm{c} \neq \mathbf{r}_\mathrm{k}$ and Eqs. (43) will detect the inconsistency.

Equation (43) is the FLT for ring element (c). By adding over elements energy-work equations, one gets

$$\mathcal{M}\mathrm{d}[\gamma_v \gamma(\omega R) c^2] = \delta W^{\mathrm{ext}} \,, \tag{44}$$

$$\mathcal{M}[\gamma_{v_\mathrm{f}} \gamma(\omega_\mathrm{f} R) - \gamma_{v_\mathrm{I}} \gamma(\omega_\mathrm{I} R)] c^2 = W^{\mathrm{ext}} \,, \tag{45}$$

with $W^{\mathrm{ext}} = \Sigma_\mathrm{k}(\mathbf{F}_\mathrm{k} \cdot \mathbf{u}_\mathrm{k})\Delta t$, for constant forces and almost constant velocities, $u_\mathrm{k}\Delta t = \Delta x_{\mathrm{ci}} + r_\mathrm{k}\Delta\theta$.

These equations properly describe processes in rigid T&R (i.e., rigid rolling). If on the contrary, this set of equations do not sufficiently describe the process, the process is ill-posed.

Four-tensor momenta Eq. (26) contains all the necessary information to fully solve the type of processes posed, which evolve at the conservation of mechanical energy. On the one hand, Eq. (26) does not consider thermal effects, a characteristic feature of dissipative forces. When thermal effects occur during the process, the momentum four-tensor associated with thermal photons exchanged with thermal surroundings must be included (see below). On the other hand, Eq. (26) can incorporate thermodynamic forces $\mathbf{F}_\xi$ coming from a chemical reaction (see below).

5.2 Standard configuration observer

When studying a problem under the relativistic approach, only those formalisms explicitly in agreement with the principle of relativity can be implemented. This demand translates into the need to find an equation (ansatz) that automatically transforms between inertial frames, S and $\bar{\mathrm{S}}$ in the standard configuration with velocity $\mathbf{V} = (V, 0, 0)$, for example, by simply applying an operator, ensuring that the equation remains with its functional form unchanged. Assuming this operator is the Lorentz transformation given by Eq. (10), the allowed equations must involve four-vectors or four-tensors [44].

Therefore, a four-vector, A^μ, transforms as $\bar{A}^\mu = \mathcal{L}^\nu_\mu(V)A^\nu$, and a four-tensor, $B^{\mu\nu}$, transforms as $\bar{B}^{\mu\nu} = \mathcal{L}^\mu_\delta(V)B^{\delta\sigma}\mathcal{L}^\nu_\sigma(-V)$. When dealing with four-vector equations, with multiple elements involved, one has:

$$\begin{aligned} &\Sigma_\mathrm{c}\mathcal{L}^\mu_\nu(V)\mathrm{d}E^\nu_\mathrm{c} = \Sigma_\mathrm{c}\mathcal{L}^\mu_\nu(V)\delta W^\nu_\mathrm{c} \,, \\ &\mathrm{d}\mathcal{L}^\mu_\nu(V)\mathrm{d}E^\nu = \mathcal{L}^\mu_\nu(V)\delta W^\nu \rightarrow \mathrm{d}\bar{E}^\nu = \delta\bar{W}^\nu \,. \end{aligned} \tag{46}$$

For a four-tensor-moment equation, $\mathrm{d}J^{\mu\nu} = M^{\mu\nu}\mathrm{d}t$,where t is the proper time in frame S, with the ground and thermal reservoir at rest, one has: [45, pp. 17-19]:

$$\begin{aligned} &\Sigma_\mathrm{c}\mathcal{L}^\mu_\nu(V)r^\nu_\mathrm{c} \otimes \mathcal{L}^\mu_\nu(V)\mathrm{d}E^\nu_\mathrm{c} = \Sigma_\mathrm{c}\mathcal{L}^\mu_\nu(V)r^\nu_\mathrm{c} \otimes \mathcal{L}^\mu_\nu(V)\delta W^\nu_\mathrm{c} \,, \\ &\mathrm{d}\big[\mathcal{L}^\mu_\delta(V)J^{\delta\sigma}\mathcal{L}^\nu_\sigma(-V)\big] = \big[\mathcal{L}^\mu_\delta(V)M^{\delta\sigma}\mathcal{L}^\nu_\sigma(-V)\big]\mathrm{d}t \rightarrow \mathrm{d}\bar{J}^{\mu\nu} = \bar{M}^{\mu\nu}\mathrm{d}t \,. \end{aligned} \tag{47}$$

The four-tensor momenta equation is covariant, expressed in relativistic space-time language (as Wheeler demanded).

6 Horizontal rolling ring

Figure 1 sketches, in frame S (x, y, z, t) where the ground is at rest and x-axis parallel to the floor, a process for a ring rolling under the action of force $\mathbf{F} = (F, 0, 0)$ [46], exerted on a string, previously wrapped around a circle of radius r, which unwinds like a spool. Gravity $\mathbf{G} = (0, -\mathcal{M}g, 0)$ and normal $\mathbf{N} = (0, N, 0)$ forces are exerted on the ring, with $\mathbf{G} + \mathbf{N} = (0, 0, 0)$; so, they will not be considered. On the ring edge, at its contact point with the floor, the force demanded by the rolling condition is $\mathbf{F}_{\mathrm{D}} = (-F_{\mathrm{D}}, 0, 0)$ and must be obtained.

Four-vector fundamental equation and Poinsot-Euler rotation equation. This exercise is solved first using the four-vector fundamental equation plus the Poinsot-Euler rotation equation.

Constant forces are applied during time interval $[t_0, t_0 + t_1]$. Speeds v_1, ω_1, ring (ci) displacement x_1 and angle θ_1 must be obtained, by the rolling condition $v_1 = R\omega_1$, and $x_1 = R\theta_1$. The four-vector fundamental equation for this process is [25]: $E^{\mu}_{\mathrm{f}} - E^{\mu}_{\mathrm{I}} = W^{\mu}_{F} + W^{\mu}_{\mathrm{D}}$. In its matrix (1D) form:

$$\begin{pmatrix} c\gamma_{v_1}\gamma(\omega_1 R)\mathcal{M}\mathbf{v}_1 \\ \gamma_{v_1}\gamma(\omega_1 R)\mathcal{M}c^2 \end{pmatrix} - \begin{pmatrix} 0 \\ \mathcal{M}c^2 \end{pmatrix} = \begin{pmatrix} c(\mathbf{F} - \mathbf{F}_{\mathrm{D}})t_1 \\ F(x_1 + r\theta_1) \end{pmatrix} .$$

Matching by components, NSL and FLT equations are given by:

$$\begin{aligned} \gamma_{v_1}\mathcal{M}v_1 &\approx (F - F_{\mathrm{D}})t_1 \, , \\ \gamma_{v_1}\gamma(\omega_1 R)\mathcal{M}c^2 - \mathcal{M}c^2 &= F(x_1 + r\theta_1) \, . \end{aligned}$$

For the NSL-CDR equation one has:

$$(\gamma_{v_1} - 1)\mathcal{M}c^2 = (F - F_{\mathrm{D}})x_1 \, , \tag{48}$$

with pseudo-work term $\mathrm{p}W_{\mathrm{D}} = F_{\mathrm{D}}x_1$.

The relativistic Poinsot-Euler equation is assumed to be $I_{\mathrm{R}}\mathrm{d}[\gamma(\omega R)\omega] = \Gamma^{\mathrm{ex}}\mathrm{d}t$. Then, for constant torques applied during time interval $[t_0, t_0 + t_1]$, PER equation and its PER-CDR equation become:

$$\begin{aligned} I_{\mathrm{R}}[\gamma(\omega_1 R)\omega_1] &\approx (Fr + F_{\mathrm{D}}R)t_1 \, , \\ \big[\gamma(\omega_1 R) - 1\big]\mathcal{M}c^2 &= (Fr + F_{\mathrm{D}}R)\theta_1 \, . \end{aligned}$$

Four-tensor momenta equation. For ring elements (c) and ($\tilde{\mathrm{c}}$) one has four-tensor momenta equations:

$$\mathrm{d}\begin{pmatrix} R\cos\theta_{\mathrm{c}} \\ R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}[v + \omega R\sin\theta_{\mathrm{c}}] \\ -c\gamma(\omega R)\mathcal{M}_{\mathrm{s}}[\omega R\cos\theta_{\mathrm{c}}] \\ 0 \\ \gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}c^2 \end{pmatrix} = \begin{pmatrix} R\cos\theta_{\mathrm{c}} \\ R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c(f + f_{\mathrm{r|c}}\sin\theta_{\mathrm{c}} - f_{\mathrm{D}} + f_{\mathrm{D|r}}\sin\theta_{\mathrm{c}}) \\ -c(f_{\mathrm{r|c}}\cos\theta_{\mathrm{c}} + f_{\mathrm{D|r}}\cos\theta_{\mathrm{c}}) \\ 0 \\ (fv + f_{\mathrm{r|c}}R\omega) \end{pmatrix} \mathrm{d}t \, .$$

$$\mathrm{d}\begin{pmatrix} -R\cos\theta_{\mathrm{c}} \\ -R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c\gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}[v - \omega R\sin\theta_{\mathrm{c}}] \\ c\gamma(\omega R)\mathcal{M}_{\mathrm{s}}[\omega R\cos\theta_{\mathrm{c}}] \\ 0 \\ \gamma_v\gamma(\omega R)\mathcal{M}_{\mathrm{s}}c^2 \end{pmatrix} = \begin{pmatrix} -R\cos\theta_{\mathrm{c}} \\ -R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c(f - f_{\mathrm{r|c}}\sin\theta_{\mathrm{c}} - f_{\mathrm{D}} - f_{\mathrm{D|r}}\sin\theta_{\mathrm{c}}) \\ c(f_{\mathrm{r|c}}\cos\theta_{\mathrm{c}} + f_{\mathrm{D|r}}\cos\theta_{\mathrm{c}}) \\ 0 \\ (fv + f_{\mathrm{r|c}}R\omega) \end{pmatrix} \mathrm{d}t \, .$$

Analasing these relationships it is not difficult to verify which forces contribute to ring centre linear translation (by $\dot{W}_{\rm c}^{\mu} + \dot{W}_{\tilde{\rm c}}^{\mu}$) and which to ring rotation (by $M_{\rm c}^{\mu\nu} + M_{\tilde{\rm c}}^{\mu\nu}$).

For this process, with angular momentum four-tensor $J^{\mu\nu}$ and torque four-tensor $M^{\mu\nu} = M_F^{\mu\nu} + M_{\rm D}^{\mu\nu}$, the four-tensor momenta equation is:

$$\mathrm{d}J^{\mu\nu} \quad = \quad (M_F^{\mu\nu} + M_{\rm D}^{\mu\nu})\mathrm{d}t\,. \tag{49}$$

From Eq. (49), the following equations are obtained:

PER. Angular-impulse–angular-momentum variation equation $\mathrm{d}J_z = M_z\mathrm{d}t$ ($\gamma_v \approx 1$):

$$\mathcal{M}R^2\mathrm{d}[\gamma(\omega R)\omega] \approx (Fr + F_{\rm D}R)\mathrm{d}t\,. \tag{50}$$

This equation, which was previously written as a hypothesis, is now obtained from the four-tensor momenta equation formalism.

PER-CDR. From Eq. (50) by relationship Eq. (29) one obtains pseudo-work–rotational kinetic energy variation equation:

$$\mathcal{M}\mathrm{d}[\gamma(\omega R)c^2] = (Fr + F_{\rm D}R)\mathrm{d}\theta\,. \tag{51}$$

By integration, with initial conditions $\omega_{\rm I} = 0$, $\theta_{\rm I} = 0$ (and in the low speed limit $v/c \to 0$),

$$\begin{aligned} I_{\rm R}\gamma(\omega_1 R)\omega_1 \quad &= \quad (Fr + F_{\rm D}R)t_1 \;\to\; I_{\rm R}\omega_1 = (Fr + F_{\rm D}R)t_1. \\ [\gamma(\omega_1 R) - 1]\mathcal{M}c^2 \quad &= \quad (Fr + F_{\rm D}R)\theta_1 \to \frac{1}{2}I_{\rm R}\omega_1^2 = (Fr + F_{\rm D}R)\theta_1\,. \end{aligned}$$

As it must be demanded, the classical equations for the process are obtained from relativistic equations in the low speed limit $\omega R/c \to 0$.

NSL. Matching components with common factor ct_0 in Eq. (34), the linear-impulse–linear-momentum variation equation for ring centre (centre-of-inertia) is $[\gamma(\omega R) \approx 1]$:

$$ct_0\big\{\mathcal{M}\mathrm{d}(\gamma_v v) \approx (F - F_{\rm D})\mathrm{d}t\big\}\,, \tag{52}$$

which constitutes NSL for the process. This equation checks that forces involved in the process are applied during time interval $[t_0, t_0 + \mathrm{d}t]$.

NSL-CDR. From Eq. (52) and by relationship Eq. (40) one obtains

$$\mathcal{M}\mathrm{d}(\gamma_v c^2) = (F - F_{\rm D})\mathrm{d}x\,. \tag{53}$$

This pseudo-work ring (centre-of-inertia) kinetic energy variation is the NSL-CDR equation for the process. Integrating, with initial conditions $v_{\rm I} = 0$, $x_{\rm I} = 0$ (and taking the limit $v/c \to 0$) one obtains:

$$\begin{aligned} \gamma_{v_1}\mathcal{M}v_1 \quad &= \quad (F - F_{\rm D})t_1 \;\to\; \mathcal{M}v_1 = (F - F_{\rm D})t_1 \\ (\gamma_{v_1} - 1)\mathcal{M}c^2 \quad &= \quad (F - F_{\rm D})x_1 \to \frac{1}{2}\mathcal{M}v_1^2 = (F - F_{\rm D})x_1\,. \end{aligned}$$

The following condition is found for $F_{\rm D}$, demanded by the rolling condition, with $\gamma_{v_1} = \gamma(\omega_1 R)$:

$$F_{\rm D} = F\frac{R - r}{2R}\,. \tag{54}$$

When $r = R$, $F_{\rm D} = 0$ and the rolling condition is satisfied just by force $\mathbf{F}$: the ring spins as a whole around point $\bar{\rm O}$.

FLT. In this process Eq. (34) there is an equation per element with common factor $\mathbf{r}_{\rm c} = R\cos\theta_{\rm c}\mathbf{i} + R\sin\theta_{\rm c}\mathbf{j}$. Matching four-tensors components with $\mathbf{r}_{\rm c}$, one obtains:

$$\mathbf{r}_{\rm c}\big\{\mathcal{M}_{\rm s}{\rm d}[\gamma_v\gamma(\omega R)c^2]\big\} = \mathbf{r}_{\rm c}\big\{[(\mathbf{f}_{\rm t}\cdot\mathbf{v}) + (\mathbf{f}_{\rm r|c}\cdot\omega R)\big]{\rm d}t\,\big\}\,.$$

This relationship checks that the locality principle is met in the process description. Then, the energy equation for a ring generic element (c) can be expressed as:

$$\mathcal{M}_{\rm s}{\rm d}[\gamma_v\gamma(\omega R)c^2] = n_{\rm r}^{-1}[(\mathbf{F}\cdot\mathbf{v}) + (rF/R)\omega R]{\rm d}t\,,$$

and ring energy equation is obtained as:

$$\mathcal{M}{\rm d}[\gamma_v\gamma(\omega R)c^2] = F(v + r\omega){\rm d}t\,. \tag{55}$$

By integration from initial $(v_{\rm I} = 0, \omega_{\rm I} = 0)$, $(x_{\rm I} = 0, \theta_{\rm I} = 0)$, to final (to be determined) states $[v_1 \equiv v_{\rm ci}(t_1),\, \omega_1 \equiv \omega(t_1)]$, with $x_1 \equiv x_{\rm ci}(t_1)$, $\theta_1 \equiv \theta(t_1)$, one obtains:

$$[\gamma_{v_1}\gamma(\omega_1 R) - 1]\mathcal{M}c^2 = F(x_1 + r\theta_1)\,. \tag{56}$$

In the low-speed limit $v/c \to 0)$,

$$(\gamma_{v_1} - 1)\mathcal{M}c^2 + [\gamma(\omega_1 R) - 1]\mathcal{M}c^2 \approx F(x_1 + r\theta_0) \;\to \frac{1}{2}\mathcal{M}v_1^2 + \frac{1}{2}I_{\rm R}\omega_1^2 = F(x_1 + r\theta_1)\,.$$

With $x_1 = R\theta_1$, FLT Eq. (57) is obtained as sum of NSL-CDR Eq. (54) and PER-CDR Eq. (52); i.e., the process evolves with mechanical energy conservation.

When demanded force $F_{\rm D}$ is obtained to be greater than friction force $F_{\rm R} = \mu_{\rm d}N$ (maximum force ground exerts on the ring), i.e., with $F_{\rm D} > \mu_{\rm d}\mathcal{M}g$, the rolling condition cannot be achieved, the ring slips, ground-ring force is $F_{\rm g/r} = \mu_{\rm d}\mathcal{M}g$ and thermal effects will be observed (see below).

6.1 Parallel axis theorem

Whenever a valid point is chosen to take momenta, the centre of inertia, or a point instantaneously at rest, the four-tensor momenta equation formalism directly applies the parallel axis theorem.

At instant t, the contact point between the ring edge and the incline, point $\bar{\rm O}$ in Fig. 1, is, by the rolling condition, instantaneously at rest in frame S [47]. Instead of choosing the ring centre as origin of position vectors, $r^\mu_{\rm c}$ and $r^\mu_{\tilde{\rm c}}$, it is possible to choose the instantaneous axis of rotation, passing through point $\bar{\rm O}$, as origin for linear momentum and force lever arms. From point $\bar{\rm O}$, the position four-vectors, $\bar{r}^\mu_{\rm c}$ and $\bar{r}^\mu_{\tilde{\rm c}}$, for the pair (c$\tilde{\rm c}$) are:

$$\bar{r}^\mu_{\rm c} = \begin{pmatrix} R\cos\theta_{\rm c} \\ R\sin\theta_{\rm c} + R \\ 0 \\ ct_0 \end{pmatrix}\,,\; \bar{r}^\mu_{\tilde{\rm c}} = \begin{pmatrix} -R\cos\theta_{\rm c} \\ -R\sin\theta_{\rm c} + R \\ 0 \\ ct_0 \end{pmatrix}\,. \tag{57}$$

From the four-tensor-moment fundamental equation, $\Sigma_{\rm c}(\bar{r}^\mu_{\rm c}\otimes{\rm d}E^\mu_{\rm c}) = \Sigma_{\rm c}(\bar{r}^\mu_{\rm c}\otimes\delta W^\mu_{\rm c})$, it is obtained:

Poinsot-Euler's law to point $\bar{\rm O}$ [48]. For the process sketched in Fig. 1, the PER equation to point $\bar{\rm O}$ is:

$$ {\rm d}\big\{\big[\gamma_v I_{\rm R}\gamma(\omega R) + \gamma_v\mathcal{M}R^2\gamma(\omega R)]\omega\big\} = F(r + R){\rm d}t\,. \tag{58}$$

By integrating during time interval $[t_0, t_0 + t_1]$,

$$\gamma_v \gamma(\omega R) I_0 \omega + \gamma_v \gamma(\omega R) M R^2] \omega\big\} = F(r + R) t_1 \,. \tag{59}$$

PER equation relative to Ō can be expressed as linear combination of NSL and PER relative to O:

$$\begin{aligned} \gamma_v \gamma(\omega R)(I + \mathcal{M} R^2)\omega = F(r + R)t \;\rightarrow \\ \gamma_v \gamma(\omega R) I_0 \omega = (Fr + R F_{\mathrm{D}}) t_1\big] + \\ R[\gamma_v \gamma(\omega R) \mathcal{M} v\big\} = (F - F_{\mathrm{D}}) t_1 \,. \end{aligned} \tag{60}$$

Four-tensor components with common factor ct_0 involved in NSL are not affected by this change.

Work–energy equation: Regarding components with common factor R, the work–energy equation is obtained,

$$\begin{aligned} (\mathbf{r}_{\mathrm{c}} + \mathbf{R})\big\{\mathcal{M}_{\mathrm{s}} \mathrm{d}[\gamma_v \gamma(\omega R) c^2] = (fv + f_{\mathrm{r|c}} R\omega)\mathrm{d}t\big\} \,, \\ \mathcal{M} \mathrm{d}[\gamma_v \gamma(\omega R) c^2] = F(\mathrm{d}x + r\mathrm{d}\theta) \,. \end{aligned} \tag{61}$$

Ring total kinetic energy $K + K_{\mathrm{rot}} = \mathcal{M}\gamma_v \gamma(\omega R) c^2 - \mathcal{M} c^2 = \hat{K}_{\mathrm{rot}}$, does not depend on the point (provided it is a valid point) with respect to which momenta are taken (same result as in classical physics).

From F_{D} given by Eq. (54), by using relationship

$$[\gamma(\omega_1 R)\gamma_{v_1} - 1]c^2 \approx [\gamma(\omega_1 R) - 1]c^2 + (\gamma_{v_1} - 1)c^2 \,,$$

for the translational kinetic energy one has:

$$(\gamma_{v_1} - 1)\mathcal{M} c^2 \approx F \frac{1}{2}(R + r)\theta_1 \,,$$

and

$$[\gamma(\omega_1 R) - 1]\mathcal{M} c^2 \approx F \frac{1}{2}(R + r)\theta_1 \,,$$

for the rotational kinetic energy. For point O, half of the work performed by $\mathbf{F}$ goes to translational kinetic energy (ring centre) and half to rotational kinetic energy; for point Ō, a body with moment of inertia $I_{\bar{\mathrm{O}}} = 2MR^2$ rotates around point Ō with angular velocity ω.

Newton's second law equation is not affected by this change of point about which to take momenta.

7 Thermal photons four-tensor momenta

In relativity, energy exchanged as heat, e.g., in an isothermal process, is modeled by thermal photons [9] (a descriptive advantage of quantum statistical mechanics over classical thermodynamics, to be integrated using relativistic considerations [49]). Under the four-vector fundamental equation formalism, thermal effects are considered through a heat four-vector $\delta Q^\mu = (0, 0, 0, \delta Q)$ with zero linear momentum components and energy component given by

δQ [37]. For a description using a four-tensor formalism, a four-tensor momenta must be associated to each photon involved in the process, and then, by applying statistical mechanics considerations, the thermal photons four-tensor momenta will be obtained.

Four-tensor momenta for photon four-vector. When a body at rest, with temperature T slightly above surroundings temperature T_0, radiates thermal photons, according to Stefan-Boltzmann law, energy emission is not expected to be such that the body spontaneously translates or rotates: such behaviour, even satisfying thermodynamics first law, would infringe its second law. Under the four-tensor formalism, each emitted photon must be described to ensure that the ensemble of emitted photons has: (i) zero net linear momentum and (ii) zero net angular momentum (maximum entropy principle); i.e., the emitted energy as heat is a thermal photons ensemble. These restrictions regarding thermal photons are equivalent to the implicit consideration of heat in classical physics: bulk energy with zero linear and angular momentum. In contrast, a pencil of laser photons, with non-zero net linear momentum, is regarded as energy exchanged by work (not heat) [50].

A correct heat description considers emitted photons on ring element (c), located by vector $\mathbf{r}_{\rm c}$, as a set of opposite pairs (s$\tilde{\rm s}$). Linear-momentum–energy four-vectors for these photons, with frequency ν (monochromatic approximation [51]), directions $\mathbf{u}_{\rm s} = (\cos\theta_{\rm s}, \sin\theta_{\rm s}, 0)$ and $\mathbf{u}_{\tilde{\rm s}} = -\mathbf{u}_{\rm s}$, are given, respectively, by:

$$E^{\mu}_{\rm s|c} = \begin{pmatrix} h\nu\cos\theta_{\rm s} \\ h\nu\sin\theta_{\rm s} \\ 0 \\ h\nu \end{pmatrix}, \; E^{\mu}_{\tilde{\rm s}|{\rm c}} = \begin{pmatrix} -h\nu\cos\theta_{\rm s} \\ -h\nu\sin\theta_{\rm s} \\ 0 \\ h\nu \end{pmatrix}, \tag{62}$$

with zero net linear momentum. On ring element ($\tilde{\rm c}$), with $\tilde{\mathbf{r}}_{\rm c} = -\mathbf{r}_{\rm c}$, another pair (s$\tilde{\rm s}$) is emitted, with four-vectors $E^{\mu}_{{\rm s}|\tilde{\rm c}}$ and $E^{\mu}_{\tilde{\rm s}|\tilde{\rm c}}$. Photons emitted on ring element (c) comply: (i) $\mathbf{u}_{\rm s} + \mathbf{u}_{\tilde{\rm s}} = 0$, and (ii) $\mathbf{r}_{\rm c} \times (\mathbf{u}_{\rm s} + \mathbf{u}_{\tilde{\rm s}}) = 0$; the same applies for ring element ($\tilde{\rm c}$) emitted photons. This microscopic description guarantees zero linear and angular momentum for energy emitted by heat at any level (ring elements, ring as a whole, thermal surroundings and universe).

Heat four-vector δQ^{μ} is then obtained by adding over $N_{\rm ph}/2$ opposite pairs of photons (s$\tilde{\rm s}$) emitted by ring elements,

$$\delta Q^{\mu} = \frac{1}{2} N_{\rm ph} \Sigma_{\rm c} \Sigma_{{\rm s}\tilde{\rm s}} (e^{\mu}_{\rm s} + e^{\mu}_{\tilde{\rm s}}) = (0, 0, 0, \dot{N}_{\rm ph} h\nu {\rm d}t)\,, \tag{63}$$

where $\delta Q = \dot{N}_{\rm ph} h\nu {\rm d}t$ is the energy emitted by thermal photons during time interval $[t_0, t_0 + {\rm d}t]$. Thermal photons rate emission is related to ring surface temperature $T \approx T_0$ (slightly above room's temperature) as $\dot{N}_{\rm ph}(T) \propto T^3$; with $\nu(T) \propto T$ (Wien law), power emitted is $\dot{E}_{\rm ph} = \dot{N}_{\rm ph} h\nu$, with $\dot{N}_{\rm ph} h\nu = \sigma A T^4$ (black body Stefan-Boltzmann law) [52].

Photons (s$\tilde{\rm s}$) emitted by element (c) and leaving the system have, respectively, angular impulse four-tensor:

$$M^{\mu\nu}_{\rm s|c} = r^{\mu}_{\rm c} \otimes E^{\nu}_{\rm s|c} =$$

$$= \begin{pmatrix} R\cos\theta_{\rm c} \\ R\sin\theta_{\rm c} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} h\nu\cos\theta_{\rm s} \\ h\nu\sin\theta_{\rm s} \\ 0 \\ h\nu \end{pmatrix} N_{\rm ph|c}\,, M_{\rm s|c} = \begin{pmatrix} 0 & M_{\rm z|ph;c} & 0 & G_{x|{\rm ph;c}} \\ -M_{\rm z|ph;c} & 0 & 0 & G_{y|{\rm ph;c}} \\ 0 & 0 & 0 & 0 \\ -G_{x|{\rm ph;c}} & -G_{y|{\rm ph;c}} & 0 & 0 \end{pmatrix},$$

$$M^{\mu\nu}_{\tilde{\mathrm{s}}|\mathrm{c}} = r^{\mu}_{\mathrm{c}} \otimes E^{\nu}_{\tilde{\mathrm{s}}|\mathrm{c}} =$$

$$= \begin{pmatrix} R\cos\theta_{\mathrm{c}} \\ R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} -h\nu\cos\theta_{\mathrm{s}} \\ -h\nu\sin\theta_{\mathrm{s}} \\ 0 \\ h\nu \end{pmatrix} N_{\mathrm{ph}|\mathrm{c}}\,, M_{\tilde{\mathrm{s}}|\mathrm{c}} = \begin{pmatrix} 0 & -M_{\mathrm{z}|\mathrm{ph;c}} & 0 & G_{x|\mathrm{ph;c}} \\ M_{\mathrm{z}|\mathrm{ph;c}} & 0 & 0 & G_{y|\mathrm{ph;c}} \\ 0 & 0 & 0 & 0 \\ -G_{x|\mathrm{ph;c}} & -G_{y|\mathrm{ph;c}} & 0 & 0 \end{pmatrix}.$$

Thermal photons ensemble describes radiated electromagnetic energy with zero net linear and angular momentum [53].

In frame S (thermal reservoir at rest), $M^{\mu\nu}_{\mathrm{ph}}$ four-tensor is obtained adding over $N_{\mathrm{ph}|\mathrm{c}}/2$ pairs of photons emitted by each element during time interval $[t_0, t_0 + \mathrm{d}t]$,

$$M^{\mu\nu}_{\mathrm{ph}} = \frac{1}{2} N_{\mathrm{ph}|\mathrm{c}} \Sigma_{\mathrm{c}} \Sigma_{\mathrm{s}\tilde{\mathrm{s}}} (M^{\mu\nu}_{\mathrm{s}|\mathrm{c}} + M^{\mu\nu}_{\tilde{\mathrm{s}}|\mathrm{c}}) = \begin{pmatrix} 0 & 0 & 0 & G_{x|\mathrm{ph}} \\ 0 & 0 & 0 & G_{y|\mathrm{ph}} \\ 0 & 0 & 0 & 0 \\ -G_{x|\mathrm{ph}} & -G_{y|\mathrm{ph}} & 0 & 0 \end{pmatrix}, \qquad (64)$$

with $N_{\mathrm{ph}} = \dot{N}_{\mathrm{ph}}(T)\mathrm{d}t$, $N_{\mathrm{ph}} = n_{\mathrm{r}} N_{\mathrm{ph}|\mathrm{c}}$, and four-tensor components

$$\begin{aligned} M_{z|\mathrm{ph}} &= 0\,, \\ G_{x|\mathrm{ph}} &= \Sigma_{\mathrm{c}} R\cos\theta_{\mathrm{c}} N_{\mathrm{ph}|\mathrm{c}} h\nu\,, \\ G_{y|\mathrm{ph}} &= \Sigma_{\mathrm{c}} R\sin\theta_{\mathrm{c}} N_{\mathrm{ph}|\mathrm{c}} h\nu\,, \end{aligned}$$

with $G_{x|\mathrm{ph}}\mathbf{i} + G_{y|\mathrm{ph}}\mathbf{j} = \Sigma_{\mathrm{c}} \mathbf{r}_{\mathrm{c}} N_{\mathrm{ph}|\mathrm{c}} h\nu$. Thermal photons do not contribute to the angular momentum equation (PER) or the linear momentum equation (NSL). They only contribute to the energy equation (FLT), locating the energy emitted as heat on each ring element.

8 Four-tensor momenta equation

The four-tensor momenta equation for a general T&R process is given by:

$$\mathrm{d}\Sigma^{\mathrm{c}=\mathrm{n_r}}_{\mathrm{c}=1} J^{\mu\nu}_{\mathrm{c}} = \Sigma^{\mathrm{c}=\mathrm{n_r}}_{\mathrm{c}=1} M^{\mu\nu}_{\mathrm{c}} \mathrm{d}t + \Sigma^{\mathrm{c}=\mathrm{n_r}}_{\mathrm{c}=1} M^{\mu\nu}_{\mathrm{ph}|\mathrm{c}} \mathrm{d}t\,,$$

which can be expressed as:

$$\mathrm{d}J^{\mu\nu} = (M^{\mu\nu} + M^{\mu\nu}_{\mathrm{ph}})\mathrm{d}t\,. \qquad (65)$$

In case these summations cannot be performed, the process cannot be reduced to a one-body process and the exercise is ill-posed. Eq. (65) in matrix form is as follows:

$$\mathrm{d}\begin{pmatrix} 0 & J_z & 0 & N_x \\ -J_z & 0 & 0 & N_y \\ 0 & 0 & 0 & 0 \\ -N_x & -N_y & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & M_z & 0 & G_x + G_{x|\mathrm{ph}} \\ -M_z & 0 & 0 & G_y + G_{y|\mathrm{ph}} \\ 0 & 0 & 0 & 0 \\ -G_x - G_{x|\mathrm{ph}} & -G_y - G_{y|\mathrm{ph}} & 0 & 0 \end{pmatrix} \mathrm{d}t$$

The ansatz given by Eq. (65) as a four-tensor momenta equation is a cause-effect relationship, enough to describe a well-posed rolling process. All the derived equations (NSL, PER, FLT) do not need to be written by hand, appearing naturally by matching components in the four-tensor momenta equation. In addition, Eq. (65) simultaneously relates:

(i) Linear impulse exerted on the system by external forces to linear momentum variation. Every force exerts linear impulse although net linear momentum could be zero.

(ii) Angular impulse (torque) exerted on the system by external forces to angular momentum variation. Some forces may not exert torque.

(iii) Work performed by external forces to system+surroundings energy variation. It is worth noting that surroundings exchange energy by work and heat with the system and that some forces may not perform work. In some (dissipative) processes, work performed is not completely transformed into mechanical energy, being dissipated as heat (see below).

About mechanics-thermodynamics equations. Problem-solving in (fresh and sophomore) physics courses barely recognises that the linear momentum variation of a system (NSL) determines its translational kinetic energy variation (by NSL-CDR) as a whole (centre-of-mass translational kinetic energy) and its angular momentum variation (PER) determines its rotational kinetic energy variation (by PER-CDR). Describing these mechanical aspects involves considering each force exerted on the system (conservative, restriction or dissipative forces). Thermal internal energy variation is not considered in these mechanics equations.

The following situations may arise when comparing work W performed on the system by conservative forces with mechanical kinetic energy variations ($\Delta K_{\rm cm}$ and $\Delta K_{\rm rt}$):

(i) Work performed on the system equals its mechanical energy variation $\Delta K_{\rm cm}+\Delta K_{\rm rt} = W$. Dissipative forces are absent and the process evolves with mechanical energy conservation. Energy variation in mechanical potential $E_{\rm p}$ (work reservoir) – exerting forces and supplying work as $W = -\Delta E_{\rm p}$ –, equals, with opposite sign, system kinetic energies variations, with $\Delta K_{\rm cm} + \Delta K_{\rm rt} + \Delta E_{\rm p} = 0$.

(ii) Work performed on the system is greater than system mechanical energy variation. Dissipative forces are present. The work performed by the work reservoir (mechanical potential) is dissipated by heat (or transformed into system internal thermal energy, varying its temperature), with $Q = {\rm p}W$.

(iii) Work performed on the system by an identifiable mechanical potential is less than system mechanical energy variation. Additional work comes, in particular, from a thermodynamic potential free energy, e.g., a decrease $-\Delta G$ in Gibbs free enthalpy function due to the presence of chemical reactions, i.e., by a thermal engine, a thermodynamics work reservoir, with $W \leq -\Delta G$ and $\Delta K_{\rm cm} + \Delta K_{\rm rt} \leq -\Delta G$ (see below).

In this description, the pseudo-work concept (pW), in linear translation and rotation, plays an essential role in the mechanical characterization of the process, and it should not be confused with genuine work performed by a mechanical potential energy variation or by a thermal engine.

Although tidy to apply (at least the first time), the four-tensor momenta equation articulates all these considerations; first working vertically, building one by one all four-vectors intervening in the process, and the corresponding four-tensors in order to pose the four-tensor momenta equation. Then, the characteristic equations (NSL, PER and FLT) will naturally appear horizontally. This formalism provides the correct solutions for well-posed problems with a systematic approach, avoiding an intuitive solving.

9 Rotating ring placed on floor

When the contact point between a rolling body and the floor has non-zero speed (relative to the ground), a frictional force $\mathbf{F}_{\rm R}$, to be phenomenologically described, comes into play. Force $\mathbf{F}_{\rm R}$ opposes to body-ground contact point displacement. Part of body mechanical energy is dissipated and thermal effects emerge. In our case, these effects will be considered as energy emitted by heat, increasing the entropy of the universe.

When the rolling condition is not fulfilled in a T&R process, thermal effects take place, with mechanical energy dissipated as heat [54]. Fig. 6 shows a sketch of a process in which a ring spinning with initial angular speed $\omega_{\rm I}(t_0)$, is placed on the floor, with zero linear velocity $v_{\rm I}(t_0) = 0$. The floor exerts force $F_{\rm R} = \mu_{\rm d} N$ (Amontons-Coulomb), and time interval $[t_0, t_0 + t^*]$ elapses until the ring reaches the rolling condition: linear speed v_0 and angular speed $\omega_0 < \omega_{\rm I}$ with $v_0 = R\omega_0$. During time interval $[t_0, t_0 + t^*]$, the ring centre travels distance x^*, with angular displacement θ^* and an amount of mechanical energy Q^* is dissipated as heat.

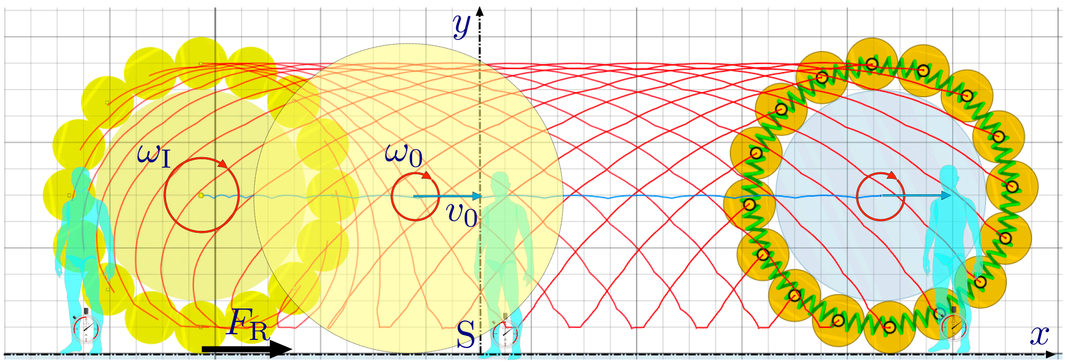

Figure 6: A spinning ring with angular velocity $\omega_{\rm I}$ is placed on the floor, with initial linear velocity $v_{\rm I} = 0$. The dynamical friction coefficient (ring-floor) is $\mu_{\rm d}$. The ground exerts frictional force $F_{\rm R}$. At time t^*, $v(t^*) \equiv v_0$, $\omega(t^*) \equiv \omega_0$, the ring reaches the rolling condition $v_0 = R\omega_0$. During the process, mechanical energy is dissipated. At $t > t^*$, no horizontal forces are exerted on the ring, with $v_0 = R\omega_0$.

In this problem, there are seven unknown magnitudes (i.e., the answers provided by nature to the posed problem): t^*, ω_0, v_0, θ^*, x^*, N, $F_{\rm R}$, Q^* (magnitudes with asterisk are those increasing during transient); by the rolling condition $v_0 = \omega_0 R$, six independent equations are needed (NSL equation provides two independent equations, one for spatial component, x and y; for y, $N = \mathcal{M}g$); the formalism must allow to obtain the remaining five equations and unknown quantities.

During time interval $[t_0, t_0 + t^*]$ there is relative displacement between ring edge-ground contact point and a frictional forces appears. The frictional force is opposed to ring rotation and favors ring linear translation; this force is considered to be an Amontons-Coulomb kind frictional force $\mathbf{F}_{\rm R} = (\mu_{\rm d} N, 0, 0)$ and increases ring centre linear speed, one case in which a frictional force promotes the movement of a body as a whole. Frictional force is parallel to ring centre motion direction, exerting linear and angular momentum, and not performing work. Since $\mathbf{G} + \mathbf{N} = 0$, gravitational and normal forces are not directly considered.

Four-vector fundamental equation and Poinsot-Euler rotation equation. For this process, the four-vector fundamental equation is: $E^{\mu}_{\rm f} - E^{\mu}_{\rm I} = W^{\mu}_{\rm R} + Q^{\mu}$. In its matrix form:

$$\begin{pmatrix} c\gamma_{v_0}\mathcal{M}\mathbf{v}_0 \\ \gamma_{v_0}\gamma(\omega_0 R)\mathcal{M}c^2 \end{pmatrix} - \begin{pmatrix} 0 \\ \mathcal{M}\gamma(\omega_{\rm I} R)c^2 \end{pmatrix} \approx \begin{pmatrix} c\mu_{\rm d}\mathcal{M}\mathbf{g}t^* \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Q_0 \end{pmatrix}. \qquad (66)$$

Until reaching the rolling condition, thermal effects (modeled as thermal photons) take place. From Eq. (66), NSL and FLT equations are obtained:

$$\begin{aligned}\gamma_{v_0}\mathcal{M}v_0 \approx \mu_{\rm d}\mathcal{M}gt^*\,,\\ \gamma_{v_0}\gamma(\omega_0 R)\mathcal{M}c^2-\gamma(\omega_{\rm I}R)\mathcal{M}c^2 \;\;=\;\; Q^*\,.\end{aligned}$$

For the NSL-CDR equation one has:

$$(\gamma_{v_0}-1)\mathcal{M}c^2=\mu_{\rm d}\mathcal{M}gx^*\,. \tag{67}$$

Finally, PER and its PER-CDR equations are:

$$\begin{aligned}I_{\rm R}[\gamma(\omega_0 R)\omega_0-\gamma(\omega_{\rm I}R)\omega_{\rm I}] \;\;&\approx\;\; -\mu_{\rm d}\mathcal{M}gRt^*\,,\\ \mathcal{M}[\gamma(\omega_0 R)-\gamma(\omega_{\rm I}R)]c^2 \;\;&=\;\; -\mu_{\rm d}\mathcal{M}gR\theta^*\,.\end{aligned}$$

Then $Q^*=-\mu_{\rm d}\mathcal{M}g(R\theta^*-x^*)$.

Four-tensor momenta equation. Phenomenological friction force $\mathbf{F}_{\rm R}$ is assumed to be distributed among ring elements in terms of two forces:

(i) Force $\mathbf{f}_{\rm R|t}=(f_{\rm R|t},0,0)$, with $f_{\rm R|t}=n_{\rm r}^{-1}\mu_{\rm d}\mathcal{M}g$, identical for each element, contributing to ring linear (centre) translation.

(ii) Force $\mathbf{f}_{\rm R|r;c}=(-f_{\rm R|r;c}\sin\theta_{\rm c}, f_{\rm R|r;c}\cos\theta_{\rm c},0)$, with $f_{\rm R|r;c}=f_{\rm R|t}$, which depends on element (c), contributing to ring rotation.

The force-power four-vectors associated to friction force $\mathbf{F}_{\rm R}$ distributed forces $\mathbf{f}_{\rm R|t}$ and $\mathbf{f}_{\rm R|r;c}$ are respectively:

$$\begin{aligned}\delta W^{\mu}_{\rm R|t}\equiv \dot W^{\mu}_{\rm R|t}{\rm d}t=\big(cf_{\rm R|t},0,0,0\big)\,{\rm d}t\,,\\ \delta W^{\mu}_{\rm R|r;c}\equiv \dot W^{\mu}_{\rm R|r;c}=\big(-cf_{\rm R|t}\sin\theta_{\rm c},\quad cf_{\rm R|t}\cos\theta_{\rm c},0,0\big)\,.\end{aligned} \tag{68}$$

These forces do not perform work.

Four-tensor momenta equation elements (c) and (c̃). Considering how forces have been distributed on ring elements, elements (c) and (c̃) four-tensor momenta equations (forces $\mathbf{G}$ and $\mathbf{N}$ are not considered) are given by:

$${\rm d}\begin{pmatrix}R\cos\theta_{\rm c}\\ R\sin\theta_{\rm c}\\ 0\\ ct_0\end{pmatrix}\otimes\begin{pmatrix}c\gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}[v+\omega R\sin\theta_{\rm c}]\\ -c\gamma(\omega R)\mathcal{M}_{\rm s}[\omega R\cos\theta_{\rm c}]\\ 0\\ \gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}c^2\end{pmatrix}=\begin{pmatrix}R\cos\theta_{\rm c}\\ R\sin\theta_{\rm c}\\ 0\\ ct_0\end{pmatrix}\otimes\begin{pmatrix}c(-f_{\rm R}-f_{\rm R|r}\sin\theta_{\rm c})\\ cf_{\rm R|r}\cos\theta_{\rm c}\\ 0\\ \dot N_{\rm ph|c}h\nu\end{pmatrix}{\rm d}t\,.$$

$${\rm d}\begin{pmatrix}-R\cos\theta_{\rm c}\\ -R\sin\theta_{\rm c}\\ 0\\ ct_0\end{pmatrix}\otimes\begin{pmatrix}c\gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}[v-\omega R\sin\theta_{\rm c}]\\ c\gamma(\omega R)\mathcal{M}_{\rm s}[\omega R\cos\theta_{\rm c}]\\ 0\\ \gamma_v\gamma(\omega R)\mathcal{M}_{\rm s}c^2\end{pmatrix}=\begin{pmatrix}-R\cos\theta_{\rm c}\\ -R\sin\theta_{\rm c}\\ 0\\ ct_0\end{pmatrix}\otimes\begin{pmatrix}c(-f_{\rm R}+f_{\rm R|r}\sin\theta_{\rm c})\\ -cf_{\rm R|r}\cos\theta_{\rm c}\\ 0\\ \dot N_{\rm ph|c}h\nu\end{pmatrix}{\rm d}t\,.$$

Observing these relationships it is not difficult to verify which forces contribute to ring centre linear translation (by $\dot W^{\mu}_{\rm c}+\dot W^{\mu}_{\tilde{\rm c}}$) and which to ring rotation (by $M^{\mu\nu}_{\rm c}+M^{\mu\nu}_{\tilde{\rm c}}$).

Adding over pairs (cc̃), the four-tensor momenta equation for the process described in Fig. 6 is:

$${\rm d}J^{\mu\nu}=(M^{\mu\nu}_{\rm R}+M^{\mu\nu}_{\rm ph}){\rm d}t\,. \tag{69}$$

From Eq. (69) the following equations are obtained:

PER. Angular-impulse–angular-momentum variation equation is:

$$\mathcal{M}R^2{\rm d}[\gamma(\omega R)\omega]\approx -\mu_{\rm d}\mathcal{M}gR{\rm d}t\,. \tag{70}$$

The corresponding PER-CDR equation is given by:

$$\mathcal{M}\mathrm{d}[\gamma(\omega R)c^2] = -\mu_\mathrm{d}\mathcal{M}gR\mathrm{d}\theta\,. \tag{71}$$

By integration of Eqs. (70)-(71) (and in the low-speed limit):

$$\begin{aligned} I_\mathrm{R}[\gamma(\omega_0 R)\omega_0 - \gamma(\omega_\mathrm{I} R)\omega_\mathrm{I})] &= -\mu_\mathrm{d}\mathcal{M}gRt^* \;\to\; I_\mathrm{R}(\omega_0 - \omega_\mathrm{I}) = -\mu_\mathrm{d}\mathcal{M}gRt^* & (72)\\ \mathcal{M}[\gamma(\omega_0 R) - \gamma(\omega_\mathrm{I} R)]c^2 &= -\mu_\mathrm{d}\mathcal{M}gR\theta^* \to \frac{1}{2}I_\mathrm{R}(\omega_0^2 - \omega_\mathrm{I}^2) = -\mu_\mathrm{d}\mathcal{M}gR\theta^*\,, & (73)\end{aligned}$$

with $\omega_0 < \omega_\mathrm{I}$. Initial ring rotational kinetic energy is transformed into ring translational kinetic energy by the frictional force; during transient interval, mechanical energy is dissipated as heat.

NSL. From Eq. (34), matching components with common factor ct_0, the linear-impulse–linear-momentum variation equation is as follows:

$$\mathcal{M}\mathrm{d}(\gamma_v v) \approx \mu_\mathrm{d}\mathcal{M}g\mathrm{d}t\,. \tag{74}$$

Its NSL-CDR is given by:

$$\mathcal{M}\mathrm{d}(\gamma_v c^2) = \mu_\mathrm{d}\mathcal{M}g\mathrm{d}x\,. \tag{75}$$

Integrating (low speed limit):

$$\begin{aligned} \mathcal{M}\gamma_{v_0}v_0 &= \mu_\mathrm{d}\mathcal{M}gt^* \;\to\; \mathcal{M}v_0 = \mu_\mathrm{d}\mathcal{M}gt^* & (76)\\ (\gamma_{v_0} - 1)\mathcal{M}c^2 &= \mu_\mathrm{d}\mathcal{M}gx^* \to \frac{1}{2}\mathcal{M}v_0^2 = \mu_\mathrm{d}\mathcal{M}gx^*\,. & (77)\end{aligned}$$

From Eqs. (72) and (76), by using the rolling condition $v_0 = R\omega_0$, the time lapse until reaching the rolling condition and final linear v_0 and angular ω_0 velocities are obtained. From Eqs. (73) and (77), the distance traveled by the ring centre and the angle rotated are obtained.

With moment of inertia $I_\mathrm{R} = \mathcal{M}R^2$ and $x_0 = R\theta_0$, then $\omega_0 \approx \omega_\mathrm{I}/2$.

$$K_\mathrm{rt}(\omega_\mathrm{I}) - K_\mathrm{rt}(\omega_0) = \frac{3}{4}\frac{1}{2}I_\mathrm{R}\omega_\mathrm{I}^2\,,\; K_\mathrm{rt}(\omega_0) = \frac{1}{4}\frac{1}{2}I_\mathrm{R}\omega_\mathrm{I}^2\,,\; K(v_0) = \frac{1}{4}\frac{1}{2}I_\mathrm{R}\omega_\mathrm{I}^2\,. \tag{78}$$

Final angular velocity ω_0 does not depend on friction coefficient. Time lapse t^*, distance x^* and angle θ^* until reaching the rolling condition are given, respectively, by [55]:

$$t^* \approx \frac{1}{2}\frac{\omega_\mathrm{I} R}{\mu_\mathrm{d} g}\,;\; x^* = \frac{1}{8}\frac{\omega_\mathrm{I}^2 R^2}{\mu_\mathrm{d} g}\,,\; \theta^* = \frac{3}{8}\frac{\omega_\mathrm{I}^2 R}{\mu_\mathrm{d} g}\,. \tag{79}$$

Time lapse $[t_0, t_0 + t^*]$ is inversely proportional to friction coefficient μ_d. It is worth mentioning that $R\theta^* > x^*$: initially the ring rotates with almost no translation (Fig. 6). For $t > t_0 + t^*$, the ring-floor contact force is zero: the ring-ground contact point speed is zero, with $v_0 - \omega_0 R = 0$: the ring rolls with constant linear and angular speeds (Galileo's principle of inertia or Newton's first law).

FLT. From Eq. (34), matching components with common factor $\mathbf{r}_\mathrm{c}$, one has:

$$\mathbf{r}_\mathrm{c}\left[\mathcal{M}_\mathrm{s}\mathrm{d}(\gamma_v\gamma(\omega R)c^2) = \delta Q_\mathrm{c}\right]. \tag{80}$$

Adding over ring elements:

$$\Sigma_\mathrm{c}\mathcal{M}_\mathrm{s}\mathrm{d}[\gamma_v\gamma(\omega R)c^2] = \Sigma_\mathrm{c}\delta Q_\mathrm{c}\,, \tag{81}$$

one obtains:

$$\mathcal{M}\mathrm{d}[\gamma_v \gamma(\omega R) c^2] = \delta Q \,. \tag{82}$$

By integration, with angular speed initial condition ω_I, the energy equation is:

$$\gamma_{v_0}\gamma(\omega_0 R)\mathcal{M}c^2 - \gamma(\omega_\mathrm{I} R)\mathcal{M}c^2 = Q^* \,. \tag{83}$$

By approaching:

$$\begin{aligned}\{[\gamma_{v_0}\gamma(\omega_0 R) - 1] - [\gamma(\omega_\mathrm{I} R) - 1]\}\mathcal{M}c^2 &\approx [\gamma(\omega_0 R) - \gamma(\omega_\mathrm{I} R)]\mathcal{M}c^2 + (\gamma_{v_0} - 1)\mathcal{M}c^2 \approx \\ &\approx \frac{1}{2} I_\mathrm{R}(\omega_0^2 - \omega_\mathrm{I}^2) + \frac{1}{2}\mathcal{M}v_0^2 \,,\end{aligned}$$

one has (in the low speed limit):

$$[\gamma(\omega_0 R) - \gamma(\omega_\mathrm{I} R)]\mathcal{M}c^2 + (\gamma_{v_0} - 1)\mathcal{M}c^2 \approx Q^* \;\to\; \frac{1}{2}\mathcal{M}v_0^2 + \frac{1}{2} I_\mathrm{R}(\omega_0^2 - \omega_\mathrm{I}^2) = Q^* \,, \tag{84}$$

with $Q^* \approx -\mu_\mathrm{d}\mathcal{M}g(R\theta^* - x^*)$.

For mechanical energy dissipated as heat during transient interval $[t_0, t_0 + t^*]$, one has [56]

$$Q^* = -\mu_\mathrm{d}\mathcal{M}g(R\theta^* - x^*) = -\frac{1}{4}\mathcal{M}\omega_\mathrm{I}^2 R^2 = -\frac{1}{2}\frac{1}{2} I_\mathrm{R}\omega_\mathrm{I}^2 \,, \tag{85}$$

with $R\theta^* > x^*$. Therefore, for the entropy of the universe variation $\Delta S^*_\mathrm{U} = -Q^*/T > 0$. Half of the initial rotational kinetic energy is dissipated as heat; a quarter transforms into translational kinetic energy and a quarter remains as rotational kinetic energy [54].

The process is irreversible: it is not allowed to completely transform all decreasing rotational kinetic energy into ring centre-of-inertia translational kinetic energy: part of this initial mechanical energy is dissipated as heat.

10 Fireworks ring ascending an incline

Dynamical and energetic aspects of rolling processes involving chemical reactions can be described as mechanical energy production processes through variations of thermodynamic potentials. In a chemical reaction taking place spontaneously – e.g., gunpowder consumption into the cartridges attached to a fireworks wheel (see Fig. 7)–, the decrease of chemical reaction Gibbs free enthalpy function could be fully used to obtain mechanical energy. The second law of thermodynamics states the potentiality and limits for this transformation, without disclosing how the actual thermodynamics potential decreasing-mechanical energy conversion can be carried out (see below).

Figure 7 sketches a diagram of a process for a ring attached with two cartridges (in opposite configuration) in which chemical reactions take place, the ring spinning and ascending the incline. The process takes place within pressure P atmosphere. Initial velocity conditions are $v_\mathrm{I} = 0$ and $\omega_\mathrm{I} = 0$. Chemical reaction could be: $2H_2 + O_2 \to 2H_2O$. Inside cartridges, n_ξ H_2 moles and $n_\xi/2$ O_2 moles are stored. Data for chemical reaction (molar internal energy variation Δu_ξ, volume variation Δv_ξ, enthalpy variation $\Delta h_\xi = \Delta u_\xi + P\Delta v_\xi$ and entropy variation Δs_ξ are available.

The incline-ring friction coefficient is $\mu_\mathrm{d} \geq \frac{1}{3}\mathrm{tg}\alpha$. The surface of the incline exerts force F_D on the ring, fulfilling the rolling condition. If this force is greater than the gravitational force component $G_x = \mathcal{M}g\sin\alpha$, which will depend on the chemical origin forces F_ξ exerted

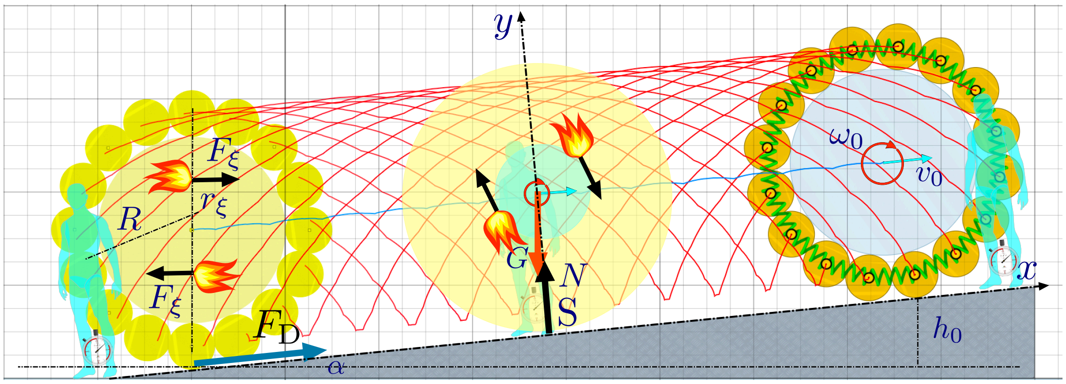

Figure 7: A ring ascending an angle α incline due to chemical origin forces F_ξ. When n_ξ moles of fuel have been consumed, ring centre speed is v_0, with angular velocity $\omega_0 = v_0/R$, distance traveled x_0 and, ascending height $h_0 = x_0 \sin\alpha$.

and, ultimately, on fuel consumption, and less than the maximum friction force component $R_x = \mu_\text{d}\mathcal{M}g\cos\alpha$, the ring will ascend the incline.

The ring mechanical state changes by varying its linear momentum – through the linear impulse applied on it – its angular momentum – through the angular impulse exerted on it – and its energy – through work performed on it –. The four-tensor momenta equation will relate the cause and effect of this process.

The following considerations should be taken into account in the analysis of this process:

(i) Final linear-momentum–energy four vector $E^\mu_\text{f}(v_0, \omega_0)$ depends on speeds v_0 and ω_0.

(ii) Initial linear-momentum–energy four vector $E^\mu_\text{f}(v_\text{I}, \omega_\text{I})$ depends on speeds v_I and ω_I.

(iii) Forces are simultaneously applied during time interval $[t_0, t_0 + t_\xi]$, until fuel exhaustion.

(iv) Force $\mathbf{G} = (-\mathcal{M}g\sin\alpha, -\mathcal{M}g\cos\alpha, 0)$ exerts linear impulse, no angular impulse and performs (negative) work. The linear-impulse–work four-vector associated to gravitational force depends on distance x_ξ traveled by ring centre, $W^\mu_\text{G}(x_\xi)$.

(v) Normal force $\mathbf{N} = (0, N, 0)$, with $N = \mathcal{M}g\cos\alpha$ exerts linear impulse, no angular impulse and performs no work.

Chemical forces. The closed end of a cartridge is hit by chemical reaction products moving at high speed (see Fig. 8). Since the cartridge is attached to the ring, chemical origin force F_ξ in each cartridge exerts linear and angular impulse and performs work on the ring. Given the cartridge's opposite configuration, their net linear impulse is zero, with no contribution to ring linear momentum variation. On the other hand, cartridges' angular impulse exerted and work performed on ring must be added.

From enthalpy variation $\Delta H_\xi = n_\xi \Delta h_\xi$ produced by chemical reactions – internal energy variation ΔU_ξ minus expansion work $W_P = -P\Delta V_\xi$ against the atmosphere –, a minimum amount of heat $Q^\text{mn} = n_\xi T\Delta s_\xi$ must be transferred to the thermal reservoir surrounding ring cartridges to ensure a non decreasing entropy of the universe variation during the process, with $n_\xi \Delta s_\xi + |Q^\text{mn}|/T = 0$. For a process in which heat exchanged is Q^mn, maximum available work $W_\xi = -n_\xi \Delta f_\xi$ is obtained, being $\Delta f_\xi = \Delta u_\xi - T\Delta s_\xi$ molar Helmholtz free energy function variation, and maximum mechanical energy production $\Delta E_\text{m} = -n_\xi \Delta g_\xi$ is achieved, being $\Delta g_\xi = \Delta u_\xi + P\Delta v_\xi - T\Delta s_\xi$ (i.e., $\Delta g_\xi = \Delta f_\xi + P\Delta v_\xi$) molar Gibbs free enthalpy function ($\Delta g_\xi = \Delta h_\xi - T\Delta s_\xi$) variation. The chemical force F_ξ magnitude depends

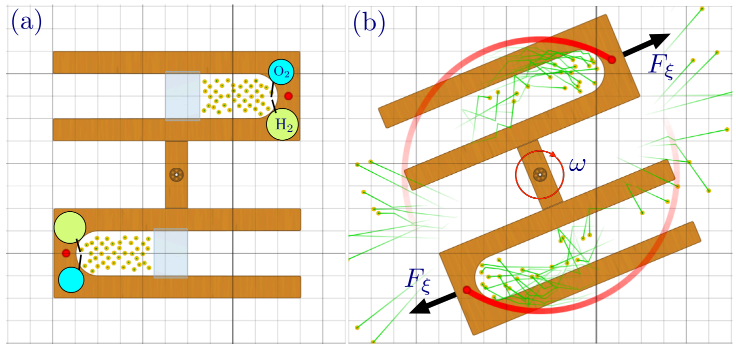

Figure 8: Two cartridges in which chemical reactions take place. (a) The reactants, stored inside two containers, are injected into the combustion chamber. (b) The forces exerted by chemical reaction products on the walls exert a torque, causing ring rotation.

on how chemical reactions are delivered inside cartridges. For example, two reservoirs, with chemicals H_2 and O_2 in stoichiometric proportions, are injected into the combustion chamber. By sparks, chemical reactions are produced. It is possible to imagine a process carried out with injection rates such that maximum work $-n_\xi \Delta f_\xi = 2F_\xi^{\rm mx} r_\xi \theta_{\rm mx} - P\Delta v_\xi$ is performed, or maximum mechanical energy variation on the ring is obtained as [12]:

$$2F_\xi^{\rm mx} r_\xi \theta_{\rm mx} = -n_\xi \Delta g_\xi \,, \tag{86}$$

with $d_\xi = r_\xi \theta_{\rm mx}$ distance traveled by force F_ξ application point, where $W_\xi = 2F_\xi^{\rm mx} d_\xi$ is work performed by chemical forces.

(vi) Chemical force $F_\xi^{\rm mx}$ exerts angular impulse, no linear impulse and performs work. The linear-impulse–work four-vector associated to chemical force depends on force F_ξ, with $W_\xi^\mu \equiv W_\xi^\mu(F_\xi^{\rm mx})$.

(vii) The rolling condition demanded force $F_{\rm D}$ exerts linear impulse, angular impulse and does not perform work. The linear-impulse–work four-vector associated to demanded force depends on force $F_{\rm D}$, with $W_\xi^\mu \equiv W_\xi^\mu({\rm D})$.

Torque four-tensors. In this case, gravitational and normal forces do not cancel each other out. Thus, their linear-impulse-work four-vectors must be obtained.

Gravitational+normal forces. Gravitational and normal forces only contribute to ring centre linear translation. To simplify calculations, force $\mathbf{F}_{\rm (G+N)}$ is defined, $f_{\rm (G+N)} = -\mathcal{M}_{\rm s} g \sin\alpha$, with linear-impulse–work four-vector:

$$\begin{aligned} \mathbf{f}_{\rm (G+N)|c} &= (-\mathcal{M}_{\rm s} g \sin\alpha, 0, 0)\,, \\ \delta W^\mu_{\rm (G+N)|c} \equiv \dot{W}^\mu_{\rm (G+N)|c} {\rm d}t &= (-c\mathcal{M}_{\rm s} g \sin\alpha, 0, 0, -\mathcal{M}_{\rm s} g v \sin\alpha){\rm d}t\,. \end{aligned}$$

Chemical origin forces. Chemical origin force $\mathbf{F}_\xi$ contributes to ring rotation. They are assumed to be distributed over ring elements such that $f_\xi = n_{\rm r}^{-1} R^{-1}(F_\xi^{\rm mx} r_\xi)$, with:

$$\begin{aligned} \mathbf{f}_{\xi|\rm c} &= (f_\xi \sin\theta_{\rm c}, -f_\xi \cos\theta_{\rm c}, 0)\,, \\ \delta W^\mu_{\xi|\rm c} \equiv \dot{W}^\mu_{\xi|\rm c} {\rm d}t &= (cf_\xi \sin\theta_{\rm c}, -cf_\xi \cos\theta_{\rm c}, 0, f_\xi R\omega){\rm d}t\,. \end{aligned}$$

Demanded force. This force $\mathbf{F}_{\rm D}$, demanded by the rolling condition, is assumed to be distributed over ring elements in terms of two forces:

1. Force $\mathbf{f}_{\mathrm{D}} = (f_{\mathrm{D}}, 0, 0)$, with modulus $f_{\mathrm{D}} = n_{\mathrm{r}}^{-1} F_{\mathrm{D}}$, identical for each element, contributing to ring centre translation,

$$\delta W^{\mu}_{\mathrm{D|t}} \equiv \dot{W}^{\mu}_{\mathrm{D|t}} \mathrm{d}t \quad = \quad (c f_{\mathrm{D}}, 0, 0, 0) \mathrm{d}t \,.$$

2. Force $\mathbf{f}_{\mathrm{D|r;c}} = (f_{\mathrm{D|r}} \sin\theta_{\mathrm{c}}, -f_{\mathrm{D|r}} \cos\theta_{\mathrm{c}}, 0)$, with modulus $f_{\mathrm{D|r}} = n_{\mathrm{r}}^{-1} F_{\mathrm{D}}$, depending on element (c), contributing to ring rotation.

$$\delta W^{\mu}_{\mathrm{D|r;c}} \equiv \dot{W}^{\mu}_{\mathrm{D|r;c}} \mathrm{d}t \quad = \quad (c f_{\mathrm{D|r}} \sin\theta_{\mathrm{c}}, -c f_{\mathrm{D|r}} \cos\theta_{\mathrm{c}}, 0, 0) \mathrm{d}t \,.$$

Four-vector fundamental equation and Poinsot-Euler equation. The four-vector fundamental equation for this process is: $E^{\mu}_{\mathrm{f}} - E^{\mu}_{\mathrm{I}} = W^{\mu}_{\mathrm{G+N}} + 2W^{\mu}_{\xi} + W^{\mu}_{\mathrm{D}}$, with

$$\begin{pmatrix} c\gamma_{v_{\mathrm{mx}}} \mathcal{M} \mathbf{v}_{\mathrm{mx}} \\ \gamma_{v_{\mathrm{mx}}} \gamma(\omega_{\mathrm{mx}} R) \mathcal{M} c^2 \end{pmatrix} - \begin{pmatrix} 0 \\ \mathcal{M} c^2 \end{pmatrix} \approx \begin{pmatrix} c(\mathbf{F}_{\mathrm{D}} - \mathcal{M} \mathbf{g} \sin\alpha) t_{\xi} \\ 2F^{\mathrm{mx}}_{\xi} r_{\xi} \theta_{\mathrm{mx}} - \mathcal{M} g \sin\alpha x_0 \end{pmatrix} ,$$

where t_{ξ} is fuel compsumption time interval $[t_0, t_0 + t_{\xi}]$.

Equations for NSL and FLT are:

$$\begin{aligned} \gamma_{v_{\mathrm{mx}}} \mathcal{M} v_{\mathrm{mx}} \quad &\approx \quad (F_{\mathrm{D}} - \mathcal{M} g \sin\alpha) t_{\xi} \,, \\ \big[\gamma_{v_{\mathrm{mx}}} \gamma(\omega_{\mathrm{mx}} R) - 1\big] \mathcal{M} c^2 \quad &= \quad 2F^{\mathrm{mx}}_{\xi} r_{\xi} \theta_{\mathrm{mx}} - \mathcal{M} g \sin\alpha x_{\mathrm{mx}} \,, \end{aligned}$$

For the NSL-CDR equation one has:

$$(\gamma_{v_{\mathrm{mx}}} - 1) \mathcal{M} c^2 \approx (F_{\mathrm{D}} - \mathcal{M} g \sin\alpha) x_{\mathrm{mx}} \,, \tag{87}$$

For this process PER equation and its PER-CDR equation are:

$$\begin{aligned} I_{\mathrm{R}} [\gamma(\omega_{\mathrm{mx}} R) \omega_{\mathrm{mx}}] \quad &\approx \quad (2F^{\mathrm{mx}}_{\xi} r_{\xi} - F_{\mathrm{D}} R) t_{\xi} \,, \\ \mathcal{M} [\gamma(\omega_{\mathrm{mx}} R) - 1] c^2 \quad &= \quad (2F^{\mathrm{mx}}_{\xi} r_{\xi} - F_{\mathrm{D}} R) \theta_{\mathrm{mx}} \,, \end{aligned}$$

with $2F^{\mathrm{mx}}_{\xi} r_{\xi} \theta_{\mathrm{mx}} = -n_{\xi} \Delta g_{\xi}$.

Four-tensor momenta equation. Considering the ring as the system, and considering the chemical reaction taking place through F_{ξ} forces, the angular momentum four-tensor for the ring will be $J^{\mu\nu}$, as stated above.

For fireworks ring elements (c) and (c) one has

$$\mathrm{d} \begin{pmatrix} R\cos\theta_{\mathrm{c}} \\ R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c\gamma_v \gamma(\omega R) \mathcal{M}_{\mathrm{s}} [v + \omega R \sin\theta_{\mathrm{c}}] \\ -c\gamma(\omega R) \mathcal{M}_{\mathrm{s}} [\omega R \cos\theta_{\mathrm{c}}] \\ 0 \\ \gamma_v \gamma(\omega R) \mathcal{M}_{\mathrm{s}} c^2 \end{pmatrix} = \begin{pmatrix} R\cos\theta_{\mathrm{c}} \\ R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c(f_{\mathrm{D}} - \mathcal{M}_{\mathrm{s}} g \sin\alpha + f_{\xi|\mathrm{r}} \sin\theta_{\mathrm{c}} - f_{\mathrm{D|r}} \sin\theta_{\mathrm{c}}) \\ c(-f_{\xi|\mathrm{r}} \cos\theta_{\mathrm{c}} + f_{\mathrm{D|r}} \cos\theta_{\mathrm{c}}) \\ 0 \\ f_{\xi} R\omega - \mathcal{M}_{\mathrm{s}} g v \sin\alpha \end{pmatrix} \mathrm{d}t \,.$$

$$\mathrm{d} \begin{pmatrix} -R\cos\theta_{\mathrm{c}} \\ -R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c\gamma_v \gamma(\omega R) \mathcal{M}_{\mathrm{s}} [v - \omega R \sin\theta_{\mathrm{c}}] \\ c\gamma(\omega R) \mathcal{M}_{\mathrm{s}} [\omega R \cos\theta_{\mathrm{c}}] \\ 0 \\ \gamma_v \gamma(\omega R) \mathcal{M}_{\mathrm{s}} c^2 \end{pmatrix} = \begin{pmatrix} -R\cos\theta_{\mathrm{c}} \\ -R\sin\theta_{\mathrm{c}} \\ 0 \\ ct_0 \end{pmatrix} \otimes \begin{pmatrix} c(f_{\mathrm{D}} - \mathcal{M}_{\mathrm{s}} g \sin\alpha - f_{\xi|\mathrm{r}} \sin\theta_{\mathrm{c}} + f_{\mathrm{D|r}} \sin\theta_{\mathrm{c}}) \\ c(f_{\xi|\mathrm{r}} \cos\theta_{\mathrm{c}} - f_{\mathrm{D|r}} \cos\theta_{\mathrm{c}}) \\ 0 \\ f_{\xi} R\omega - \mathcal{M}_{\mathrm{s}} g v \sin\alpha \end{pmatrix} \mathrm{d}t \,.$$

Observing these relationships it is not difficult to verify which forces contribute to ring centre linear translation (by $\dot{W}^{\mu}_{\mathrm{c}} + \dot{W}^{\mu}_{\tilde{\mathrm{c}}}$) and which to ring rotation (by $M^{\mu\nu}_{\mathrm{c}} + M^{\mu\nu}_{\tilde{\mathrm{c}}}$).

For the process described in Fig. 7, considering the ring as the system (without the fuel), and forces F_ξ as external forces to the system, the four-tensor momenta equation is given by:

$$\mathrm{d}J^{\mu\nu} = (M^{\mu\nu}_{(\mathrm{G+N})} + M^{\mu\nu}_{\xi} + M^{\mu\nu}_{\mathrm{D}})\mathrm{d}t\,. \tag{88}$$

Matching by components, the following equations are obtained.

PER. Angular-impulse–angular-momentum variation equation is:

$$\mathcal{M}R^2\mathrm{d}[\gamma(\omega R)\omega] \approx (2F^{\mathrm{mx}}_{\xi} r_\xi - F_{\mathrm{D}}R)\mathrm{d}t\,. \tag{89}$$

Its corresponding PER-CDR or pseudo-work rotational kinetic energy variation is:

$$\mathcal{M}\mathrm{d}[\gamma(\omega R)c^2] = (2F^{\mathrm{mx}}_{\xi} r_\xi - F_{\mathrm{D}}R)\mathrm{d}\theta\,. \tag{90}$$

By integrating (and in the low-speed limit):

$$\begin{aligned} I_{\mathrm{R}}\gamma(\omega_{\mathrm{mx}}R)\omega_{\mathrm{mx}} &= (2F^{\mathrm{mx}}_{\xi} r_\xi - F_{\mathrm{D}}R)t_\xi \;\rightarrow\; I_{\mathrm{R}}\omega_{\mathrm{mx}} = (2F^{\mathrm{mx}}_{\xi} r_\xi - F_{\mathrm{D}}R)t_\xi \\ \mathcal{M}[\gamma(\omega_{\mathrm{mx}}R) - 1]c^2 &= (2F^{\mathrm{mx}}_{\xi} r_\xi - F_{\mathrm{D}}R)\theta_{\mathrm{mx}} \rightarrow \frac{1}{2}I_{\mathrm{R}}\omega^2_{\mathrm{mx}} = (2F^{\mathrm{mx}}_{\xi} r_\xi - F_{\mathrm{D}}R)\theta_{\mathrm{mx}}\,. \end{aligned}$$

From four-tensors components relationship $\mathrm{d}N_x\mathbf{i} + \mathrm{d}N_y\mathbf{j} = G_x\mathrm{d}t\mathbf{i} + G_y\mathrm{d}t\mathbf{j}$ one has:

NSL. Common factor ct_0. Newton's second law

$$\mathcal{M}\mathrm{d}(\gamma_v v) \approx (F_{\mathrm{D}} - \mathcal{M}g\sin\alpha)\mathrm{d}t\,. \tag{91}$$

Its corresponding NSL-CDR or pseudo-work–translational kinetic energy equation is:

$$\mathcal{M}\mathrm{d}(\gamma_v c^2) = (F_{\mathrm{D}} - \mathcal{M}g\sin\alpha)\mathrm{d}x\,. \tag{92}$$

By integrating (and in the low-speed limit):

$$\begin{aligned} \mathcal{M}\gamma_{v_{\mathrm{mx}}}v_{\mathrm{mx}} &= (F_{\mathrm{D}} - \mathcal{M}g\sin\alpha)t_\xi \;\rightarrow\; \mathcal{M}v_{\mathrm{mx}} = (F_{\mathrm{D}} - \mathcal{M}g\sin\alpha)t_\xi \\ \mathcal{M}[\gamma_{v_{\mathrm{mx}}} - 1]c^2 &= (F_{\mathrm{D}} - \mathcal{M}g\sin\alpha)x_{\mathrm{mx}} \rightarrow \frac{1}{2}\mathcal{M}v^2_{\mathrm{mx}} = (F_{\mathrm{D}} - \mathcal{M}g\sin\alpha)x_{\mathrm{mx}}\,. \end{aligned}$$

For demanding force F_{D} one obtains:

$$F_{\mathrm{D}} = F^{\mathrm{mx}}_{\xi}\frac{r_\xi}{R} + \frac{1}{2}Mg\sin\alpha\,. \tag{93}$$

If the ring is to go up the incline, then $2F^{\mathrm{mx}}_{\xi} r_\xi/R > \mathcal{M}g\sin\alpha$. The required force F_{D} must be less than the friction force $F_{\mathrm{R}} = \mu_{\mathrm{d}}\mathcal{M}g\cos\alpha$ (maximum force that the incline can exert on the ring), i.e., $F_{\mathrm{D}} < F_{\mathrm{R}}$. Otherwise, the problem would not be well defined.

FLT. Matching terms with common factor $\mathbf{r}_{\mathrm{c}}$ for element (c):

$$\mathbf{r}_{\mathrm{c}}\big\{\mathcal{M}_{\mathrm{s}}\mathrm{d}[\gamma_v\gamma(\omega R)c^2] = -\mathcal{M}_{\mathrm{s}}g\sin\alpha\, v\mathrm{d}t + 2f_\xi R\,\omega\mathrm{d}t\,\big\}\,.$$

Adding over ring elements, with $v\mathrm{d}t = \mathrm{d}x$, $\omega\mathrm{d}t = \mathrm{d}\theta$:

$$\mathcal{M}_{\mathrm{s}}\mathrm{d}[\gamma_v\gamma(\omega R)c^2] = -\mathcal{M}_{\mathrm{s}}g\sin\alpha\mathrm{d}x + 2f_\xi R\mathrm{d}t\theta\,,$$

and ring energy equation is given by:

$$\mathcal{M}\mathrm{d}[\gamma_v\gamma(\omega R)c^2] = -\mathcal{M}g\sin\alpha\mathrm{d}x + 2F^{\mathrm{mx}}_{\xi} r_\xi\mathrm{d}\theta\ .$$

By using the following approximation:

$$\mathrm{d}[\gamma_v \gamma(\omega R) c^2] \approx \gamma(\omega R)\mathrm{d}(\gamma_v - 1)] c^2 + \gamma_v \mathrm{d}[\gamma(\omega R) - 1] c^2 ,$$

and by integration, one obtains for the ring energy equation:

$$\mathcal{M}(\gamma_{v_{\mathrm{mx}}} - 1)c^2 + \mathcal{M}[\gamma(\omega_{\mathrm{mx}} R) - 1]c^2) = -\mathcal{M} g \sin\alpha x_{\mathrm{mx}} + 2F_\xi^{\mathrm{mx}} r_\xi \theta_{\mathrm{mx}} \ .$$

In the low-speed limit:

$$\begin{aligned} \mathcal{M}\{[\gamma(\omega_{\mathrm{mx}} R) - 1] + (\gamma_{v_{\mathrm{mx}}} - 1)\}c^2 + \mathcal{M} g \sin\alpha x_{\mathrm{mx}} &\approx 2F_\xi^{\mathrm{mx}} r_\xi \theta_{\mathrm{mx}} \ \rightarrow \\ \frac{1}{2}\mathcal{M} v_{\mathrm{mx}}^2 + \frac{1}{2} I_{\mathrm{R}} \omega_{\mathrm{mx}}^2 + \mathcal{M} g h_{\mathrm{mx}} &= 2F_\xi^{\mathrm{mx}} r_\xi \theta_{\mathrm{mx}} \, . \end{aligned}$$

with maximum height $h_{\mathrm{mx}} = \sin\alpha x_{\mathrm{mx}}$.

Fuel consumption equation. To relate forces F_ξ to fuel consumption n_ξ, data on the use of the reagents would be needed, how they are injected, etc. Thermodynamics allows to obtain the minimum fuel consumption to achieve a given force. In practice, more fuel will be needed than the minimum to get the same performance.

Once the amount of fuel (e.g., n_ξ moles) is known, the maximum speed v_{mx} achieved can be obtained, with $\omega_{\mathrm{mx}} R = v_{\mathrm{mx}}$, and ascended height h_{mx}. For fuel consumption, the following equation is obtained:

$$\begin{aligned} \mathcal{M}(\gamma_{v_{\mathrm{mx}}} - 1)c^2 + \mathcal{M}[\gamma(\omega_{\mathrm{mx}} R) - 1]c^2 &+ \mathcal{M} g h_{\mathrm{mx}} = -n_\xi \Delta g_\xi \ \rightarrow \\ \frac{1}{2}\mathcal{M} v_{\mathrm{mx}}^2 + \frac{1}{2} I_{\mathrm{R}} \omega_{\mathrm{mx}}^2 + \mathcal{M} g h_{\mathrm{mx}} &= -n_\xi \Delta g_\xi \, . \\ \frac{1}{2}\mathcal{M} v_{\mathrm{mx}}^2 + \frac{1}{2} I_{\mathrm{R}} \omega_{\mathrm{mx}}^2 + \mathcal{M} g h_{\mathrm{mx}} + P n_\xi \Delta v_\xi &= -n_\xi \Delta f_\xi \, . \end{aligned}$$

Mechanical energy production (i.e., translational kinetic energy K_{cm}, rotational kinetic energy K_{rt} and ring-Earth gravitational potential energy $E_{\mathrm{p}} \approx \mathcal{M} g h_{\mathrm{mx}}$) during the process comes from the decrease of chemical reaction Gibbs free enthalpy function.

As stated above, whether $F_\xi = F_\xi^{\mathrm{mx}}$, the process is reversible, with zero entropy of the universe variation. The produced mechanical energy can be used to obtain electricity and then hydrolysing n_ξ H_2O moles to obtain n_ξ H_2 and $n_\xi/2$ O_2 moles. For a real-life process, $F_\xi \leq F_\xi^{\mathrm{mx}}$. In general

$$\begin{aligned} \mathcal{M}(\gamma_{v_\xi} - 1)c^2 + \mathcal{M}[\gamma(\omega_\xi R) - 1]c^2 &+ \mathcal{M} g h_\xi \leq -n_\xi \Delta g_\xi \ \rightarrow \\ \frac{1}{2}\mathcal{M} v_\xi^2 + \frac{1}{2} I_{\mathrm{R}} \omega_\xi^2 + \mathcal{M} g h_\xi &\leq -n_\xi \Delta g_\xi \, , \end{aligned}$$

with real-life values $v_\xi \leq v_{\mathrm{mx}}$, $\omega_\xi \leq \omega_{\mathrm{mx}}$ and $h_\xi \leq h_{\mathrm{mx}}$.

11 Conclusions

The equations bequeathed to us by physics founding ancestors deserve to be updated, precisely with the knowledge they have contributed to the gain. Einstein's special theory of relativity helps present equations such as Newton's second law, the Poinsot-Euler equation for rotation and the first law of thermodynamics differently, integrating them into a single covariant four-tensor momenta equation. These equations comply with the postulates of relativity.

In classical physics, the rigid body concept, with constant inertia and moment of inertia, allows to obtain equations describing linear translation with rotation processes directly. In relativity, on the one hand, the rigid-body hypothesis is not valid. A classical rigid-body must be replaced by a set of solid elements connected by springs. The inertia of energy principle implies a variation of the inertia and moment of inertia for a body throughout a rolling process. On the other hand, being relativity a local theory, it requires an identification of all forces applied to each element of the body, or in other words, all the elements on which the forces are applied must be identified.

This paper has developed a relativistic formalism to provide a four-tensor momenta equation to describe a rolling process. This formalism has been applied to three T&R processes: 1) a ring moving horizontally by the action of a force exerting a torque, with mechanical energy conservation, 2) a spinning ring placed on the floor and rolling until the rolling condition is reached, a process evolving with mechanical energy dissipation as heat, and 3) a process in which a fireworks ring ascends an incline, in a mechanical energy production process from chemical reactions, with Gibbs free enthalpy function decreasing.

The four-tensor momenta equation formalism fulfills relativistic requirements: the concept of rigid body, recurrent in classical physics, is not used and the principle of locality, dealing with distributed forces on ring elements, is fulfilled.

Three equations are obtained:

1. The equation for vectors $\mathbf{J}$ and Γ, involving linear-momentum, through the cross-product of position and linear-momentum vectors, i.e., $\mathbf{J} \equiv \mathbf{r} \times \mathbf{p}$, and a force momentum (torque), via the cross-product of position and force vectors, i.e., $\Gamma \equiv \mathbf{r} \times \mathbf{F}$. Poinsot-Euler rotation law is obtained from this equation, relating the body's angular-momentum variation to external forces angular-impulse.

2. The equation for the ct_0-momenta of both linear-momentum $\mathbf{p}$ and resultant force $\mathbf{F}$, in which the corresponding lever arm, required to obtain momenta, is not a vector but a scalar, being its value ct_0. This equation provides Newton's second law equation for the linear-momentum variation due to resultant external force. This equation checks whether the problem-solving approach considers that forces must be applied simultaneously in the appropriate reference frame (for well-posed processes, a frame must exist where this can be done).

3. The equation for energy E and work W momenta, both scalar, whose lever arm is the element position vector. Once the fulfilment of the local character of relativity is checked (work performed by external forces is used to vary the kinetic energy of the element on which they are applied), the first law of thermodynamics (energy balance equation) is obtained by adding over elements equations.

The four-tensor formalism allows to avoid posing cause-effect equations directly for a process. Instead, linear-momentum-energy four-vectors and linear-impulse-work four-vectors are obtained vertically, and mathematical operations are performed to obtain the corresponding classical equations horizontally, by equaling four-tensors components.

Thus, when a problem is well posed and the process is developed in rigid rolling, it is possible to obtain a four-tensor momenta equation describing the process. The four-tensor momenta equation contains all the information needed to describe a process. Directly, through comparison between four-tensor components, (i) the angular-impulse–angular-momentum variation equation – the Poinsot-Euler rotation equation –, (ii) the linear-impulse–linear-momentum variation equation – Newton's second law equation –, and (iii) the energy equation – the first law of thermodynamics –, are obtained. From these, the Poinsot-Euler rotation equation complementary dynamical relationship is also obtained – i.e., the pseudo-work–rotational kinetic energy variation –, and Newton's second law complementary dynamical relationship – i.e., the pseudo-work–translational kinetic energy variation –. Heat equation, and the entropy of the universe variation equation are obtained by comparing the information provided by the previous two with the information provided by the first law of thermodynamics.

The first time this formalism is applied, it will probably seem rather tedious. This circumstance is due, on the one hand, to the relativistic demand of having to identify each component constituting the system, from a proton in an atomic nucleus to the solid elements composing the body, as well as all the forces applied to the elements composing the body, and, where necessary, each emitted photon. On the other hand, the inertia of energy principle introduces cross relations between magnitudes describing linear translation and those representing rotation.

Symmetry and considerations from the statistical mechanics can reduce a many-body problem to a one-body problem with just one linear and one angular velocity. In successive applications of the formalism, obtaining the equations describing the process will be faster and simpler.

In the low-speed limit, the classical equations for the processes are obtained, and linear translation is decoupled from rotation.

Finally, the four-tensor momenta equation formalism allows the equations describing the process to be easily obtained, (i) when the momenta origin point is changed, with direct application of the parallel axes, or (ii) when the chosen reference frame is changed to another frame moving in the standard configuration, by means of the Lorentz transformation.

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