Multiplicative problem-solving strategies used by students with autism spectrum disorder

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Students with autism spectrum disorder (ASD) are found to use less efficient problem-solving strategies than their typically developing peers. This work examines the strategies used by 17 first-to-fourth graders with ASD without intellectual disability when solving multiplicative equal-groups problems. The participants mainly resorted to incorrect or low-level strategies, such as modelling. Within the first-to-third graders, all the strategies observed were low-level ones. Within the fourth graders, four of the eight students used operation strategies to solve the multiplication problem and only three to solve the partitive division one. The measurement-division problem was the most difficult for them to solve. These results highlight the difficulties faced by students with ASD when solving multiplicative problems and could guide future design of specific instruction.

Keywords: Autism Spectrum Disorders, Problem Solving, Special Needs Students, Multiplicative Word Problems (Mathematics)

Introduction

Facing word problems requires not only mathematical skills but also reading, comprehension, modelling, abstraction and reasoning abilities (Daroczy et al., 2015). Both conceptual and procedural knowledges need to be brought into play to analyse each situation, assess reasonable options and choose the most convenient strategy to solve the problem. Typically developing (TD) children often experience difficulties when posed to a problem (Daroczy et al., 2015). These difficulties are significantly increased in those cases in which the student shows a low profile in executive function or language ability (Fuchs & Fuchs, 2007), traits that are frequently present in students with autism spectrum disorder (ASD; Hart Barnett & Cleary, 2015).

According to the *Diagnostic and Statistical Manual of Mental Disorders* (DSM-V; American Psychiatric Association, 2013), the ASD is a construct used to describe individuals who typically present: (a) persistent impairments in communication and social interactions, and (b) restrictive and repetitive patterns of behaviour, interests or activities. These symptoms manifest during the first years of life and last throughout the entire life cycle. Contrary to some beliefs about ASD and mathematical giftedness, empirical research detects that about 20% of students with ASD present some kind of mathematical learning disability (Mayes & Calhoun, 2006).

In particular, evidence-based studies show that students with ASD use less efficient strategies when solving mathematical word problems than their TD peers, which could be due to factors as atypical language development, deficits in executive functioning (such as organization, working memory or metacognition) or a lower profile of theory of mind (Polo-Blanco et al., 2022). Therefore, knowing in detail the problem-solving strategies of students with ASD gives information about their reasoning and difficulties they face, which can be taken into account for the design of adapted instructions for teaching problem solving to these students (e.g., Root et al., 2022).

Multiplicative problem-solving strategies

Traditionally, one-operation mathematical problems with a multiplicative structure (i.e., those that require a multiplication or division operation to be solved) are classified into three types, namely multiplicative equal groups, comparison and Cartesian product (Nesher, 1992). Equal-groups problems are the focus of the present work. They pose situations involving sets of the same number of elements (e.g., *Sally has 2 bags and 10 marbles in each bag, how many marbles does she have?*). Additionally, within the equal-groups problems, different types are distinguished according to the operation needed to solve them, namely multiplication, partitive division and measurement division (Mulligan & Mitchelmore, 1997). According to Nesher (1992), equal-groups problems are the easiest for the students to solve, following by the multiplicative comparison ones. Cartesian product problems are shown to be the most difficult ones for primary-school students (Ivars & Fernández, 2016; Nesher, 1992). Measurement-division problems present more difficulties than partitive division and multiplication ones for both TD children (Ivars & Fernández, 2016) and children with ASD (Goñi-Cervera et al., 2023).

Children use a wide range of informal strategies, understood as those commonly used before receiving formal education (Ginsburg & Baroody, 2003). Research with TD children on multiplicative problem-solving strategies shows a progression with age towards the use of more sophisticated strategies. Mulligan and Mitchelmore (1997) identified three levels of strategies in second and third graders when solving multiplicative problems: (1) direct modelling with counting, (2) counting and (3) use of known or derived facts. Incorrect answers were shown to be the result of adopting unreasoned strategies, such as adding the two given numbers, or inappropriate models for the situation posed by the problem. Moreover, Ivars and Fernández (2016) analysed the strategies used by students from 6 to 12 years old when solving multiplicative problems. Direct modelling with counting and counting were the most common strategies among the students aged between 6 and 8, while students in the older age group were more inclined to use algorithmic strategies. The presence of more sophisticated strategies based on known or derived multiplicative facts was not a guarantee that they correctly solved the problem, because some of them used the reverse algorithm.

Research on problem solving with students with ASD is mainly focused on instructional methodologies (e.g., Hart Barnett & Cleary, 2015; Root et al, 2022). By contrast, the strategies used by these students have been little studied (Bae et al., 2015). Polo-Blanco et al. (2022) examined relationships between mathematical problem-solving strategies and the main cognitive domains associated with mathematical learning in 6 to 12-year-old children with and without ASD. They found a higher percentage of children with ASD showing difficulties in problem solving than that obtained within the non-ASD group (57% vs. 23%). Children with ASD in the poorest performing group used less elaborated strategies (such as those based on drawing or counting) than the rest of children with ASD. Van Vaerenbergh et al., (2022) examined the strategies used by 26 students with ASD when solving Cartesian product problems. The authors concluded that the problems were proved to be difficult for the students, with the additive strategy being the most frequent incorrect one. The work by Polo-Blanco et al. (2019) describes the strategies and representations used by a student with ASD when solving equal-groups problems. Before receiving formal instruction on division, the student found measurement-division problems easier to solve than partitive division ones. In this context, the student preferred direct modelling with counting strategies.

Some of the authors presenting this paper analysed in a previous study the strategies used by 10 students with ASD, with and without intellectual disability, between 8 and 13 years old when solving equal-groups problems (Goñi-Cervera et al., 2023). Most students used low-level strategies, such as counting, with little use of formal multiplication and division. The present work aims to describe the strategies that 17 students with ASD without intellectual disability use when solving equal-groups problems, as these are the easiest to solve of those with multiplicative structure (Nesher, 1992). In particular, we pose the following research questions:

(a) Do these students with ASD use more low-level strategies than TD children as they progress through the grades?
(b) Are differences observed when they address different types of equal-groups problems according to the operation needed to solve them?

Methodology

This work is a case study that adopts a descriptive and explorative approach to analyse the problem-solving strategies used by 17 participants with ASD. It allows us to delve into the details of particular problem-solving processes in a way that would become unfeasible with larger samples.

Participants

The present study was carried out with a group of 17 students without intellectual disability (IQ ≥ 70), who had been diagnosed with ASD (and showed no evidence of another psychiatric comorbidity) in mental health units, though parental interview and patient evaluation based on DSM-V criteria. They were recruited between 2019 and 2021 from 12 different mainstream schools in the Spanish region of Cantabria. At the time of the research, they were enrolled in the first four grades of Primary Education. The students were given codes for preserving their anonymity, being assigned from S1 to S17 in increasing order of age (see Table 1). In particular, from S1 to S5 were first graders with ages from 6.25 to 6.67; S6 was the only student in second grade being 6.75 years old; while S7, S8 and S9 were third graders with ages between 8.3 and 8.5 years old. The rest of participants (from S10 to S17) were fourth graders with ages between 8.75 and 10.83 years old. All participants were males, with the exception of S3 and S8.

Within the mentioned criteria, the sample is an extension of that used in Goñi-Cervera et al. (2023). In particular, there are four students (S7, S10, S13 and S17, according to the codes given in the present study; corresponding with S1, S2, S3 and S7 in Goñi-Cervera et al., 2023) belonging to both samples, because they match the criteria and were recruited at the time this previous research was carried out. The remaining 13 students within the present sample were recruited later.

Information gathering tool

Based on previous studies (Mulligan & Mitchelmore, 1997), each student was asked to solve eight multiplicative word problems in a classroom with no distractions. A guided pencil-test was used, comprising a blend of three equal-groups, three comparison and two Cartesian-product problems with the unknown located in different places. In this work, we focus on the results from the three equal-groups problems. Depending on the location of the unknown, three types of equal-groups problems are considered:

- P1 (multiplication): *There are 2 tables in the classroom, and 4 children are sitting at each table. How many children are there in the classroom?*
- P2 (partitive division): *There are 10 children and 2 tables in the classroom. If the same number of children sits at each table, how many children are sitting at each table?*
- P3 (measurement division): *There are 15 toys in the classroom to distribute equally among several children. If each child has received 3 toys, how many children are there in the classroom?*

To help the student to focus on the task, the interviewer first gave him/her a stapled booklet with a single problem on each page and an ample blank space below to solve it. This is intended to help with possible attention deficits characteristic of people with ASD. In addition, to compensate potential verbal comprehension difficulties, the student was asked to read each problem aloud, with the help of the interviewer if needed. Then, the student was asked to find the solution and was told that he/she could write, draw, use manipulatives (interlocking blocks) or answer orally. The process was videotaped and later transcribed for analysis, together with what was collected in the booklet.

Analysis categories

Following Goñi-Cervera et al. (2023), we employ a 4-level system for classifying the strategies used to solve the equal-groups problems. Level 0 corresponds to incorrect strategies. We include here strategies both unrelated (such as supplying a random number or one of the numbers present in the statement) and inconsistent with the situation posed by the problem (e.g., the use of the inverse algorithm or inappropriate additive relationships). Level 1 is direct modelling with counting. We include here strategies involving concrete manipulatives or drawings to model the problem situation. Objects are counted with no clear reference to the multiplicative structure. Same counting strategies without making use of manipulatives belong to level 2. Finally, operation strategies make up the level 3, those in which an explicit use of the multiplication and division operations are shown.

Results

Table 1 shows the solutions provided by the students to each problem and the strategies they used.

Table 1: Students' solutions and solving strategies

	P1		P2		P3	
Code	Answer	Strategy level	Answer	Strategy level	Answer	Strategy level
S1	4*	L_0	3*	L_0	15*	L_0
S2	6*	L_0	12*	L_0	18*	L_0
S3	6*	L_0	12*	L_0	18*	L_0
S4	-	L_0	5	L_1	10*	L_0
S5	8	L_2	20*	L_0	5*	L_0
S6	4*	L_0	5	L_2	15*	L_0

S7	6*	L_0	5	L_1	12*	L_0
S8	6*	L_0	6*	L_0	18*	L_0
S9	6*	L_0	12*	L_0	17*	L_0
S10	6*	L_0	5	L_1	18*	L_0
S11	8	L_3	5	L_3	12*	L_0
S12	2+4*	L_0	12*	L_0	-	L_3
S13	8	L_3	5	L_3	45*	L_0
S14	8	L_3	20*	L_0	46*	L_0
S15	8	L_3	5	L_3	45*	L_0
S16	6*	L_0	12*	L_0	28*	L_0
S17	6*	L_0	12*	L_0	18*	L_0

Note: * = incorrect result; - = no response.

Multiplication problem

Only five students (S5, S11, S13, S14 and S15) solved correctly the problem P1. Four of them are fourth graders who did the multiplication: "4 x 2 = 8", while S5 is a first grader who used a repeated-addition reasoning: "Because they are four plus four, eight". Among the 12 wrong answers, two students (S1 and S6, first and second graders, respectively) were limited to supplying one of the numbers given in the problem statement, while other nine participants used an inappropriate addition of the two numbers given by the problem: "2 + 4 = 6", except one of them (S12) who made a mistake when performing the sum and answered 8 by chance. Finally, one participant (S4) did not give an answer. Figure 1 shows three examples of different strategies adopted to solve the problem P1.

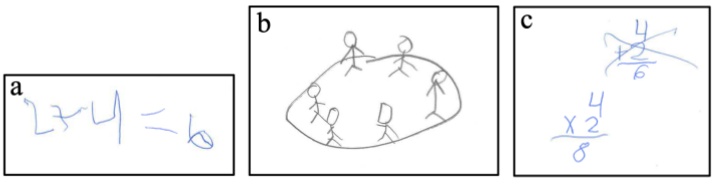

Figure 1: Examples of students' answers to problem P1, from left to right: inappropriate addition with operation (S3) and with drawings (S17), and multiplication strategy after discarding an addition (S11)

Partitive division problem

Seven students were able to solve the problem P2, of which three (S4, S7 and S10) used informal counting strategies based on concrete material. In particular, they used 10 blocks that they separated into two groups of five. Additionally, S7 drew a picture of ten people under (and in correspondence with) the row of blocks (see Figure 2a). Another student (S6) mentally calculated the answer ("every five") without providing any further explanation. The remaining three (S11, S13 and S15) were in fourth grade, and performed a division. In particular, when asked to explained his oral answer, S13 said that "five times two equals ten", and this was what he wrote in the booklet. Among the 10 wrong answers, two participants (S5, S14) applied the inverse algorithm multiplying the given numbers and six students (S2, S3, S9, S12, S16 and S17) made an inappropriate addition of the given numbers. Finally, S1 and S8 seemed to experiment the consequences of implicitly adopting this last strategy. S1 made a repeated addition with groups of three children ("three plus three, six; six plus three, nine; and only one, ten"); while S8 tried to perform a distribution by divisors but he failed because he took as the total number of children the sum of the given numbers (10 + 2). Figure 2 collects three examples of different strategies used to solve the problem P2.

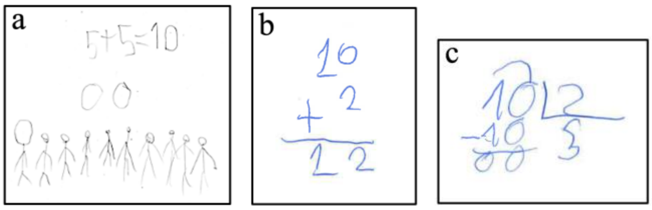

Figure 2: Examples of students' answers to problem P2, from left to right: modelling with counting with blocks (S7), an inappropriate addition (S16) and a division (S11)

Measurement division problem

None of the students managed to solve correctly the problem P3. Apparently, S12 was the only student who used an appropriate strategy. In particular, he aimed to divide 15 by three, but he wrote several solutions so that we were unable to understand his reasoning (see Figure 3b). Two students (S1 and S6) answered with a number given by the problem. Nine students made an inappropriate addition: two of them (S7 and S11) thought it necessary to subtract the given numbers, while the other seven participants (S2, S3, S8, S9, S10, S16, S17) chose to add them (S9 and S16 performed the addition incorrectly). Three fourth graders (S13, S14 and S15) used the inversed algorithm (they multiplicated 15 by 3). In particular, S14 employed the blocks to perform a multiplication and made counting errors providing an incorrect result. Finally, S4 wrote a random number, and S5 appeared to give the right answer but when asked to justify it he alleged that "15 is equal to 5, if I put a 1 to 5 it could be 15". Three examples of strategies are shown in Figure 3.

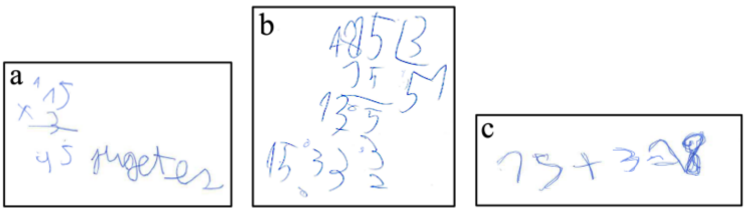

Figure 3: Examples of students' answers to problem P3, from left to right: inversed algorithm (S13), unfinished division (S12) and inappropriate addition (S3)

Discussion and conclusions

This study provides a description of the strategies used by 17 students with ASD without intellectual disability when solving equal-groups problems. Our results supplement those obtained by previous studies focused on mathematical problem solving with students with ASD (Goñi-Cervera et al., 2023; Polo-Blanco et al., 2019; 2022). The measurement-division problem revealed to be the most difficult one, which is consistent with that found with both TD children (Ivars & Fernández, 2016) and students with ASD (Goñi-Cervera et al., 2023). Only three students were able to solve correctly both P1 and P2, and they used operation strategies.

In general, in accordance with what was observed by Goñi-Cervera et al. (2023), the participants of this study generally used low-level strategies, such as modelling, or incorrect strategies. All the strategies observed within the first-to-third graders were low-level ones, but this was also observed in studies with TD students (Ivars & Fernández, 2016; Mulligan & Mitchelmore, 1997). However, the operation strategies were adopted by fewer fourth graders than expected, in particular by four of the eight fourth graders for solving the multiplication problem and only by three of them for the partitive-division problem. Among all participants, only one fourth grader managed to identify a division for solving the measurement-division problem but he failed in its resolution. Surprisingly, he was not one of the students who used an operation strategy to solve P1 and P2.

Our results are consistent with other studies which show that some students with ASD struggle with mathematical problems, which have been found to be related to characteristics of the disorder, such as low level of executive functioning or language comprehension (Bae et al., 2015; Polo-Blanco et al., 2022). The detail of the students' strategies gives information about their understanding of the multiplication and division operations and possible difficulties they face, which can shed light on the aspects in which the instruction should be focused. This analysis is limited by not taking into account cognitive factors such as IQ or level of verbal comprehension, which might provide performance profiles that could explain the difficulties observed in some of the participants. Future research could explore relationships between verbal comprehension and understanding of the key words of the statements of multiplicative structure problems (e.g., "each" or "equally").

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